Posteriors, conjugacy, and exponential families

for completely random measures

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MIT Berkeley Berkeley

Models

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- Gamma process, Poisson likelihood process (DP, CRP)

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- *Parametric* exponential family conjugacy [Diaconis & Ylvisaker 1979]

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$$p(\theta) \propto \theta^\alpha (1+\theta)^{-\alpha-\beta} = \mathrm{BetaPrime}(\theta|\alpha, \beta) \qquad \alpha > 0, \beta > 0$$

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$$p(\theta|x) \propto \theta^{\alpha+x} (1+\theta)^{-(\alpha+x)-(\beta-x+1)}$$

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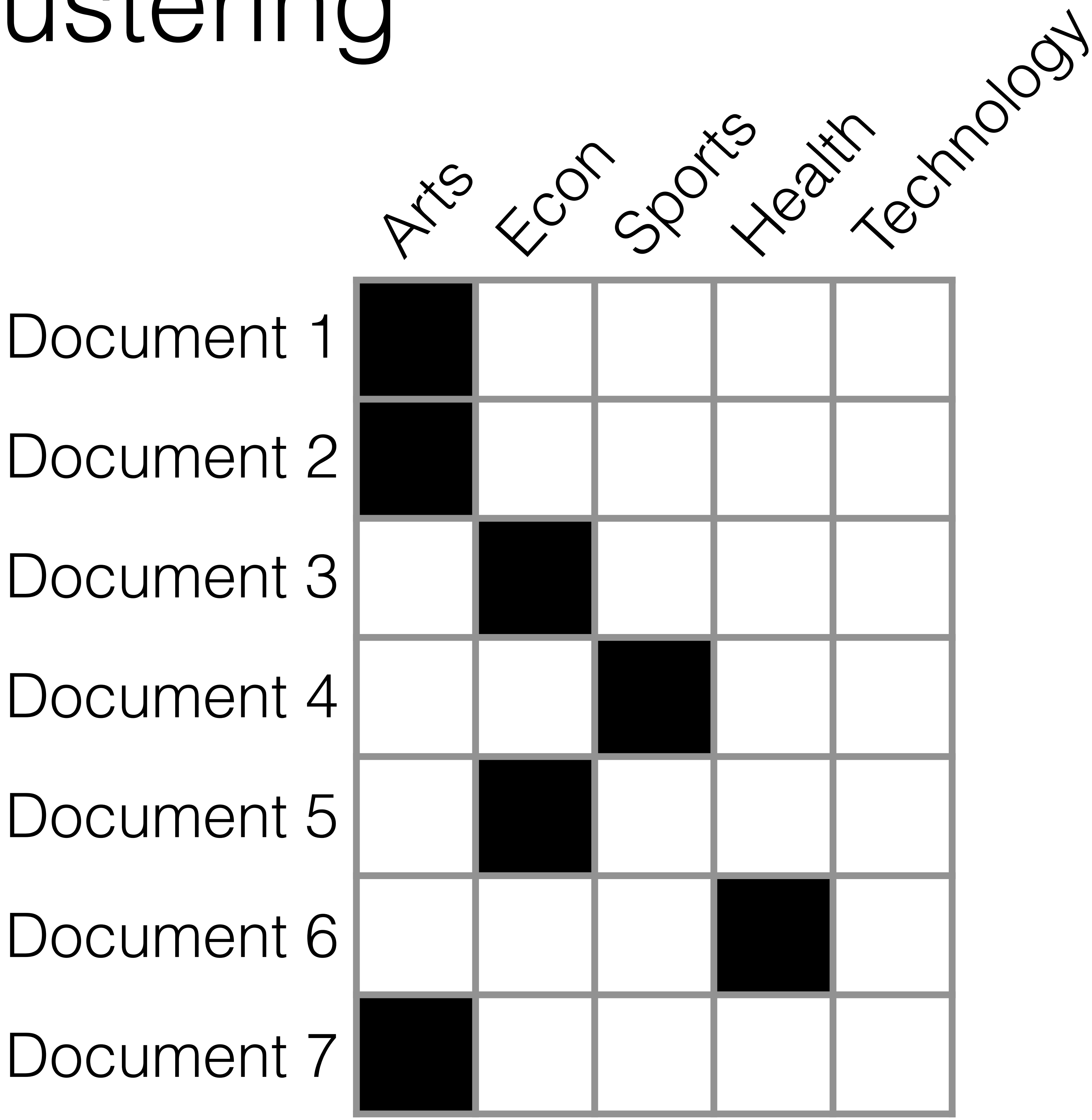
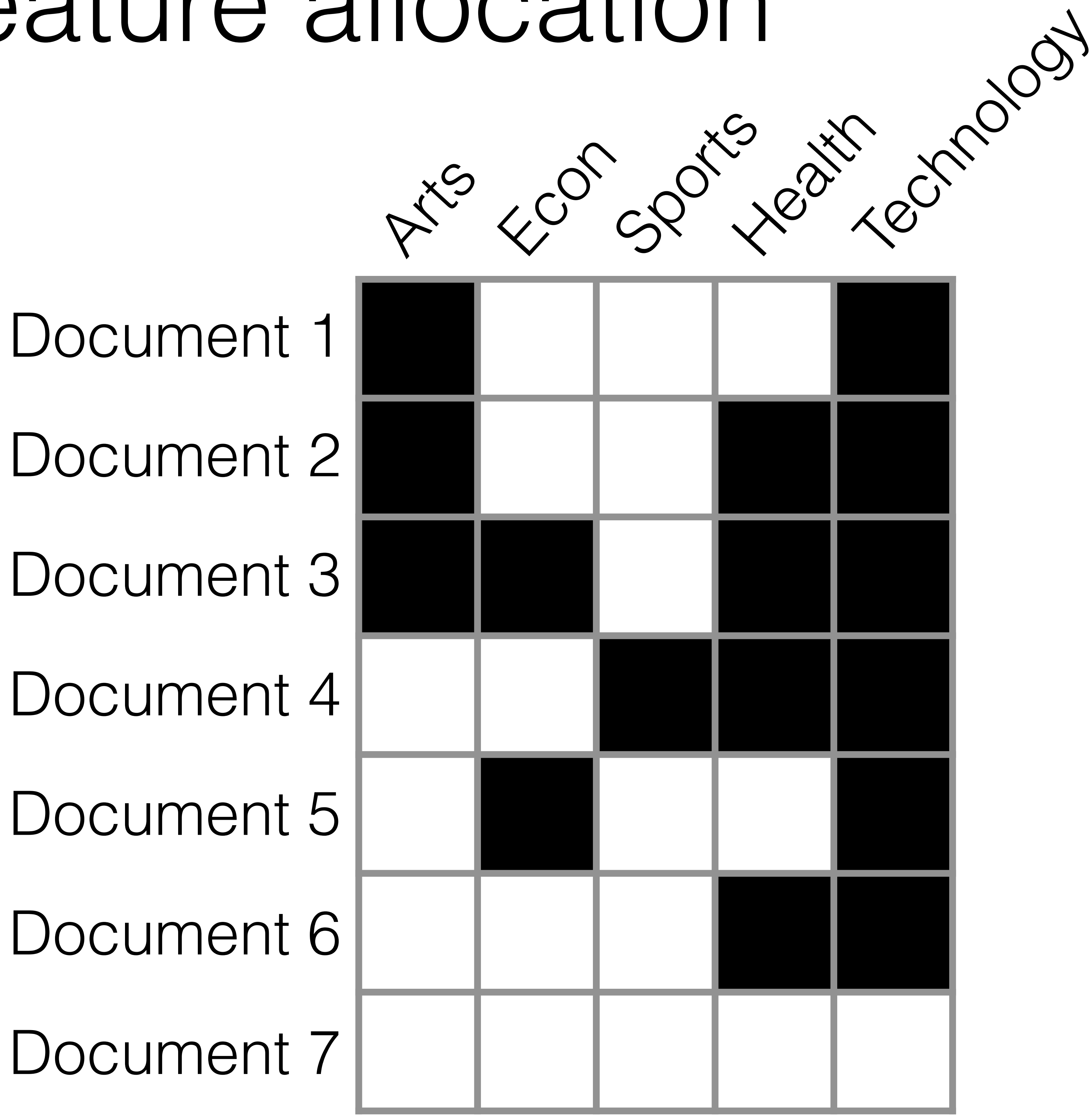
Clustering

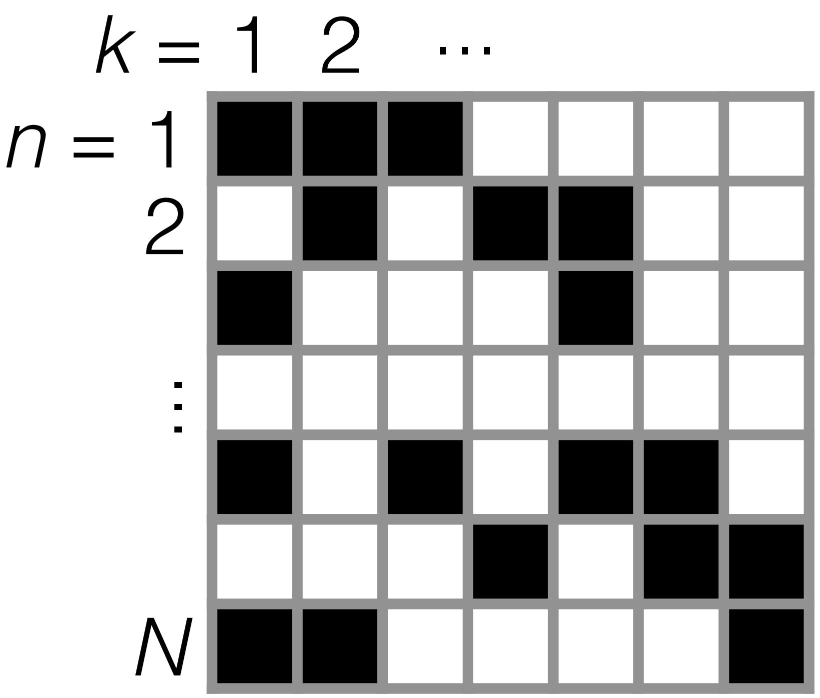


Arts
Econ
Sports
Health
Technology
Document 1
Document 2
Document 3
Document 4
Document 5
Document 6
Document 7

Feature allocation



Indian buffet process (IBP)



For $n = 1, 2, \ldots, N$

1.

2.

[Griffiths & Ghahramani 2006]

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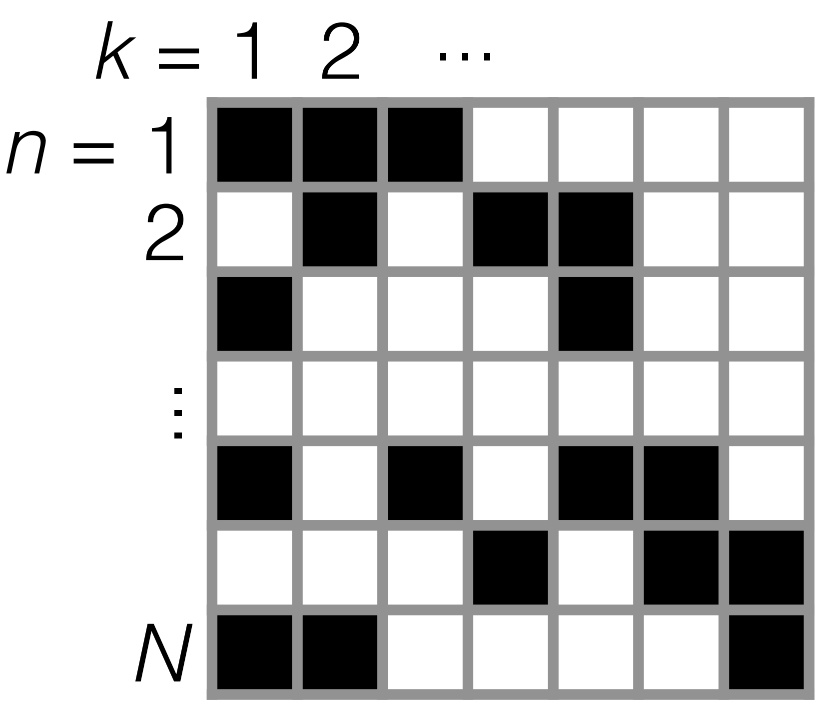

For n = 1, 2, ..., N

1. Data point n has an existing feature k that has occurred $S_{n-1,k}$ times with probability $\frac{S_{n-1,k}}{\beta + n - 1}$

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[Griffiths & Ghahramani 2006]

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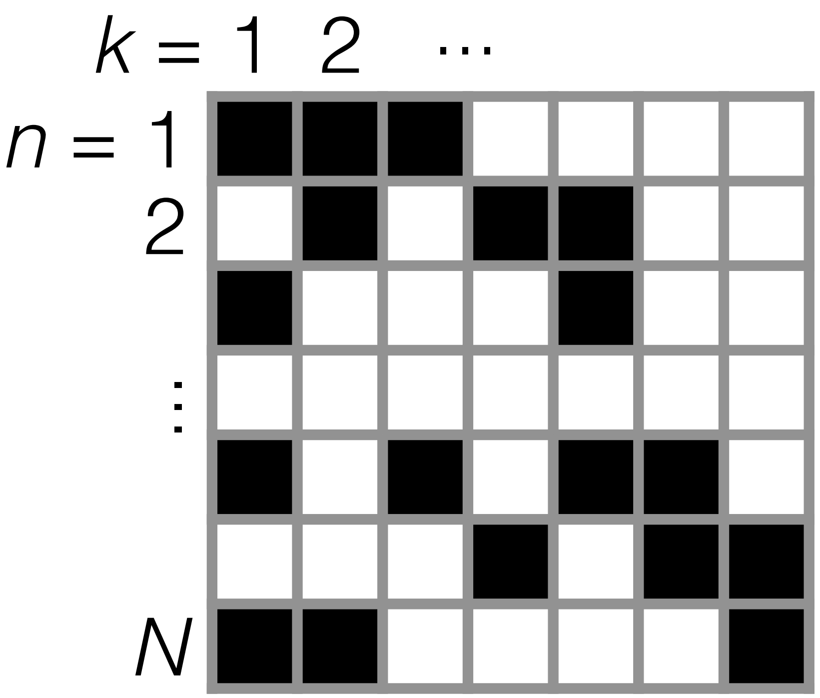

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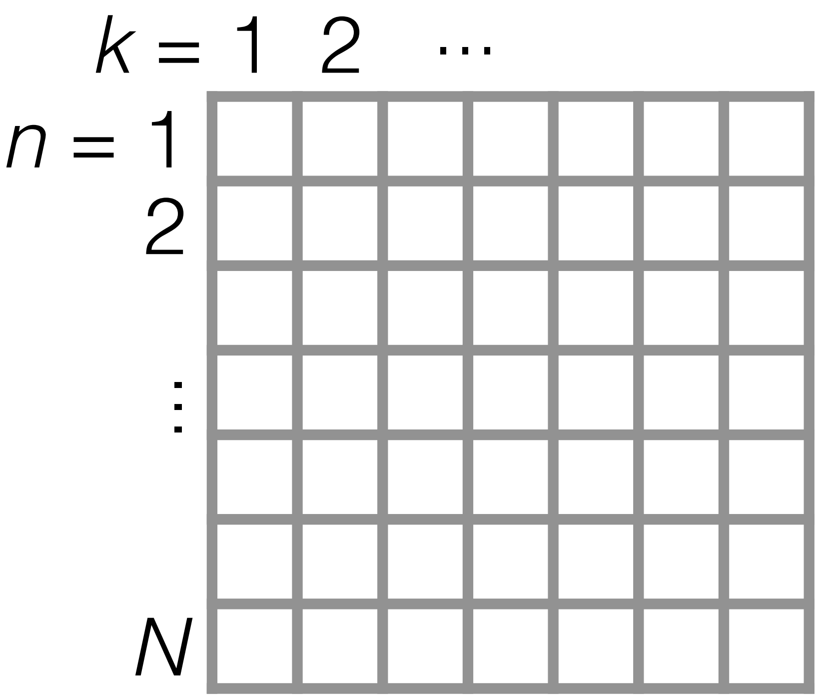

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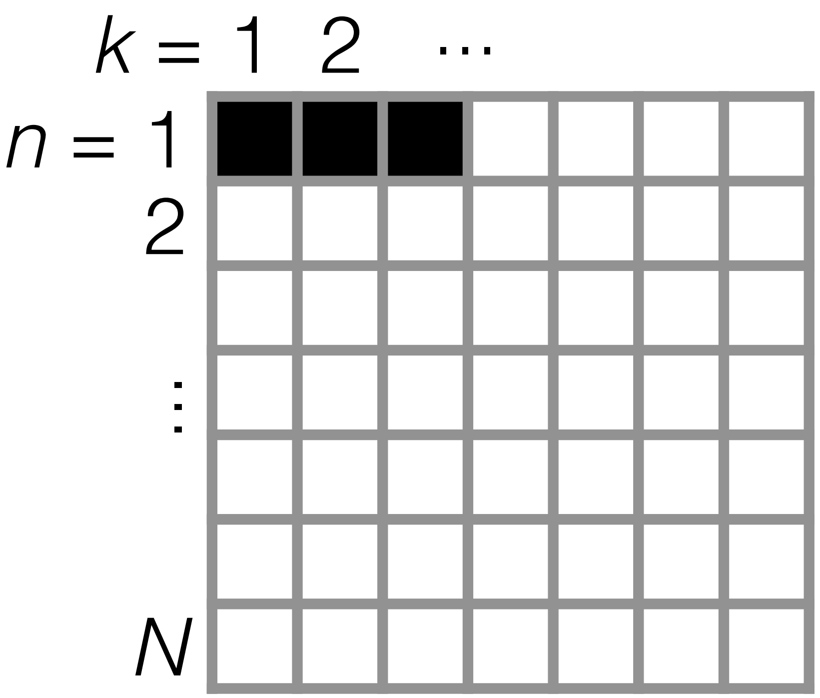

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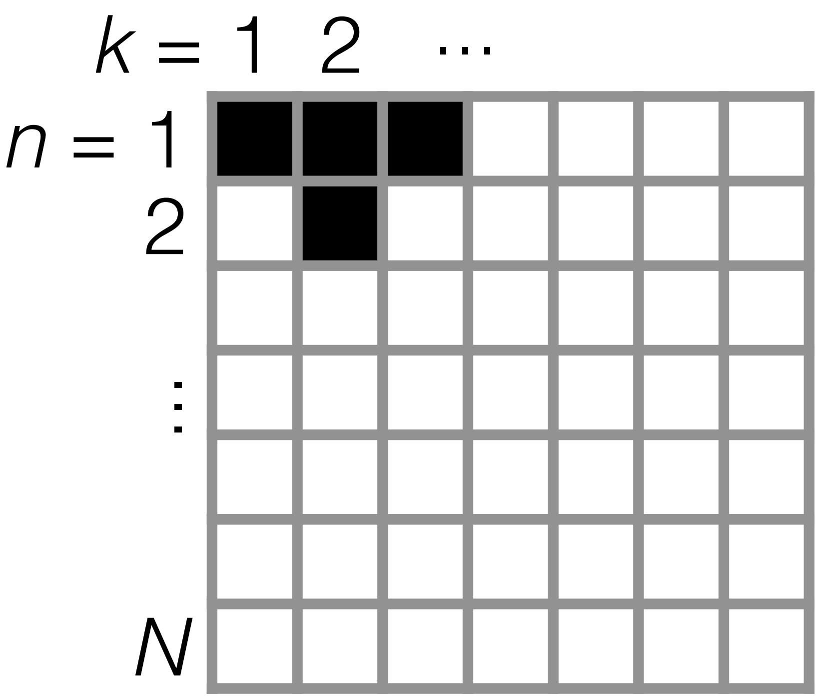

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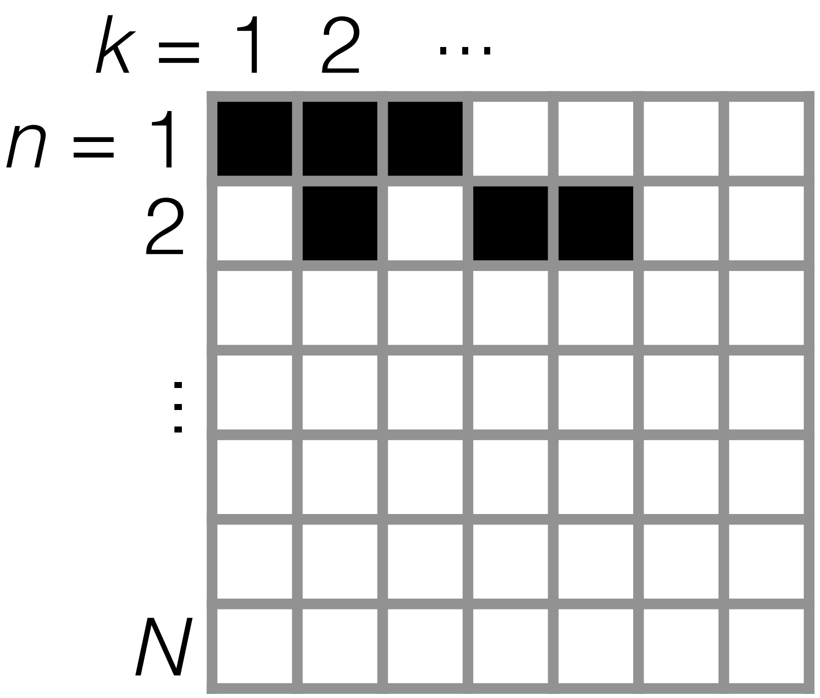

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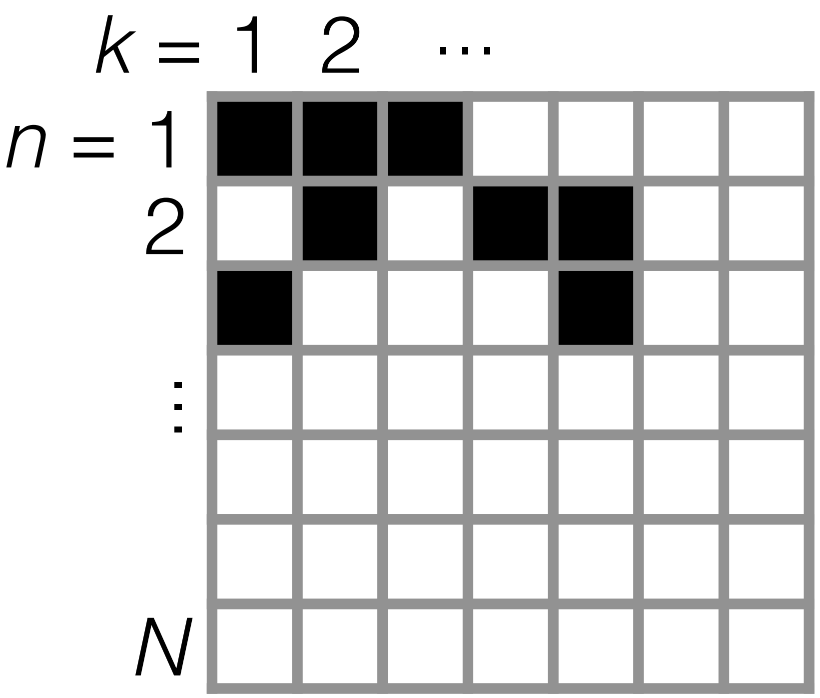

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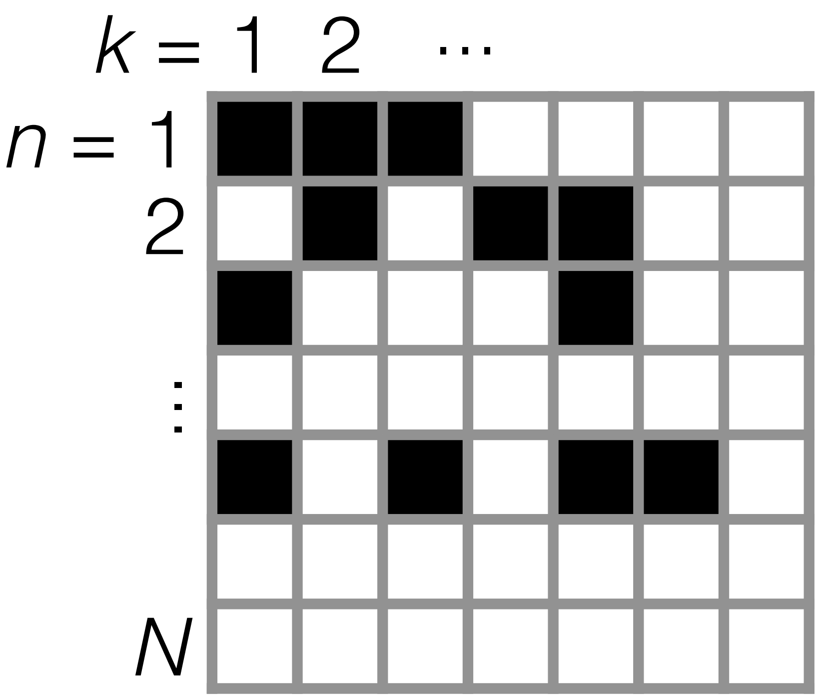

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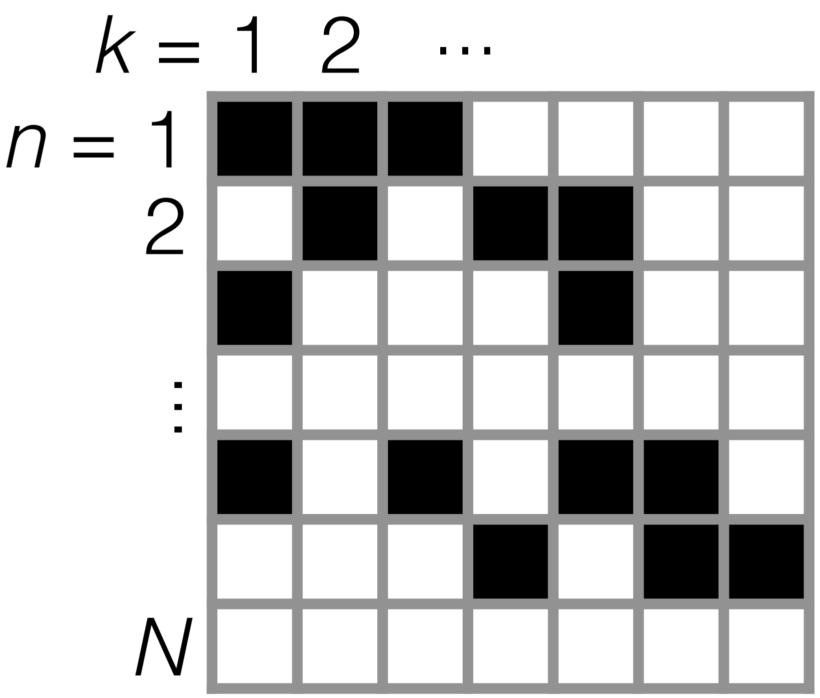

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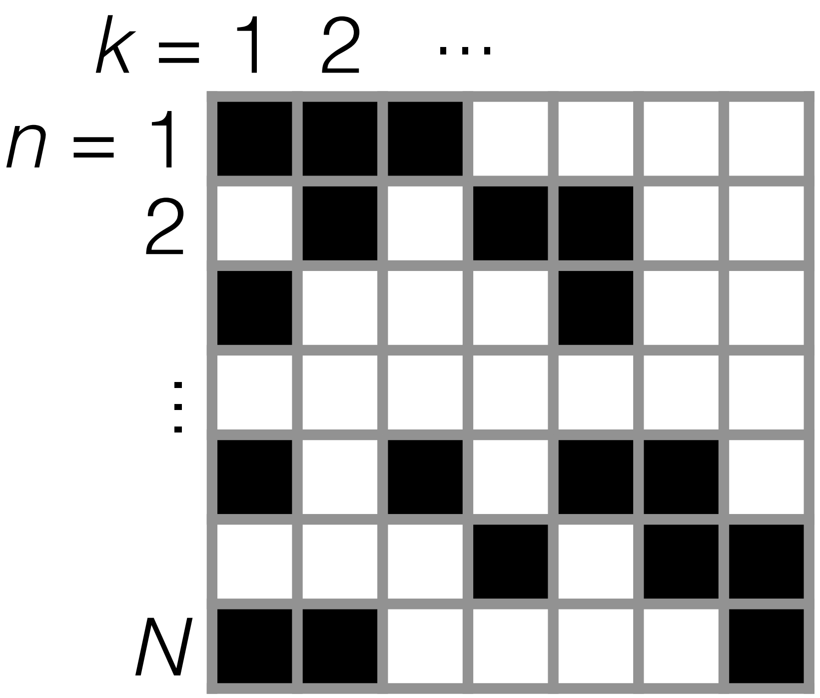

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Beta process & Bernoulli process

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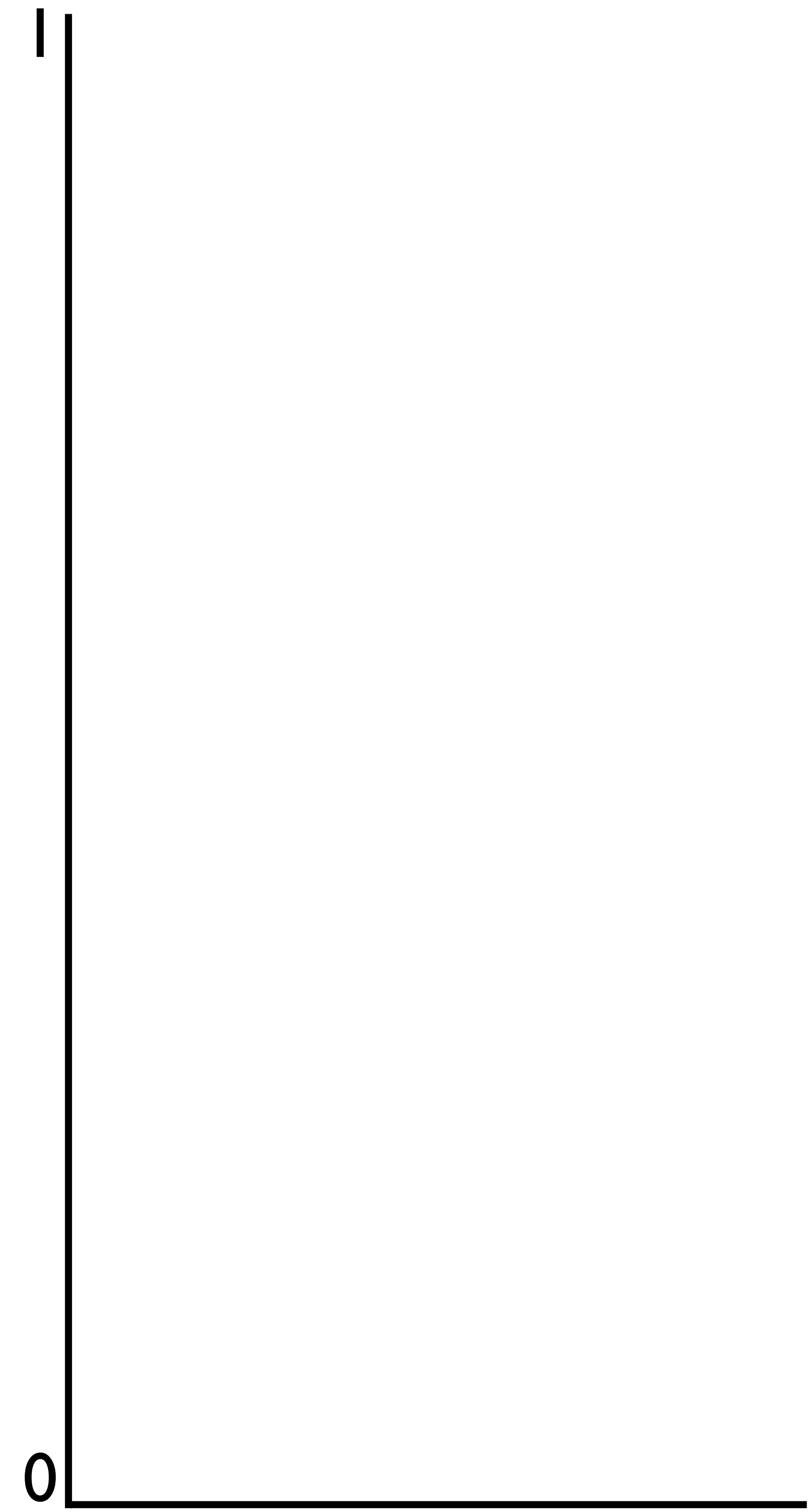

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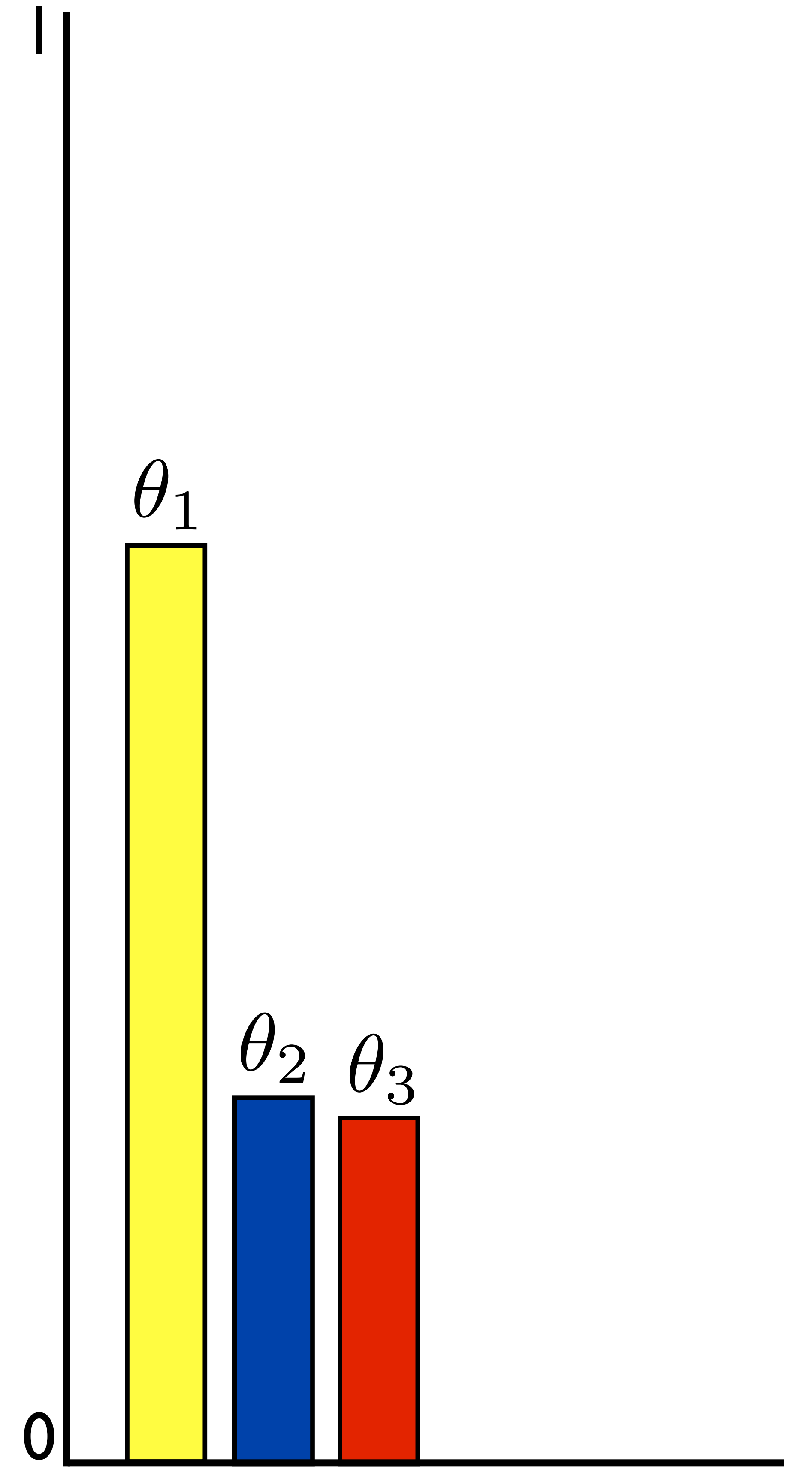

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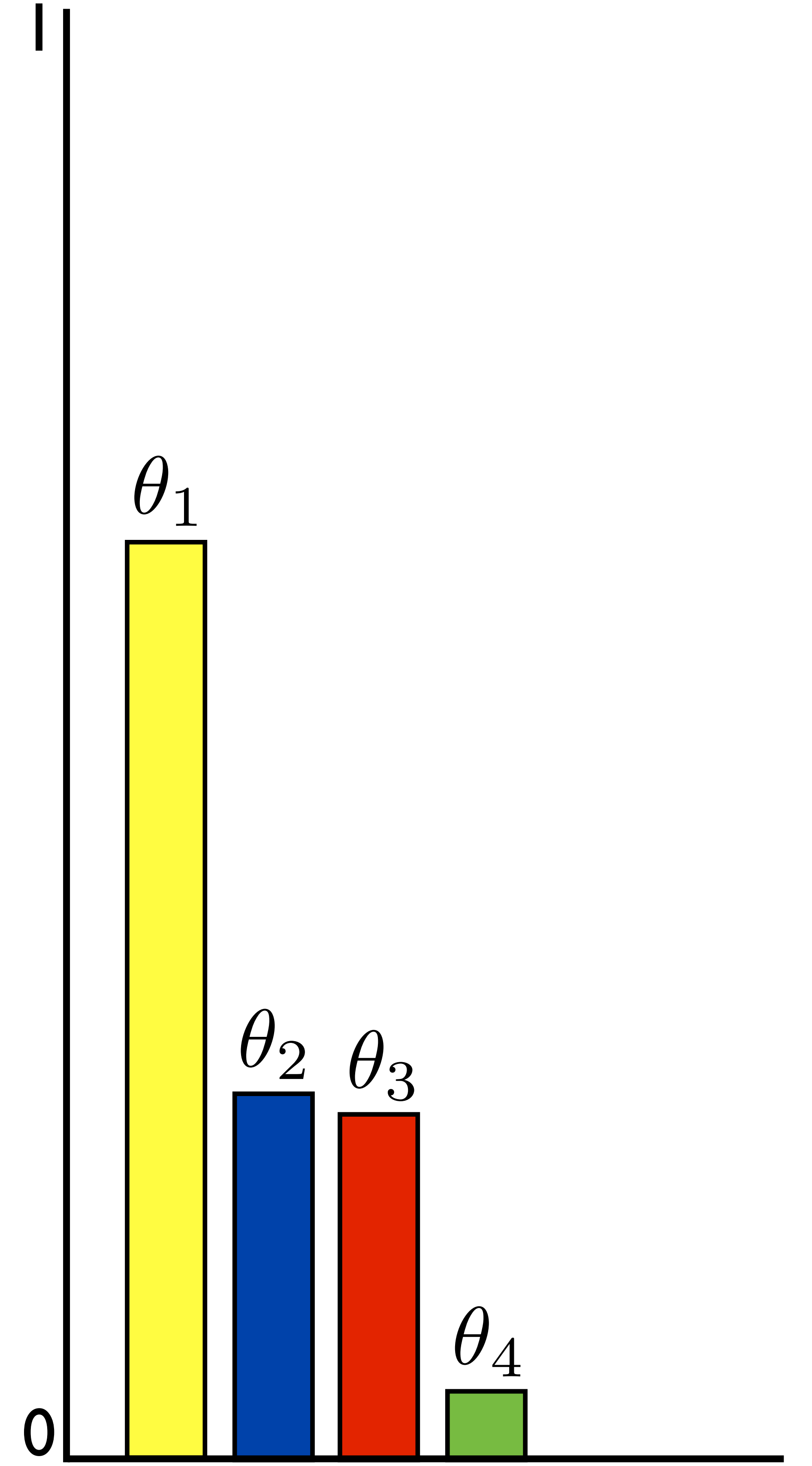

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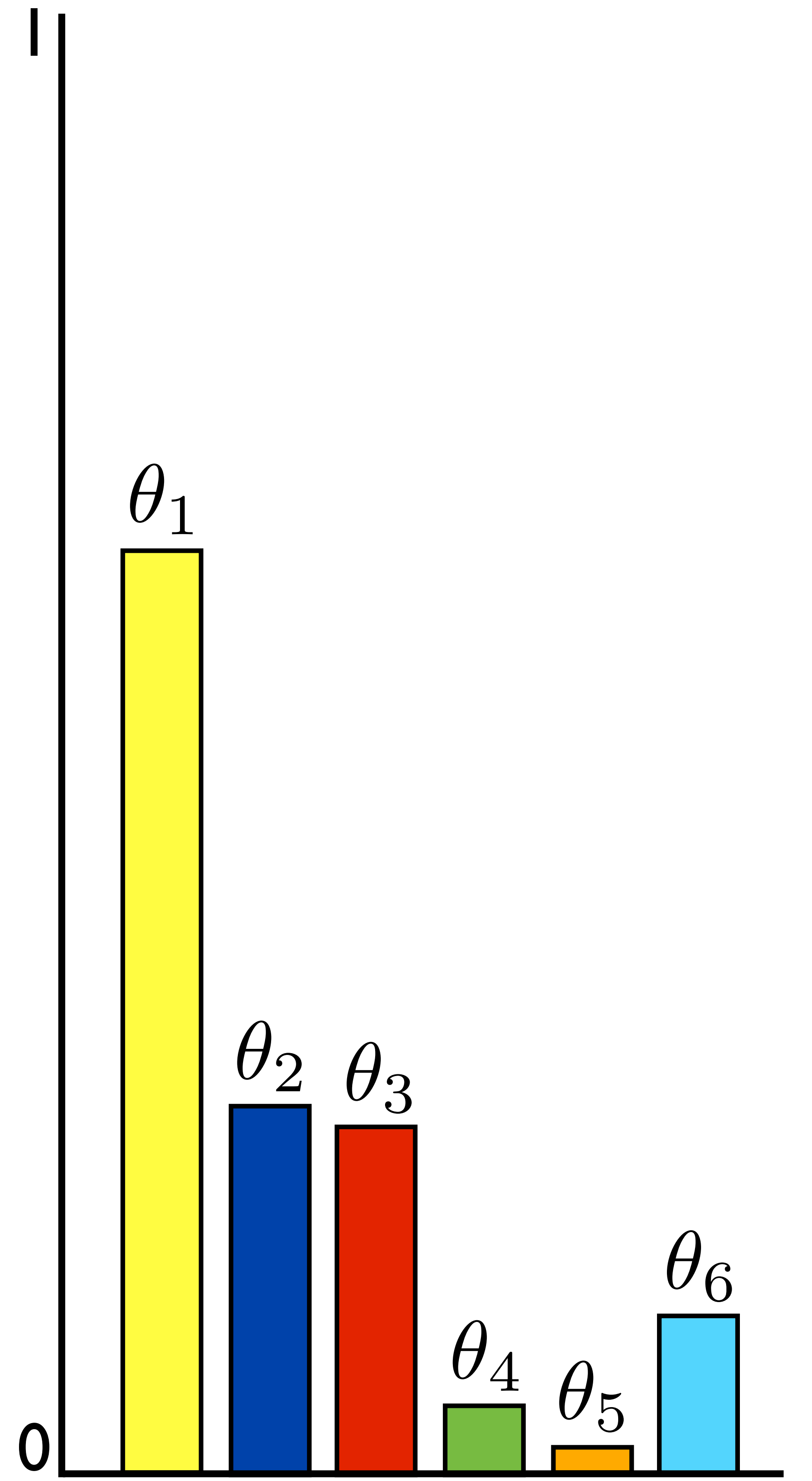

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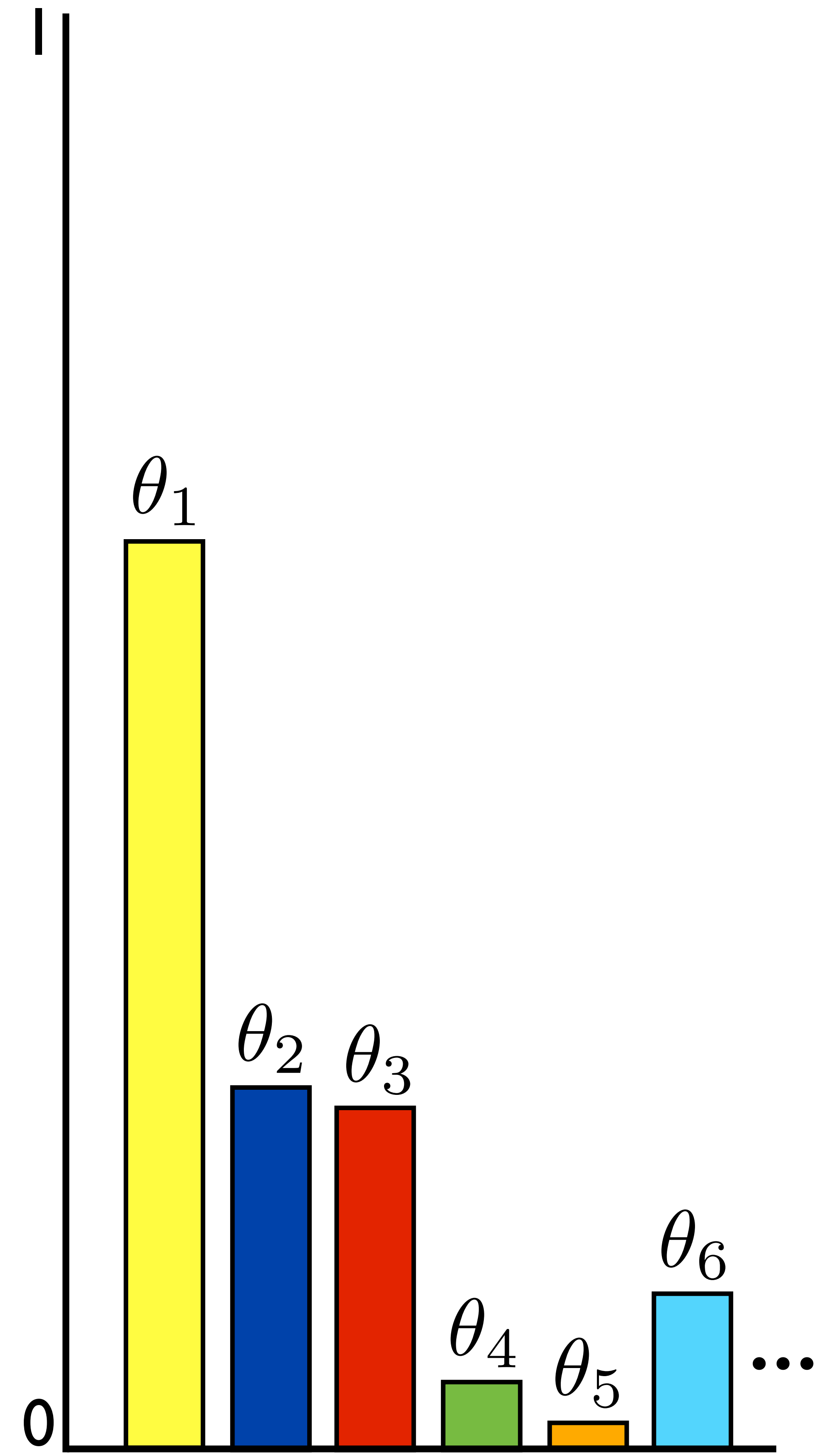

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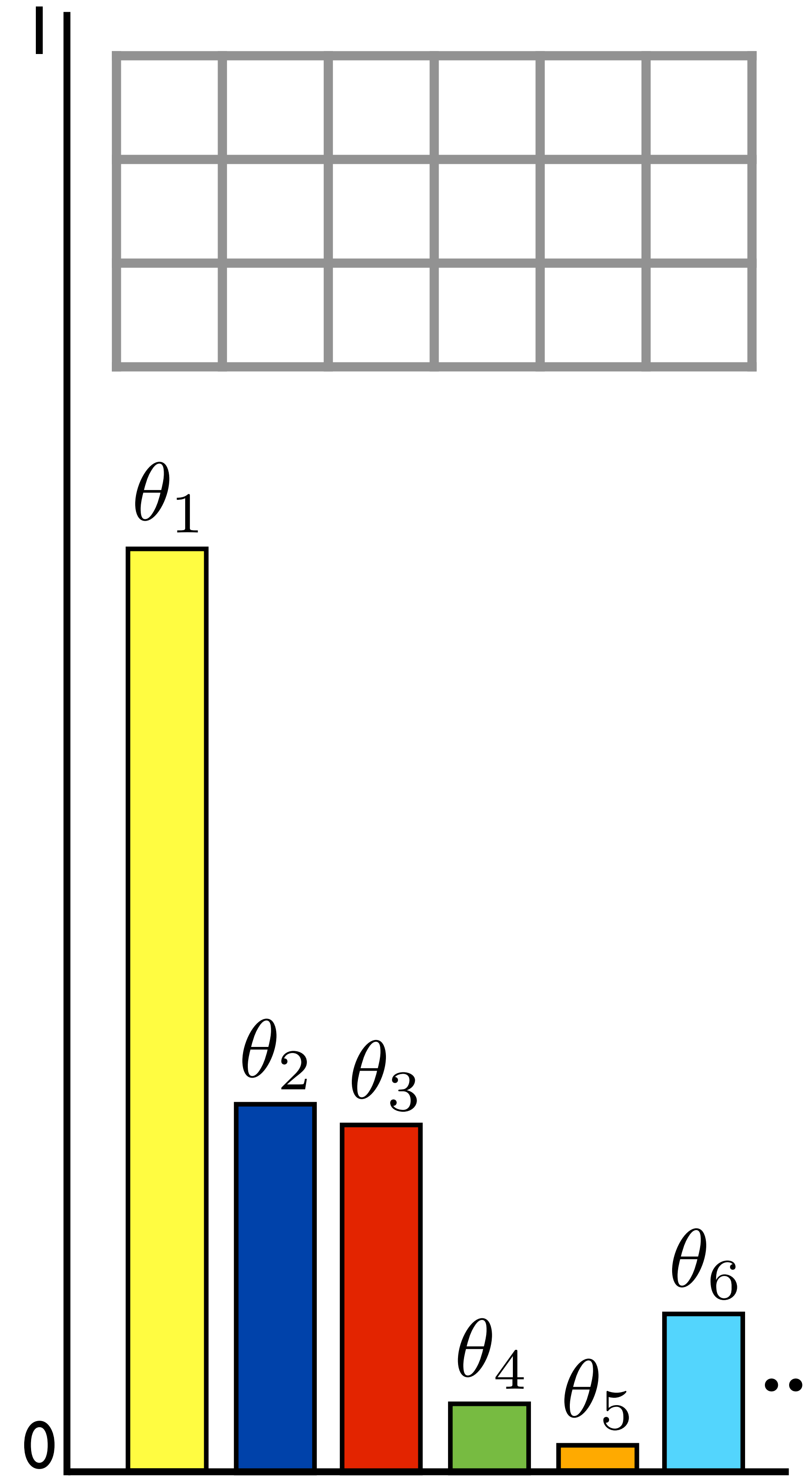

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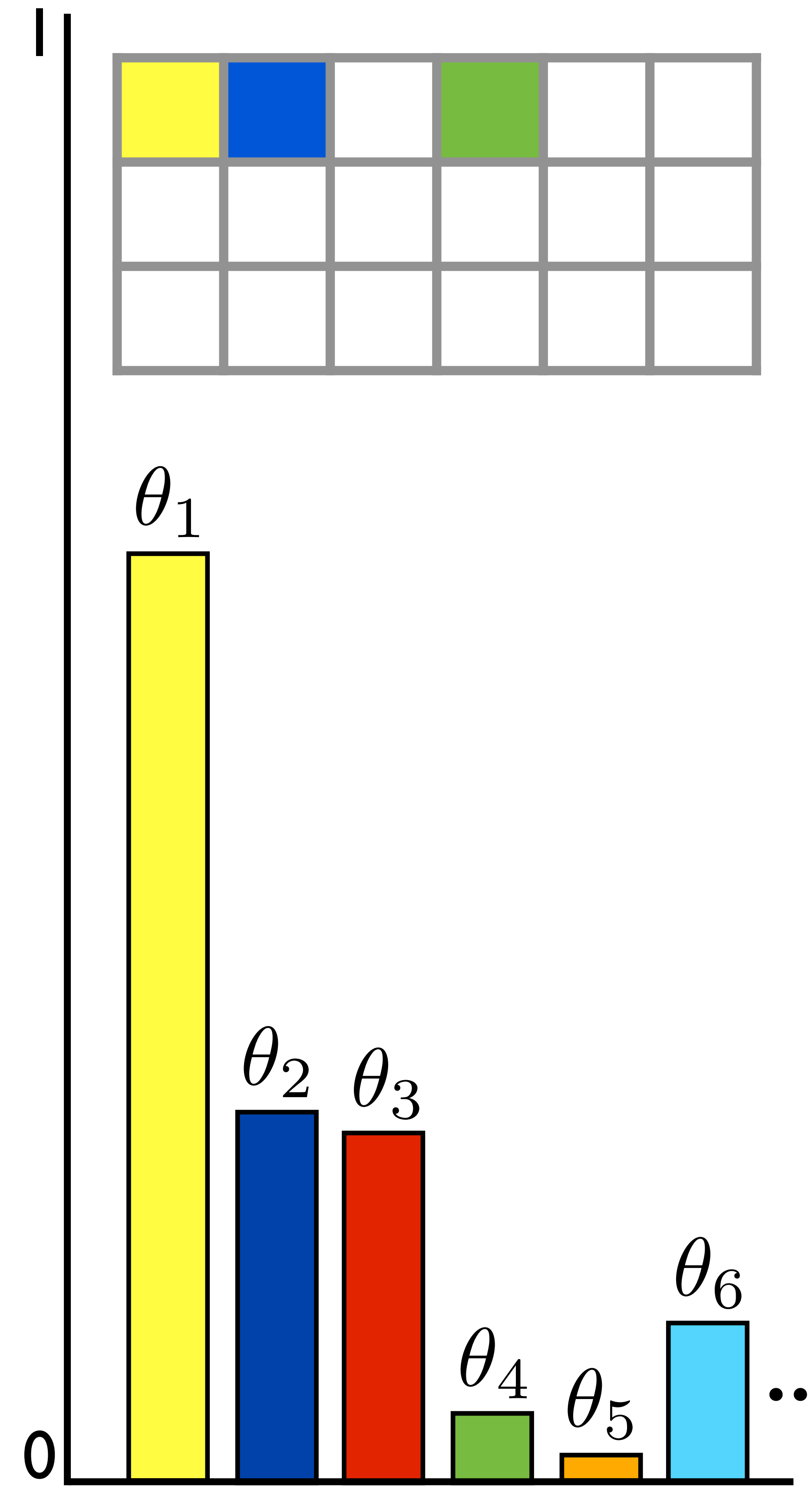

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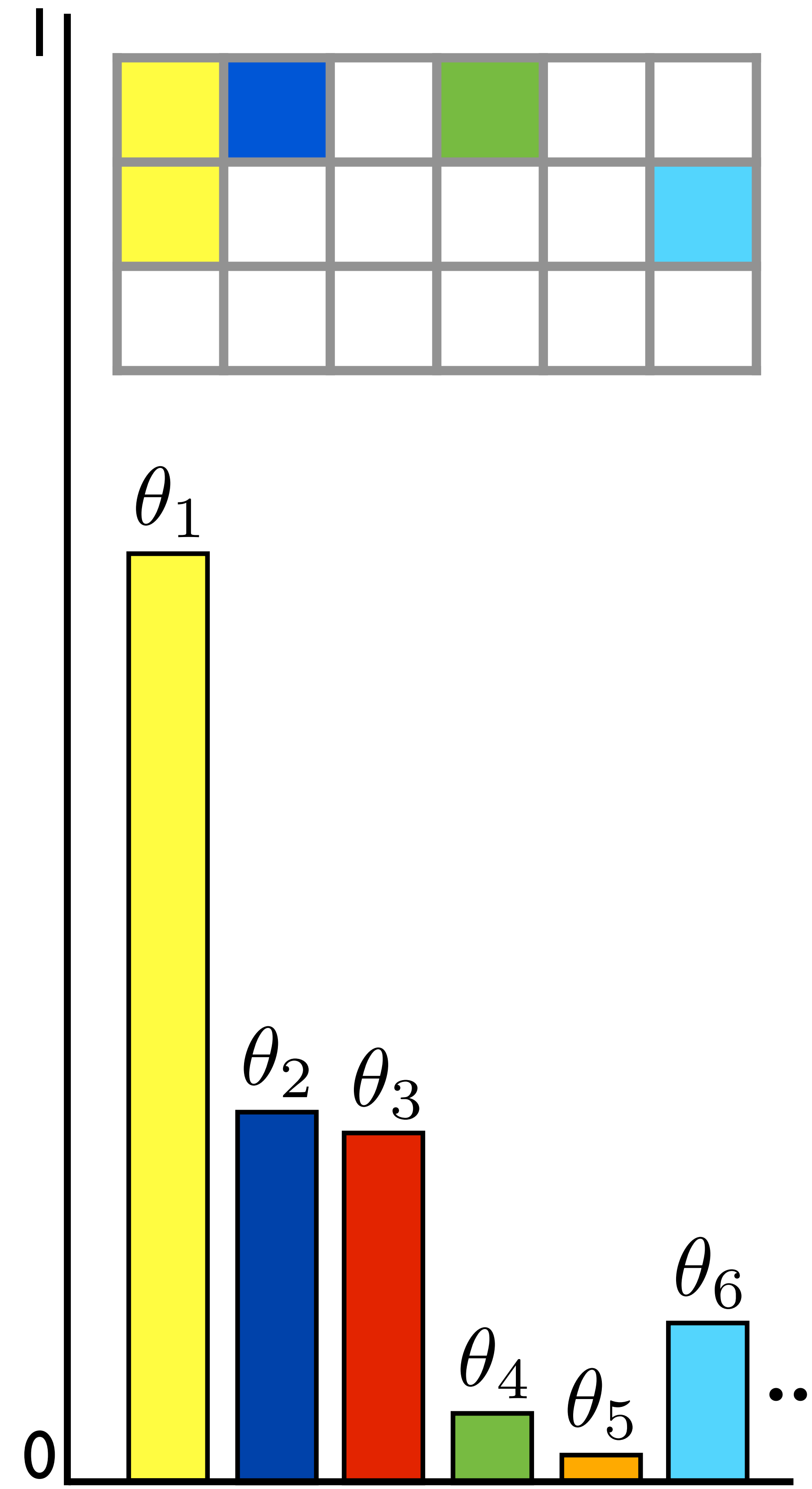

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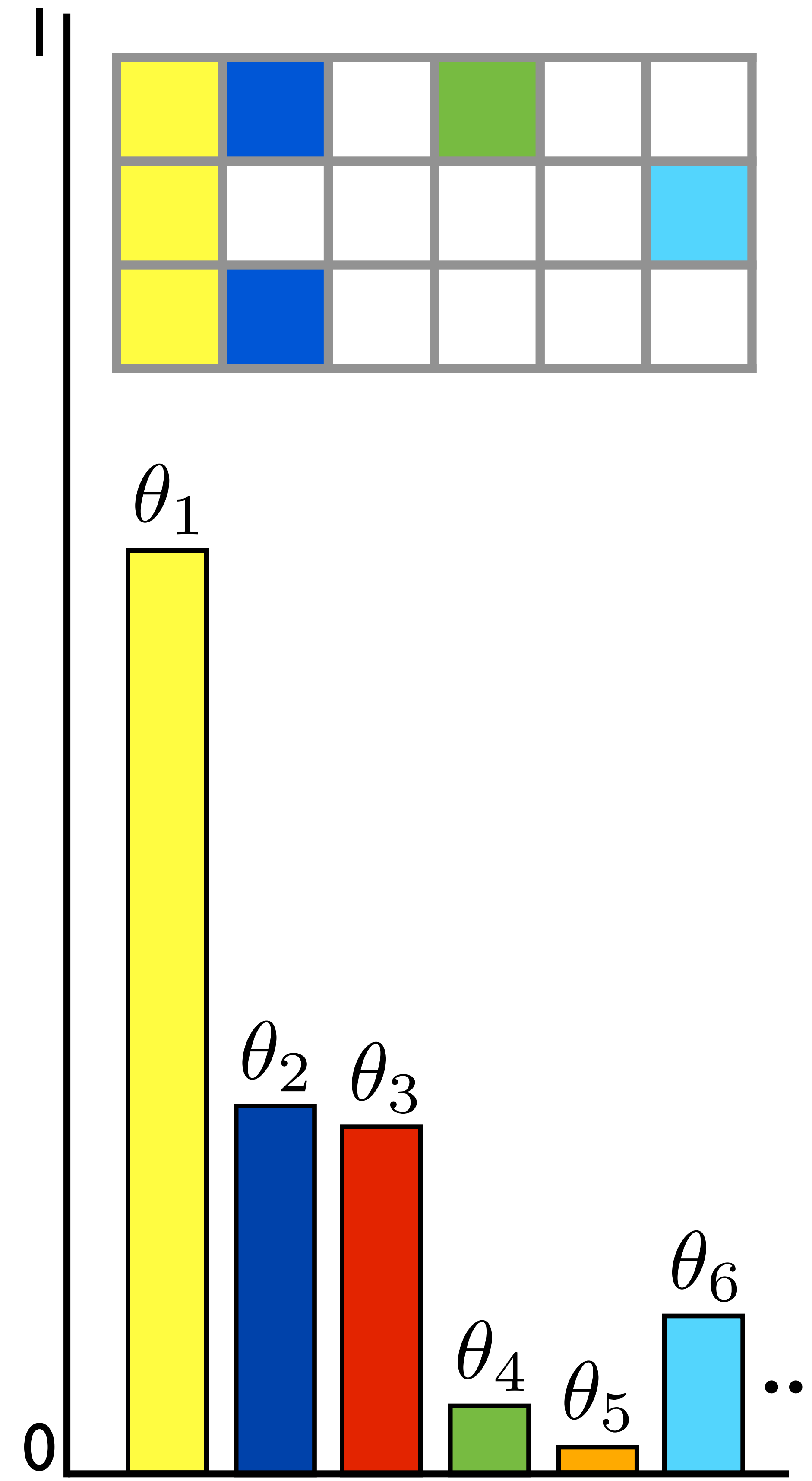

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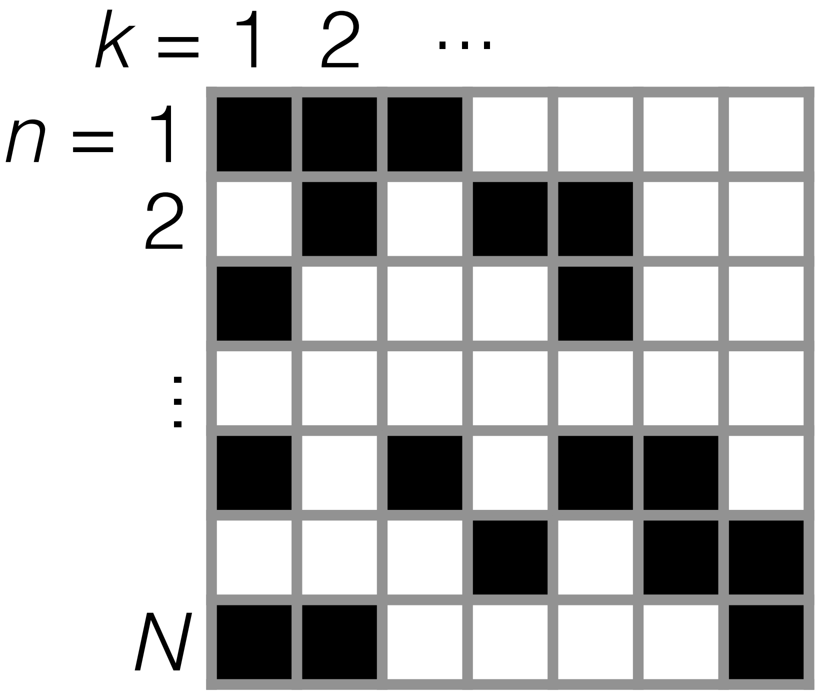

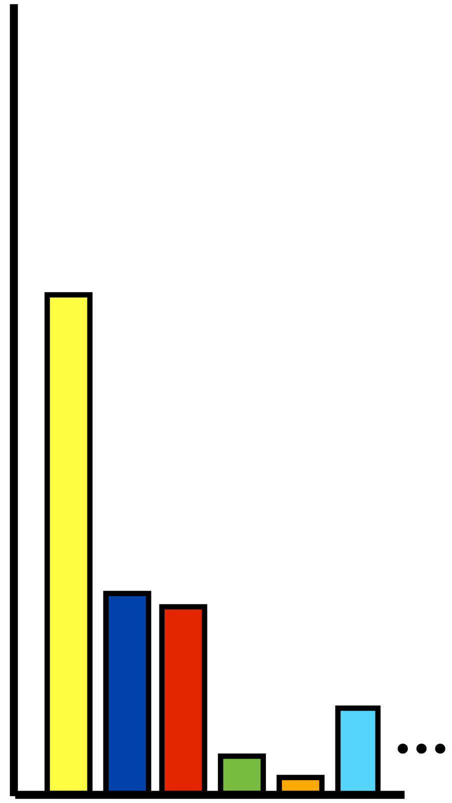

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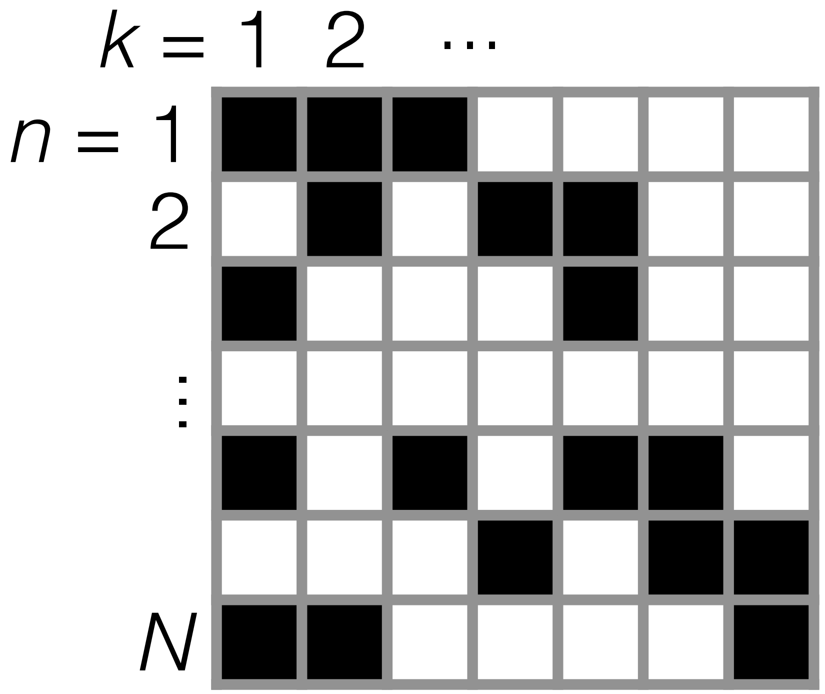

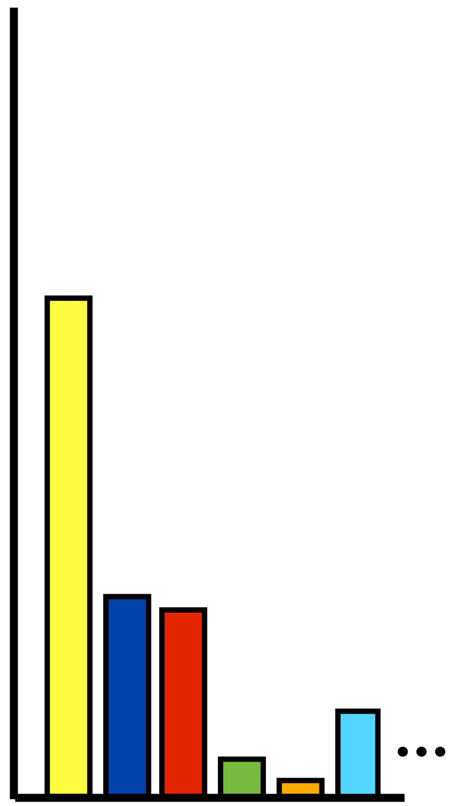

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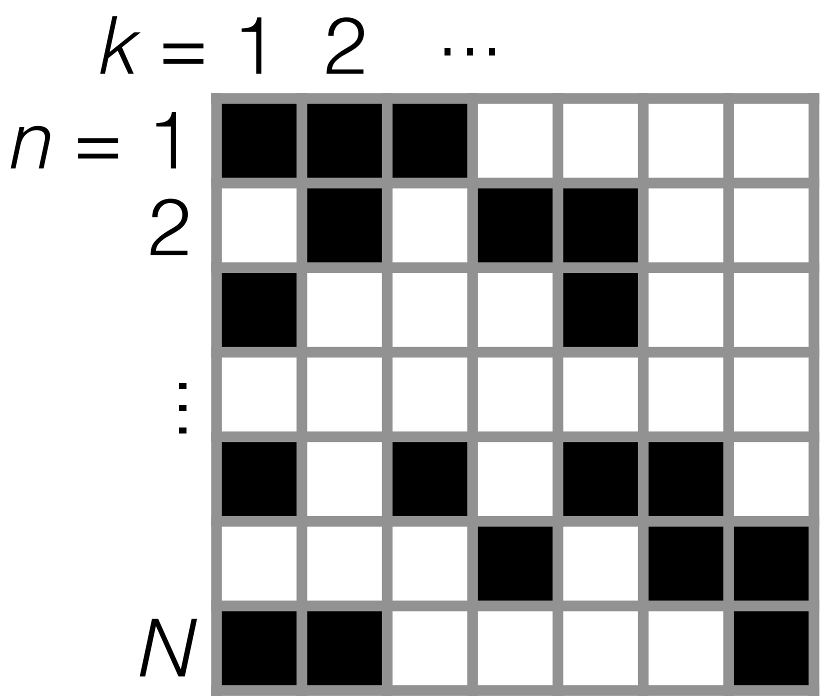

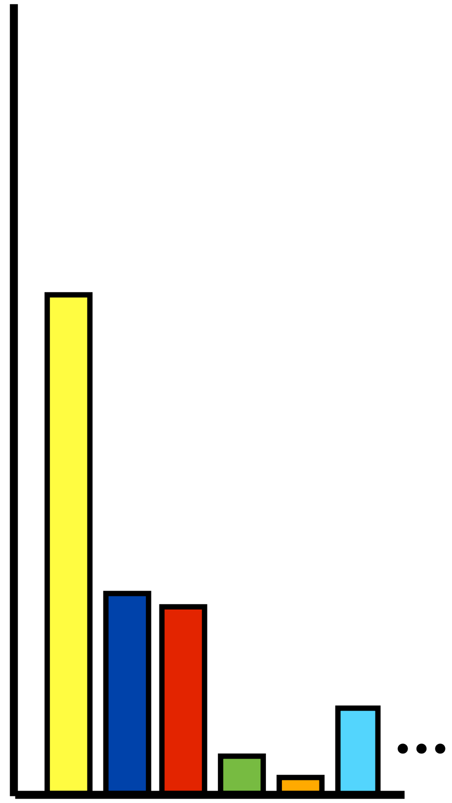

- Exchangeable (e.g. Gibbs sampling)
- Finite but unbounded

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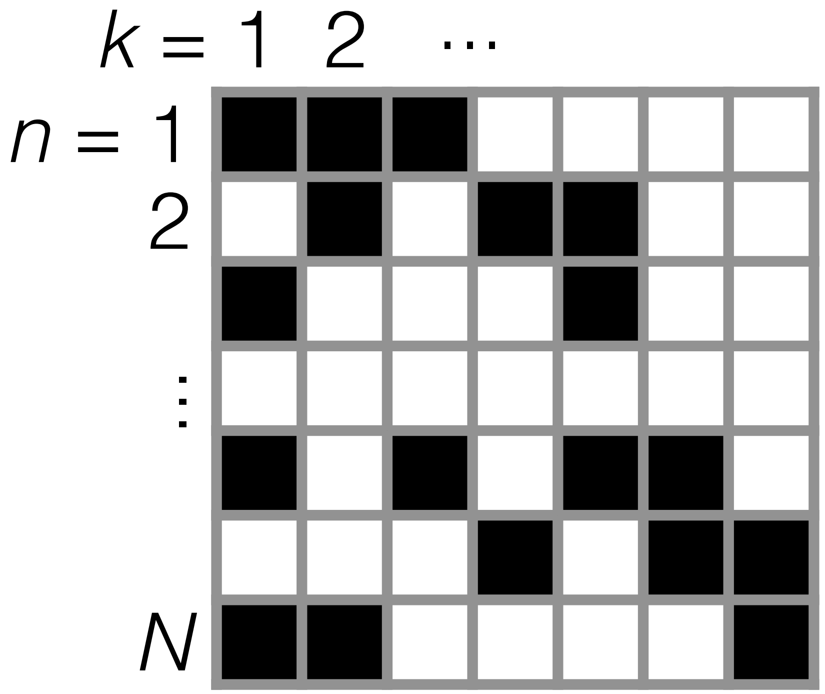

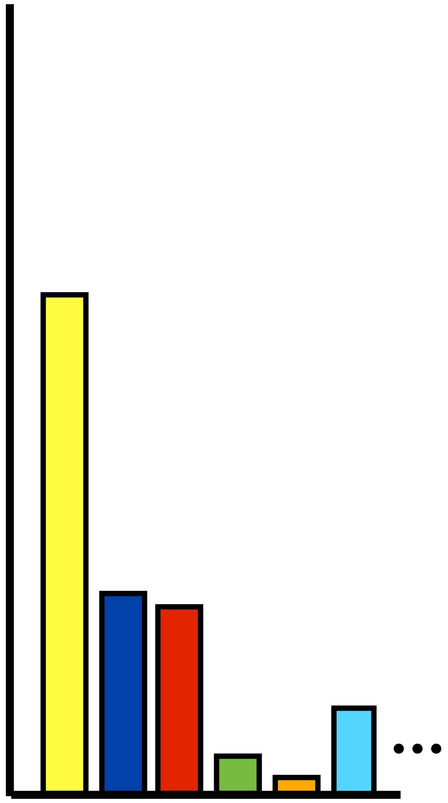

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- Hierarchical models

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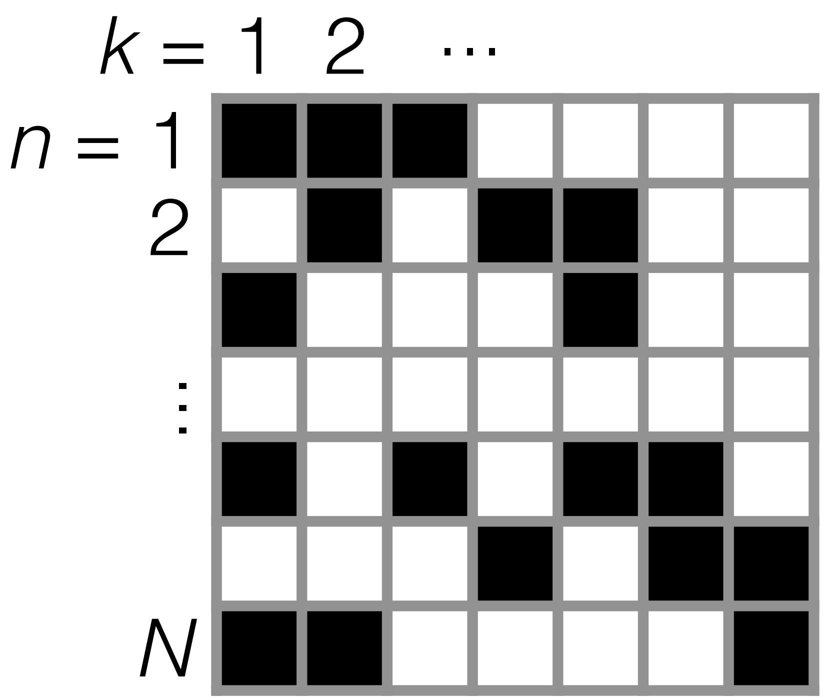

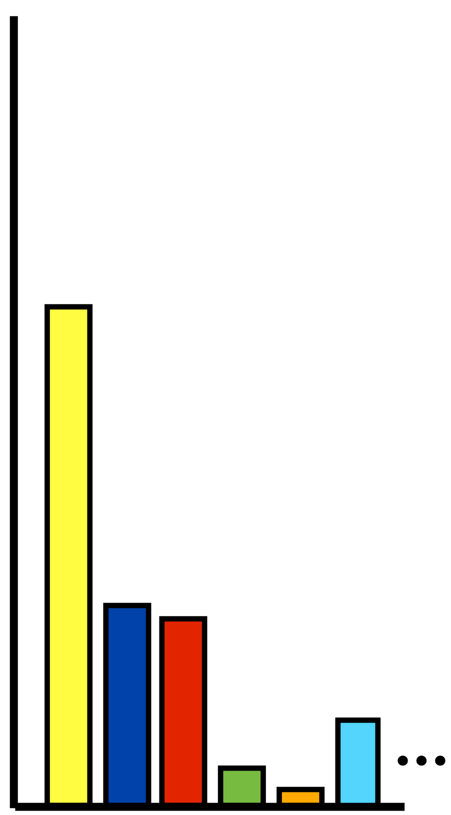

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- (Countable) sequence of finite-dimensional distributions

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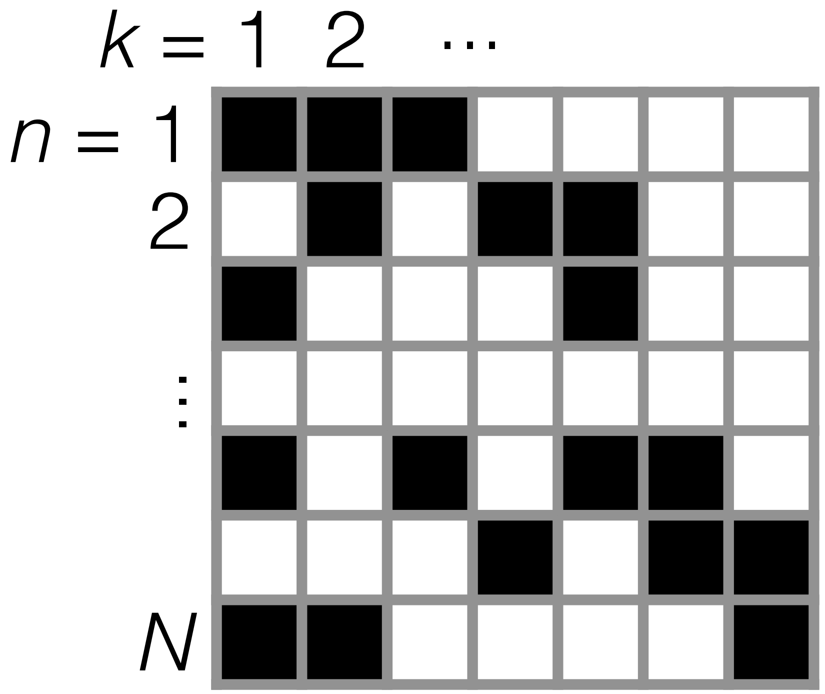

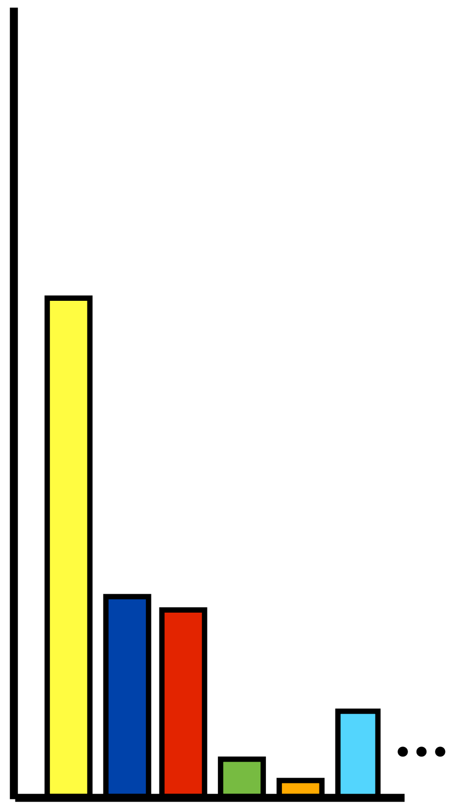

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How do we come up with these models?

One Framework

Likelihood

[Broderick,
Wilson,
Jordan 2014]

One Framework

- Conjugate prior

Likelihood → [Broderick, Wilson, Jordan 2014]

One Framework

Likelihood

[Broderick, Wilson, Jordan 2014]

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One Framework

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Likelihood
(e.g. Bernoulli)

[Broderick, Wilson, Jordan 2014]

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[Broderick, Wilson, Jordan 2014]

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One Framework

Likelihood
(e.g. Bernoulli)

[Broderick, Wilson, Jordan 2014]

- Conjugate prior *(e.g. BP)*
- Marginal *(e.g. IBP)*
- Size-biased atom sequence *(e.g. BP stick-breaking)*

Example: odds Bernoulli

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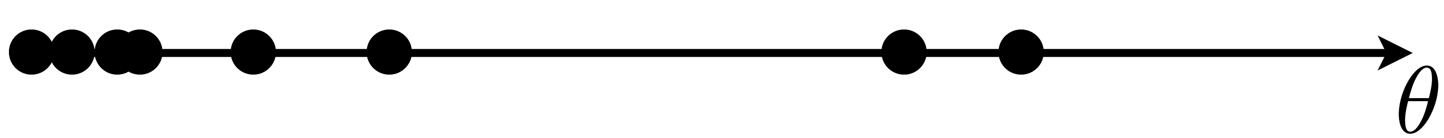

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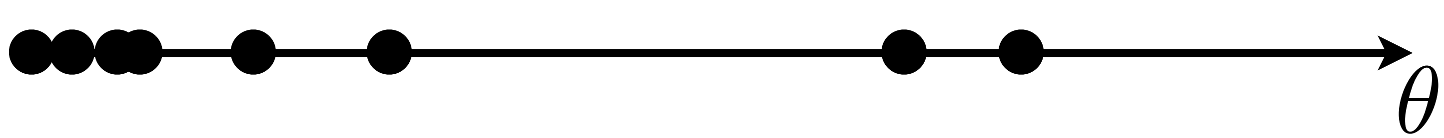

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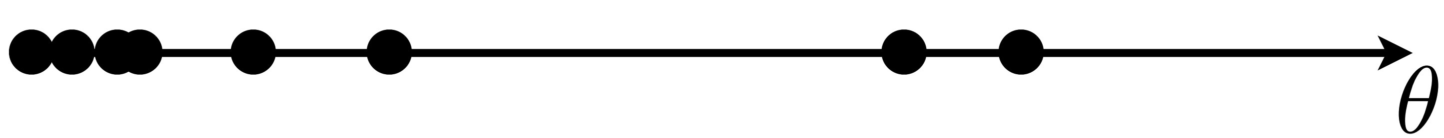

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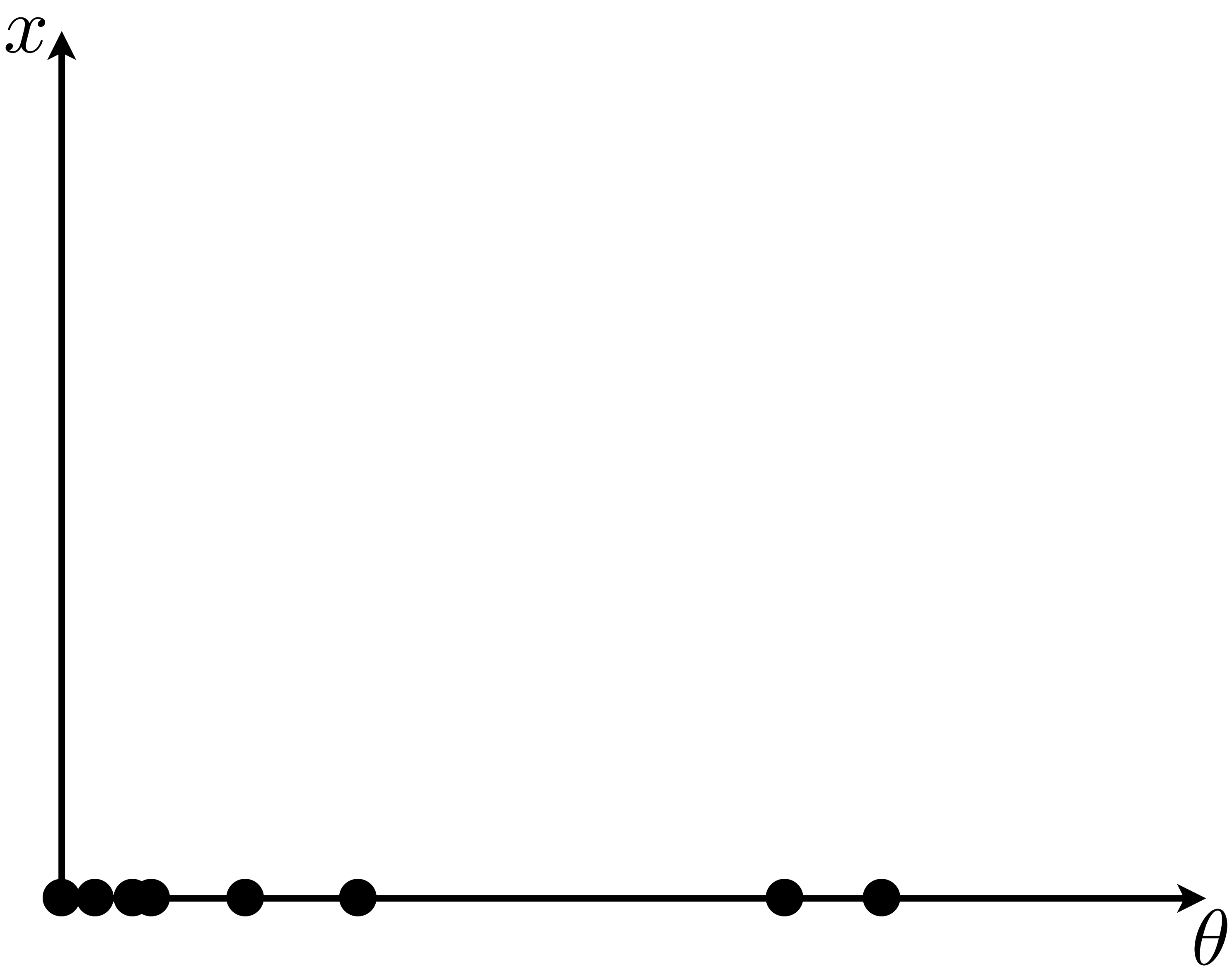

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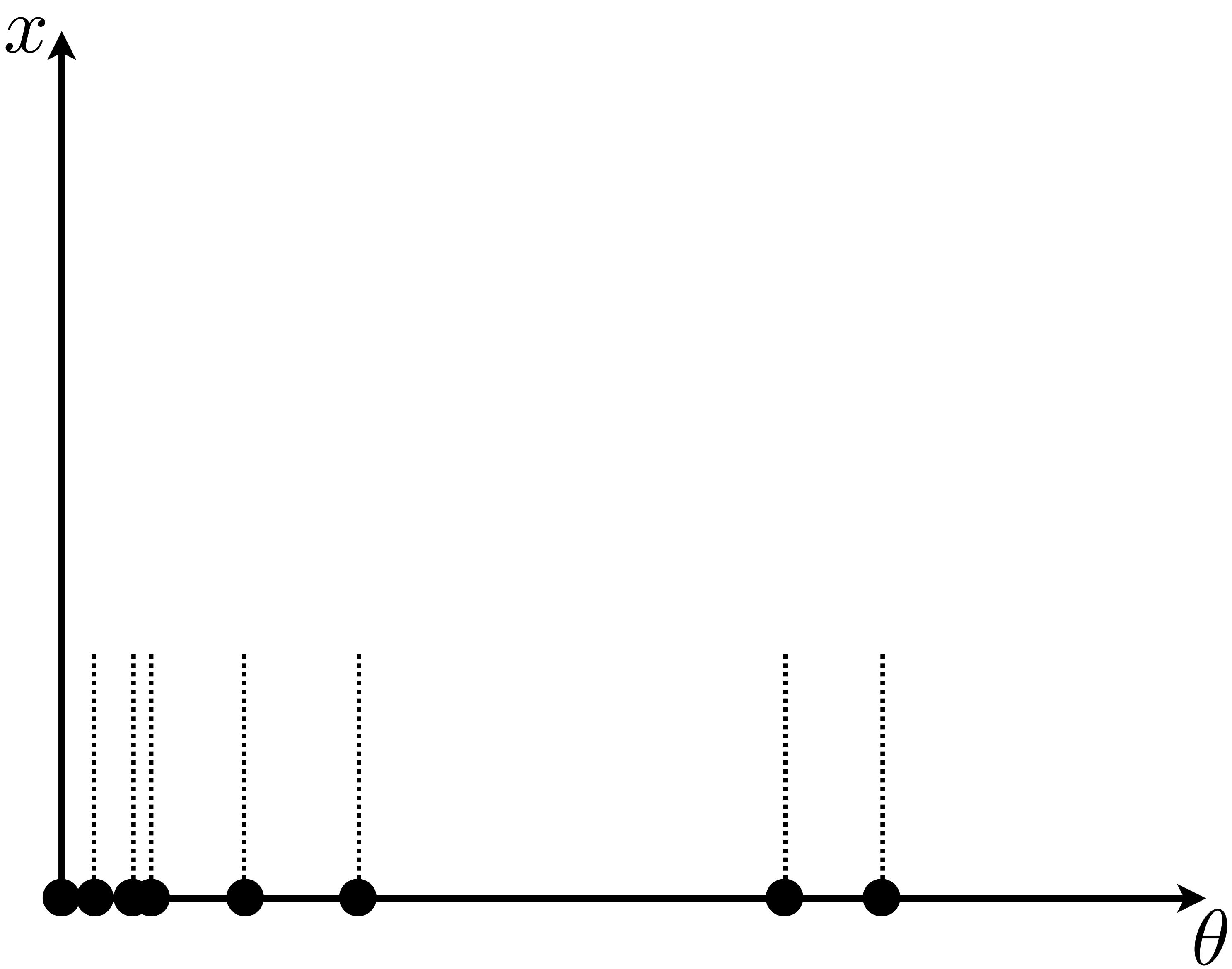

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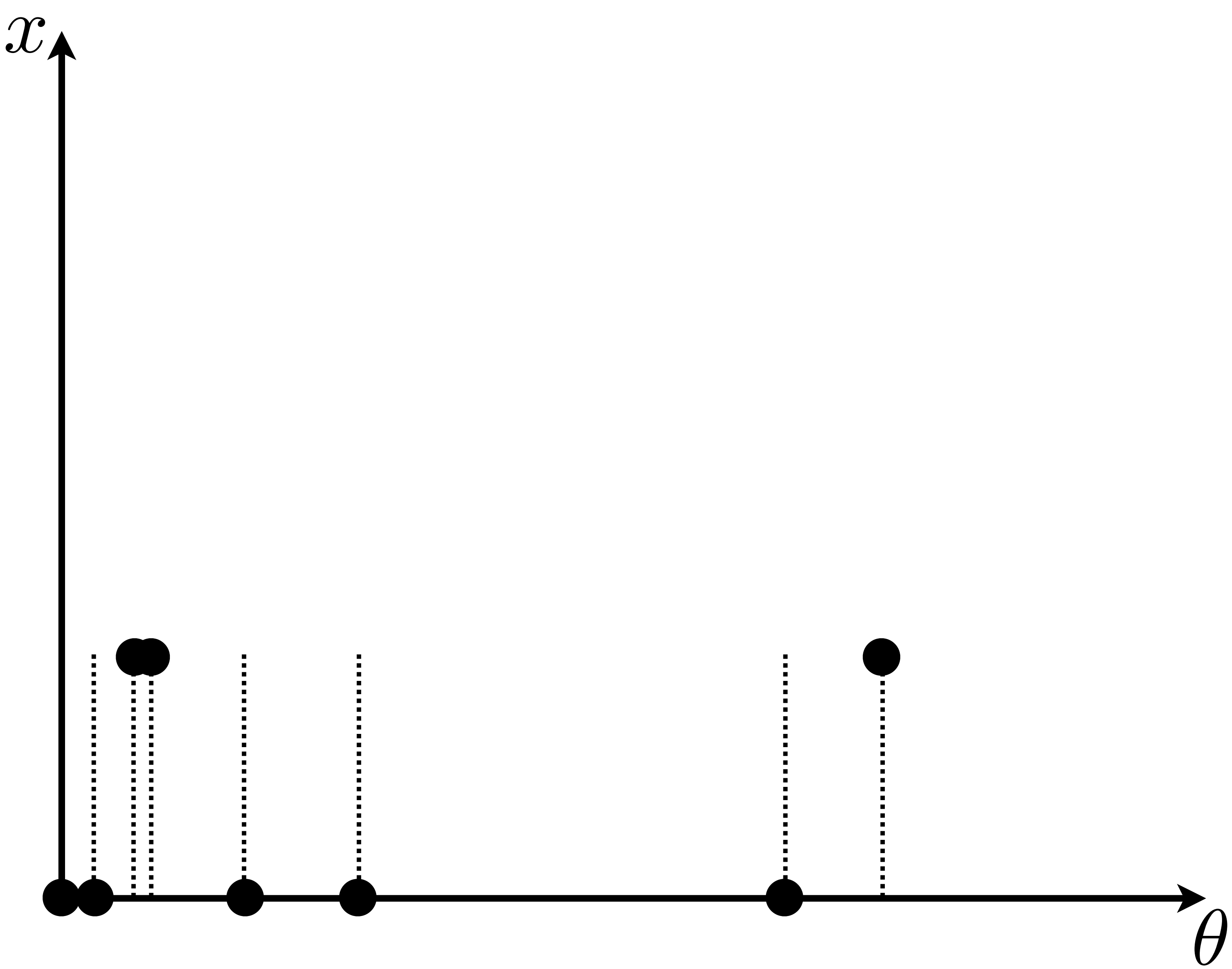

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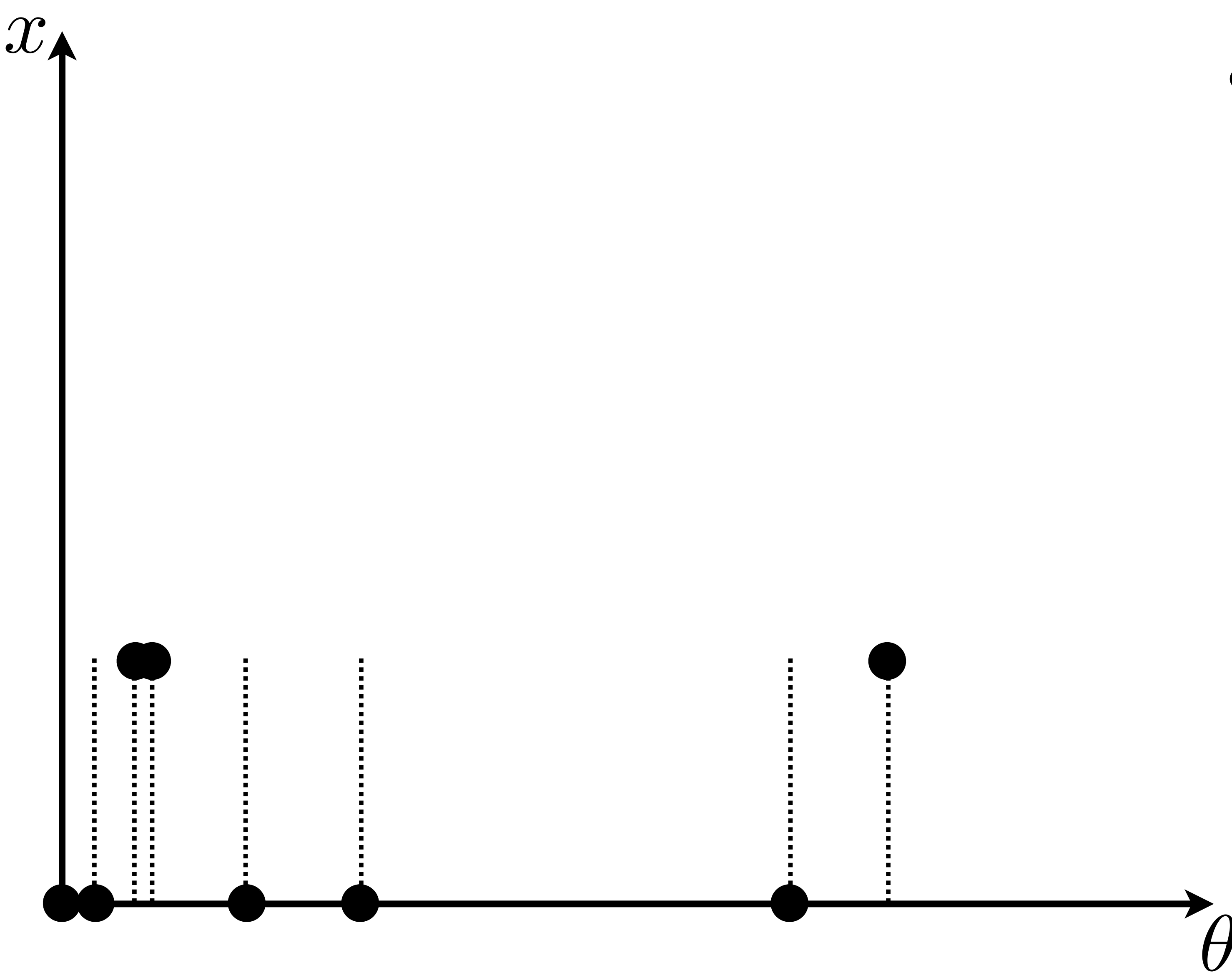

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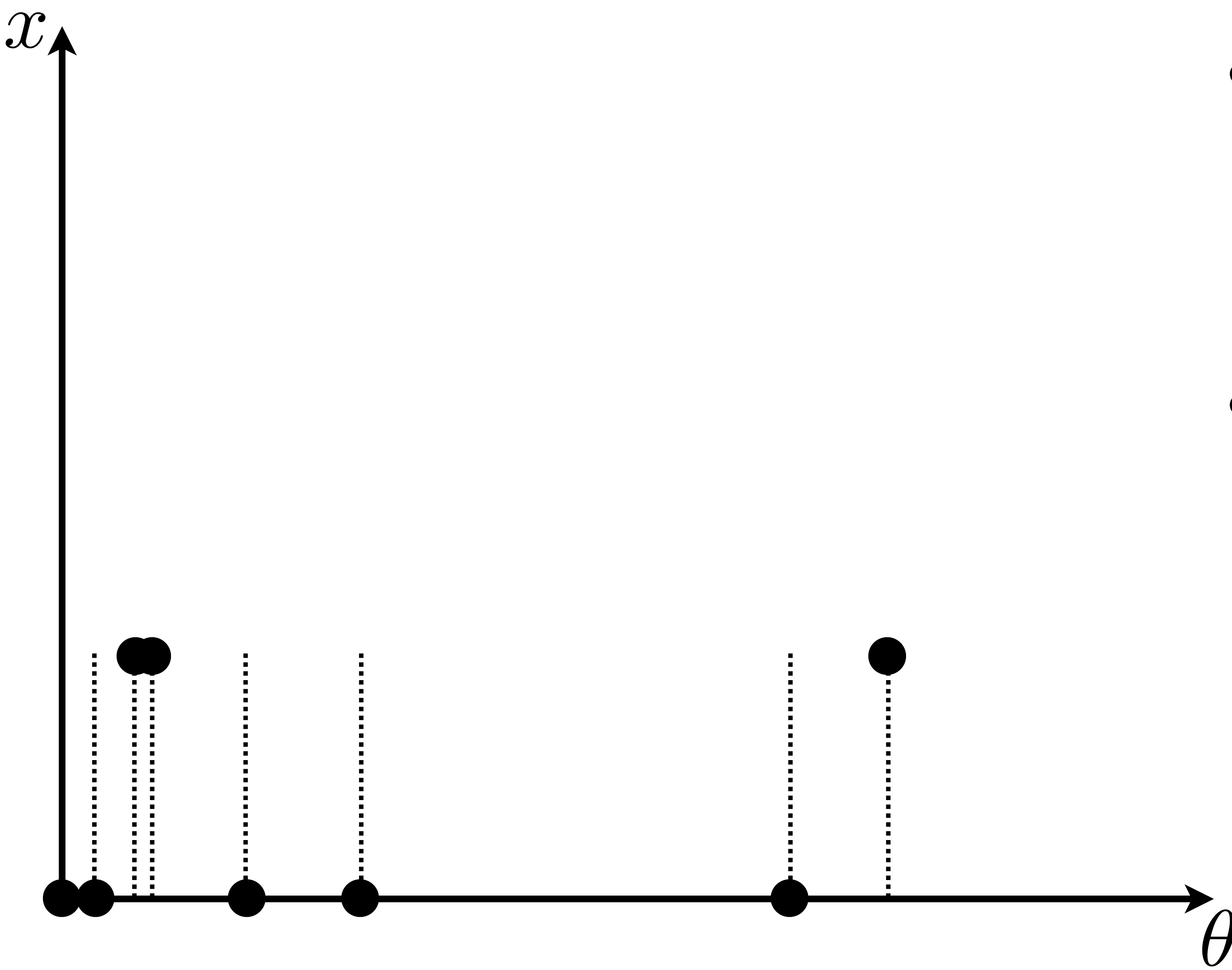

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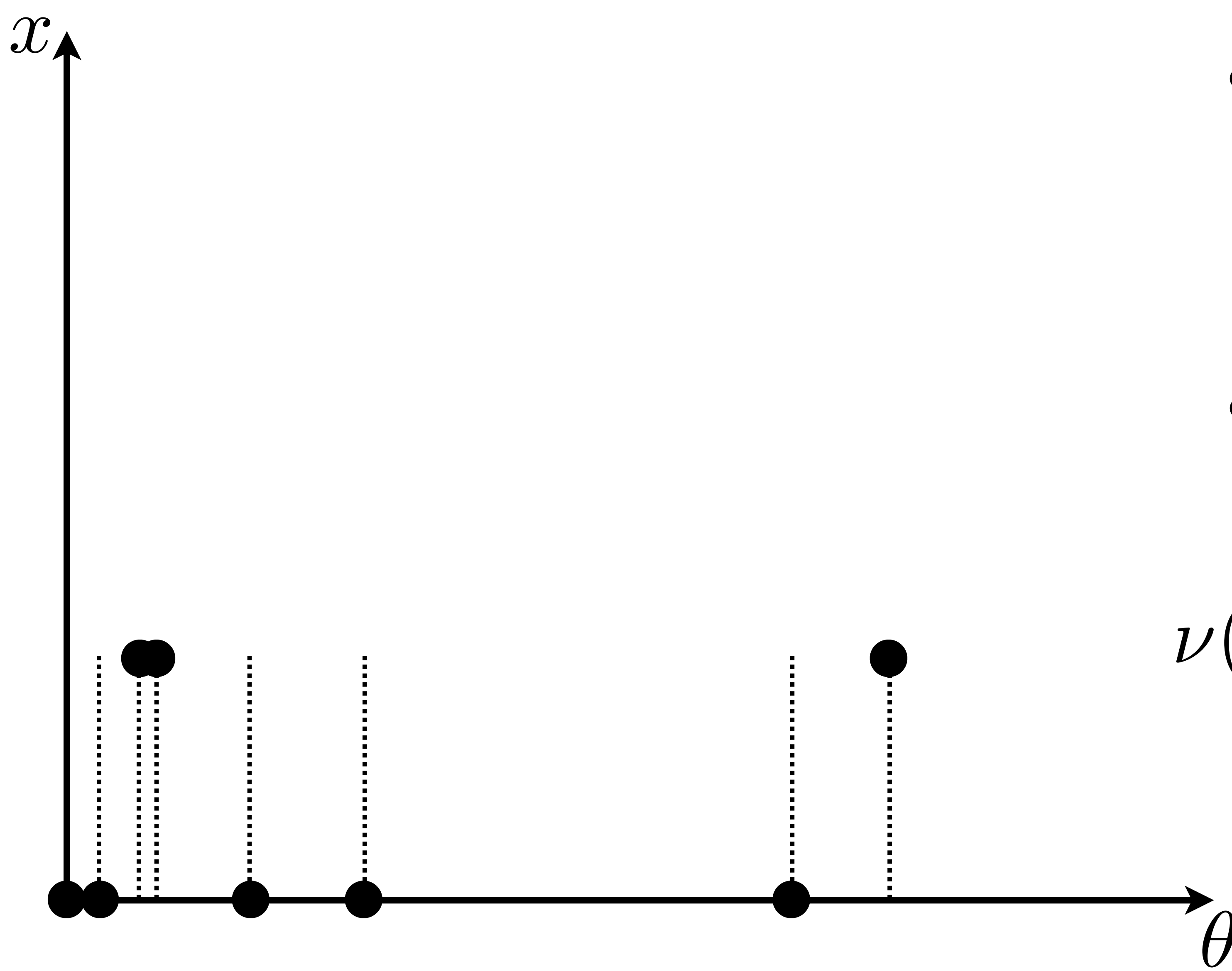

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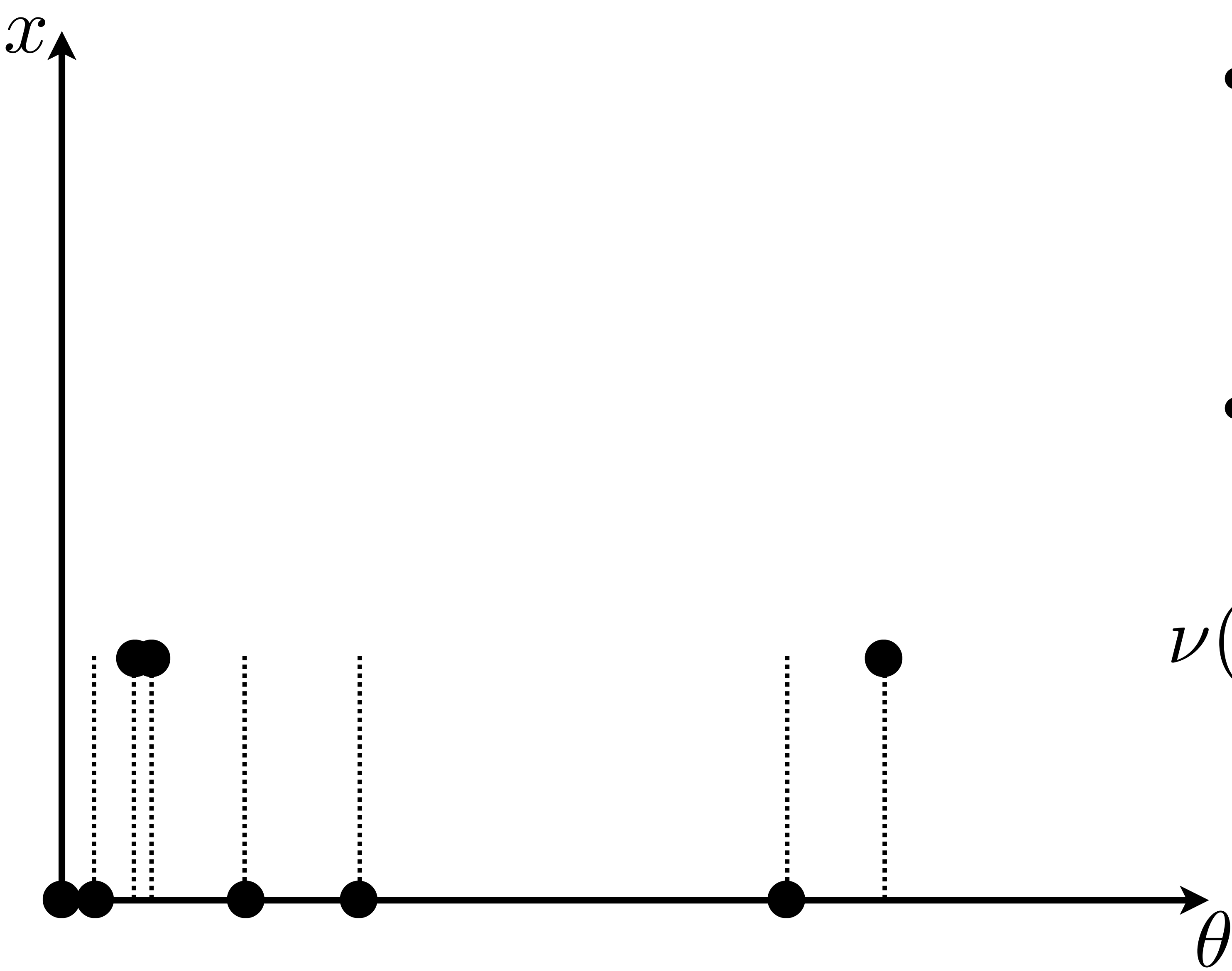

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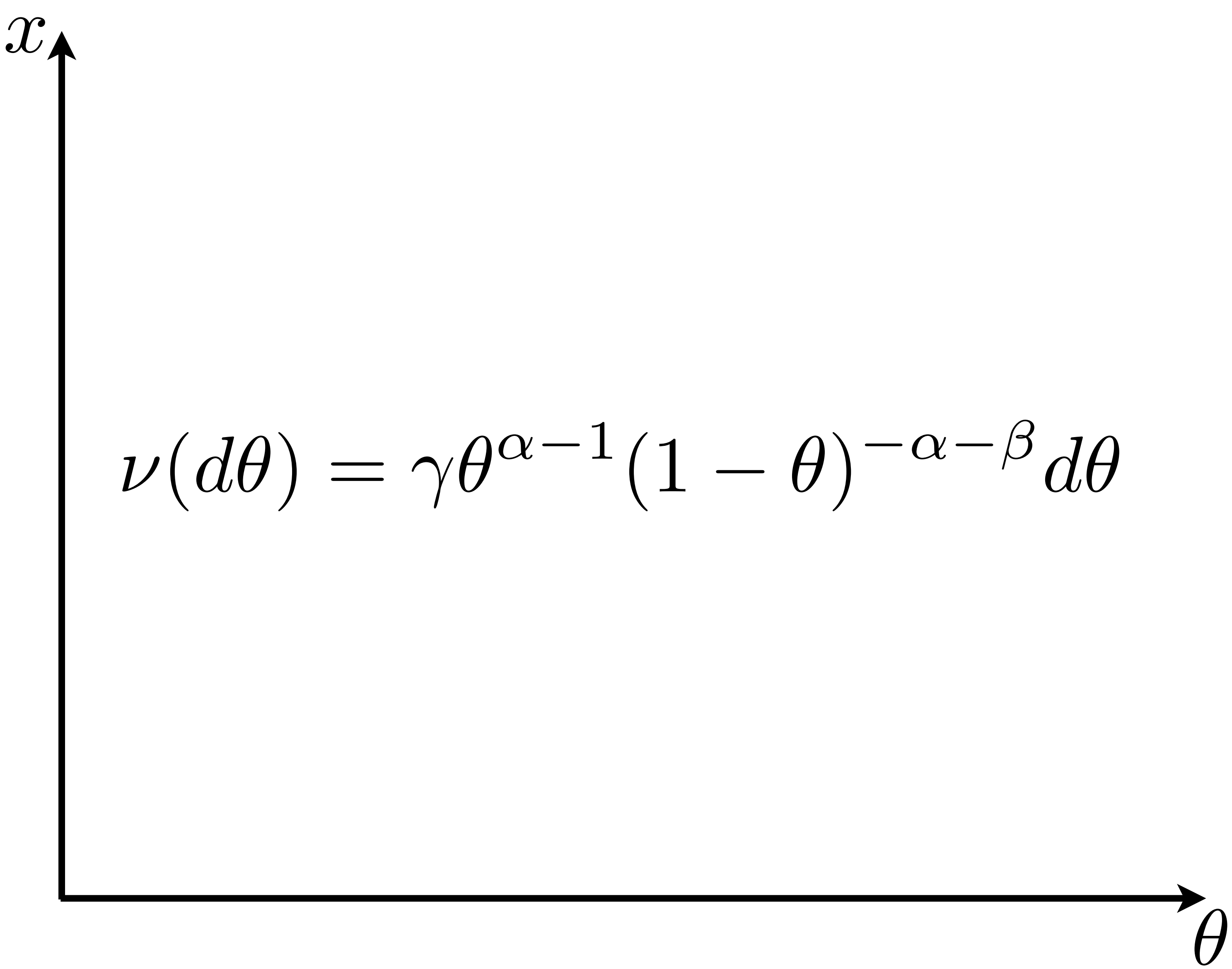

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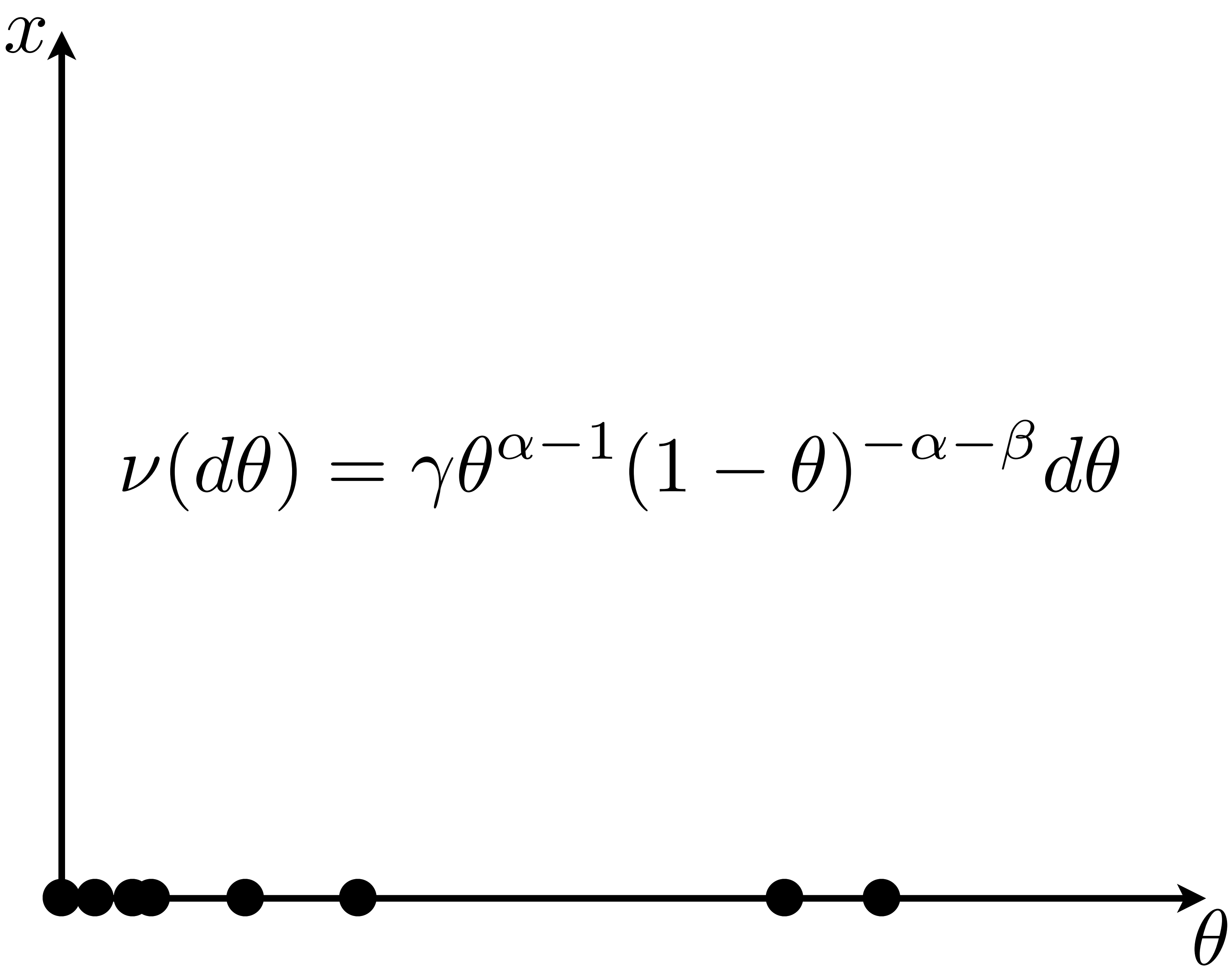

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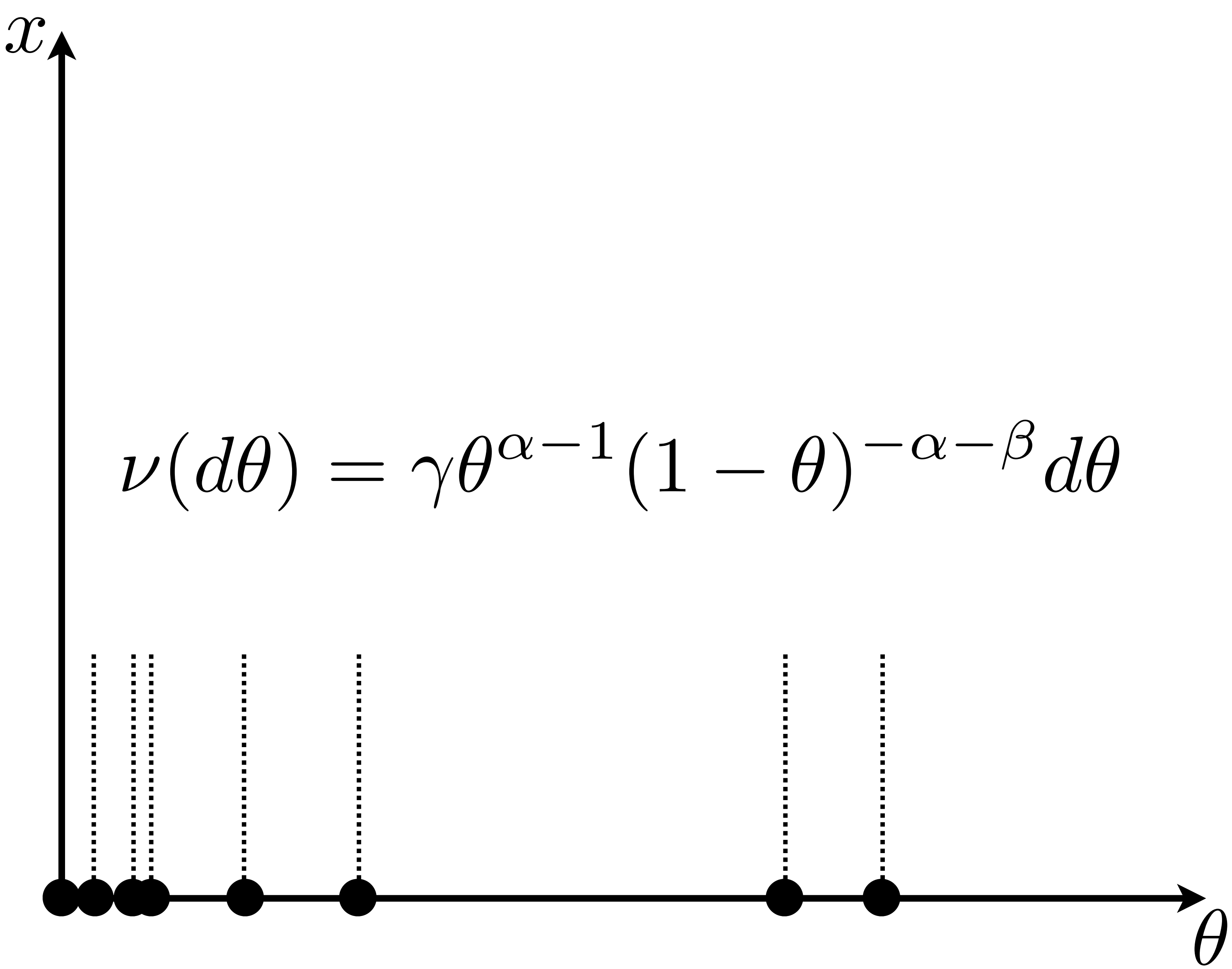

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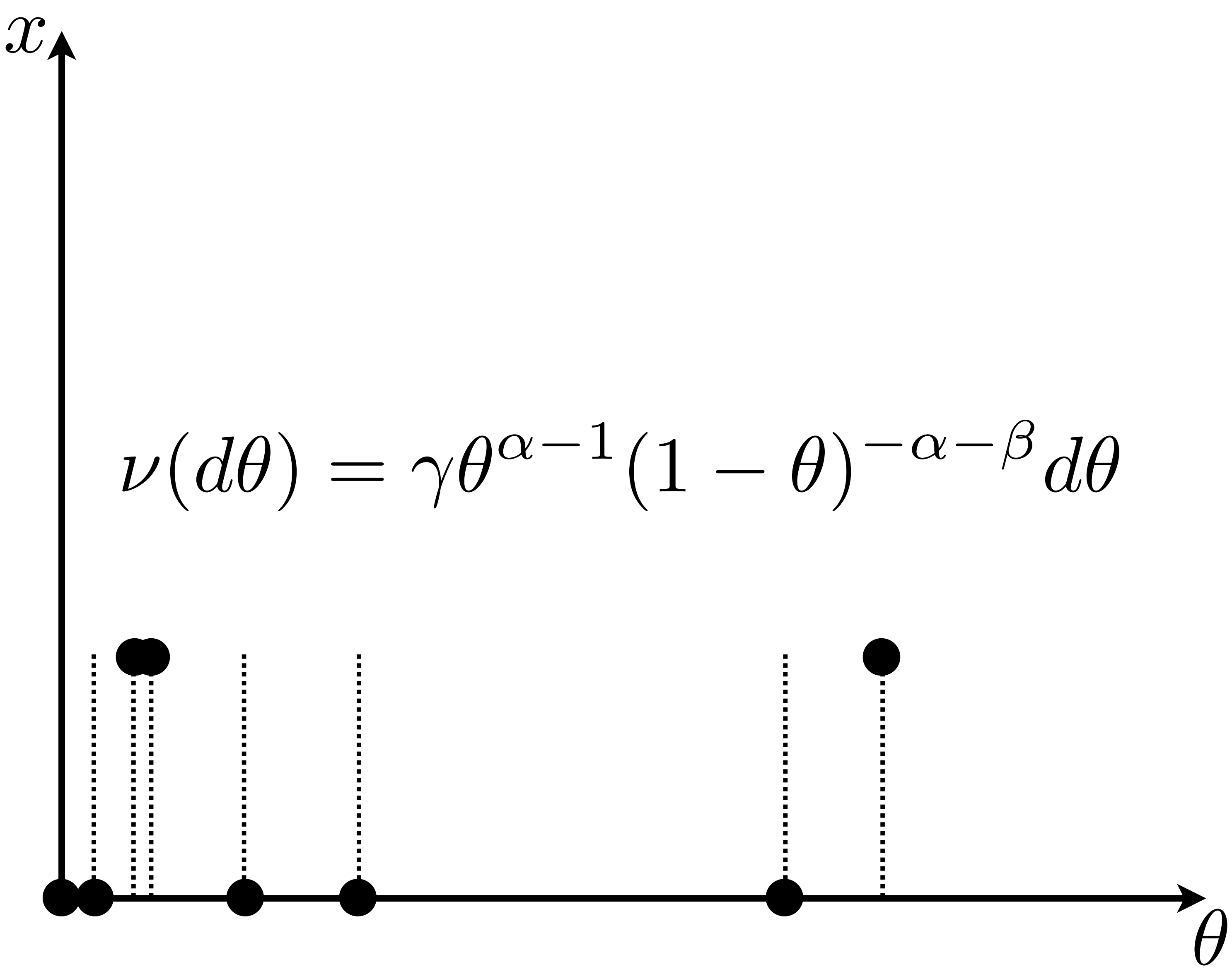

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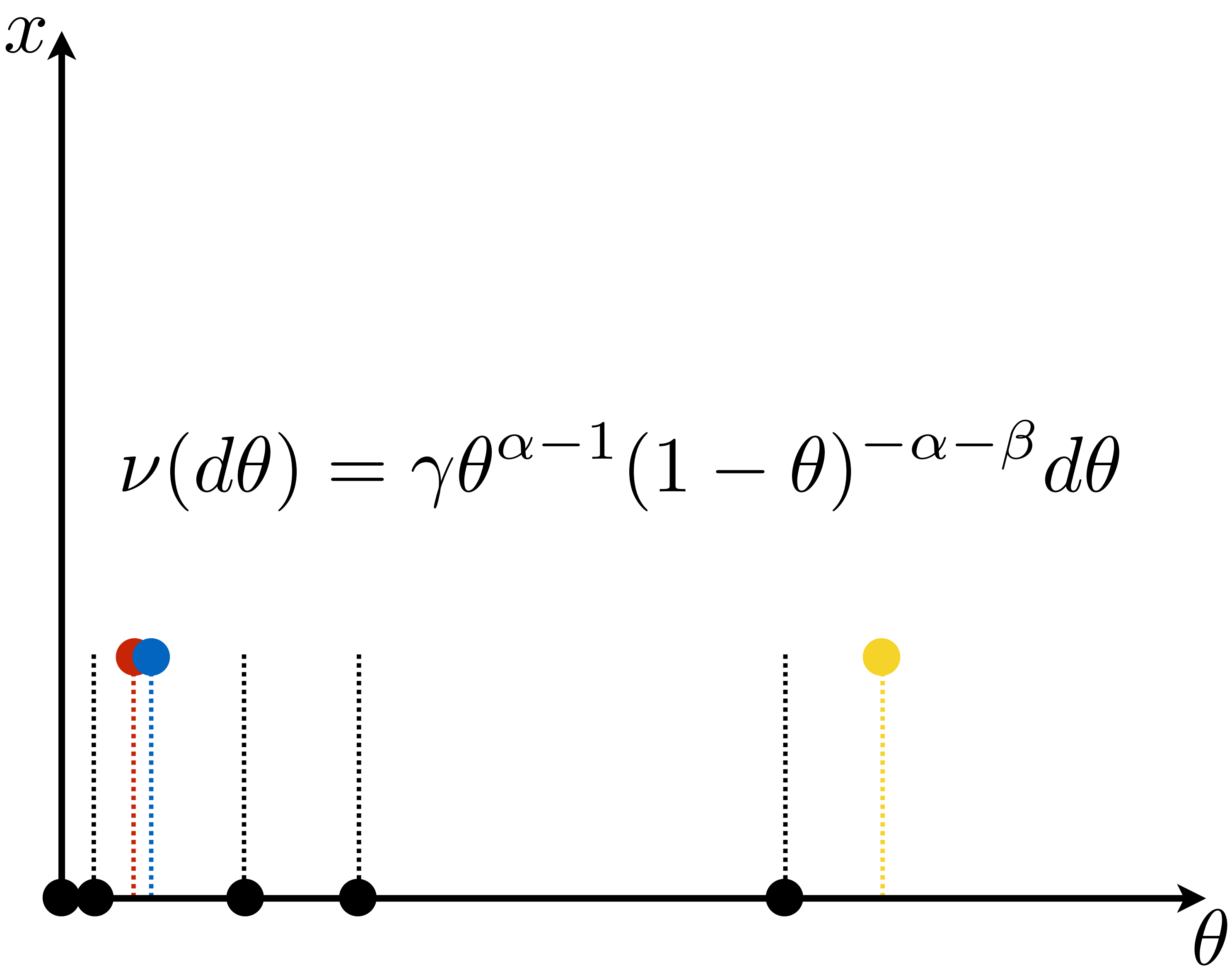

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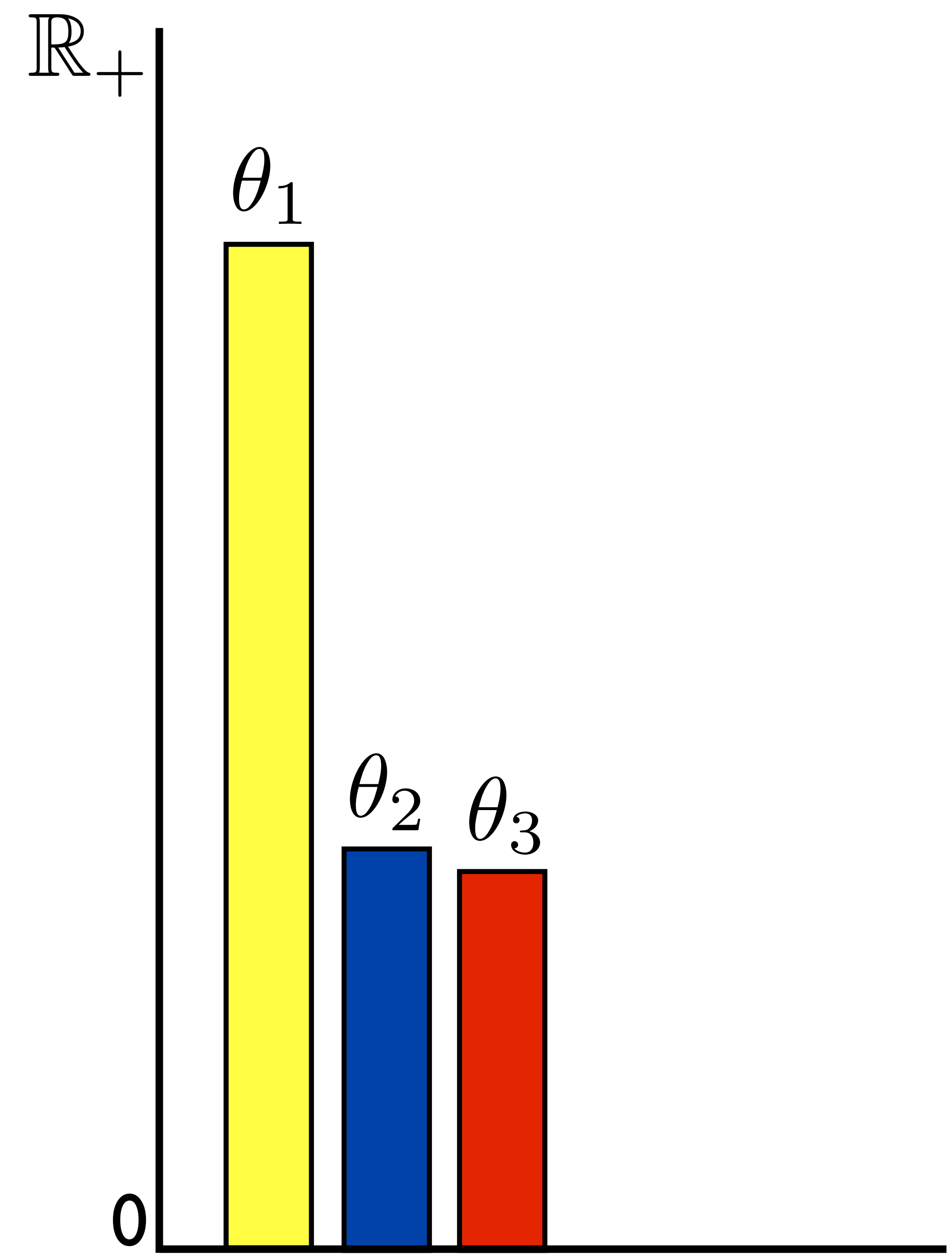

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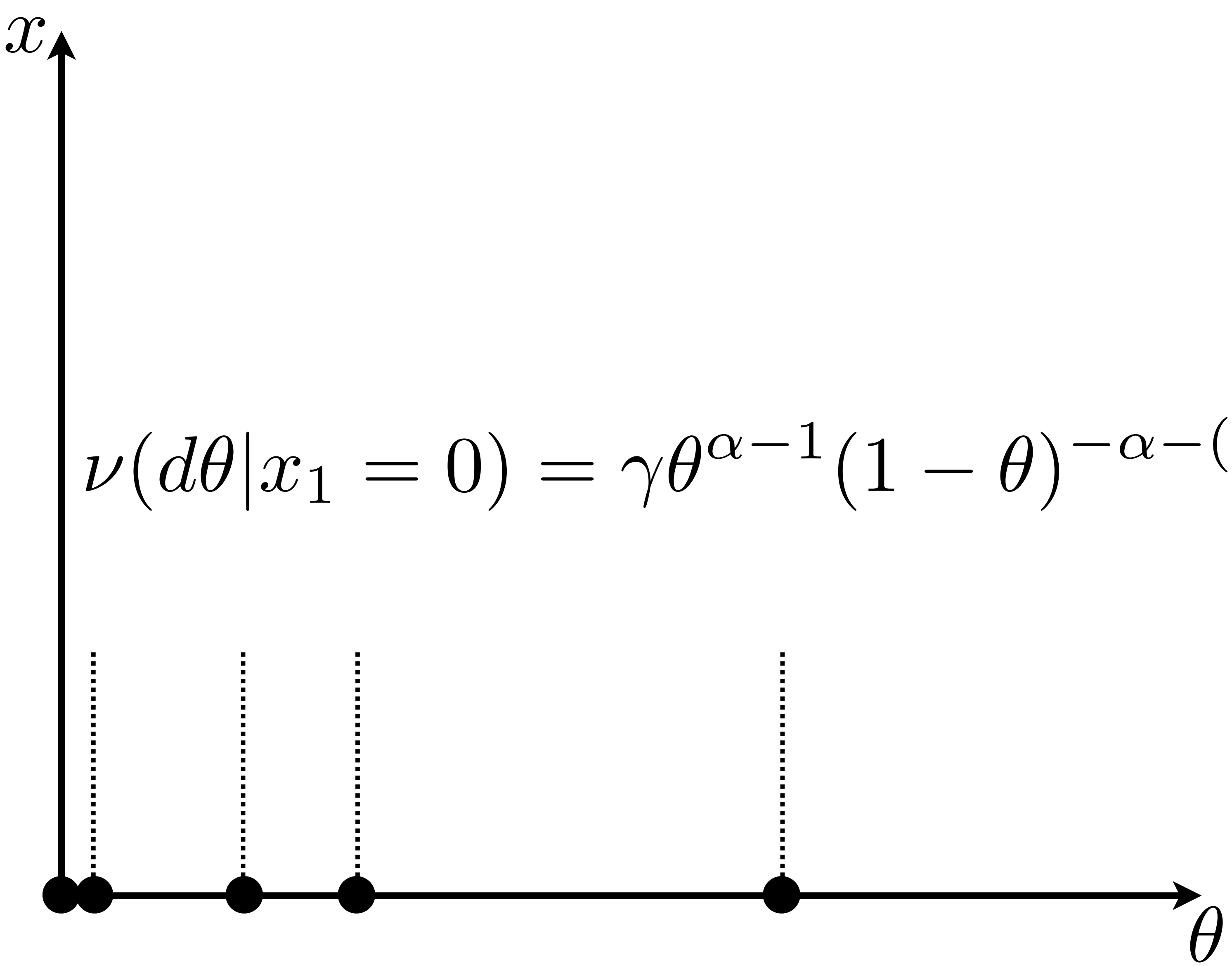

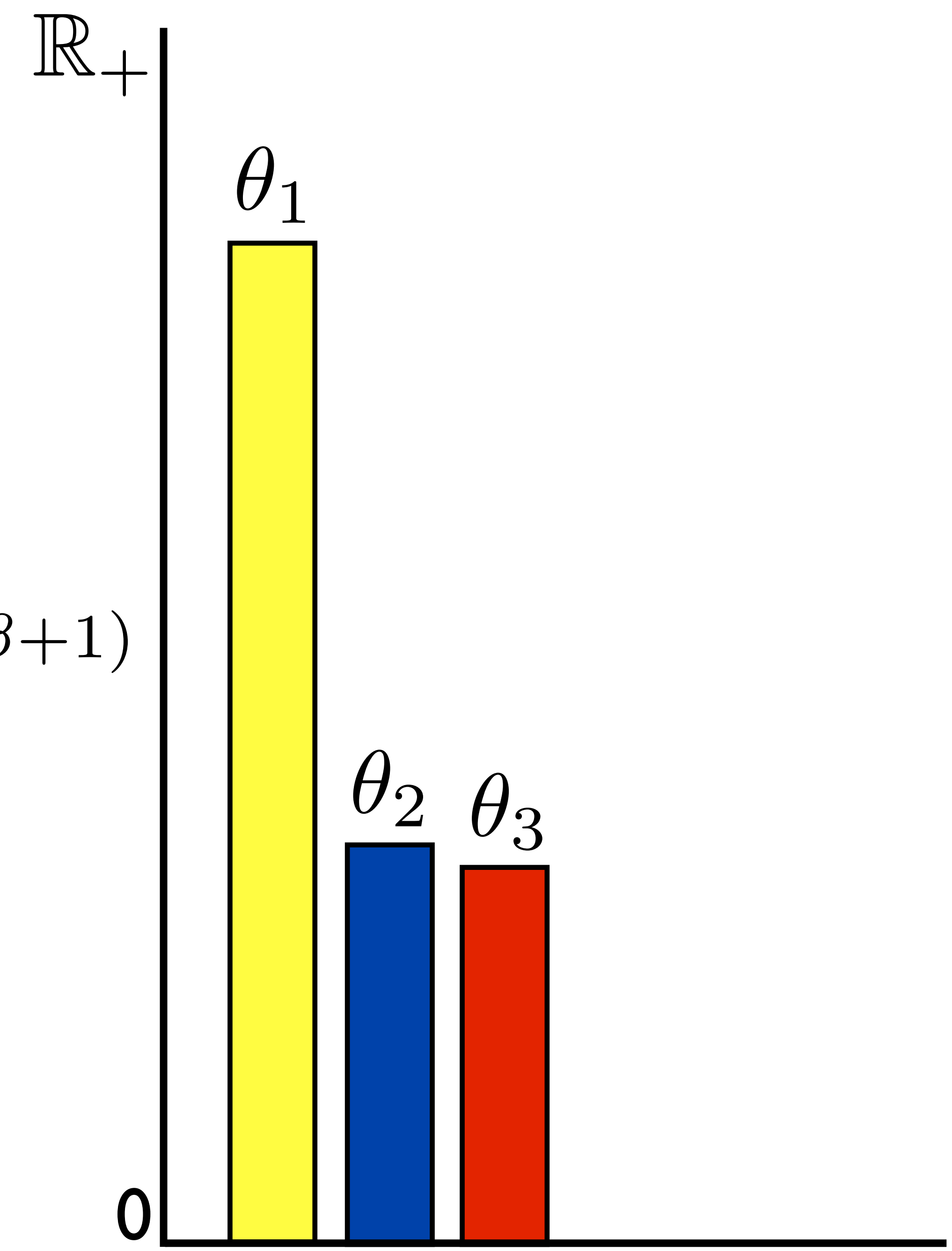

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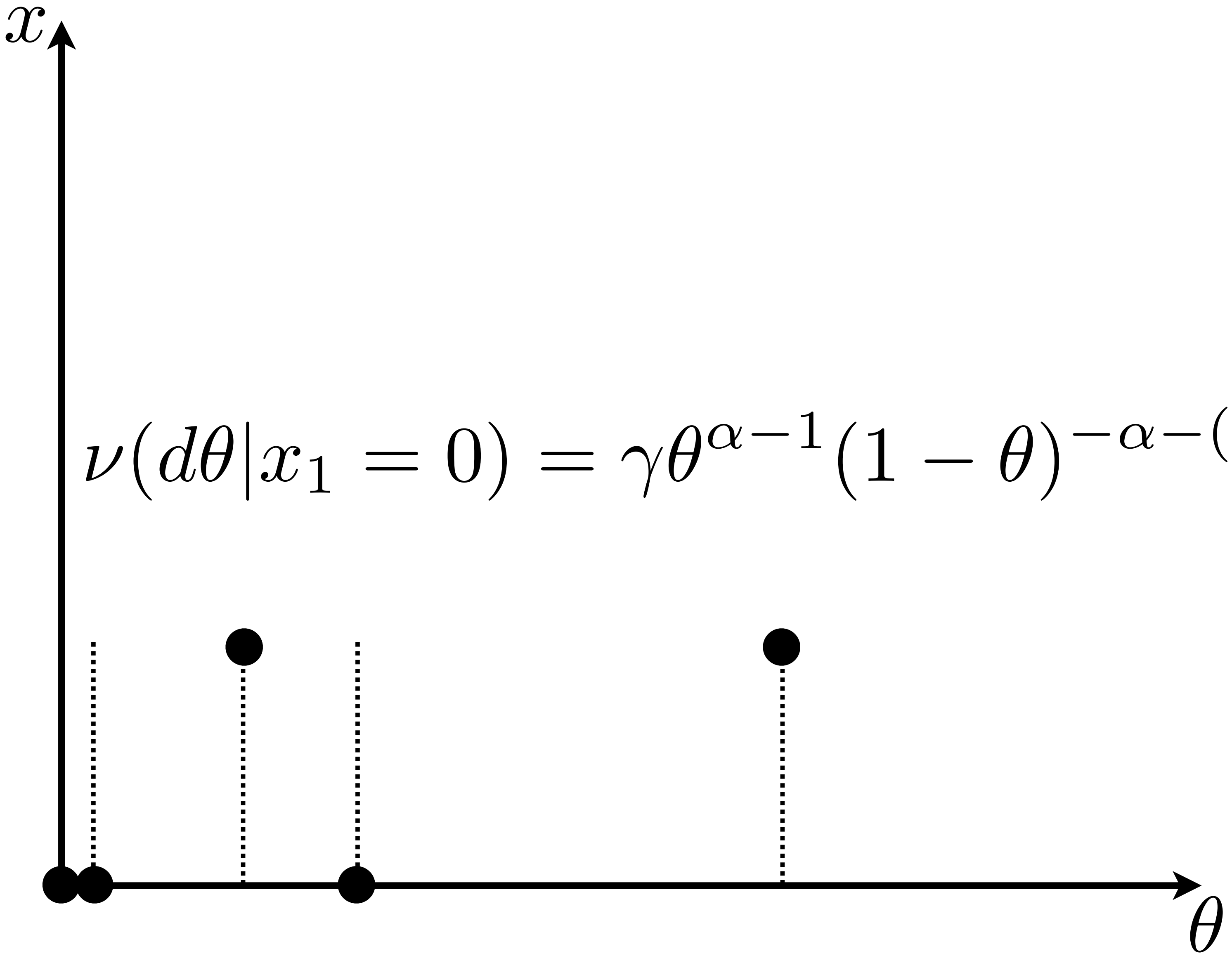

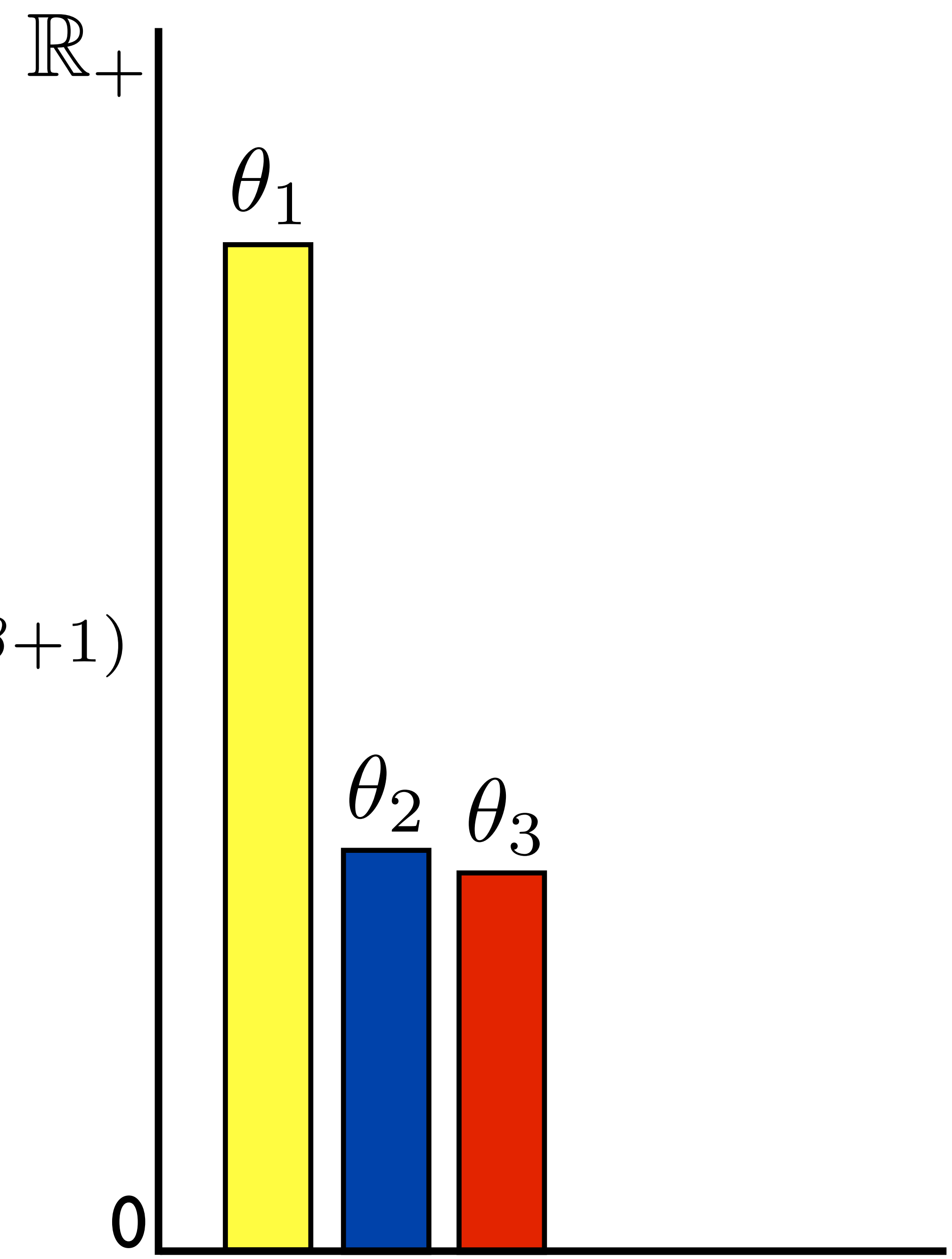

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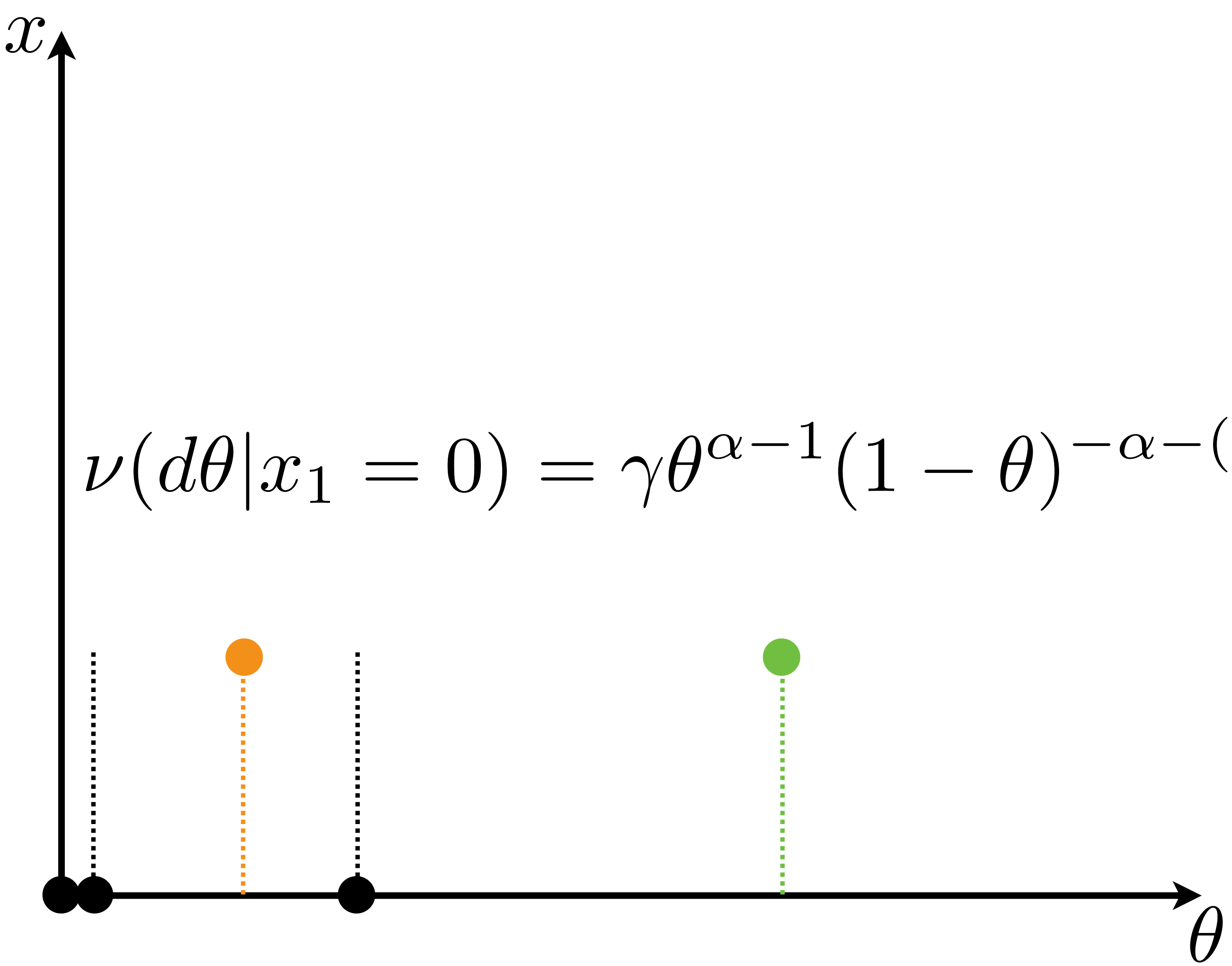

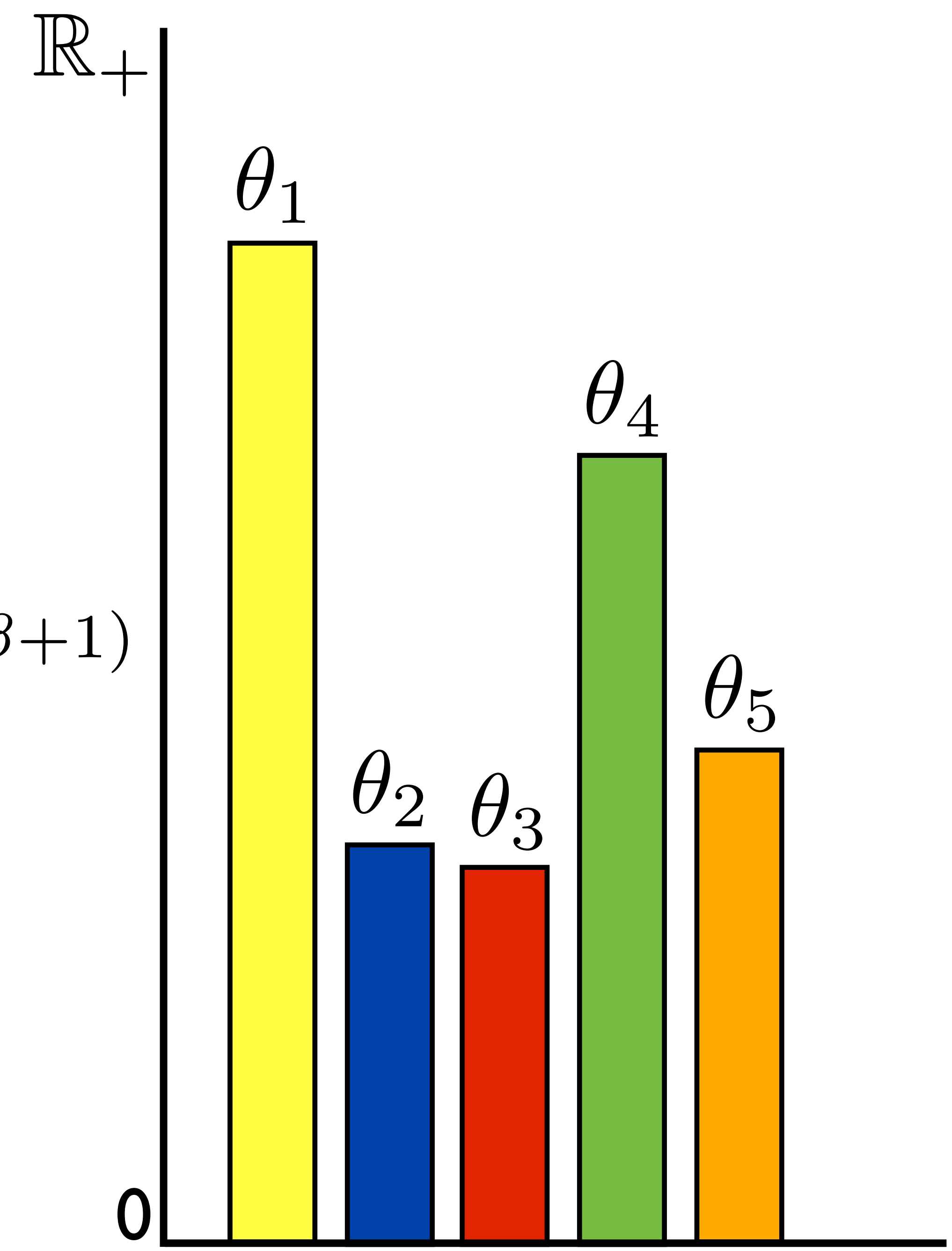

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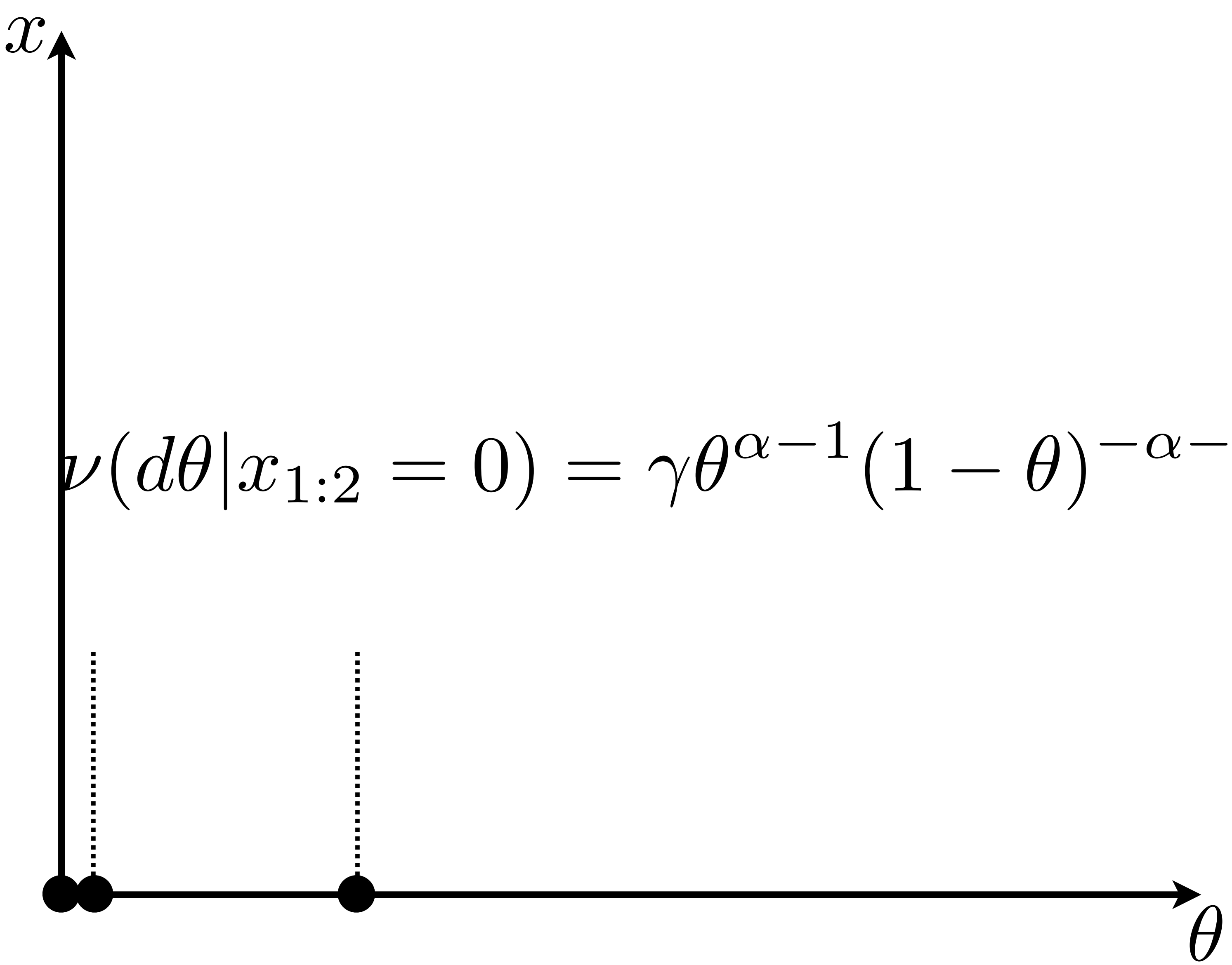

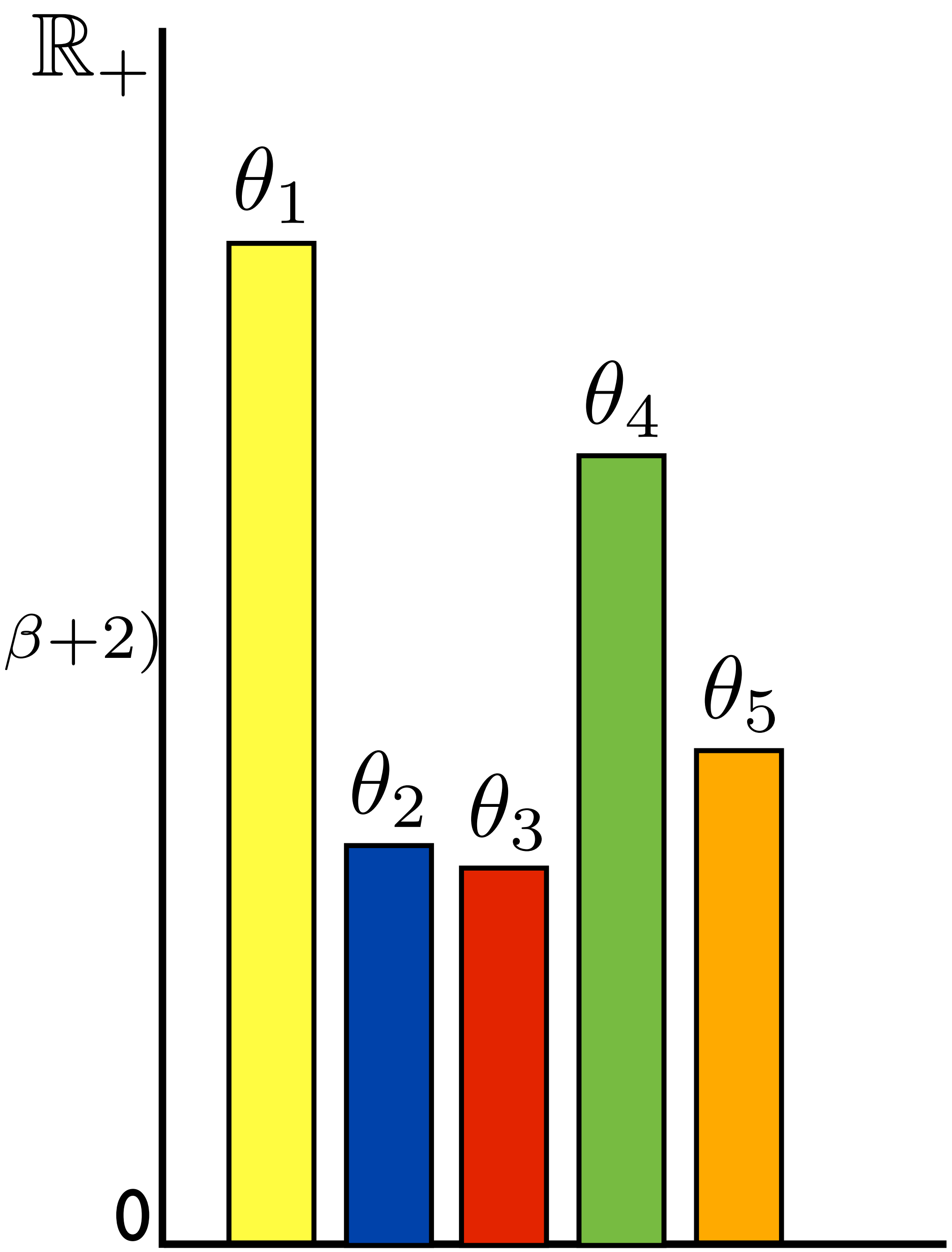

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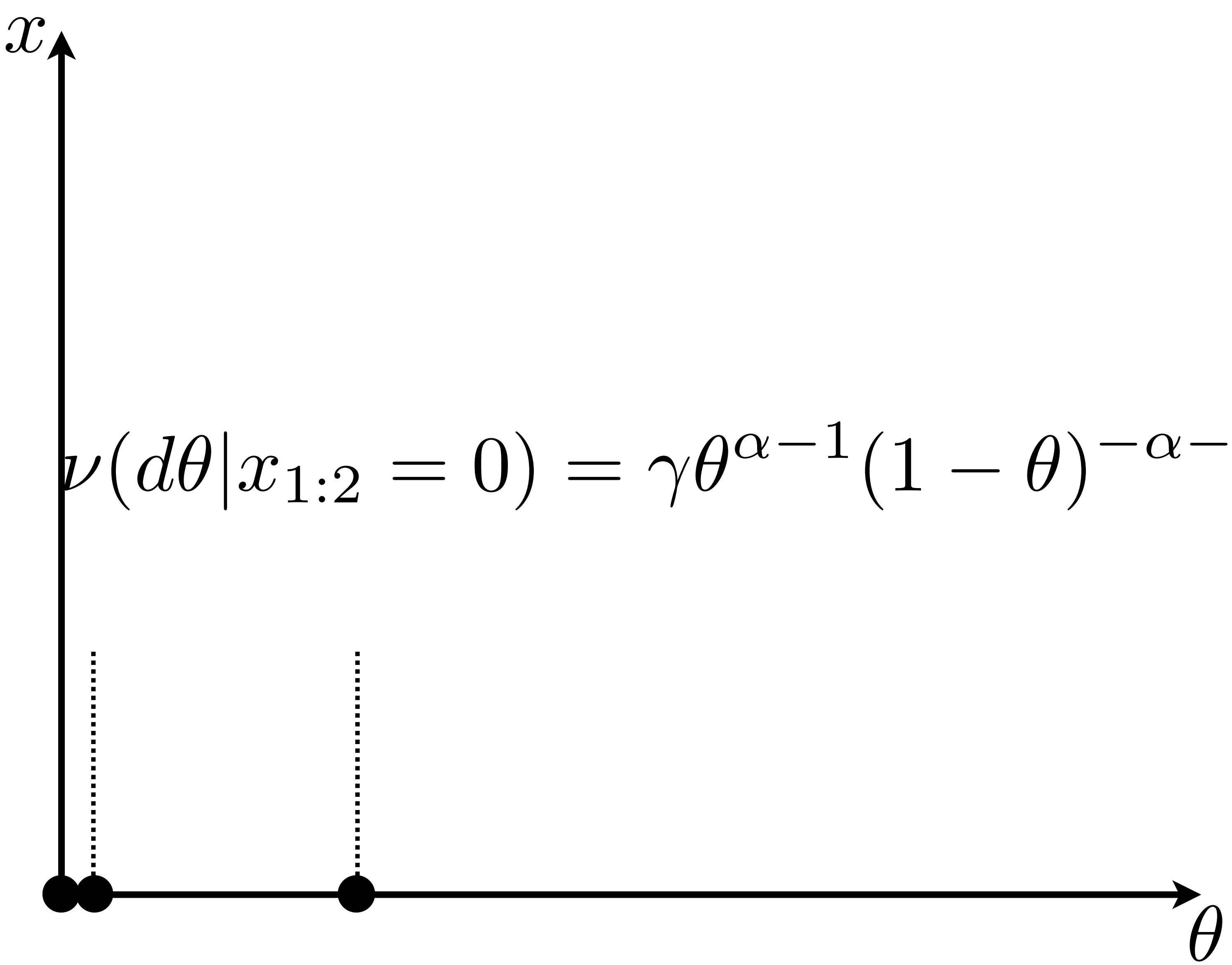
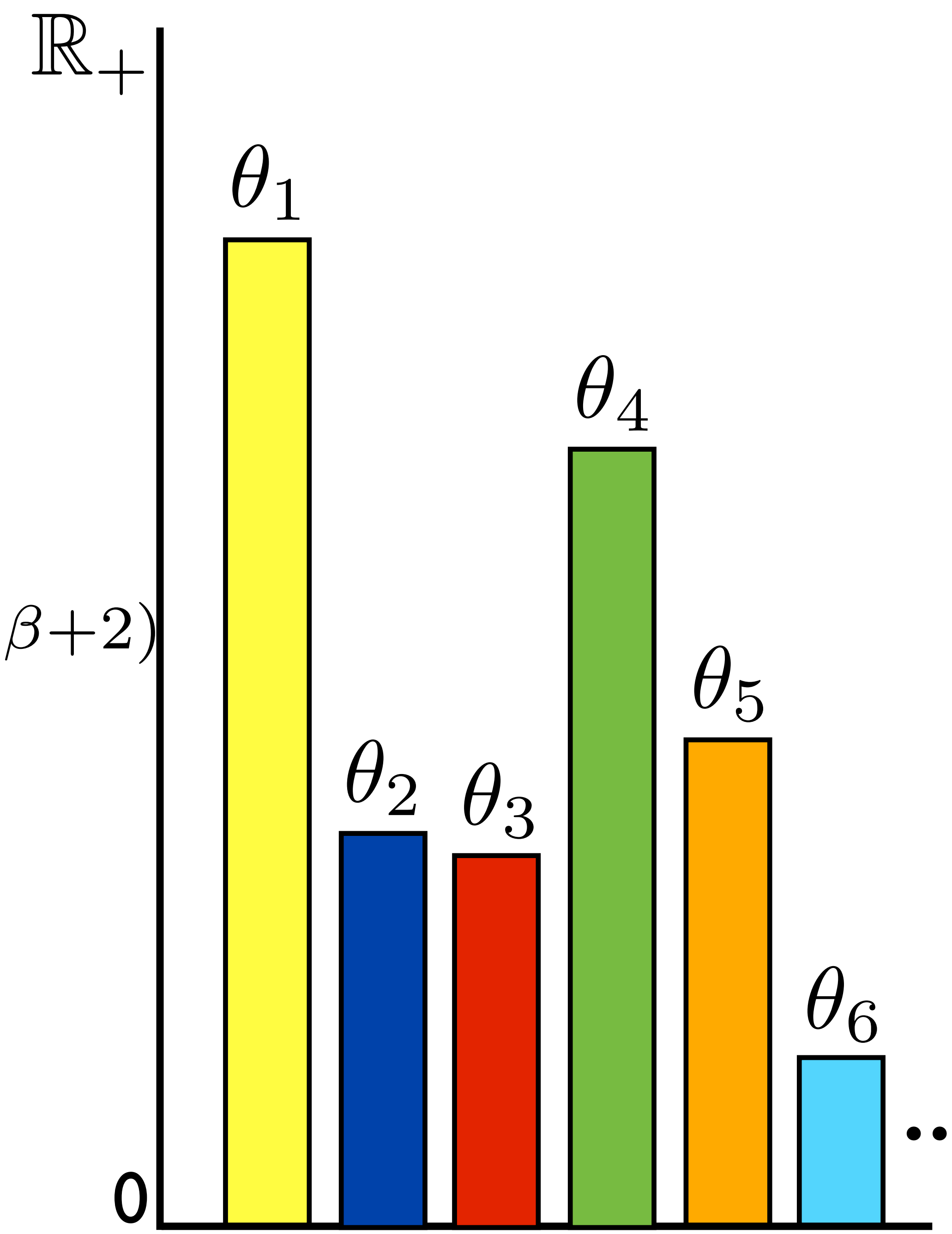




Size-biased atoms, beta prime process

$\alpha = 0$

For m = 1, 2, ...

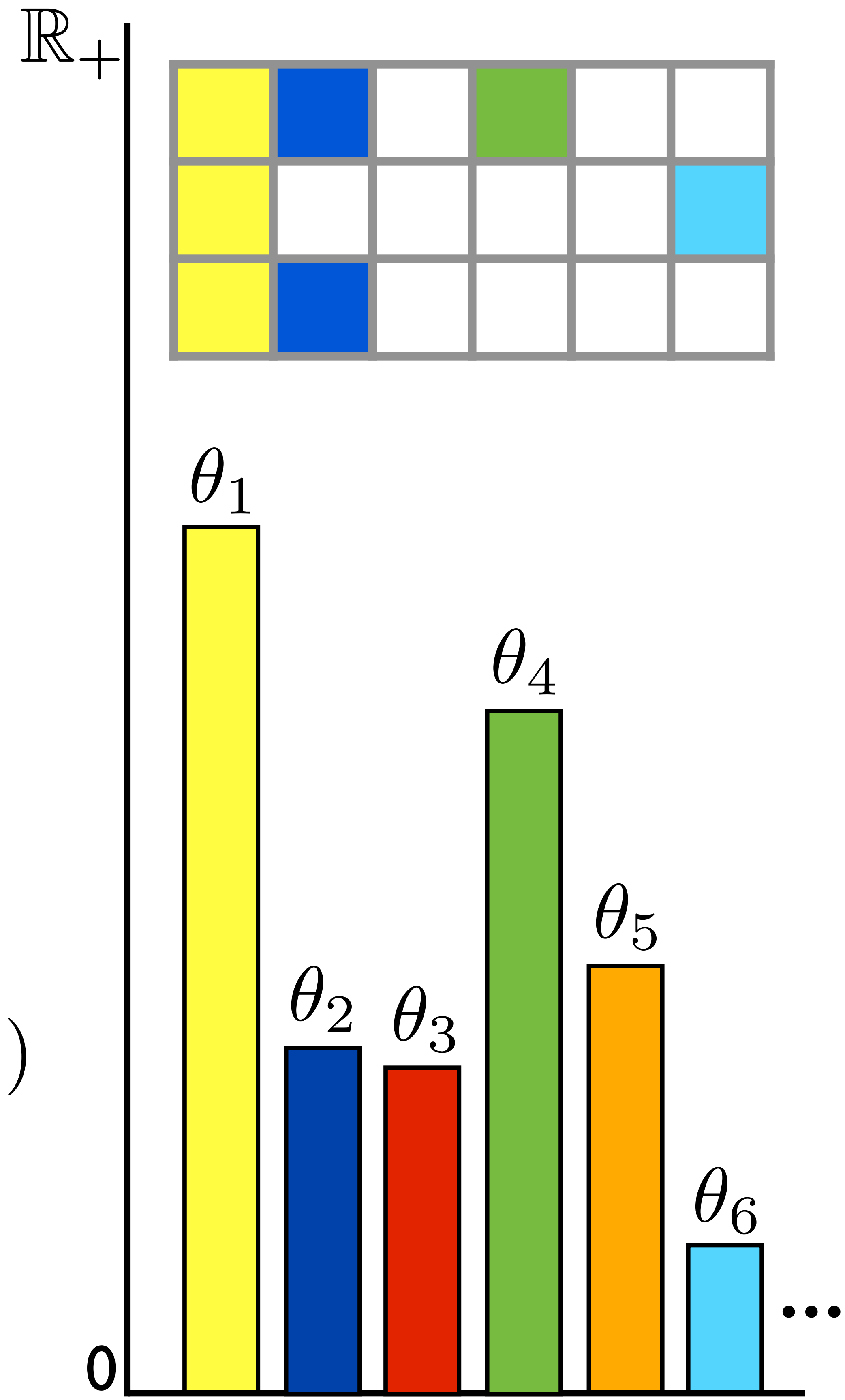
1. Draw

$$K_m^+ \sim \text{Poisson}\left(\gamma \frac{\beta}{\beta + m - 1}\right)$$

2. For $k = 1, \ldots, K_m^+$

   Draw a rate of size

$$\theta_k \sim \text{BetaPrime}(1, \beta + m - 1)$$



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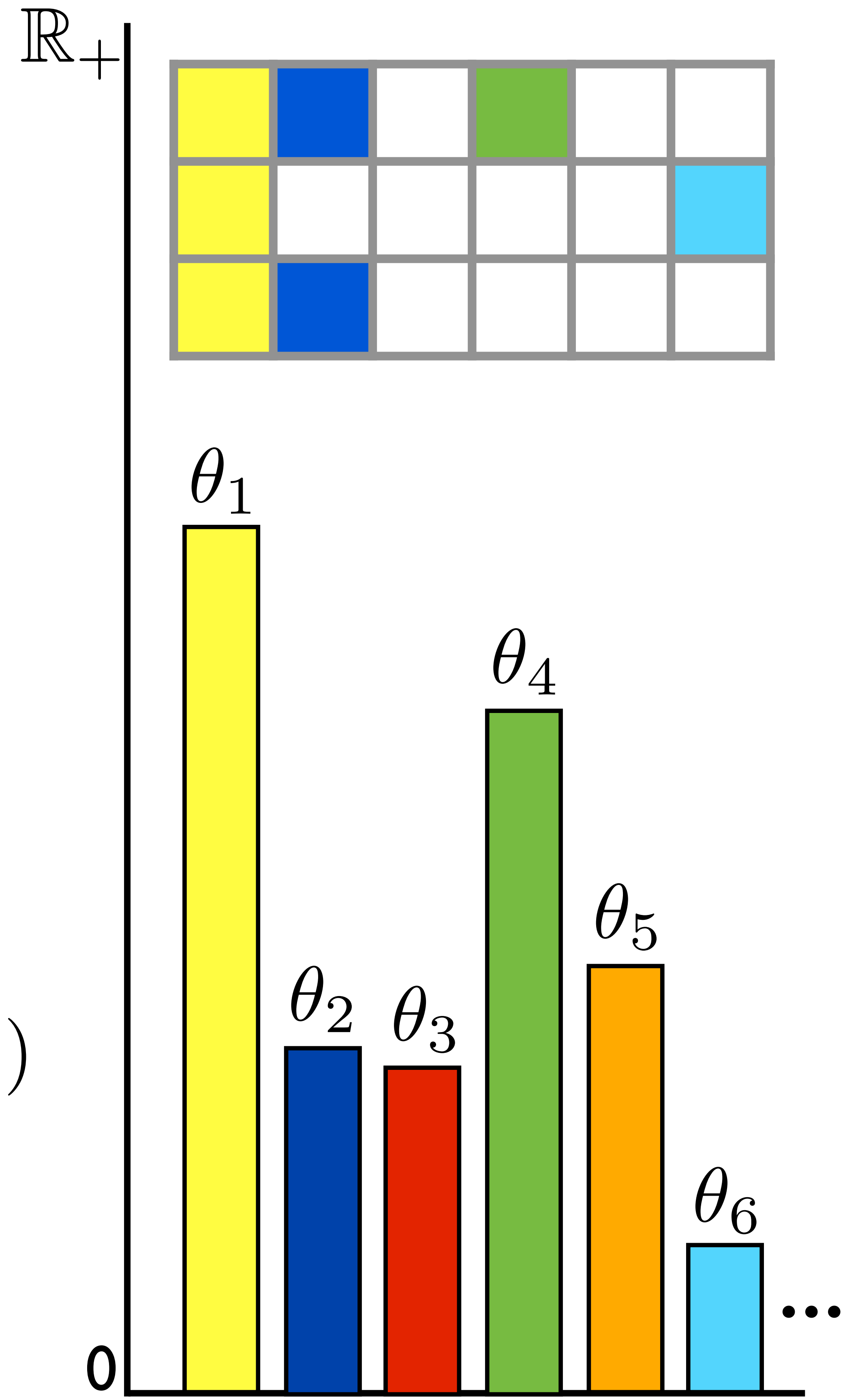

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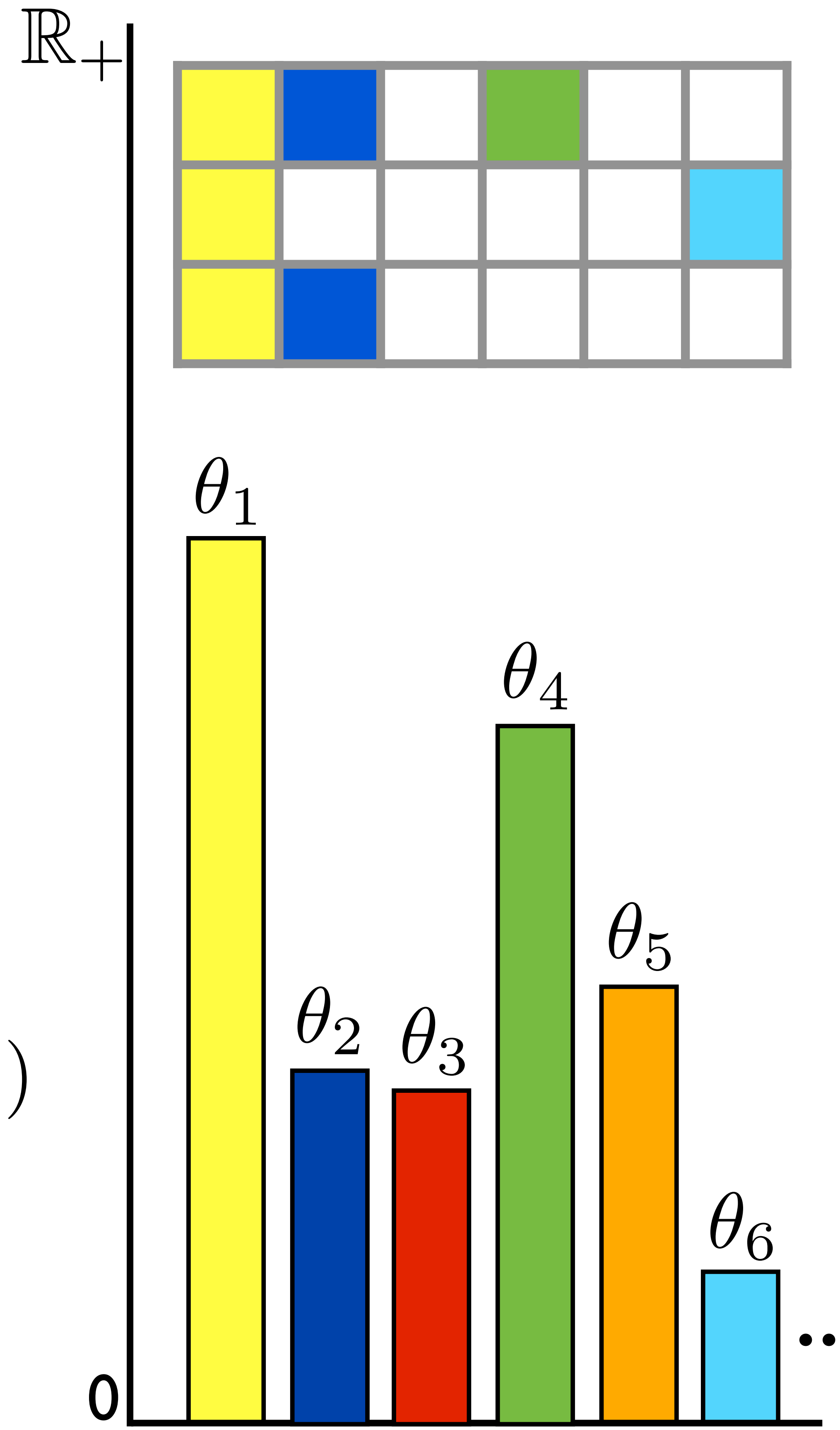

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Marginal process derivation is similar

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- Marginal process

  K_n^+ as above

$$p(x_n | x_{1:(n-1)}) = \kappa(x_n) \exp\left\{ -B(\xi + \sum_{m=1}^{n-1} x_m, \lambda + n - 1) + B(\xi + \sum_{m=1}^{n-1} x_m + x_n, \lambda + n) \right\}$$

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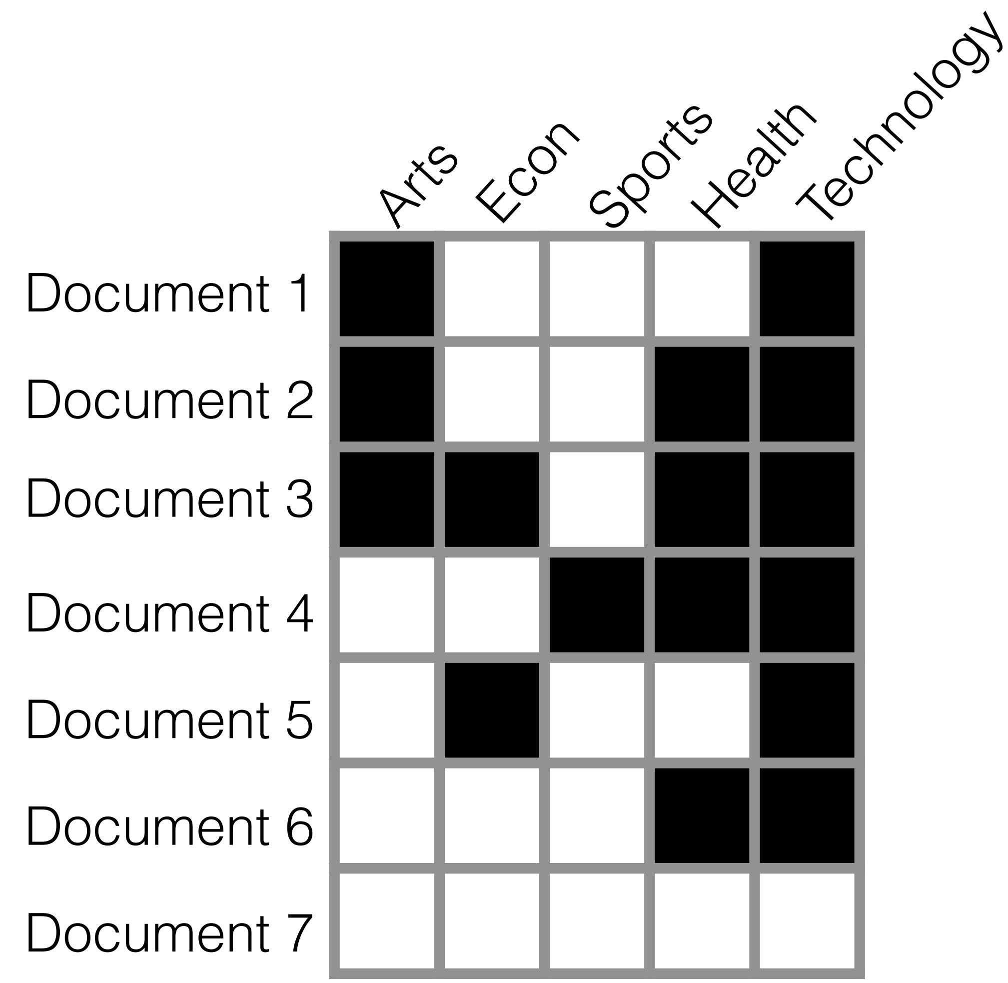

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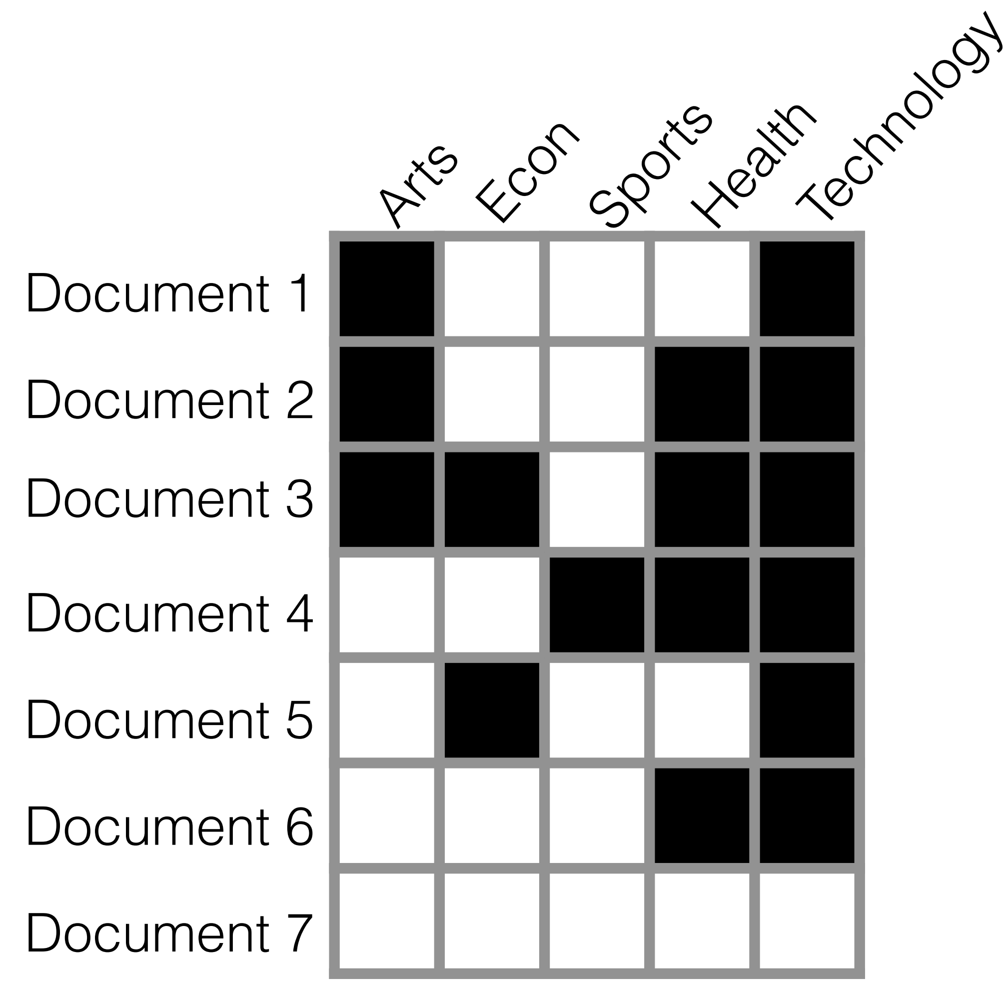

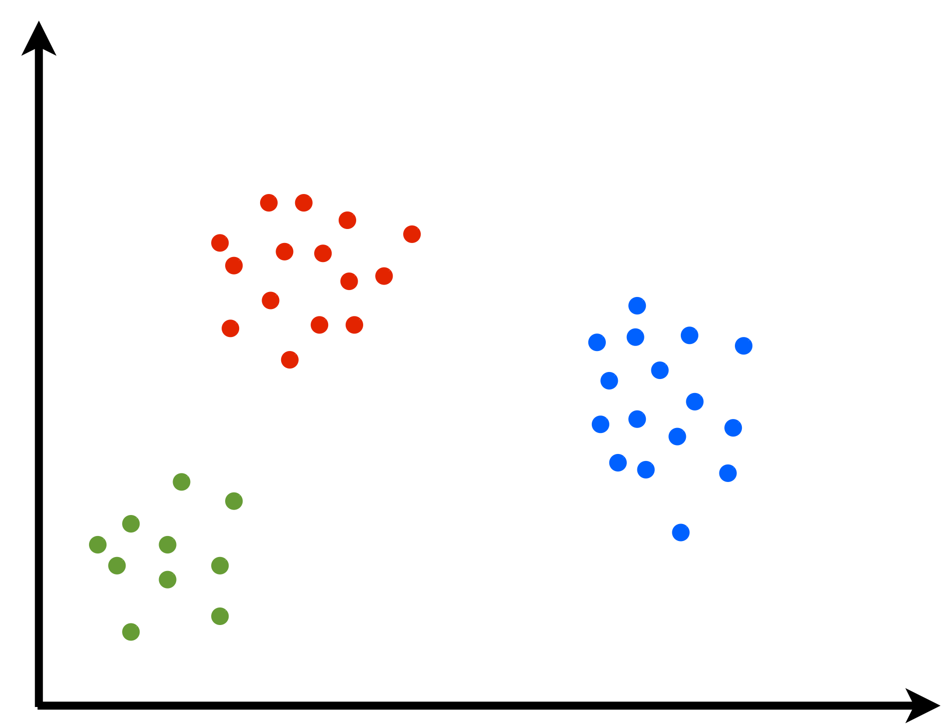

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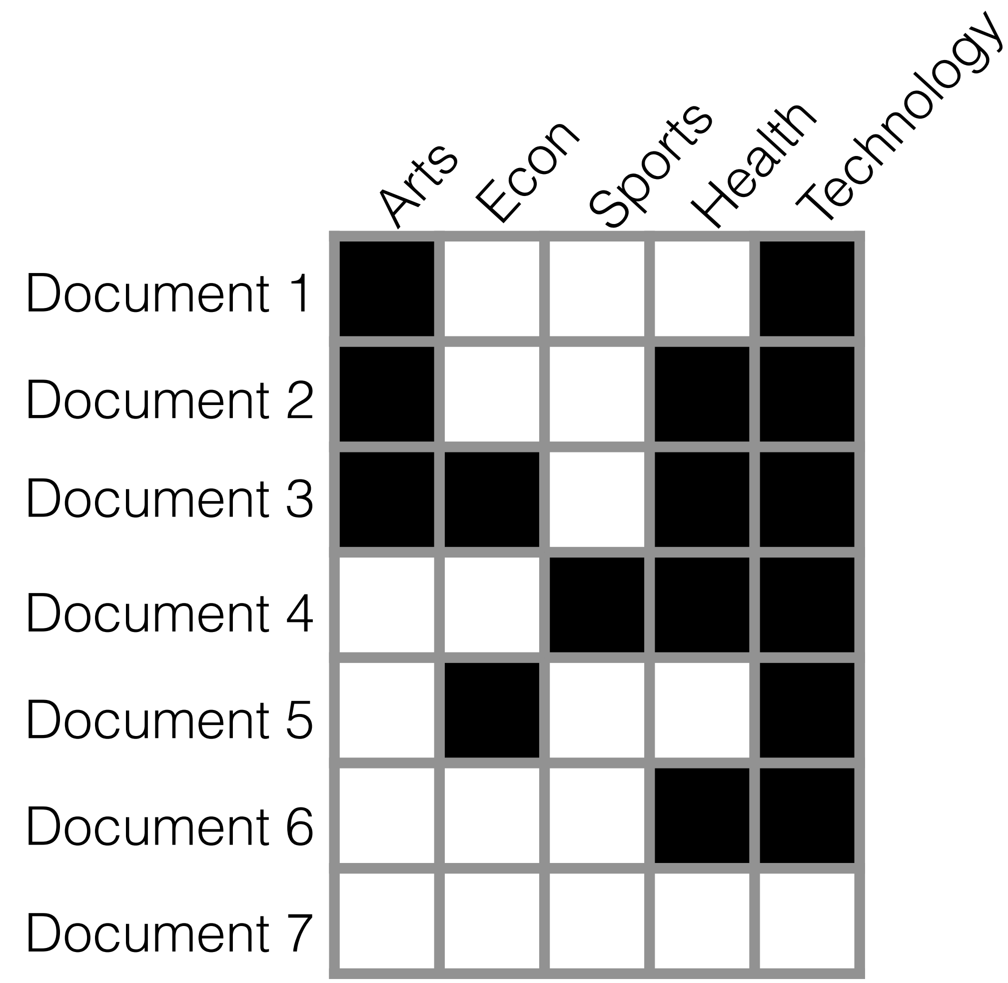
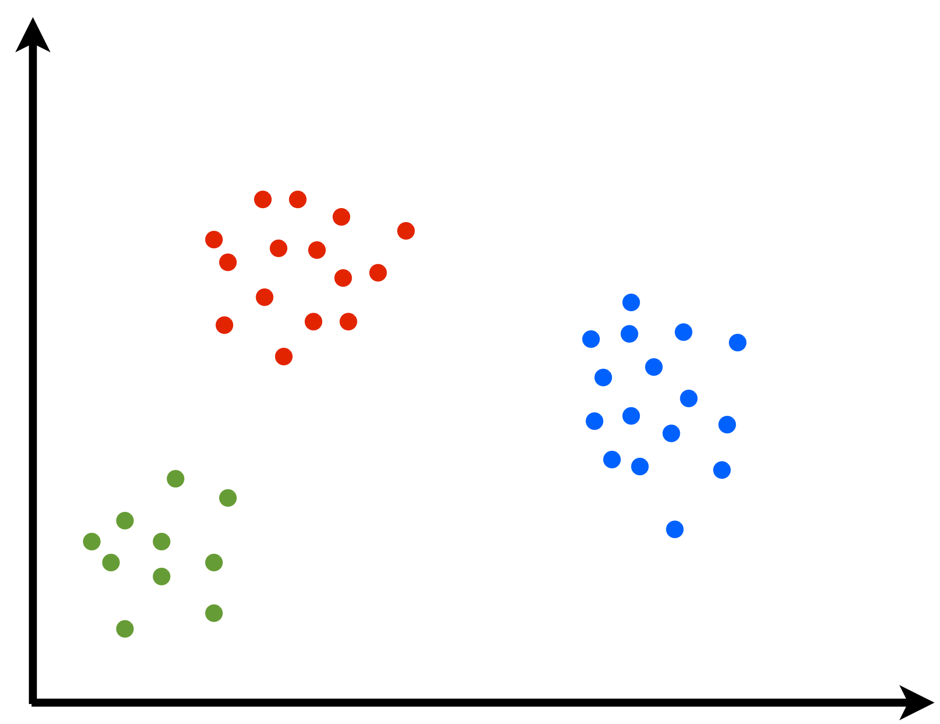
- Can be used with arbitrary (i.e., discrete, continuous, or other) data likelihood





References

T Broderick, AC Wilson, and MI Jordan. Posteriors, conjugacy, and exponential families for completely random measures. Submitted, ArXiv, 2014.

P Diaconis and D Ylvisaker. Conjugate priors for exponential families. *Annals of Statistics*, 1979.

T Griffiths and Z Ghahramani. Infinite latent features models and the Indian buffet process. In *NIPS*, 2006.

Y Kim. Nonparametric Bayesian estimators for counting processes. *Annals of Statistics*, 1999.

N Hjort. Nonparametric Bayes estimators based on beta processes in models for life history data. *Annals of Statistics*, 1990.

R Thibaux and MI Jordan. Hierarchical beta processes and the Indian buffet process. In *AISTATS*, 2007.