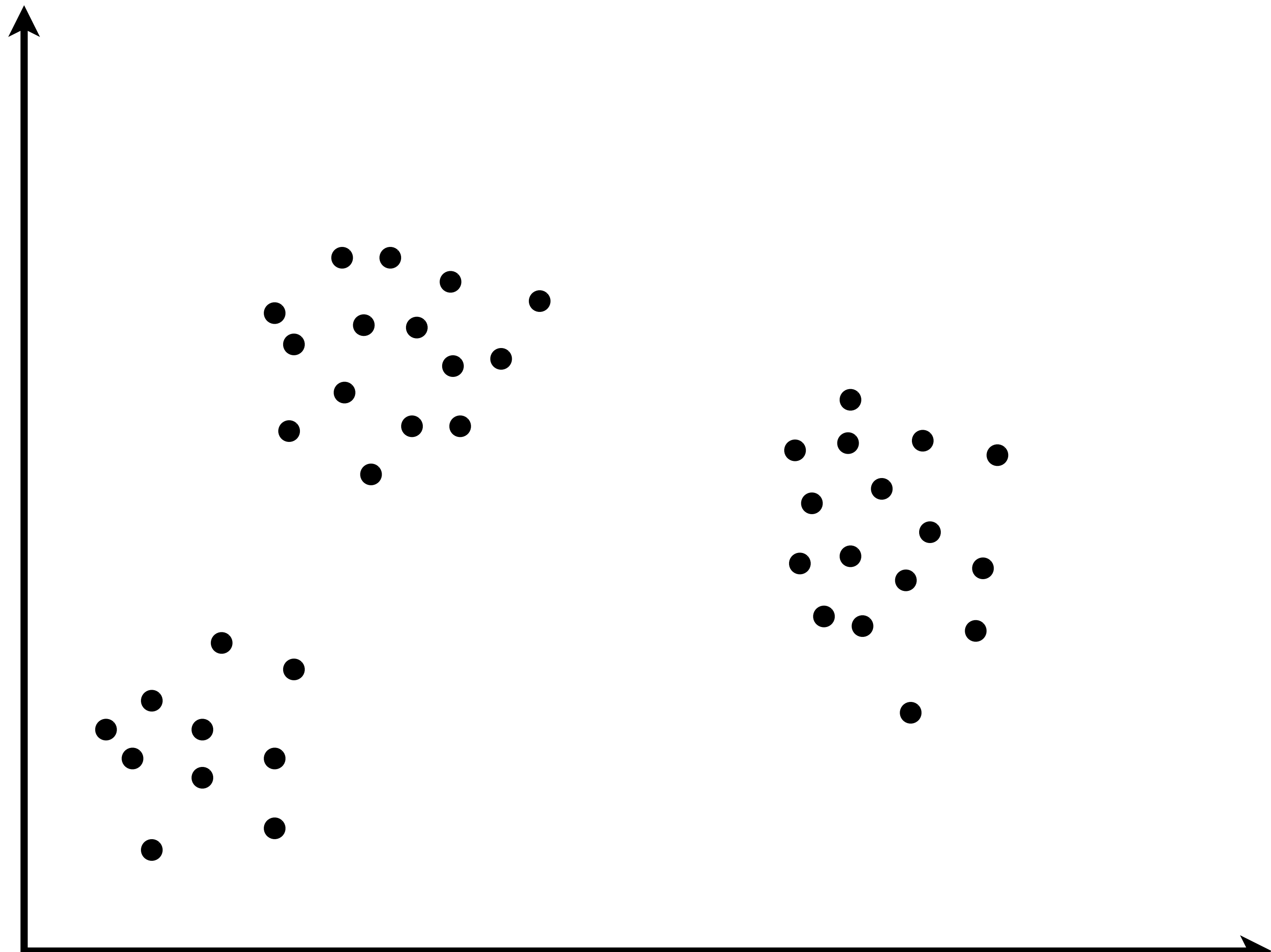
Feature allocations, probability functions, and paintboxes

Tamara Broderick
UC Berkeley
(MIT starting Spring 2015)

Clustering/Partition



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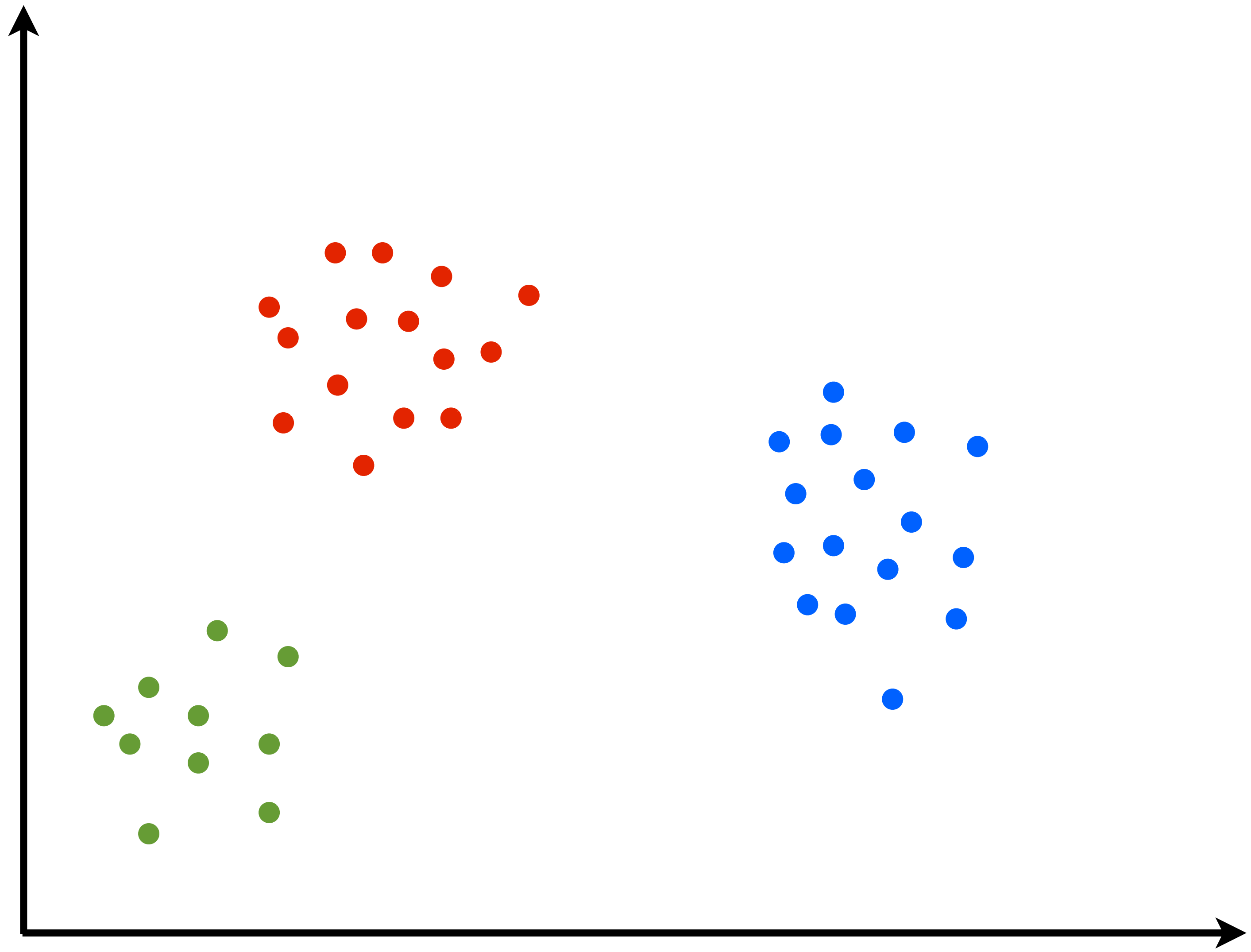
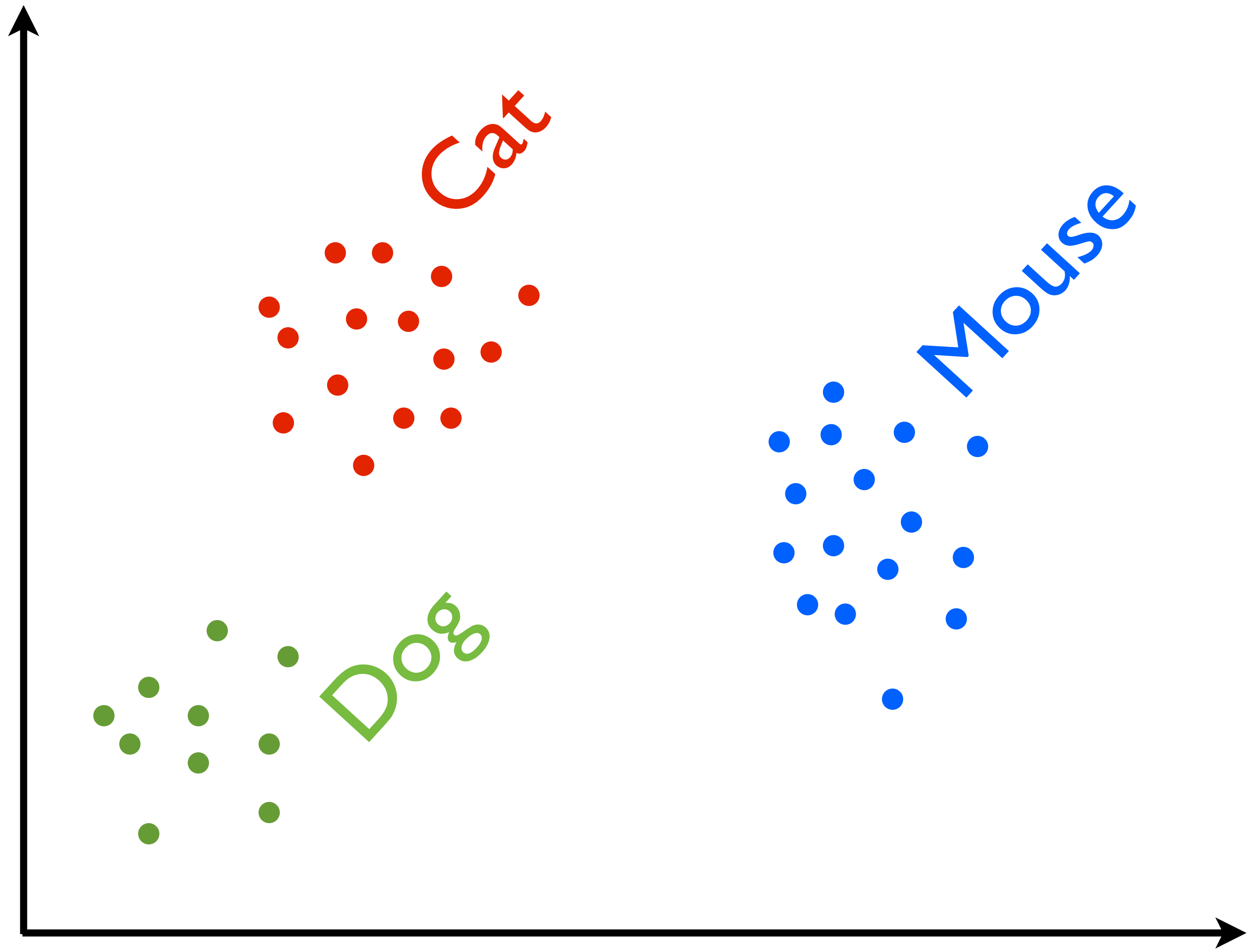


"clusters",
"classes",
"blocks (of a partition)"

Clustering/Partition



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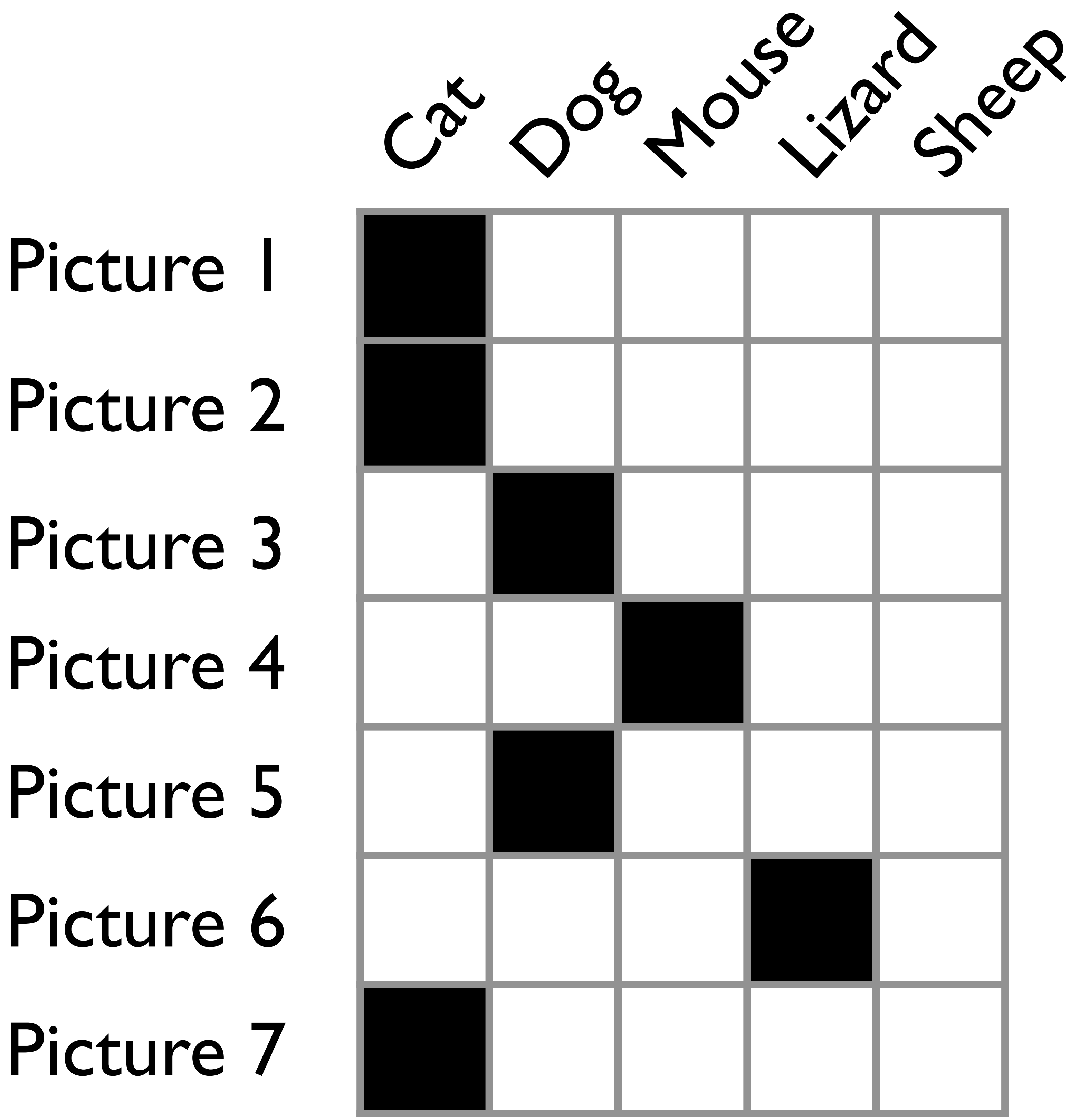
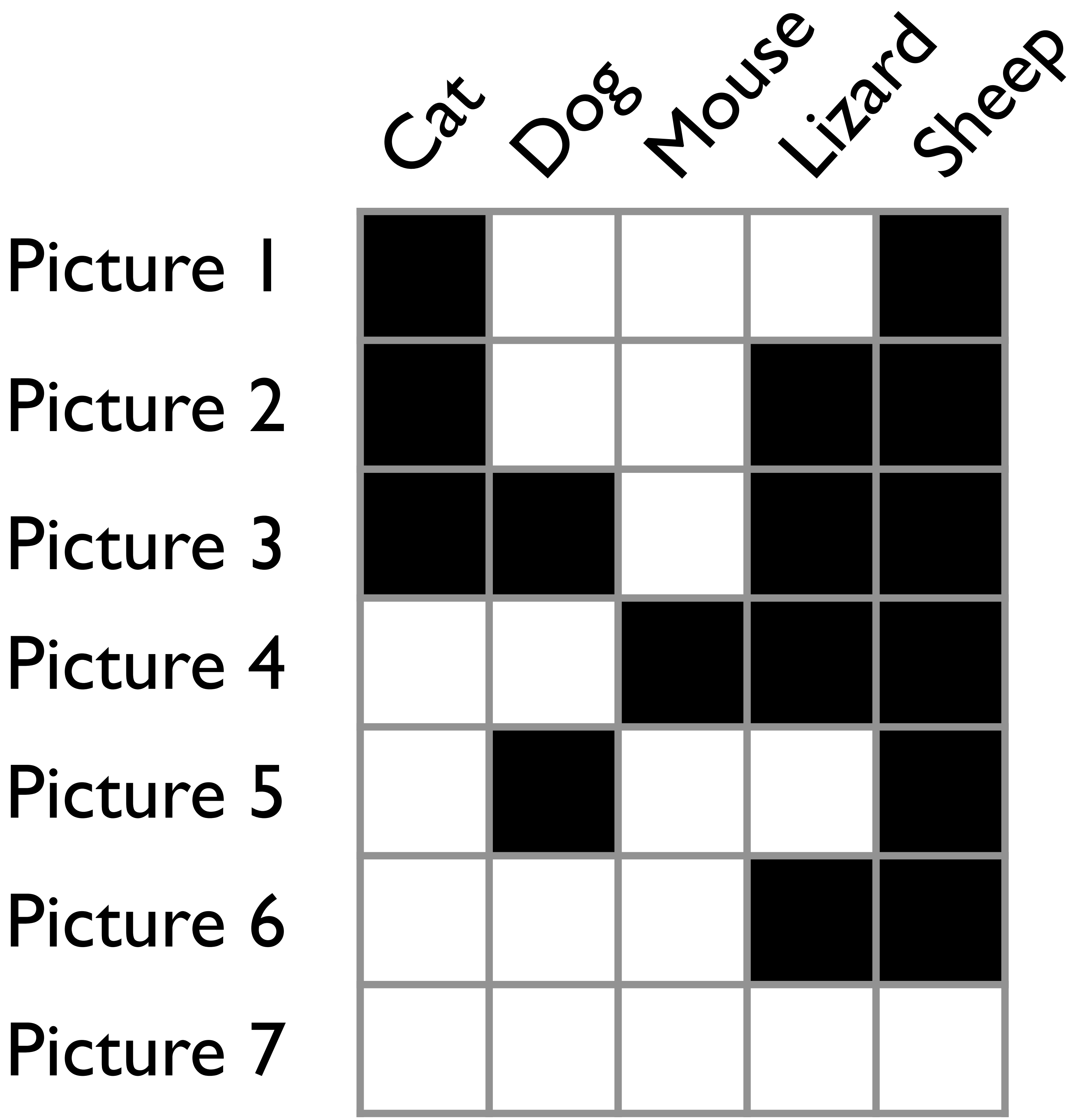
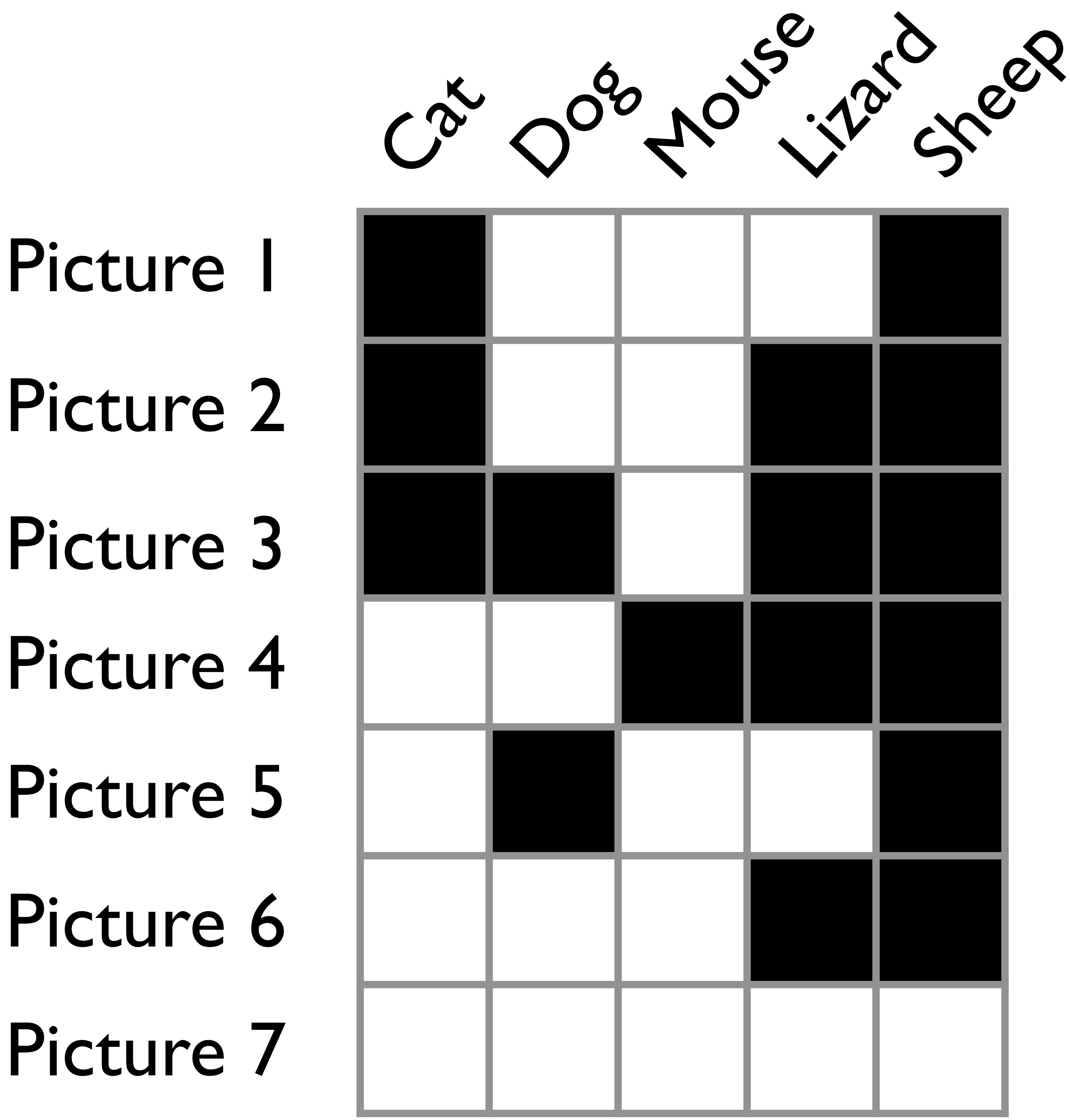


Latent feature allocation



“features”,
“topics”

Latent feature allocation



“features”,
“topics”

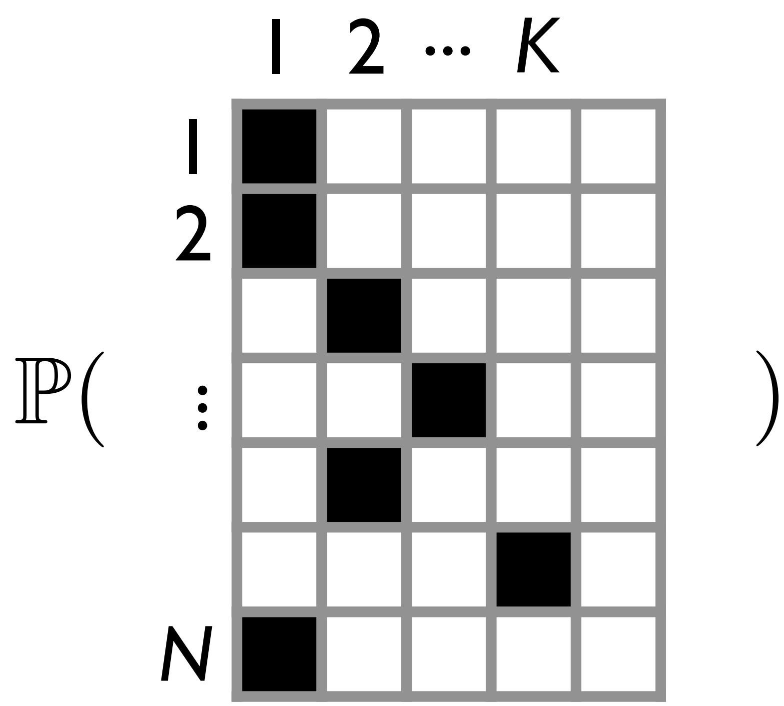
- Exchangeable
- Finite # of features per data point

Characterizations

- Exchangeable cluster distributions are characterized

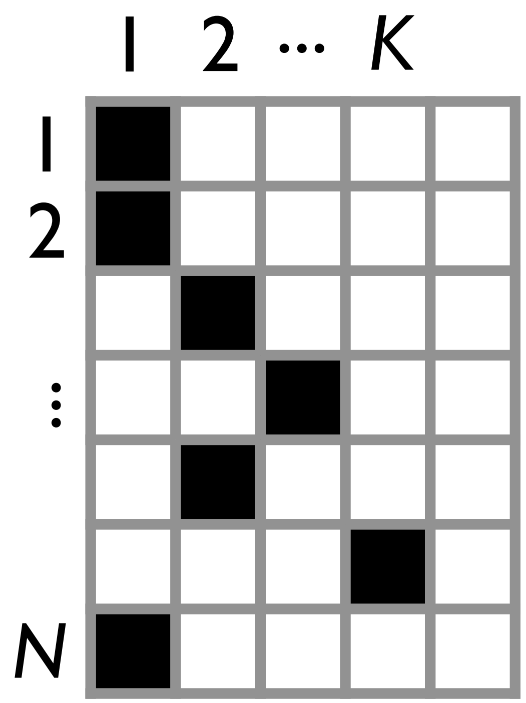
- What about exchangeable feature distributions?

Exchangeable probability functions



Exchangeable probability functions

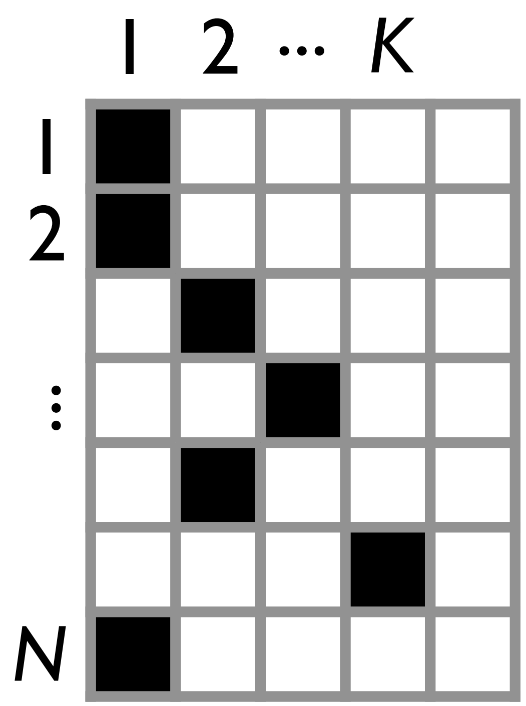
$\mathbb{P}($



$) = p\left(S_{N,1}, \ldots, S_{N,K}\right)$

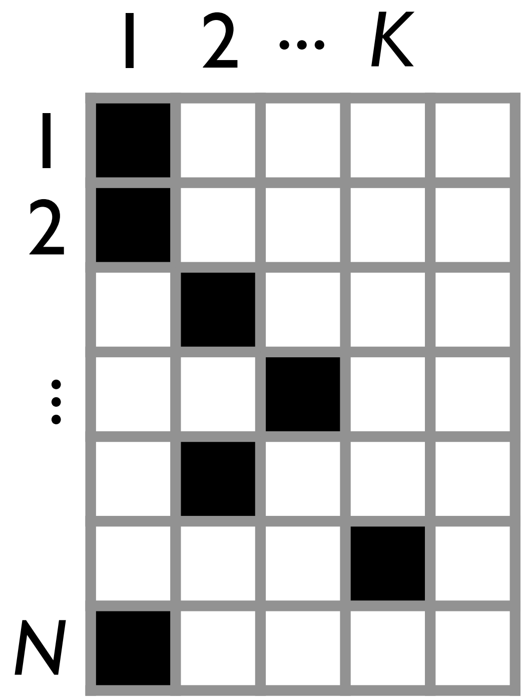
Exchangeable probability functions

Size of *K*th cluster

$\mathbb{P}($  $) = p\left(S_{N,1}, \ldots, S_{N,K}\right)$

Exchangeable probability functions

Exchangeable partition probability function (EPPF)

$\mathbb{P}($  $) = p\left(S_{N,1}, \ldots, S_{N,K}\right)$

[Pitman 1995]

Exchangeable probability functions

"Exchangeable **feature** probability function" (EFPF)?

Example: Indian buffet process

[Griffiths, Ghahramani 2006]

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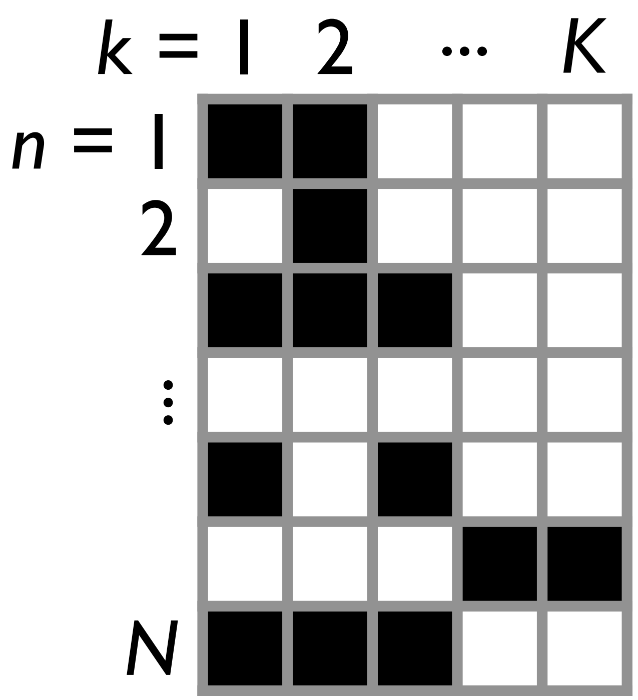

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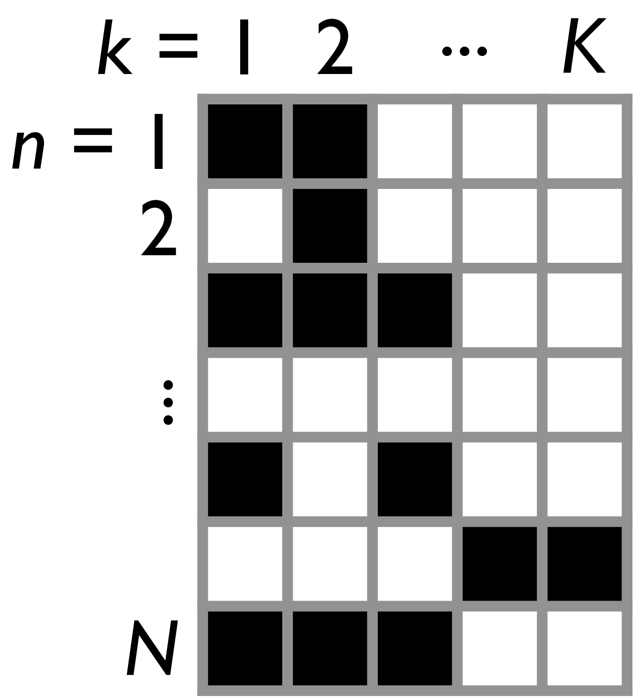

For n = 1, 2, ..., N

[Griffiths, Ghahramani 2006]

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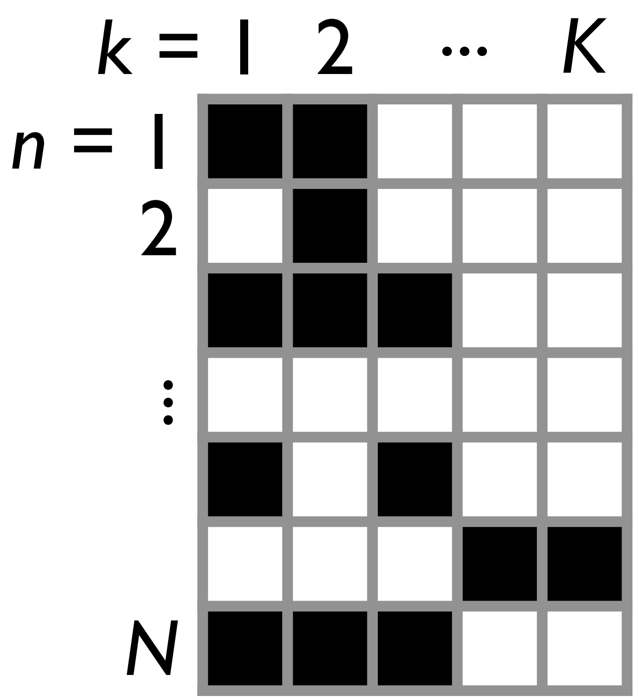

For n = 1, 2, ..., N

1. Data point n has an existing feature k that has already occurred $S_{n-1,k}$ times with probability $\dfrac{S_{n-1,k}}{\theta + n - 1}$

[Griffiths, Ghahramani 2006]

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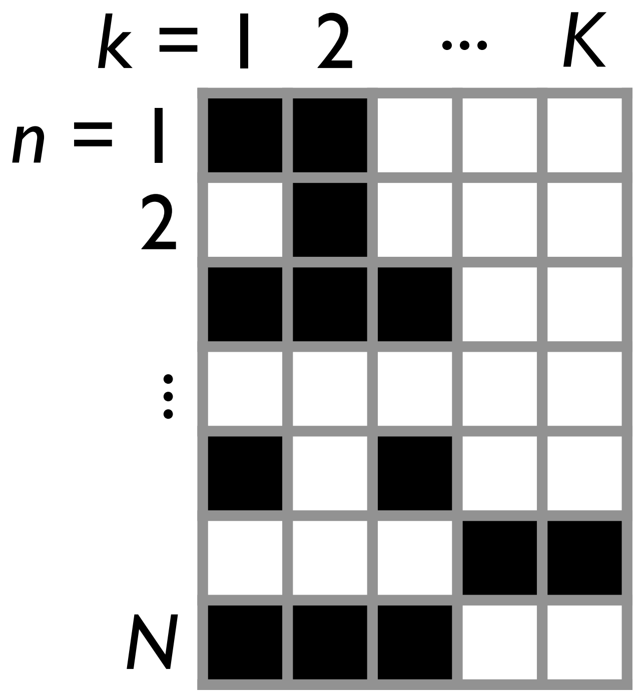

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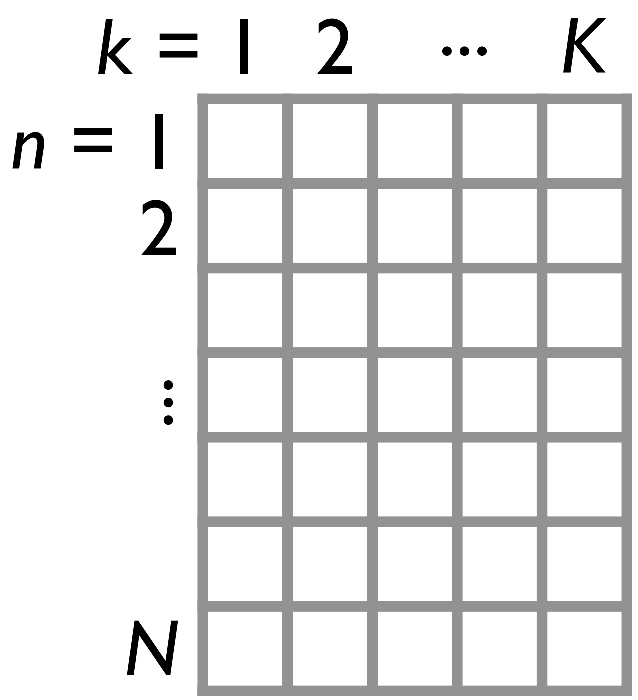

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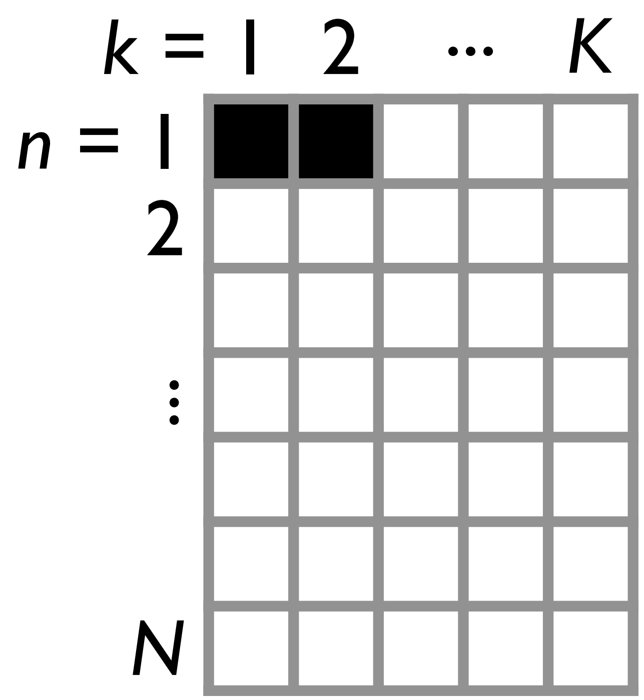

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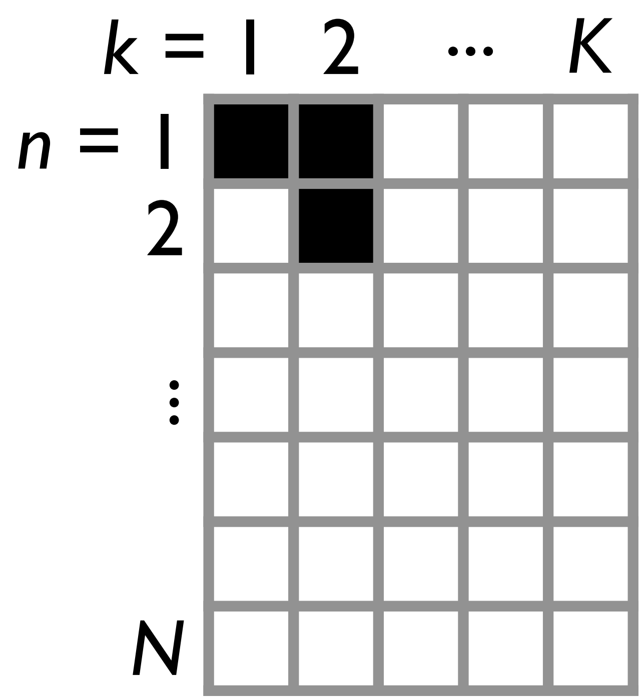

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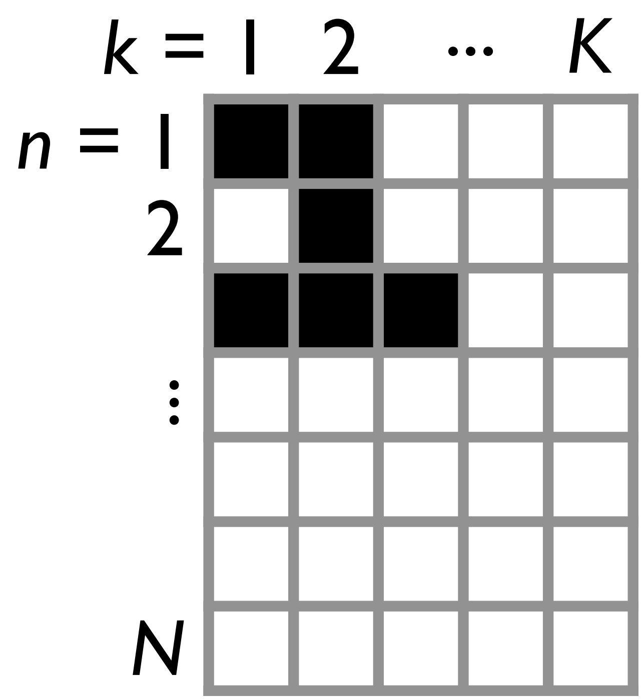

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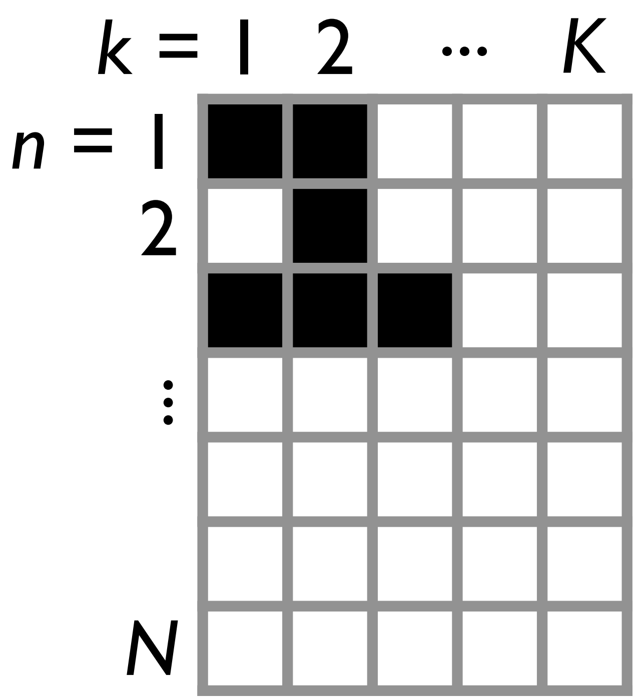

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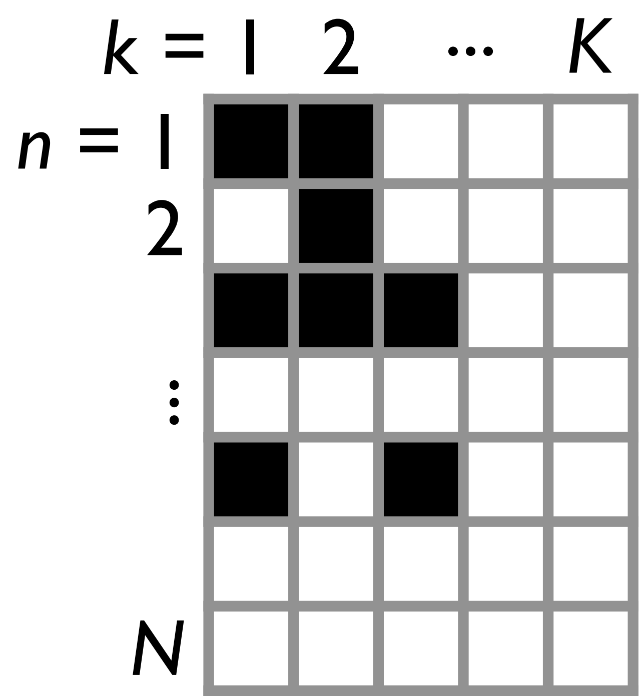

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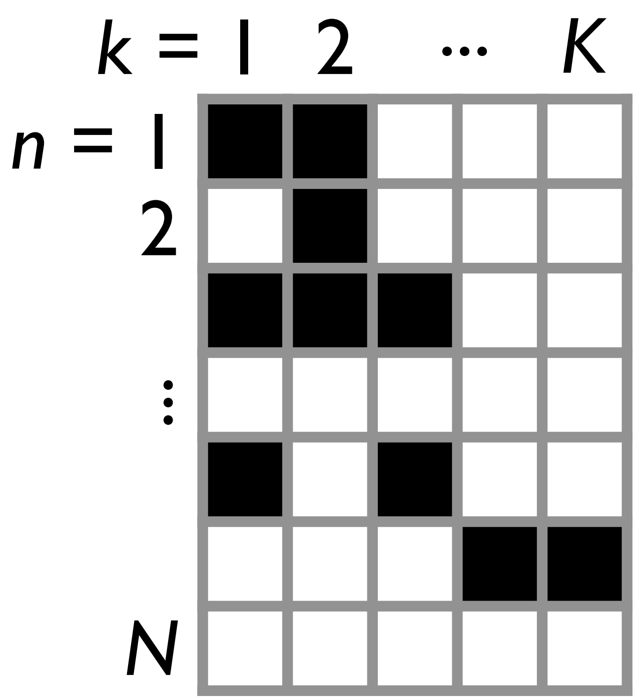

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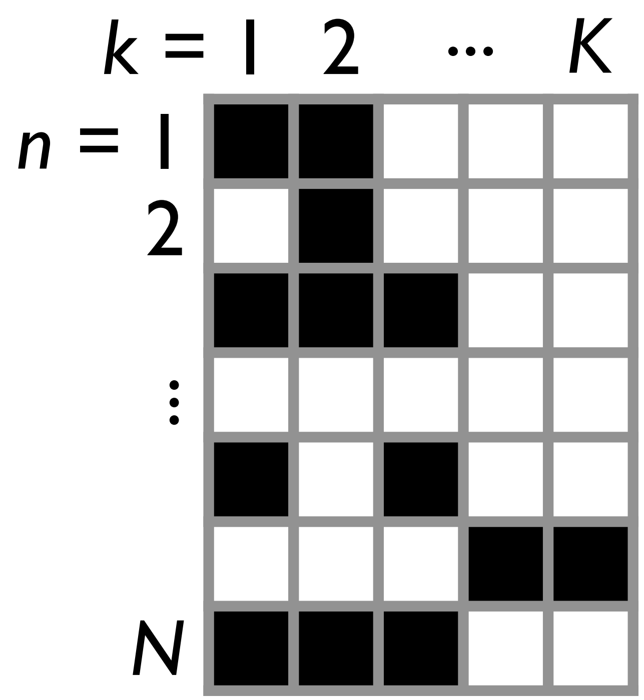

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Exchangeable probability functions

"Exchangeable feature probability function" (EFPF)?

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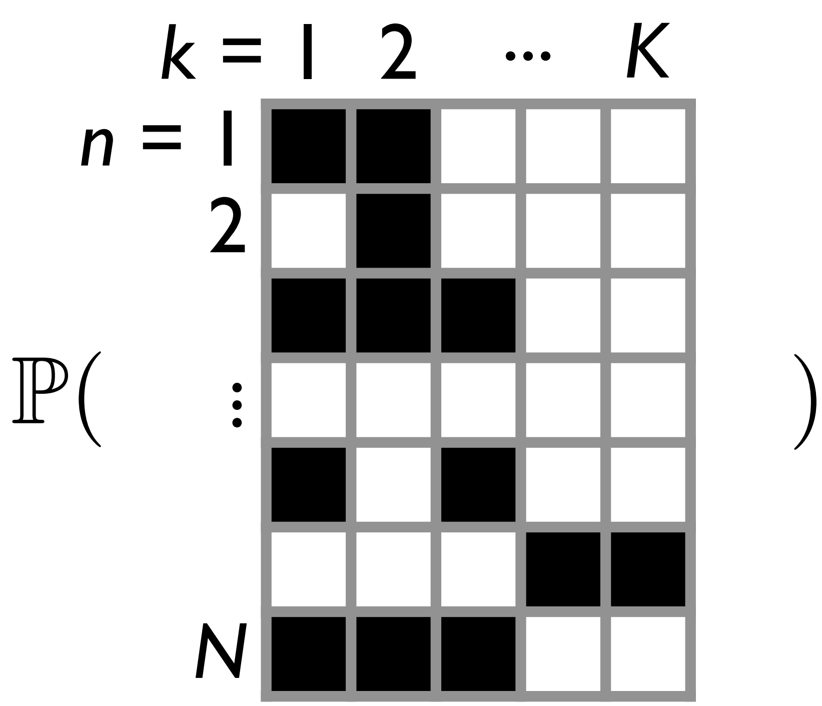
Example: Indian buffet process (IBP)



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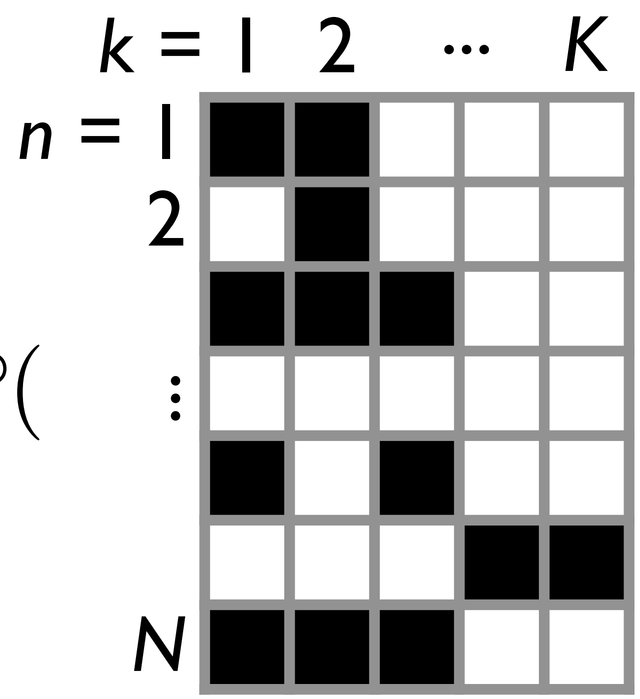
$\mathbb{P}($  $)$

$$= \frac{1}{K_N!} (\theta\gamma)^{K_N} \exp\left(-\theta\gamma \sum_{n=1}^{N} (\theta + n - 1)^{-1} \right) \prod_{k=1}^{K_N} \frac{\Gamma(S_{N,k})\Gamma(N - S_{N,k} + \theta)}{\Gamma(N + \theta)}$$

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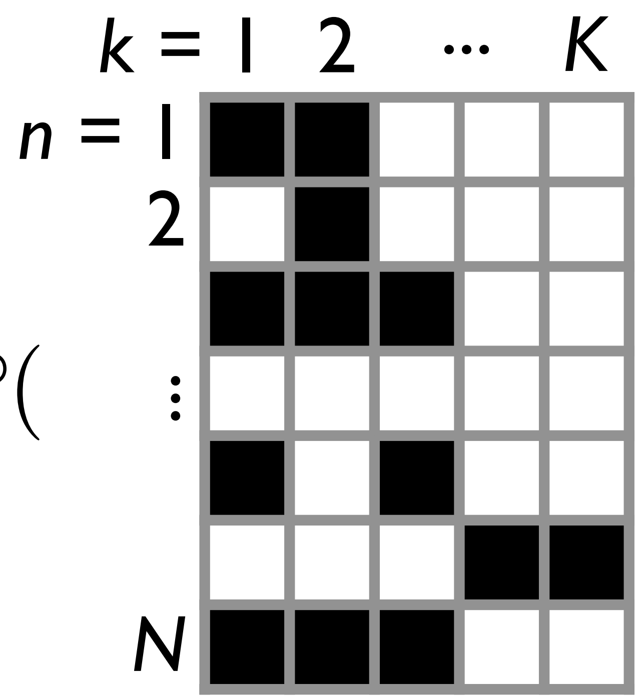
$\mathbb{P}($  $)$

Size of *k*th feature

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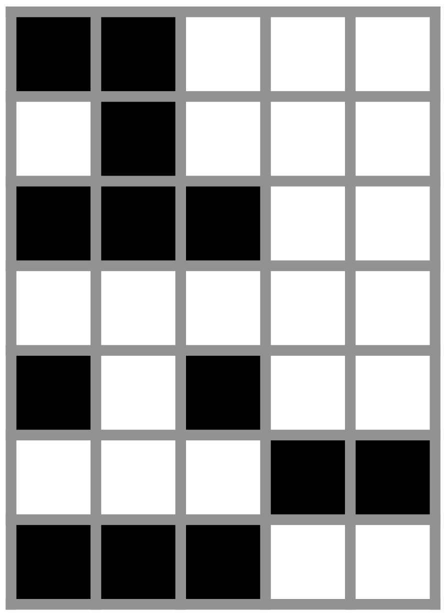

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Number of features

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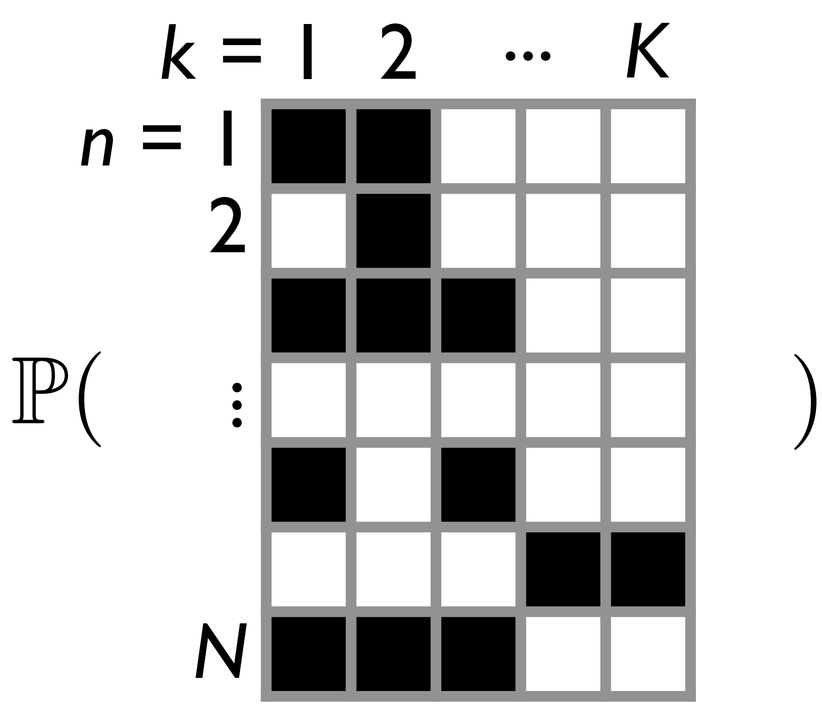

Number of data points

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$$\mathbb{P}\left(\begin{array}{c|ccccc} & k=1 & 2 & & \cdots & K \\ \hline n=1 & 1 & 1 & 0 & 0 & 0 \\ 2 & 0 & 1 & 0 & 0 & 0 \\ & 1 & 1 & 1 & 0 & 0 \\ \vdots & 0 & 0 & 0 & 0 & 0 \\ & 1 & 0 & 1 & 0 & 0 \\ & 0 & 0 & 0 & 1 & 1 \\ N & 1 & 1 & 1 & 0 & 0 \end{array} \right)$$

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[Broderick, Jordan, Pitman 2013]

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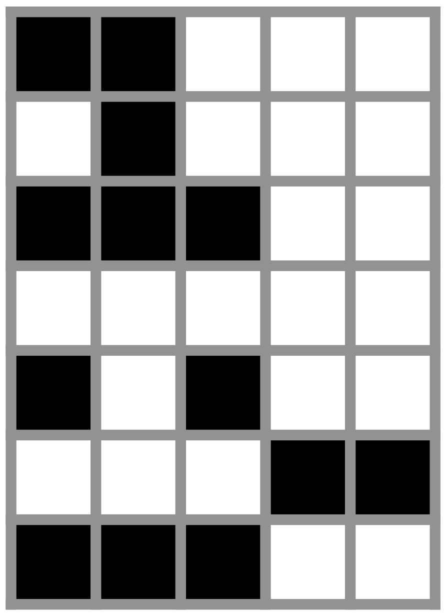
$\mathbb{P}($  $)$

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"EFPF"

[Broderick, Jordan, Pitman 2013]

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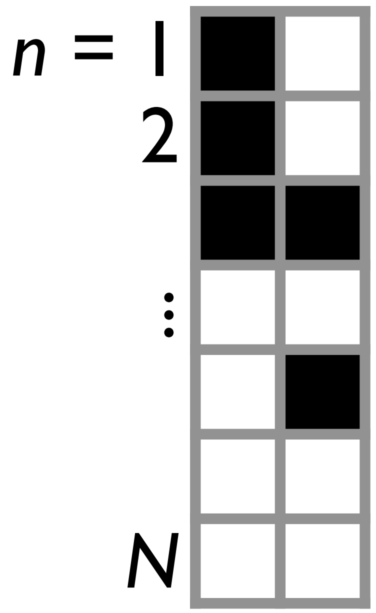
Counterexample



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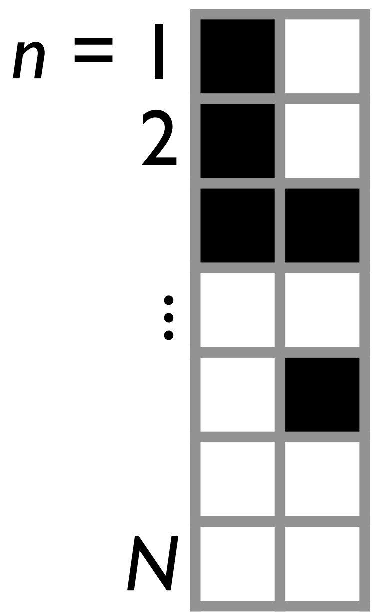
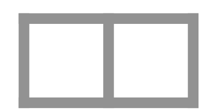


$\mathbb{P}(\text{row} =$  $) = p_1$

$\mathbb{P}(\text{row} =$  $) = p_2$

$\mathbb{P}(\text{row} =$  $) = p_3$

$\mathbb{P}(\text{row} =$  $) = p_4$

[Broderick, Jordan, Pitman 2013]

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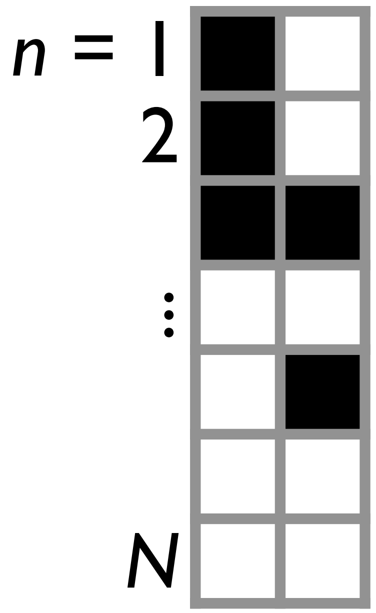

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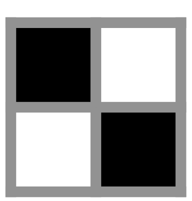

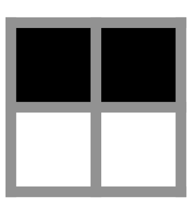

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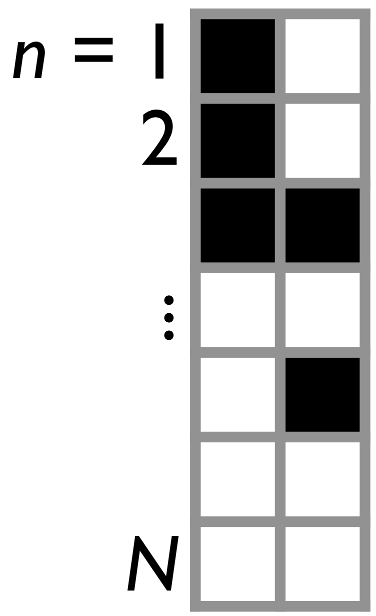

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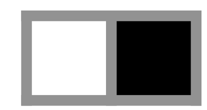
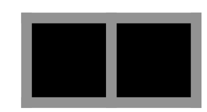
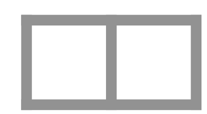
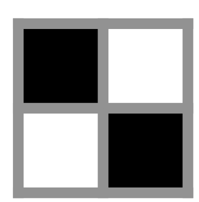
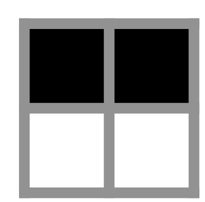
$\mathbb{P}(\text{row} =$  $) = p_2$

$\mathbb{P}(\text{row} =$  $) = p_3$

$\mathbb{P}(\text{row} =$  $) = p_4$

$\mathbb{P}($  $)$ $\mathbb{P}($  $)$

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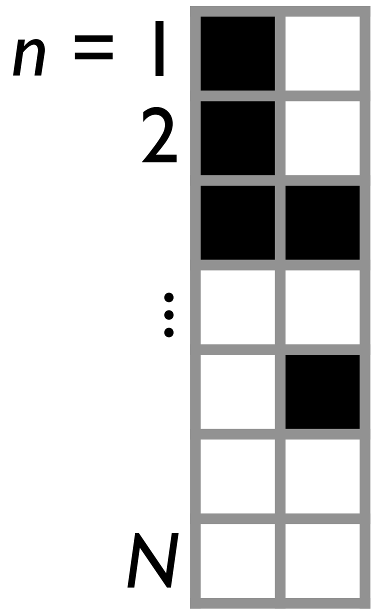

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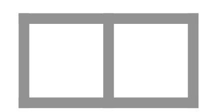
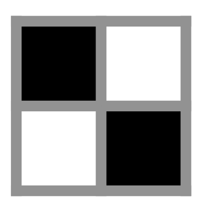
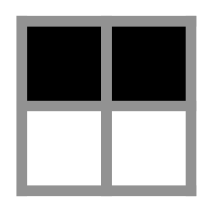
$\mathbb{P}(\text{row} =$  $) = p_4$

$\mathbb{P}($  $)$ $\mathbb{P}($  $)$

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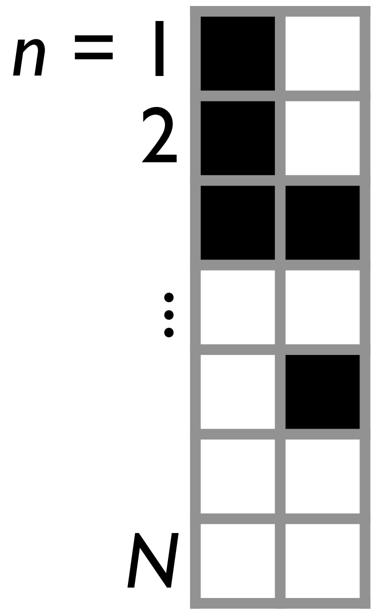

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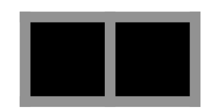
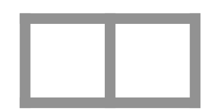
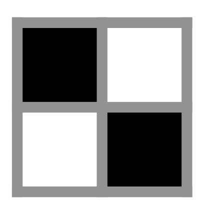
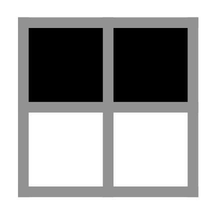
$\mathbb{P}(\text{row} =$  $) = p_3$

$\mathbb{P}(\text{row} =$  $) = p_4$

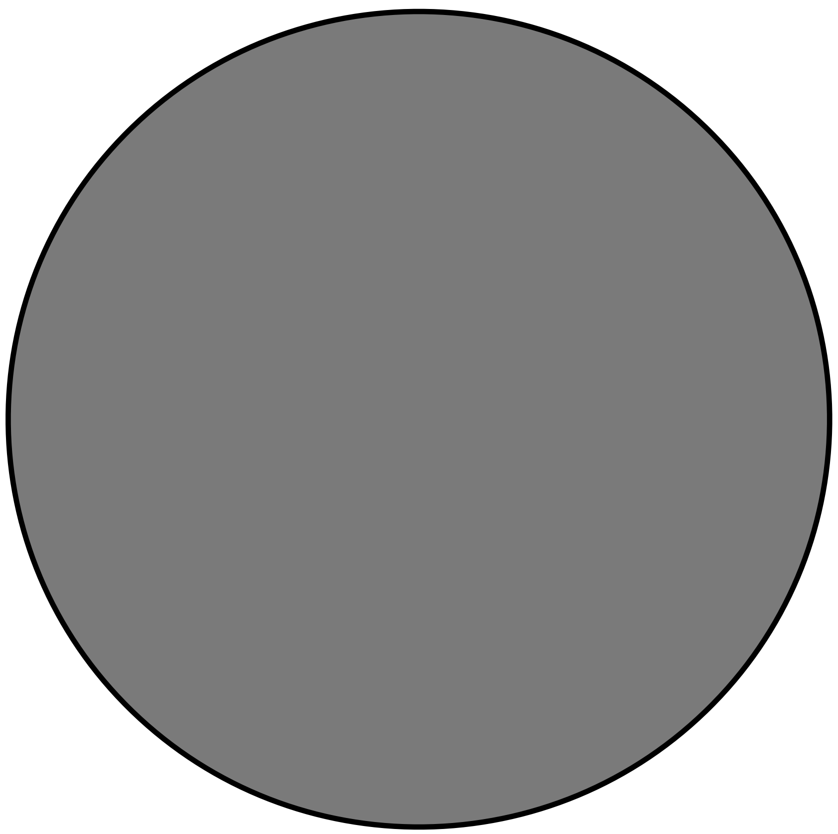
$\mathbb{P}($  $) \neq \mathbb{P}($  $)$

$$p_1 p_2 \neq p_3 p_4$$

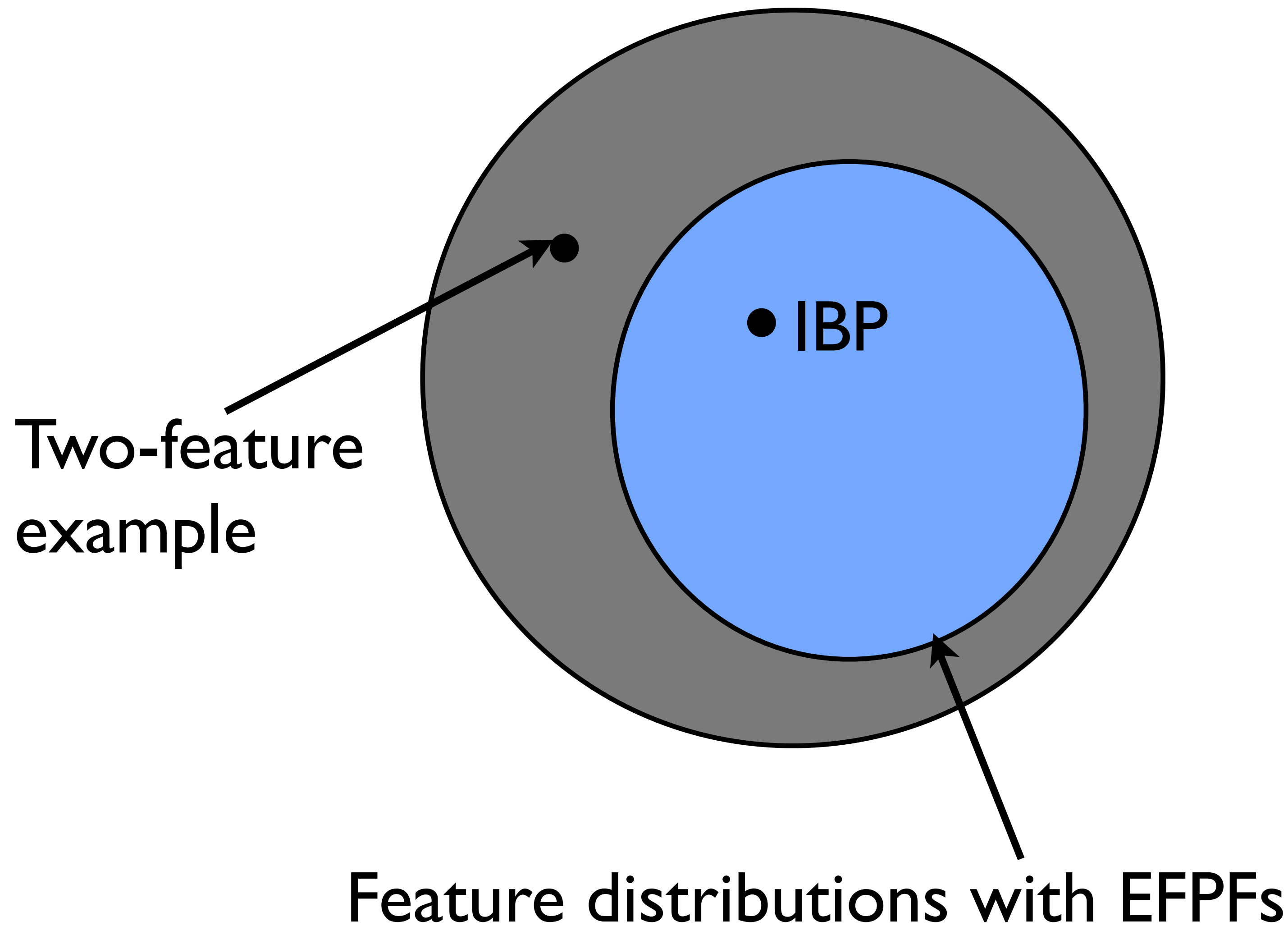
[Broderick, Jordan, Pitman 2013]

Exchangeable probability functions

Exchangeable cluster distributions
= Cluster distributions with EPPFs



Exchangeable feature distributions



[Broderick, Jordan, Pitman 2013]

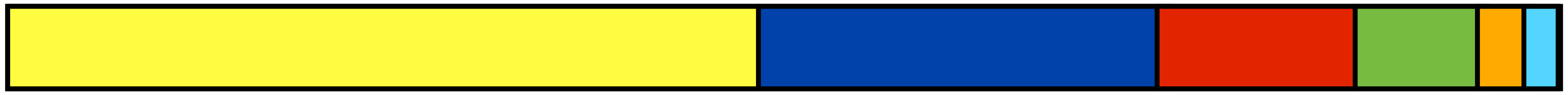
Paintboxes

Exchangeable partition: Kingman paintbox

[Kingman 1978]

Paintboxes

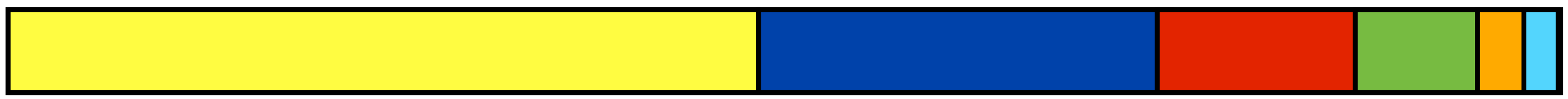
Exchangeable partition: Kingman paintbox

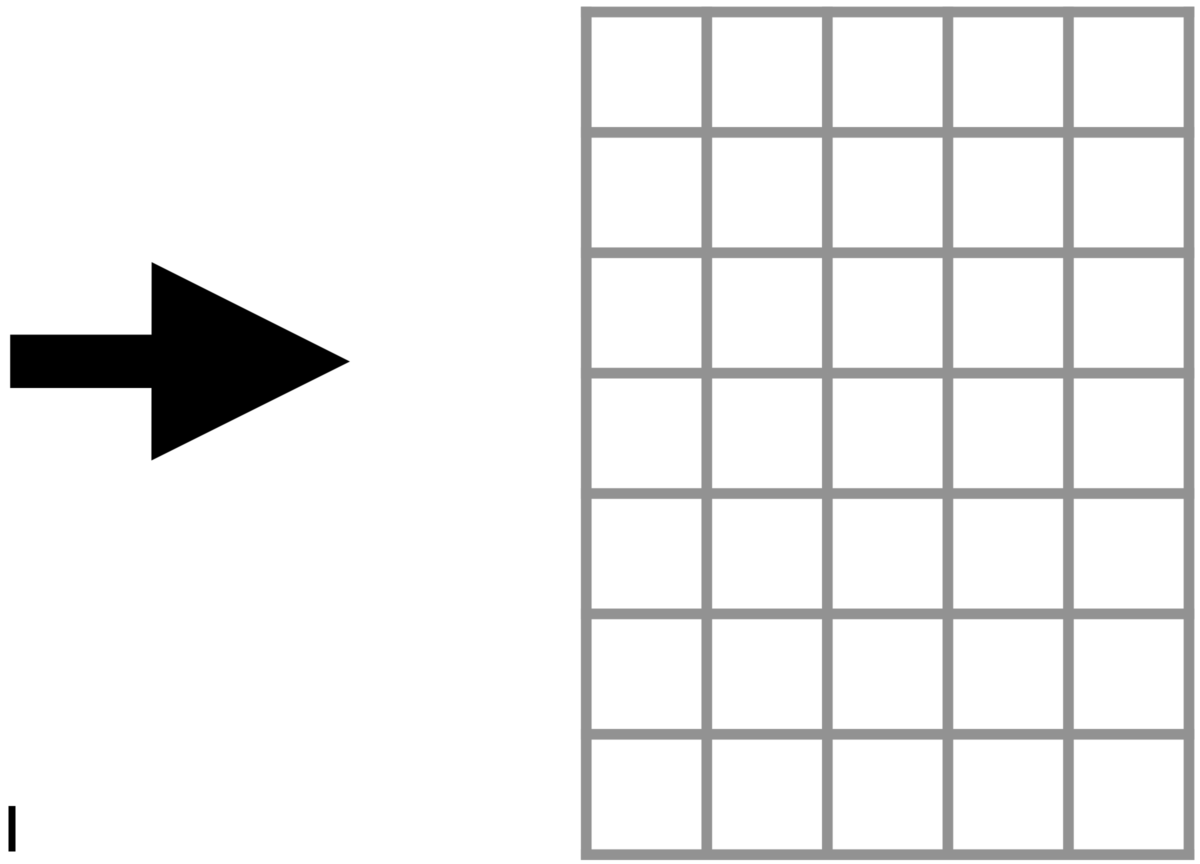


[Kingman 1978]

Paintboxes

Exchangeable partition: Kingman paintbox





[Kingman 1978]

Paintboxes

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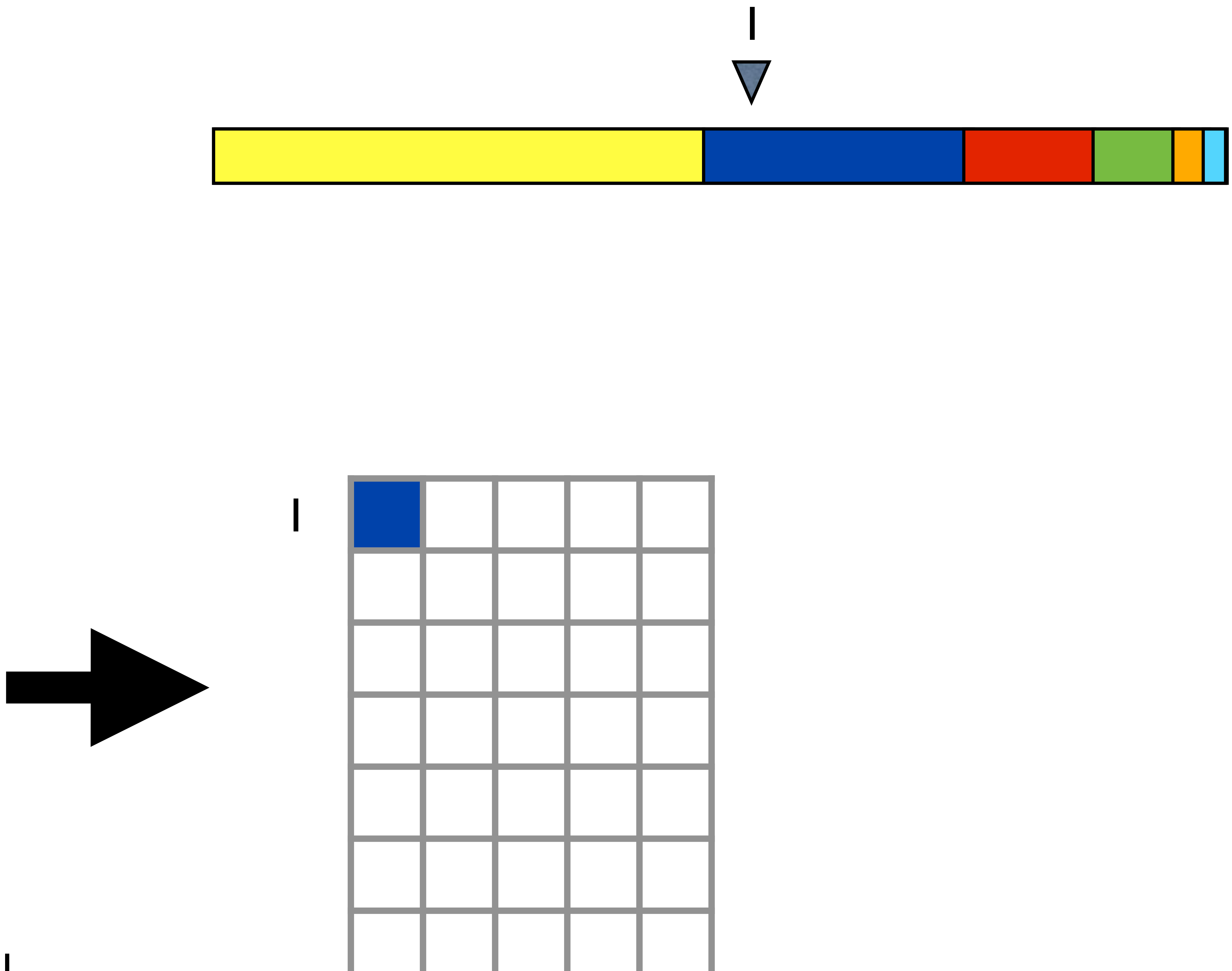

[Kingman 1978]

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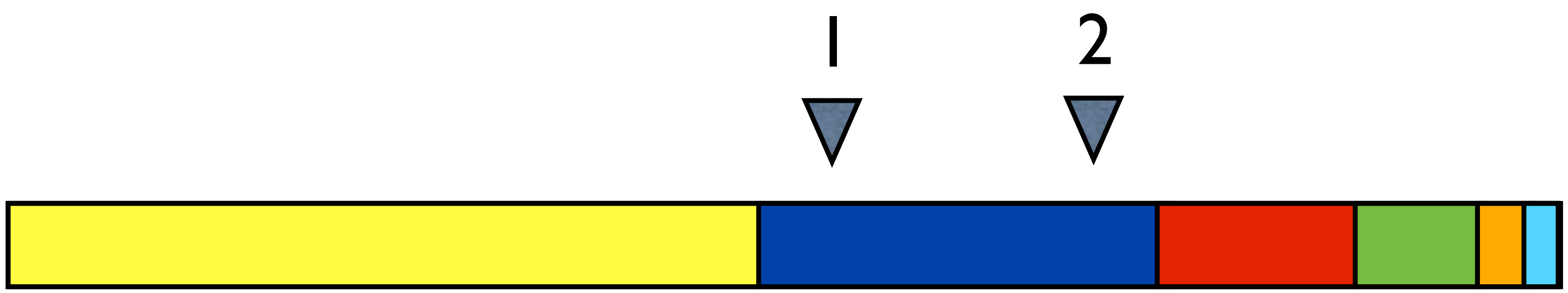
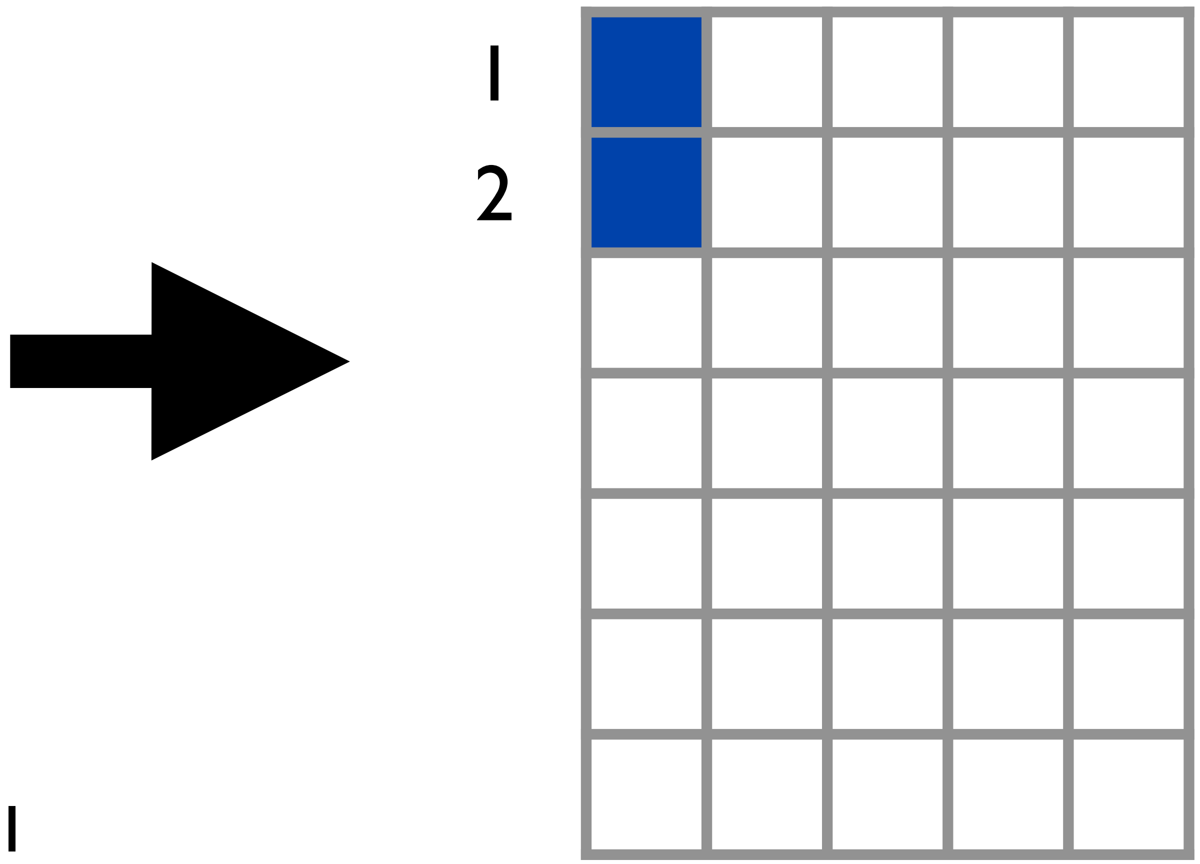
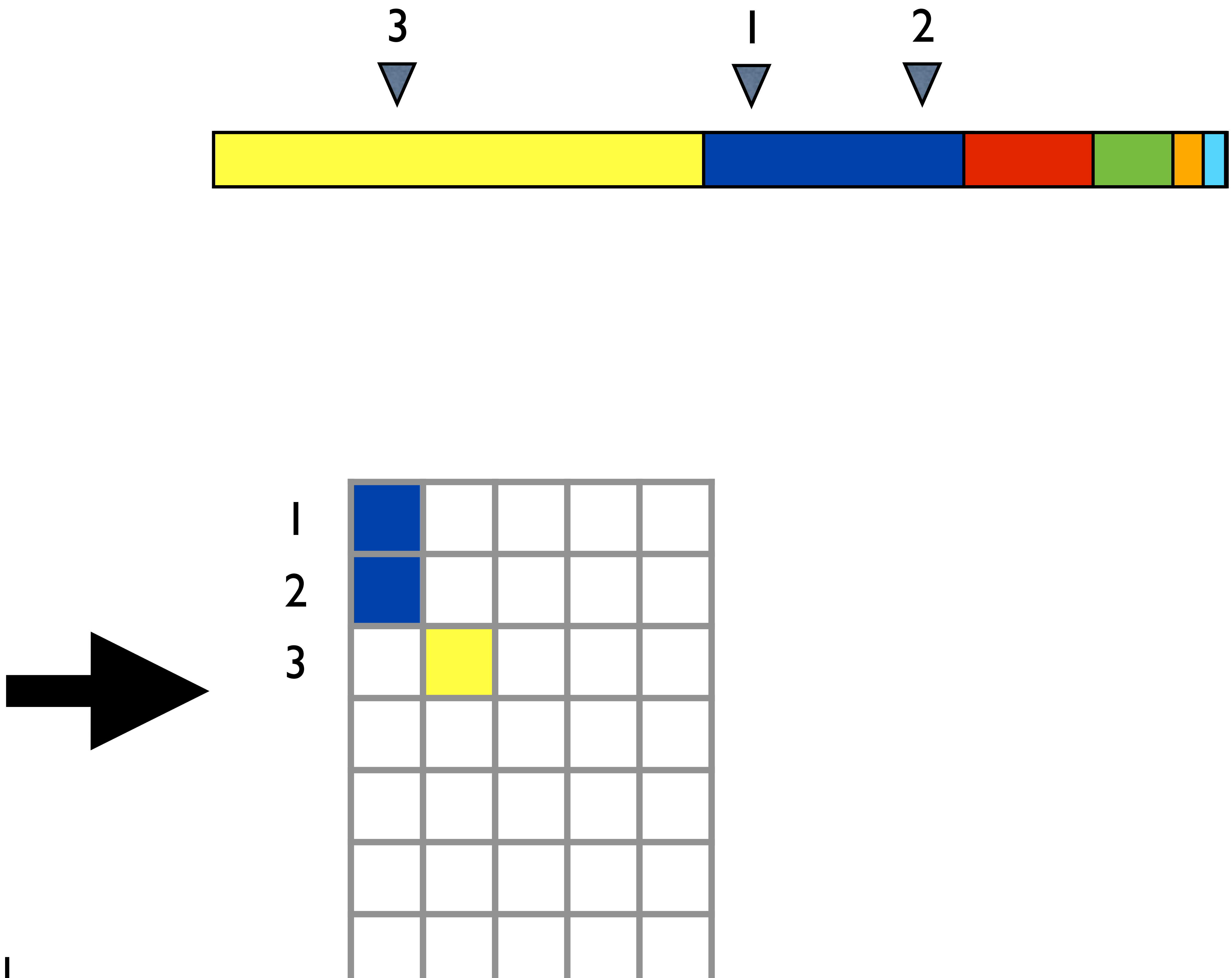




[Kingman 1978]

Paintboxes

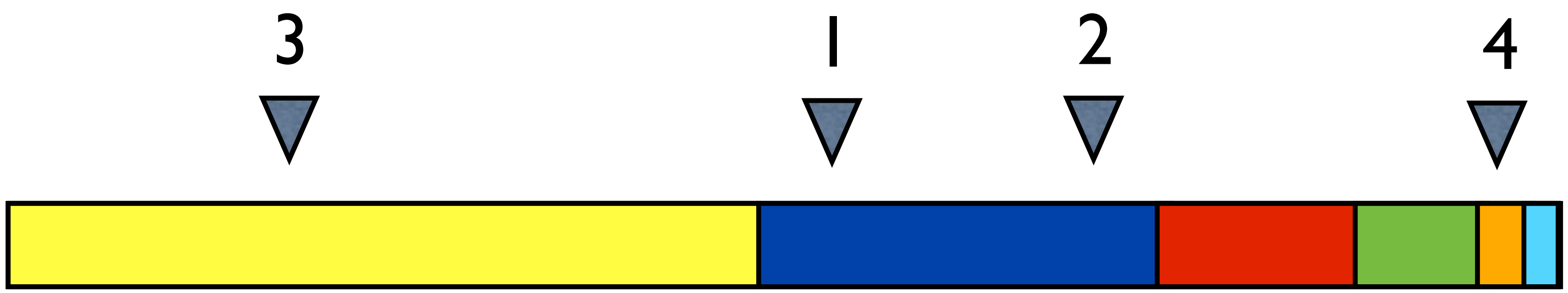
Exchangeable partition: Kingman paintbox

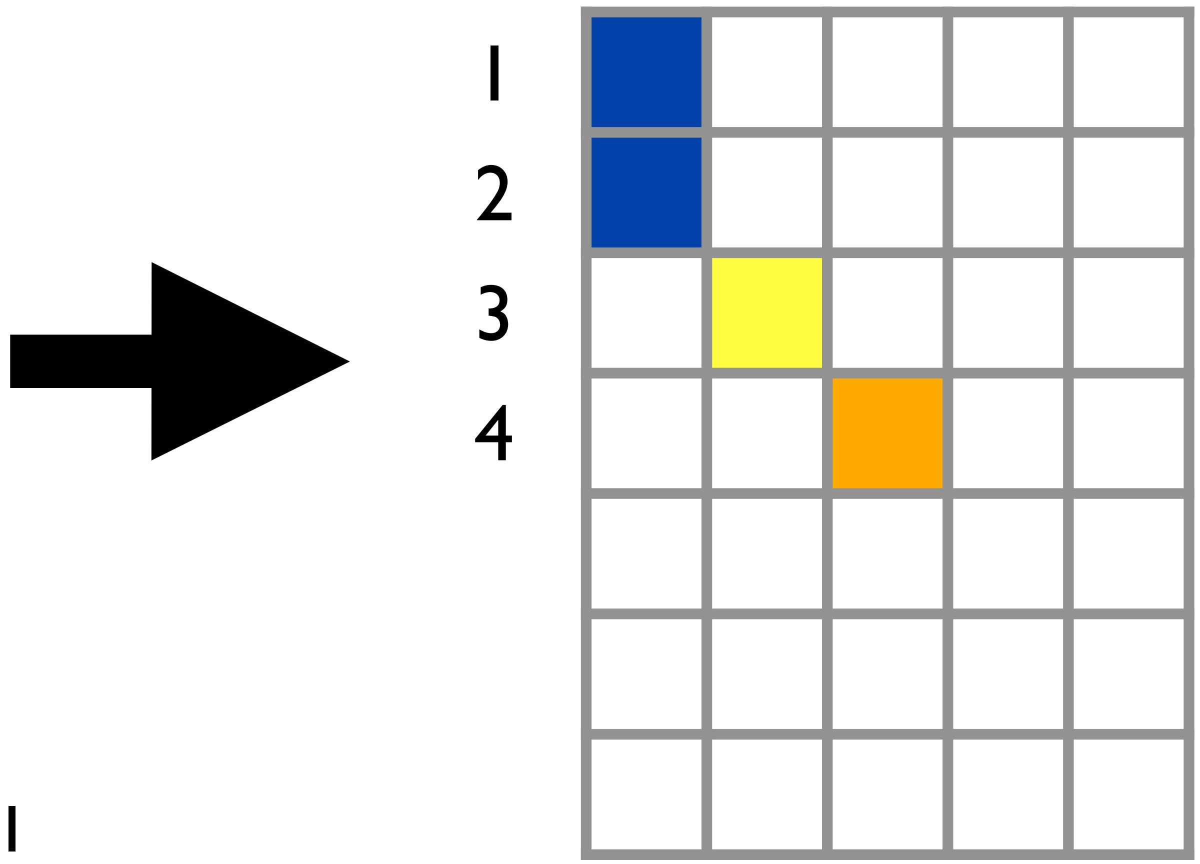


[Kingman 1978]

Paintboxes

Exchangeable partition: Kingman paintbox

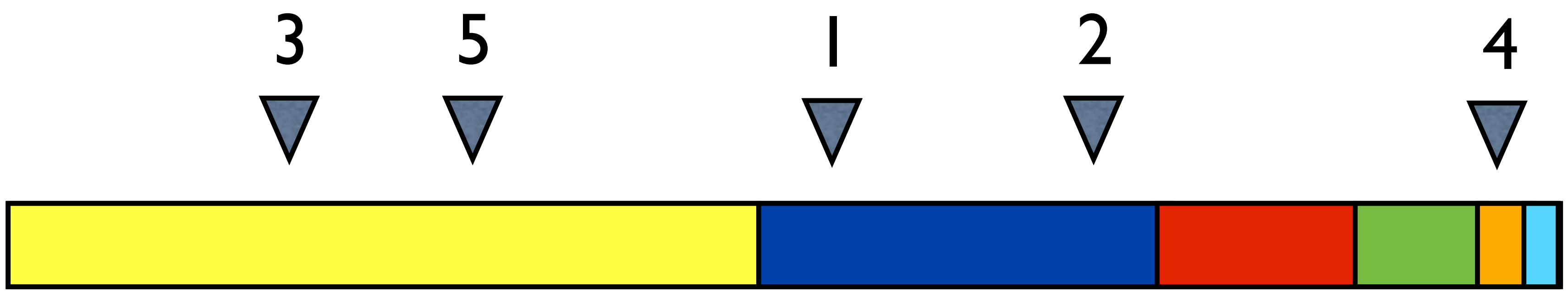


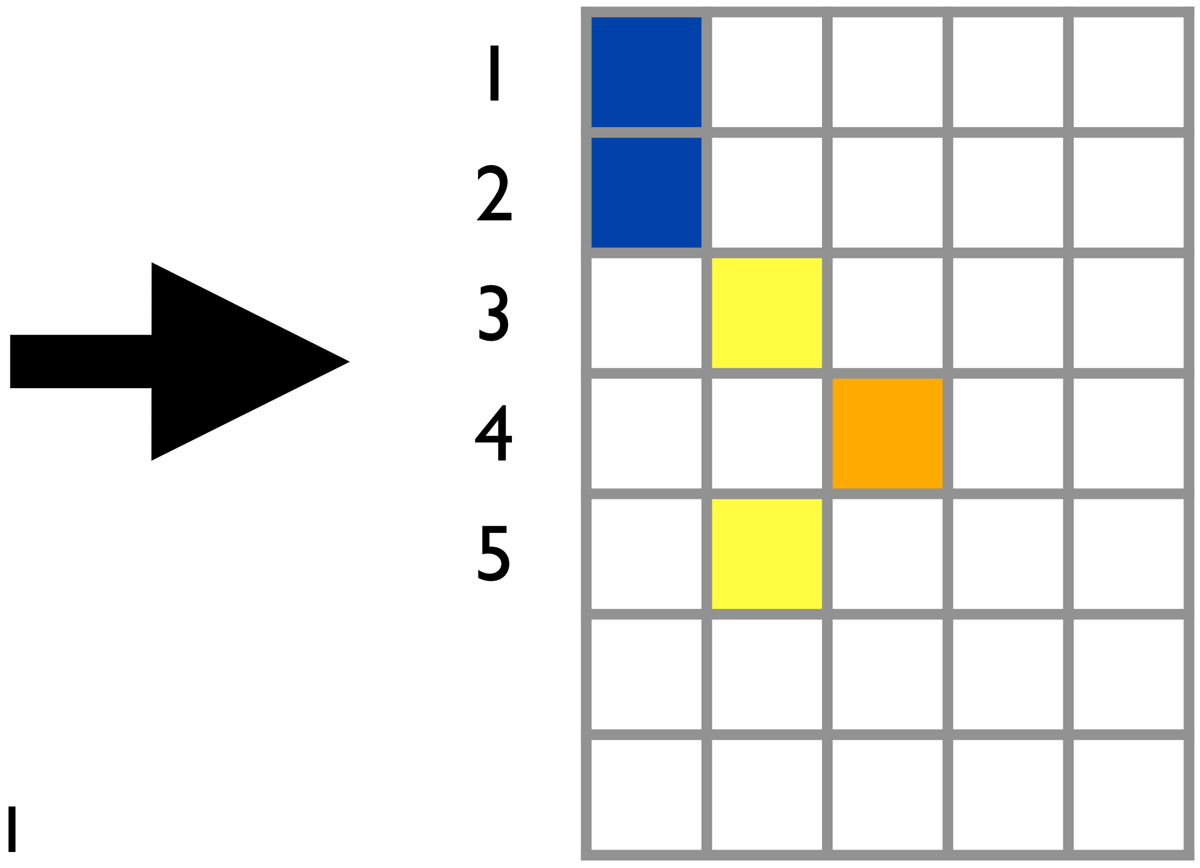


[Kingman 1978]

Paintboxes

Exchangeable partition: Kingman paintbox

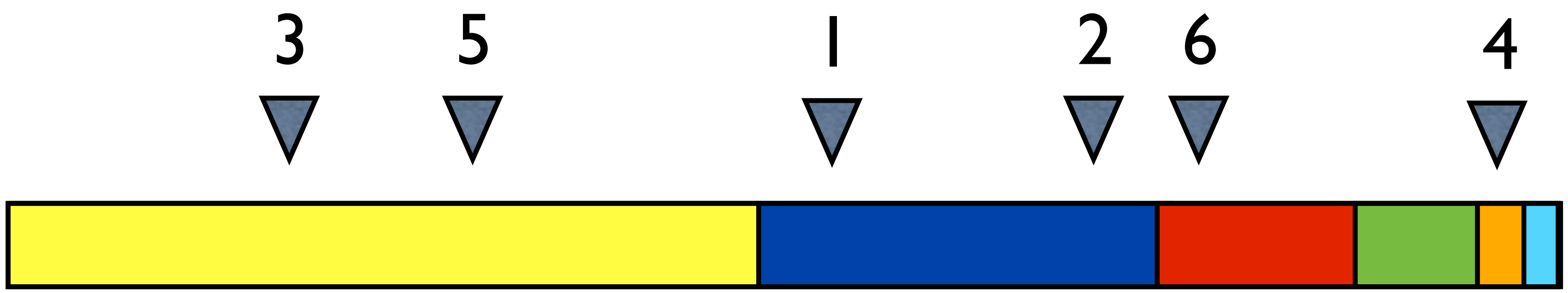


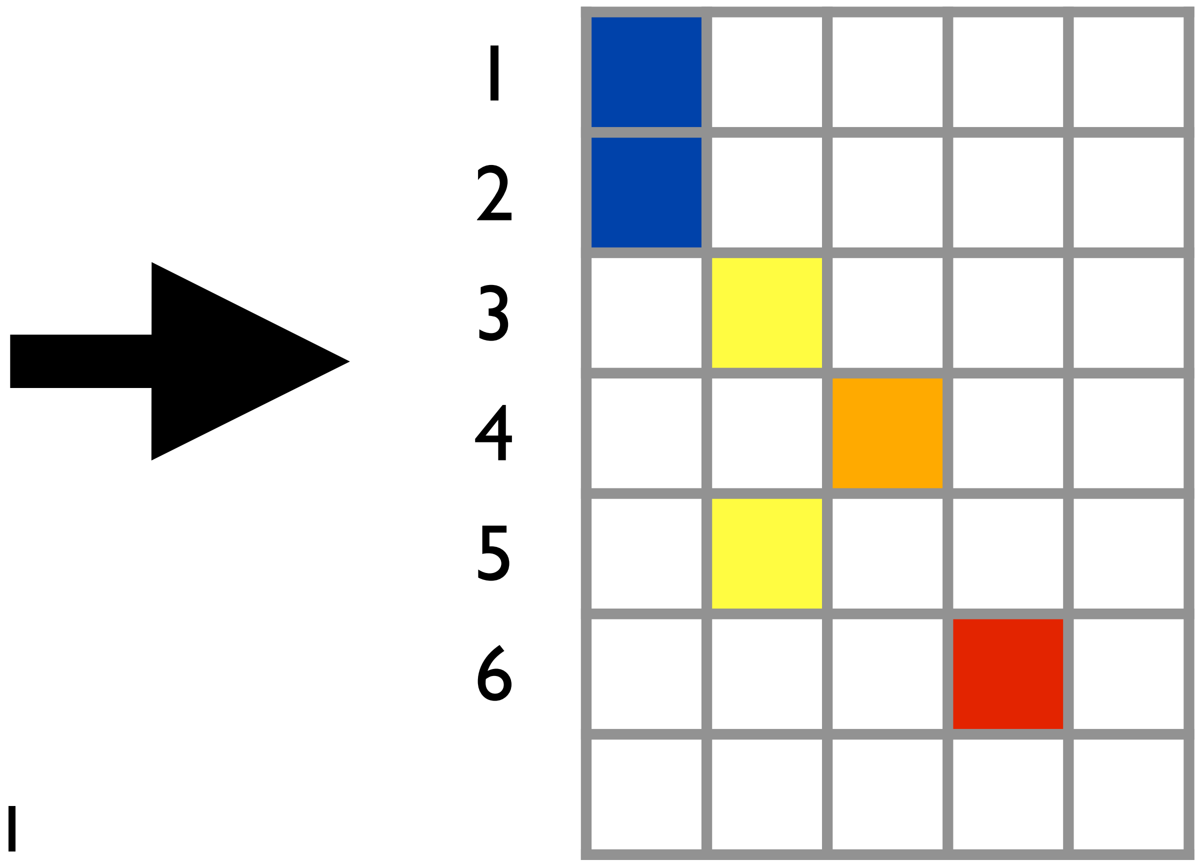


[Kingman 1978]

Paintboxes

Exchangeable partition: Kingman paintbox

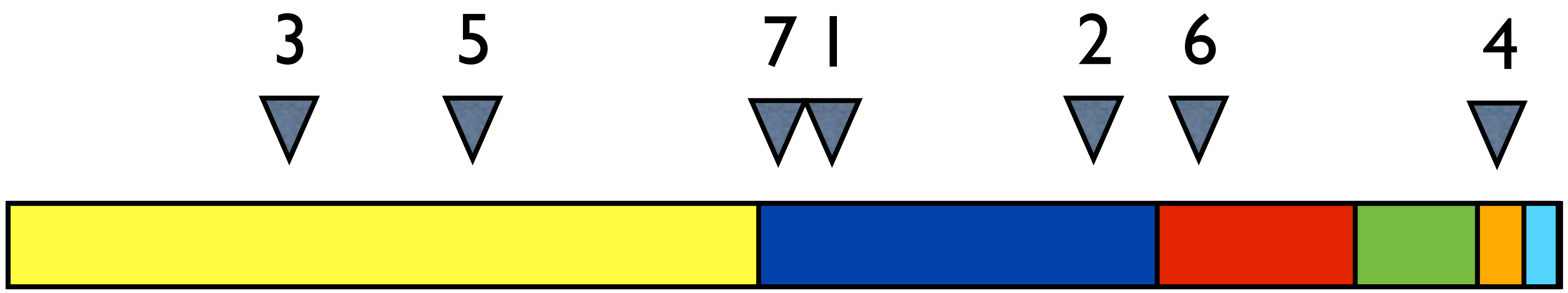


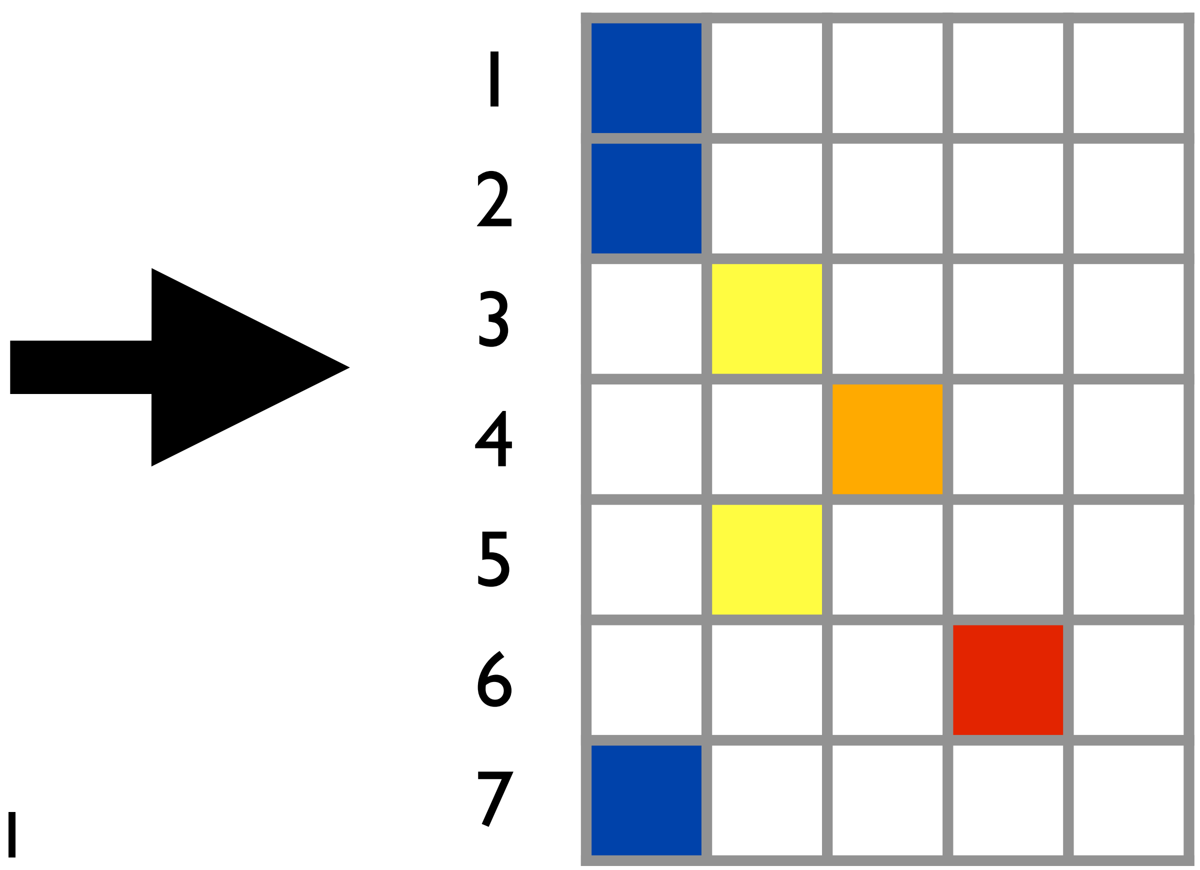


[Kingman 1978]

Paintboxes

Exchangeable partition: Kingman paintbox

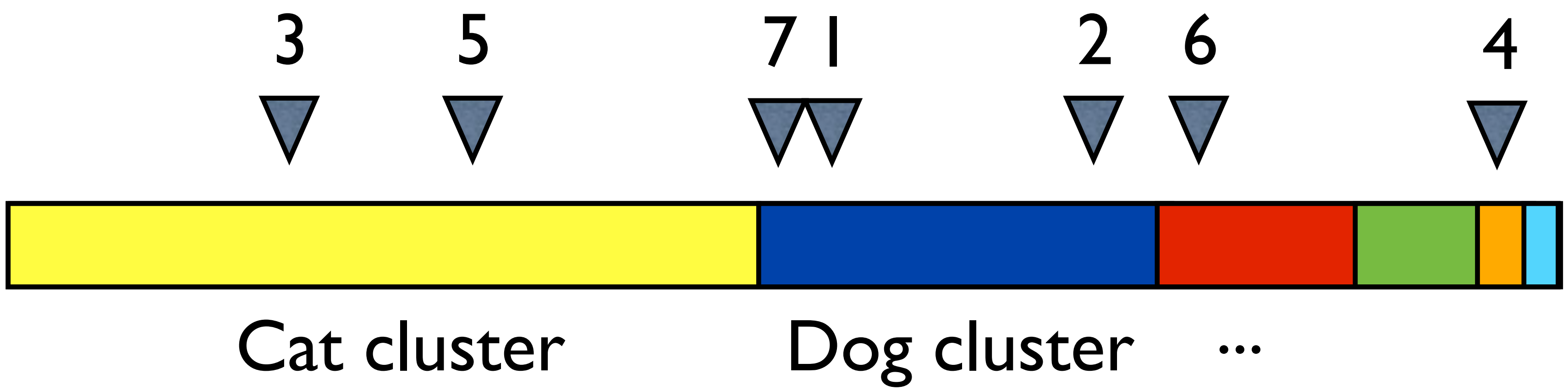


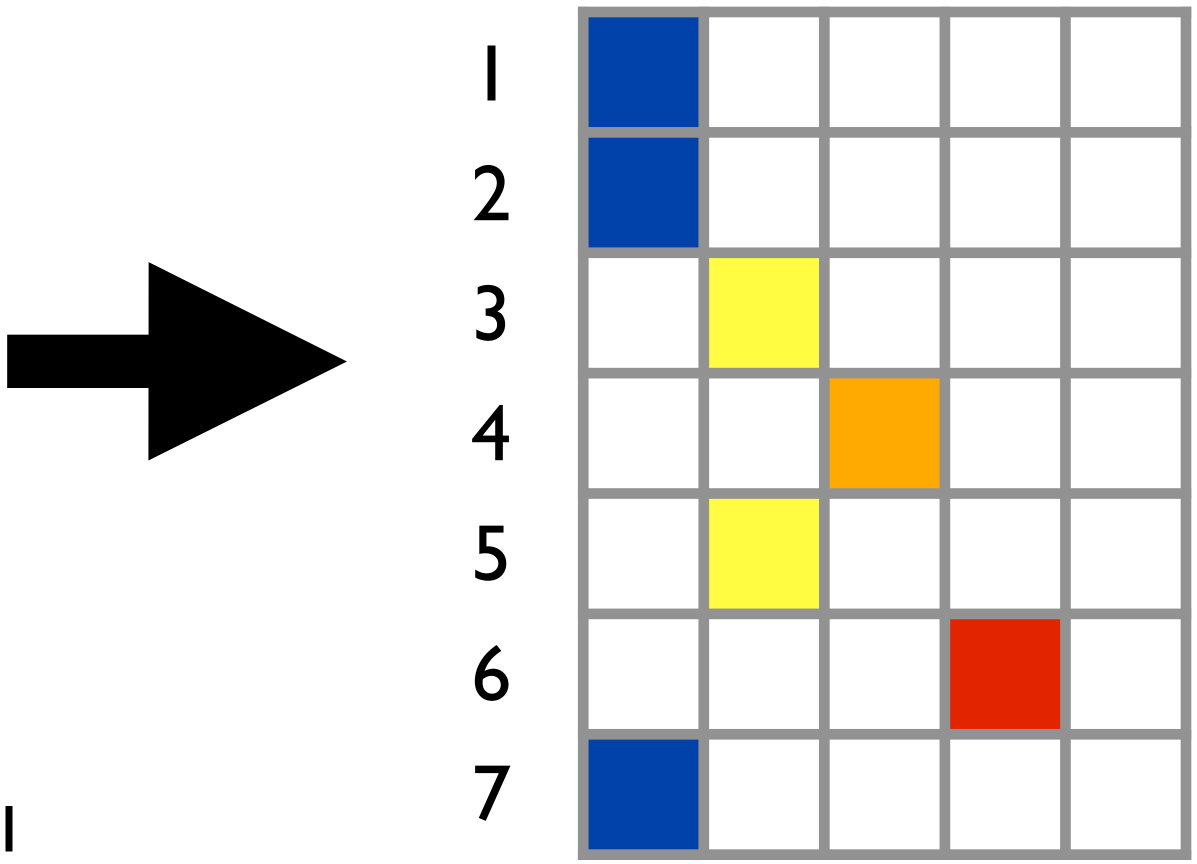


[Kingman 1978]

Paintboxes

Exchangeable partition: Kingman paintbox





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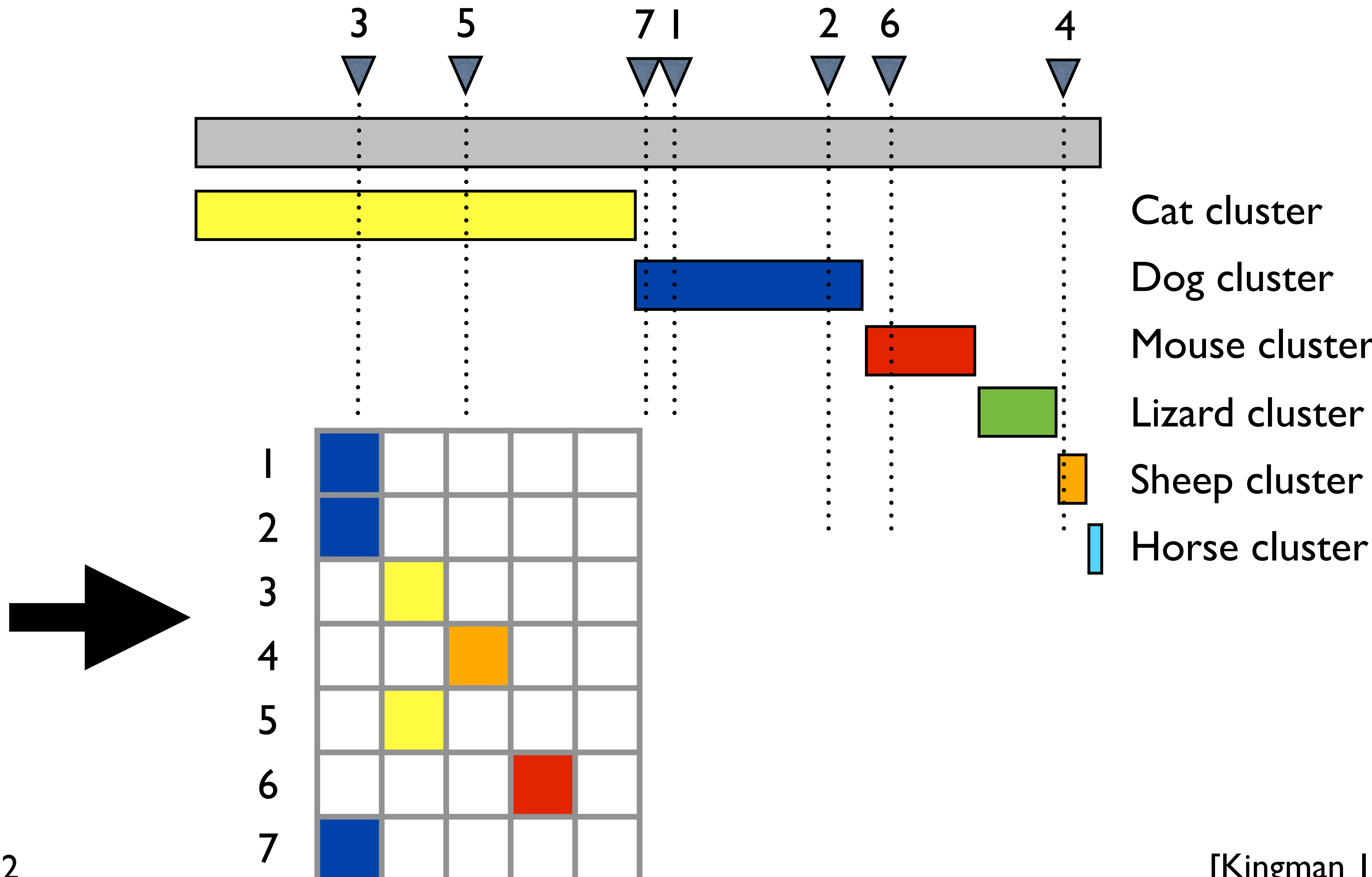
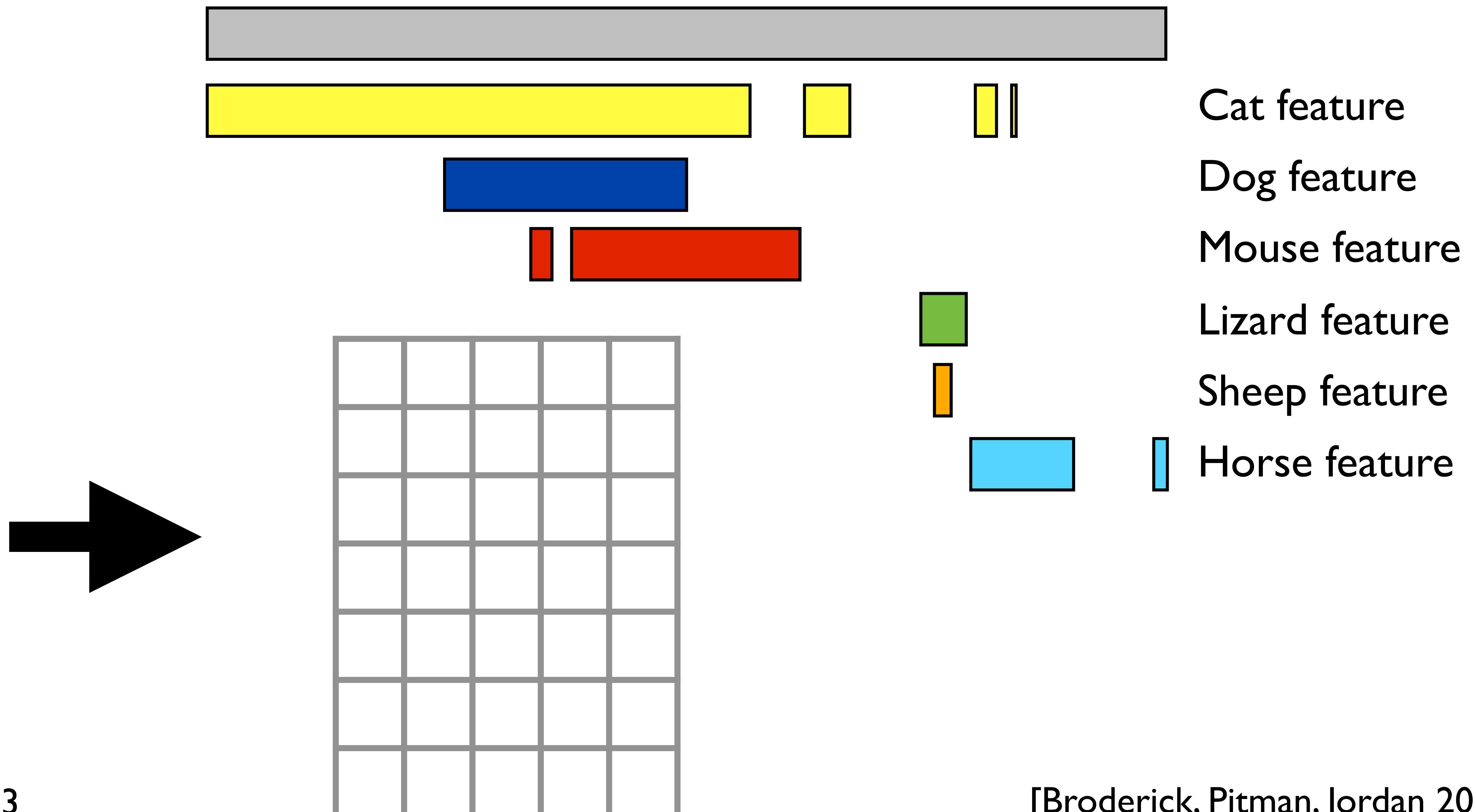
Paintboxes

Exchangeable partition: Kingman paintbox



[Kingman 1978]

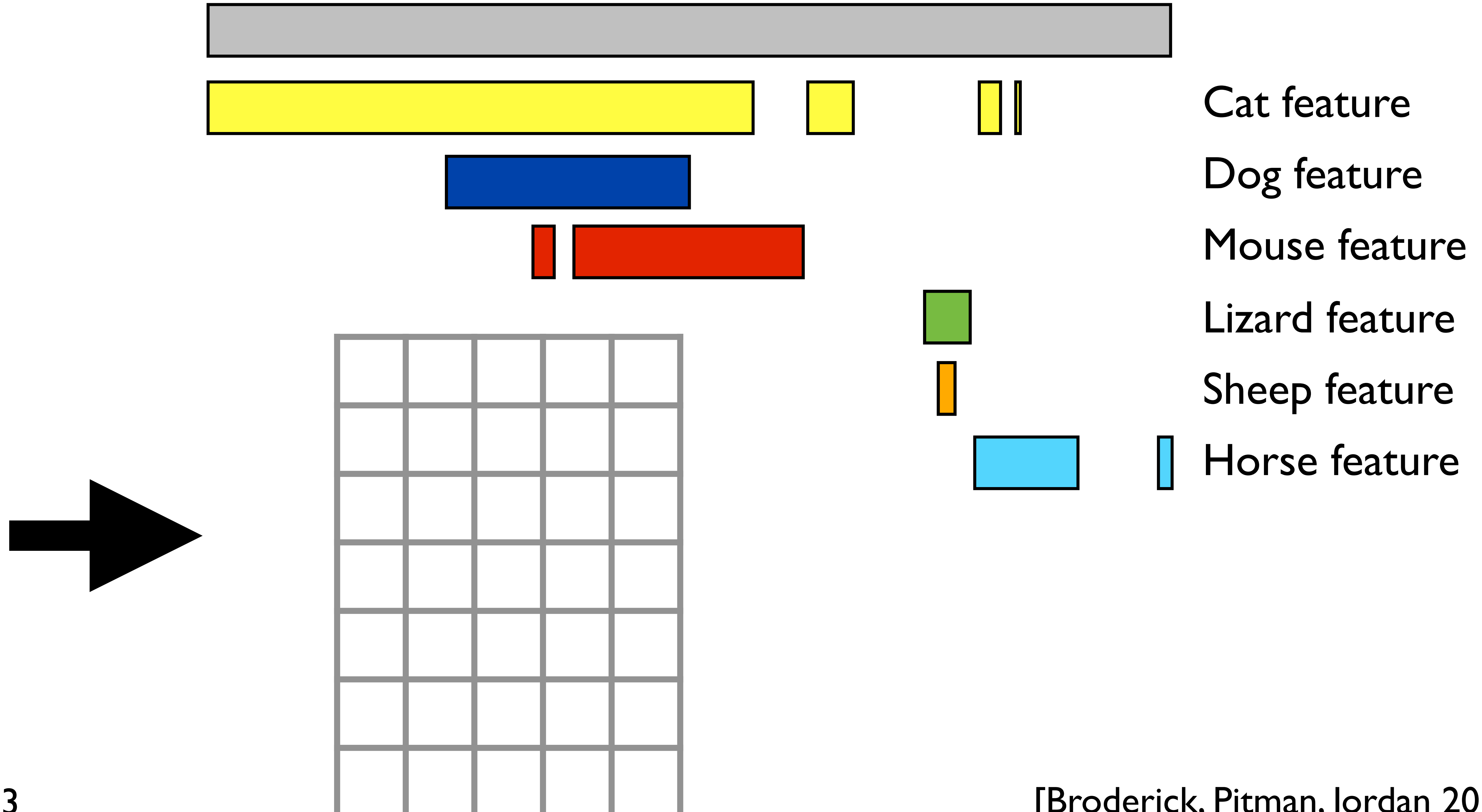
Paintboxes



[Broderick, Pitman, Jordan 2013]

Paintboxes

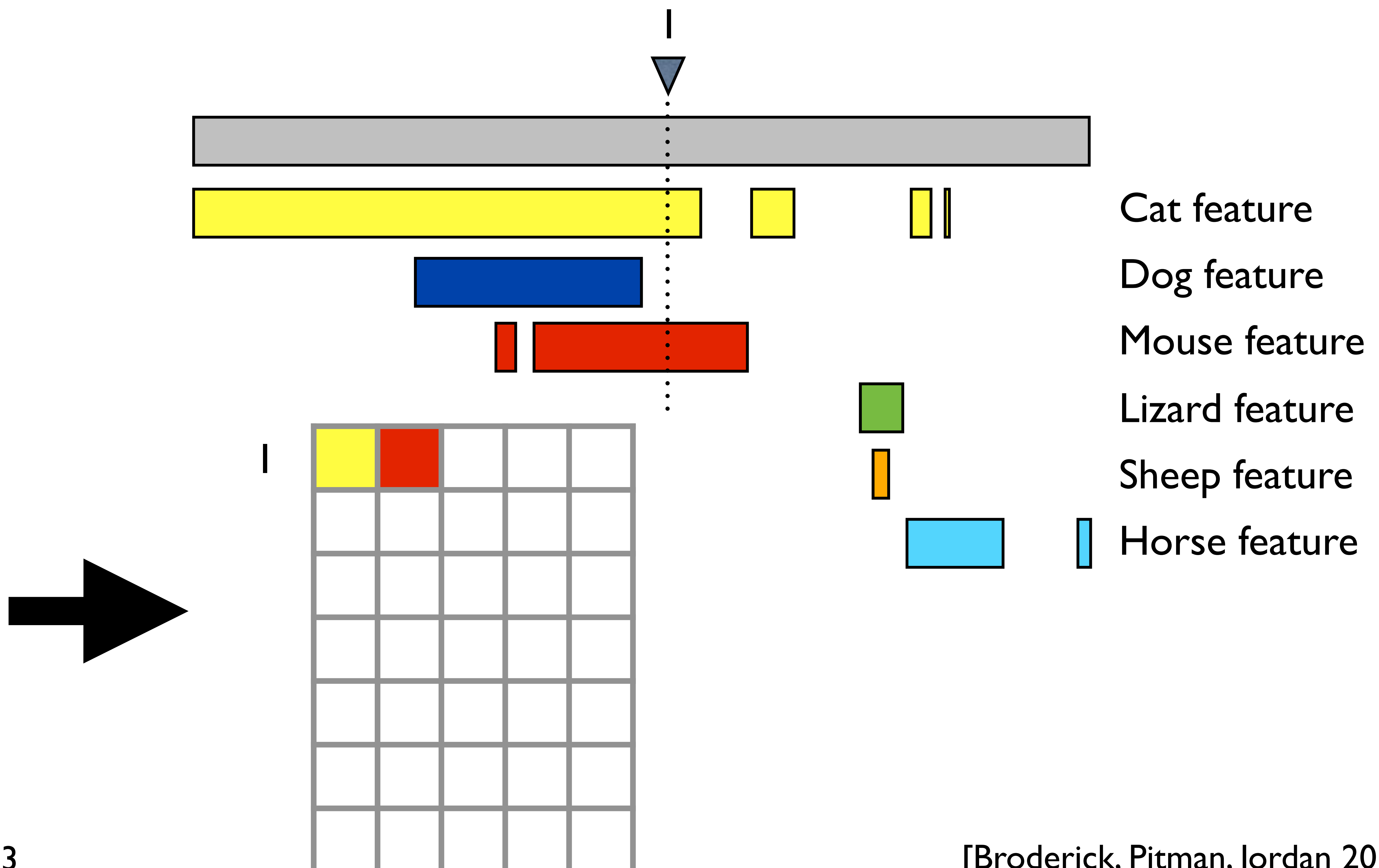
Exchangeable feature allocation: feature paintbox



[Broderick, Pitman, Jordan 2013]

Paintboxes

Exchangeable feature allocation: feature paintbox



[Broderick, Pitman, Jordan 2013]

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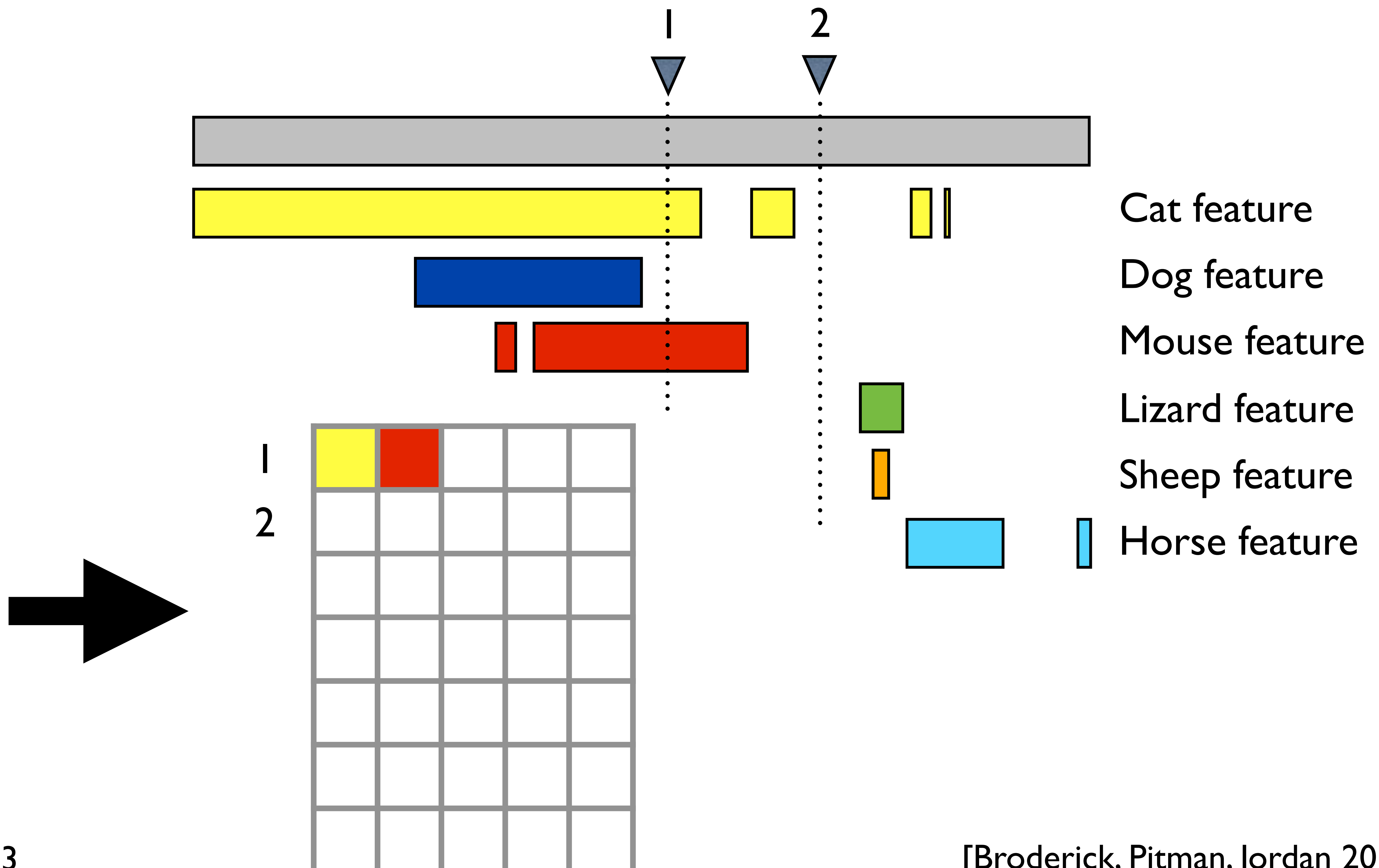
Exchangeable feature allocation: feature paintbox



[Broderick, Pitman, Jordan 2013]

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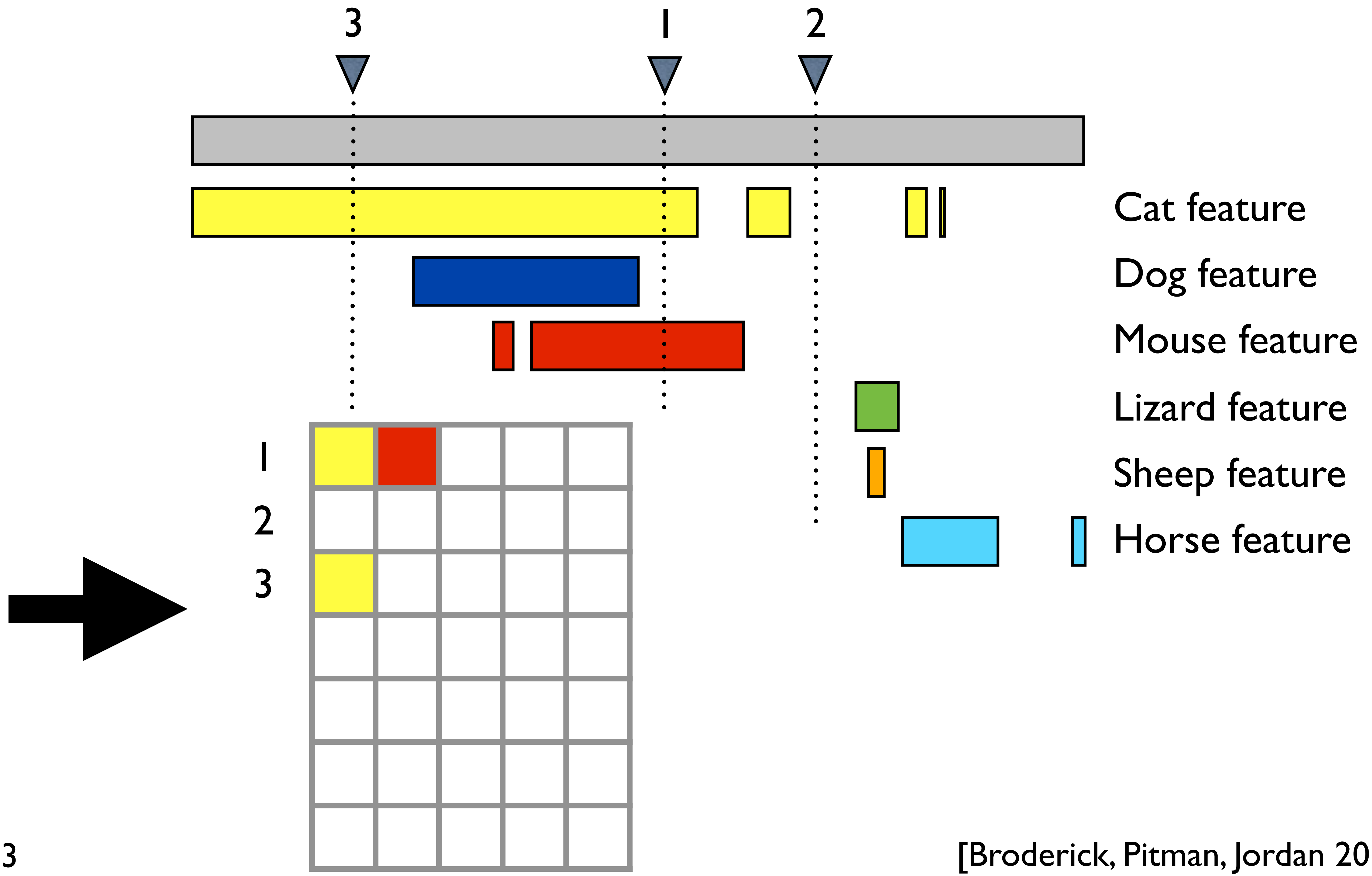
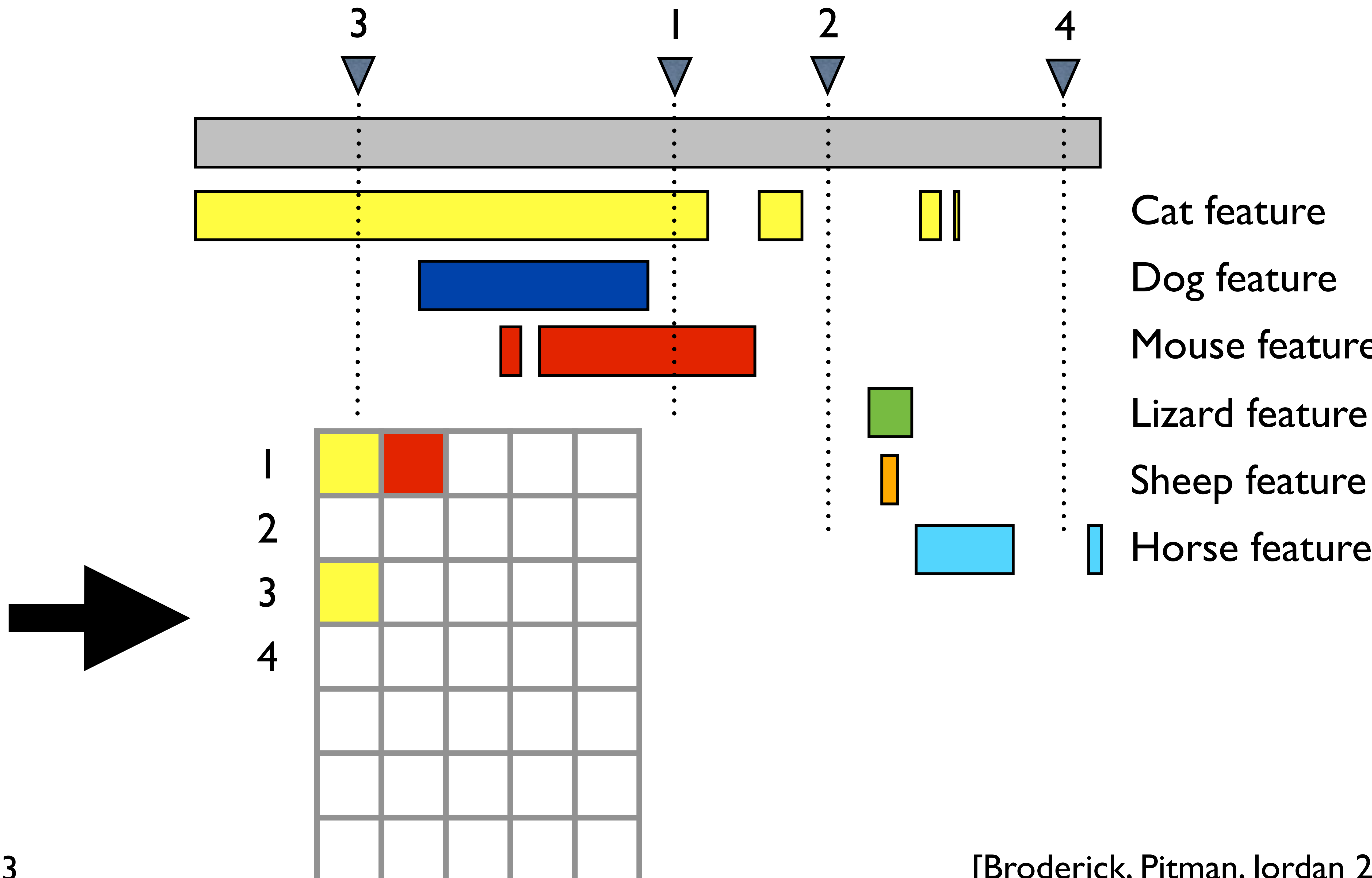
Exchangeable feature allocation: feature paintbox



[Broderick, Pitman, Jordan 2013]

Paintboxes

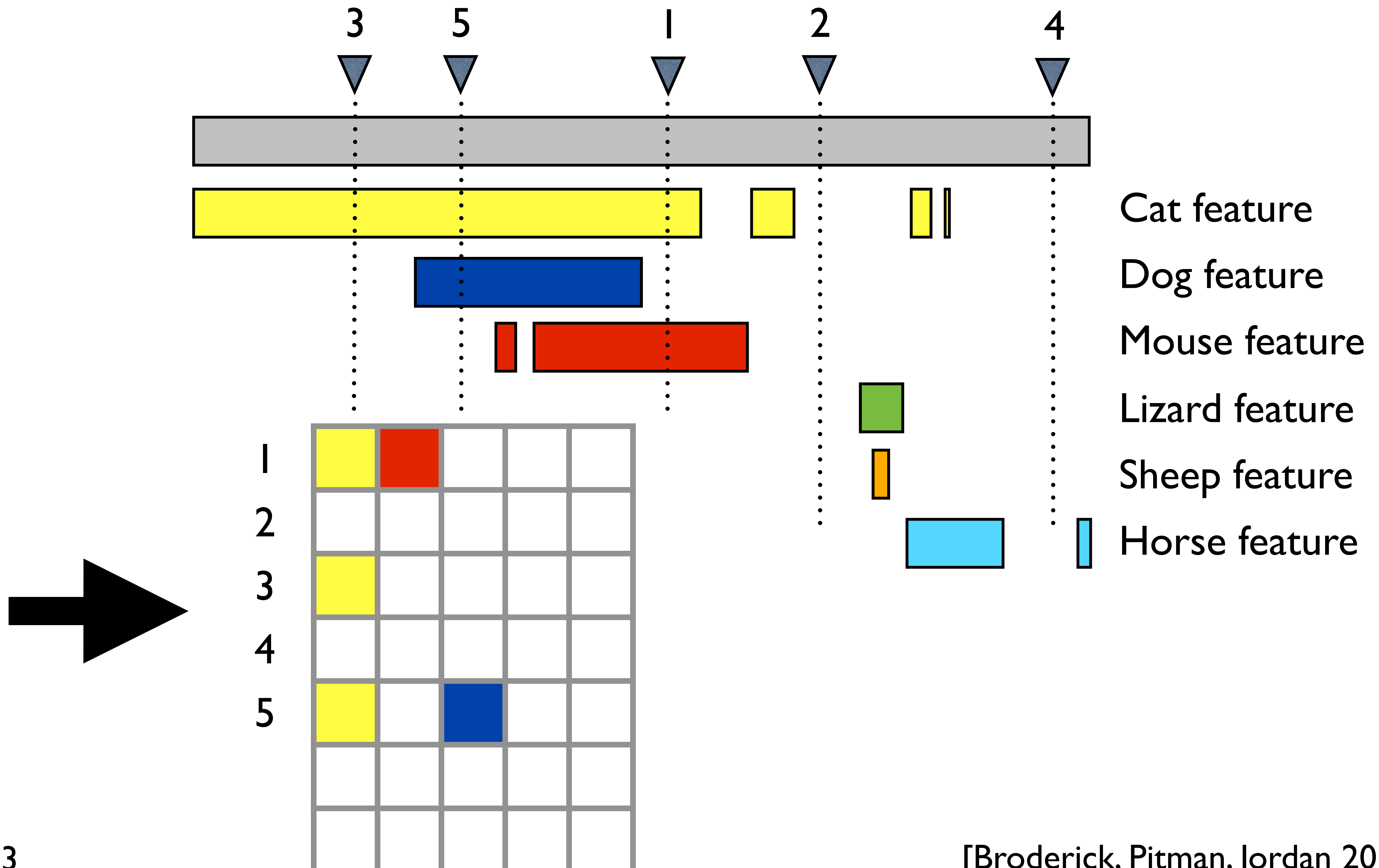
Exchangeable feature allocation: feature paintbox



[Broderick, Pitman, Jordan 2013]

Paintboxes

Exchangeable feature allocation: feature paintbox



[Broderick, Pitman, Jordan 2013]

Paintboxes

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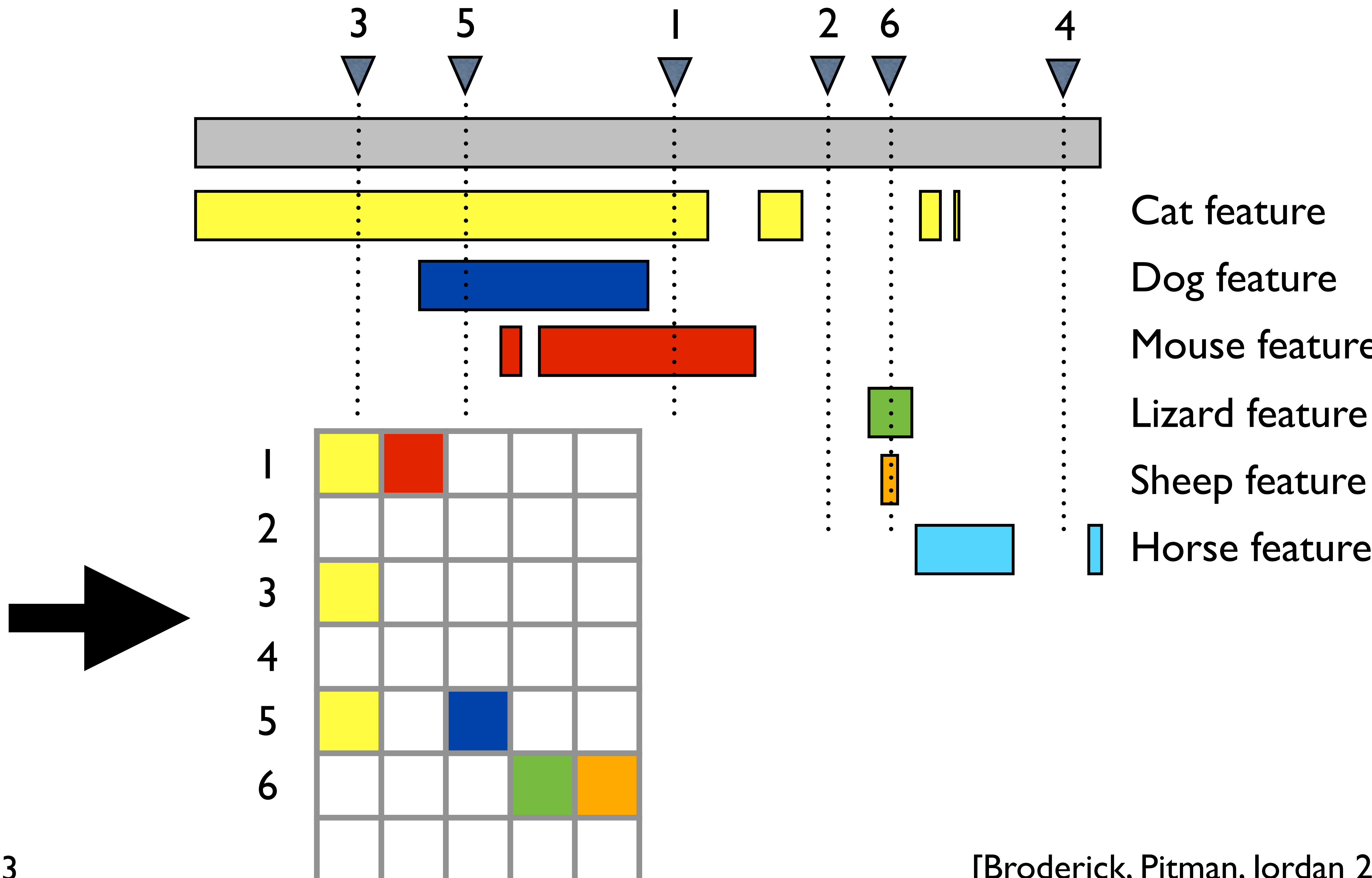

[Broderick, Pitman, Jordan 2013]

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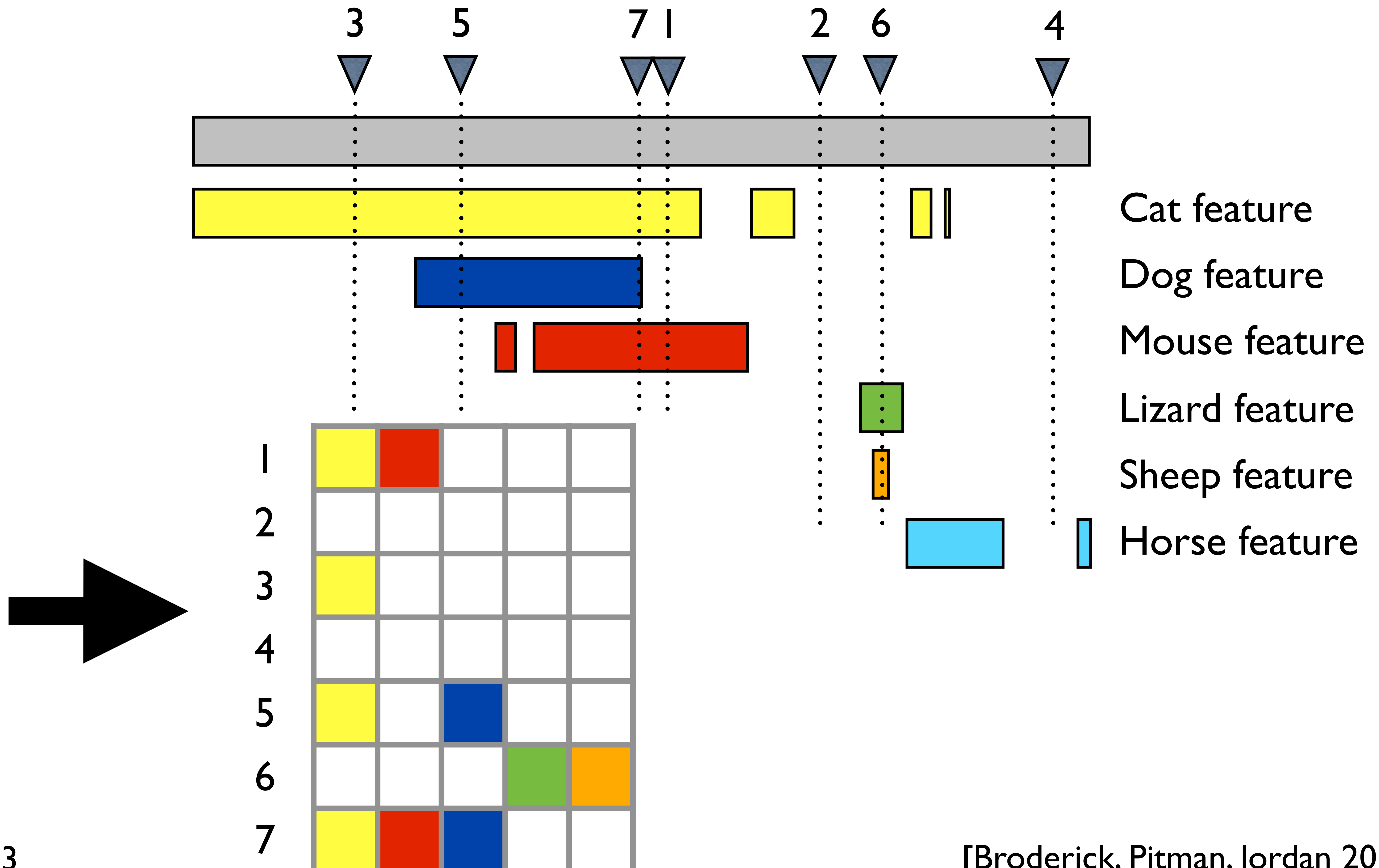
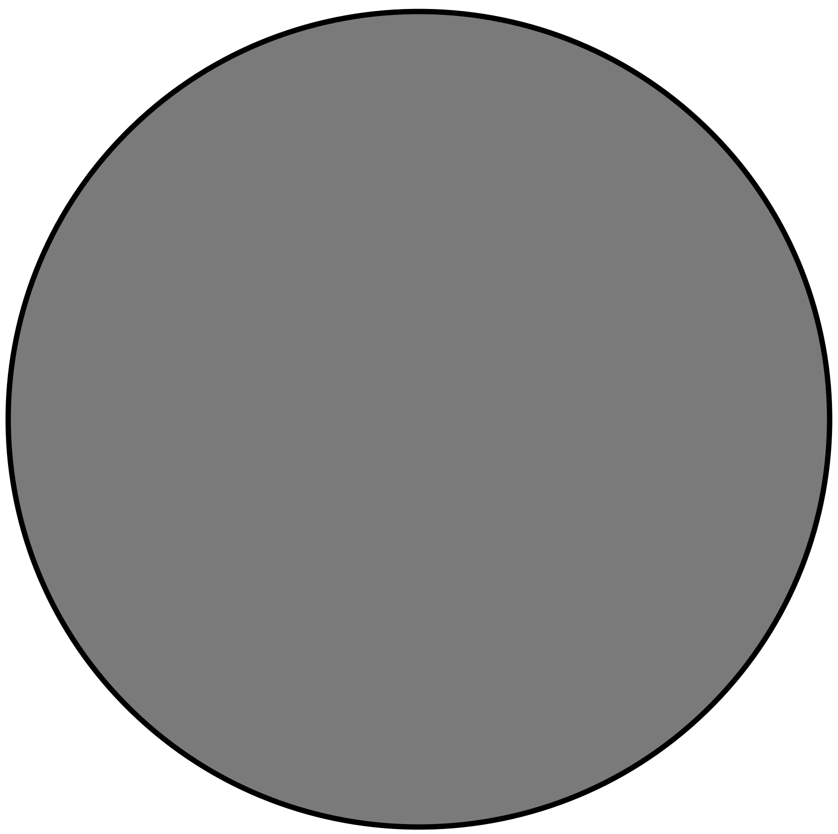


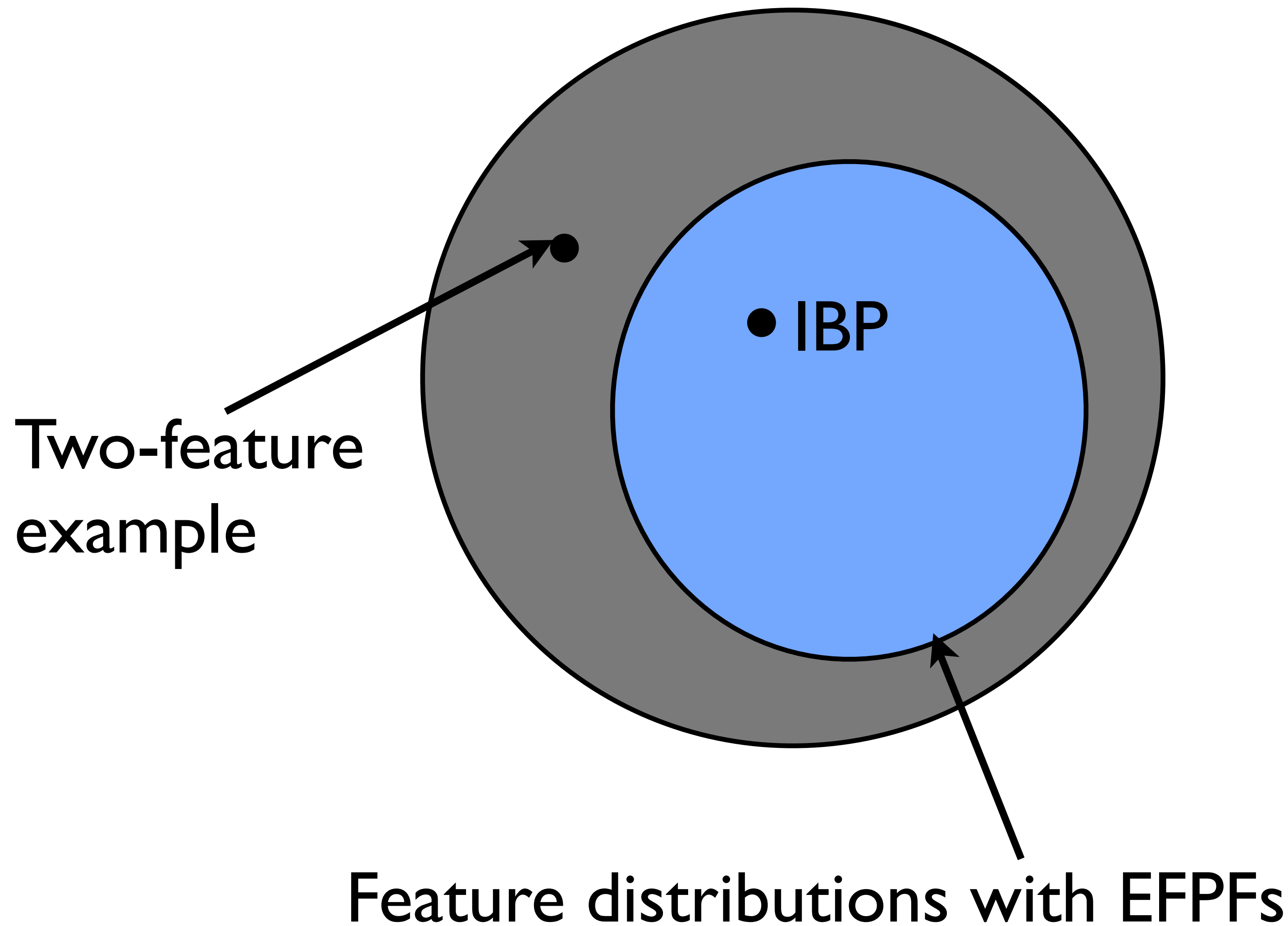
[Broderick, Pitman, Jordan 2013]

Paintboxes

Exchangeable cluster distributions
= Cluster distributions with EPPFs



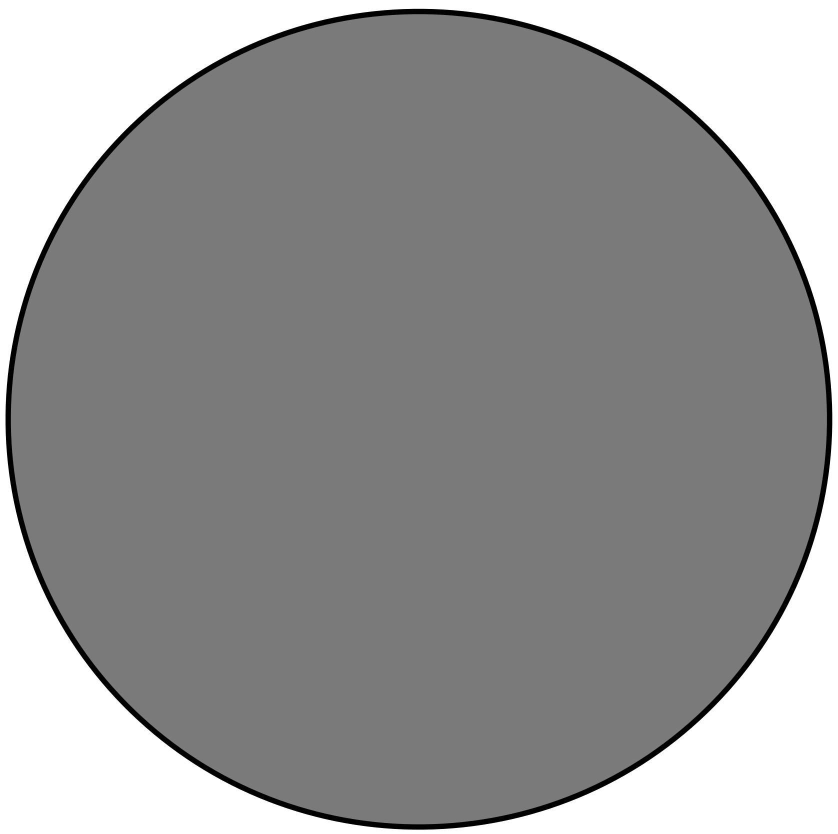
Exchangeable feature distributions



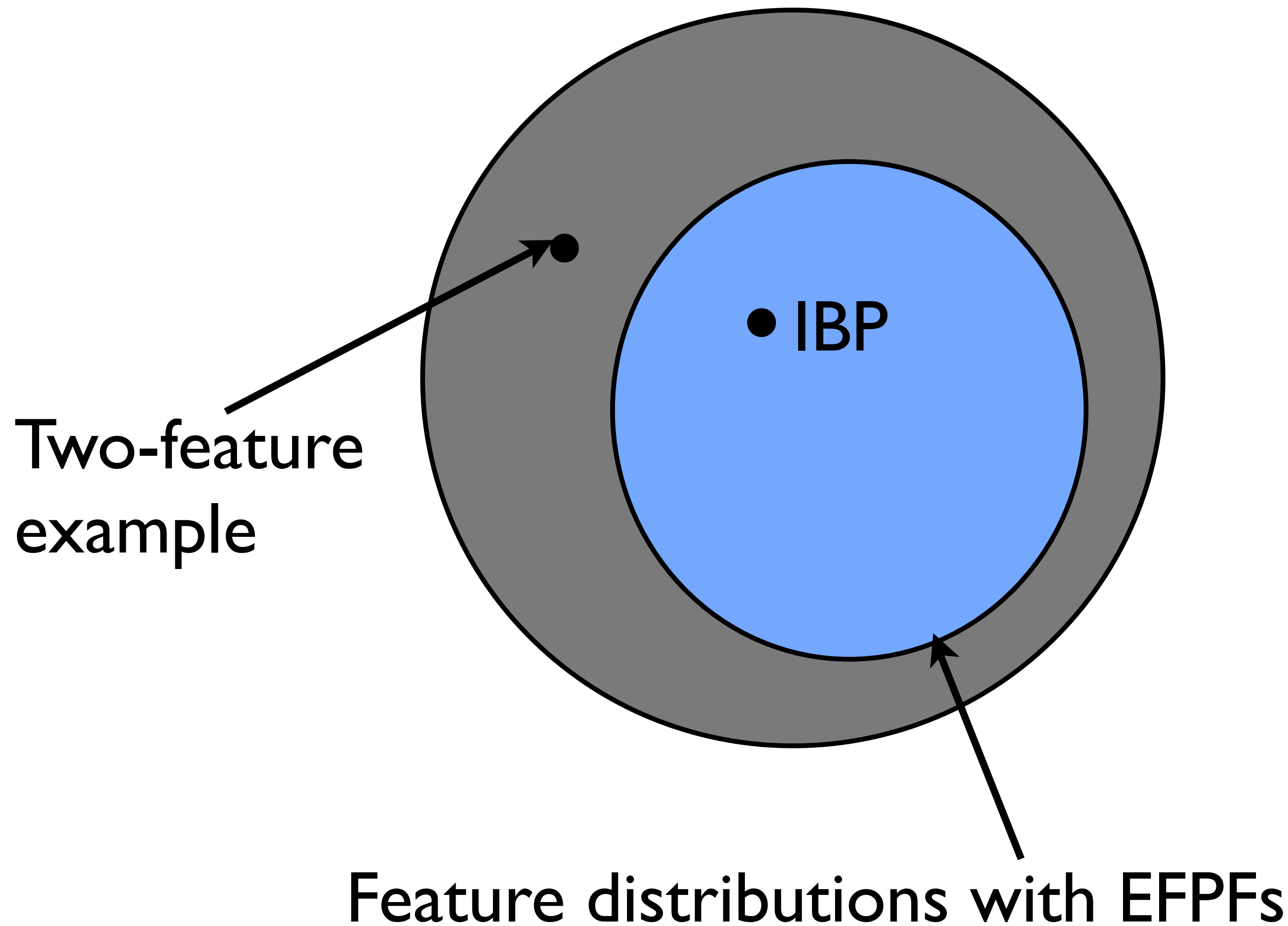
[Broderick, Pitman, Jordan 2013]

Paintboxes

Exchangeable cluster distributions
= Cluster distributions with EPPFs
= Kingman paintbox partitions



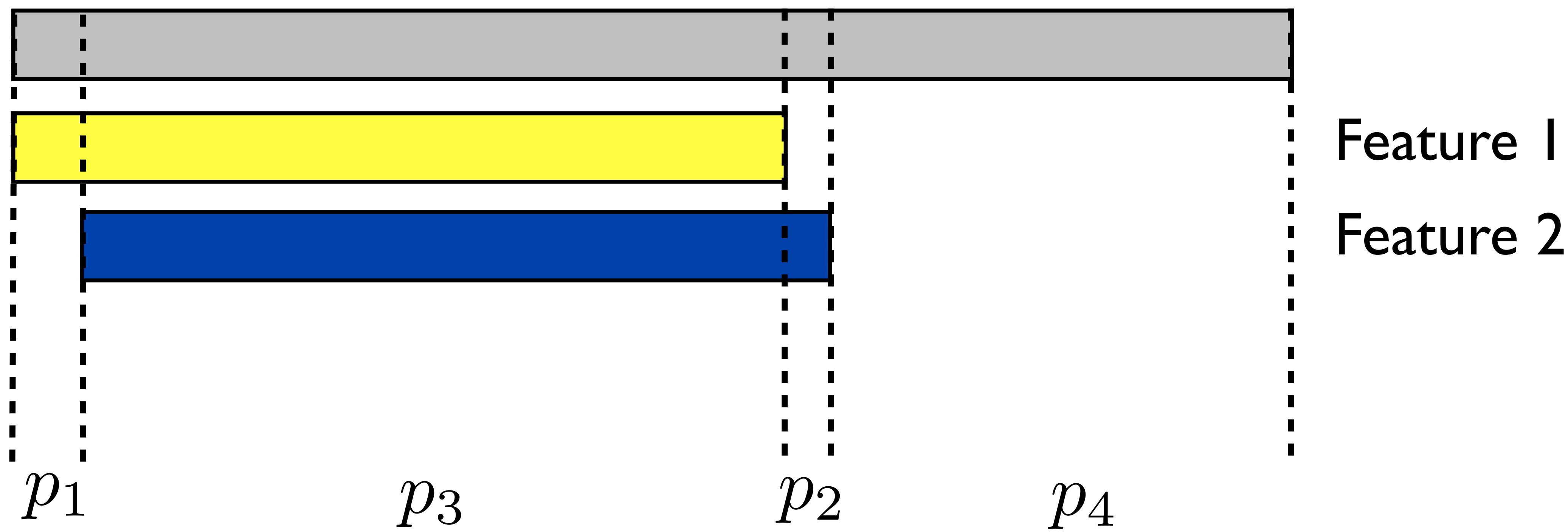
Exchangeable feature distributions
= Feature paintbox allocations



[Broderick, Pitman, Jordan 2013]

Paintboxes

Two feature example



$$\mathbb{P}(\text{row} = \blacksquare\square\,) = p_1$$

$$\mathbb{P}(\text{row} = \square\blacksquare\,) = p_2$$

$$\mathbb{P}(\text{row} = \blacksquare\blacksquare\,) = p_3$$

$$\mathbb{P}(\text{row} = \square\square\,) = p_4$$

Paintboxes

Indian buffet process: beta feature frequencies

[Thibaux, Jordan 2007]

Paintboxes

Indian buffet process: beta feature frequencies

For m = 1, 2, ...
1. Draw $K_m^+ = \mathrm{Poisson}\left(\gamma \frac{\theta}{\theta + m - 1}\right)$

[Thibaux, Jordan 2007]

Paintboxes

Indian buffet process: beta feature frequencies

For *m* = 1, 2, ...

1. Draw $K_m^+ = \mathrm{Poisson}\left(\gamma \dfrac{\theta}{\theta + m - 1}\right)$

   Set $K_m = \sum_{j=1}^{m} K_j^+$

2. For k = $K_{m-1} + 1, \ldots, K_m$

   Draw a frequency of size

$$q_k \sim \mathrm{Beta}(1, \theta + m - 1)$$

[Thibaux, Jordan 2007]

Paintboxes

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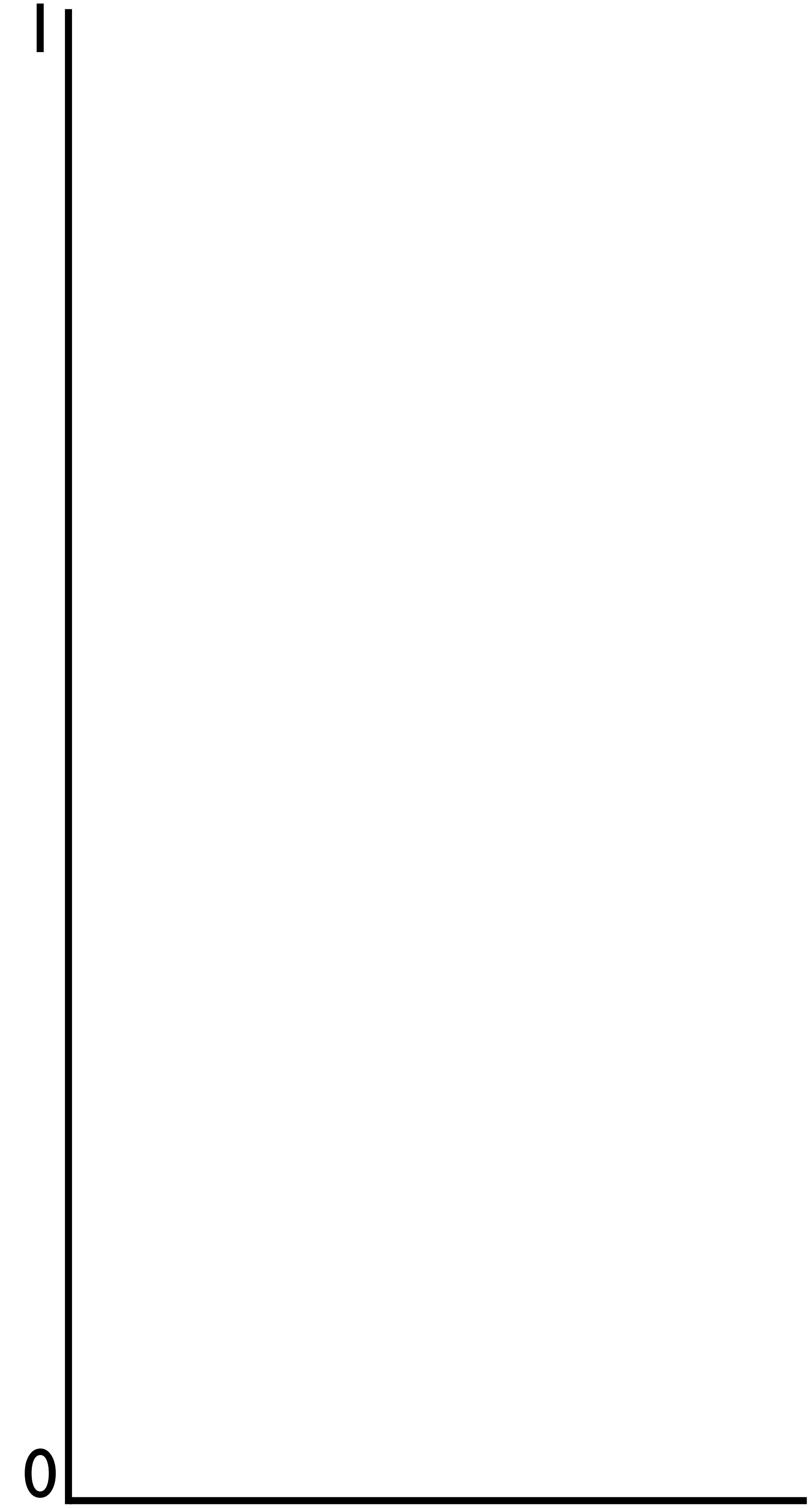

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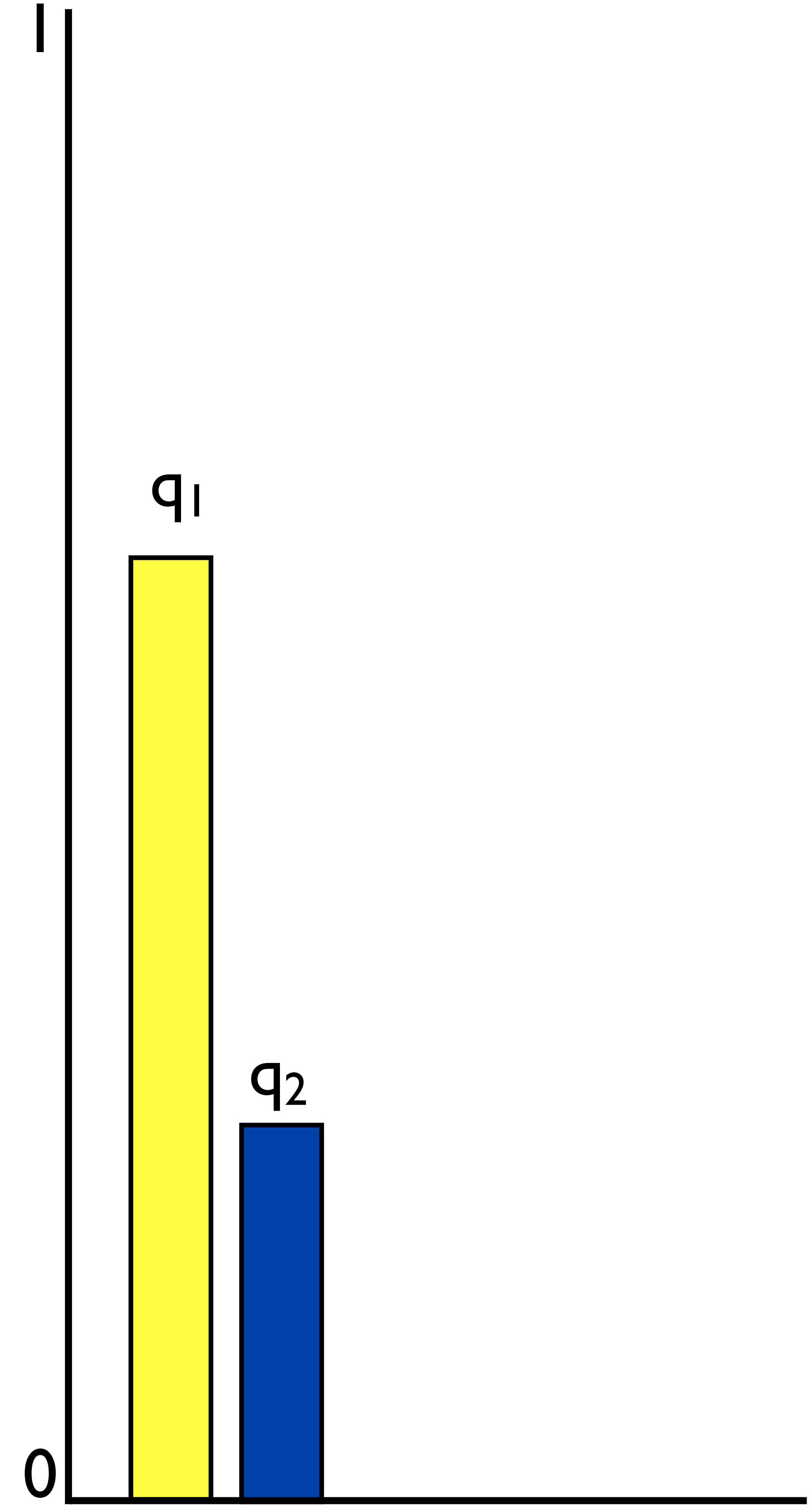

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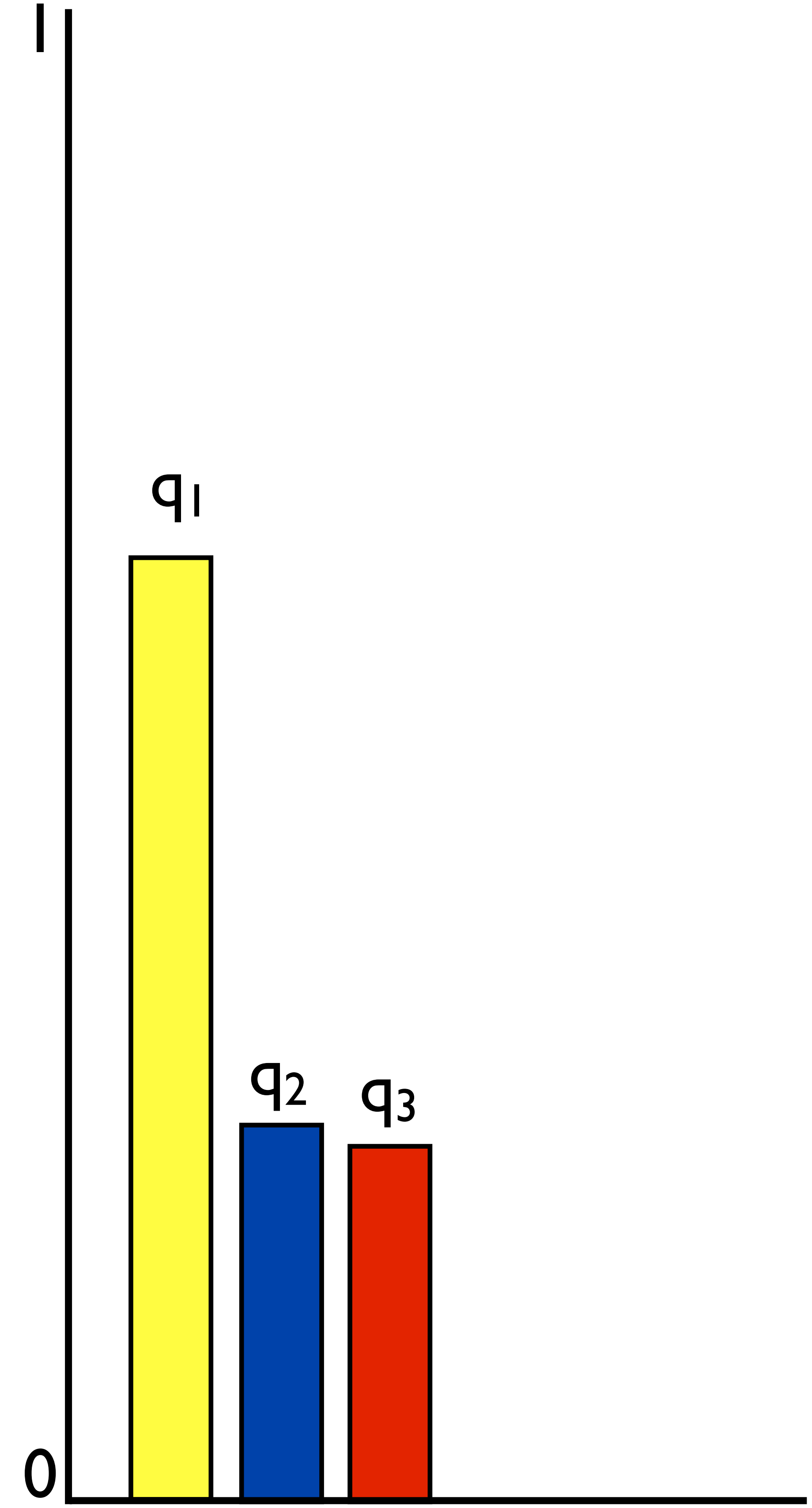

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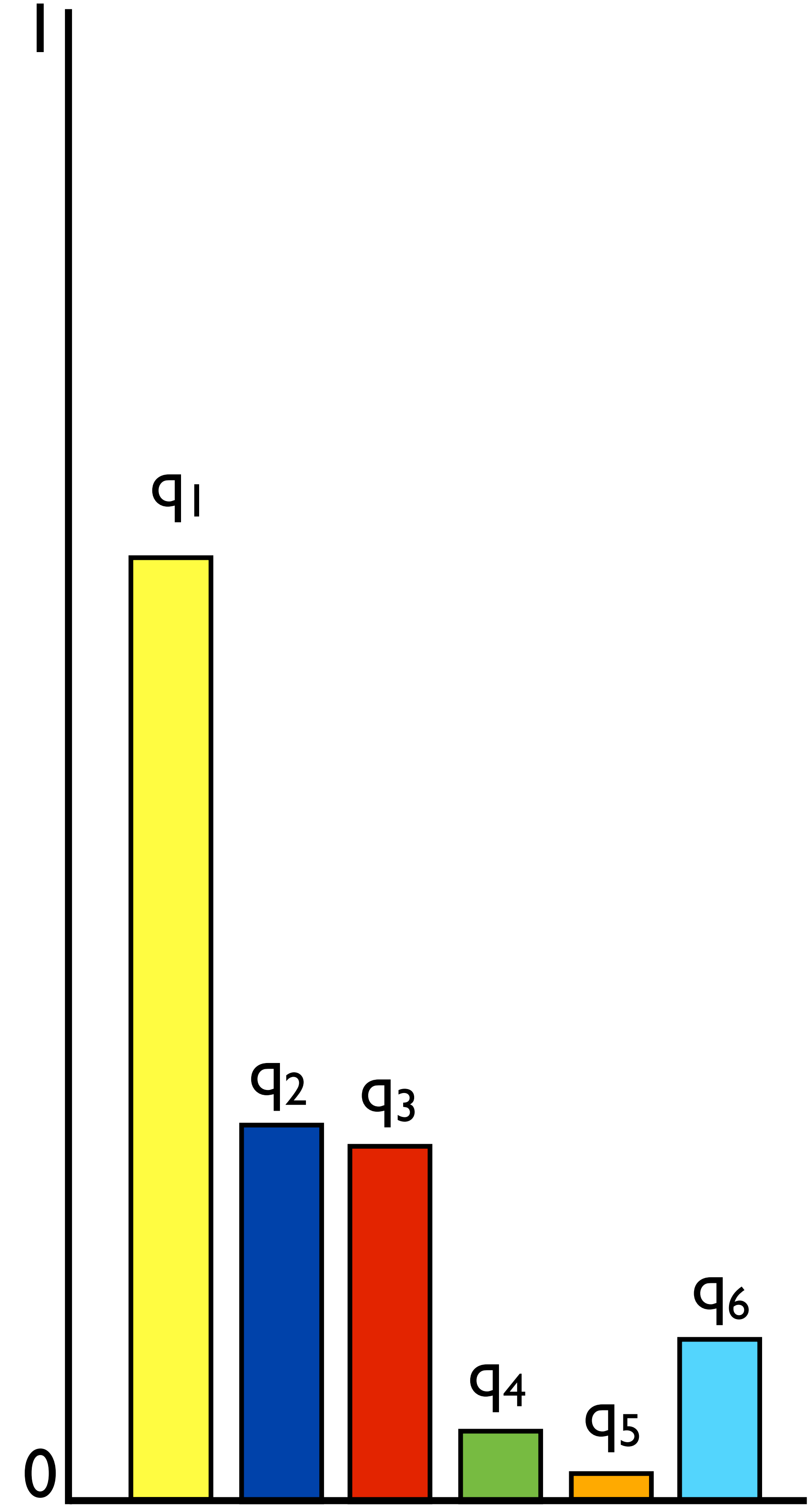

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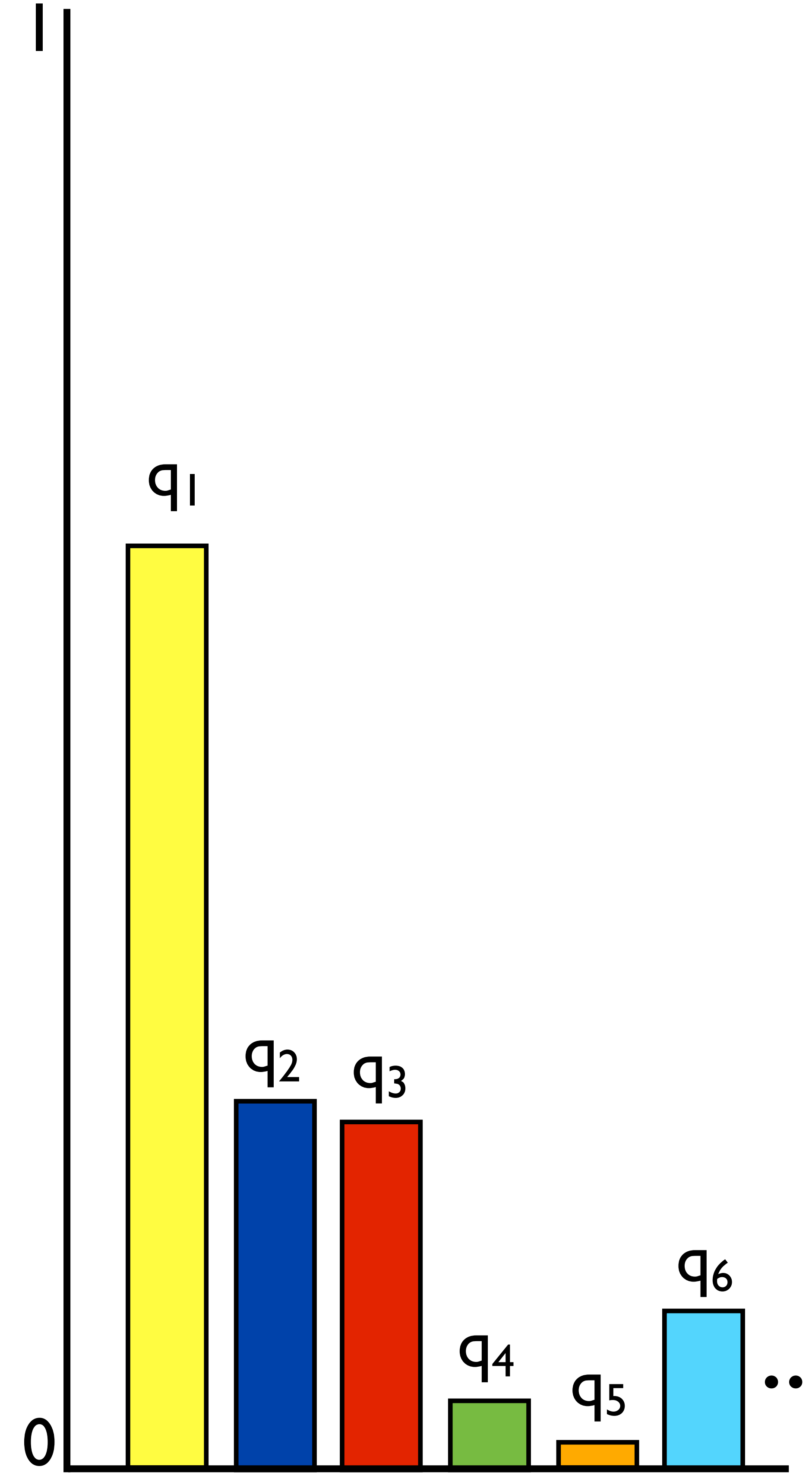

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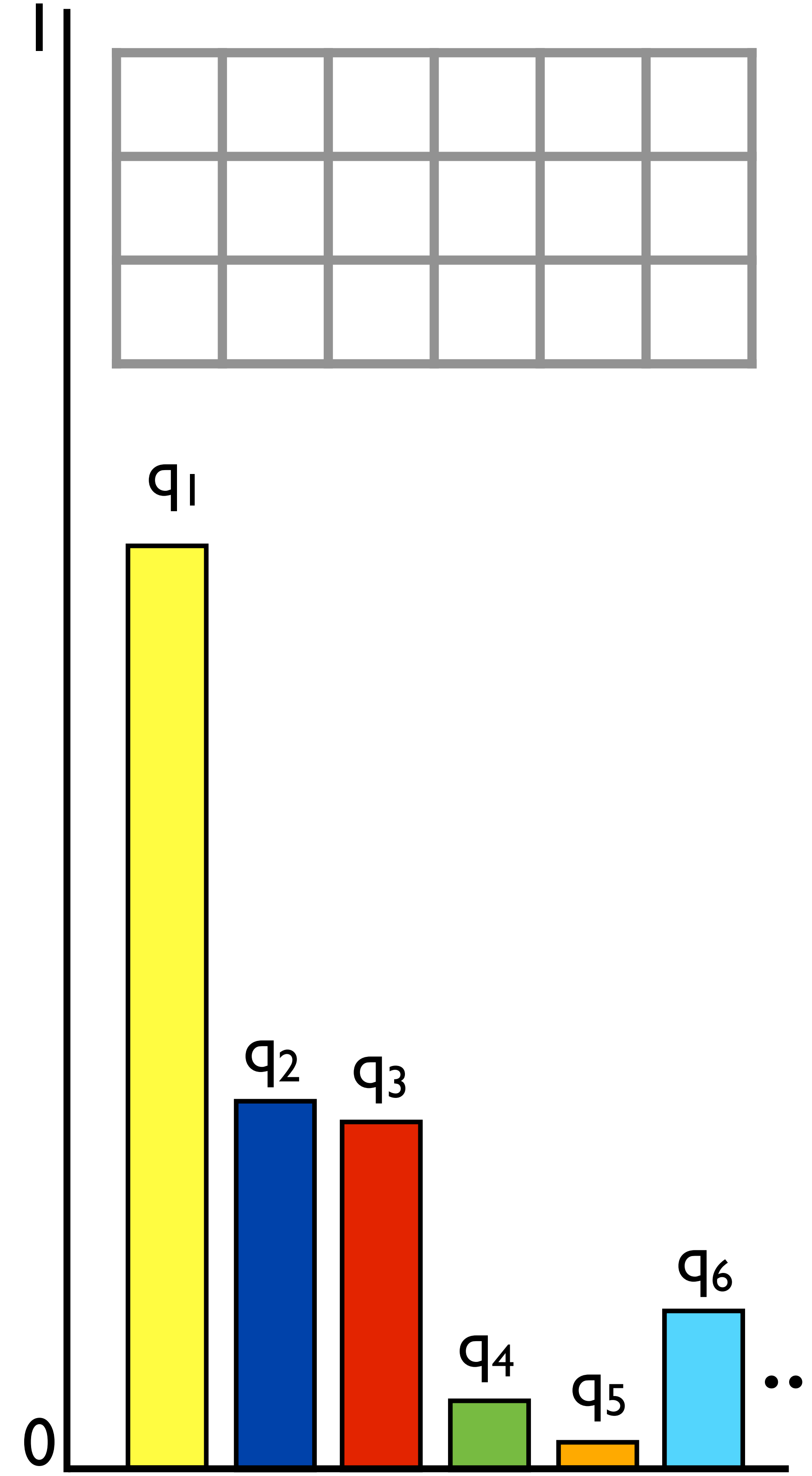

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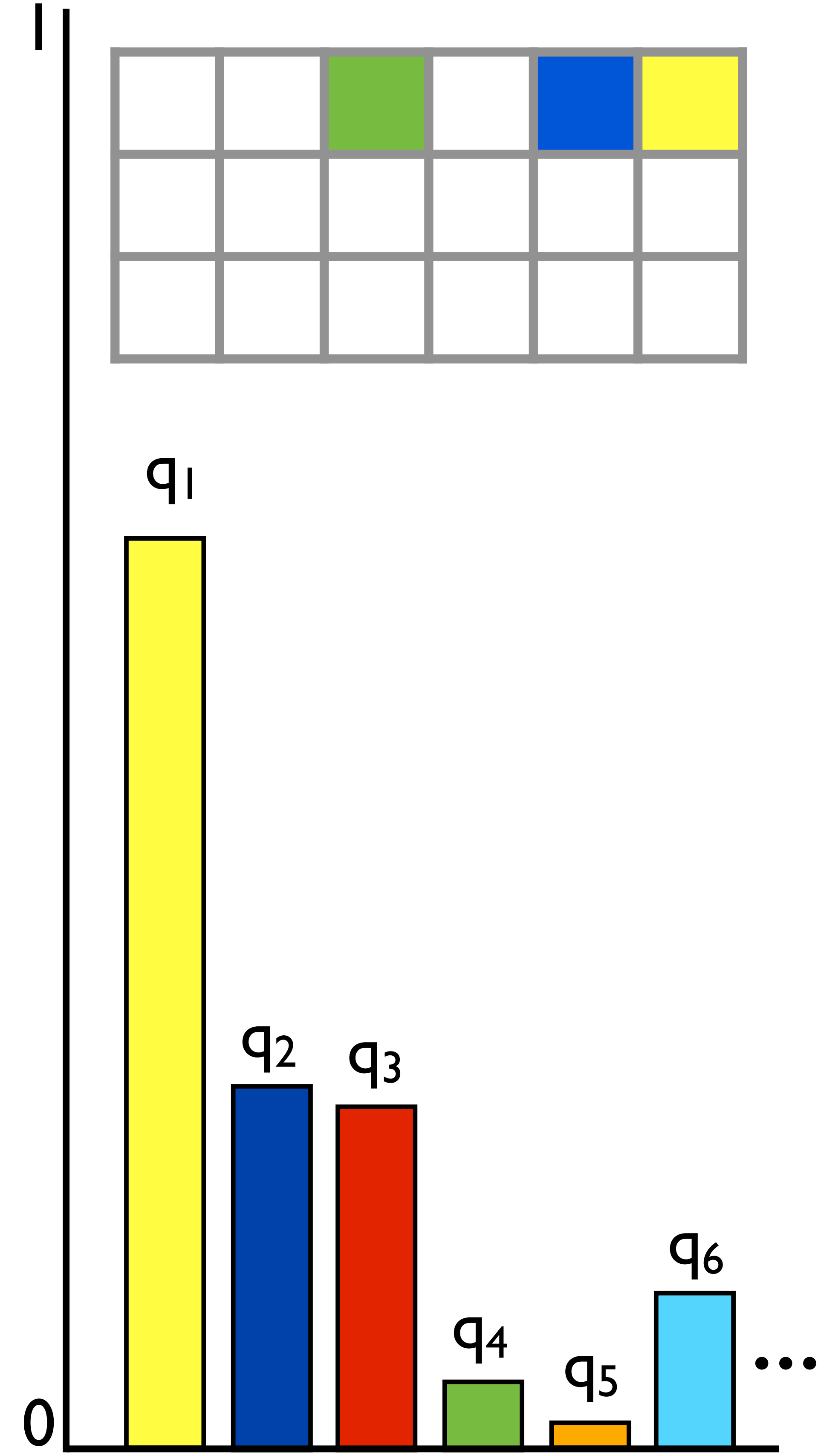

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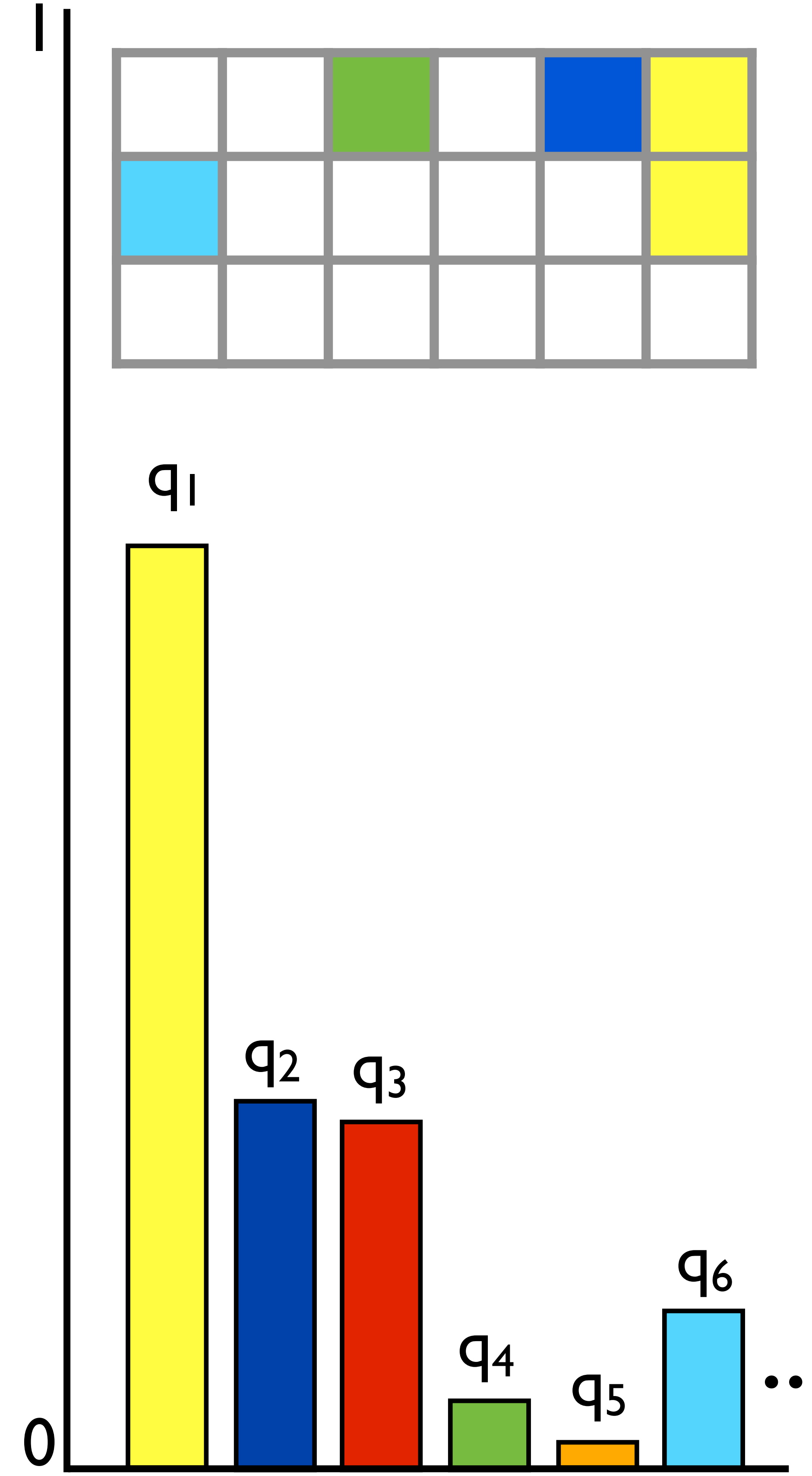

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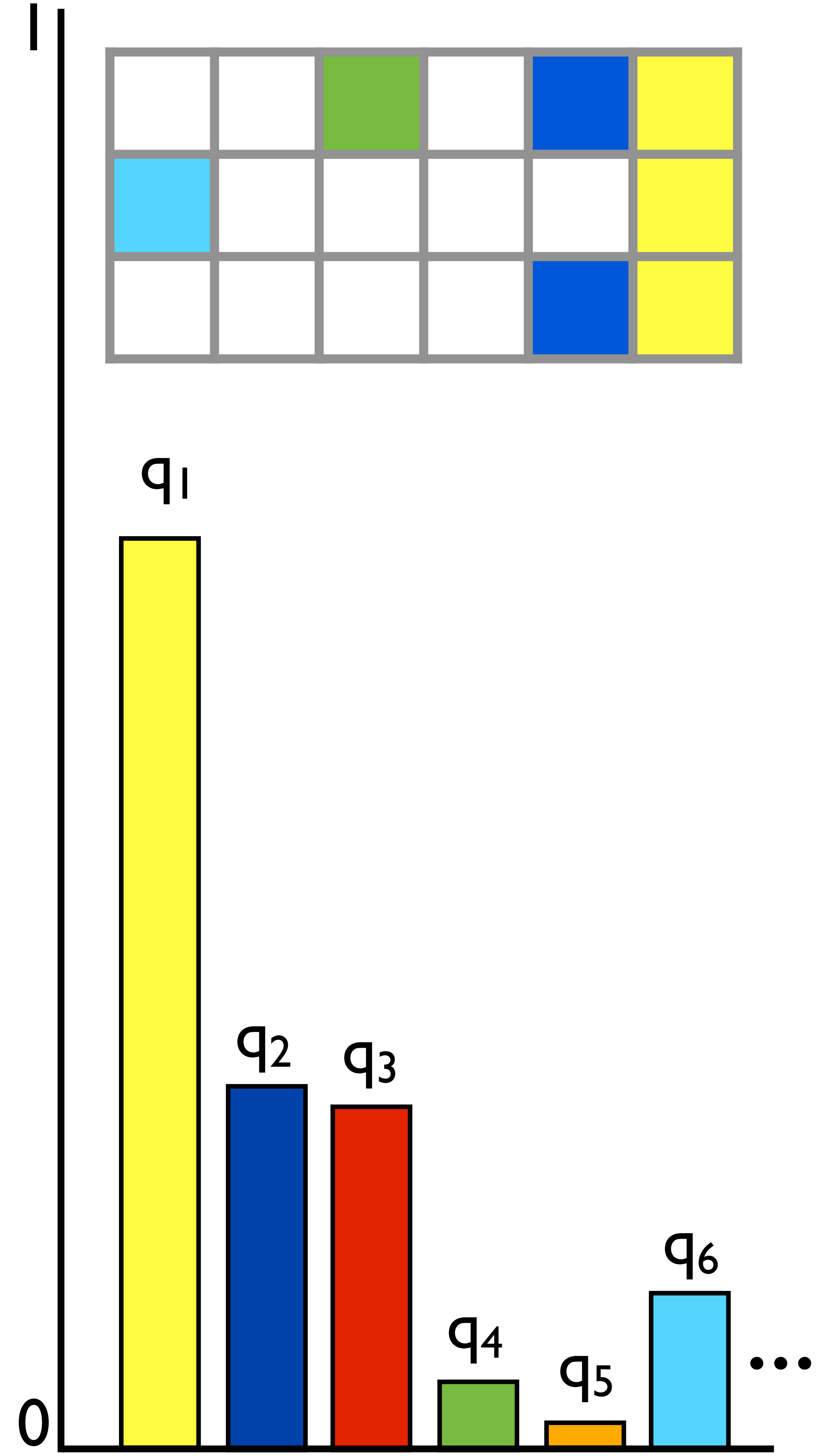
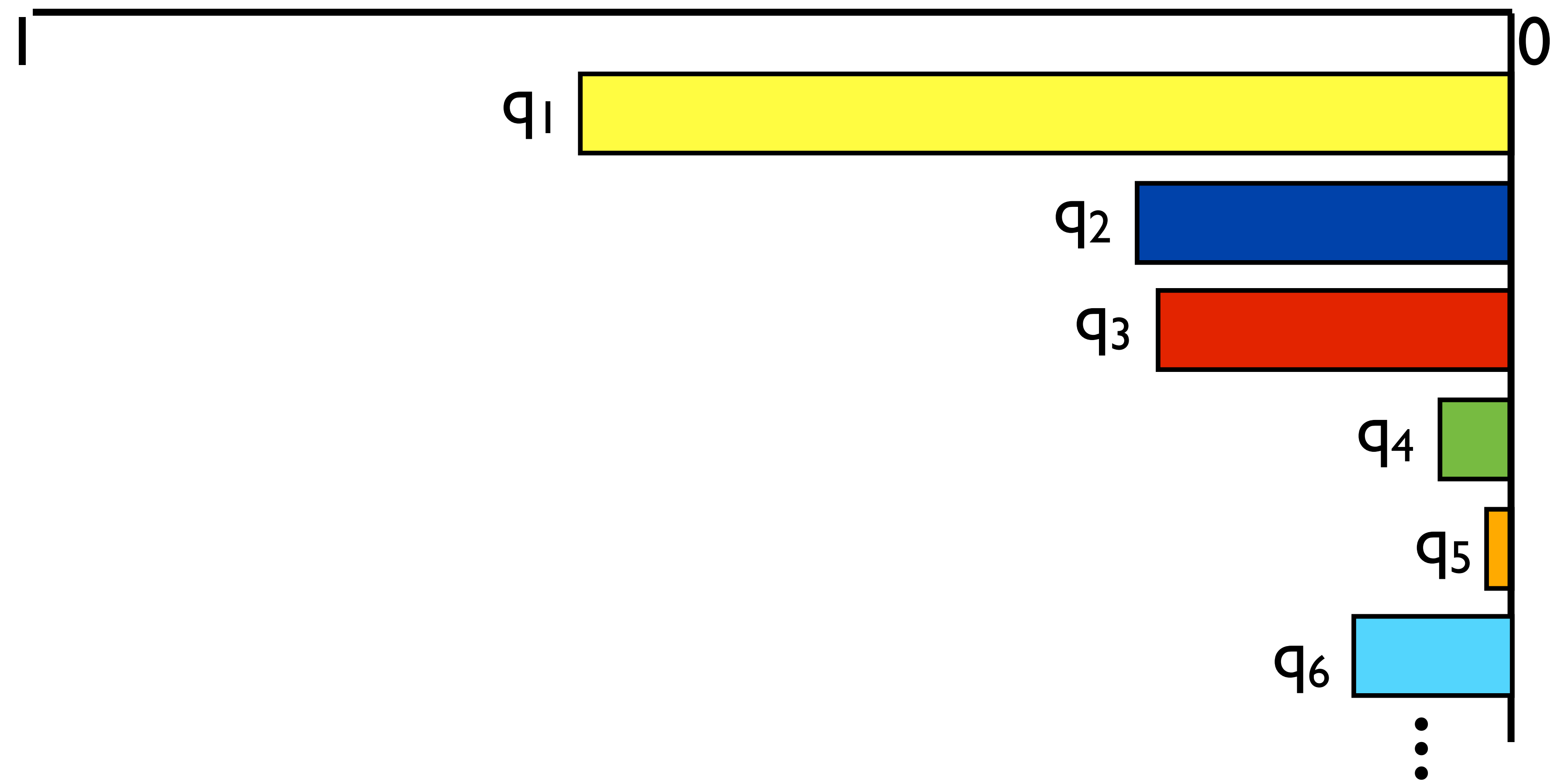


[Thibaux, Jordan 2007]

Paintboxes

Indian buffet process: beta feature frequencies

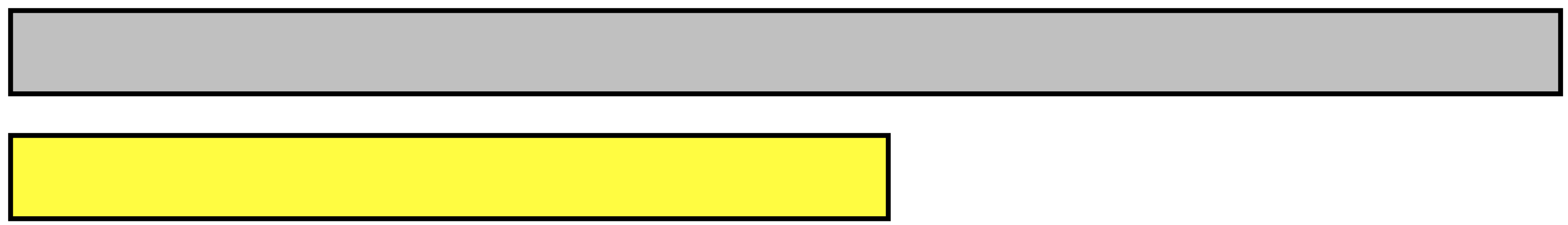


Paintboxes

Indian buffet process: beta feature frequencies

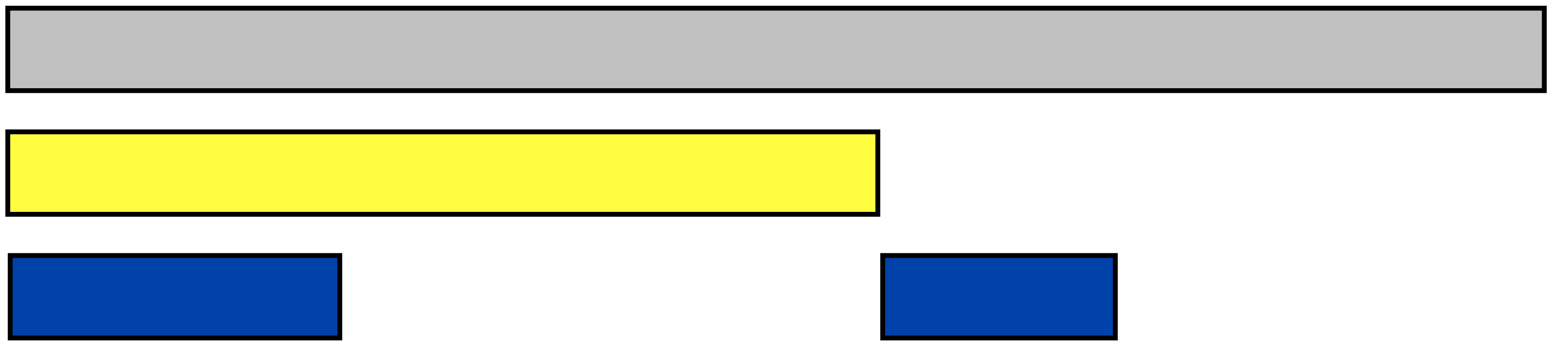
Paintboxes

Indian buffet process: beta feature frequencies



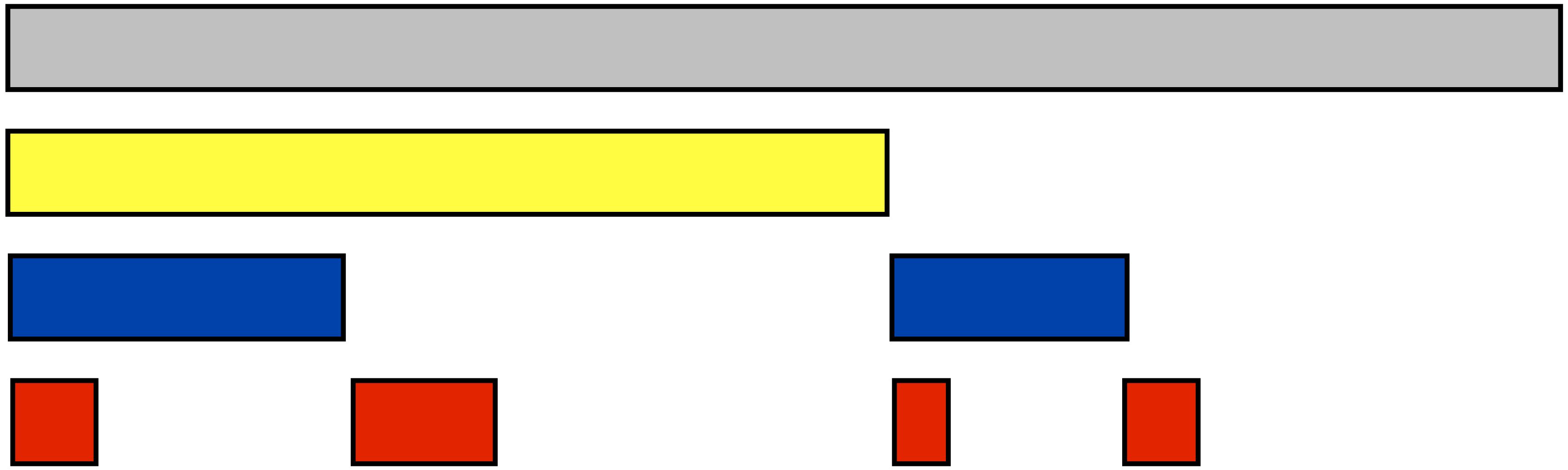
Paintboxes

Indian buffet process: beta feature frequencies



Paintboxes

Indian buffet process: beta feature frequencies



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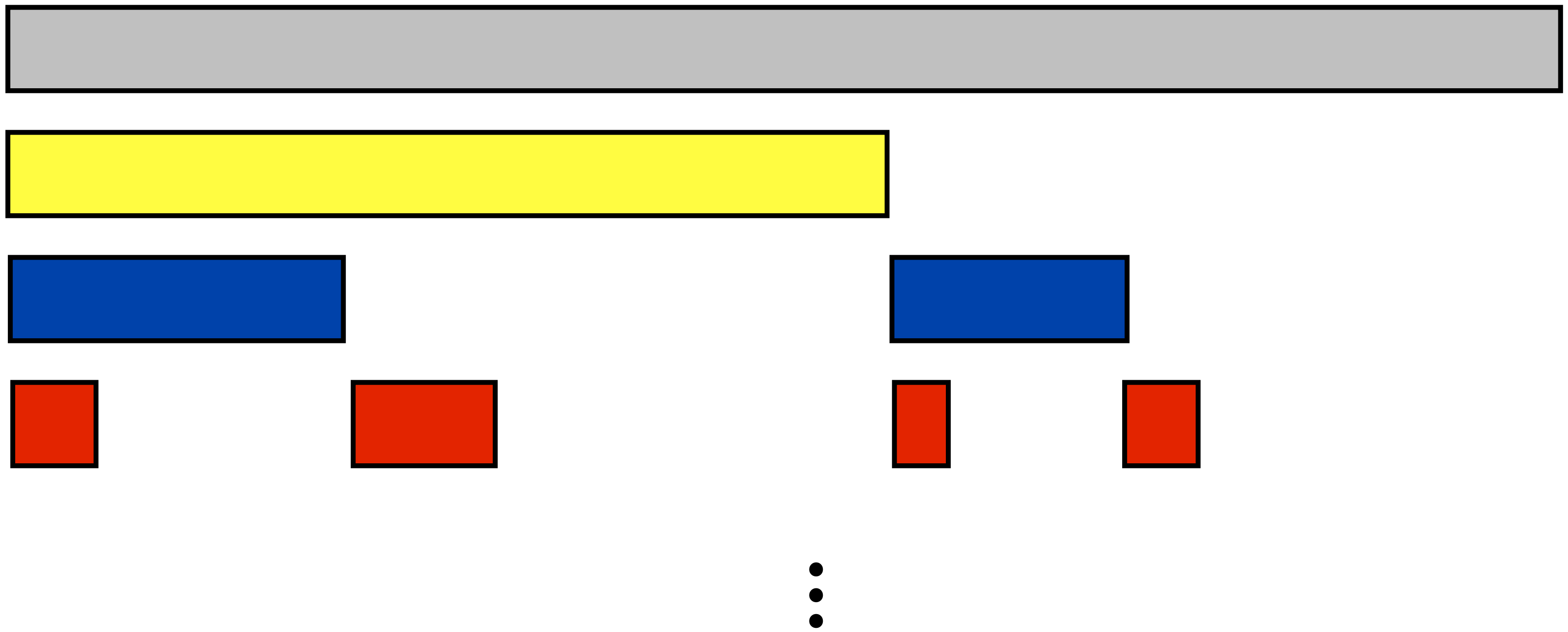
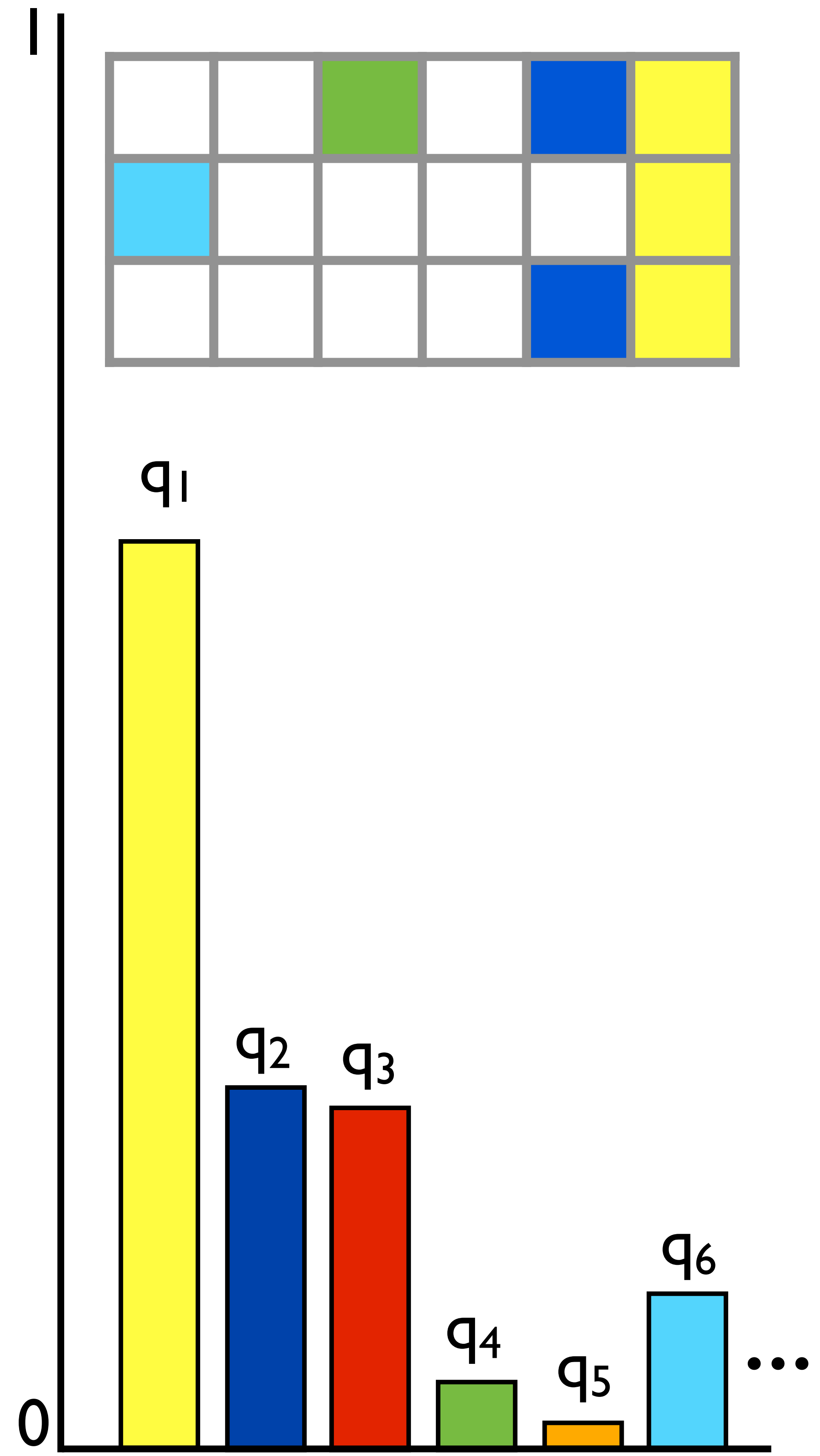
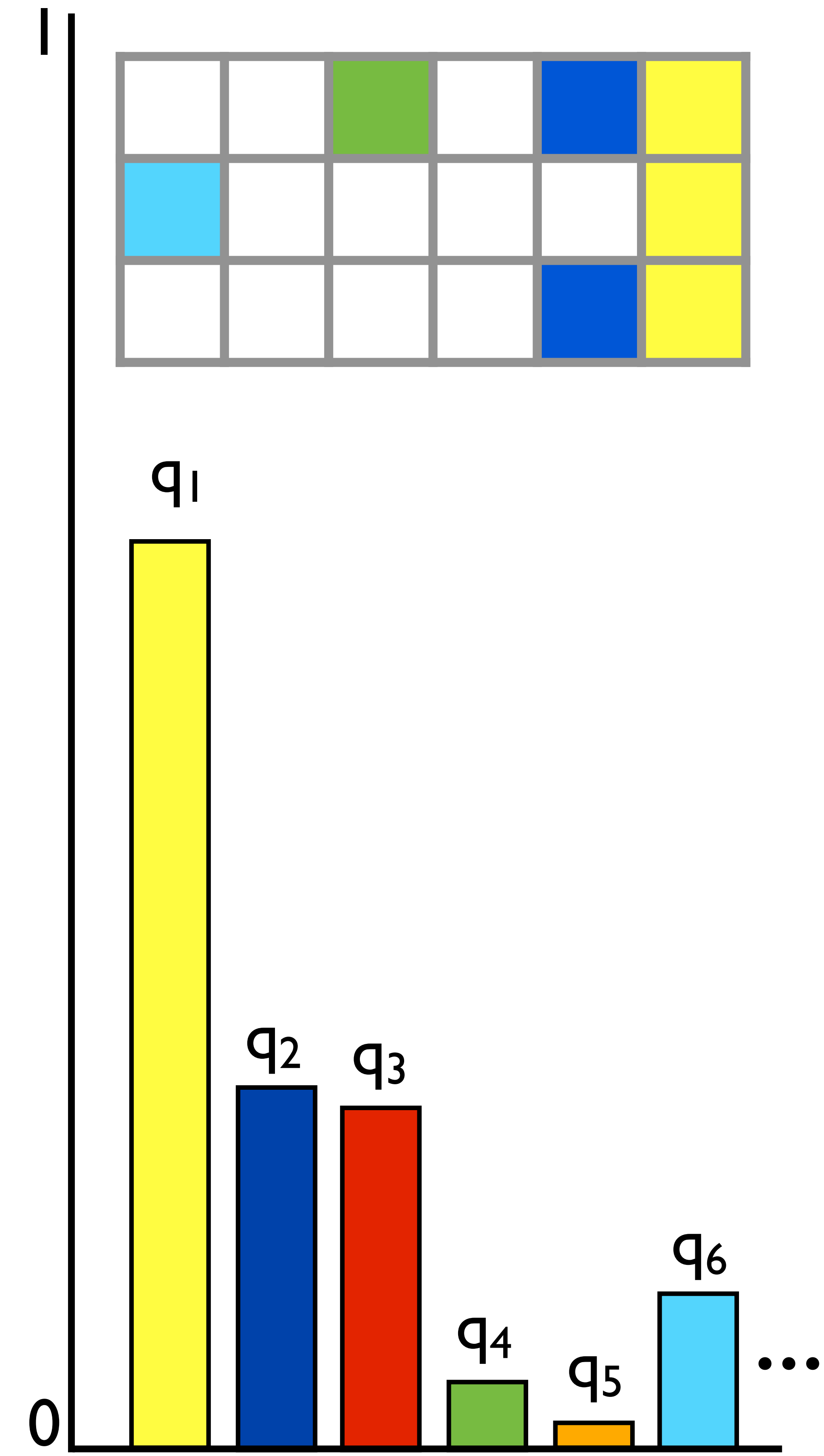


Paintboxes



Paintboxes

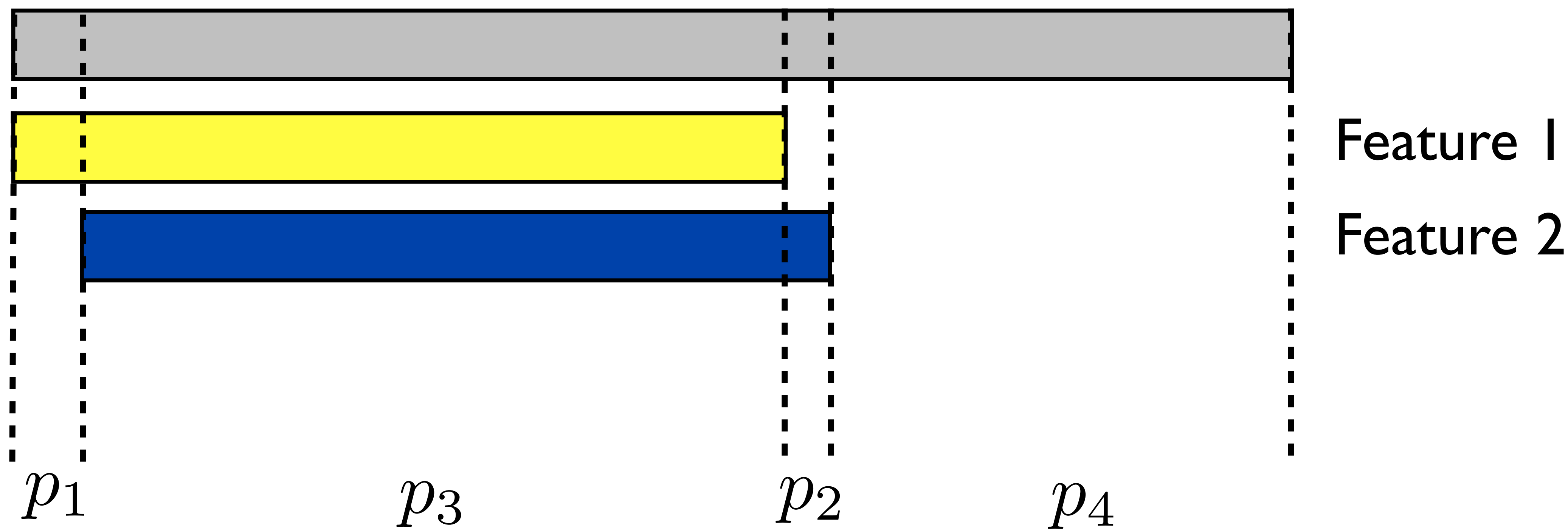
"Feature frequency models"



[Broderick, Pitman, Jordan 2013]

Paintboxes

Two feature example



$$\mathbb{P}(\text{row} = \blacksquare\square) = p_1$$

$$\mathbb{P}(\text{row} = \square\blacksquare) = p_2$$

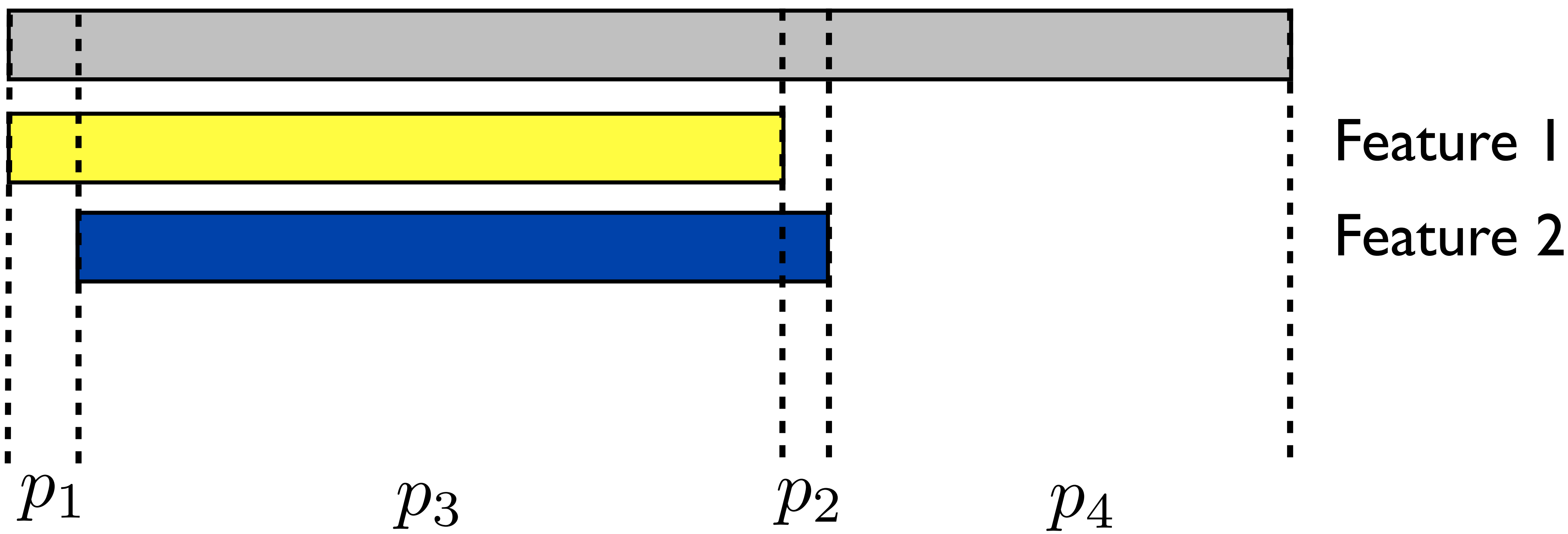
$$\mathbb{P}(\text{row} = \blacksquare\blacksquare) = p_3$$

$$\mathbb{P}(\text{row} = \square\square) = p_4$$

Paintboxes

Two feature example

Not a feature frequency model



$$\mathbb{P}(\text{row} = \blacksquare\square) = p_1$$

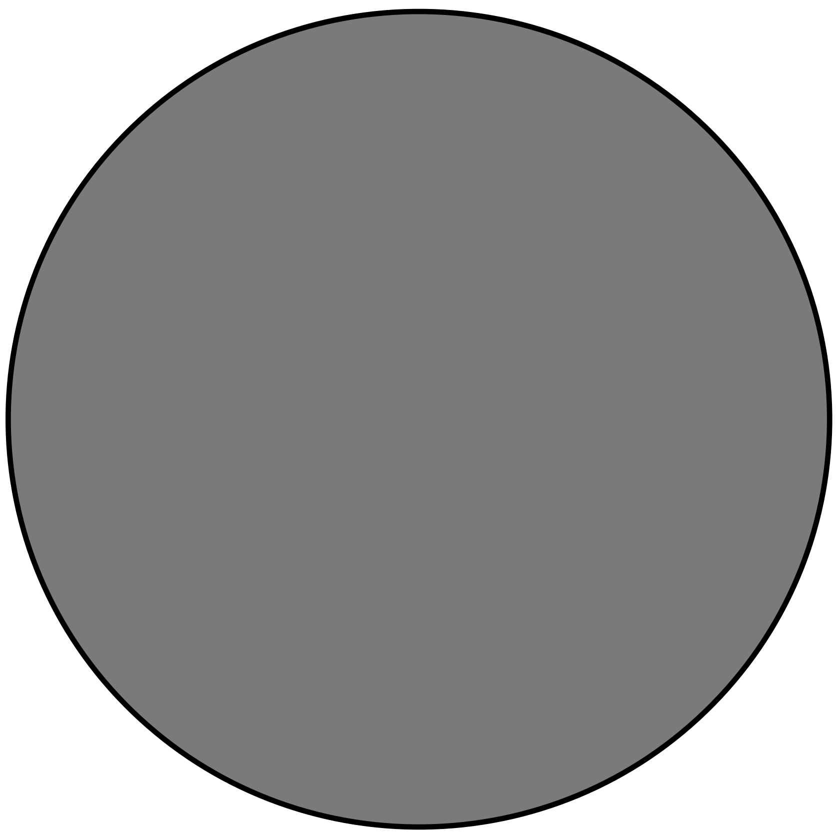
$$\mathbb{P}(\text{row} = \square\blacksquare) = p_2$$

$$\mathbb{P}(\text{row} = \blacksquare\blacksquare) = p_3$$

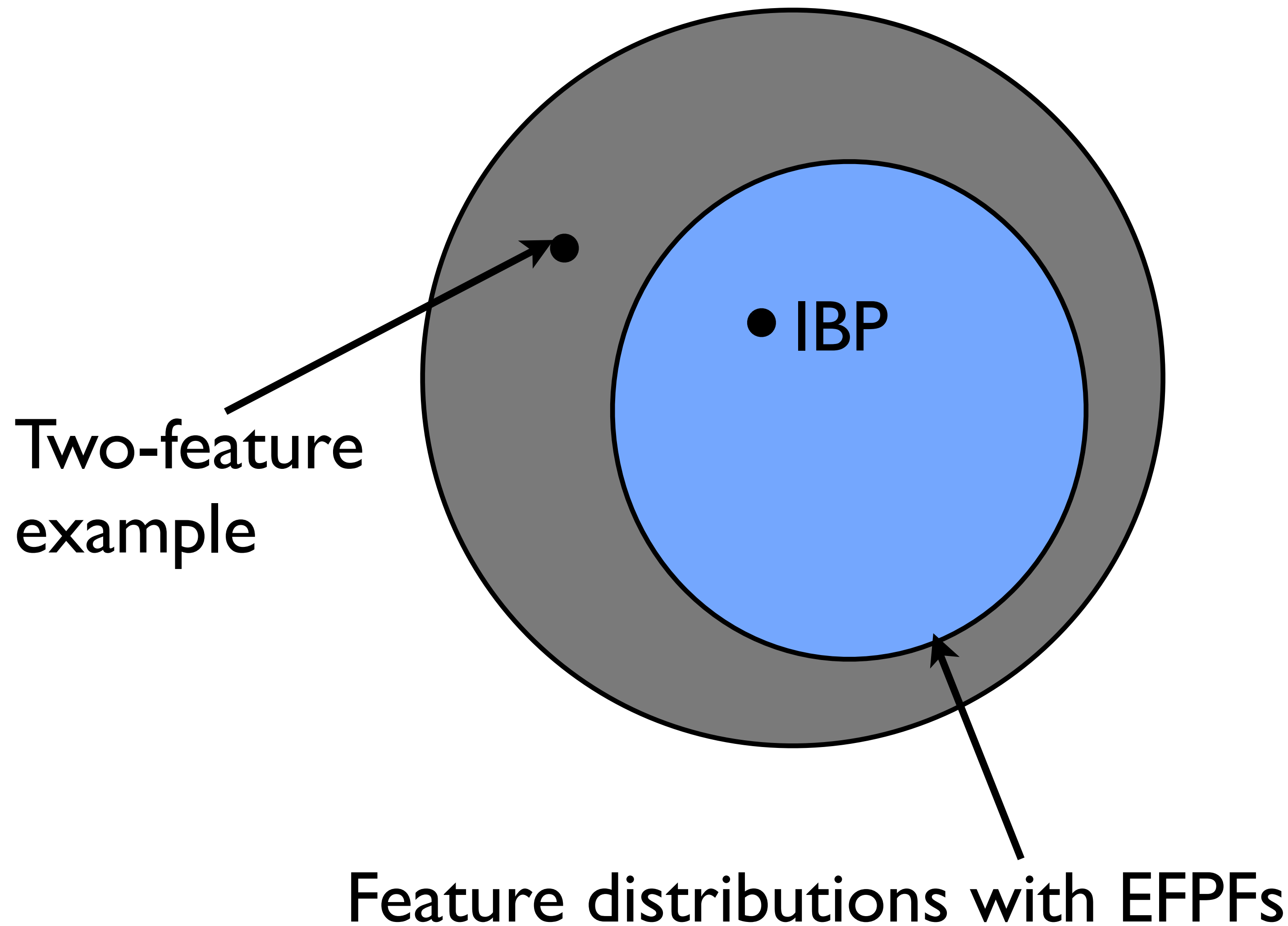
$$\mathbb{P}(\text{row} = \square\square) = p_4$$

Paintboxes

Exchangeable cluster distributions
= Cluster distributions with EPPFs
= Kingman paintbox partitions



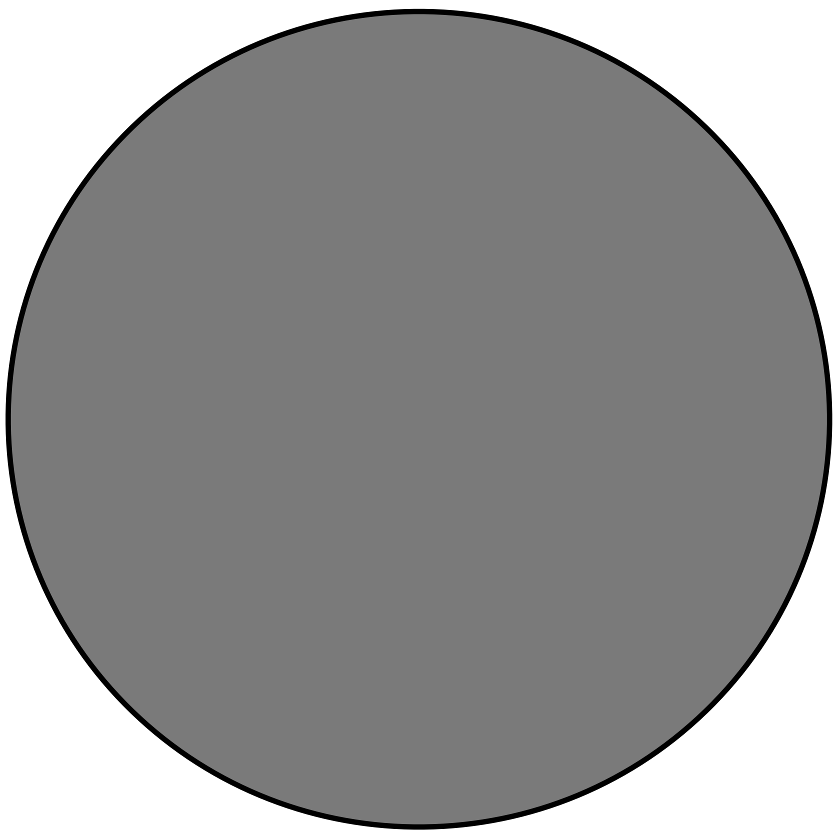
Exchangeable feature distributions
= Feature paintbox allocations



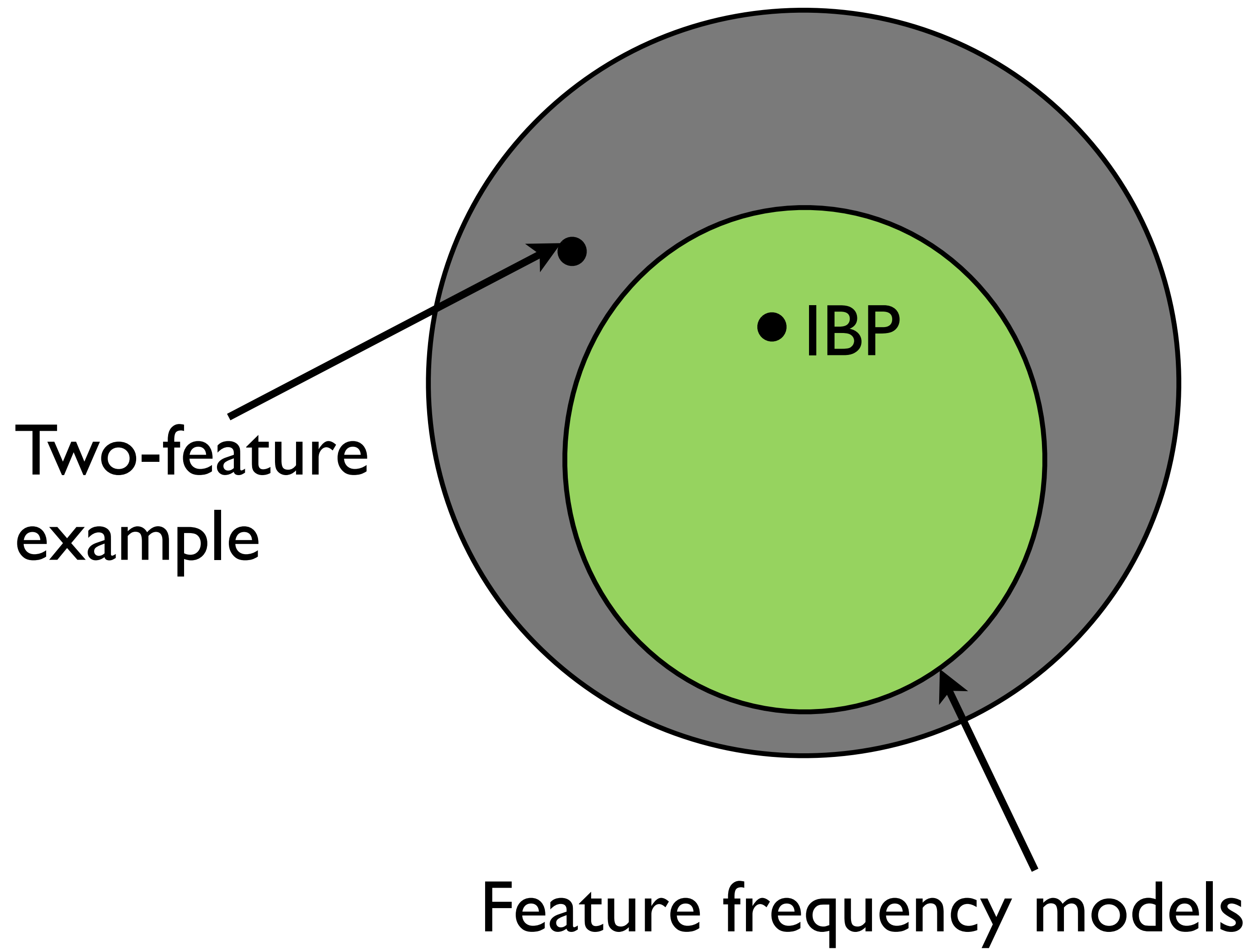
[Broderick, Pitman, Jordan 2013]

Paintboxes

Exchangeable cluster distributions
= Cluster distributions with EPPFs
= Kingman paintbox partitions

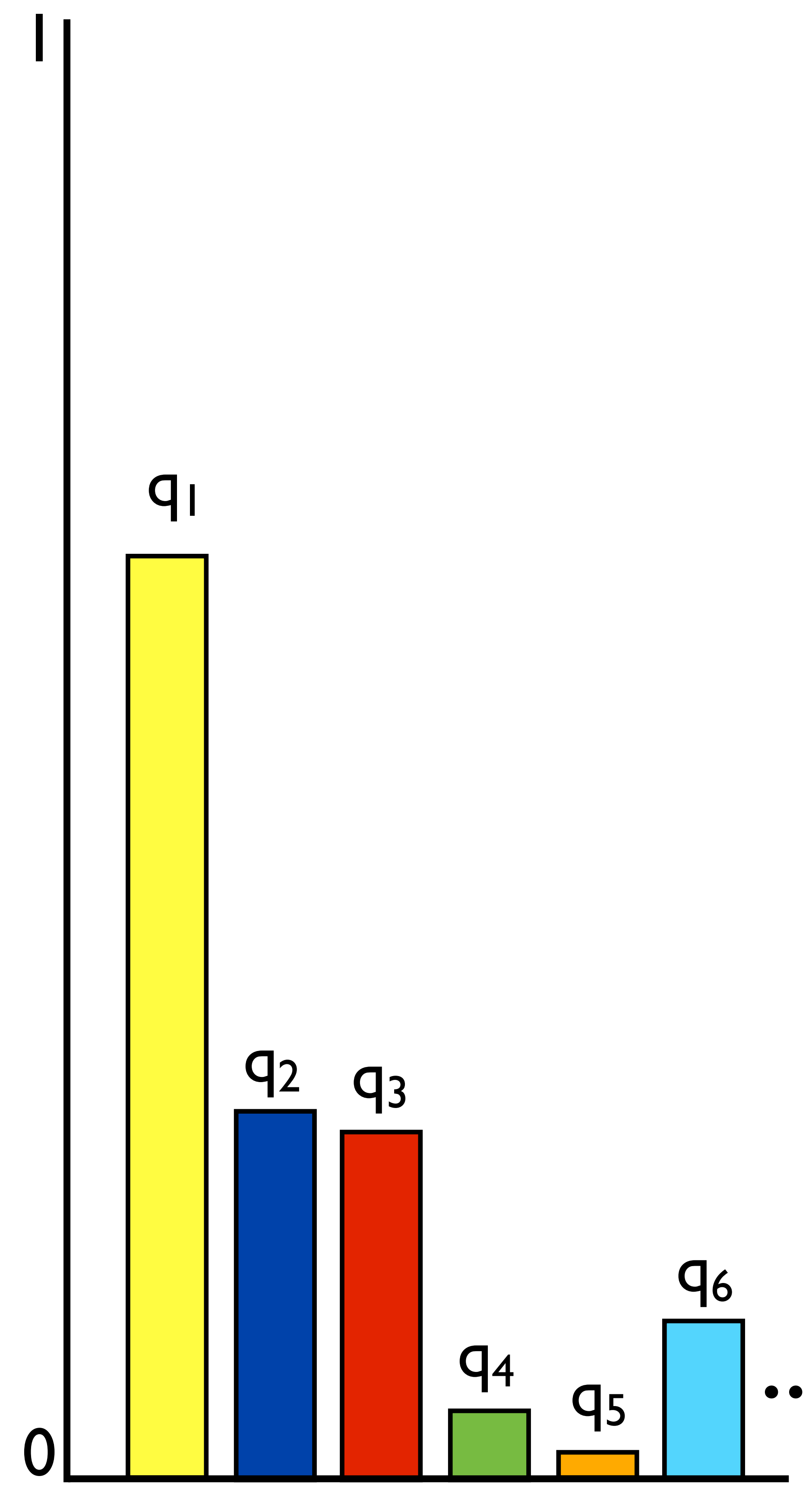


Exchangeable feature distributions
= Feature paintbox allocations

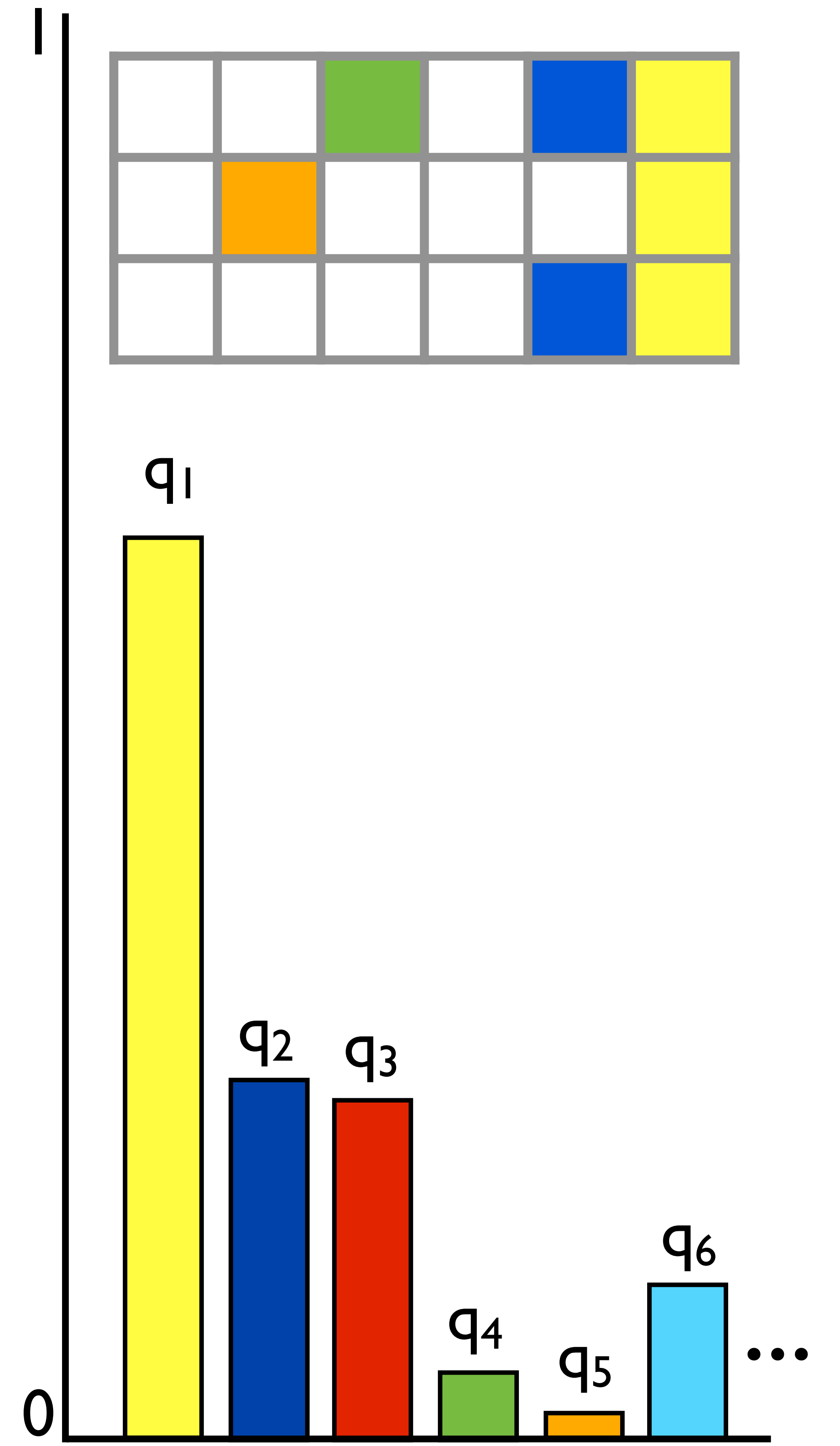


[Broderick, Pitman, Jordan 2013]

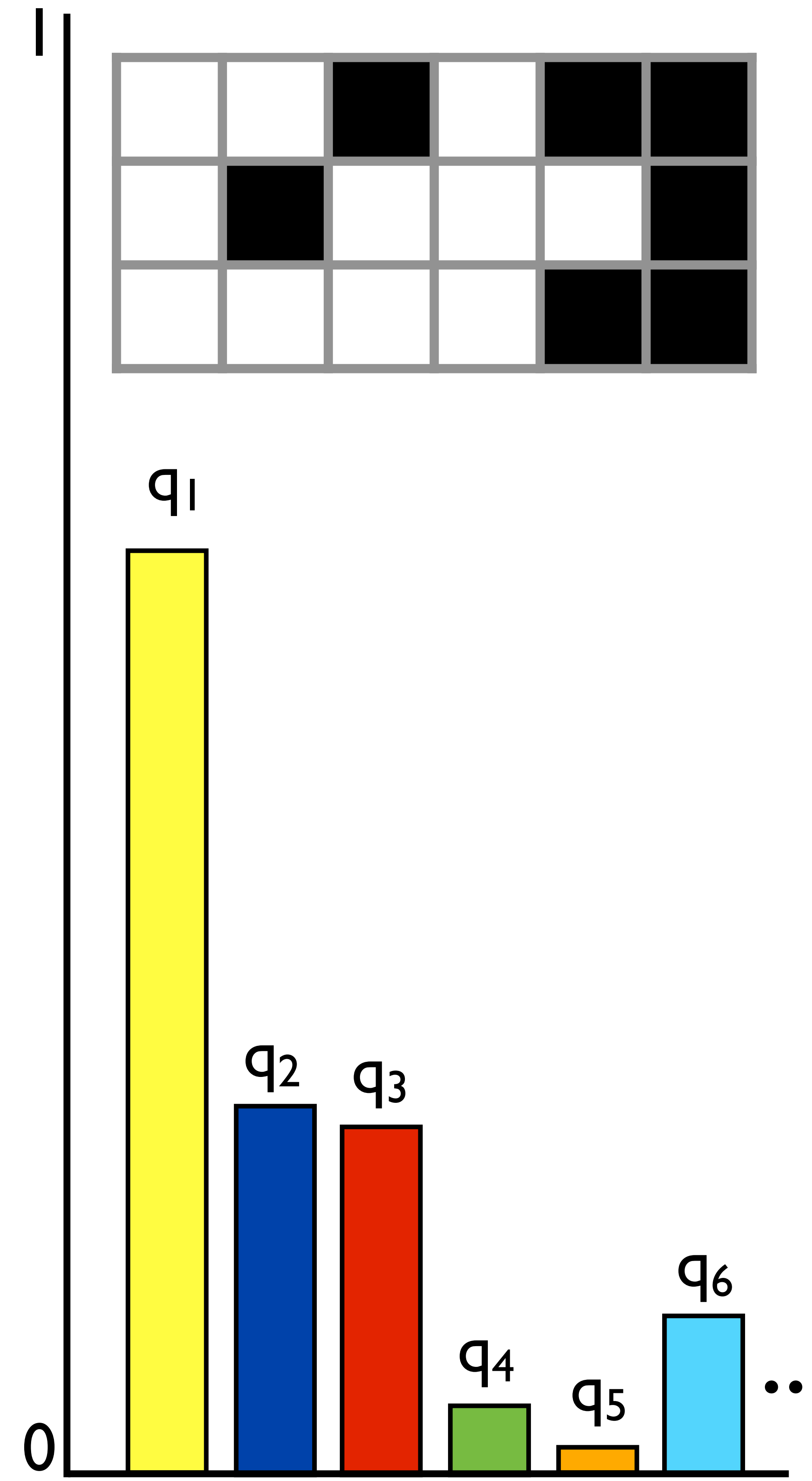
Feature frequency models: EFPFs?



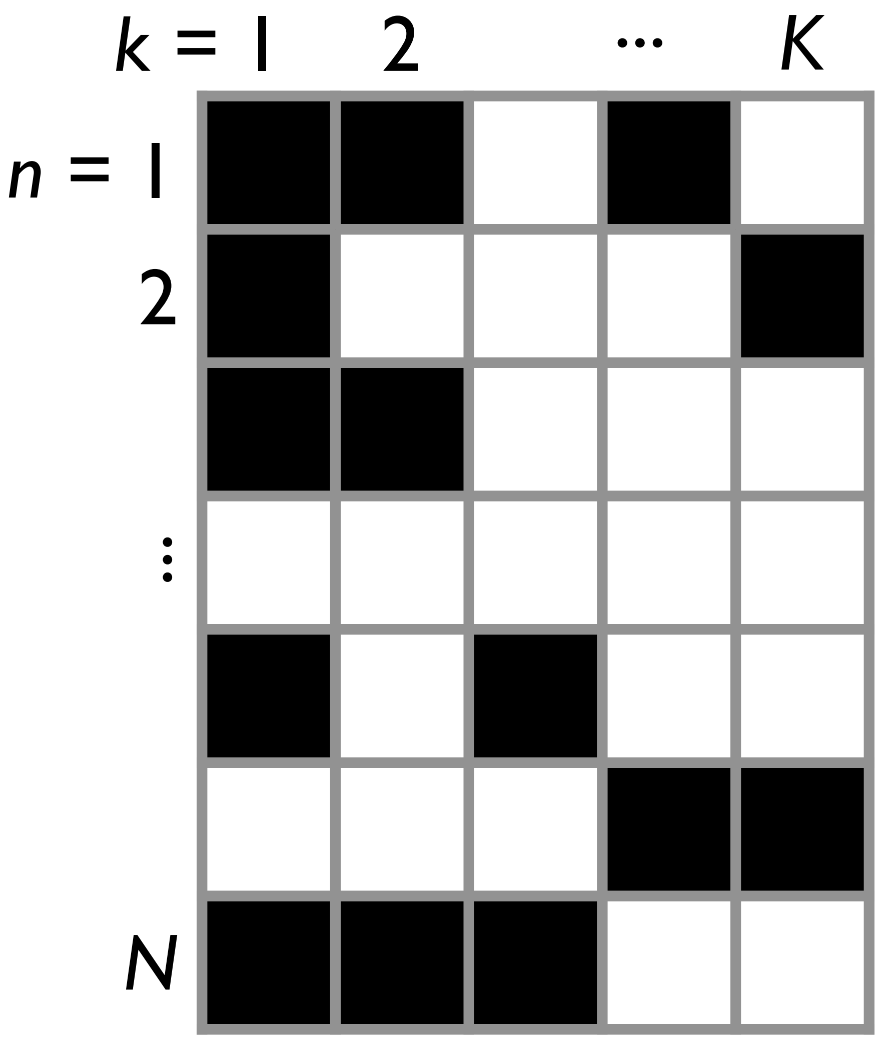
Feature frequency models: EFPFs?

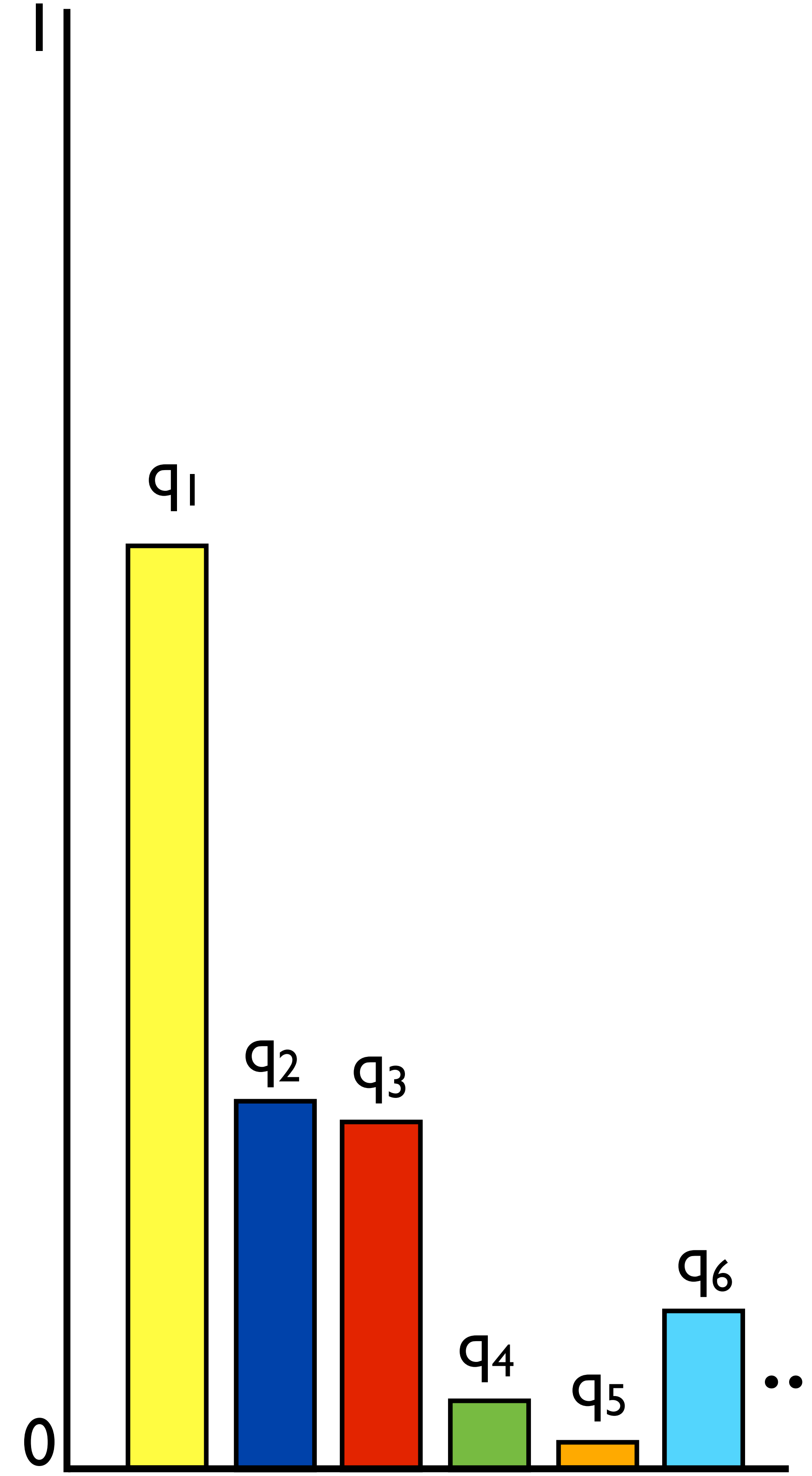


Feature frequency models: EFPFs?

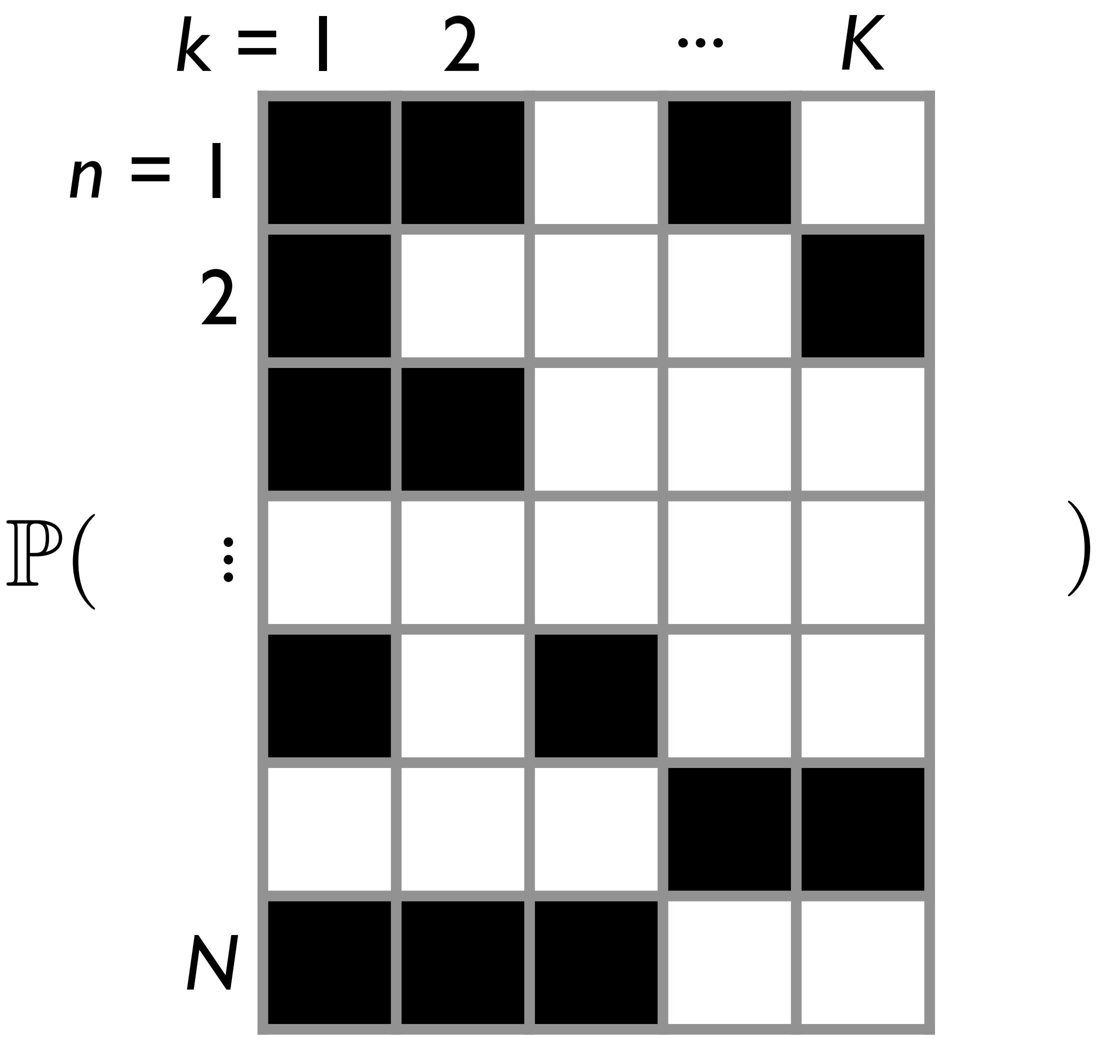


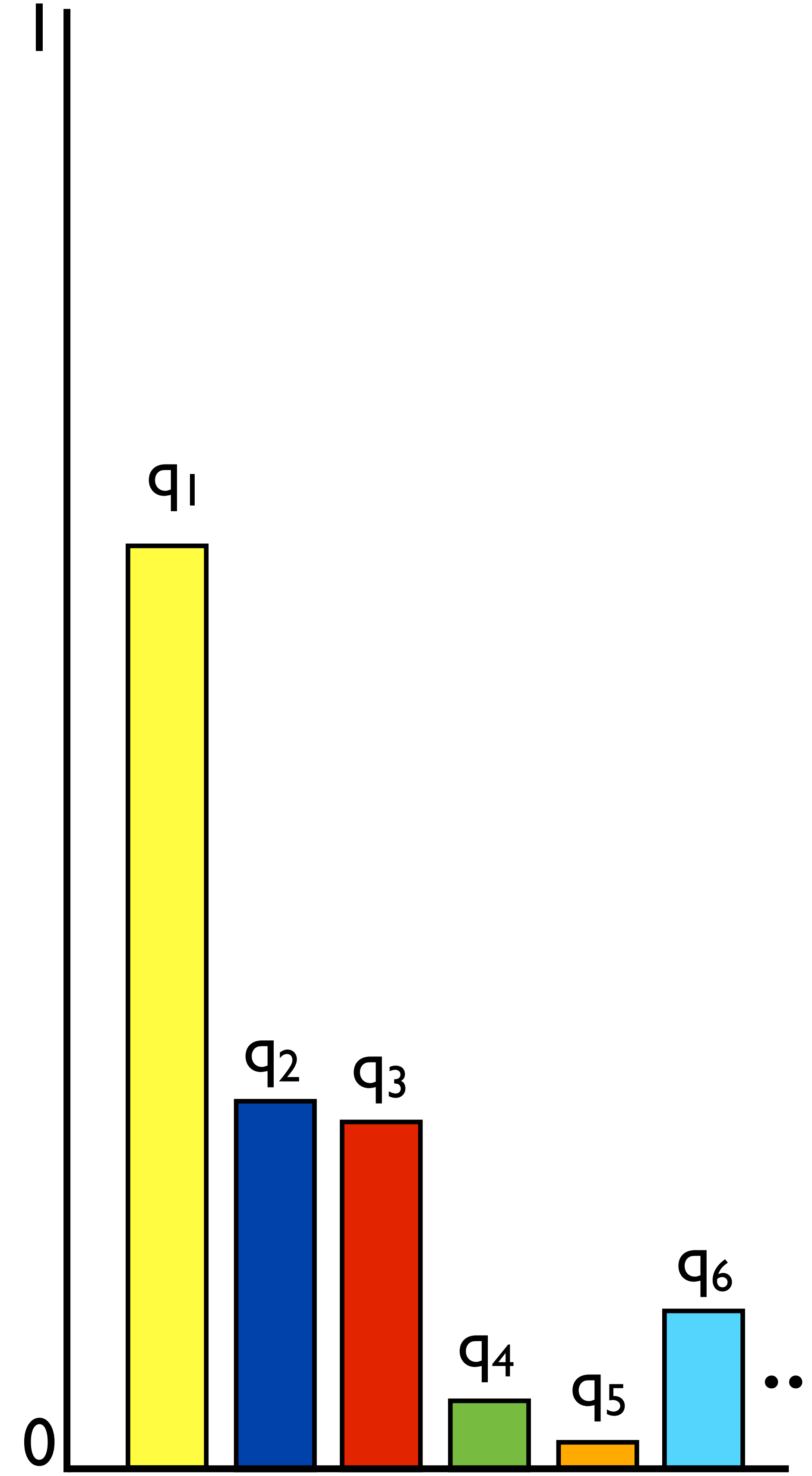
Feature frequency models: EFPFs?



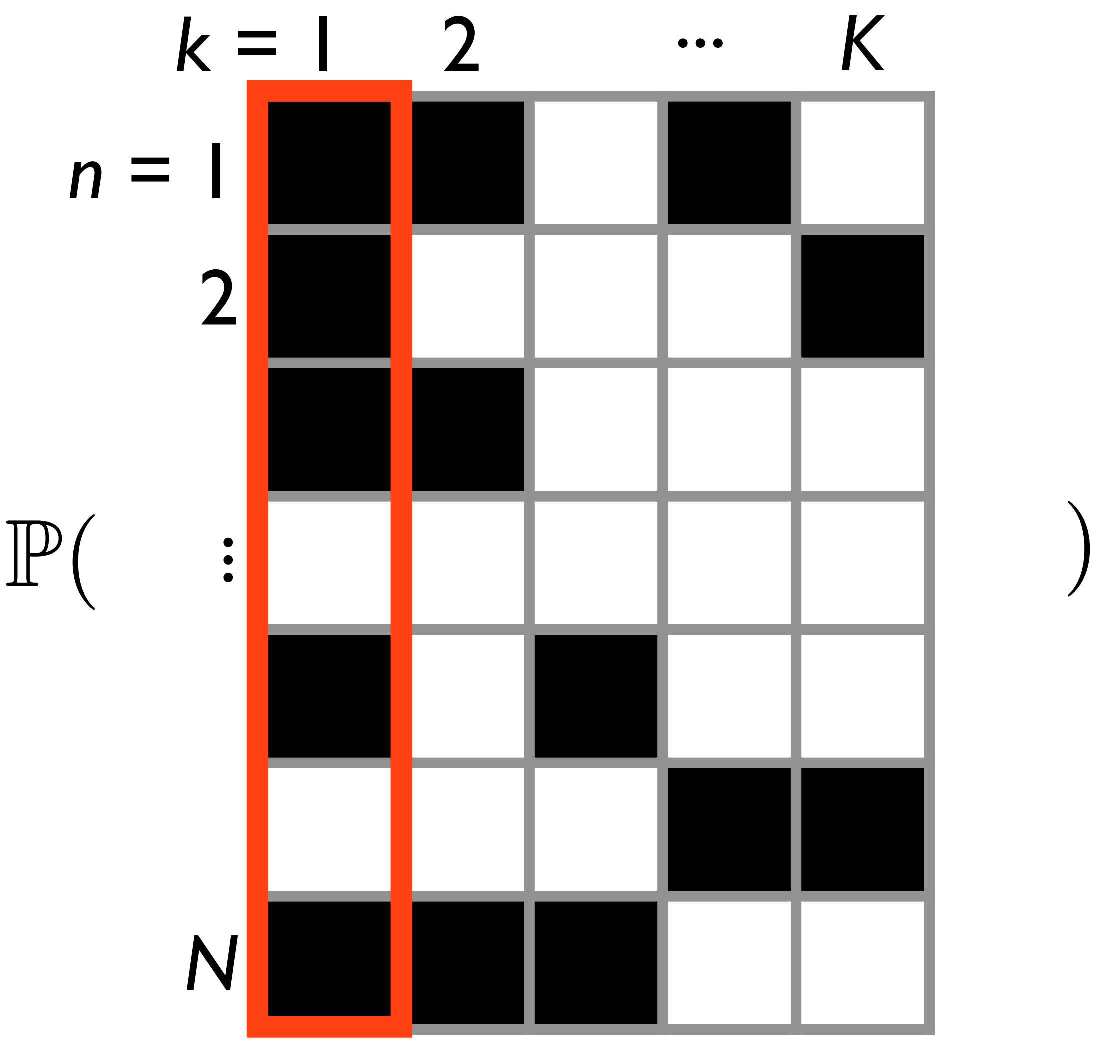


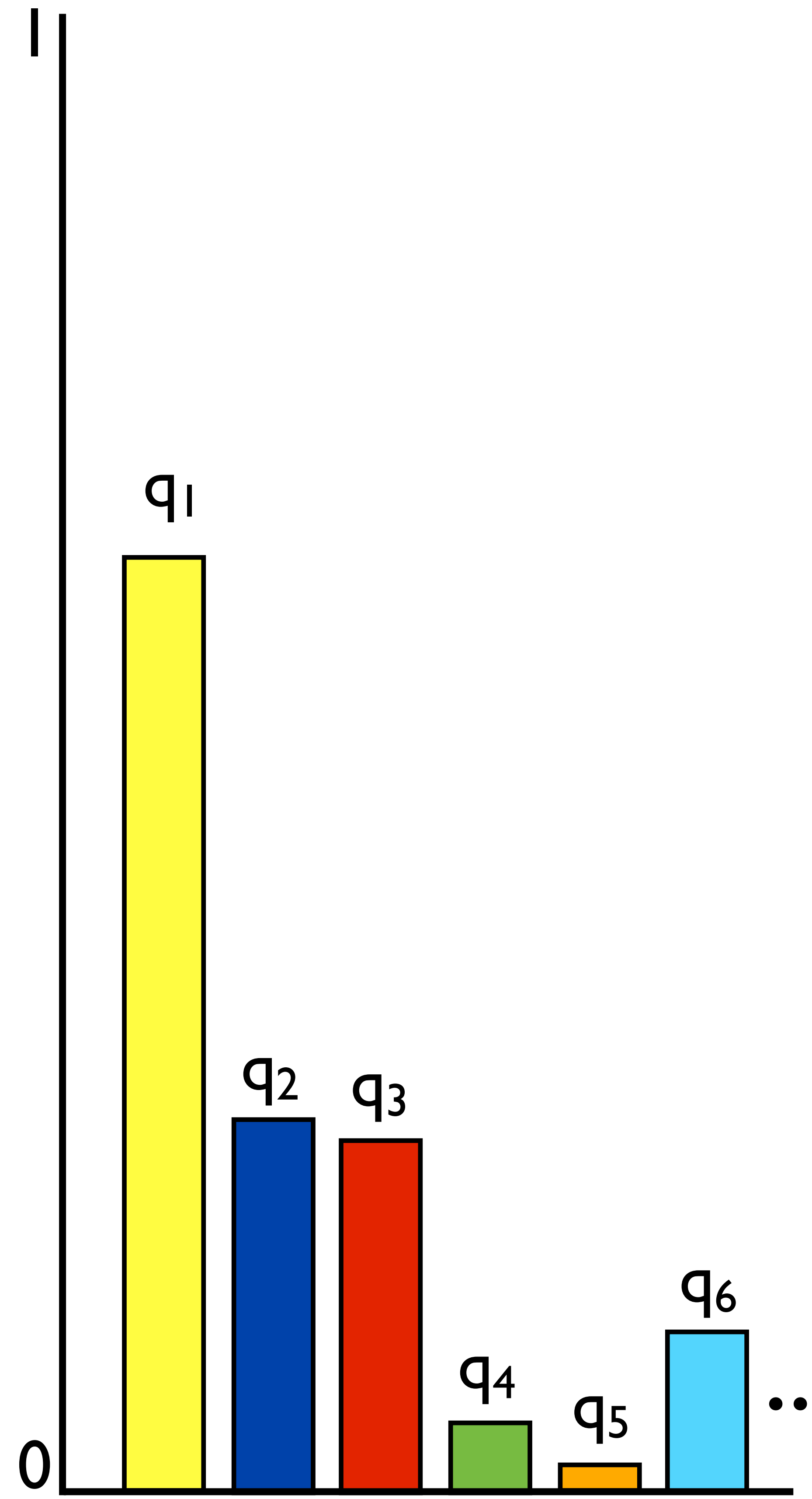
Feature frequency models: EFPFs?



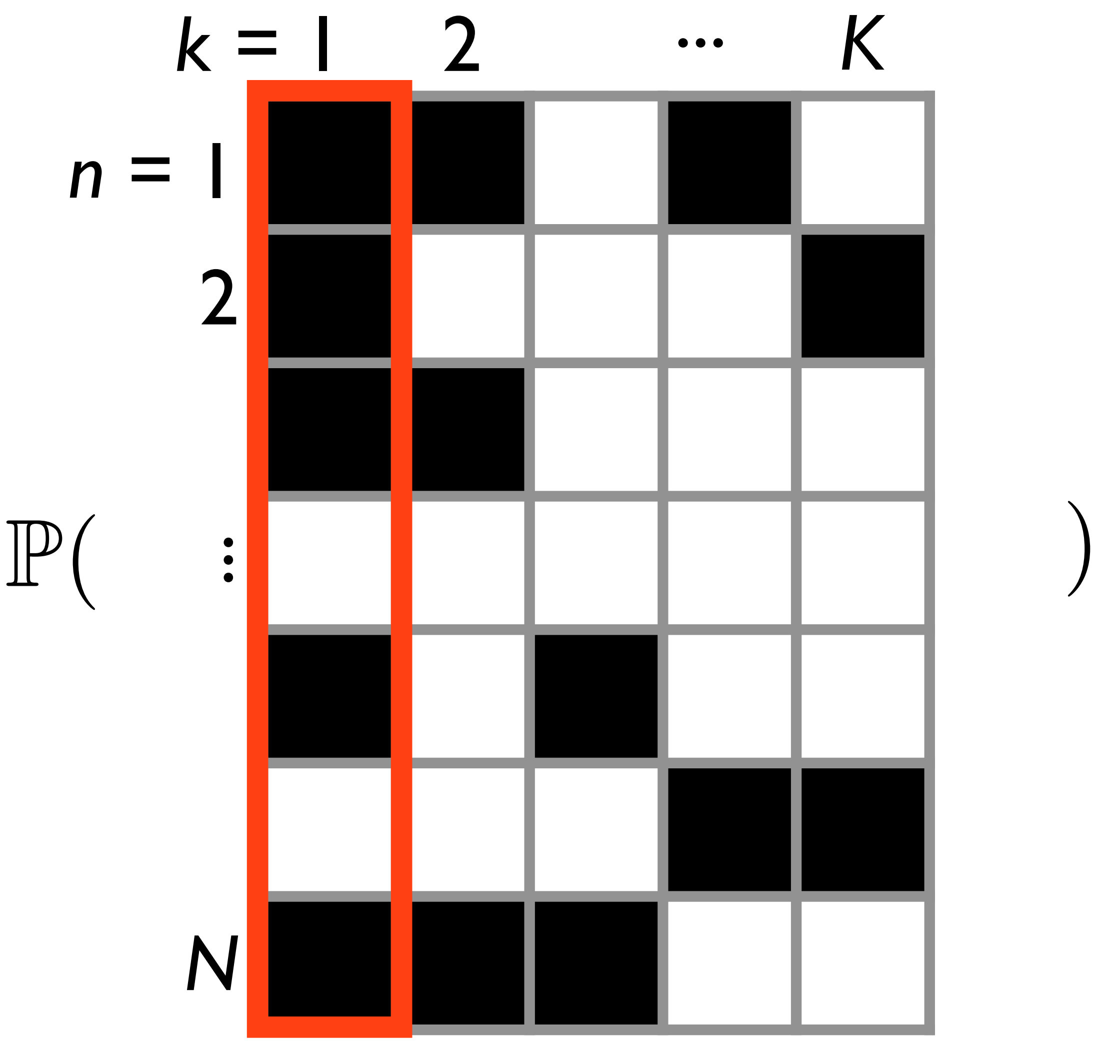


Feature frequency models: EFPFs?

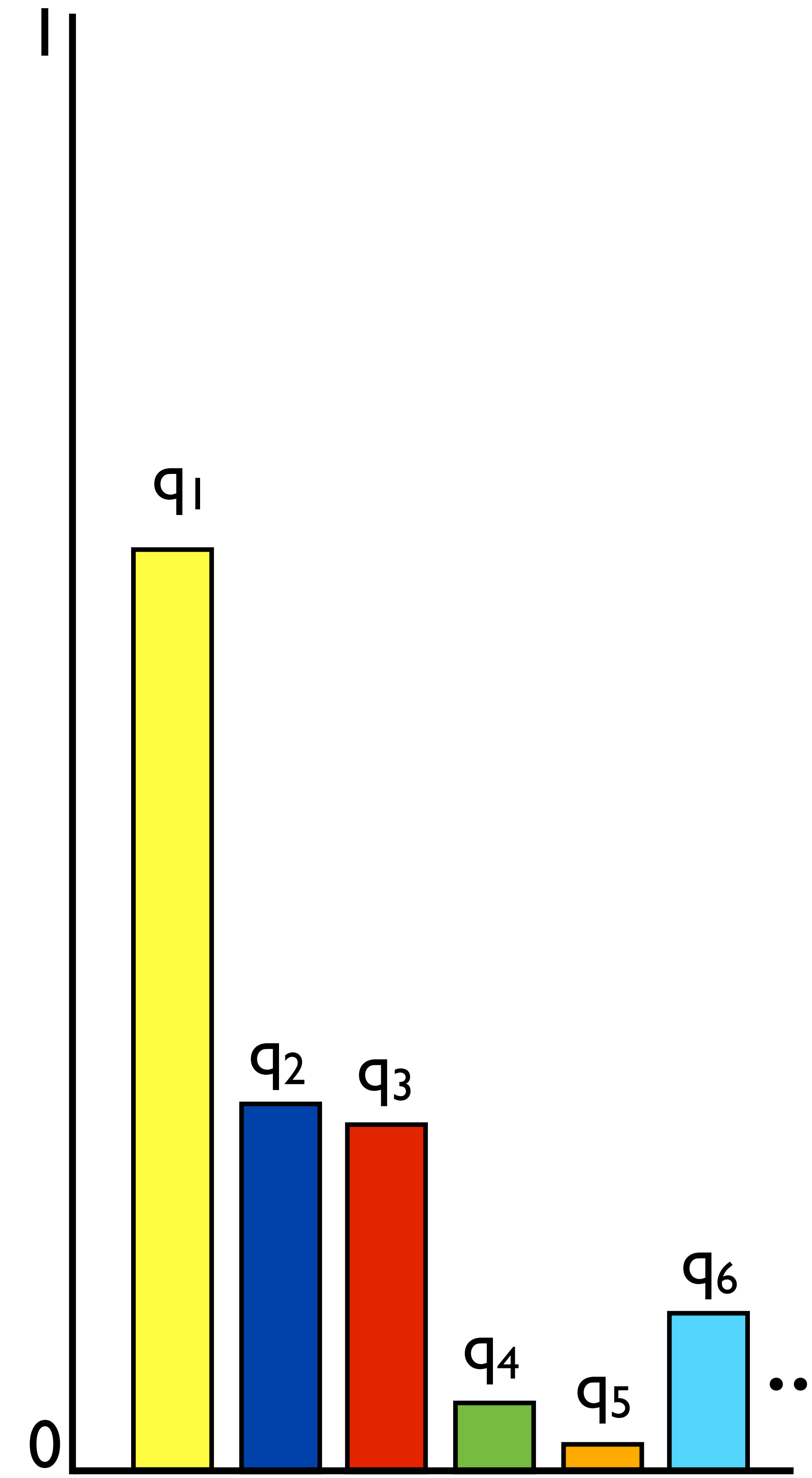




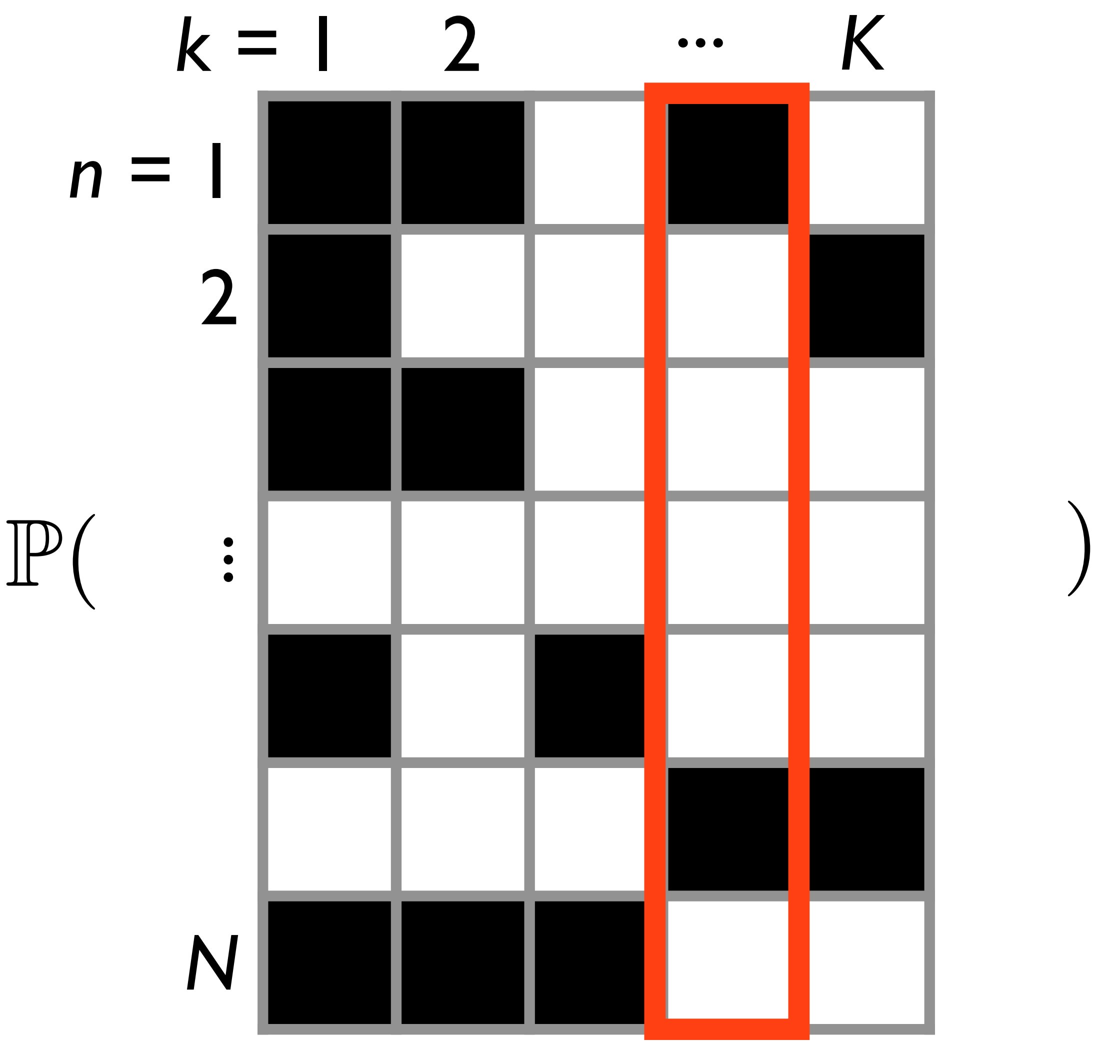
Feature frequency models: EFPFs?



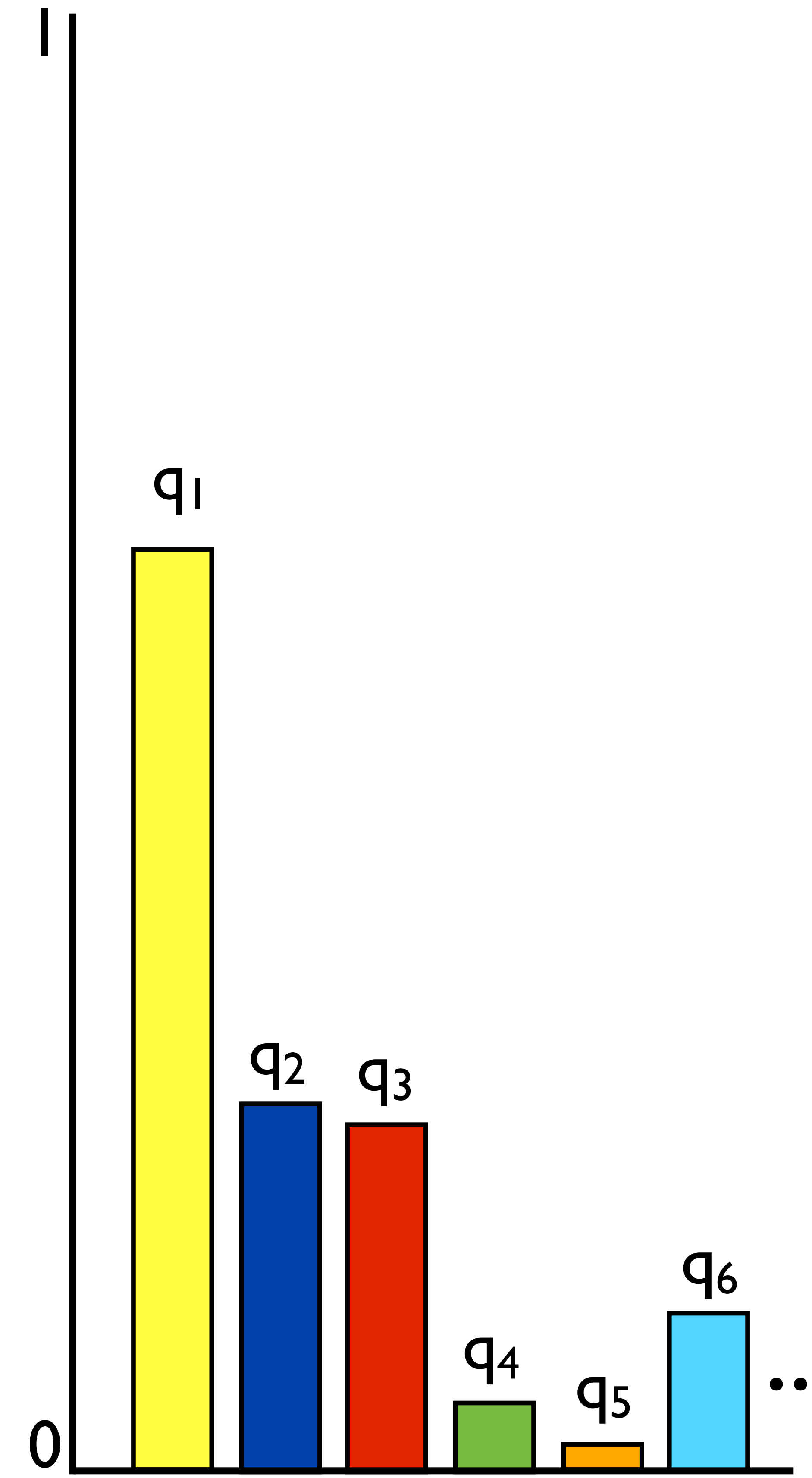
$$q_1^{S_{N,1}}(1-q_1)^{N-S_{N,1}}$$



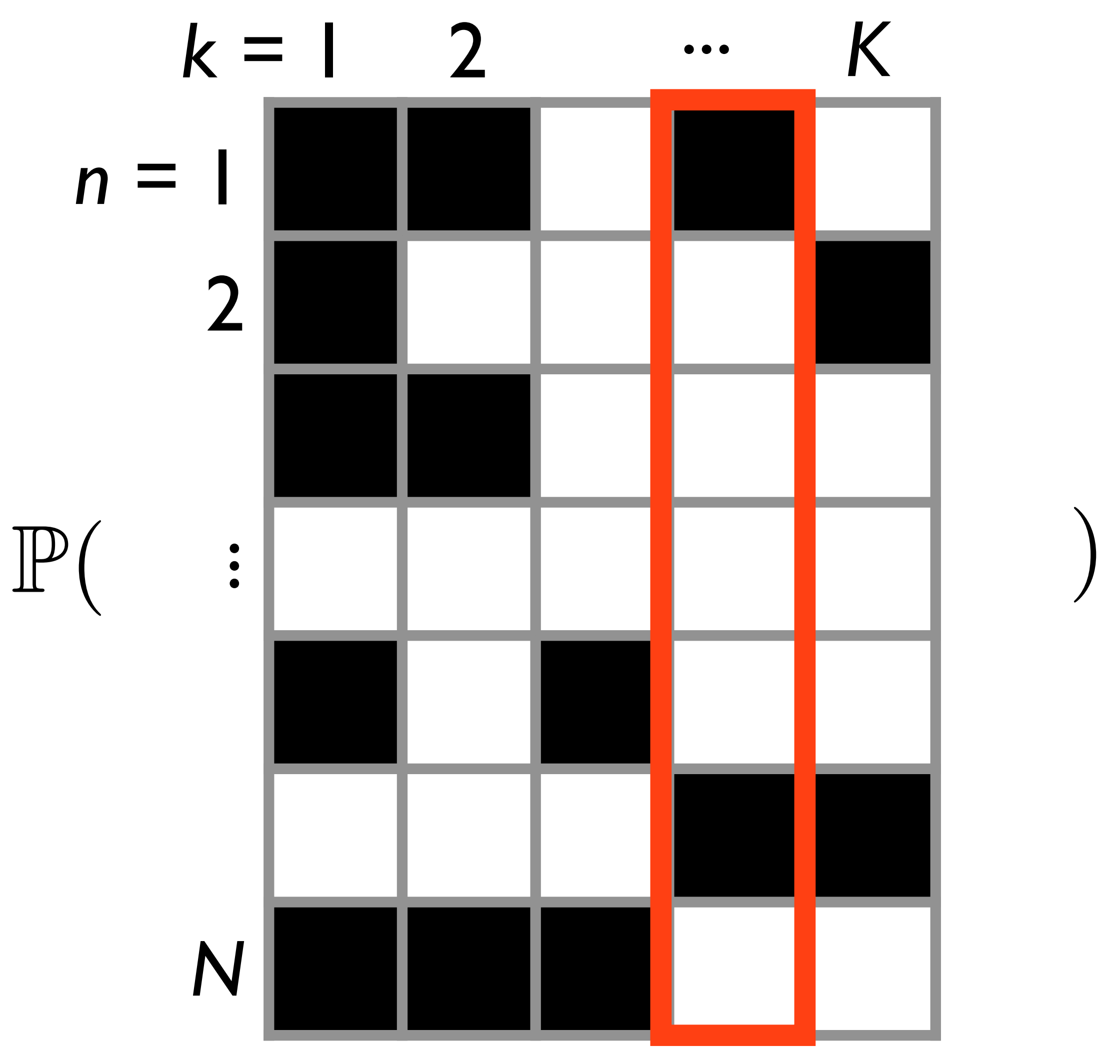
Feature frequency models: EFPFs?



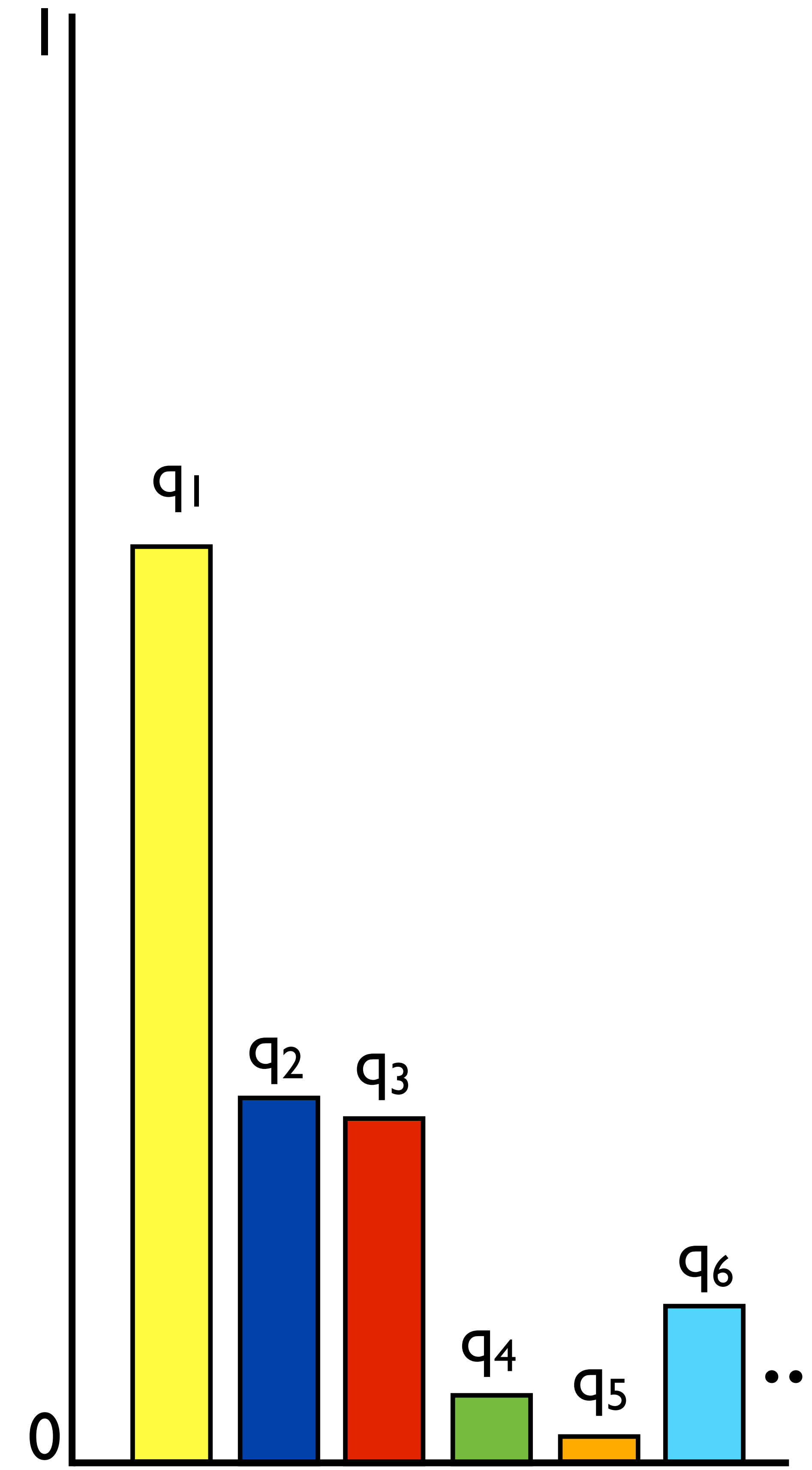
$$q_k^{S_{N,k}}(1-q_k)^{N-S_{N,k}}$$



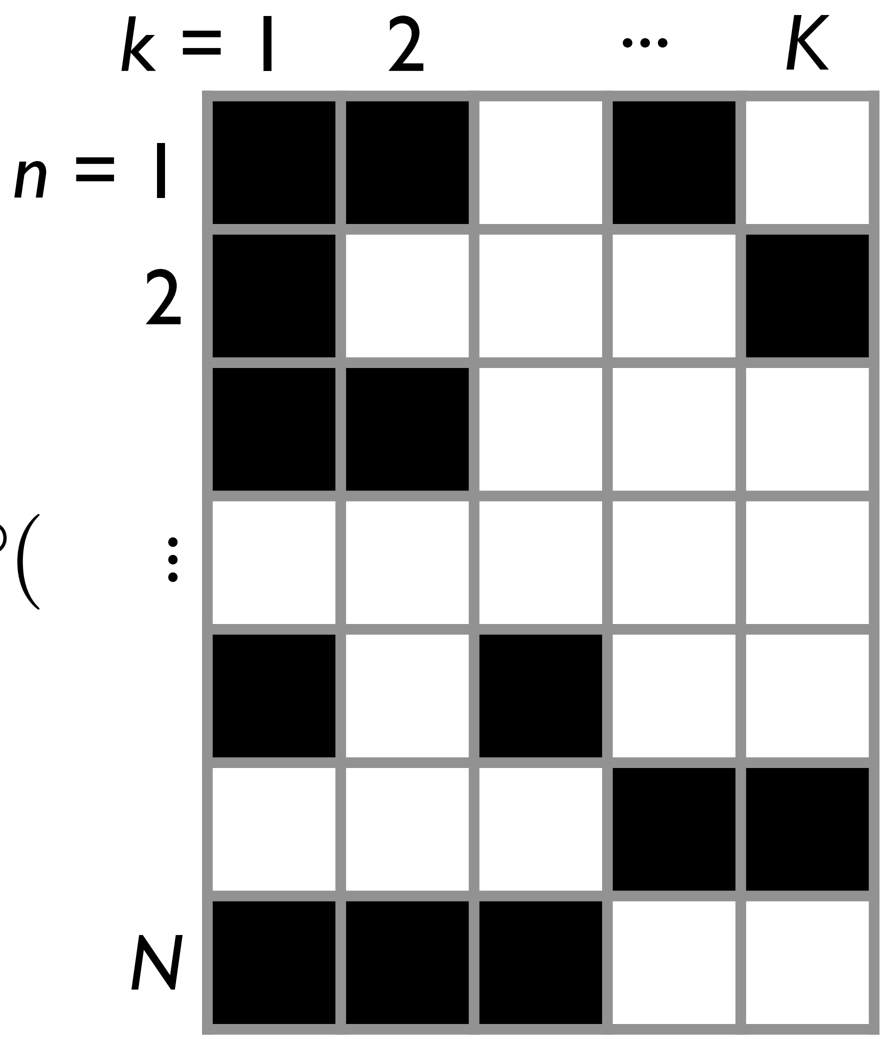
Feature frequency models: EFPFs?



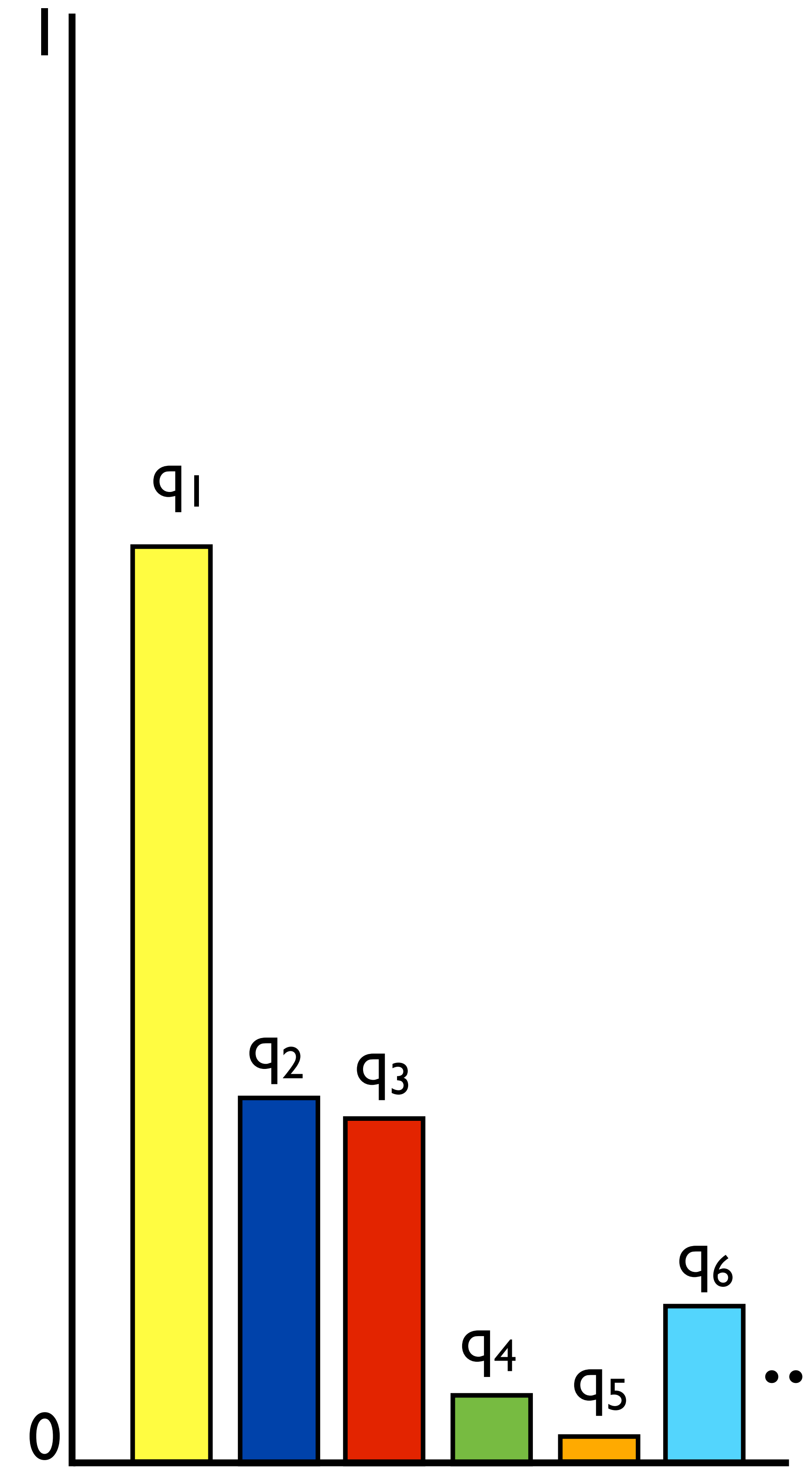
$$q_{i_k}^{S_{N,k}}(1-q_{i_k})^{N-S_{N,k}}$$



Feature frequency models: EFPFs?

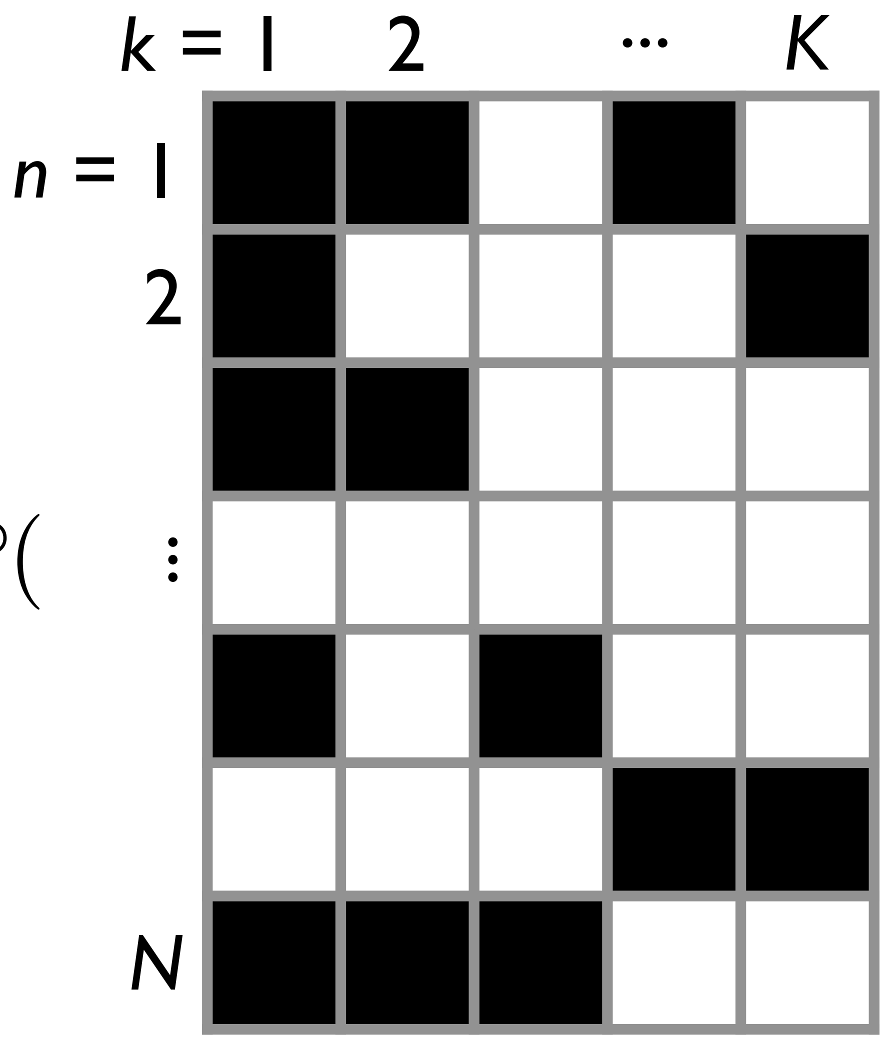
$\mathbb{P}($  $)$

$$\prod_{k=1}^{K} q_{i_k}^{S_{N,k}} (1 - q_{i_k})^{N - S_{N,k}}$$



Feature frequency models: EFPFs?

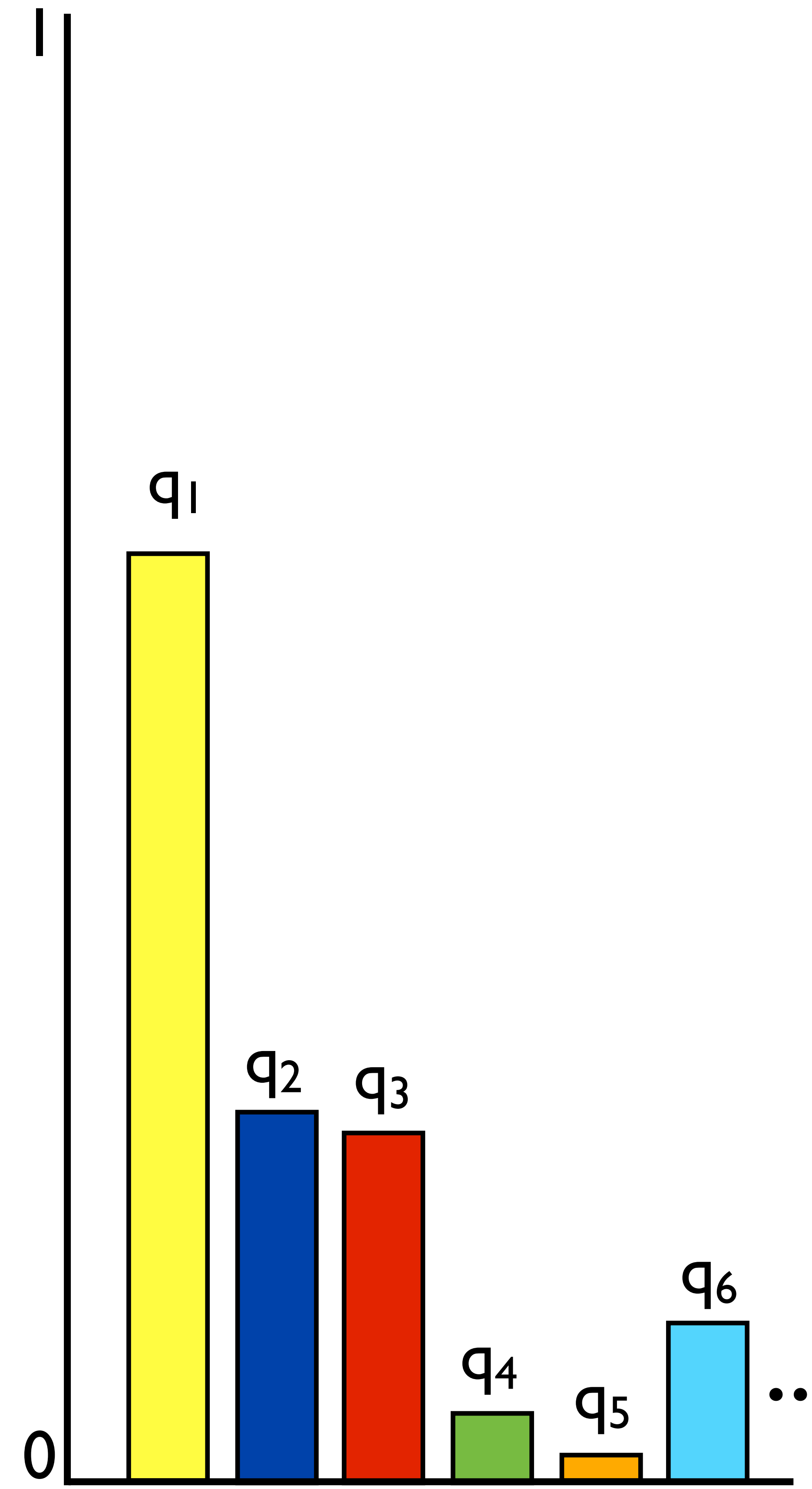
$\mathbb{P}($



$)$

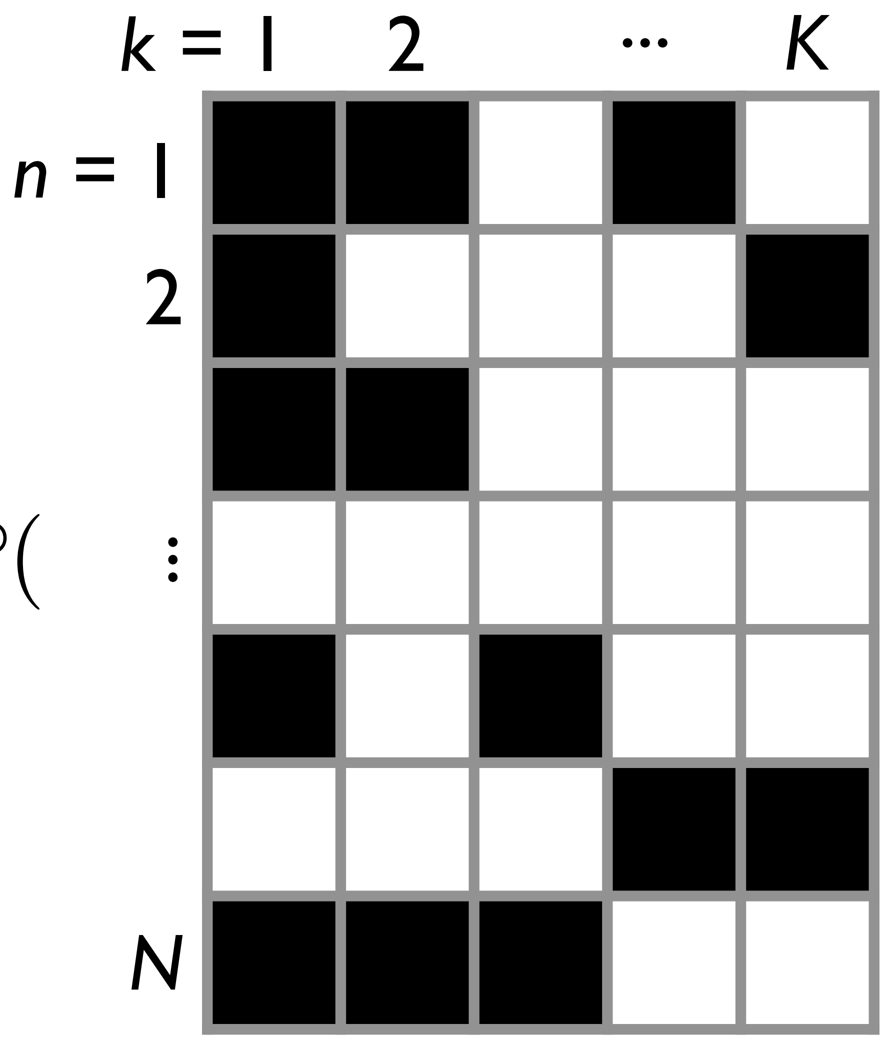
$$\prod_{k=1}^{K} q_{i_k}^{S_{N,k}} (1 - q_{i_k})^{N - S_{N,k}}$$

$$\cdot \prod_{j \notin \{i_k\}_{k=1}^{K}} (1 - q_j)^{N}$$



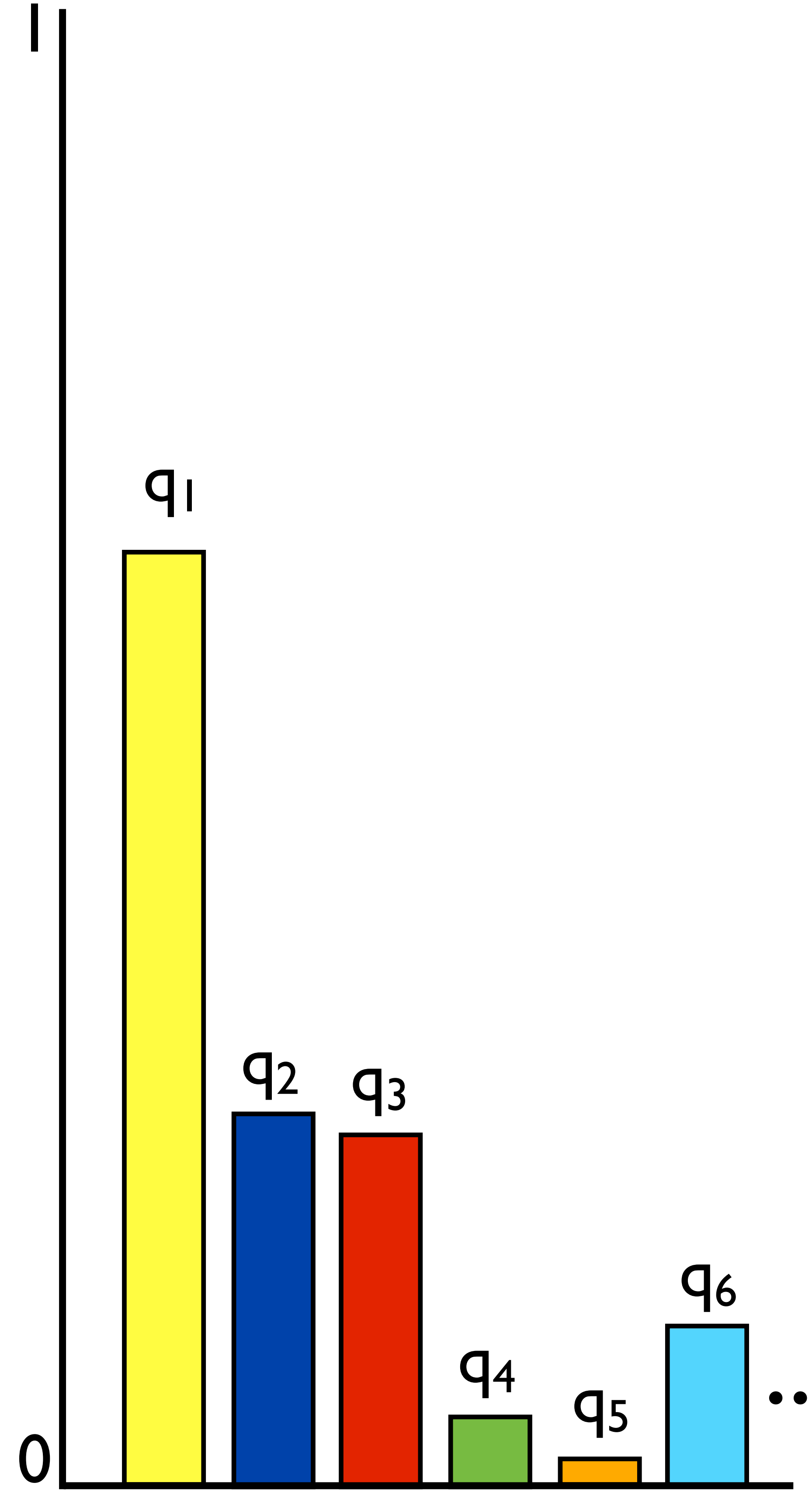
Feature frequency models: EFPFs?

$\mathbb{P}($

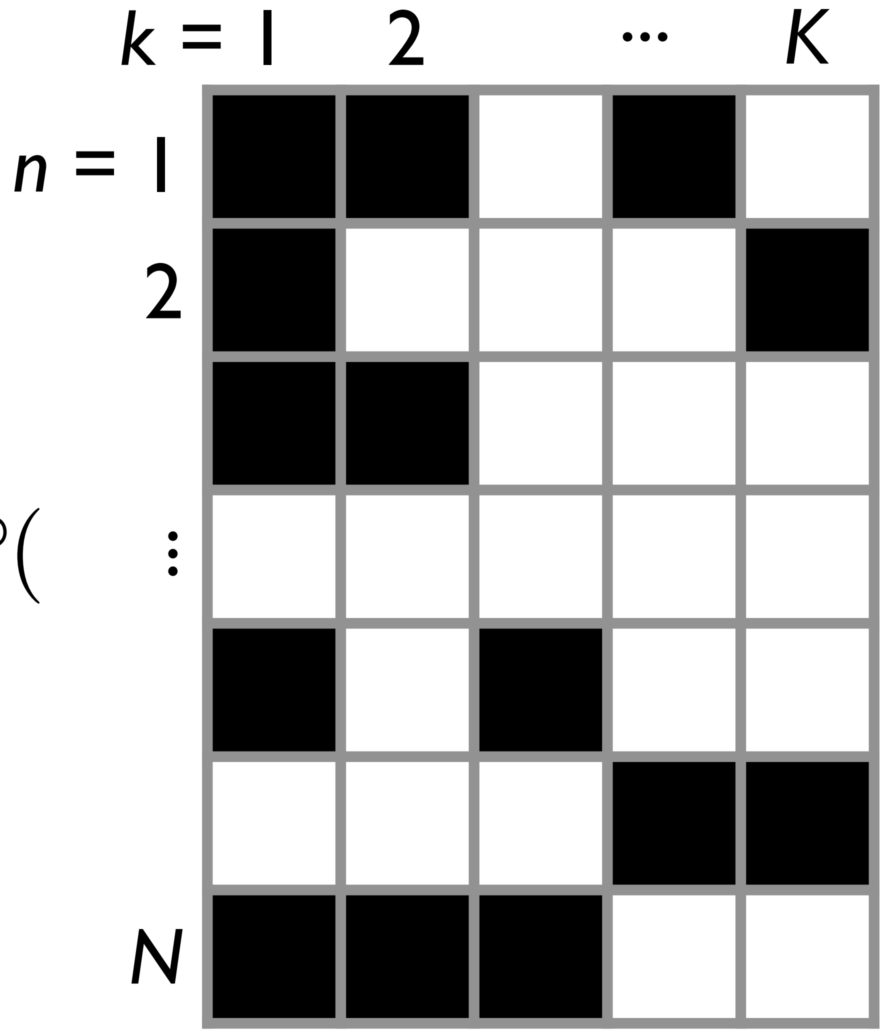


$)$

$$= \mathbb{E}\Big[\sum_{\text{distinct } i_k} \frac{1}{K!} \prod_{k=1}^{K} q_{i_k}^{S_{N,k}} (1 - q_{i_k})^{N - S_{N,k}} \cdot \prod_{j \notin \{i_k\}_{k=1}^{K}} (1 - q_j)^N \Big]$$



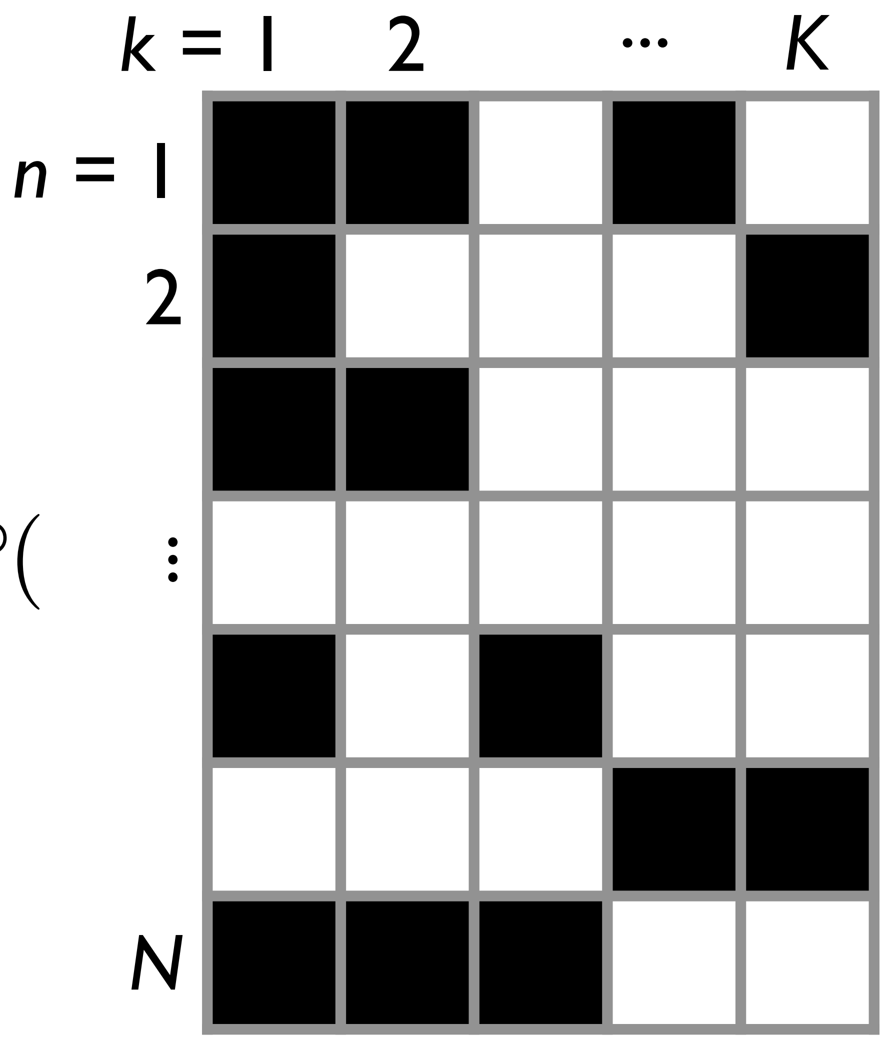
Feature frequency models: EFPFs?

$\mathbb{P}($  $)$

$$= \mathbb{E}\Big[\sum_{\text{distinct } i_k} \frac{1}{K!} \prod_{k=1}^{K} q_{i_k}^{S_{N,k}} (1 - q_{i_k})^{N - S_{N,k}} \cdot \prod_{j \notin \{i_k\}_{k=1}^{K}} (1 - q_j)^N \Big]$$

[Broderick, Pitman, Jordan 2013]

Feature frequency models: EFPFs?

$\mathbb{P}($  $)$

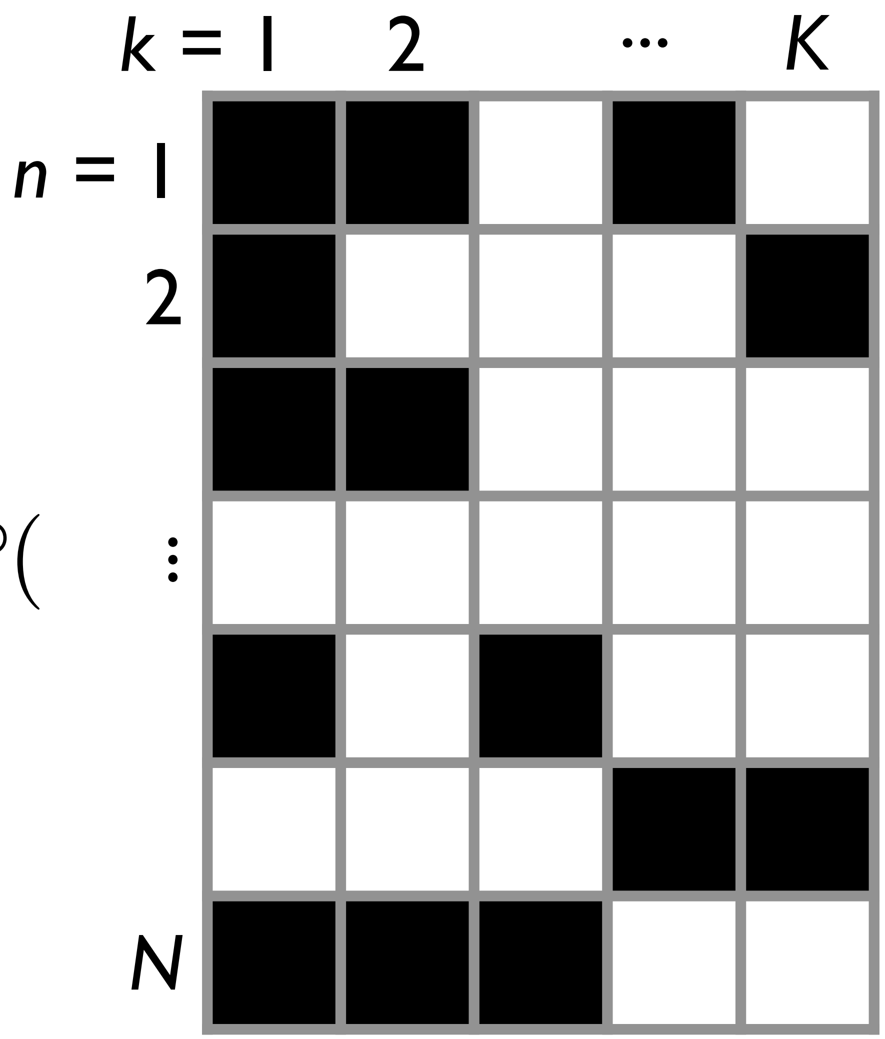
$$= \mathbb{E}\Big[\sum_{\text{distinct } i_k} \frac{1}{K!} \prod_{k=1}^{K} q_{i_k}^{S_{N,k}} (1 - q_{i_k})^{N - S_{N,k}}$$

$$\cdot \prod_{j \notin \{i_k\}_{k=1}^{K}} (1 - q_j)^{N} \Big]$$

Size of *k*th feature

[Broderick, Pitman, Jordan 2013]

Feature frequency models: EFPFs?

$\mathbb{P}($  $)$

Number of features

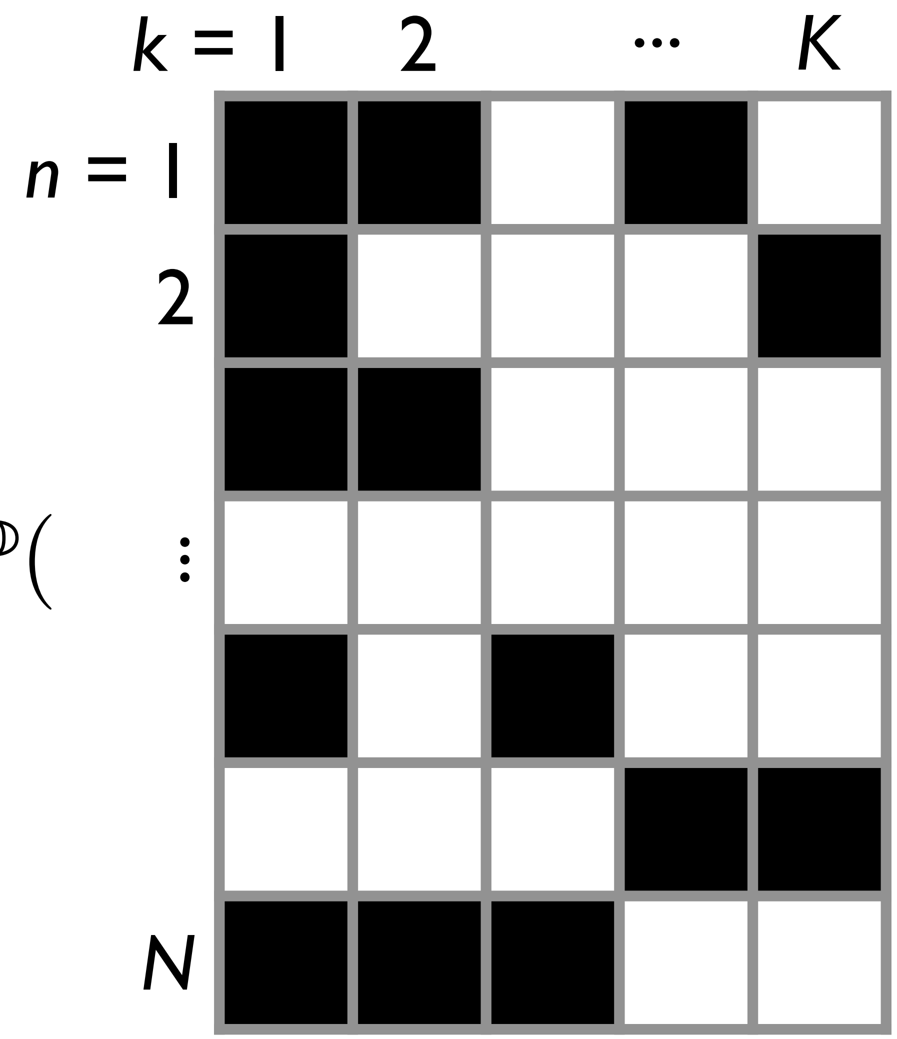
Size of kth feature

$$= \mathbb{E}\Big[\sum_{\text{distinct } i_k} \frac{1}{K!} \prod_{k=1}^{K} q_{i_k}^{S_{N,k}} (1 - q_{i_k})^{N - S_{N,k}} \cdot \prod_{j \notin \{i_k\}_{k=1}^{K}} (1 - q_j)^N \Big]$$

[Broderick, Pitman, Jordan 2013]

Feature frequency models: EFPFs?

$\mathbb{P}($



$)$

$$= \mathbb{E}\Big[\sum_{\text{distinct } i_k} \frac{1}{K!} \prod_{k=1}^{K} q_{i_k}^{S_{N,k}} (1 - q_{i_k})^{N - S_{N,k}} \cdot \prod_{j \notin \{i_k\}_{k=1}^{K}} (1 - q_j)^N \Big]$$

Number of features

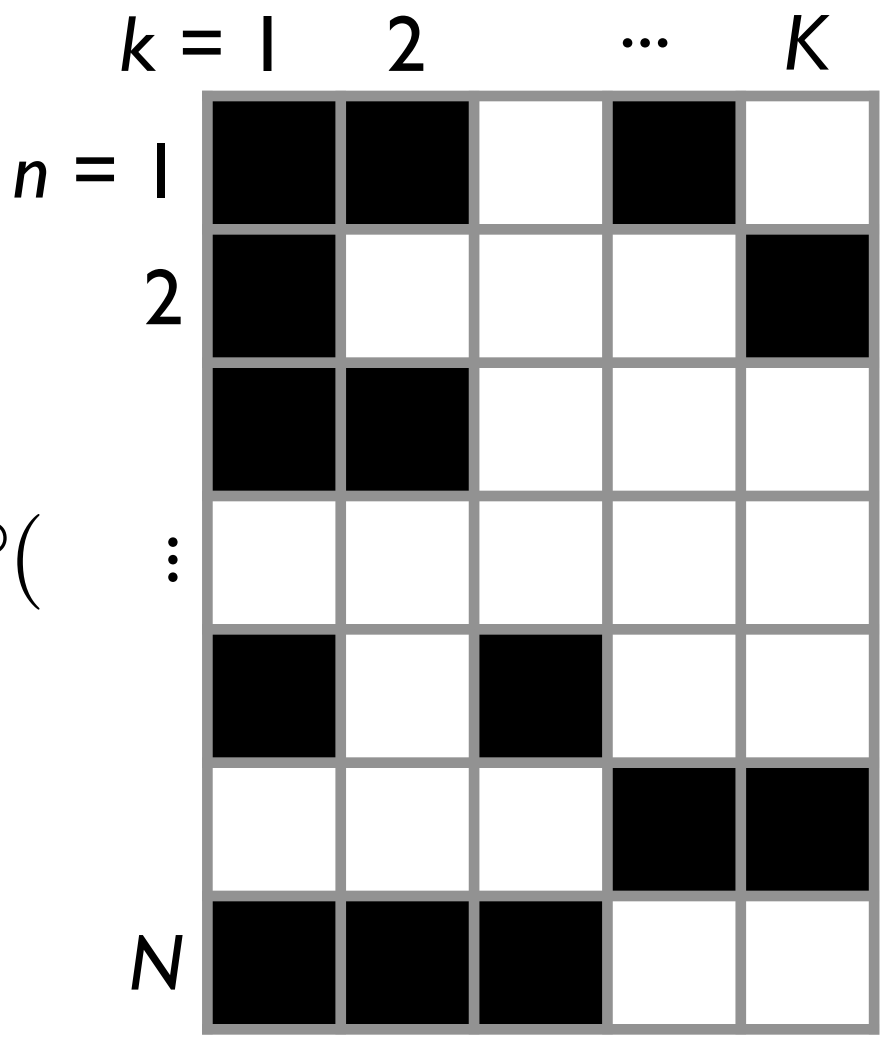
Number of data points

Size of *k*th feature

[Broderick, Pitman, Jordan 2013]

Feature frequency models: EFPFs?

$\mathbb{P}($



$)$

Number of features

Number of data points

Size of *k*th feature

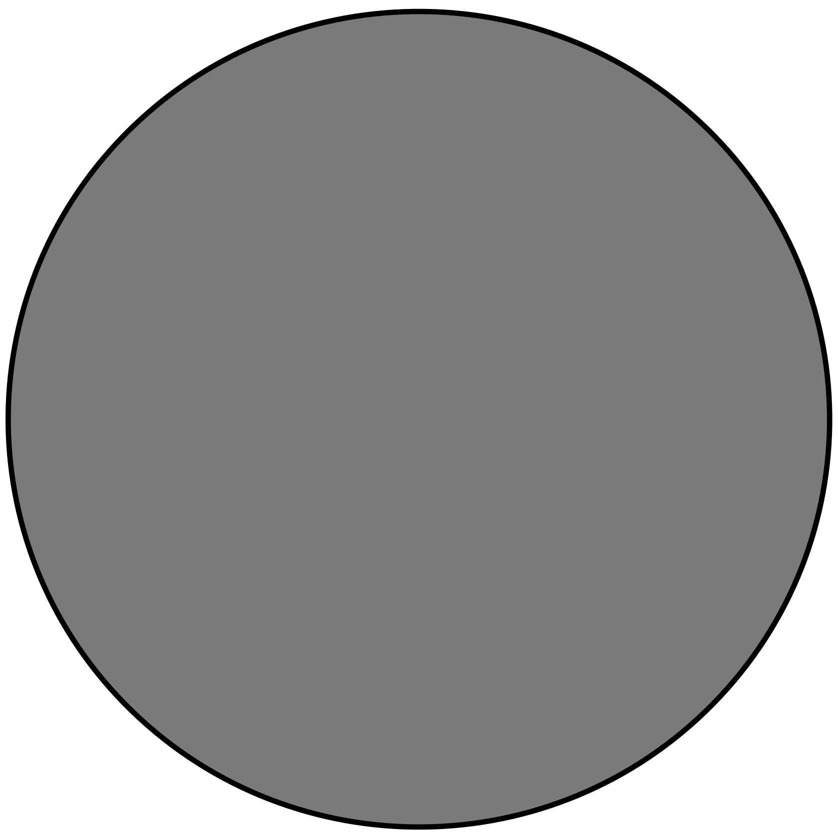
$$= \mathbb{E}\Big[\sum_{\text{distinct } i_k} \frac{1}{K!} \prod_{k=1}^{K} q_{i_k}^{S_{N,k}} (1 - q_{i_k})^{N - S_{N,k}} \cdot \prod_{j \notin \{i_k\}_{k=1}^{K}} (1 - q_j)^N \Big] = p(N; S_{N,1}, S_{N,2}, \ldots, S_{N,K})$$

EFPF

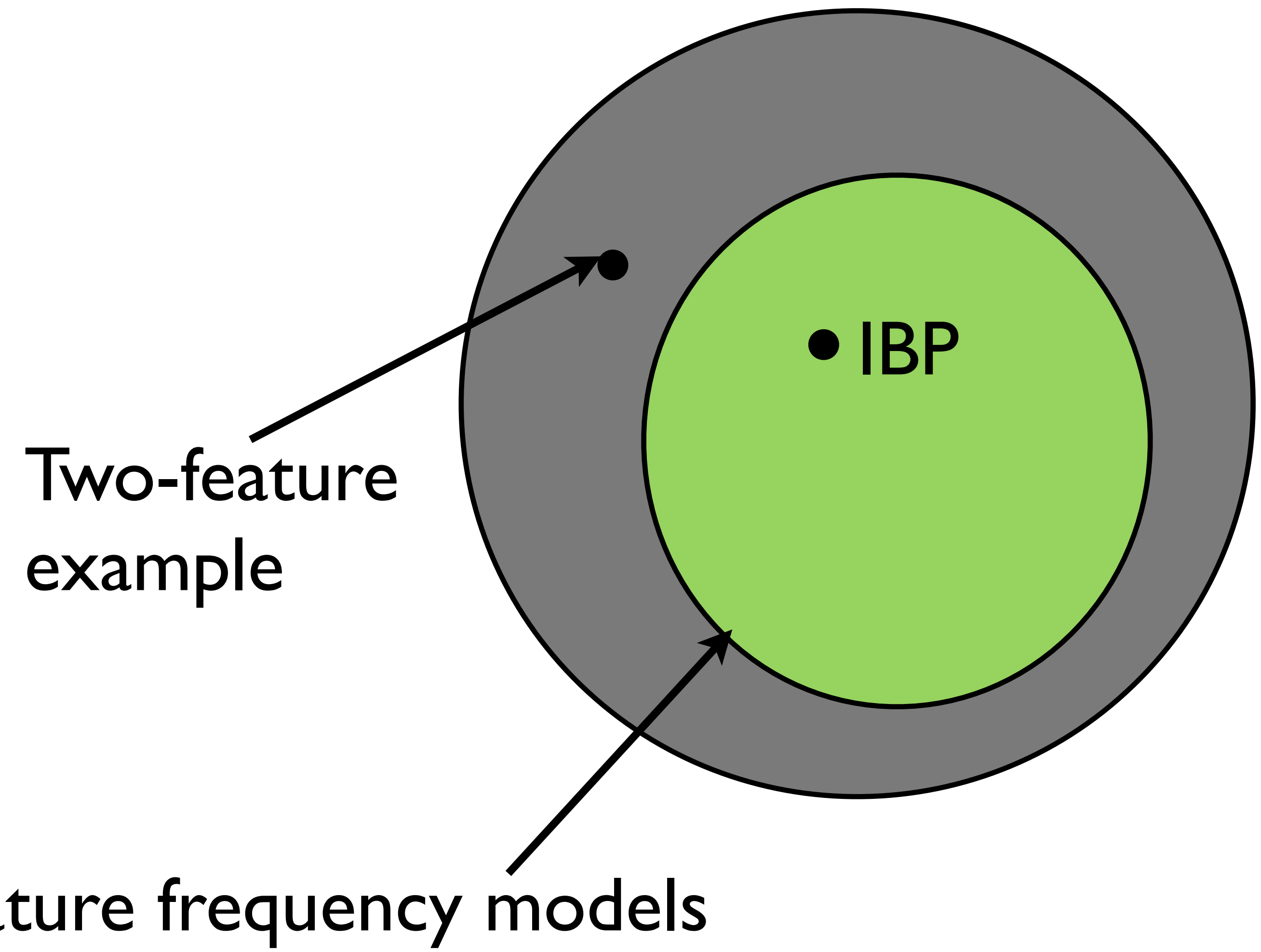
[Broderick, Pitman, Jordan 2013]

Feature frequency models: EFPFs?

Exchangeable cluster distributions
= Cluster distributions with EPPFs
= Kingman paintbox partitions



Exchangeable feature distributions
= Feature paintbox allocations



[Broderick, Pitman, Jordan 2013]

Feature frequency models: EFPFs?

Exchangeable cluster distributions
= Cluster distributions with EPPFs
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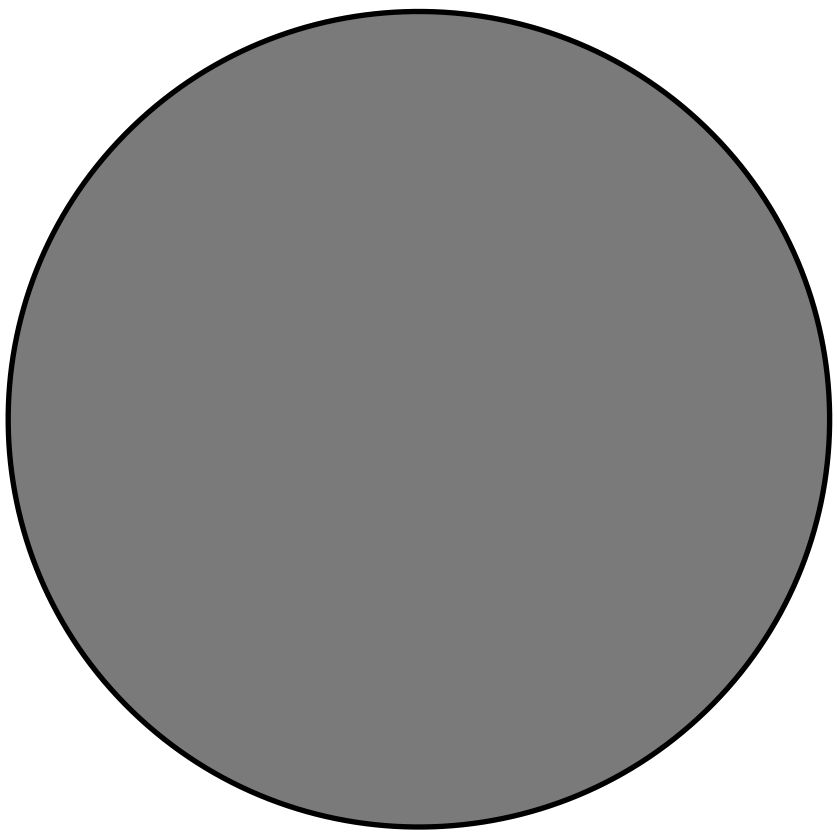

Exchangeable feature distributions
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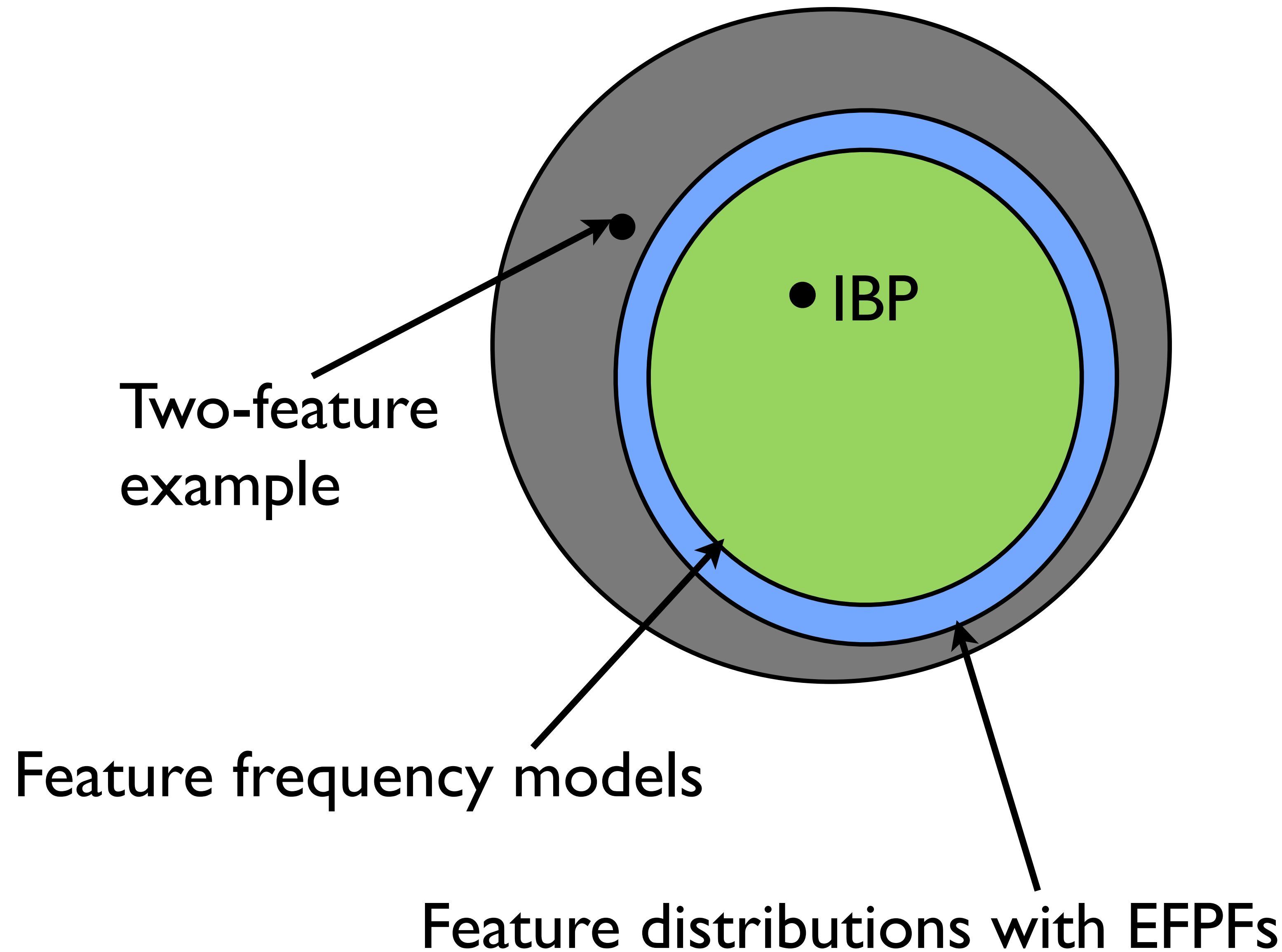


[Broderick, Pitman, Jordan 2013]

Distributions with EFPFs: frequencies?

[Broderick, Pitman, Jordan 2013]

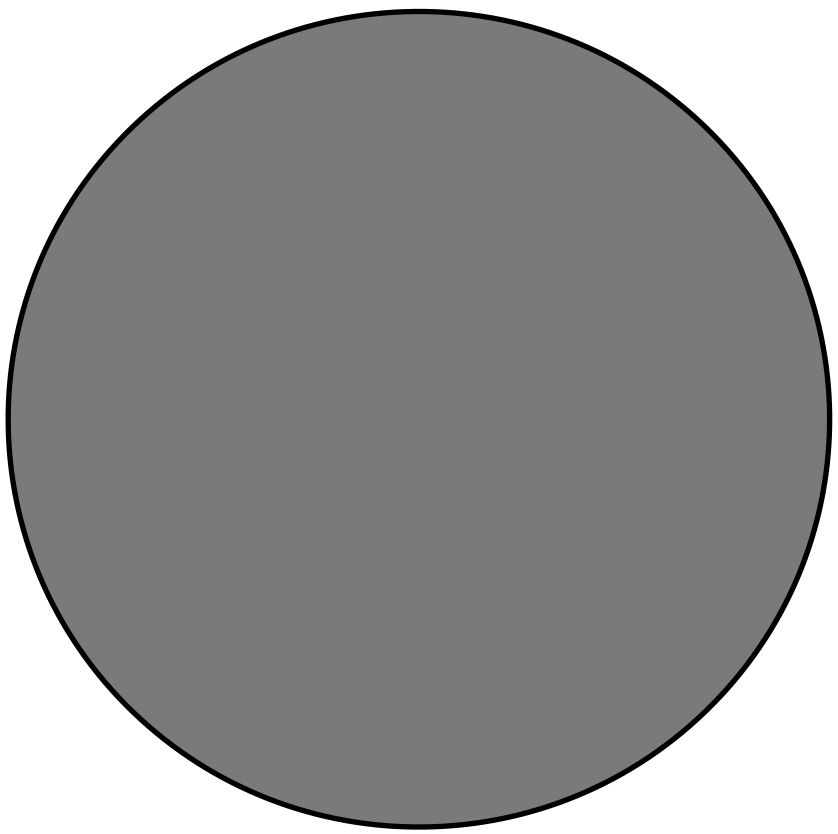
Distributions with EFPFs: frequencies?

✔ • Any number (+unbounded case) of features

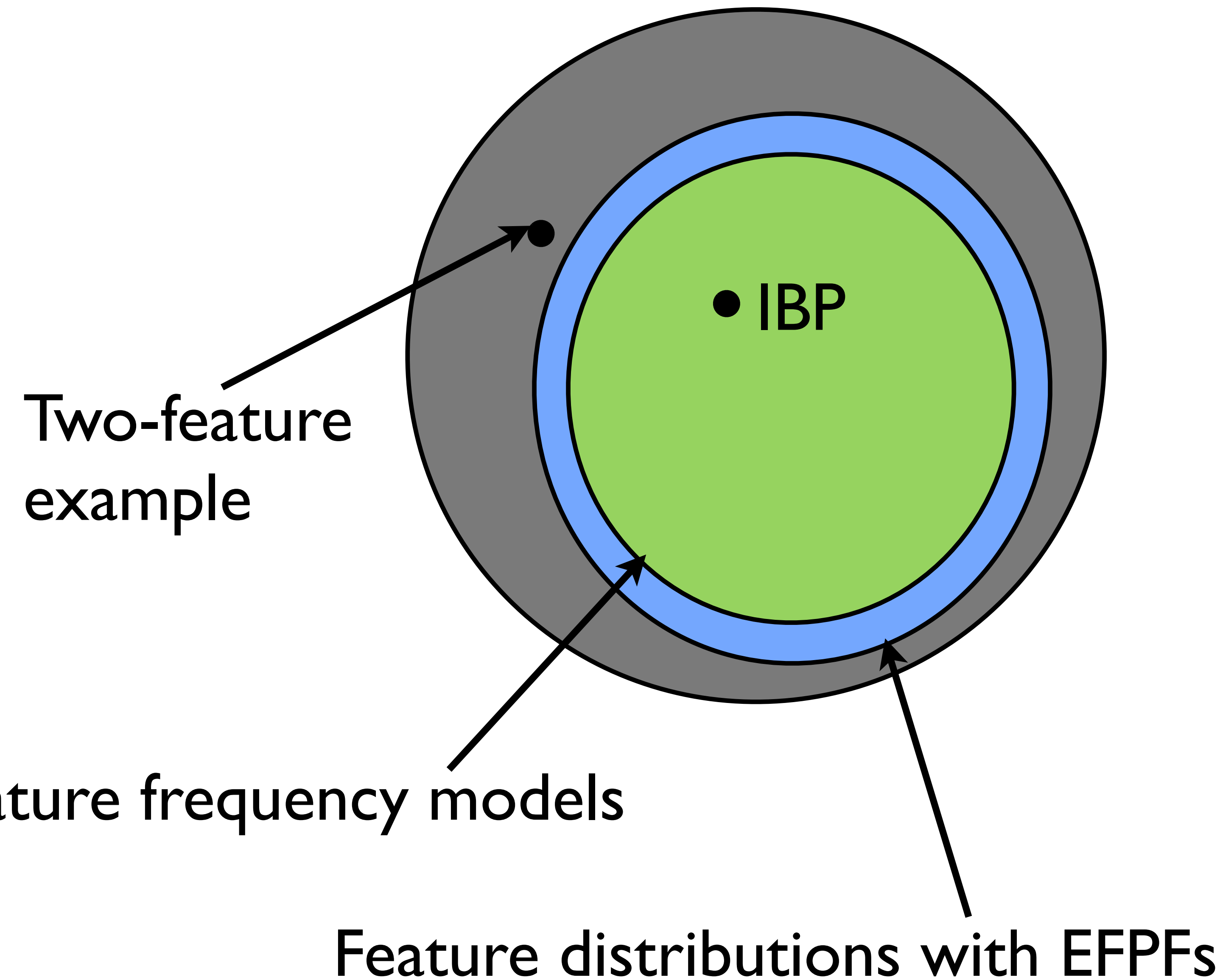
[Broderick, Pitman, Jordan 2013]

Distributions with EFPFs: frequencies?

Exchangeable cluster distributions
= Cluster distributions with EPPFs
= Kingman paintbox partitions



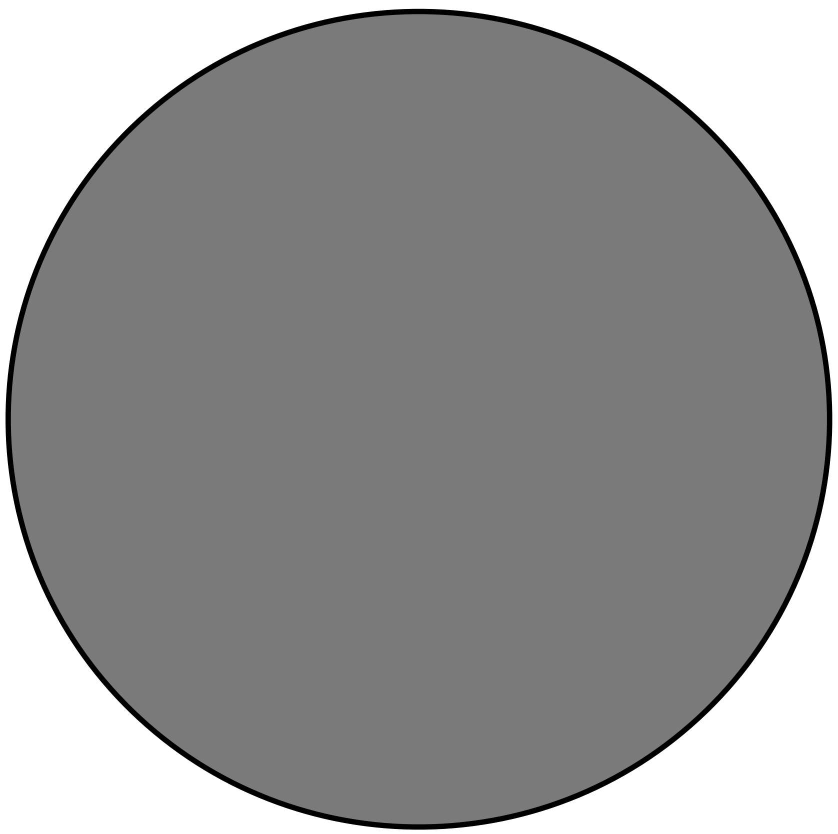
Exchangeable feature distributions
= Feature paintbox allocations



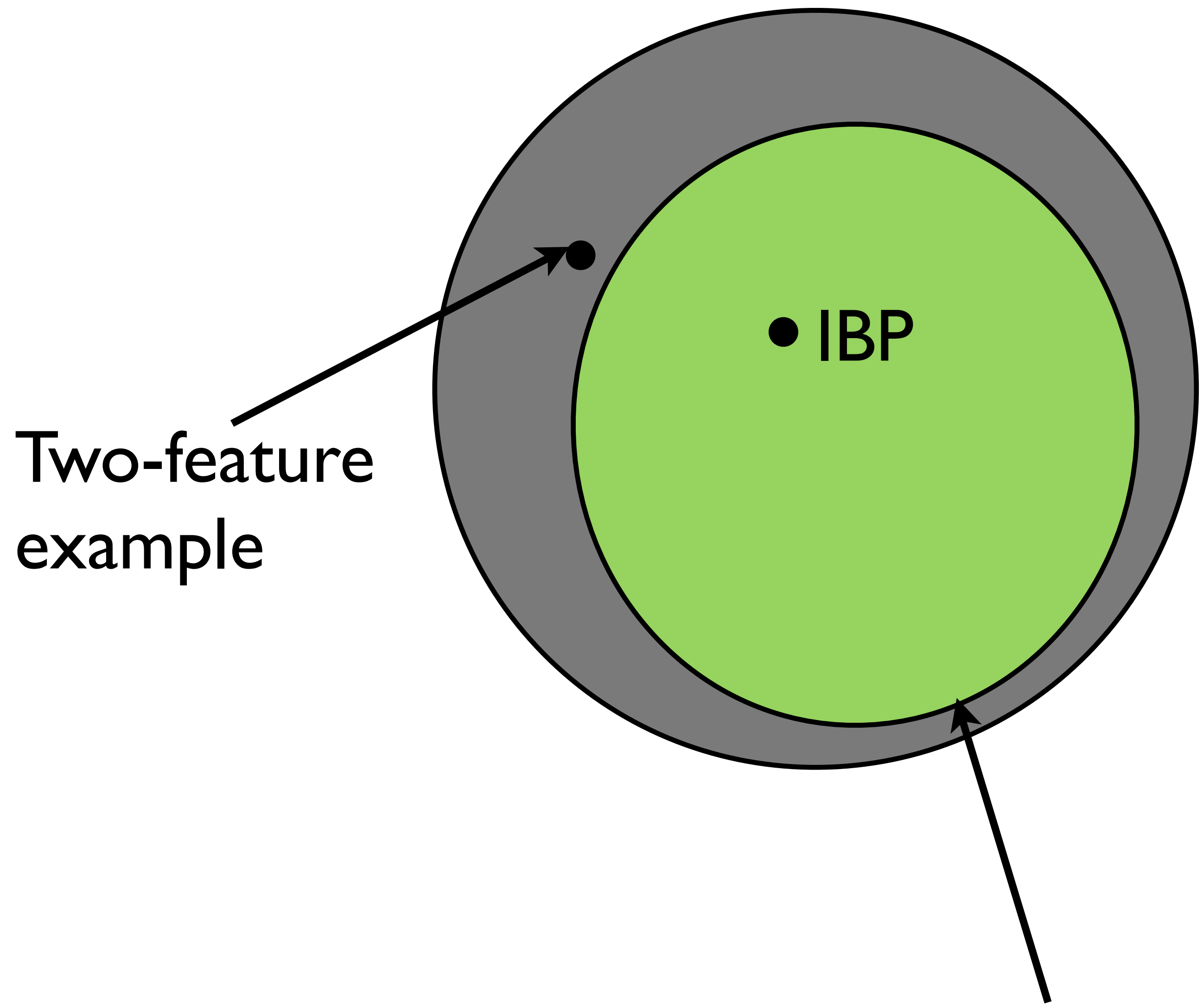
[Broderick, Pitman, Jordan 2013]

Distributions with EFPFs: frequencies?

Exchangeable cluster distributions
= Cluster distributions with EPPFs
= Kingman paintbox partitions



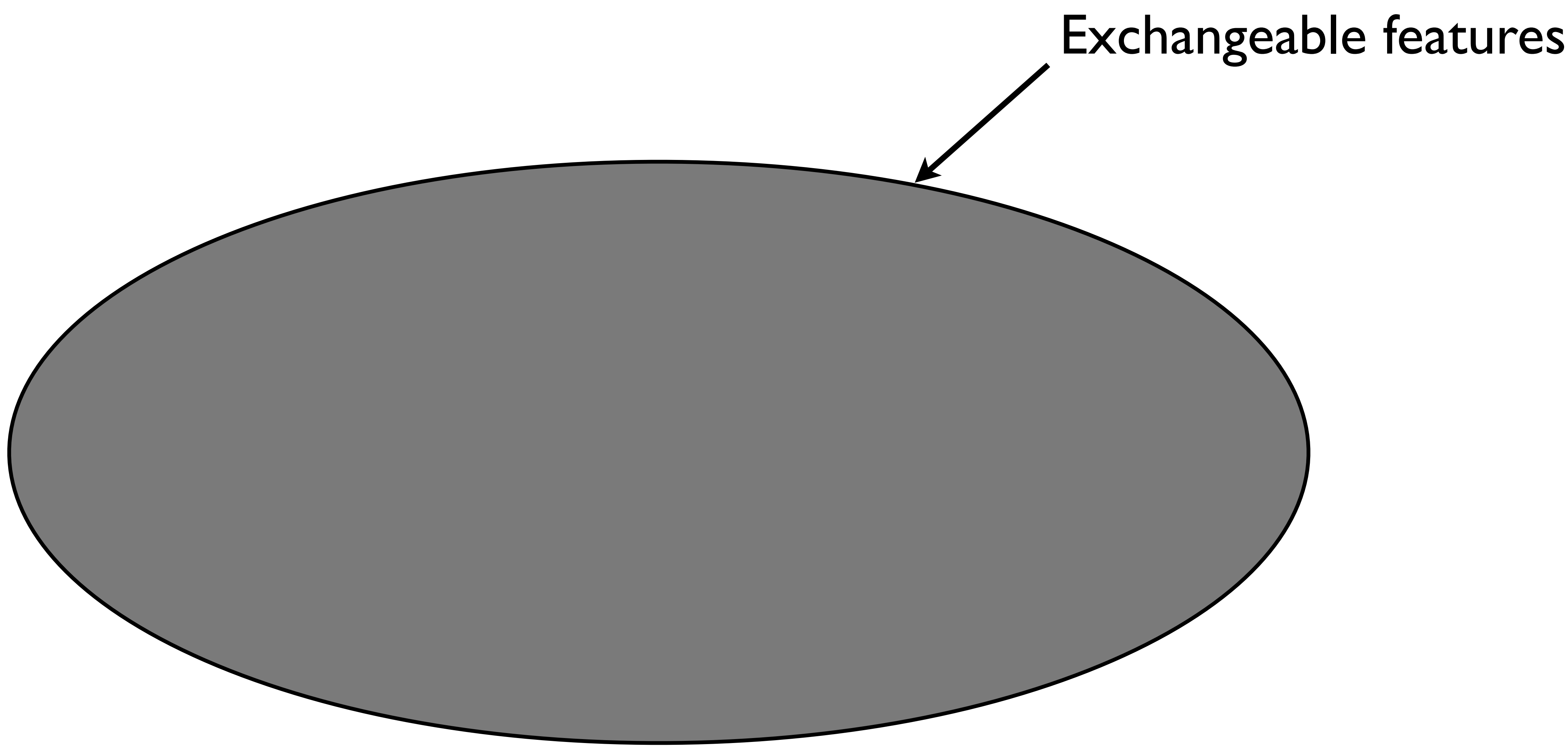
Exchangeable feature distributions
= Feature paintbox allocations



Feature distributions with EFPFs
= Feature frequency models

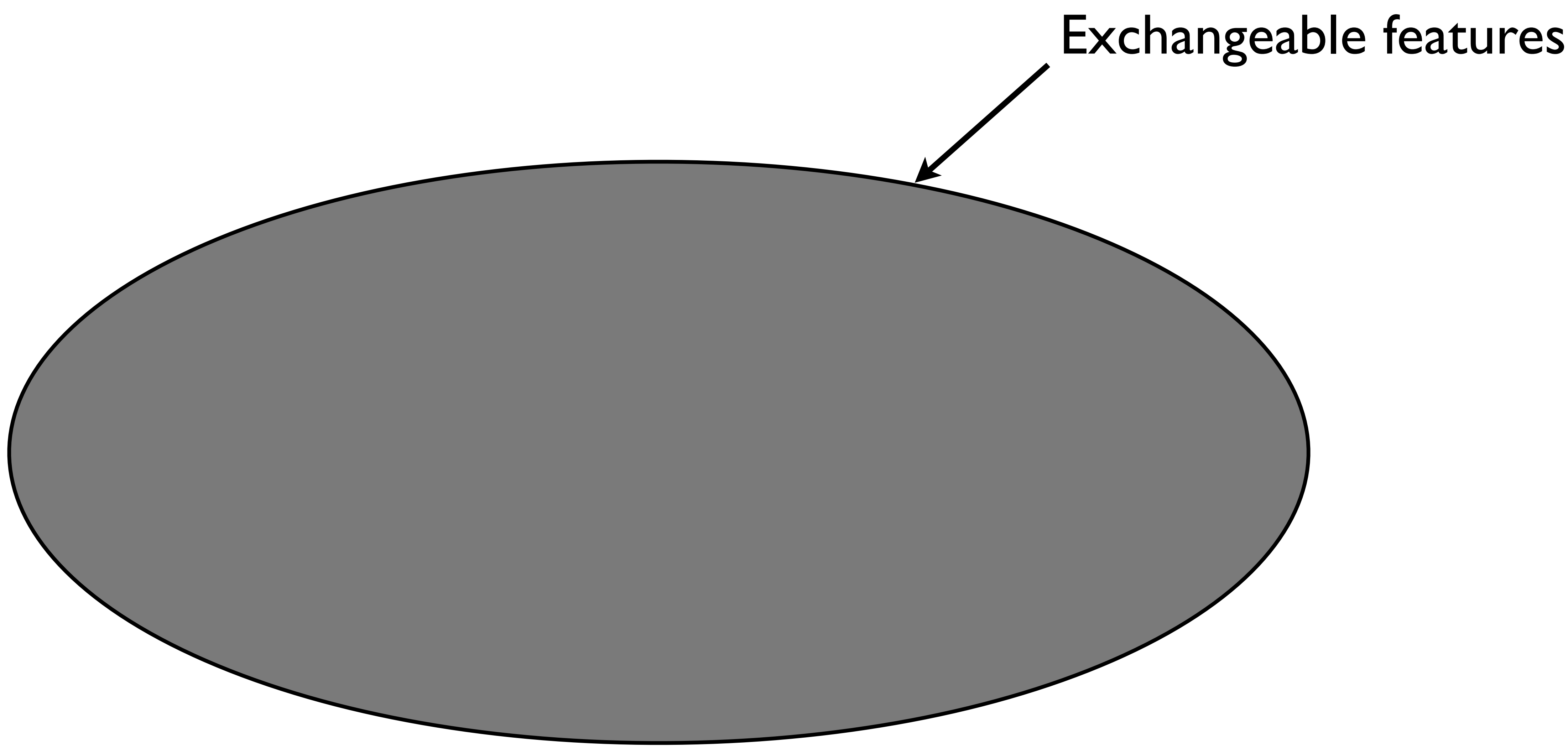
[Broderick, Pitman, Jordan 2013]

Theory conclusions



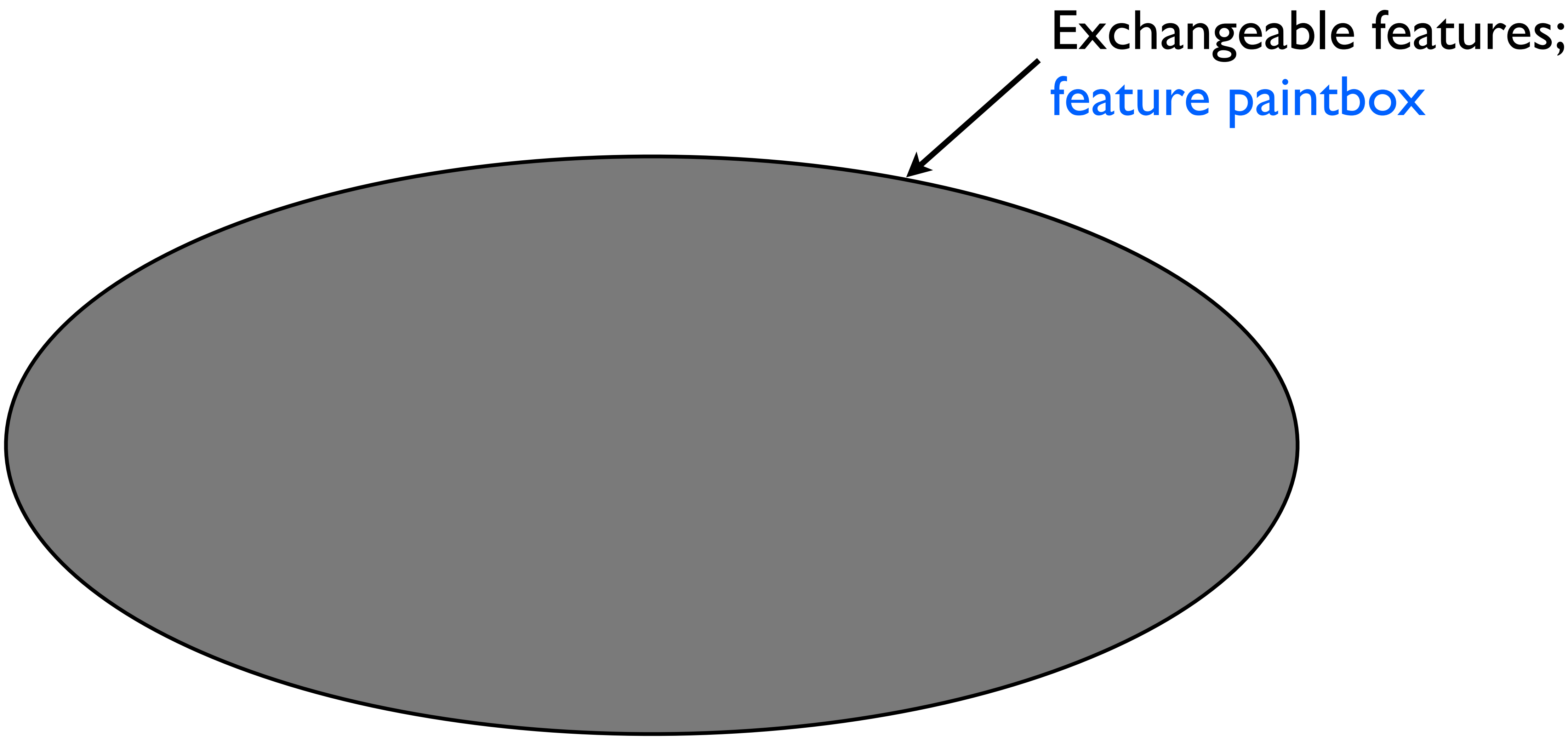
Theory conclusions

• Feature paintbox: characterization of exchangeable feature models



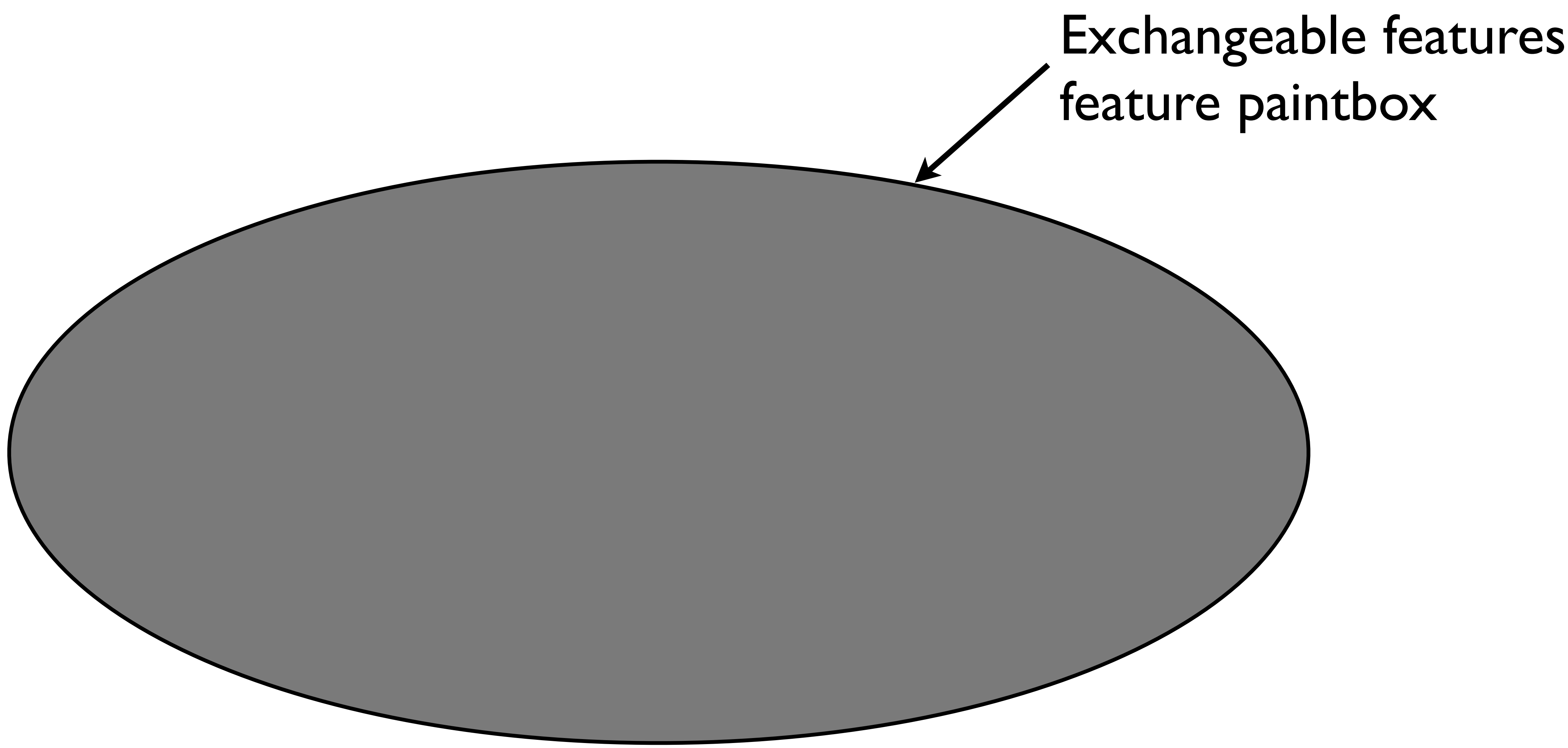
Theory conclusions

- Feature paintbox: characterization of exchangeable feature models



Theory conclusions

- Feature paintbox: characterization of exchangeable feature models
- Limits of clustering characterizations in feature case?



Theory conclusions

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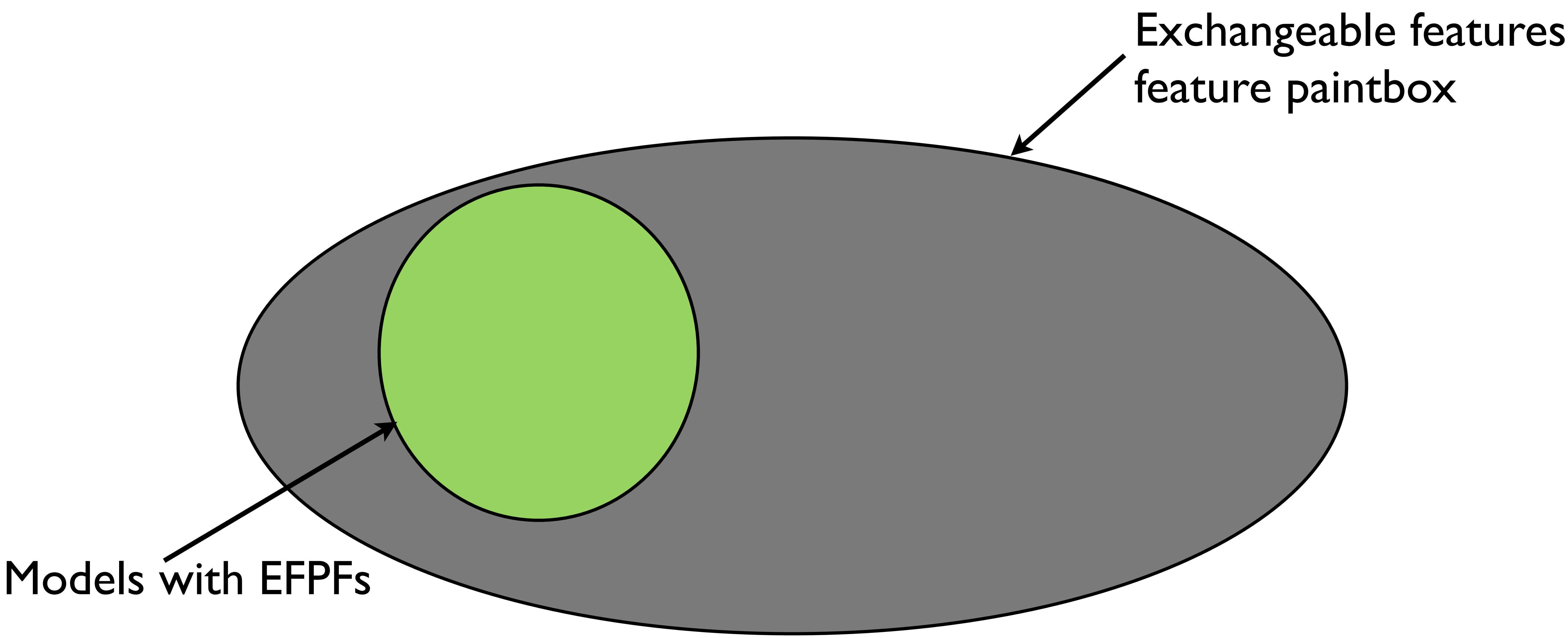

Theory conclusions

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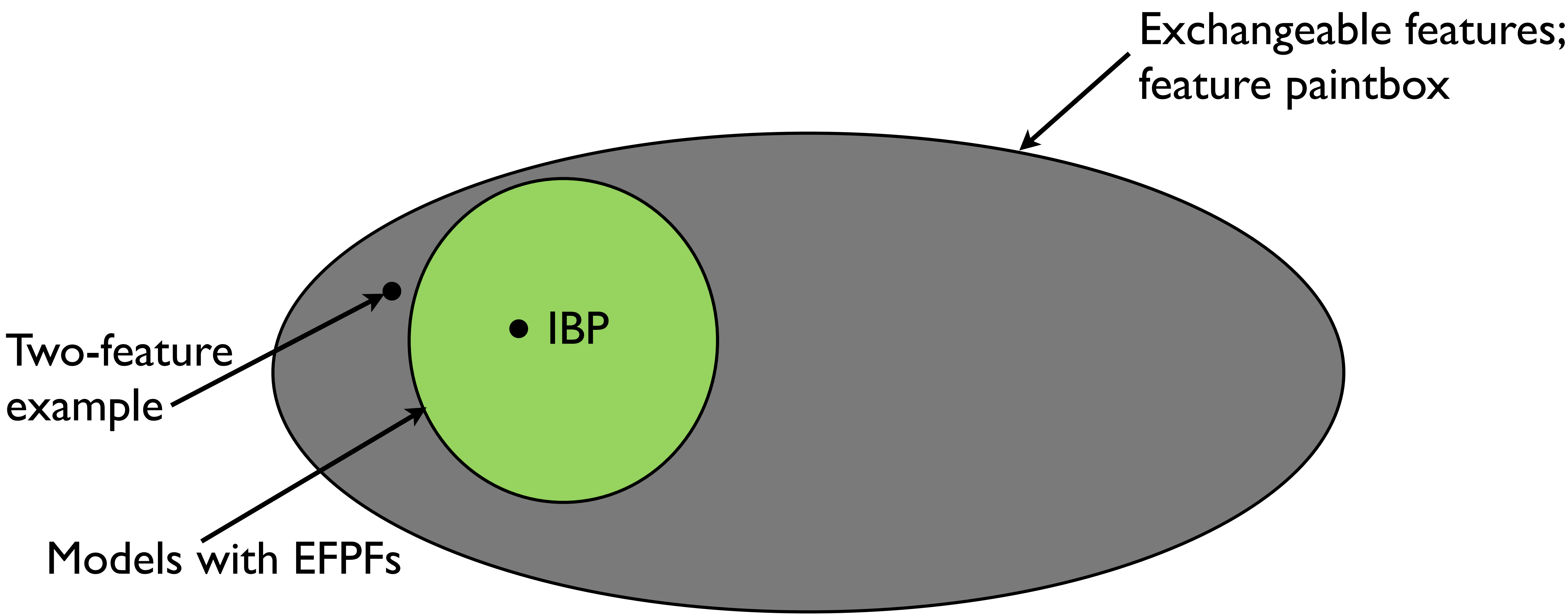

Theory conclusions

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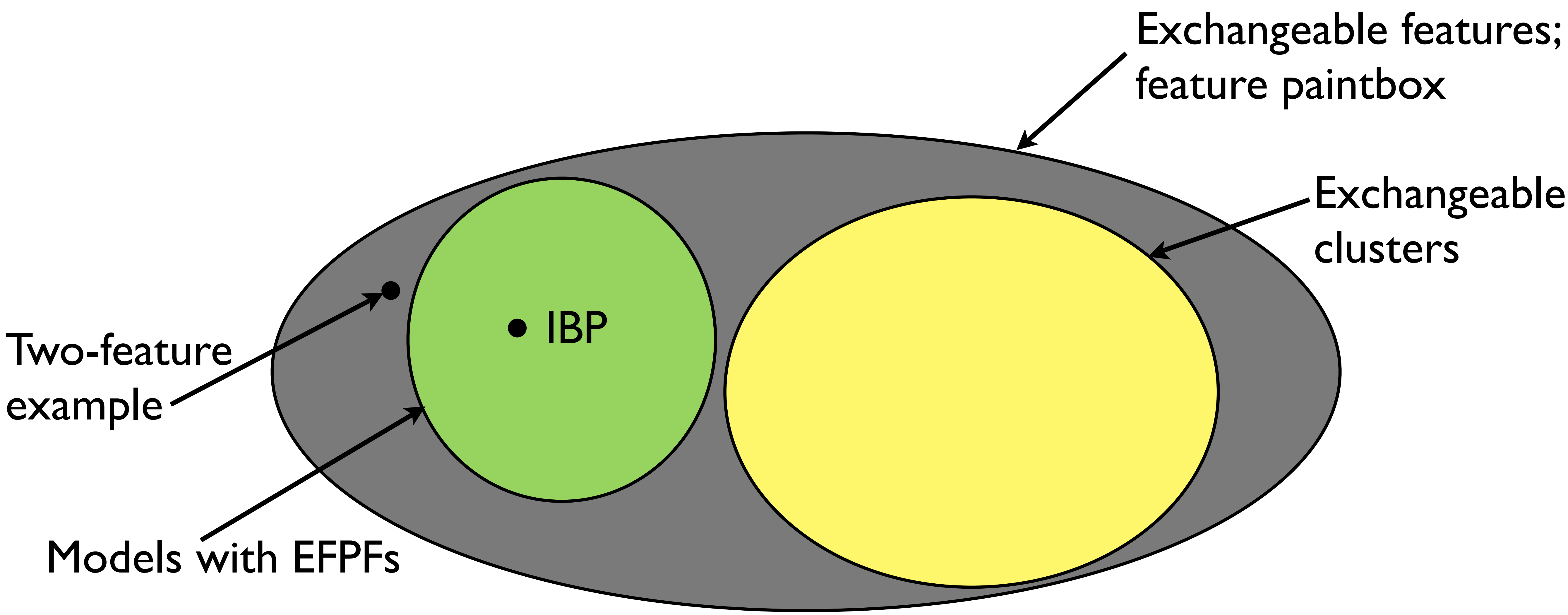
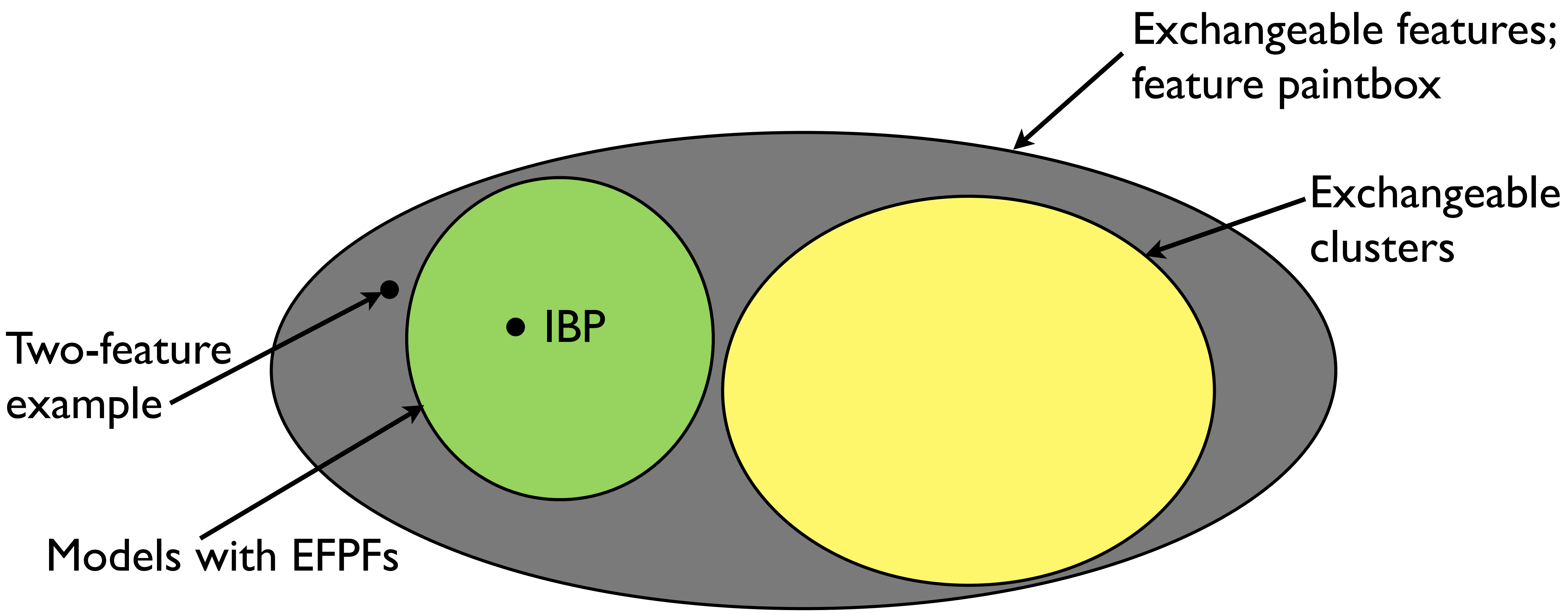


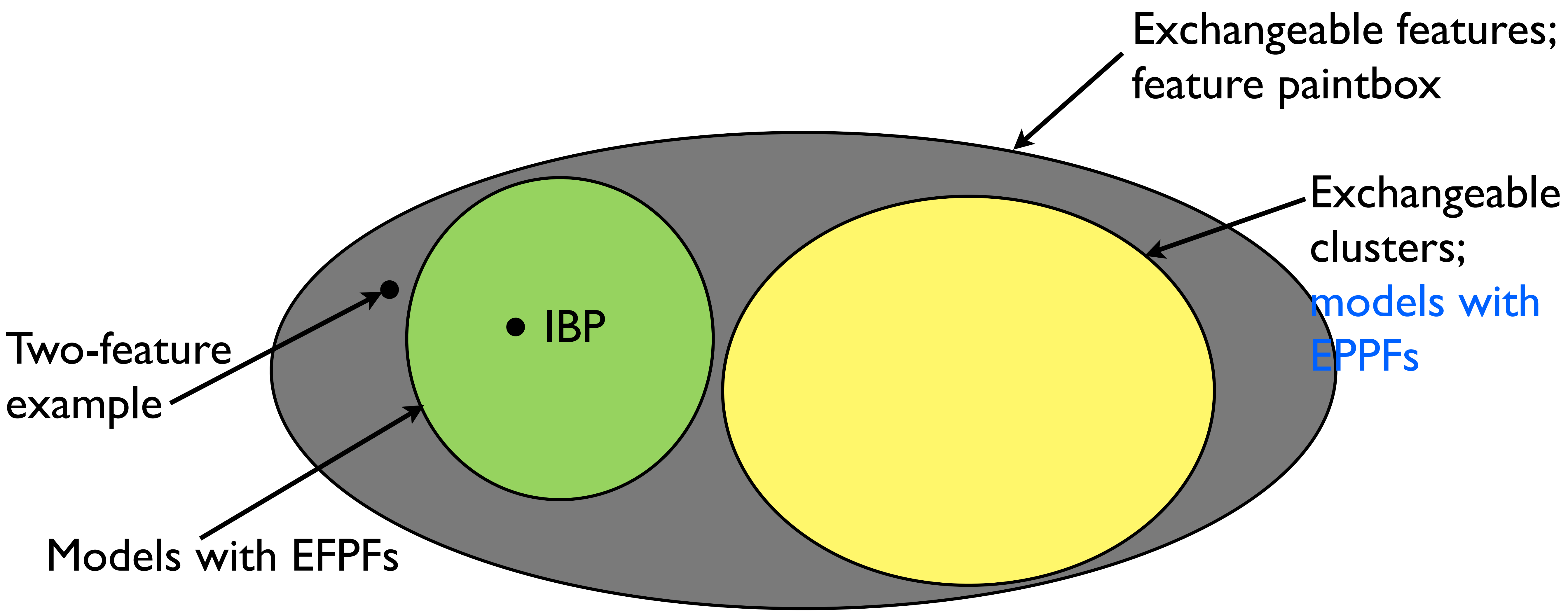
Theory conclusions

- Feature paintbox: characterization of exchangeable feature models
- Characterization of alternative correlation structure



Theory conclusions

- Feature paintbox: characterization of exchangeable feature models
- Characterization of alternative correlation structure



Theory conclusions

- Feature paintbox: characterization of exchangeable feature models
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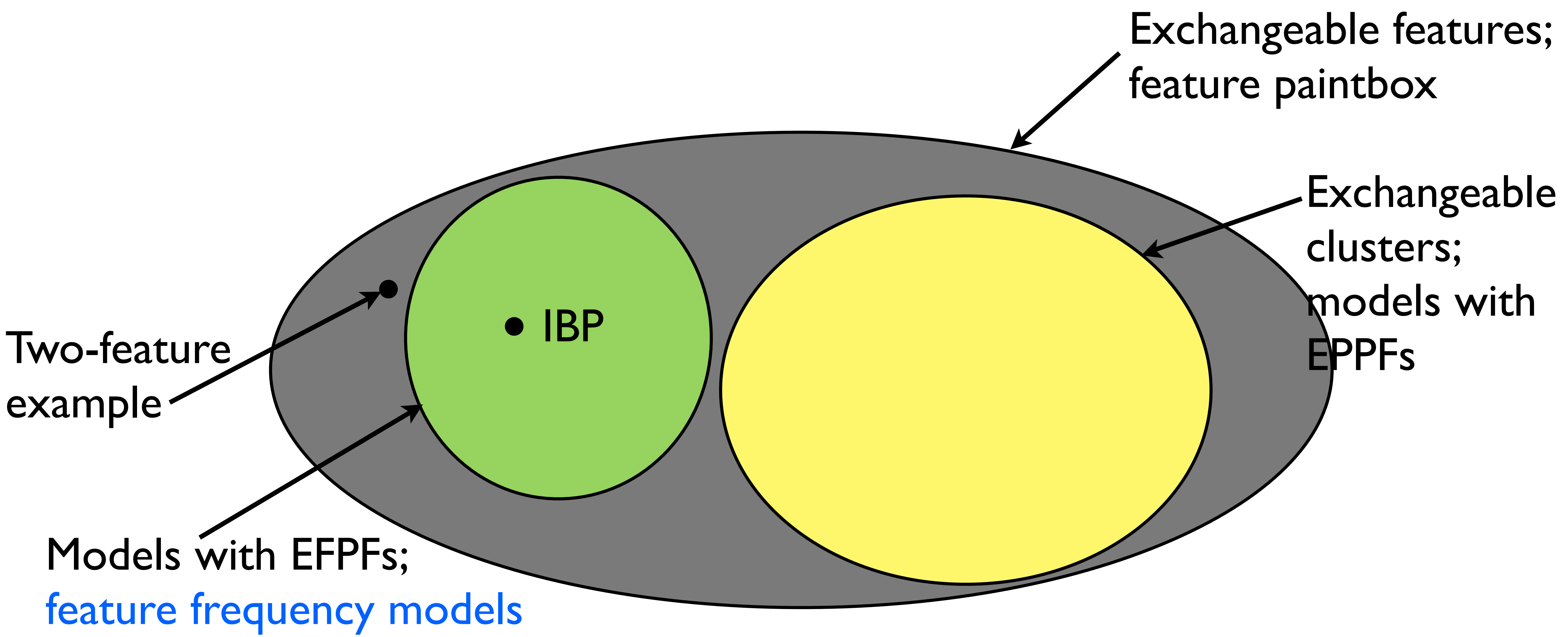
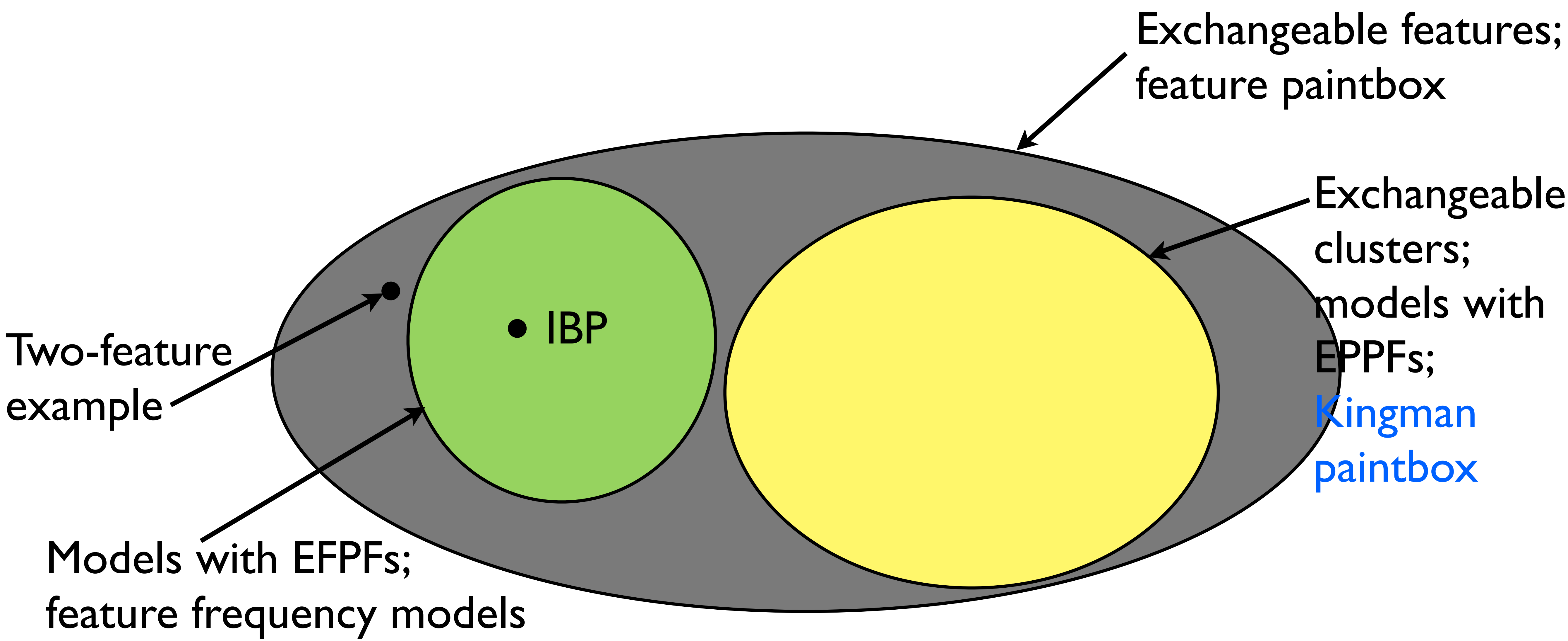


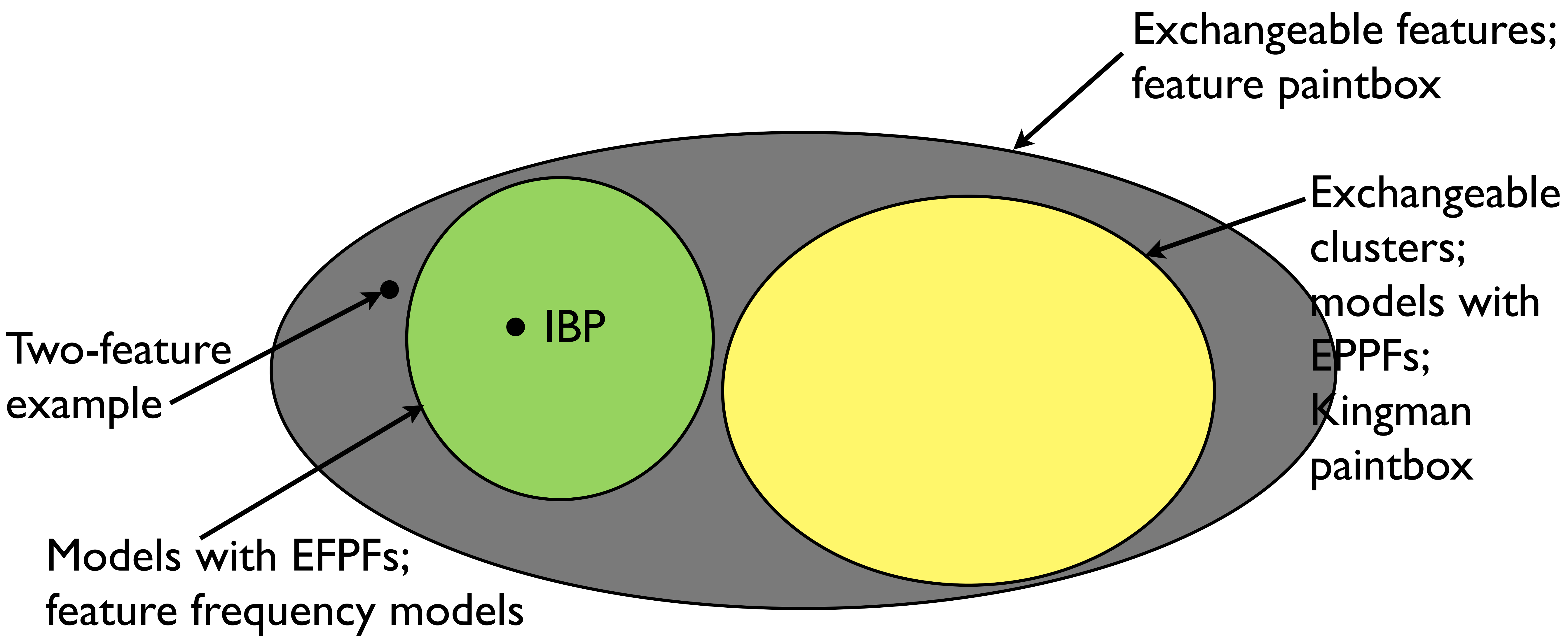
Theory conclusions

- Feature paintbox: characterization of exchangeable feature models
- Characterization of alternative correlation structure



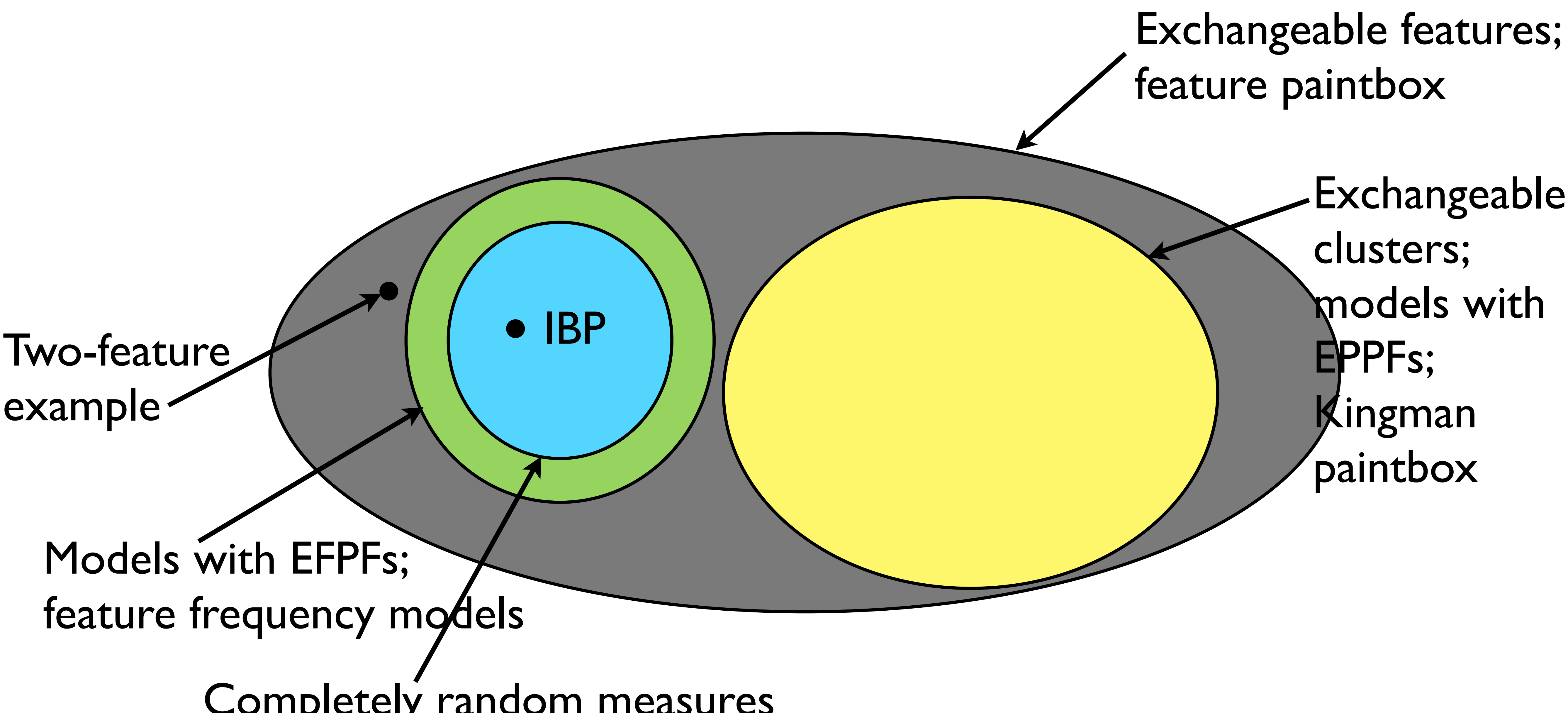
Theory conclusions

- Feature paintbox: characterization of exchangeable feature models
- Characterization of alternative correlation structure
- Remaining connections



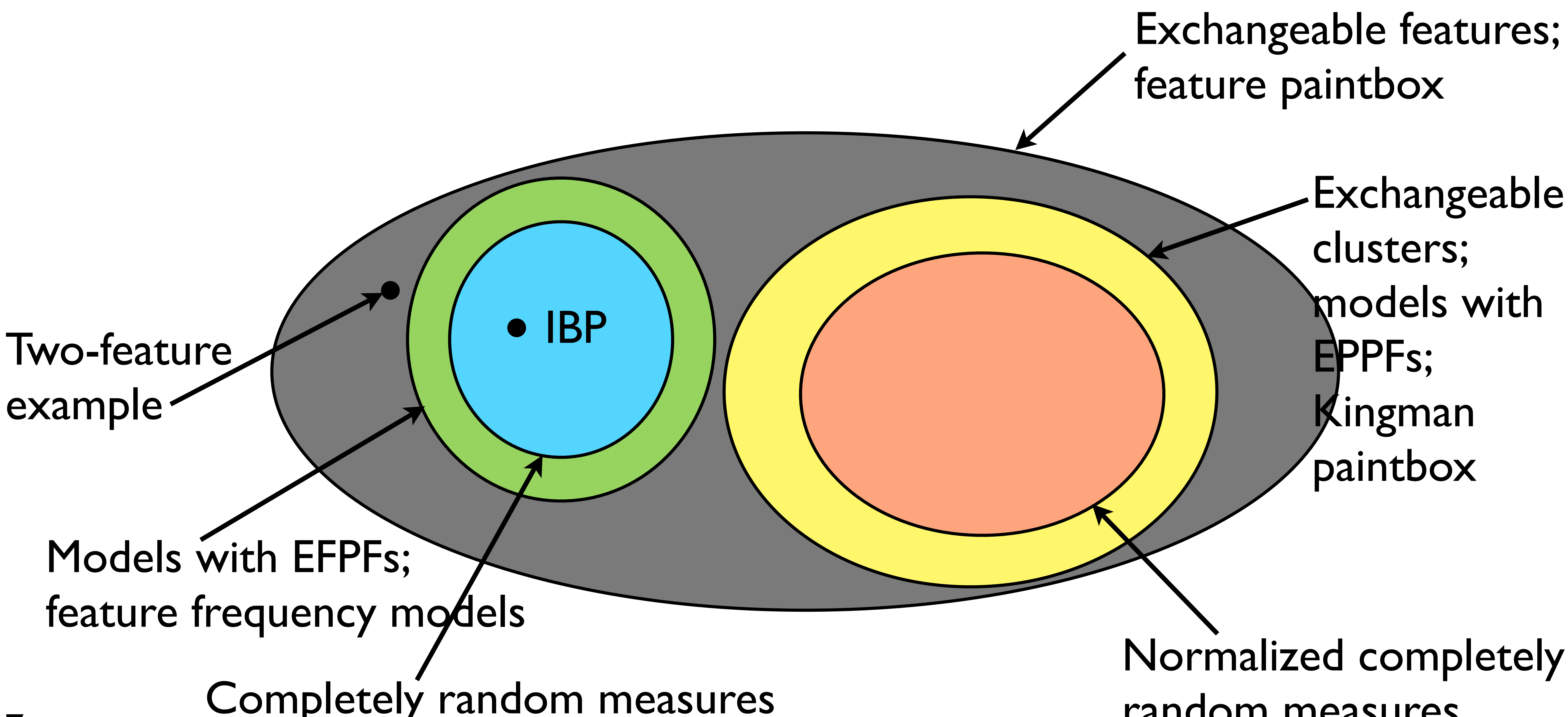
Theory conclusions

- Feature paintbox: characterization of exchangeable feature models
- Characterization of alternative correlation structure
- Remaining connections (CRMs)



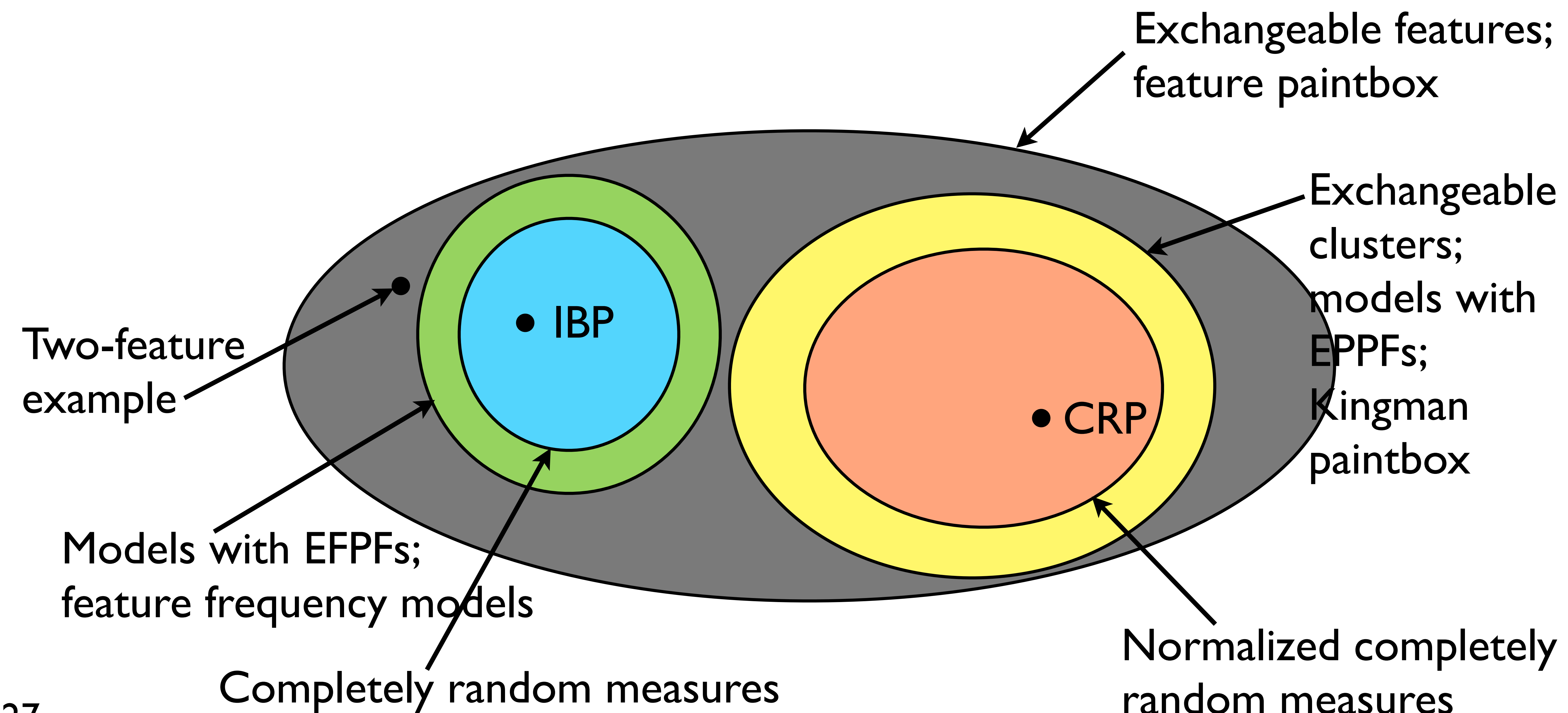
Theory conclusions

- Feature paintbox: characterization of exchangeable feature models
- Characterization of alternative correlation structure
- Remaining connections (CRMs)



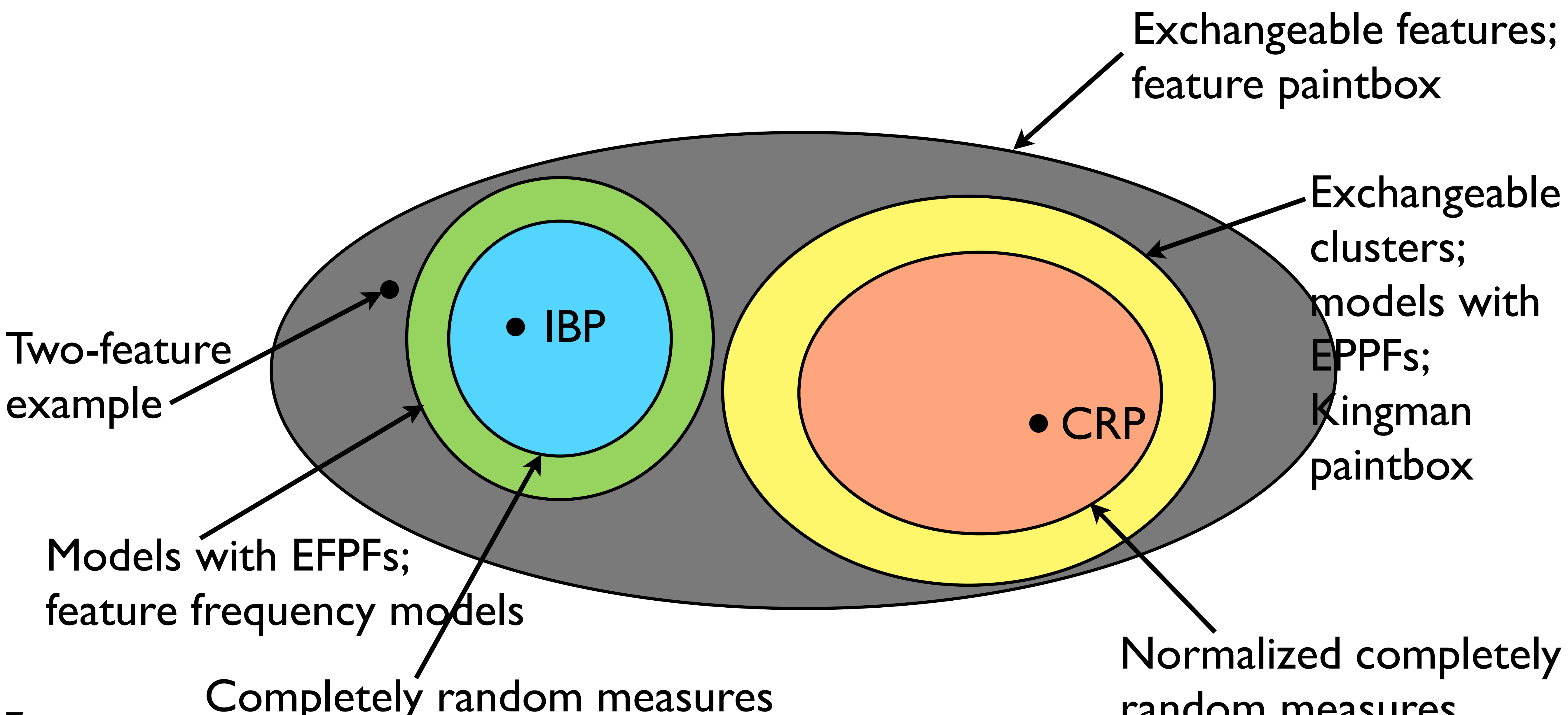
Theory conclusions

- Feature paintbox: characterization of exchangeable feature models
- Characterization of alternative correlation structure
- Remaining connections (CRMs)



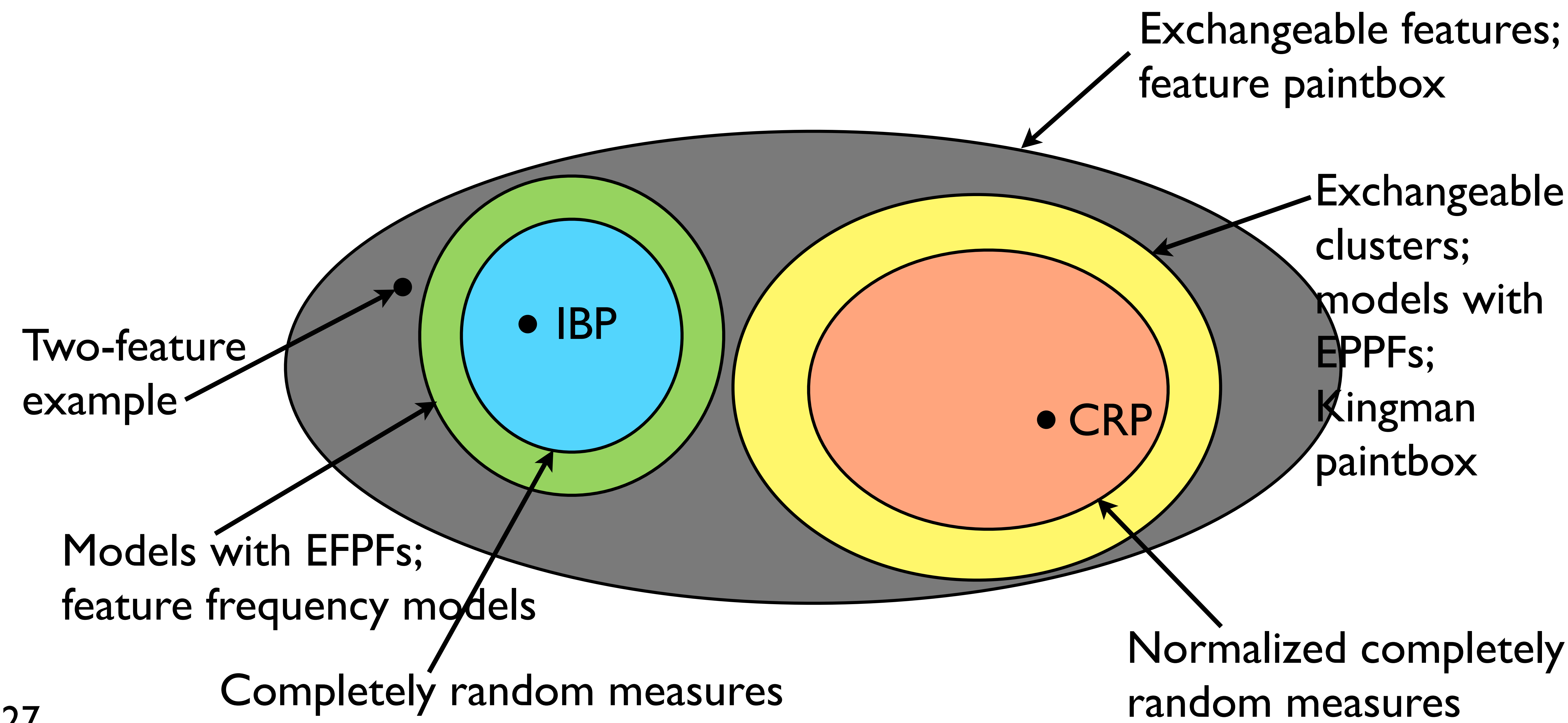
Theory conclusions

- Feature paintbox: characterization of exchangeable feature models
- Characterization of alternative correlation structure
- Remaining connections (CRMs, dust)



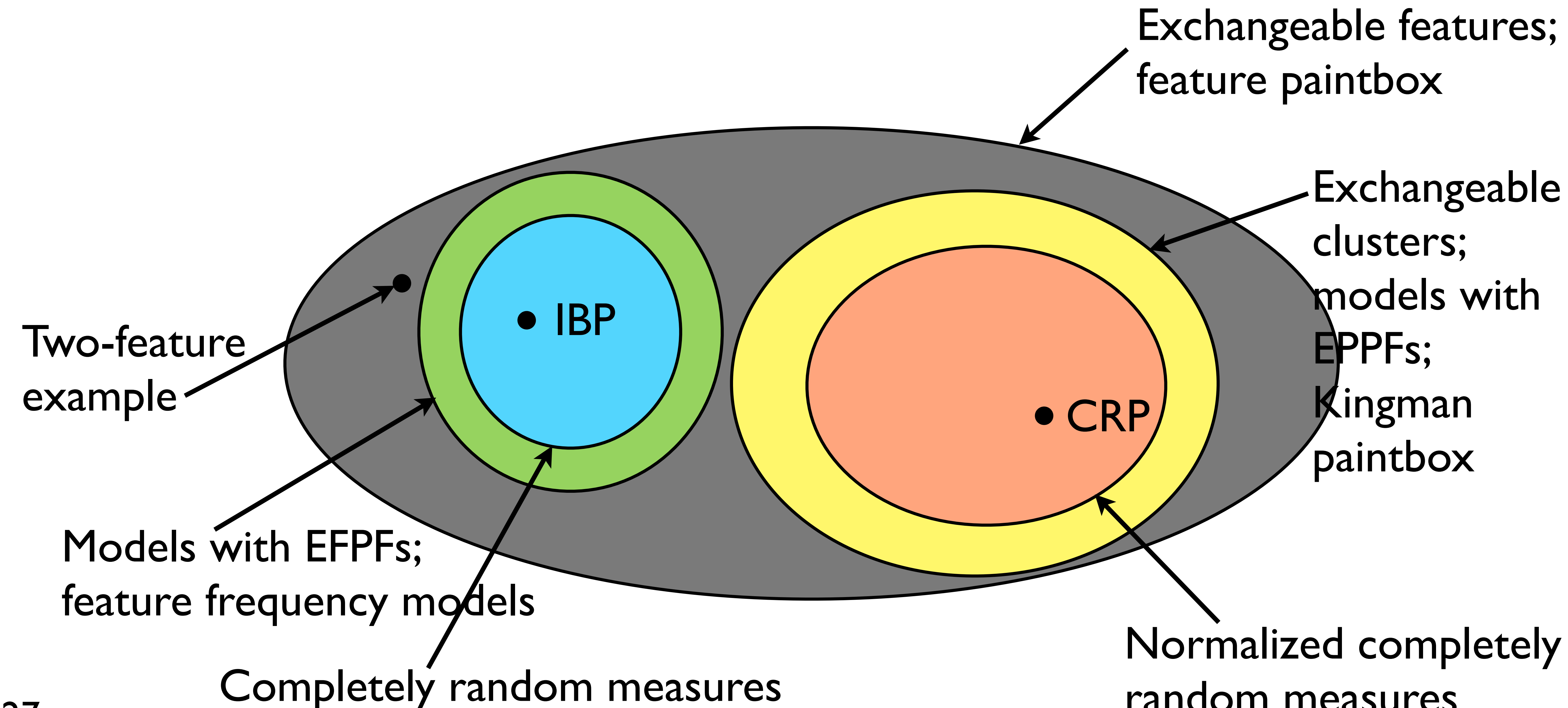
Theory conclusions

- Feature paintbox: characterization of exchangeable feature models
- Characterization of alternative correlation structure
- Remaining connections (CRMs, dust, etc)



Theory conclusions

- Feature paintbox: characterization of exchangeable feature models
- Characterization of alternative correlation structure
- Remaining connections (CRMs, dust, etc)
- Other combinatorial structures



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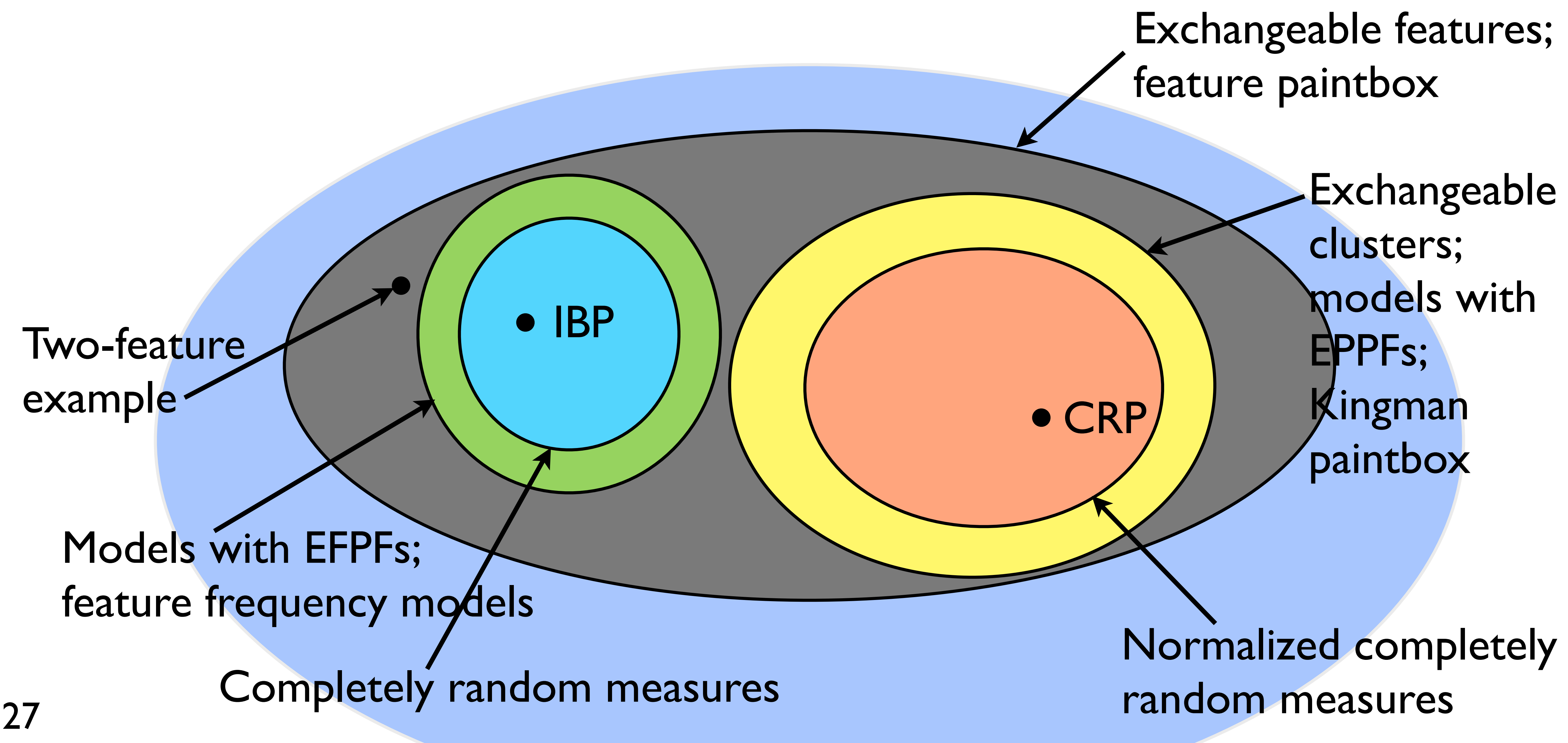

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