Nonparametric Bayes and Exchangeability: Part II

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Electrical Engineering & Computer Science
MIT

http://www.tamarabroderick.com/tutorials.html

Roadmap

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- Example problem: clustering

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- Example NPBayes model: Dirichlet process (DP)

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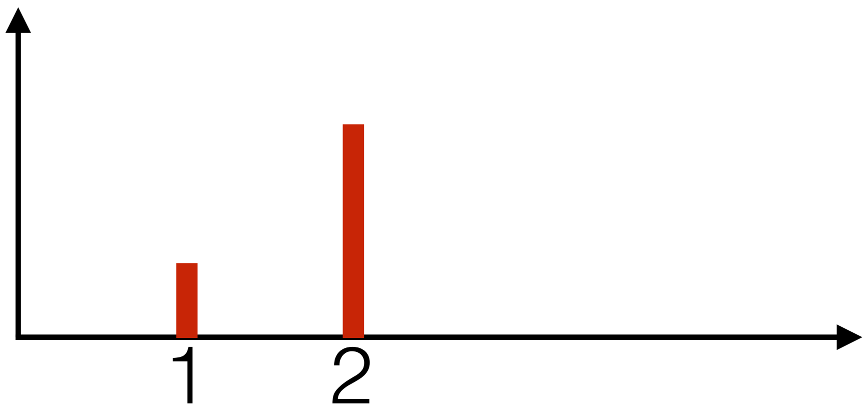

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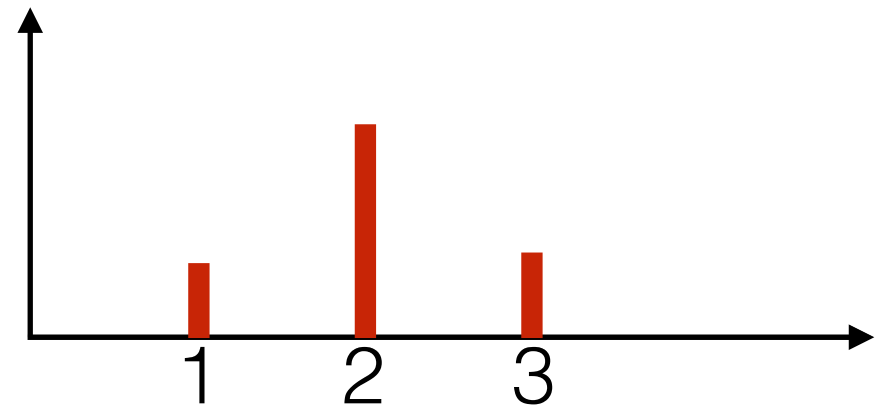

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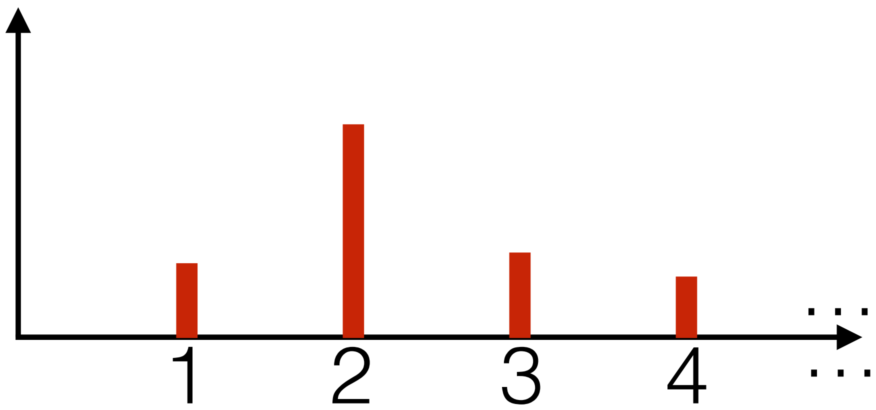

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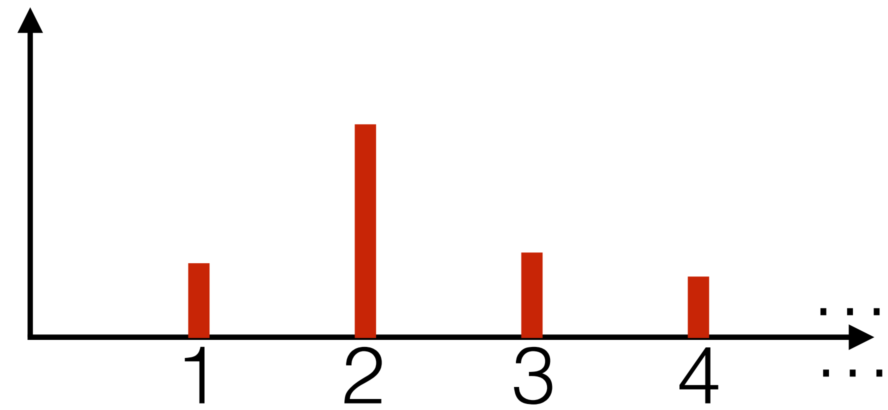

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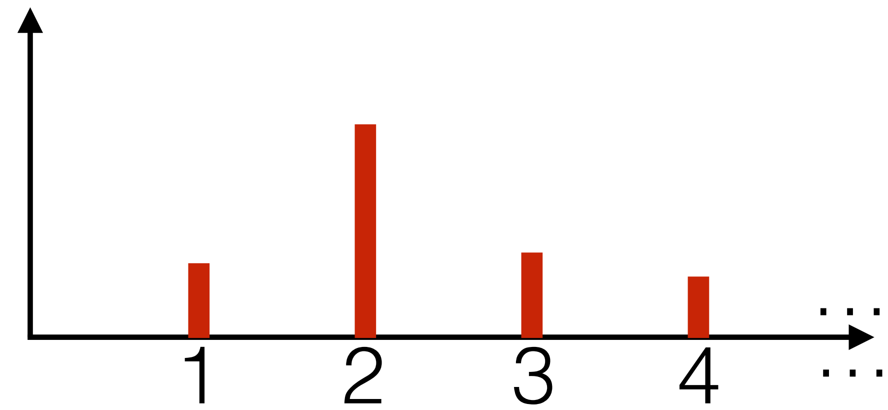

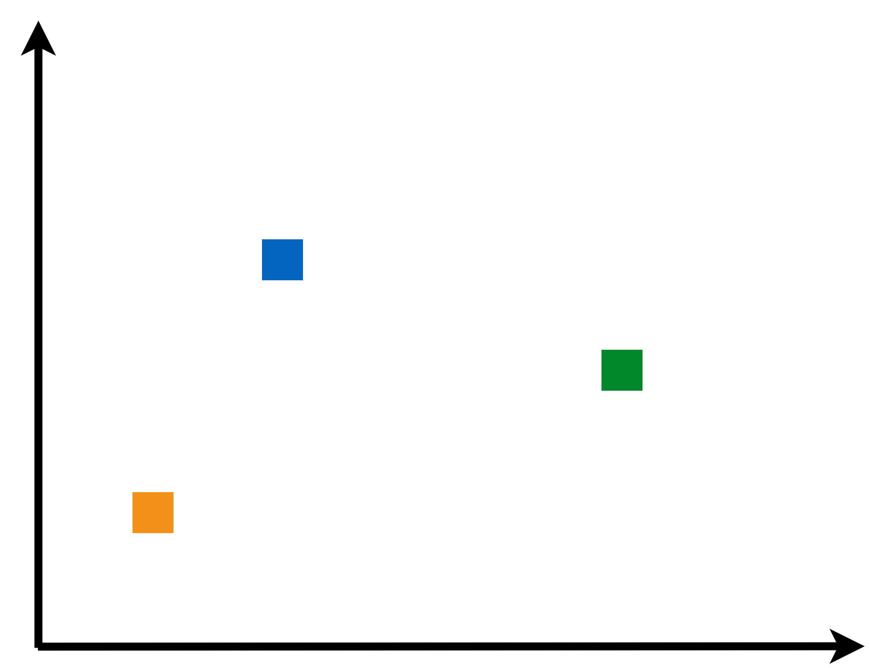

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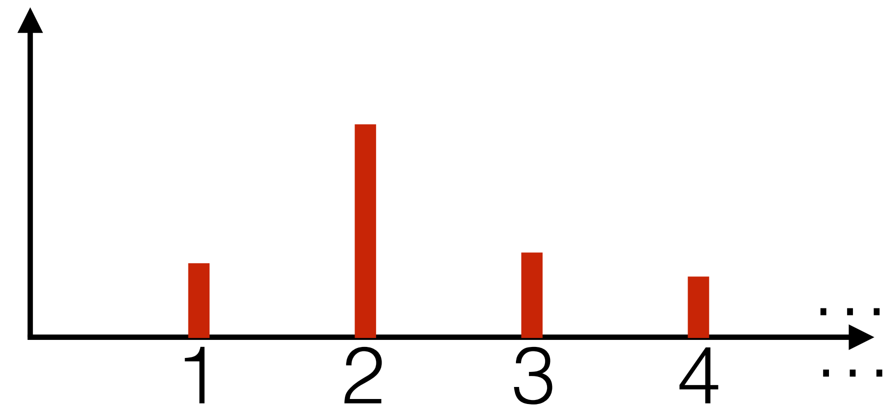

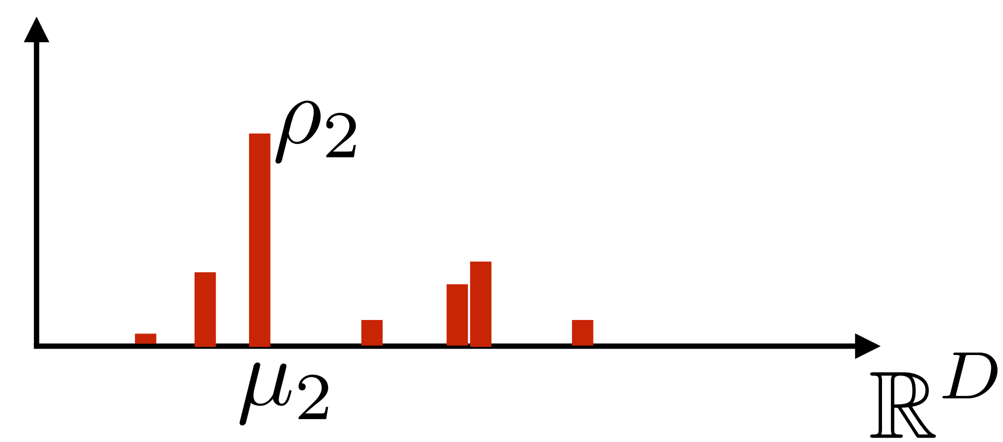

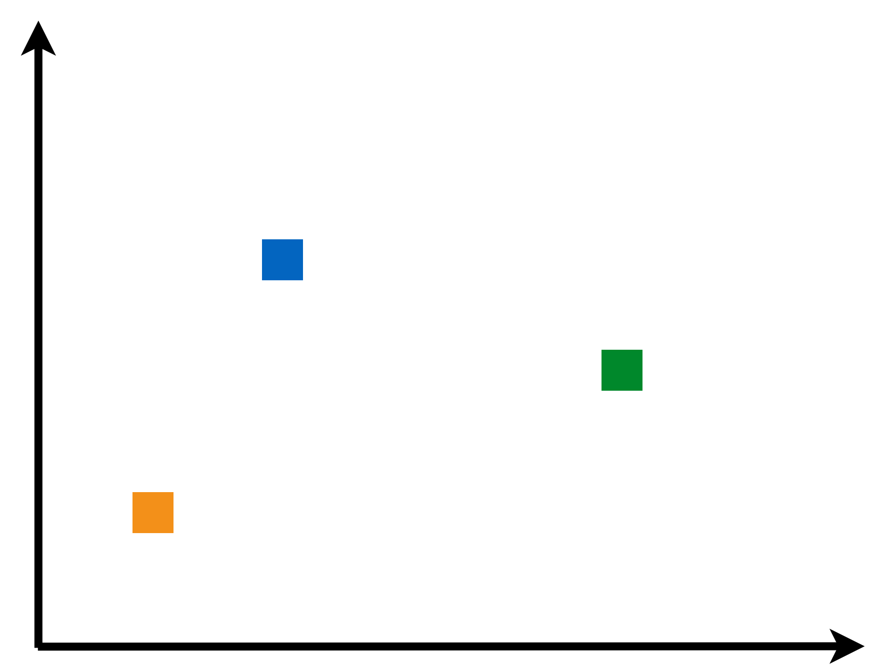

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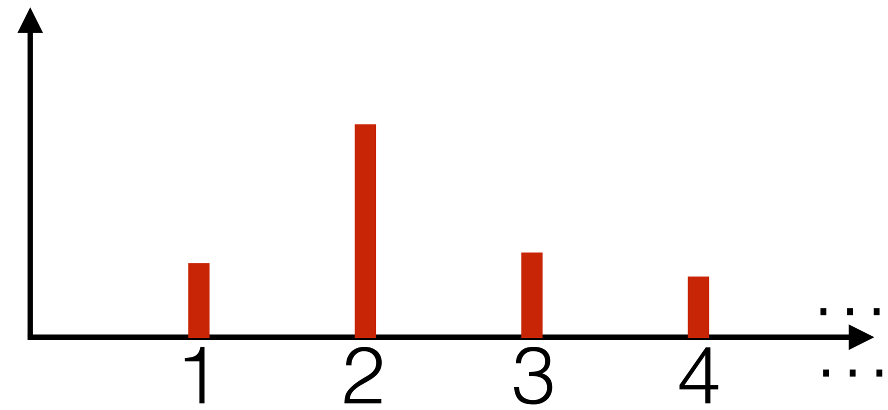

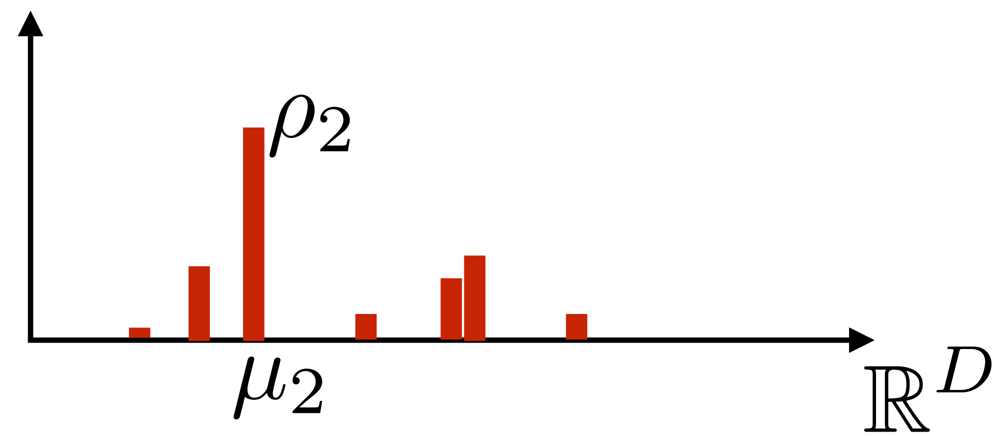

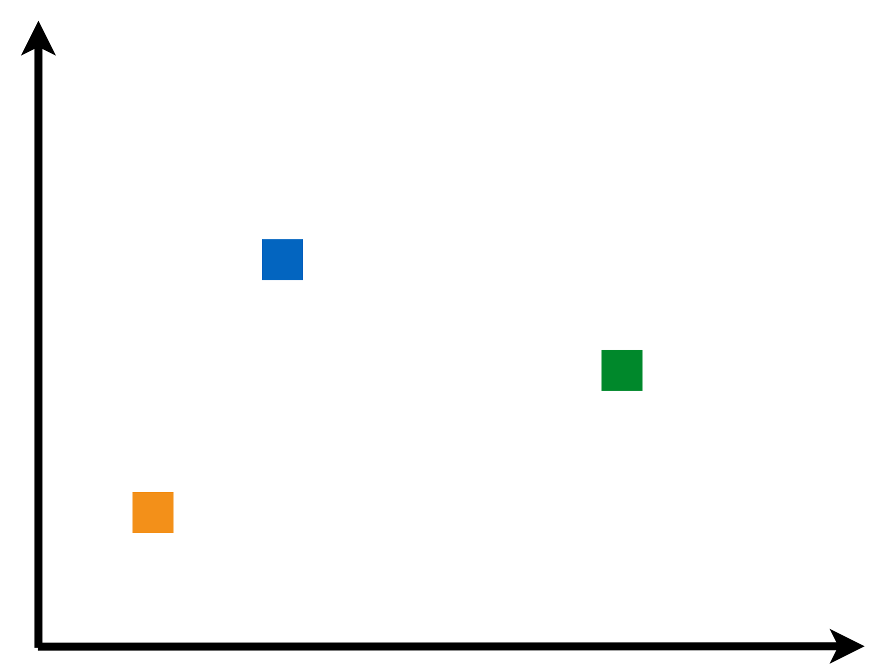

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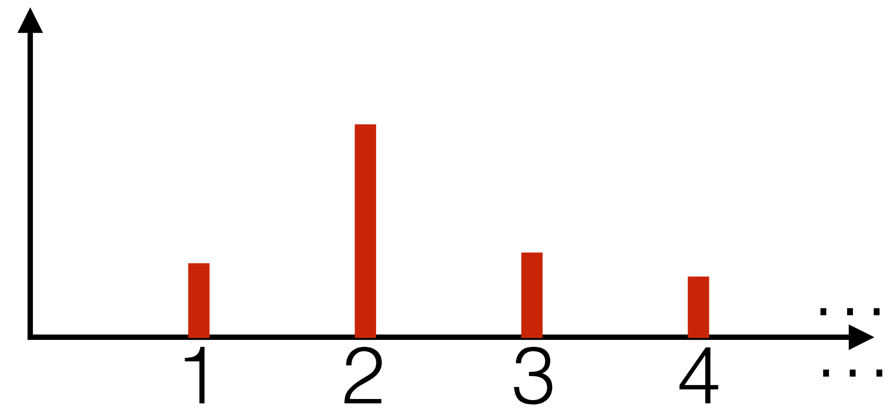

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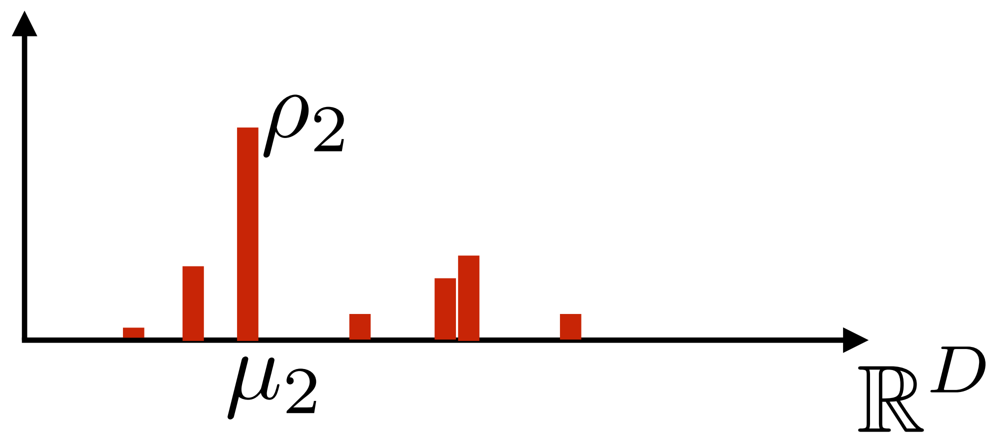

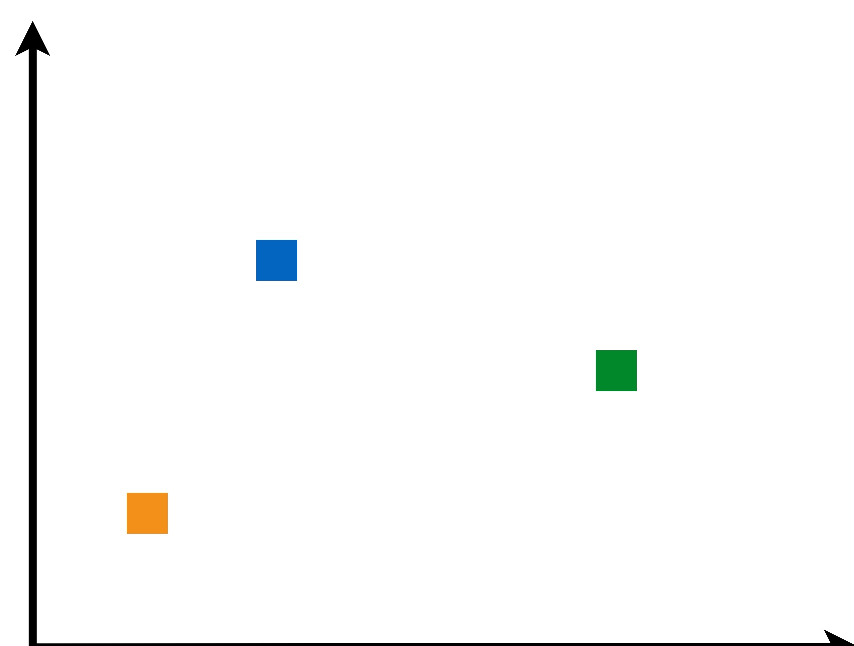

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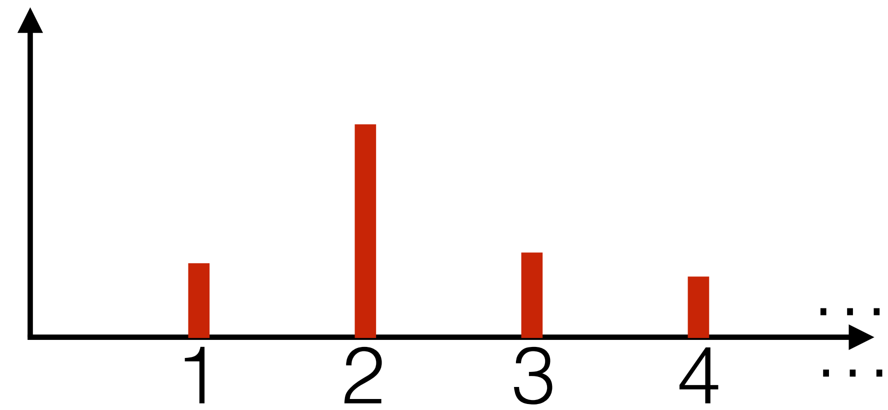

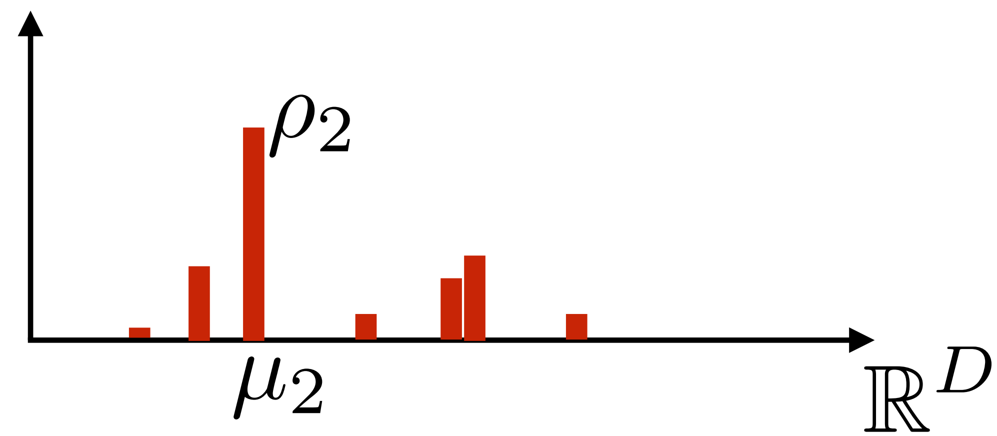

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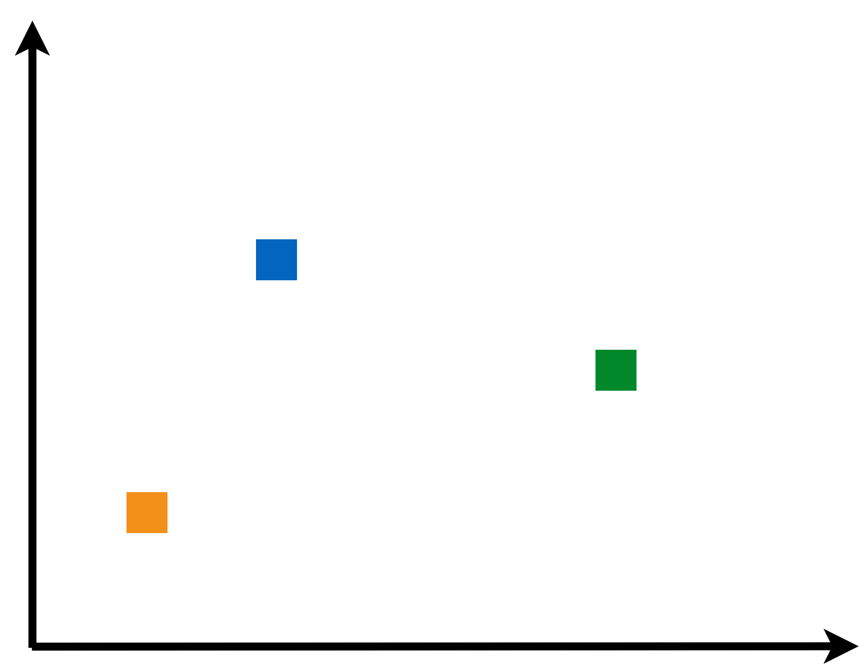

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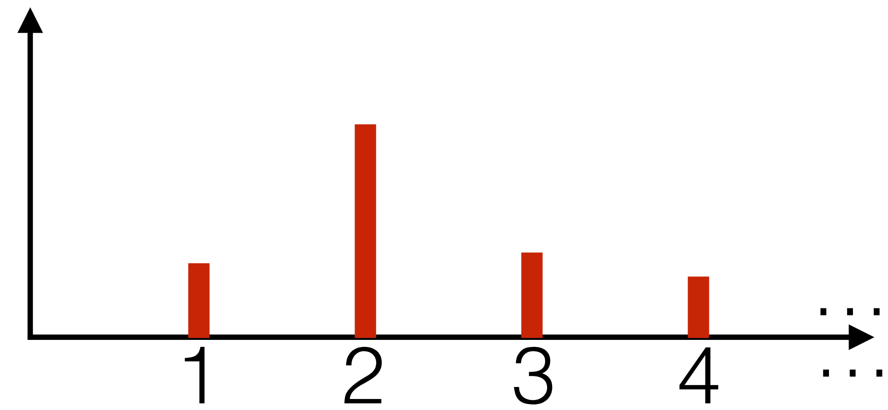

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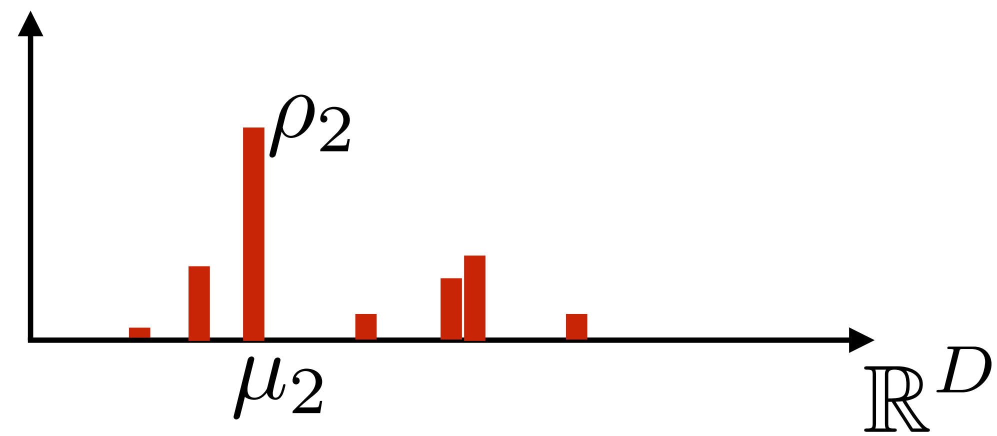

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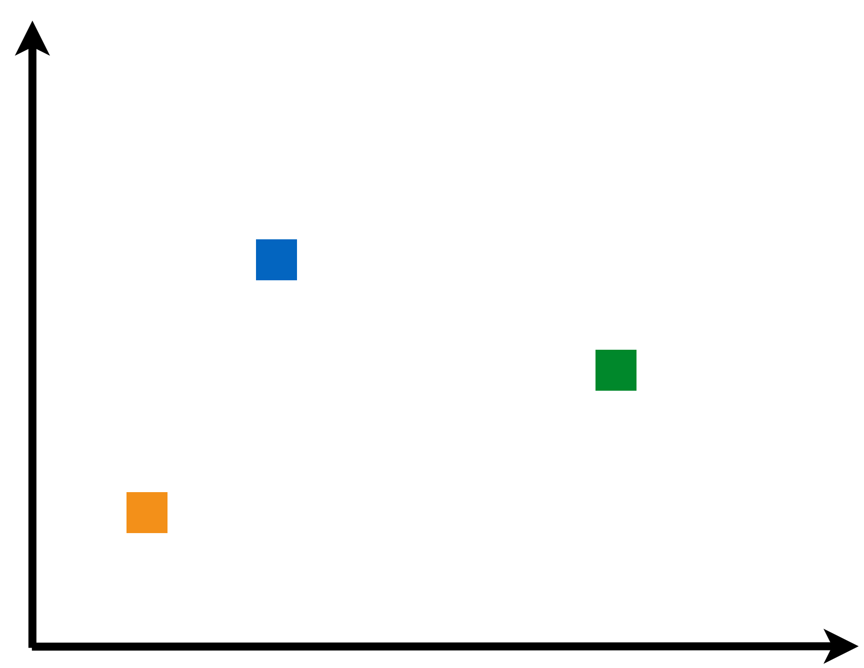

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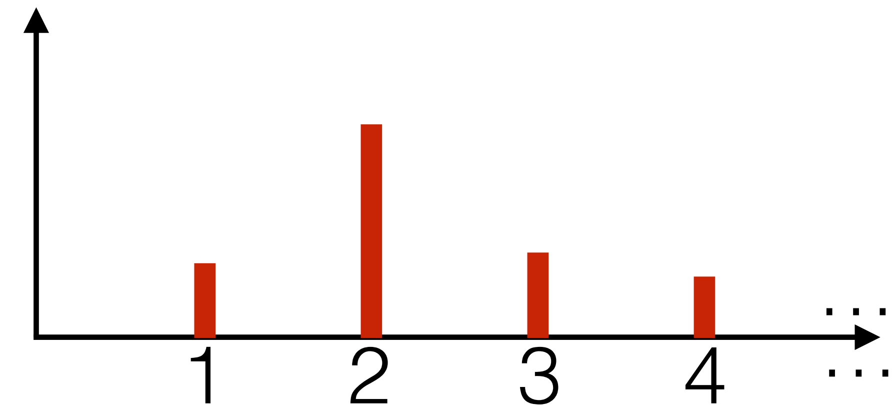

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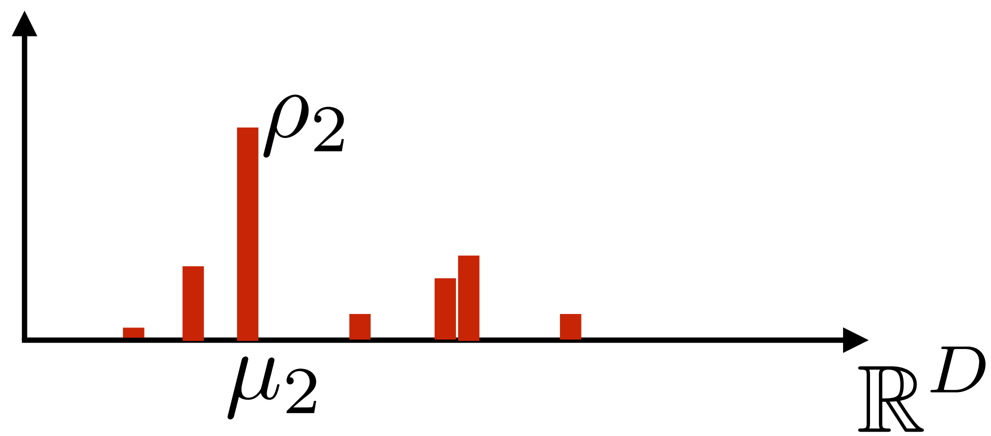

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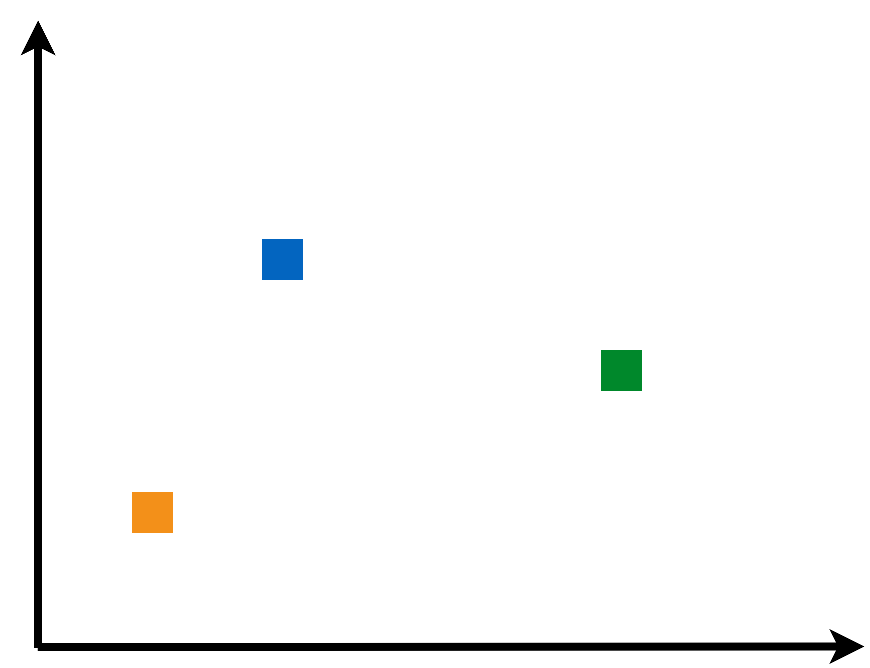

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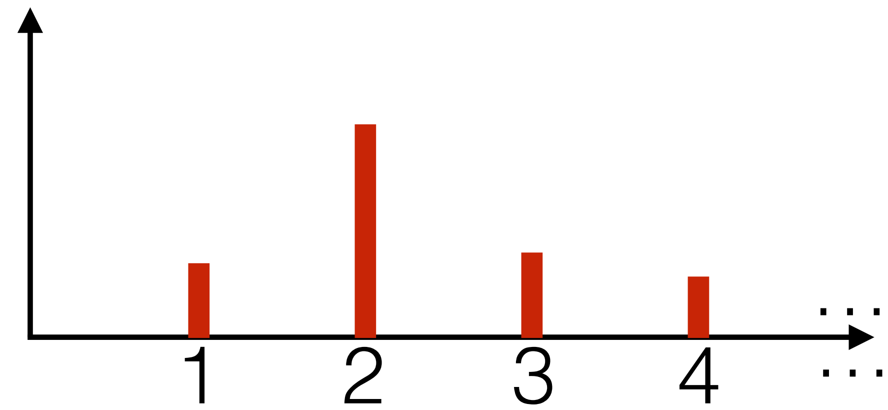

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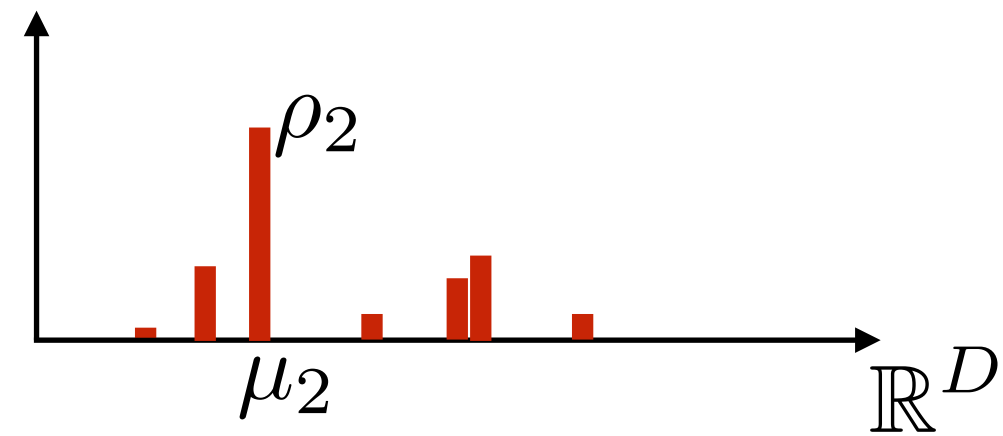

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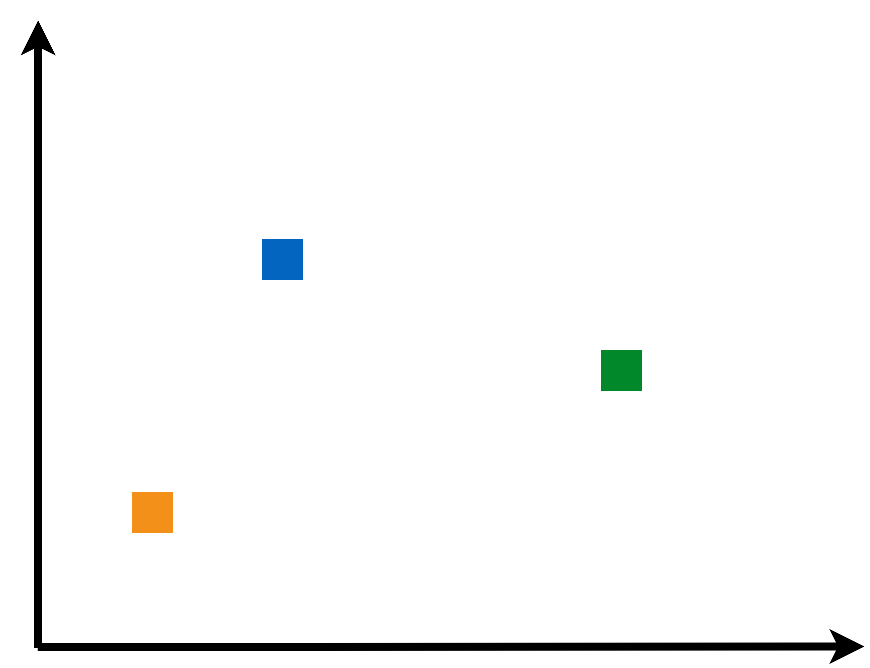

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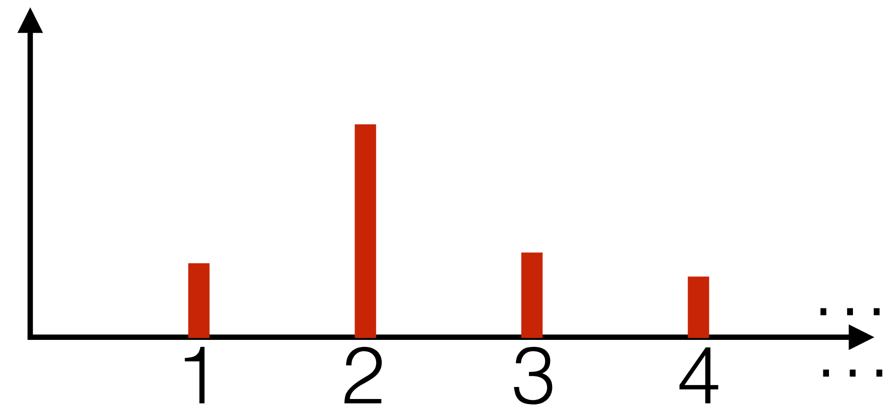

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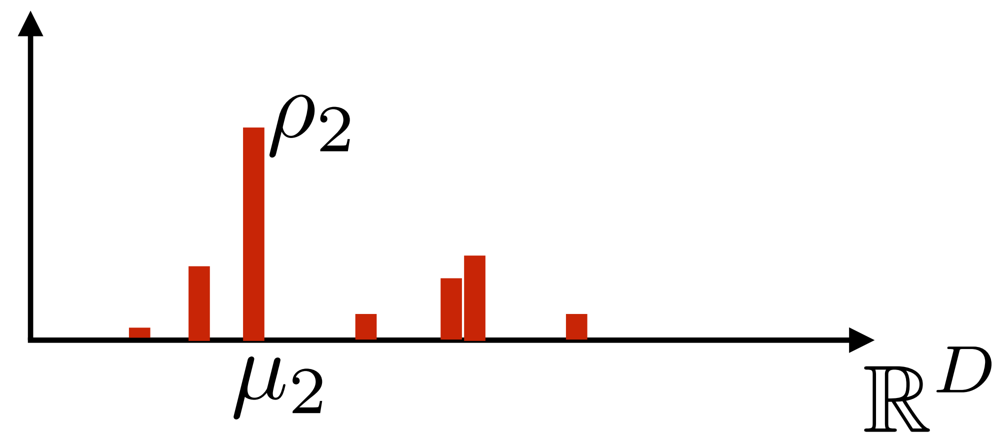

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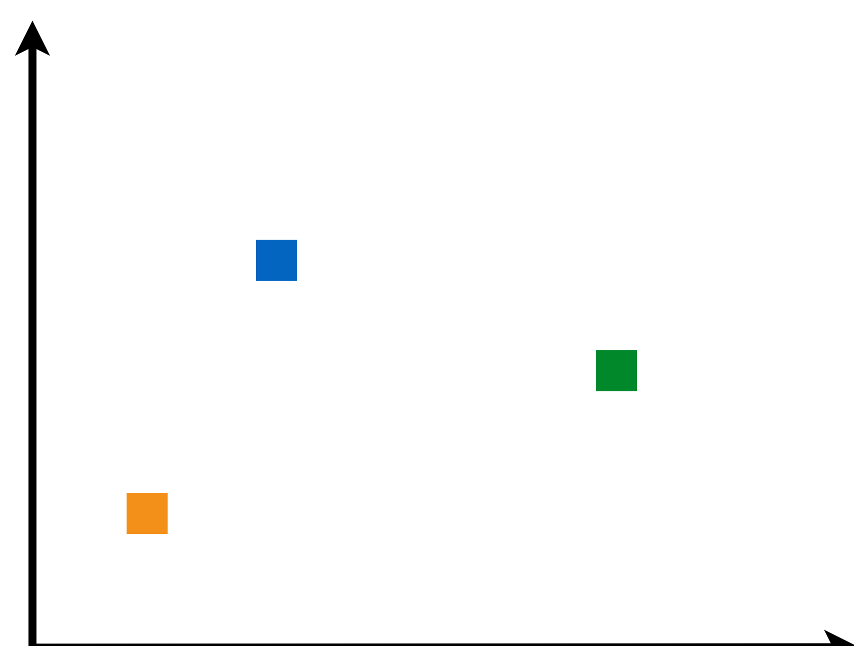

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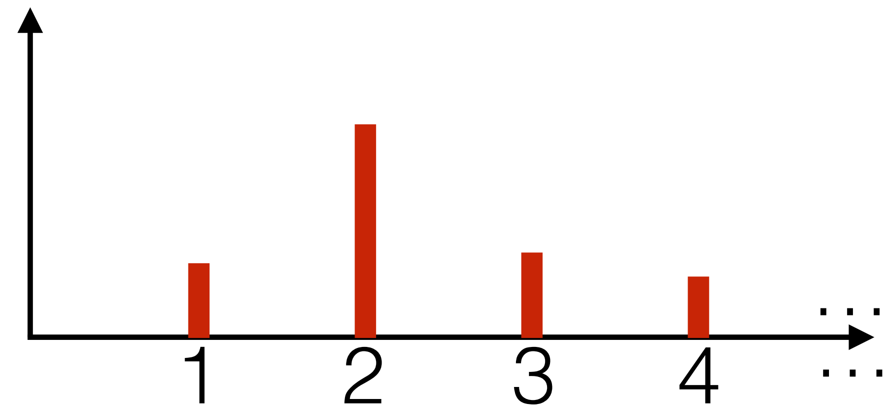

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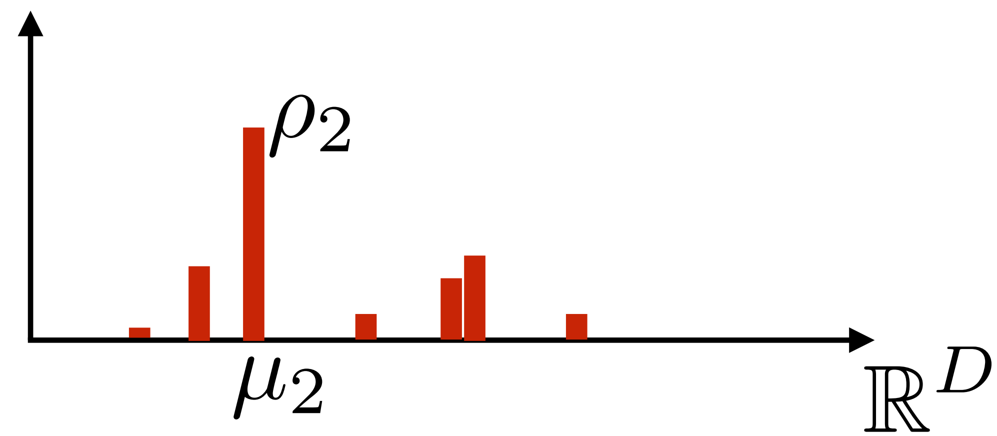

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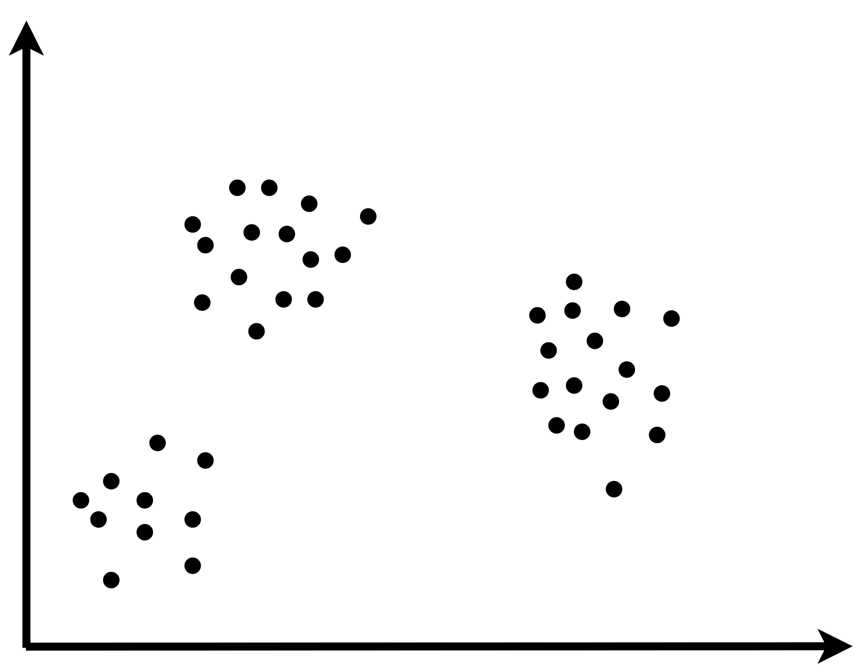

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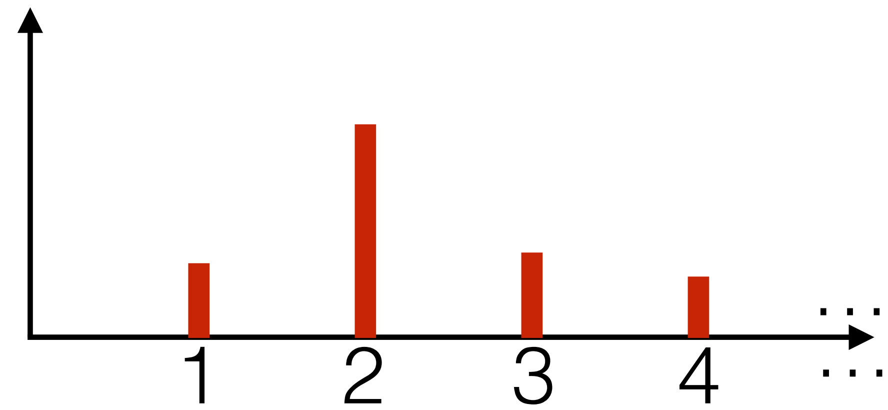

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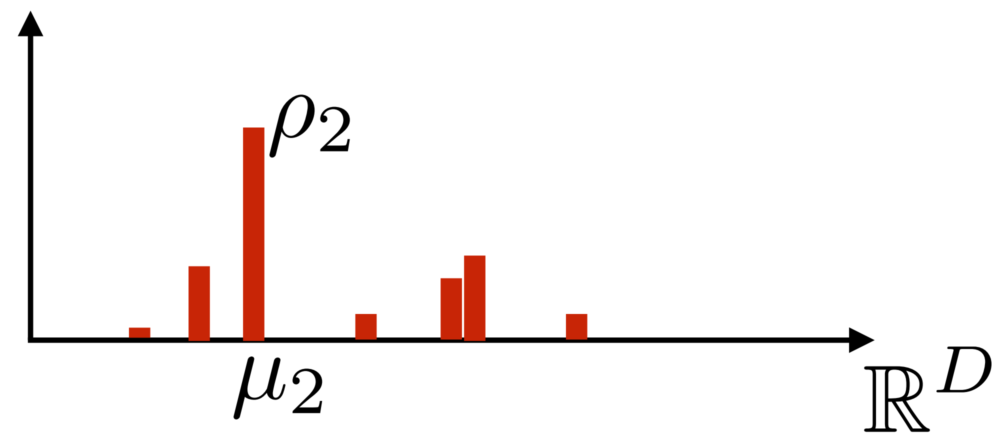

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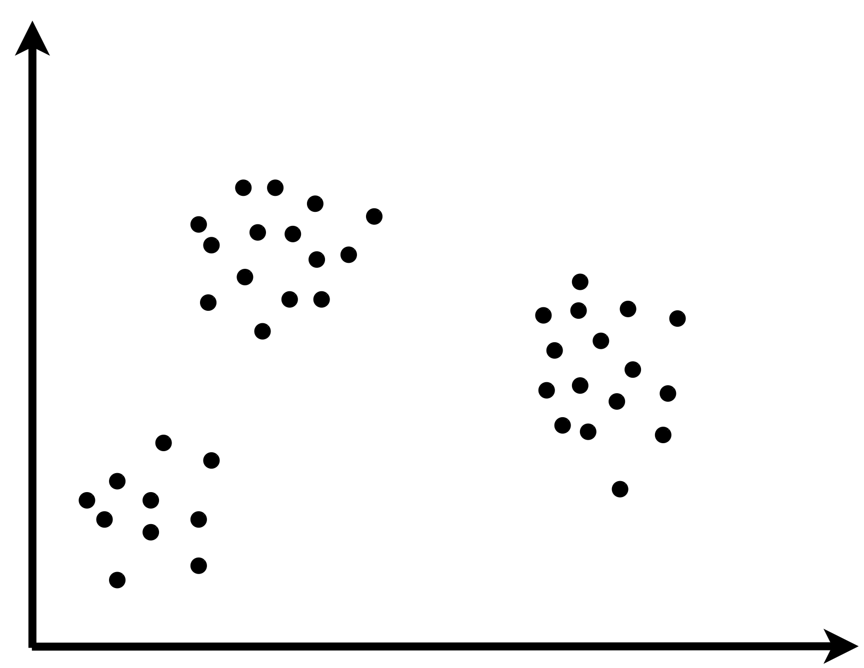

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[demo]

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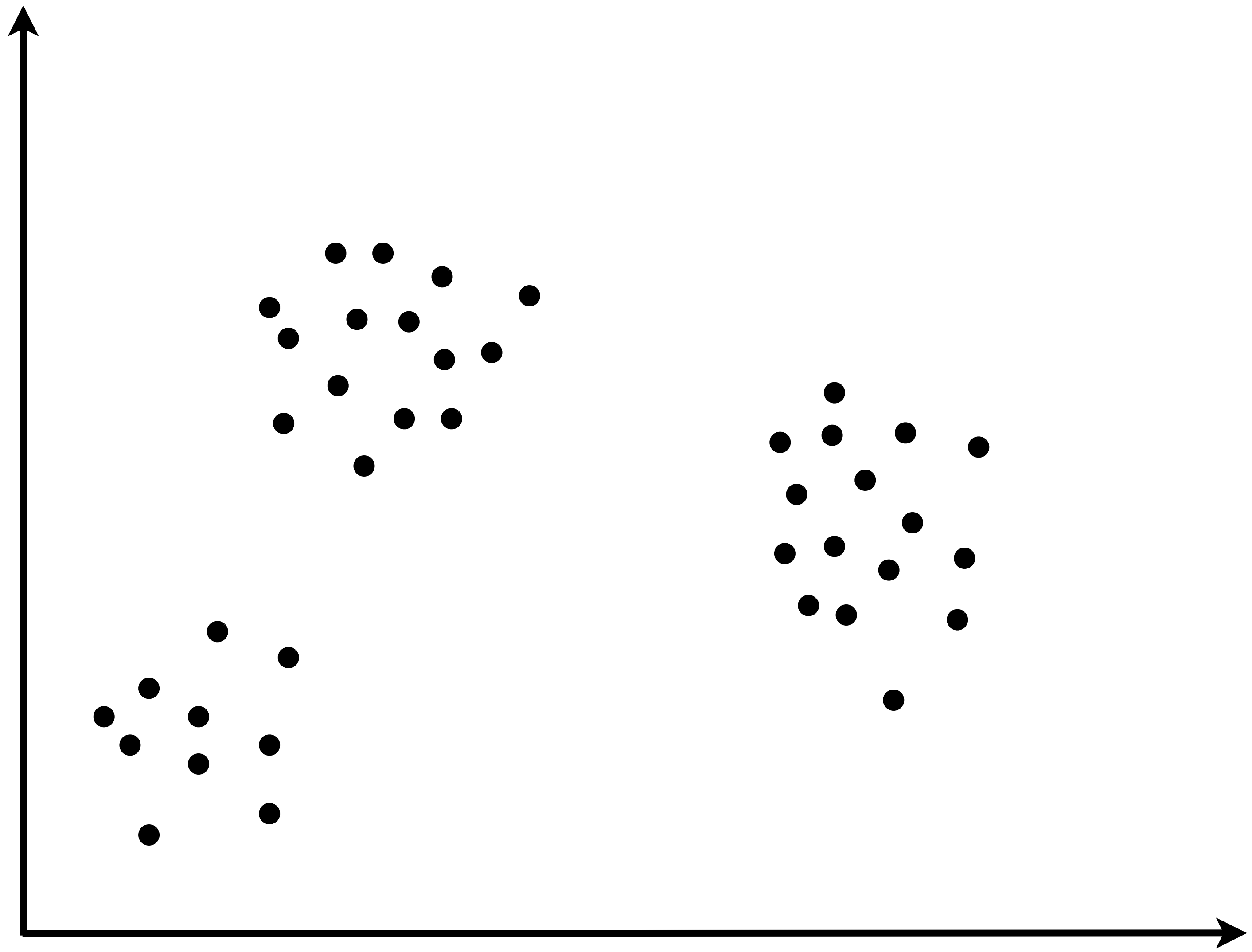
Clustering



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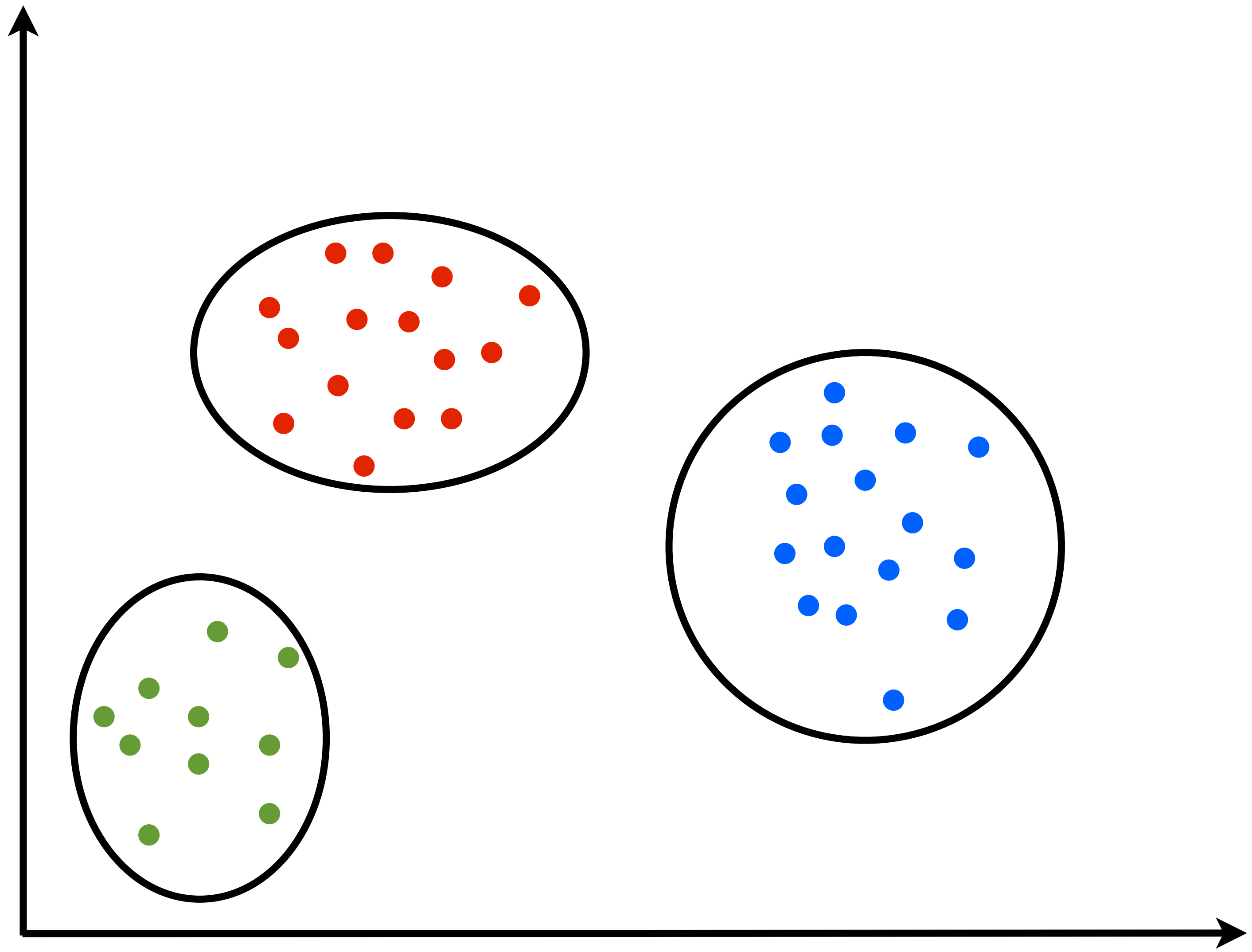
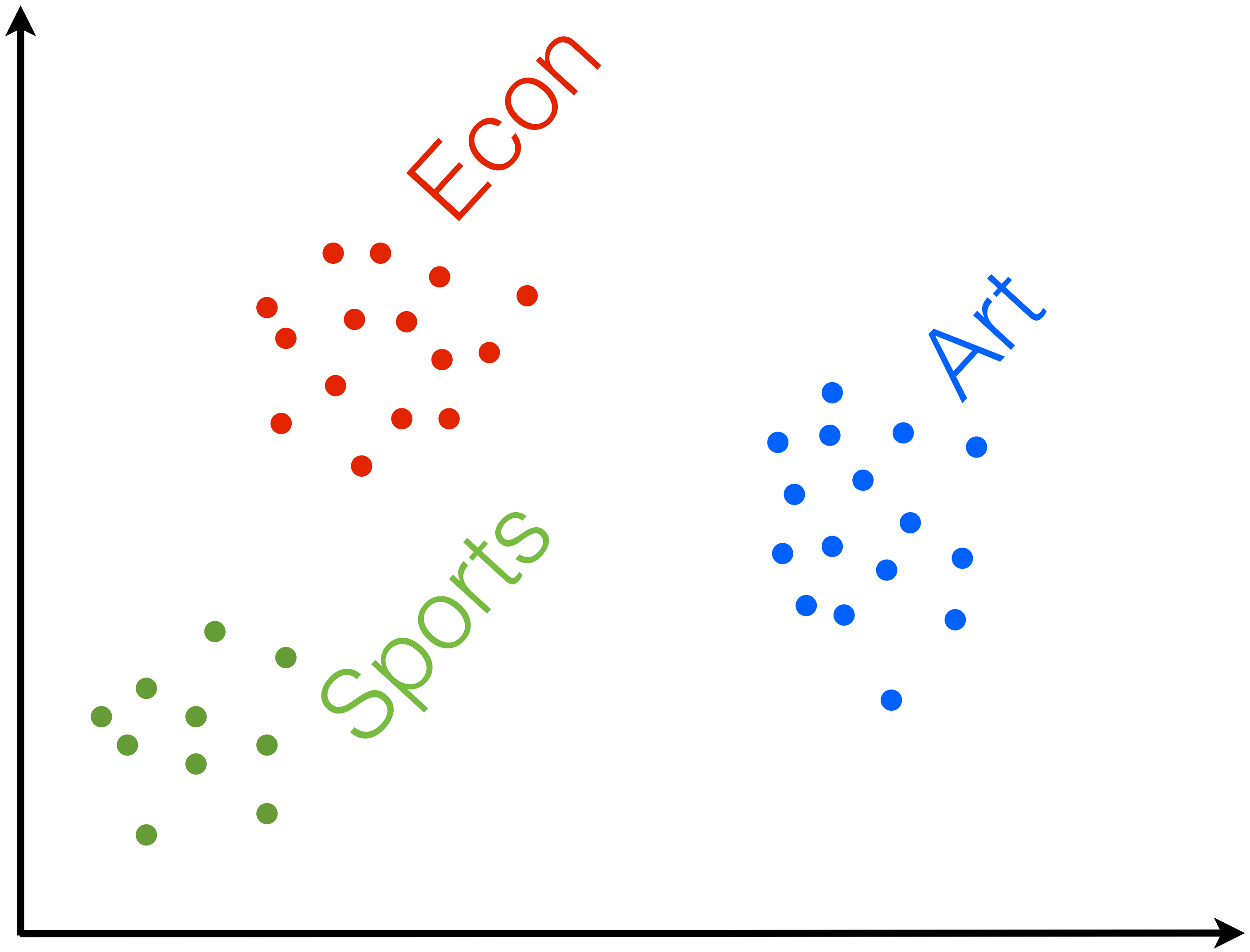


“Clusters”

Clustering



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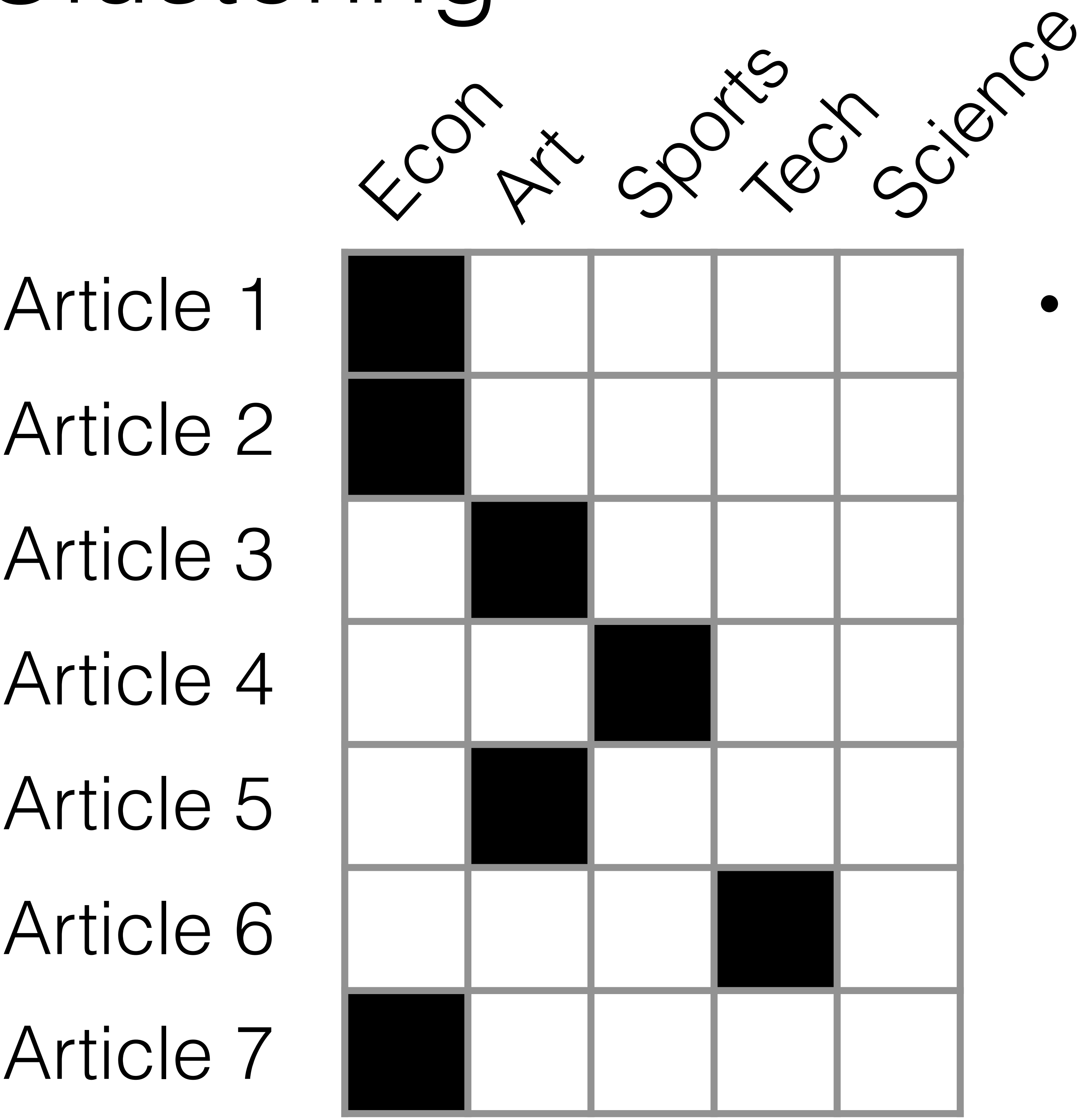
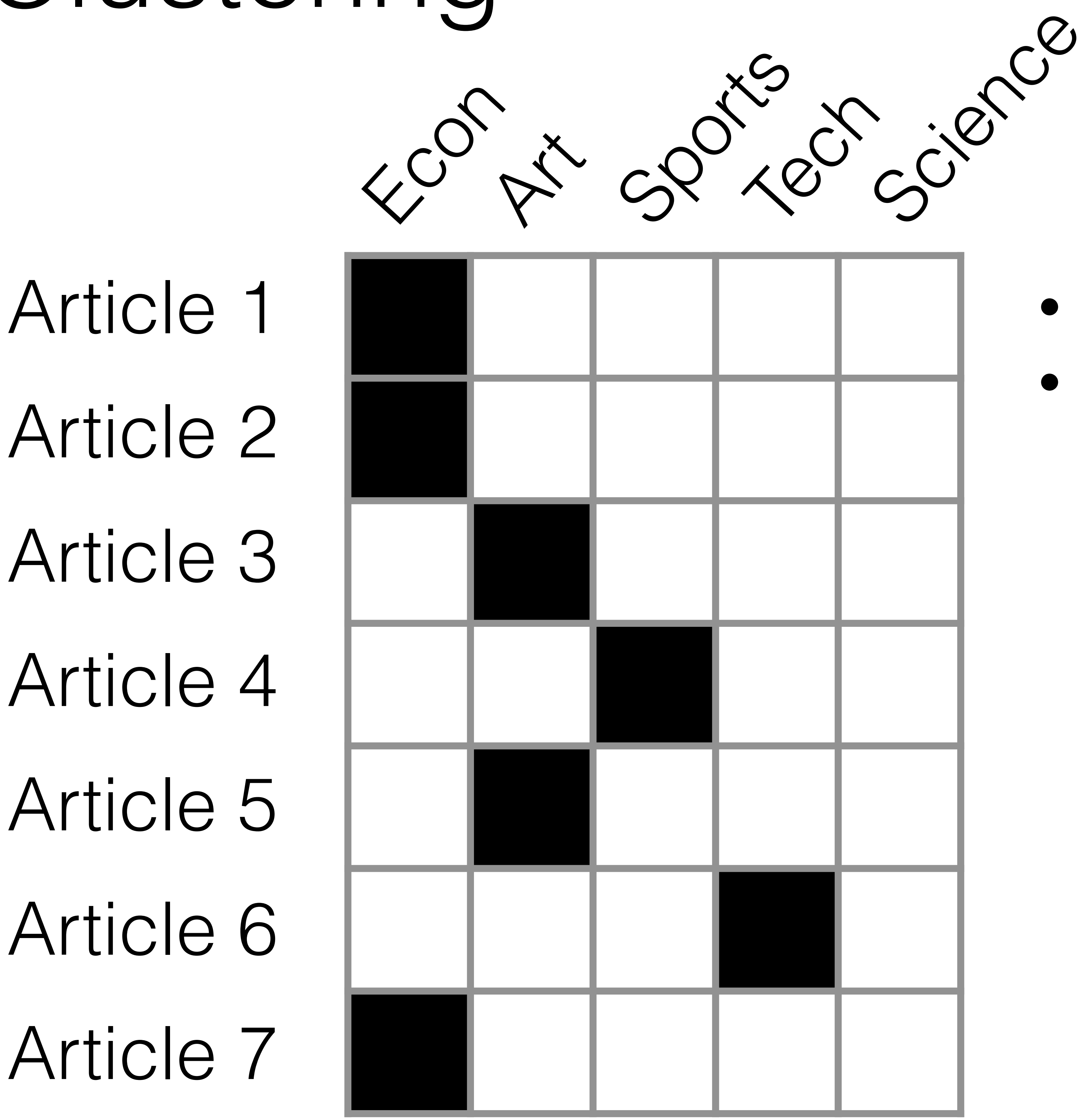


- Groups: clusters

Clustering



- Groups: clusters
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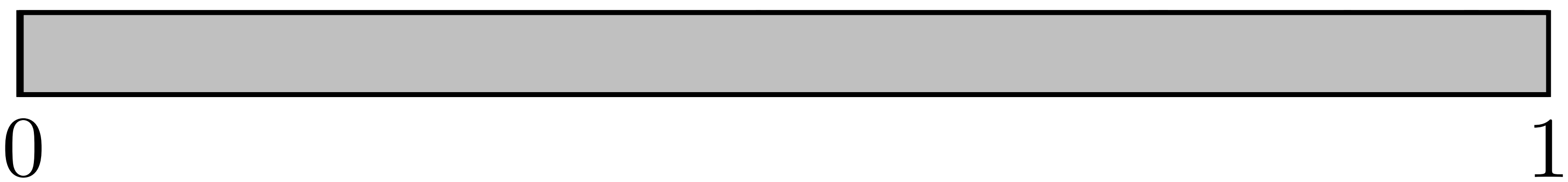
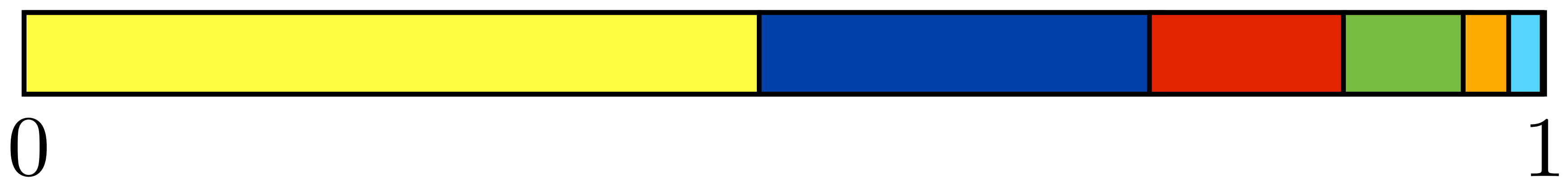


[Kingman 1978]

Clustering



[Kingman 1978]

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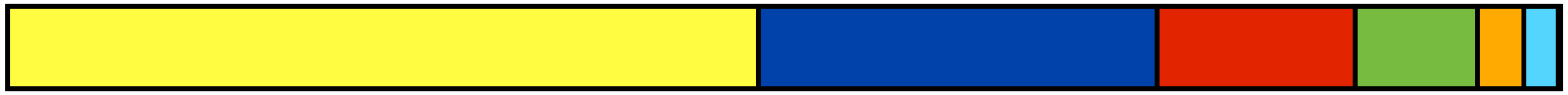

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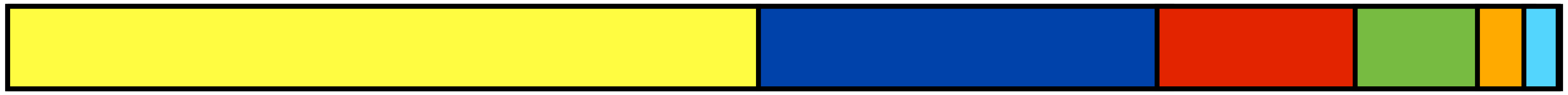
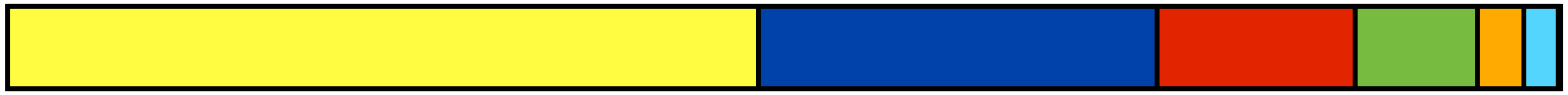


- Finite example: Dirichlet

[Kingman 1978]

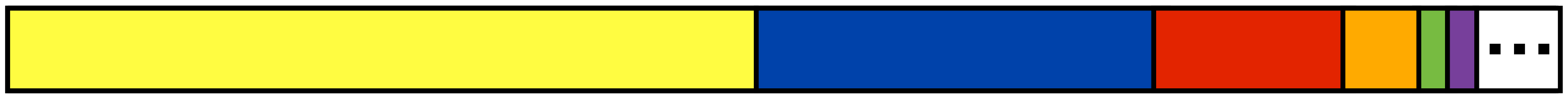
Clustering



- Finite example: Dirichlet
- Infinite example: GEM / DP

[Kingman 1978]

Clustering



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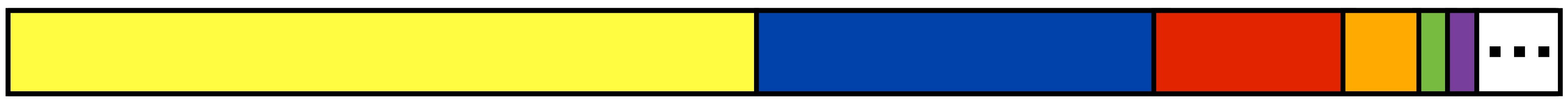

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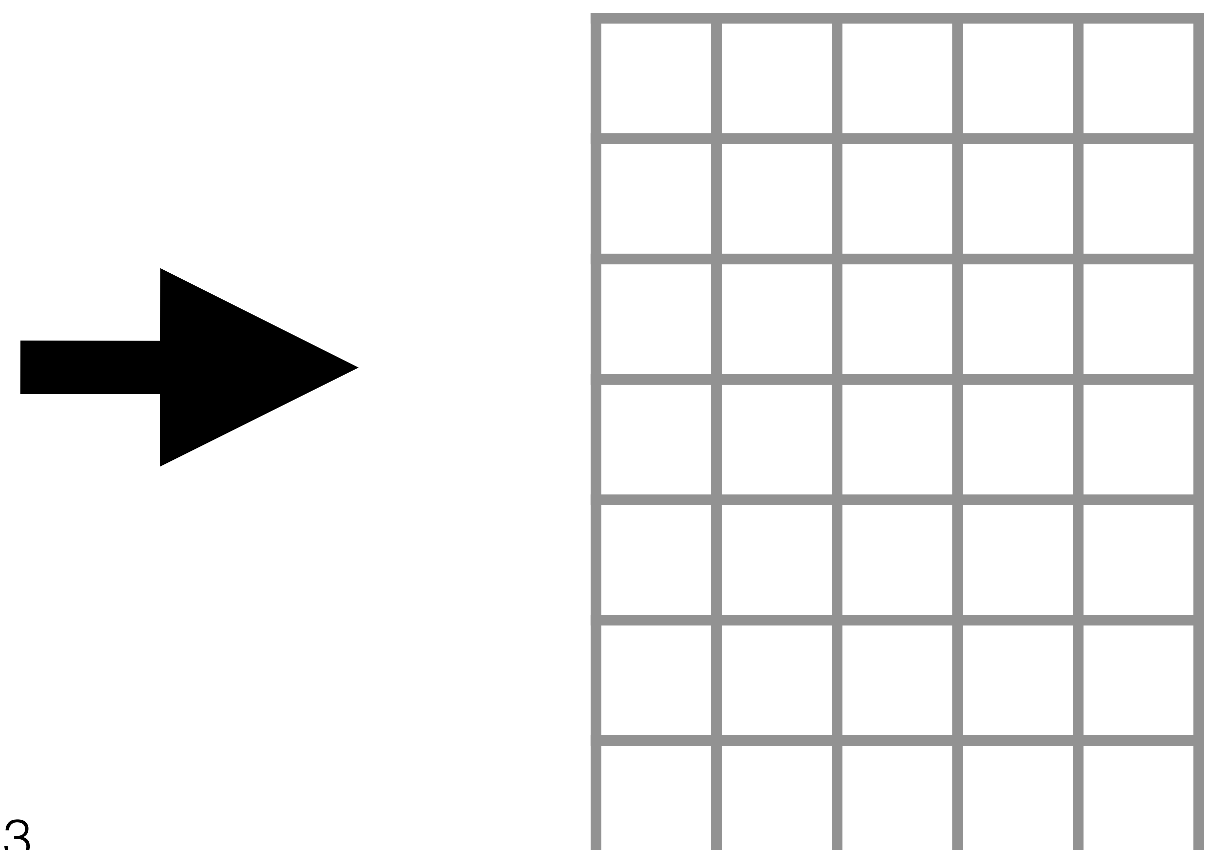

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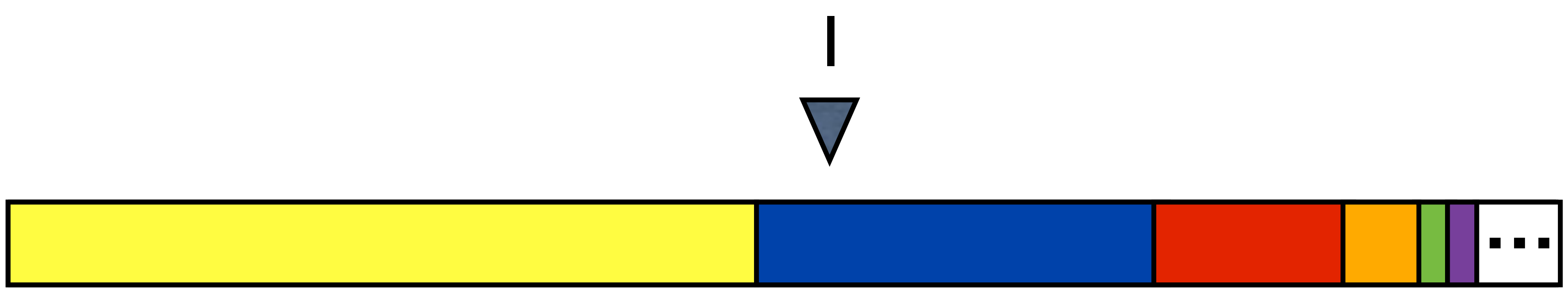

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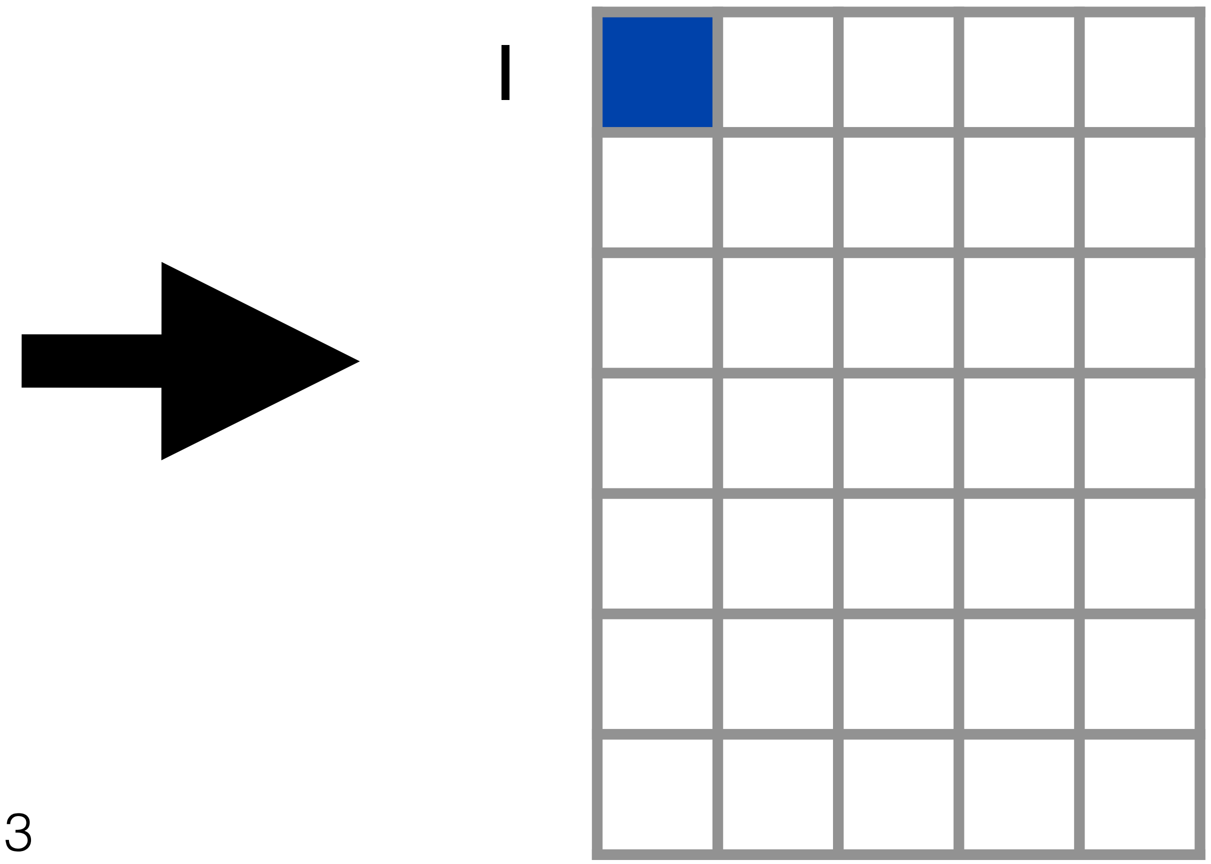

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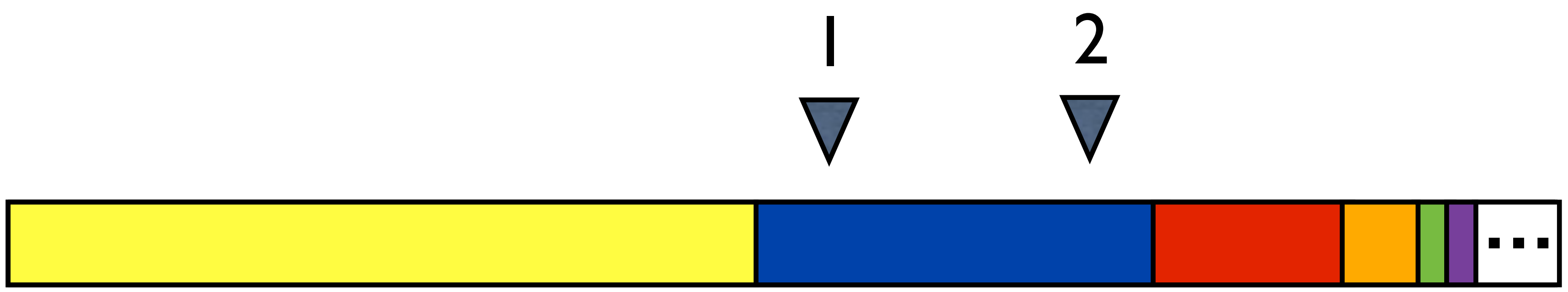

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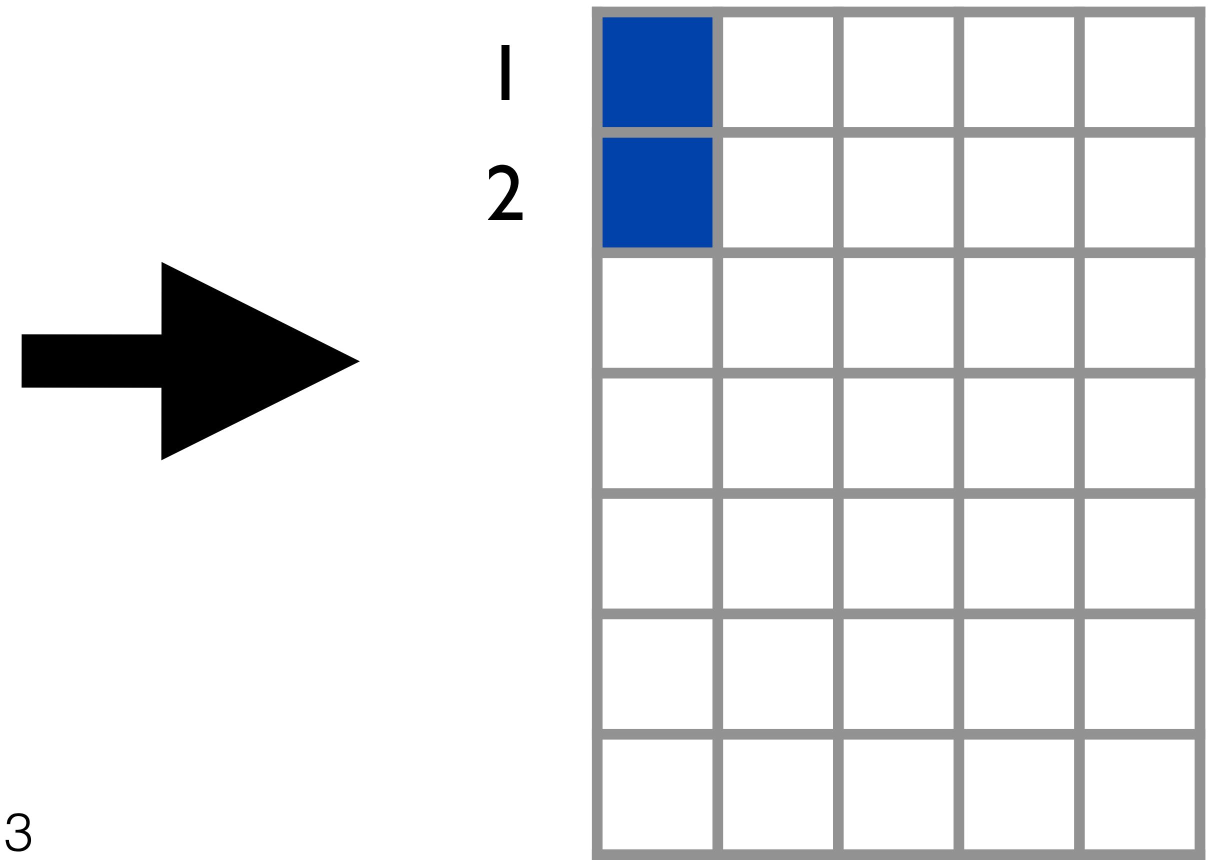

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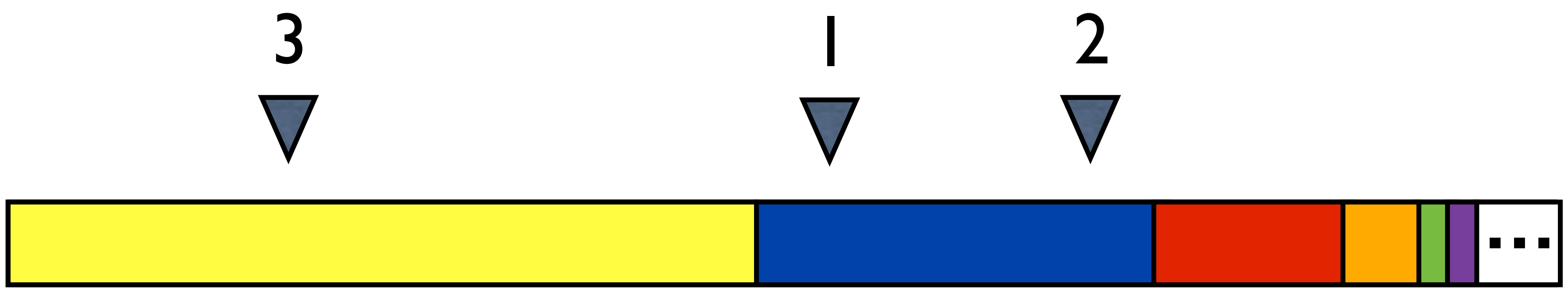

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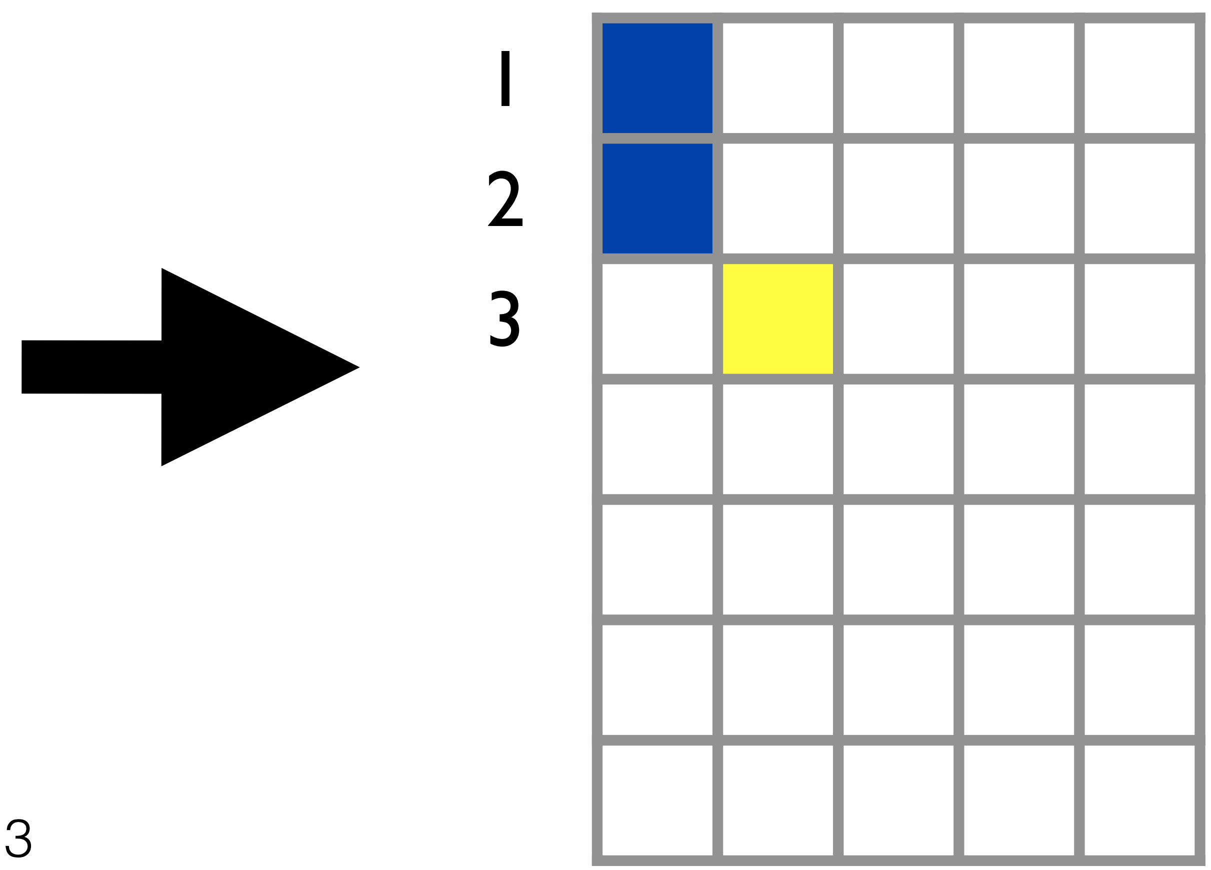

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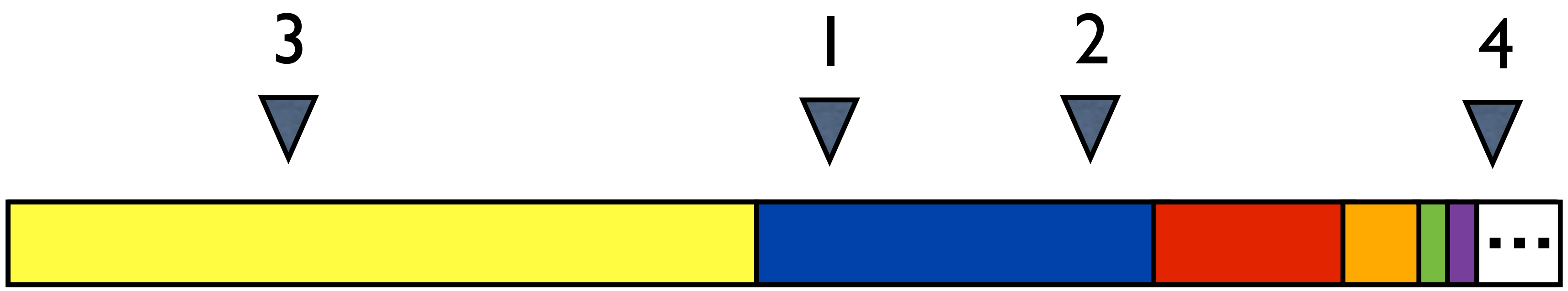

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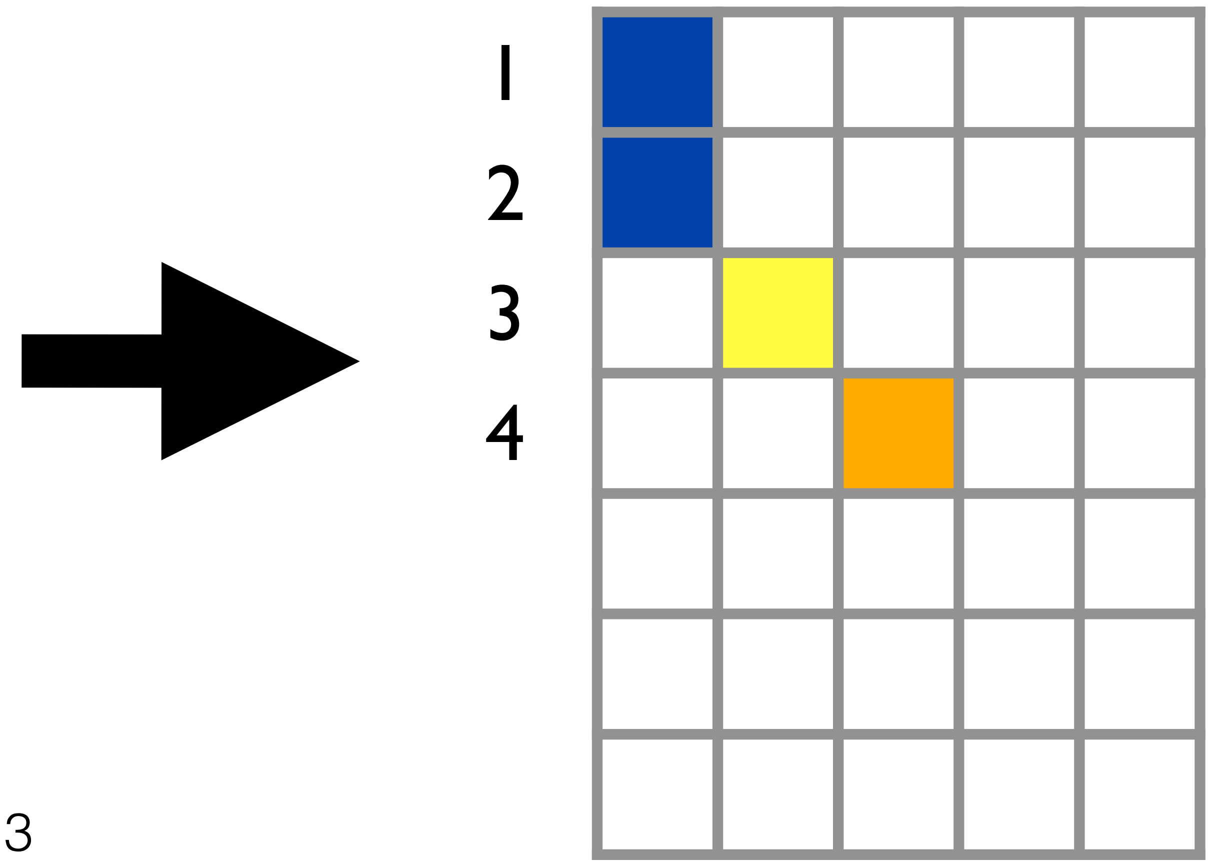

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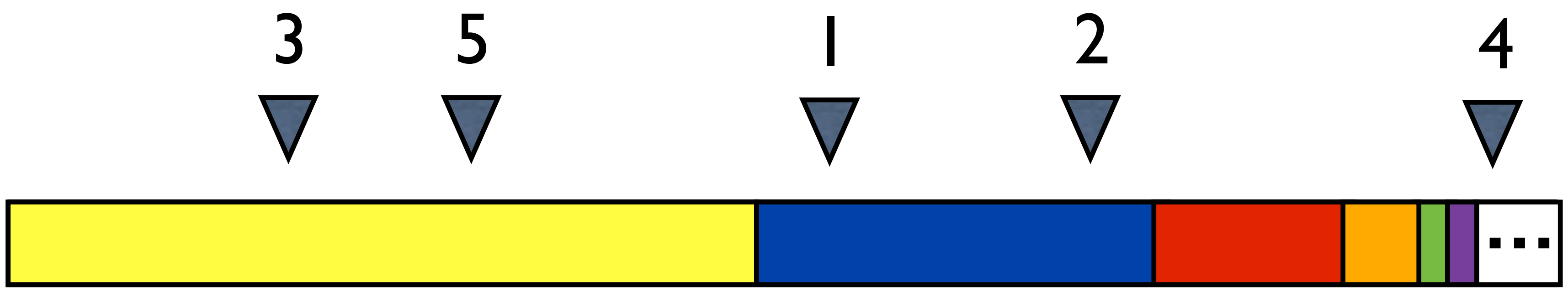

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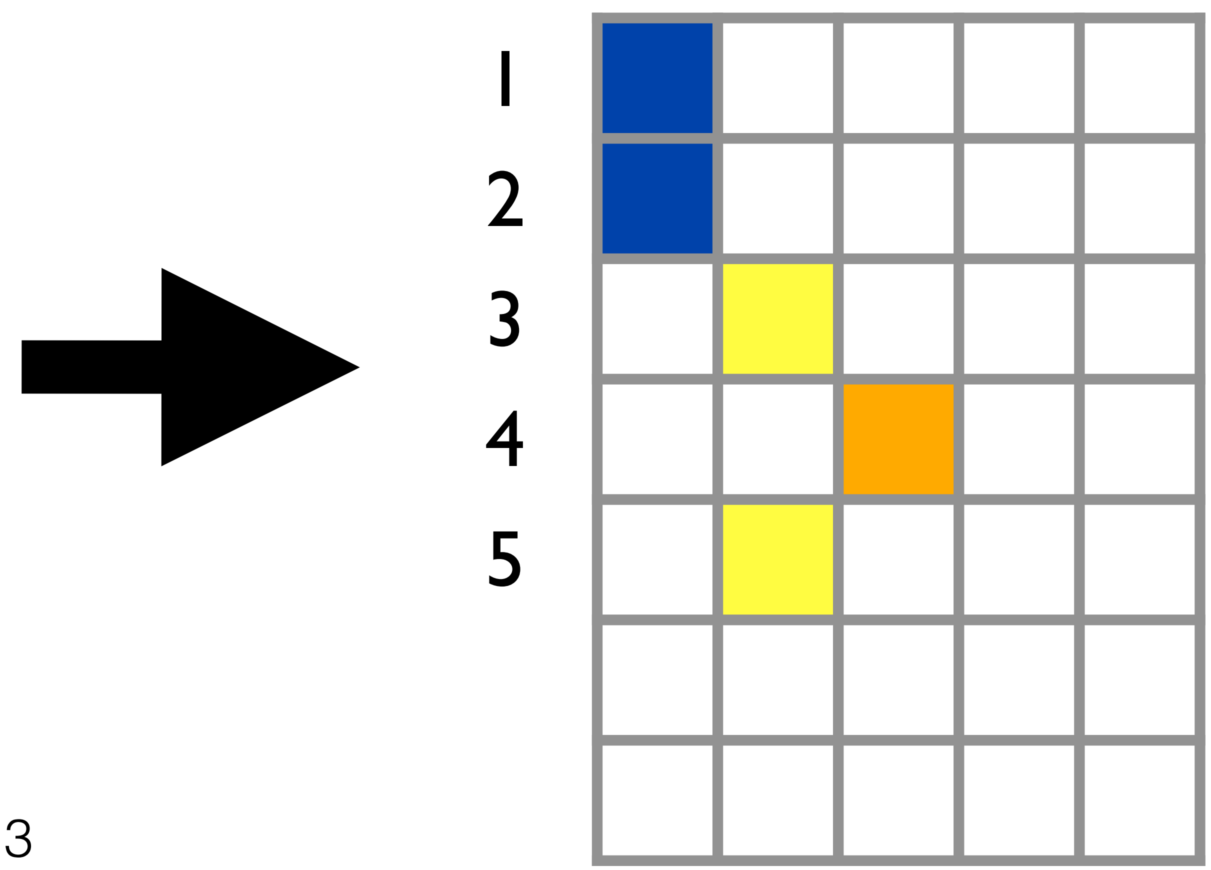

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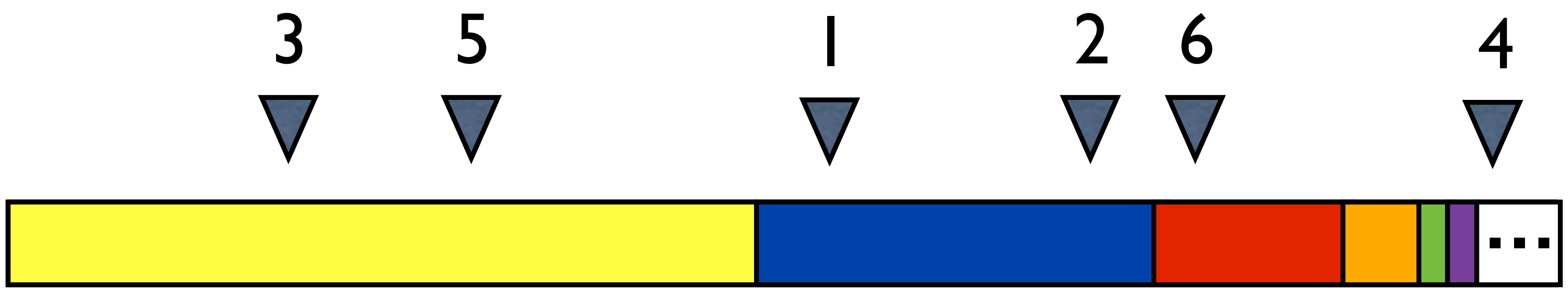

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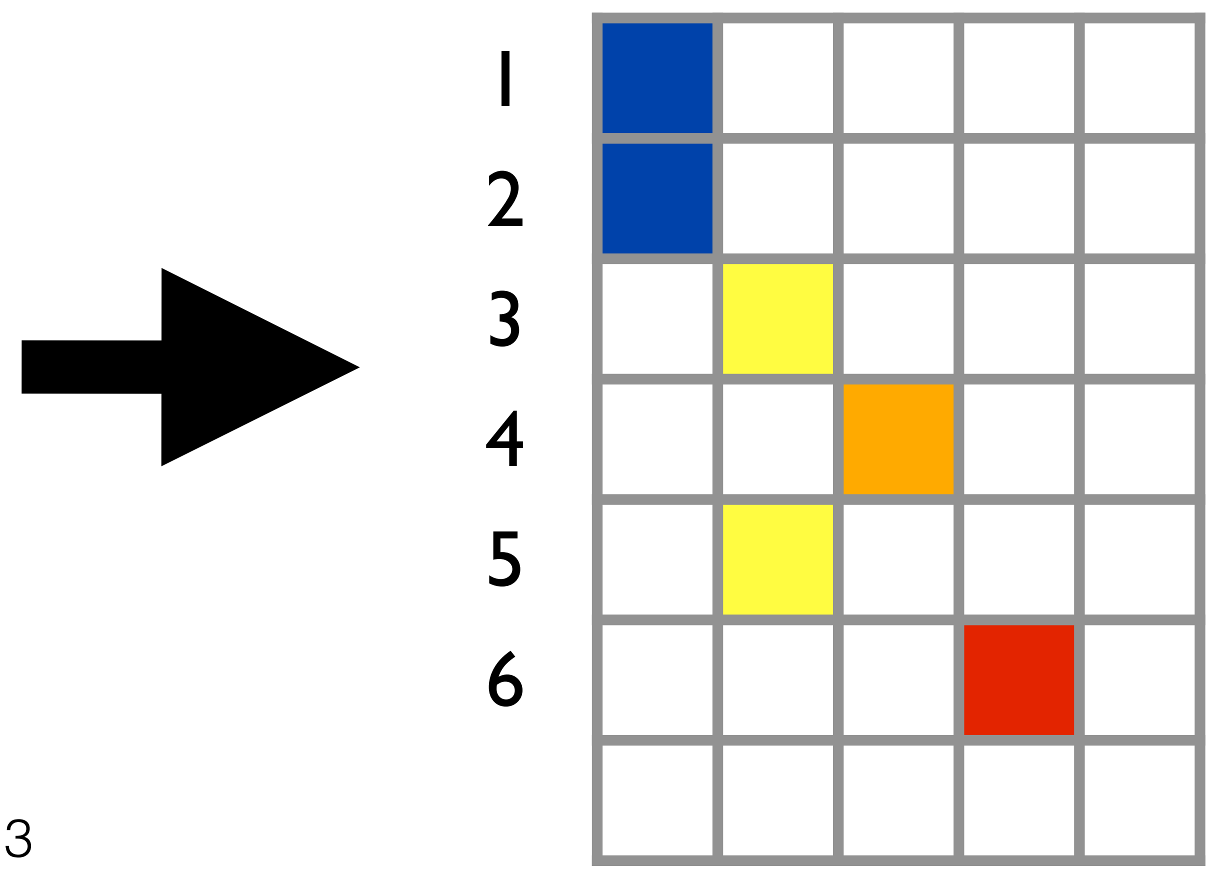

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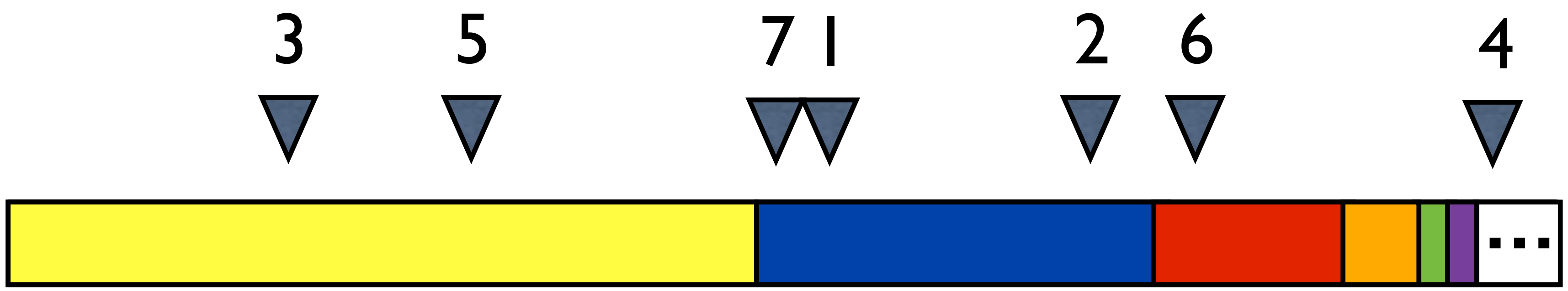

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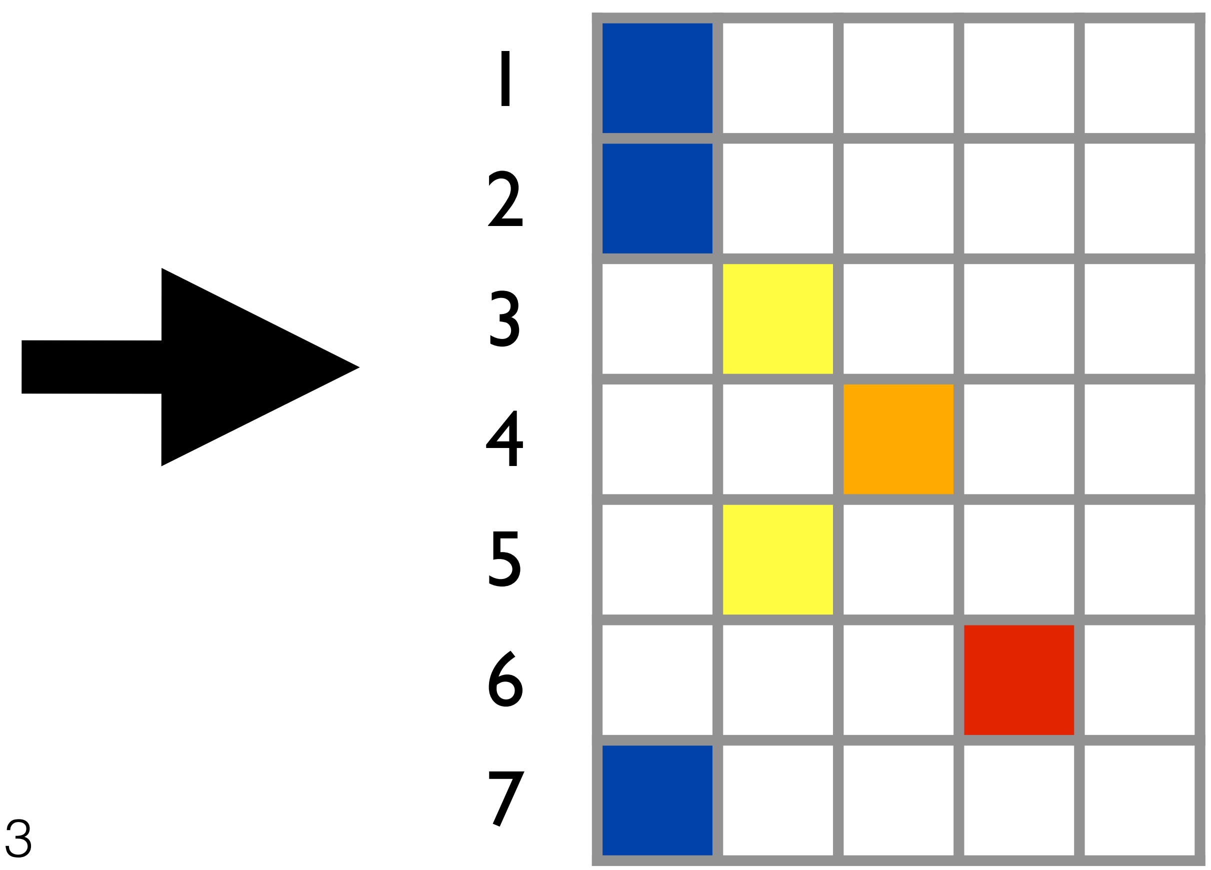

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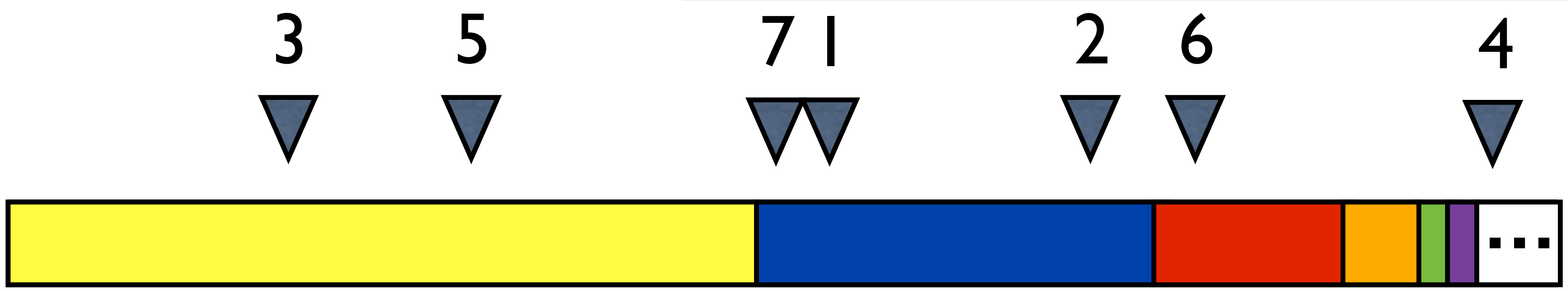
Clustering

Thm (Kingman). A clustering is exchangeable iff it has a Kingman paintbox representation



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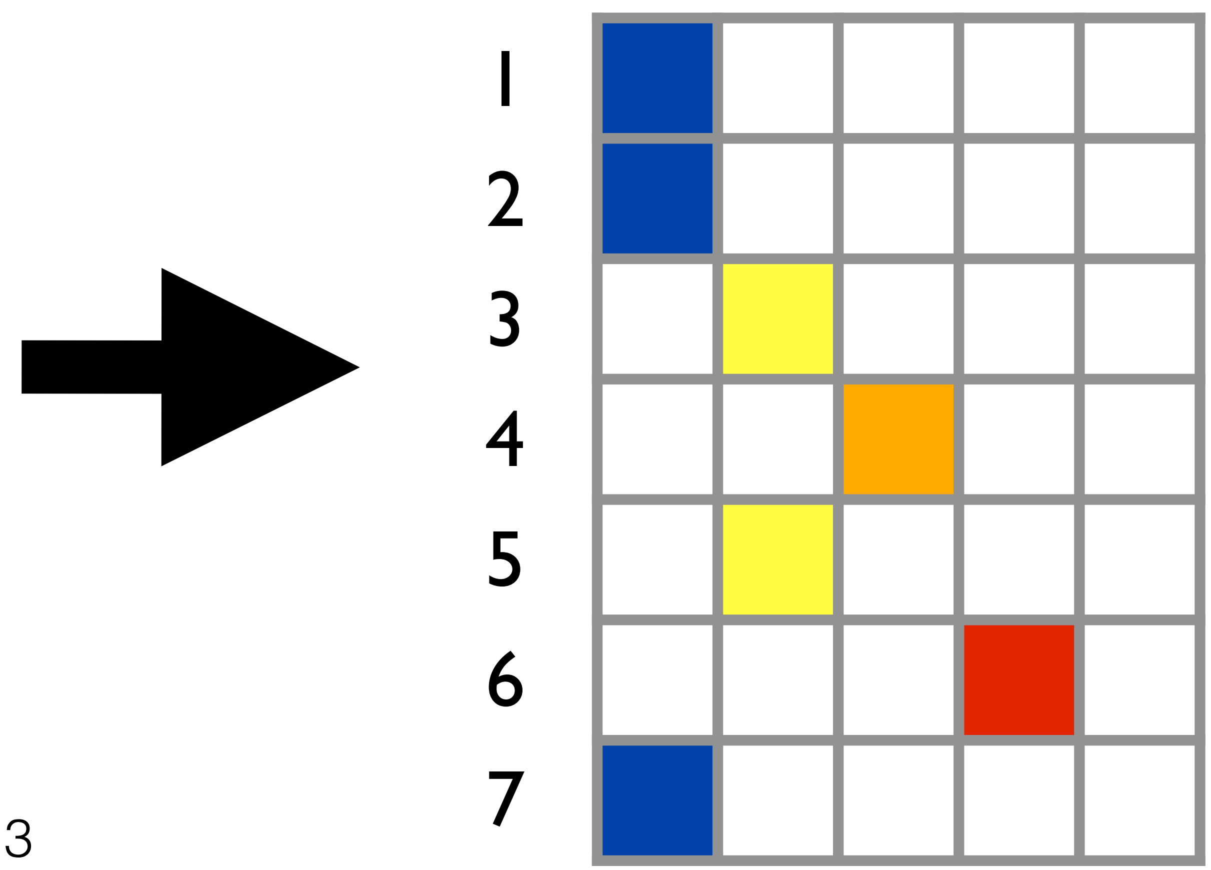

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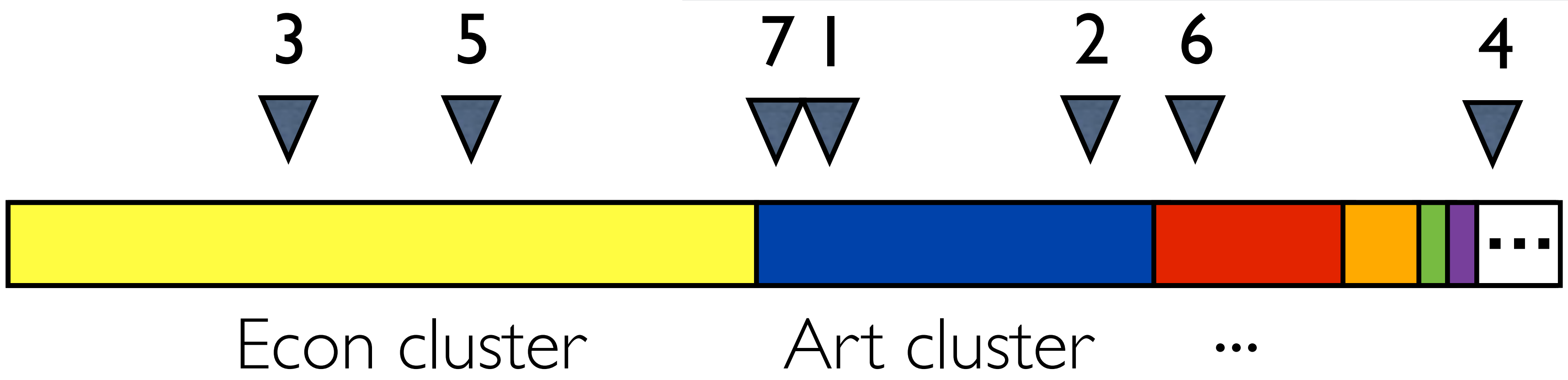

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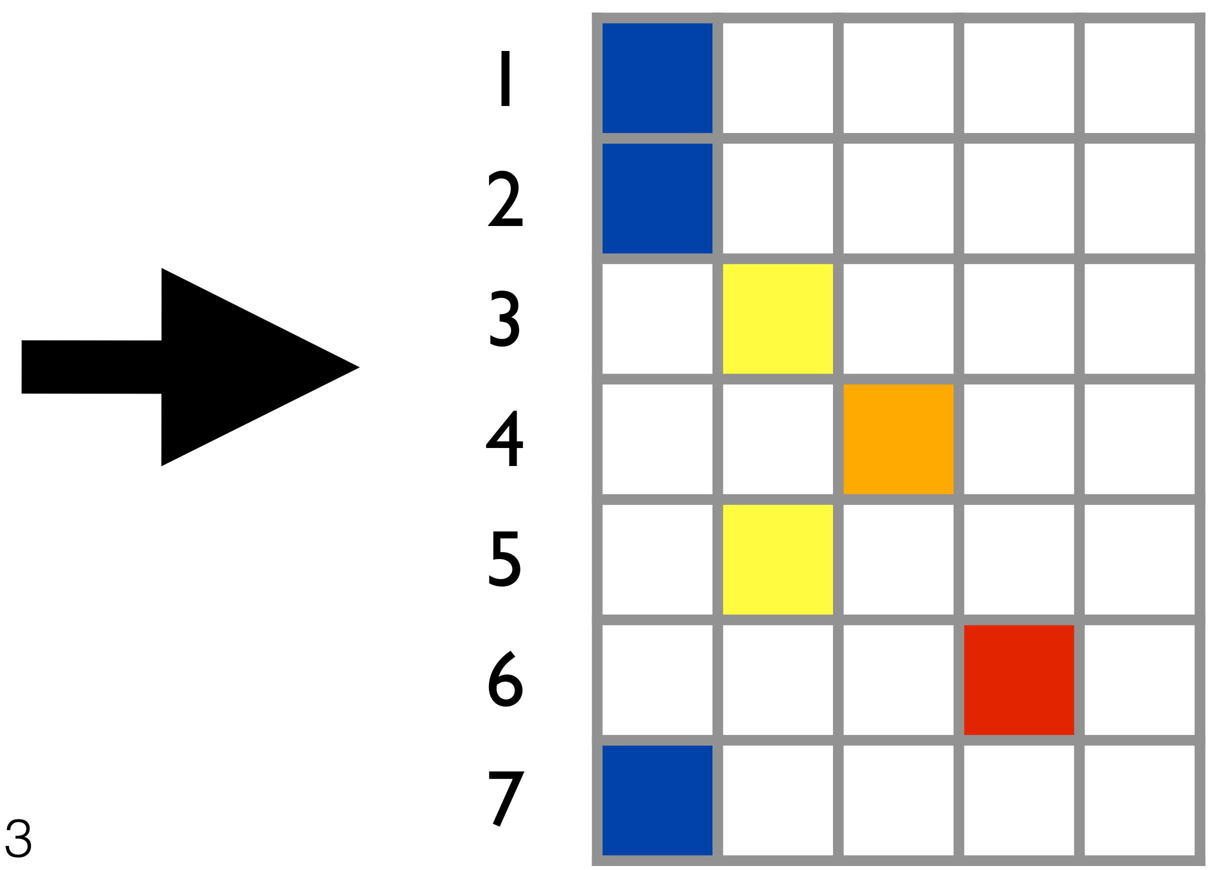

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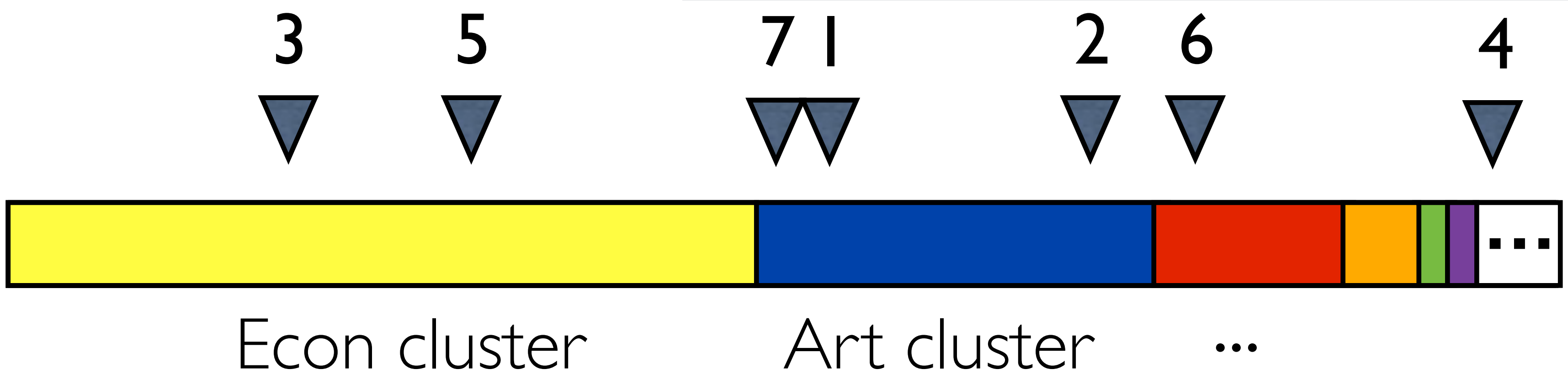

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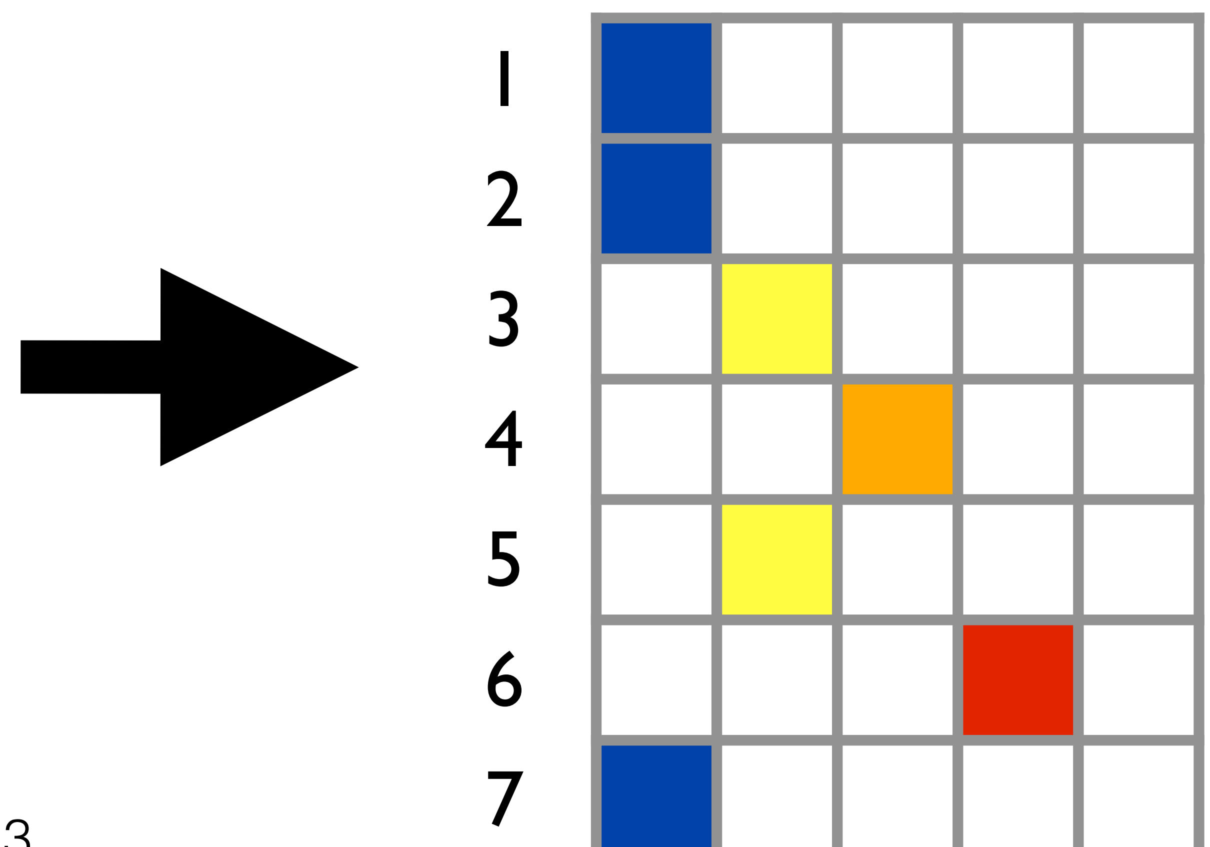

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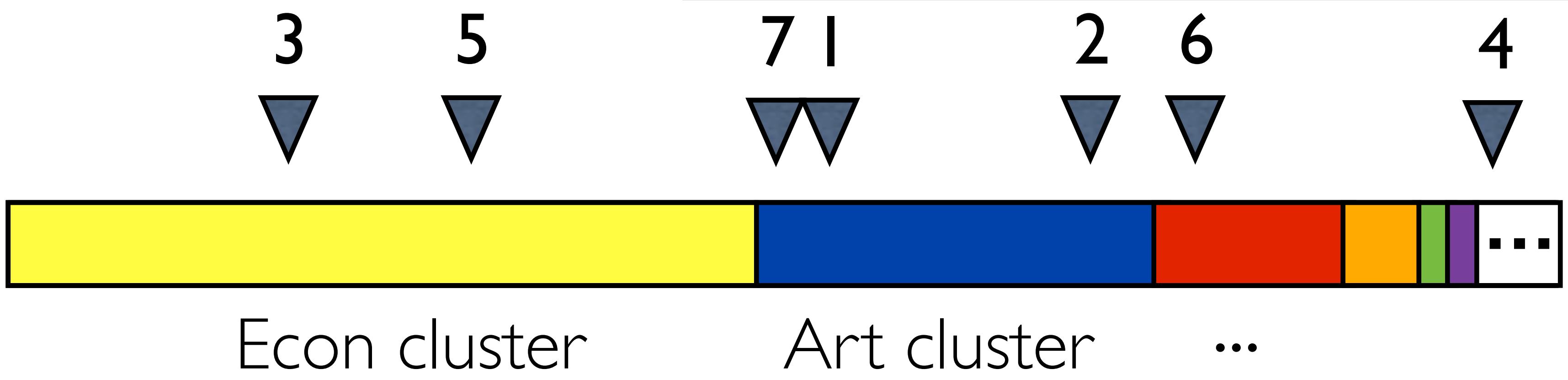

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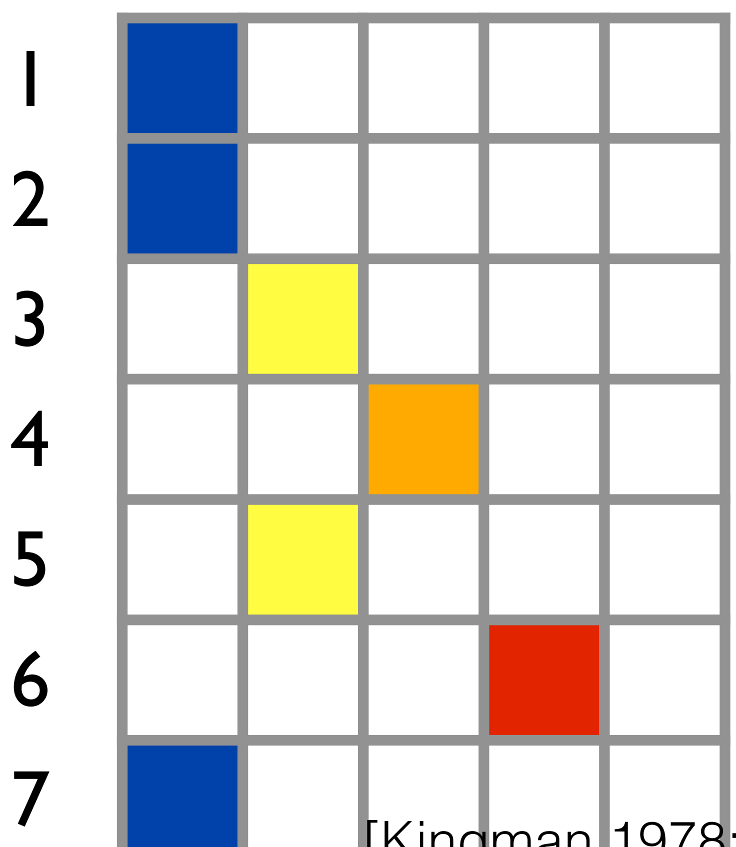
- Implications:



[Kingman 1978; Wallach et al 2010; Broderick, Steorts 2014; Miller et al 2015]

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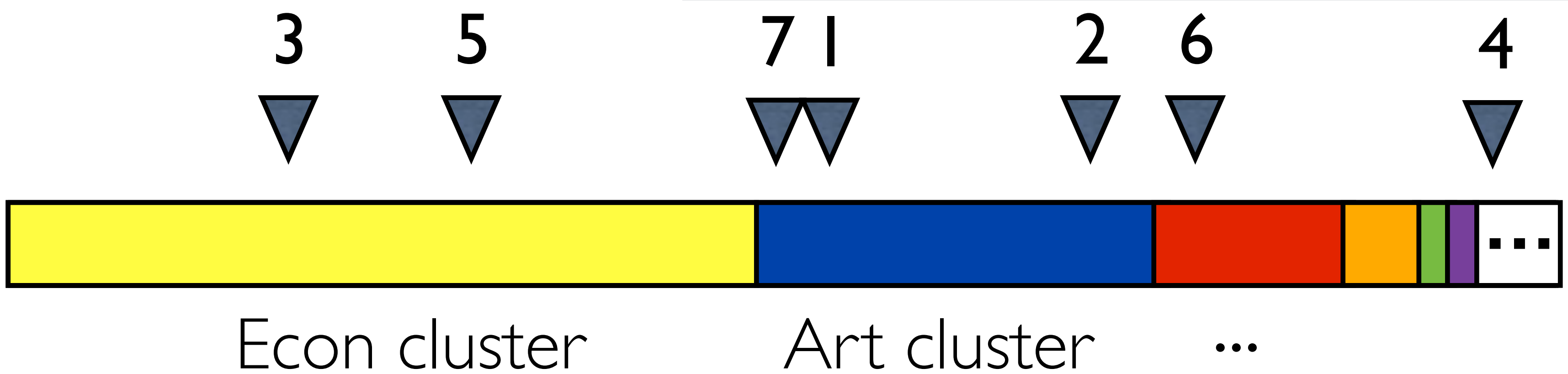

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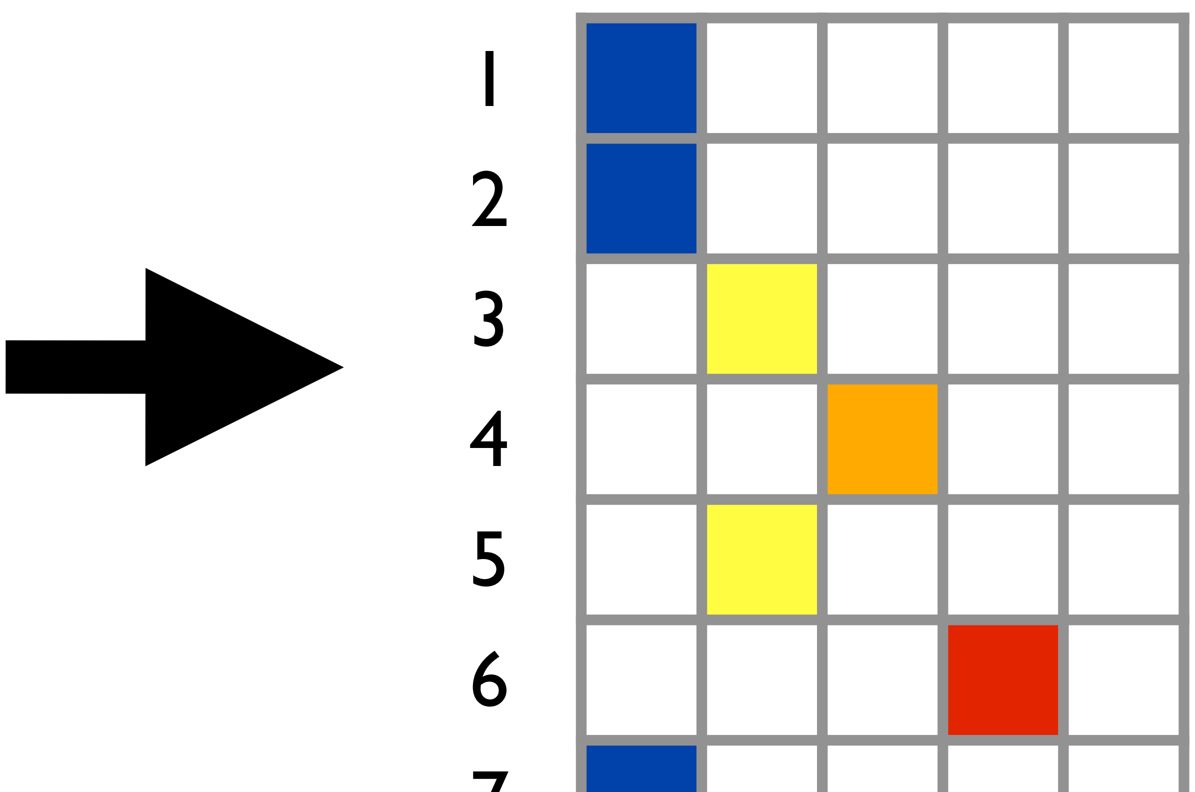
- Implications:
  - Cluster sizes grow linearly with total # data pts



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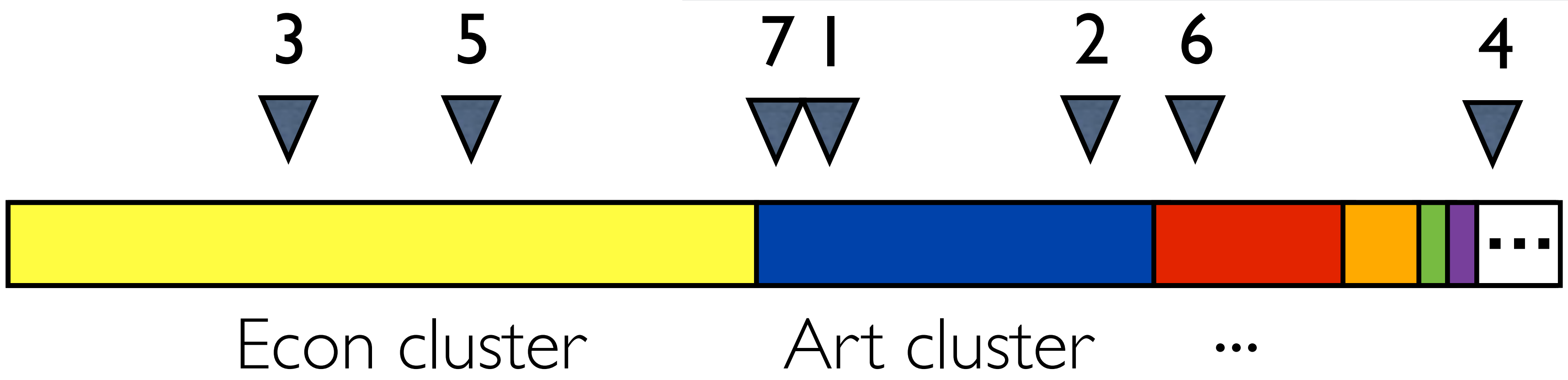

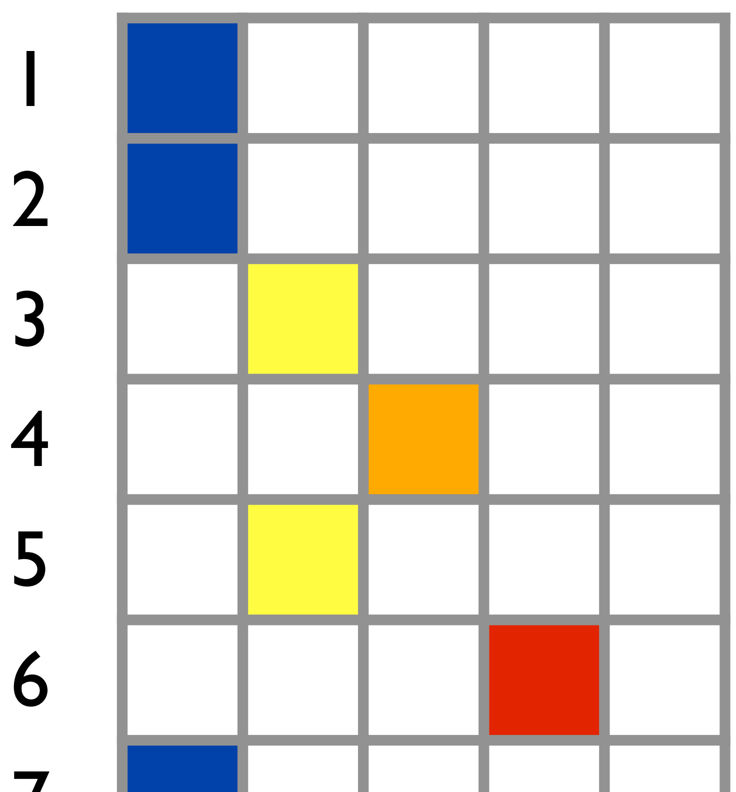

- Finite example: Dirichlet
- Infinite example: GEM / DP
- Implications:
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[Kingman 1978; Wallach et al 2010; Broderick, Steorts 2014; Miller et al 2015; Campbell et al 2019]

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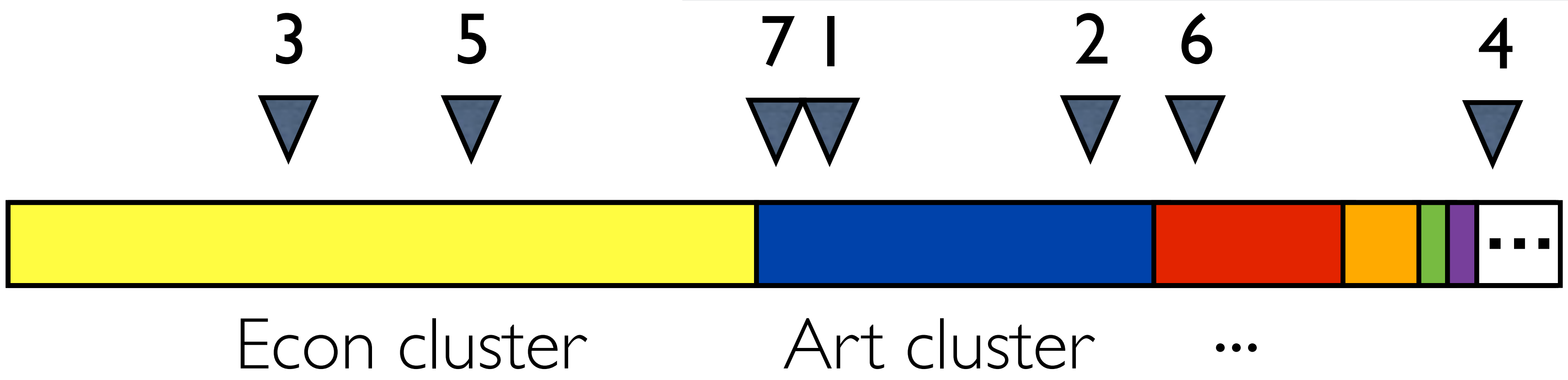

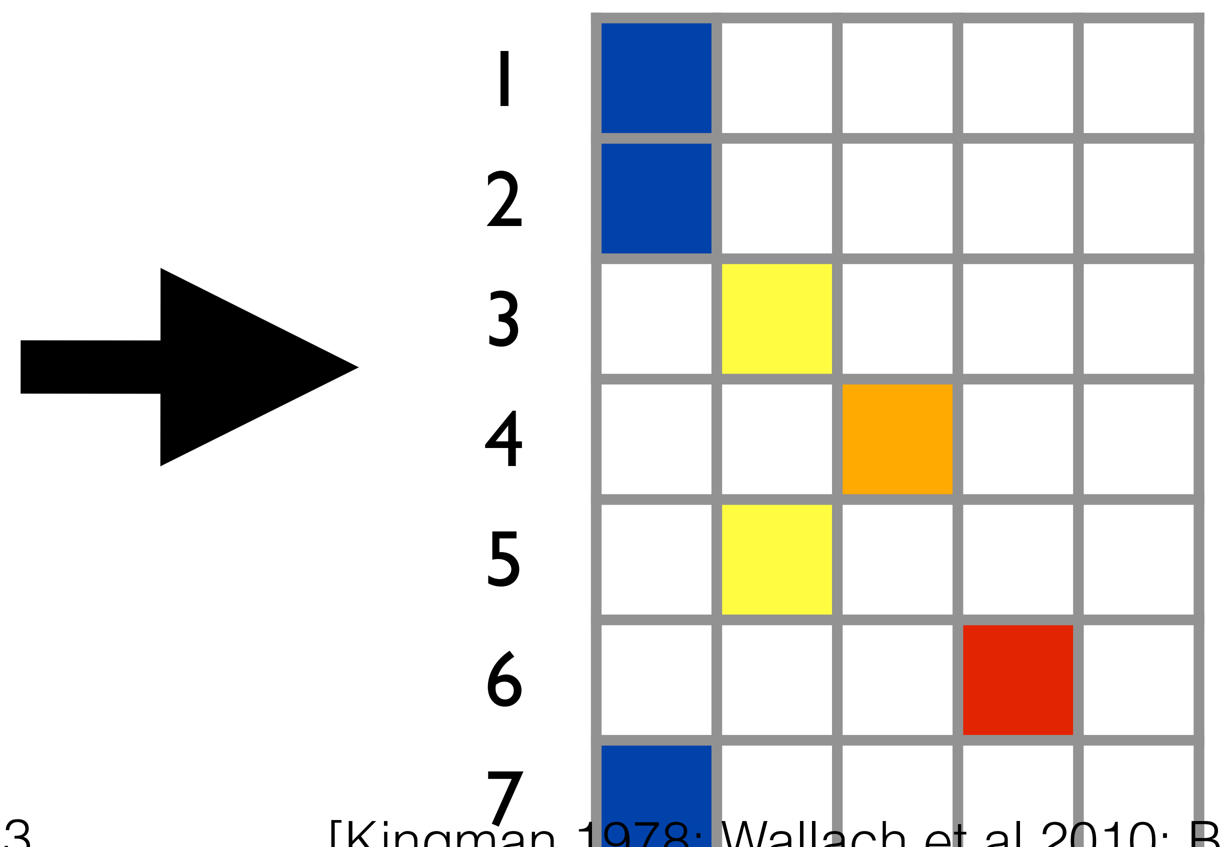

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  - See Campbell Tues 1:45 talk “Local exchangeability”

[Kingman 1978; Wallach et al 2010; Broderick, Steorts 2014; Miller et al 2015; Campbell et al 2019]

Roadmap

- Example problem: clustering
- Example NPBayes model: Dirichlet process (DP)
- De Finetti for clustering: Kingman Paintbox
- De Finetti for networks/graphs

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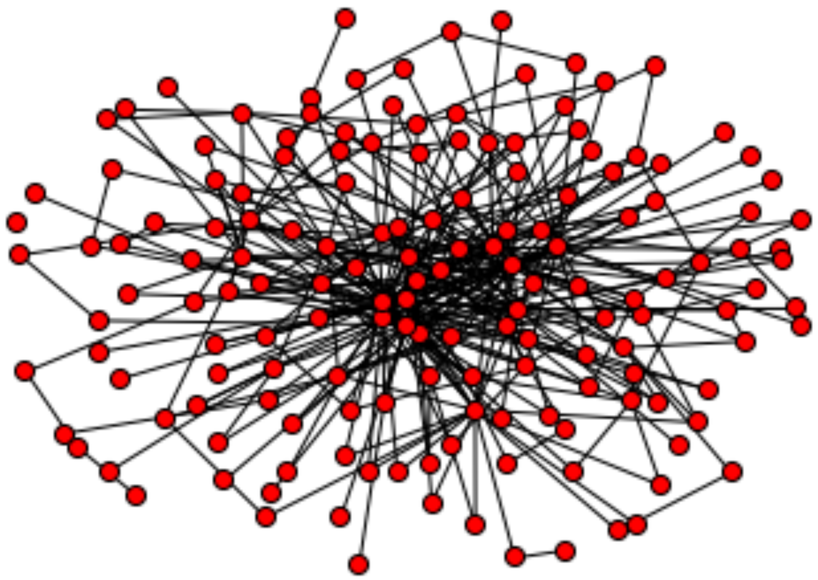
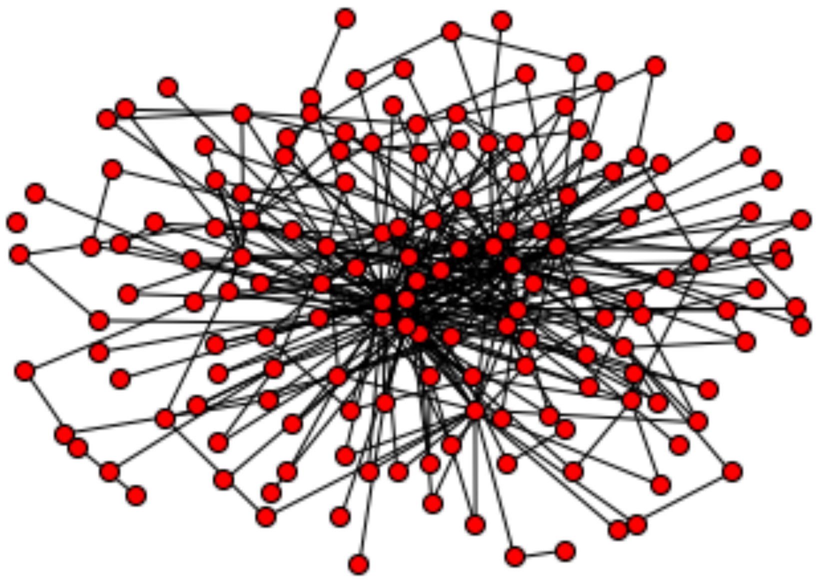


[Holland et al 1983; Kemp et al 2006; Xu et al 2007; Airoldi et al 2008; Lloyd et al 2012]



social: *Facebook, Twitter, email*
biological: *ecological, protein, gene*
transportation: *roads, railways*

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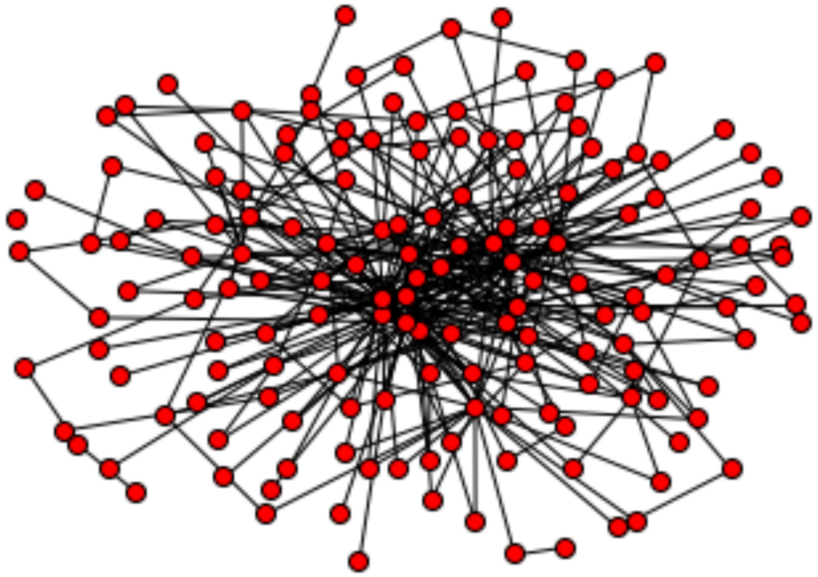
Probabilistic models for graphs

$p($  $)$

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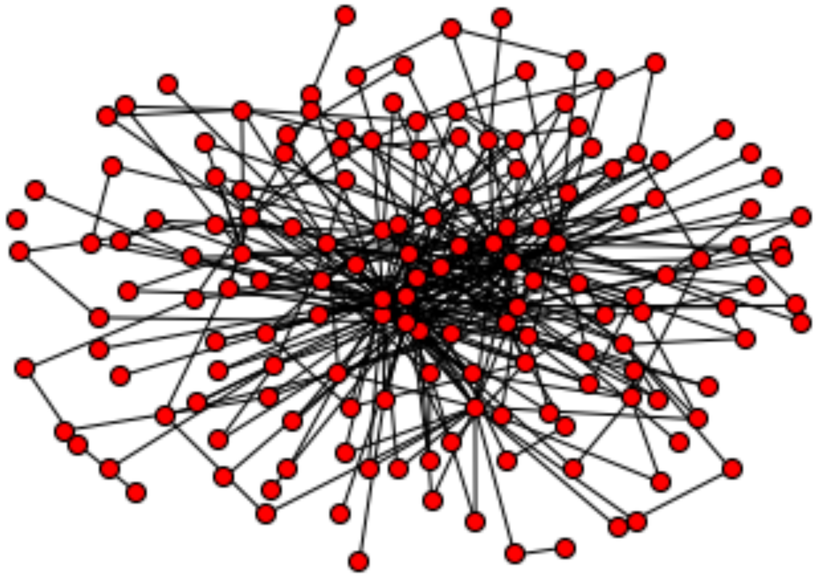
$p($  $)$

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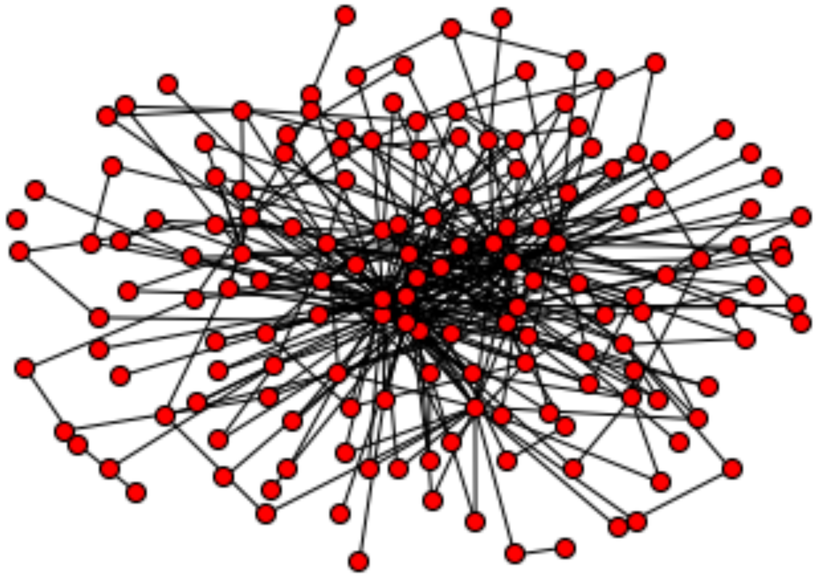
$p($  $)$

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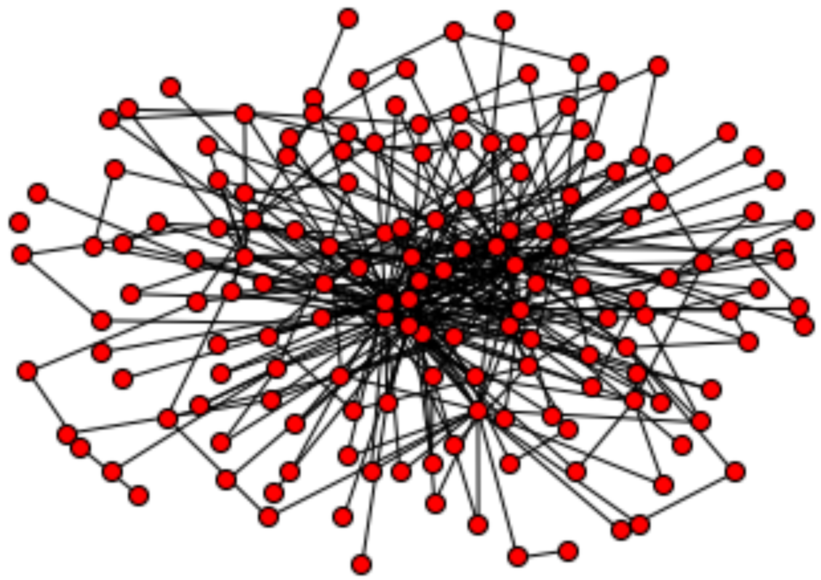
$p($  $)$

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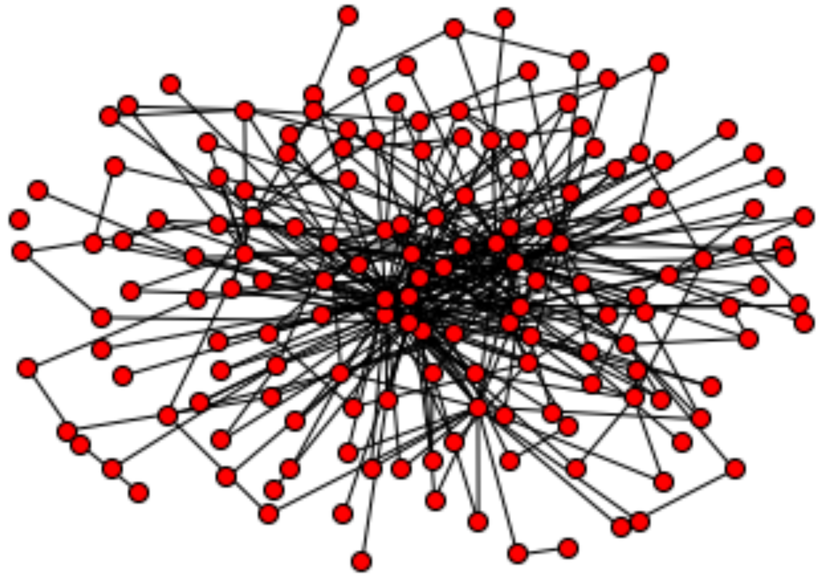
$p($  $)$

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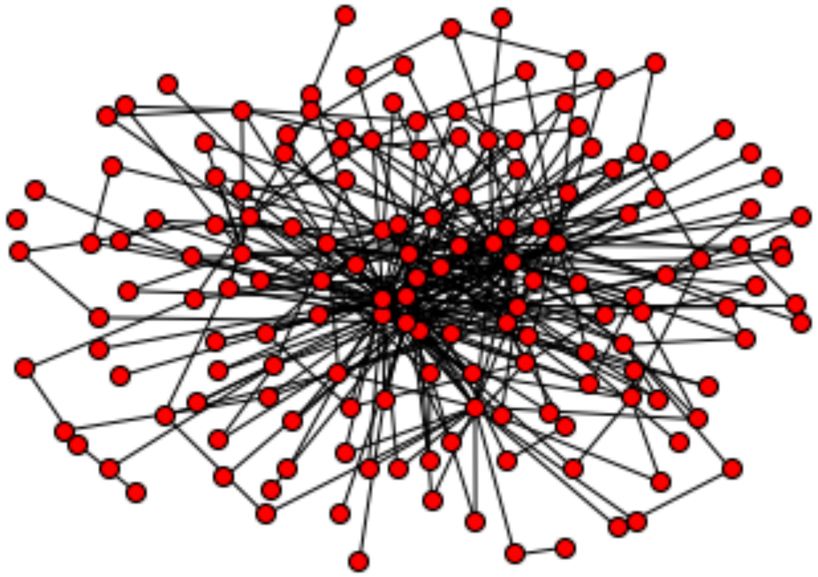
$p\Big($  $\Big)$

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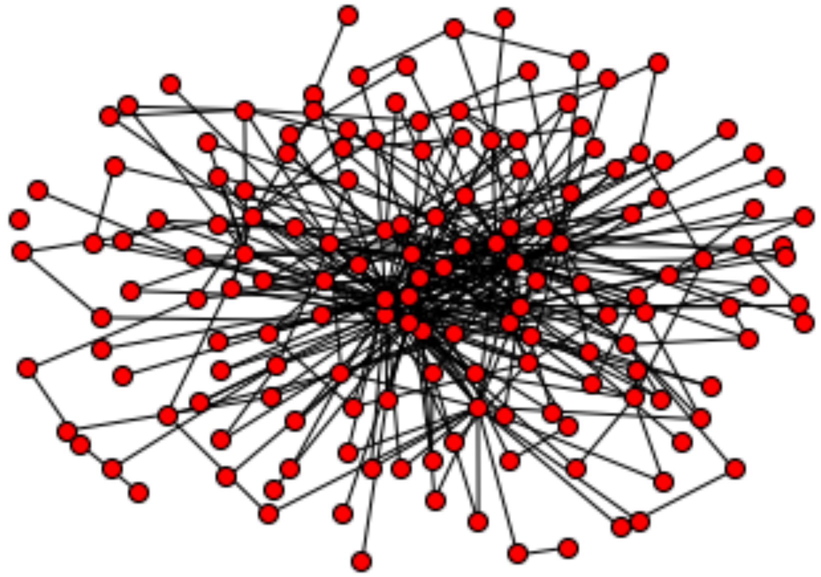
$p($  $)$

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  - Latent space model
  - Eigenmodel
  - Latent feature relational model
  - Infinite latent attribute model
  - Sparse matrix-variate Gaussian process block model
  - Random function model

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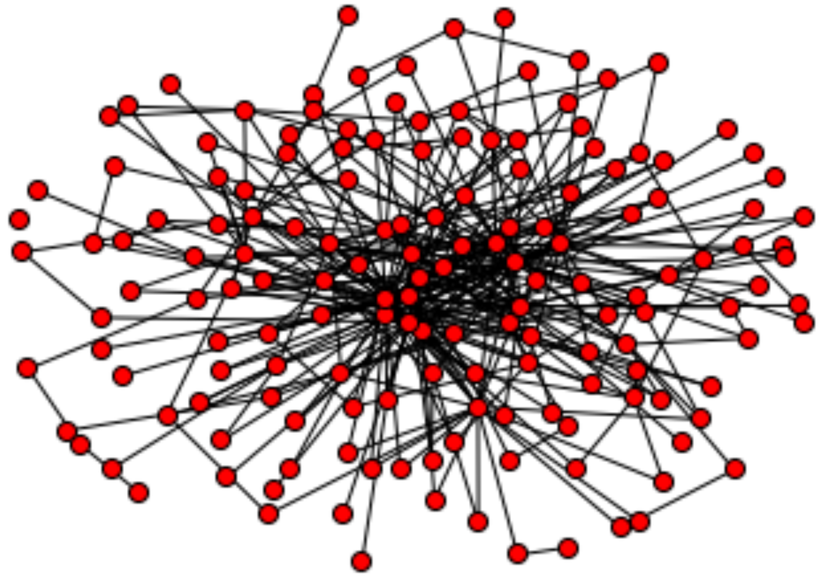
$p($  $)$

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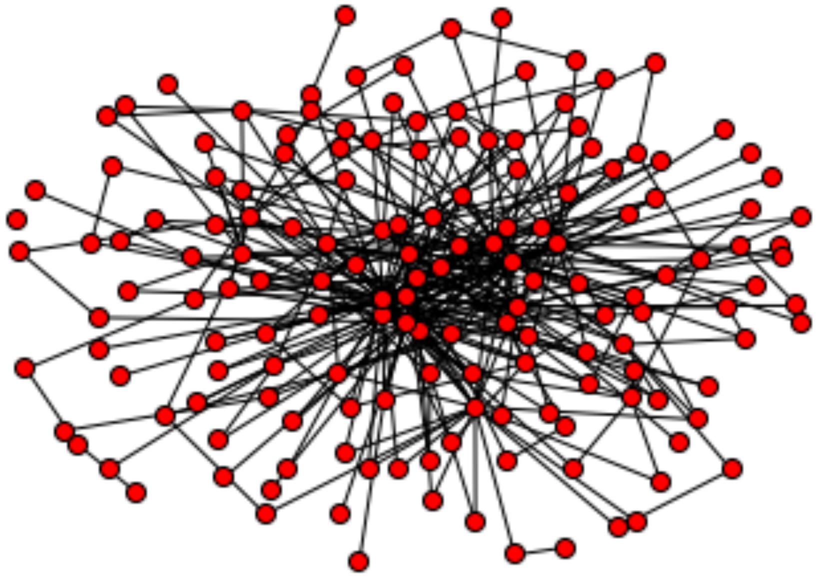
$p($  $)$

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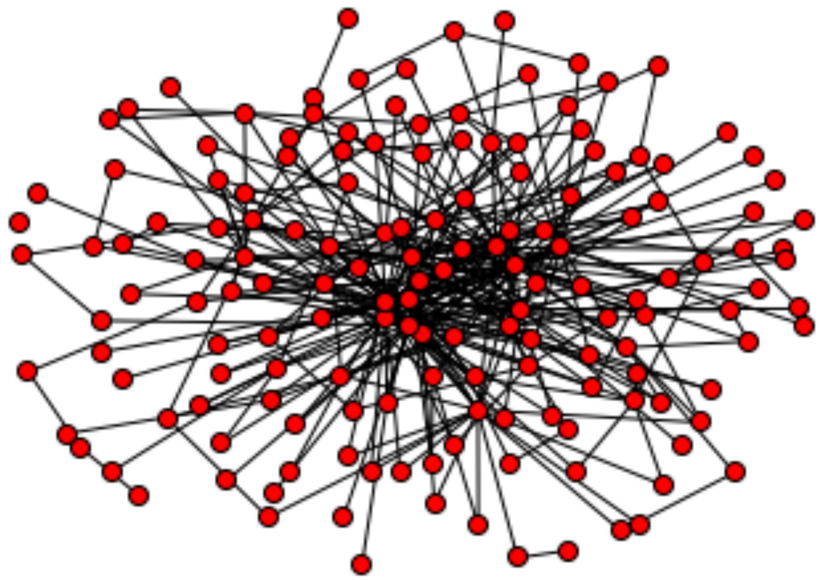
$p($  $)$

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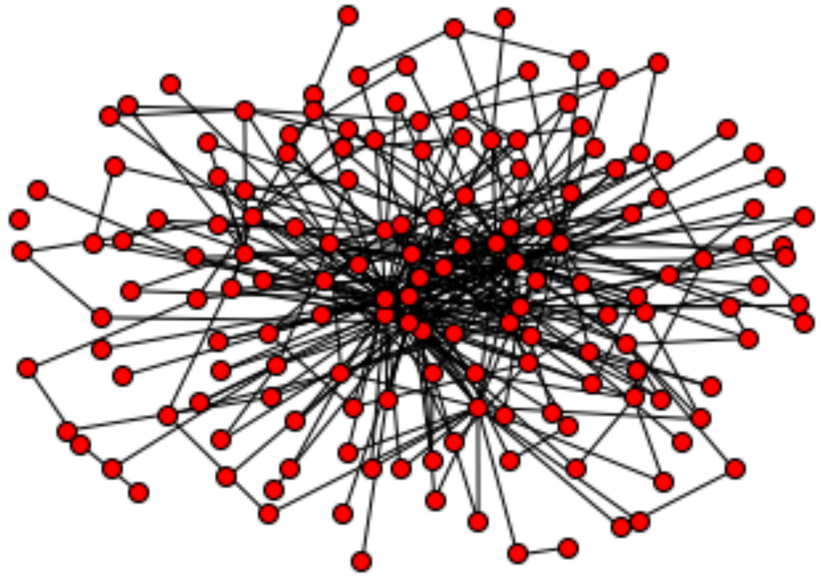
$p($  $)$

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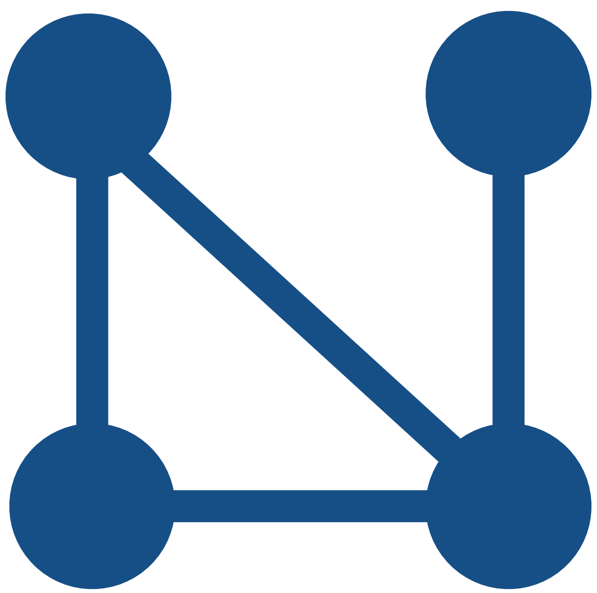
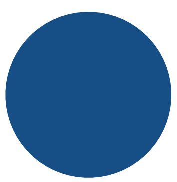
Sequence of graphs

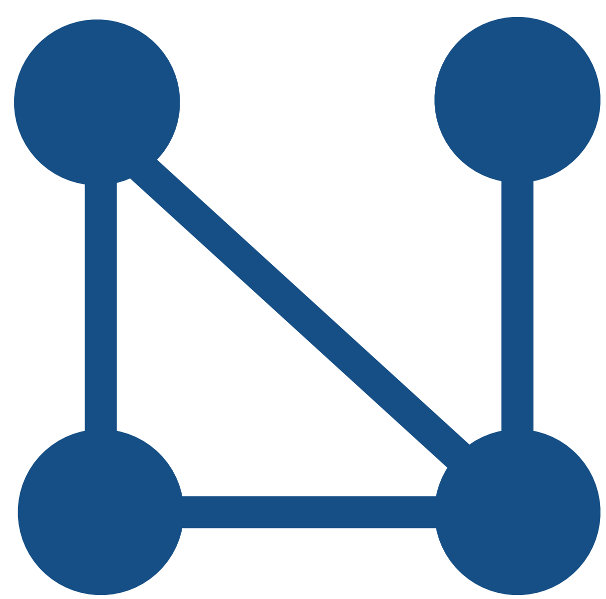


G

Sequence of graphs



G_1



G

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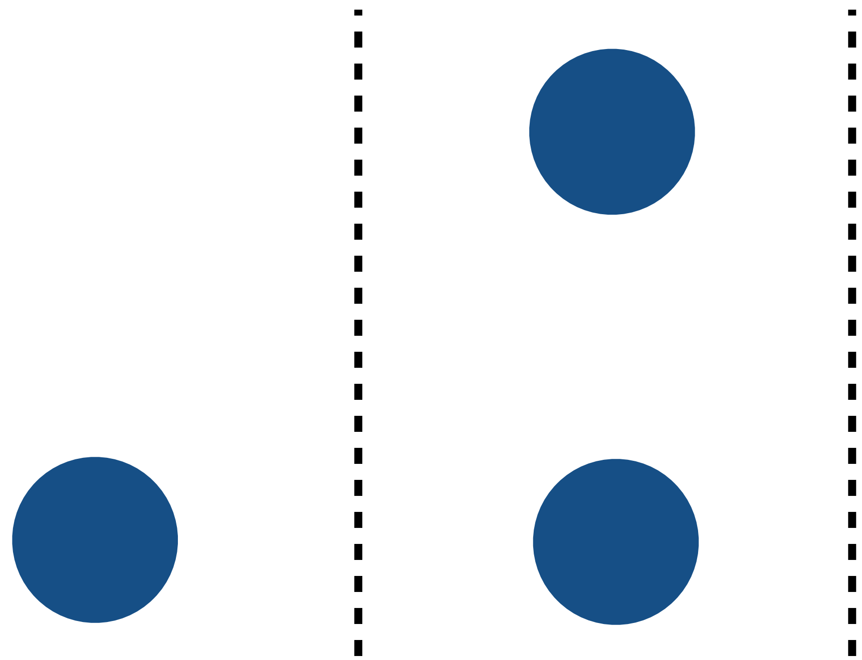
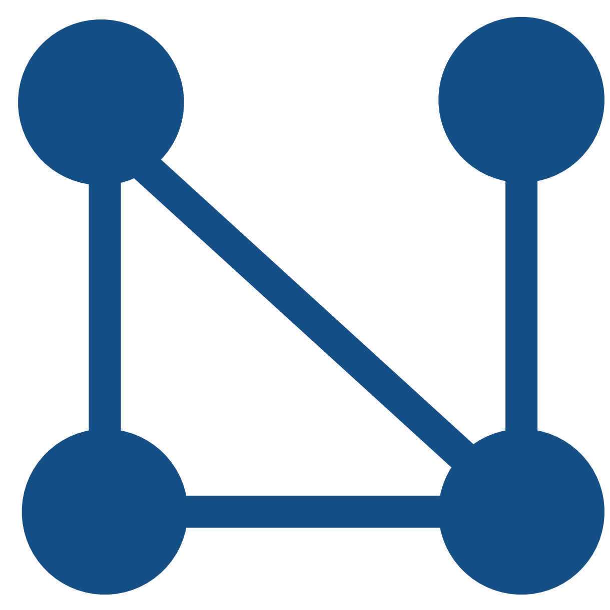


G_1 G_2



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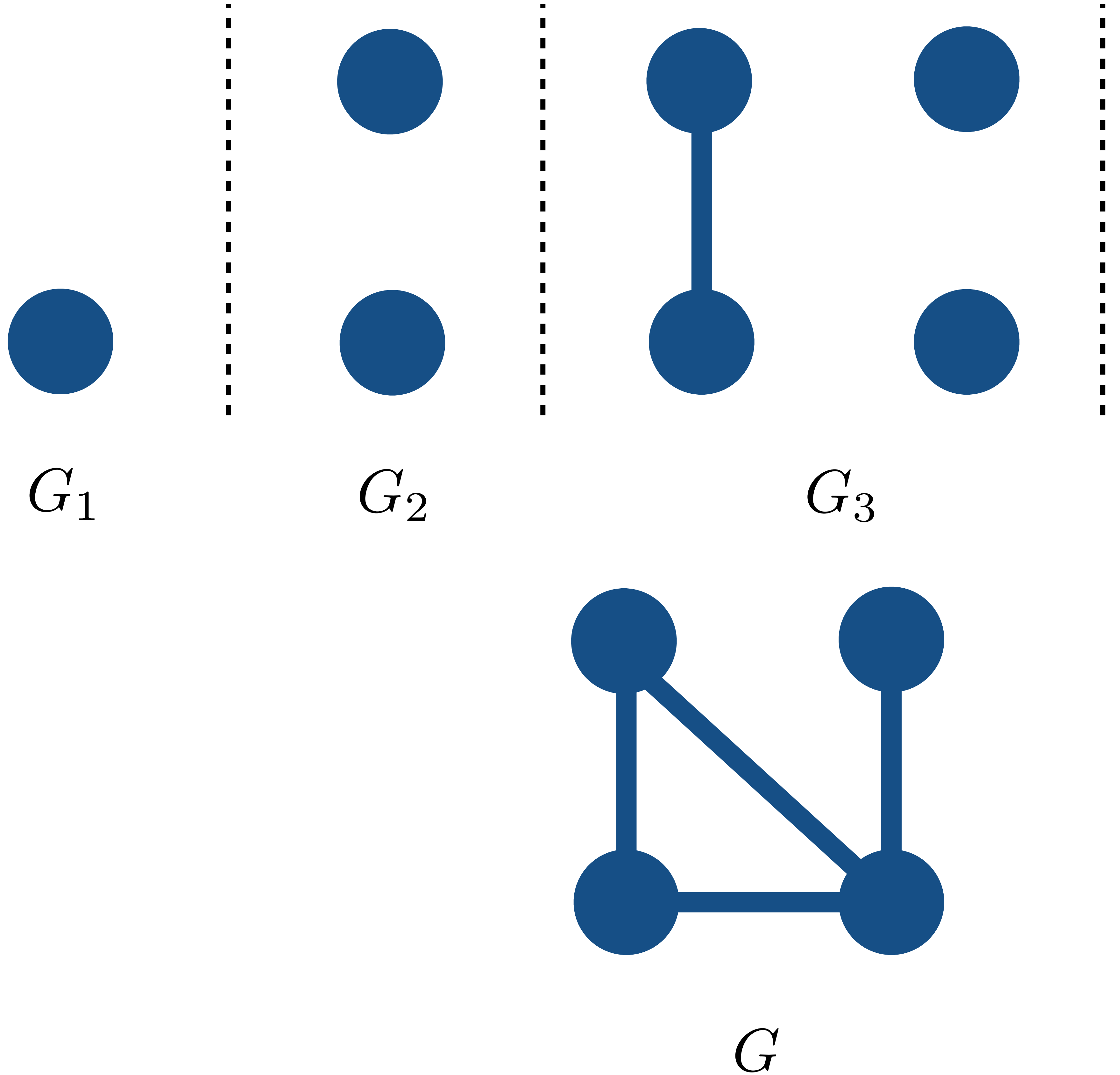

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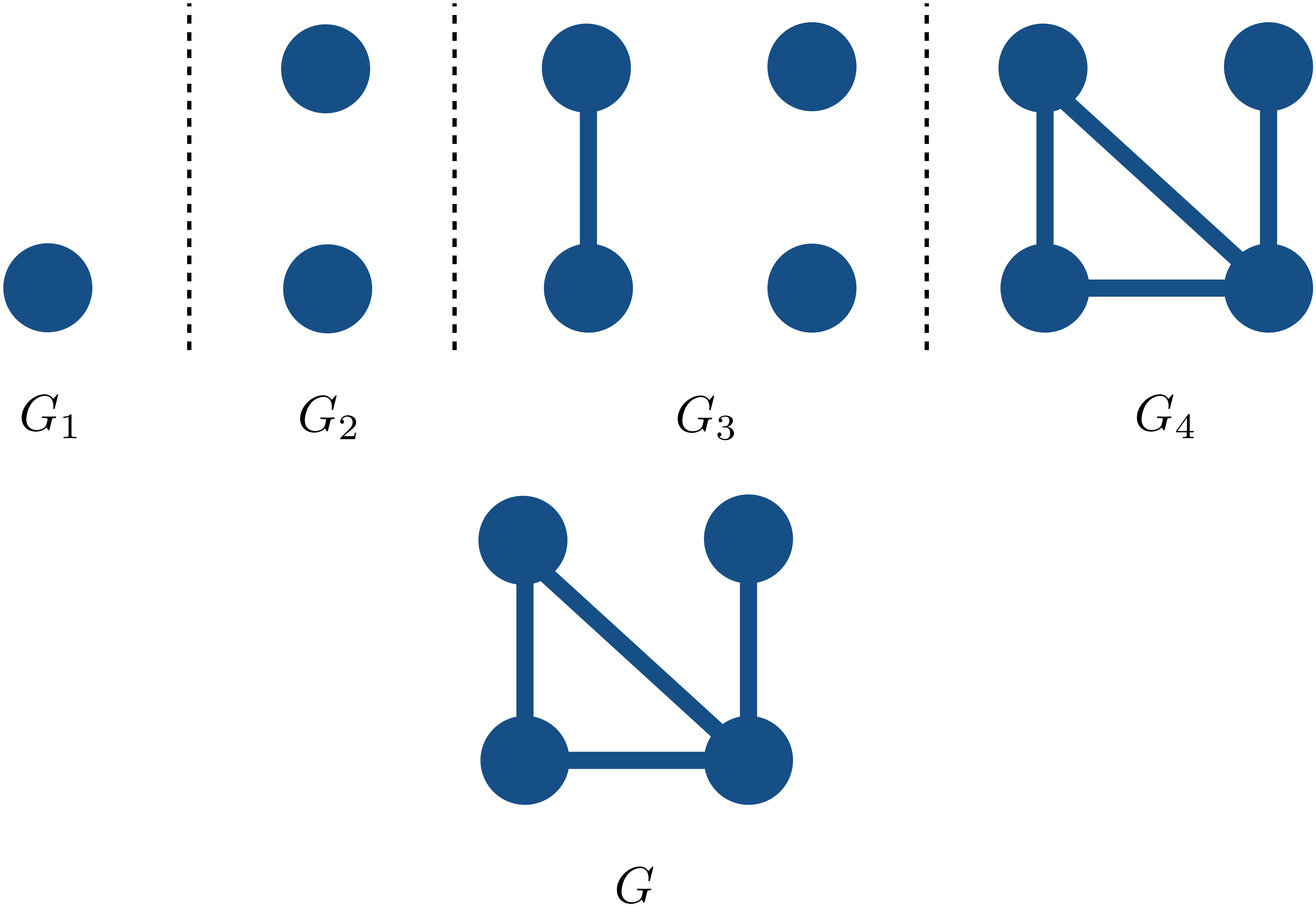

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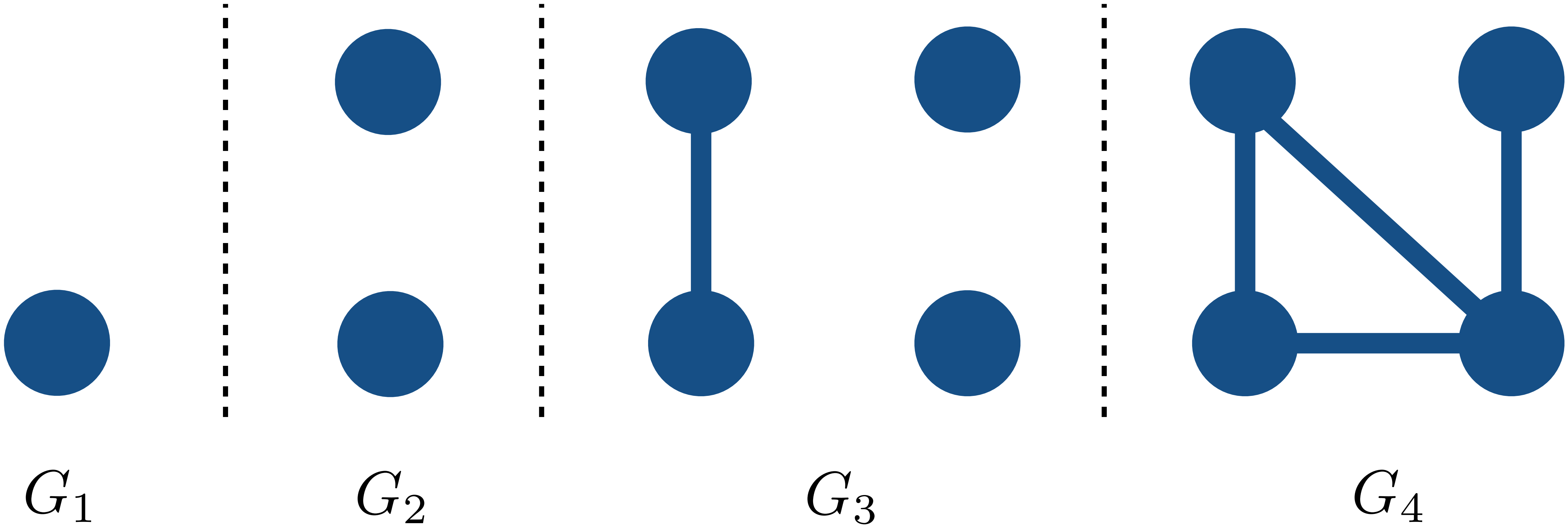

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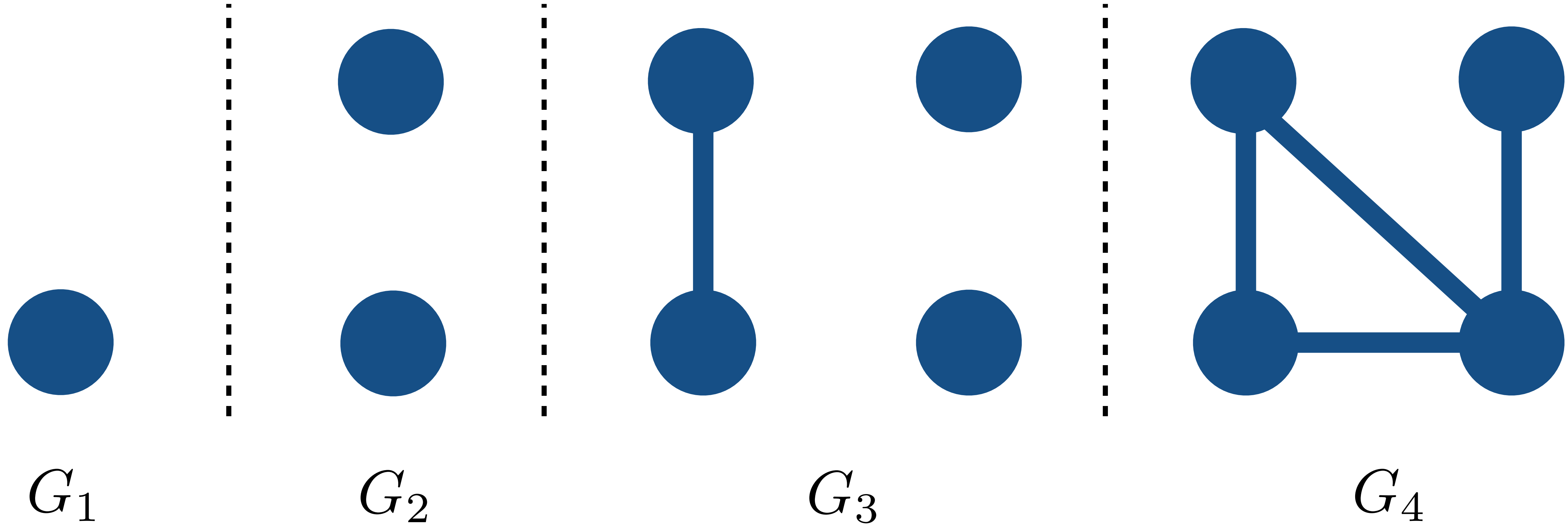
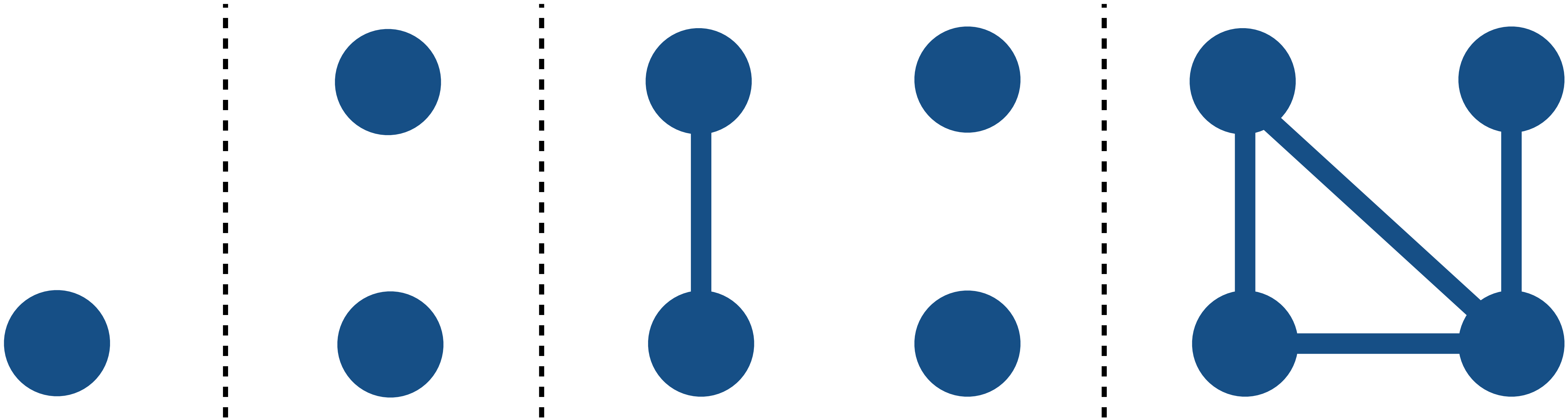


If $\#\text{nodes}(G_n) \to \infty$,

Sequence of graphs



G_1 G_2 G_3 G_4

If $\#\text{nodes}(G_n) \to \infty$,

- *Dense* graph sequence $\#\text{edges}(G_n) \geq c \cdot [\#\text{nodes}(G_n)]^2$

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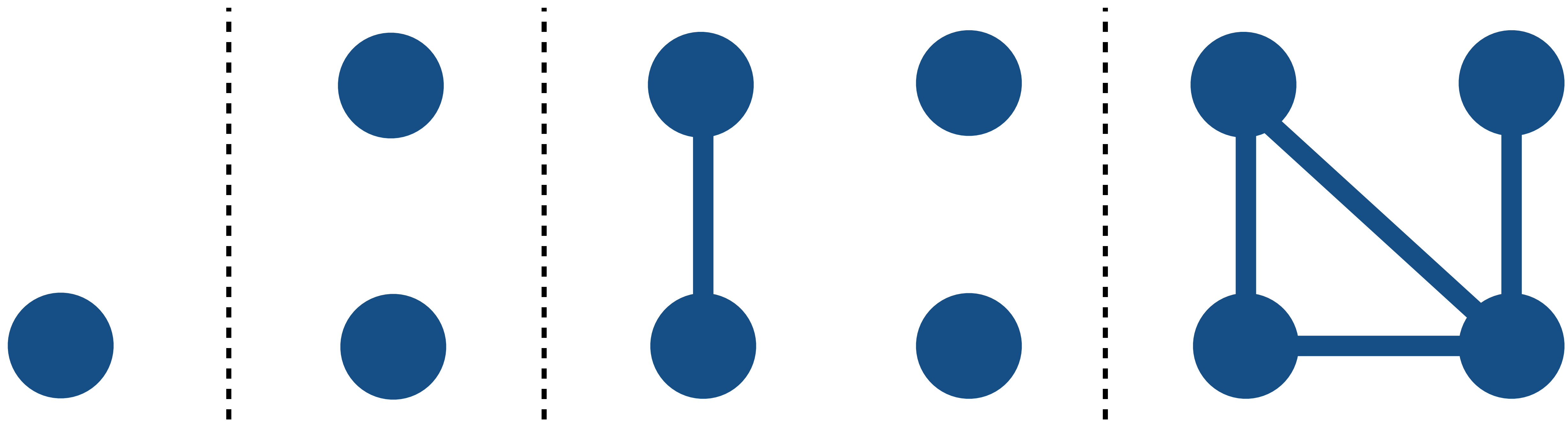
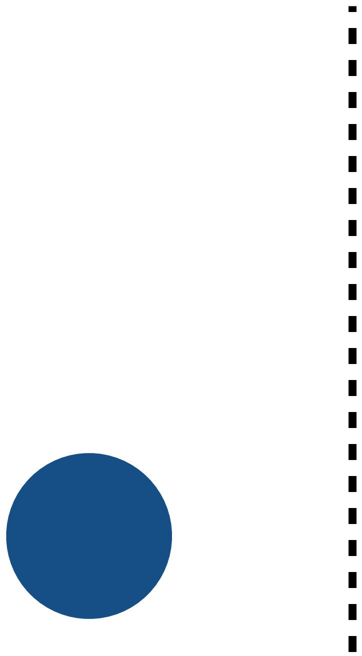


G_1 G_2 G_3 G_4

If $\#\text{nodes}(G_n) \to \infty$,

- *Dense* graph sequence $\#\text{edges}(G_n) \geq c \cdot [\#\text{nodes}(G_n)]^2$
- *Sparse* graph sequence $\#\text{edges}(G_n) \in o([\#\text{nodes}(G_n)]^2)$

The Old Way: Nodes



G_1

[Hoover 1979, Aldous 1981]

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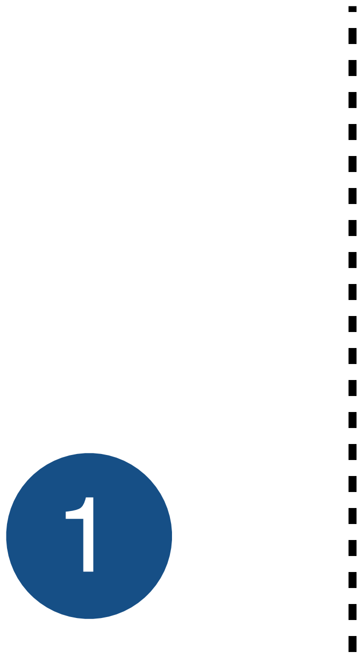

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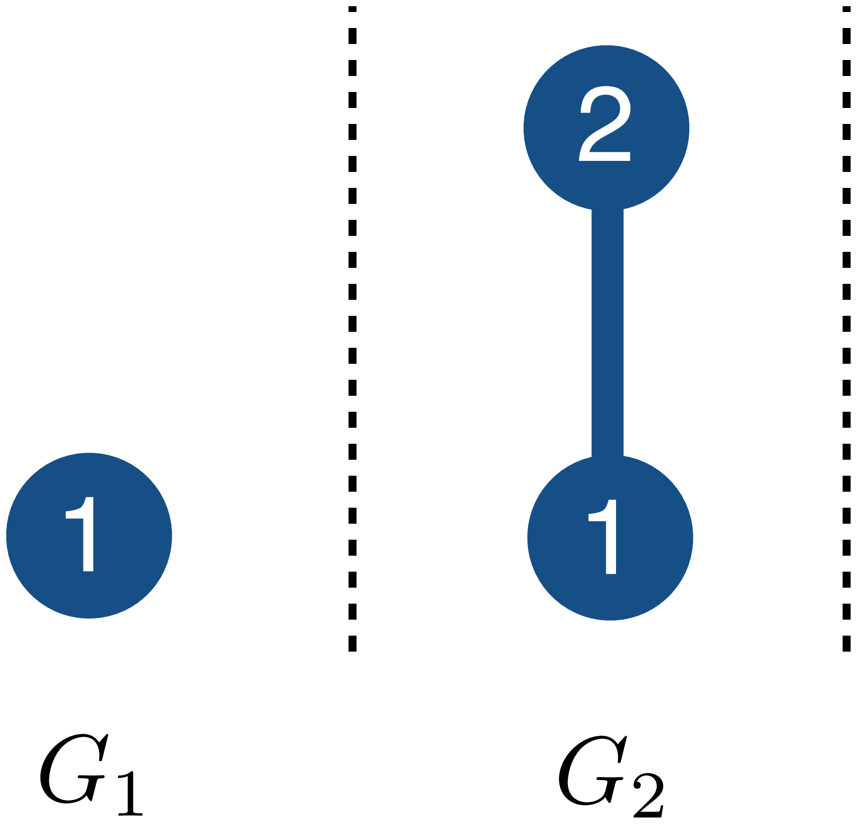

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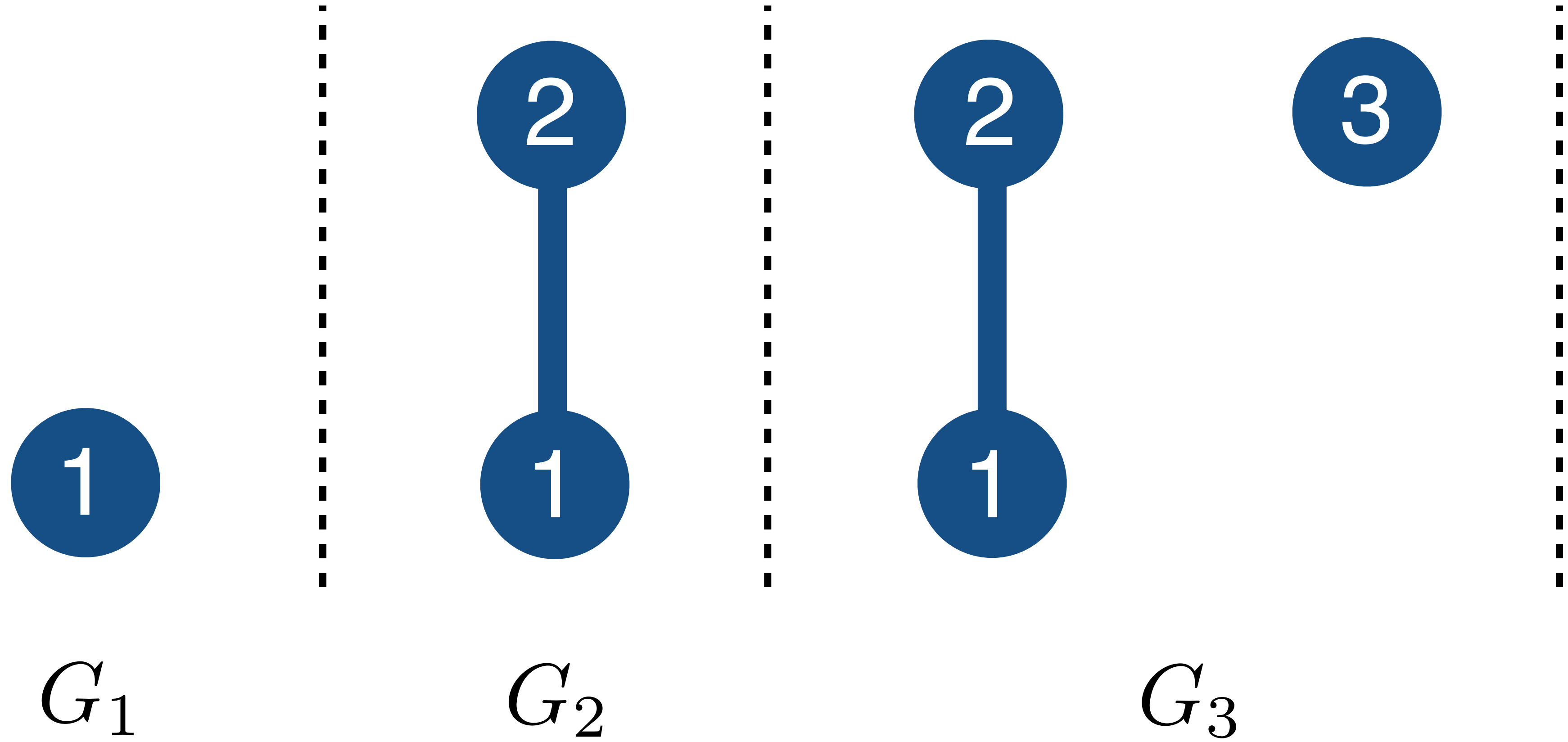

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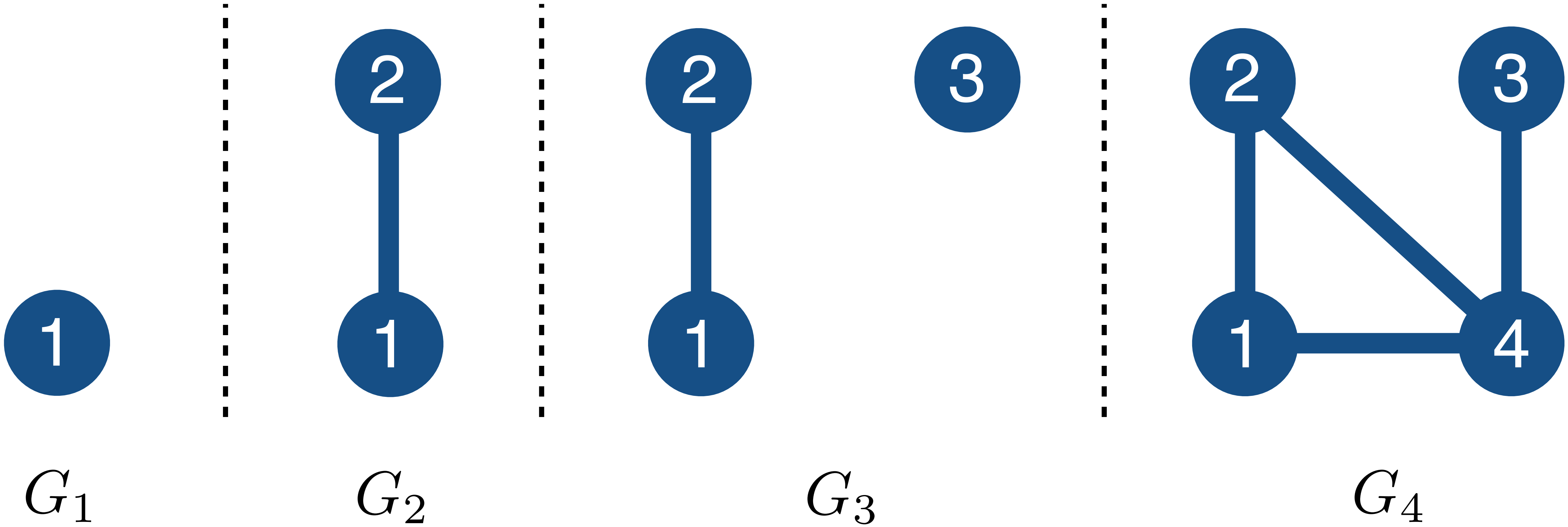
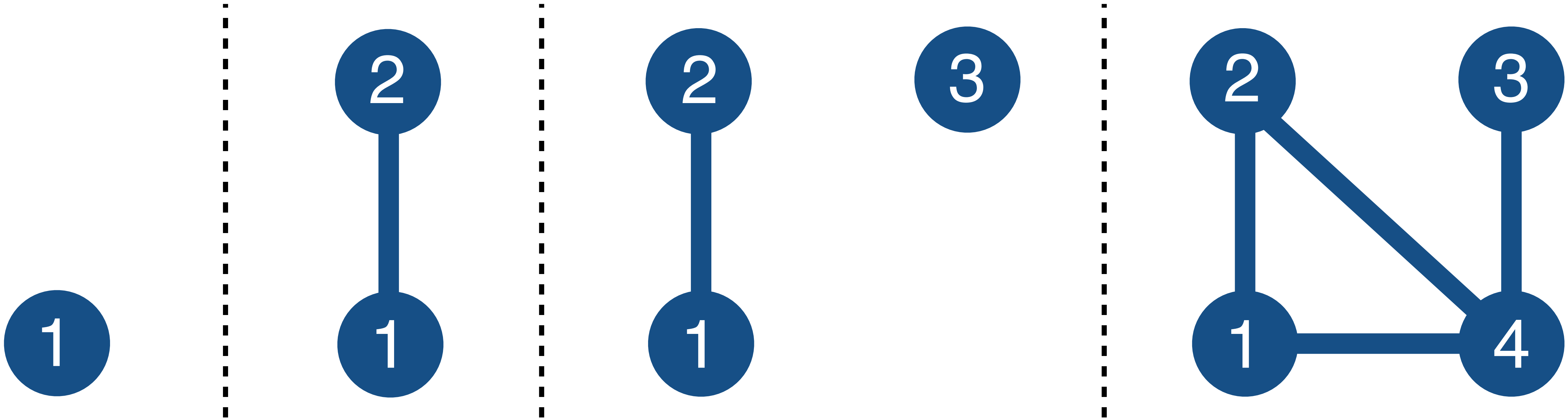


[Hoover 1979, Aldous 1981]

The Old Way: Exchangeability



G_1 G_2 G_3 G_4

[Hoover 1979, Aldous 1981]

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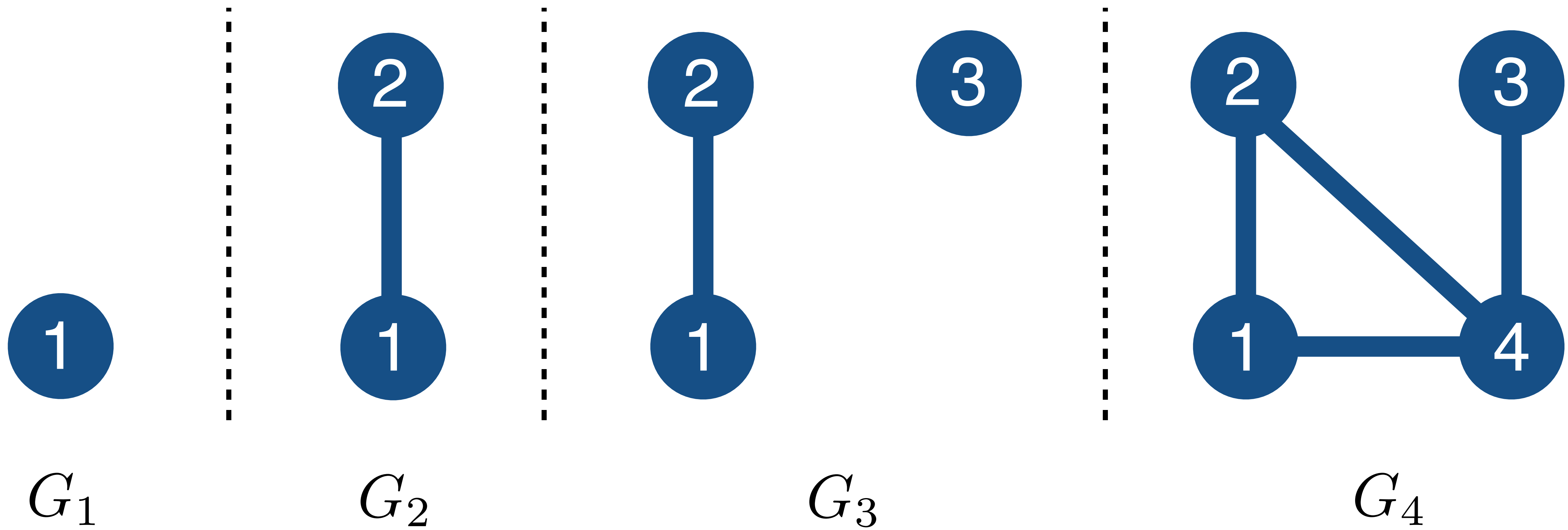

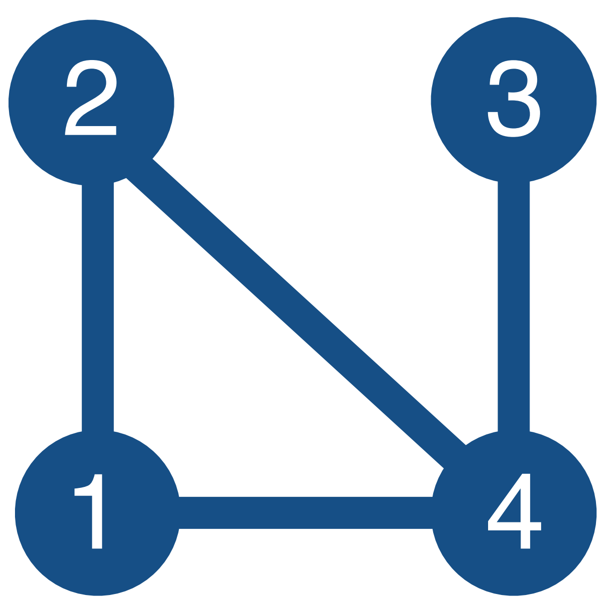

[Hoover 1979, Aldous 1981]

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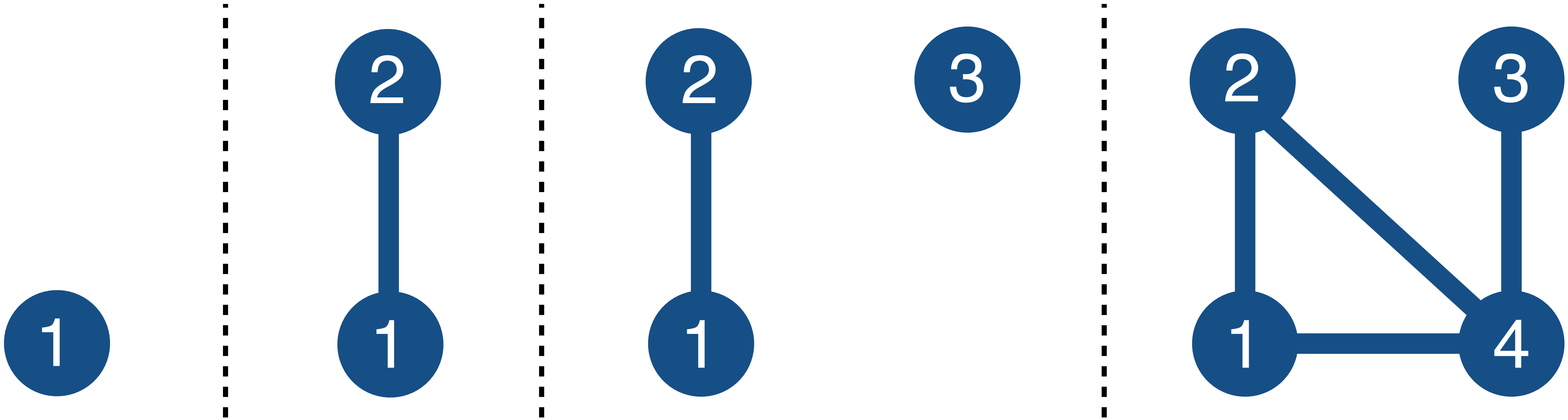

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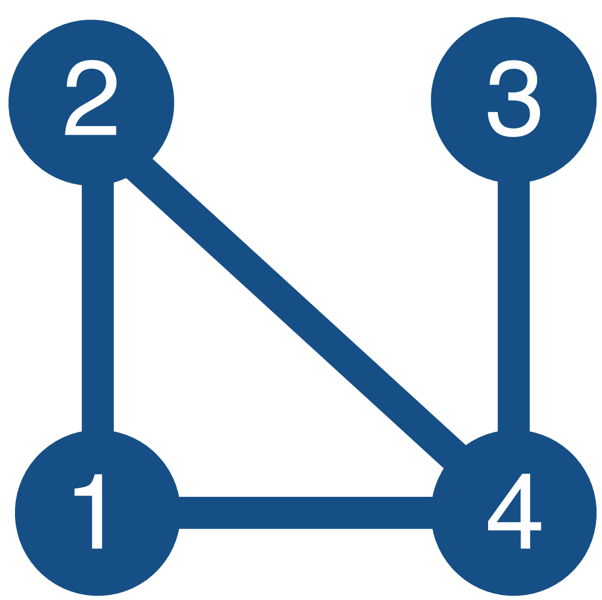
$p($  $)$

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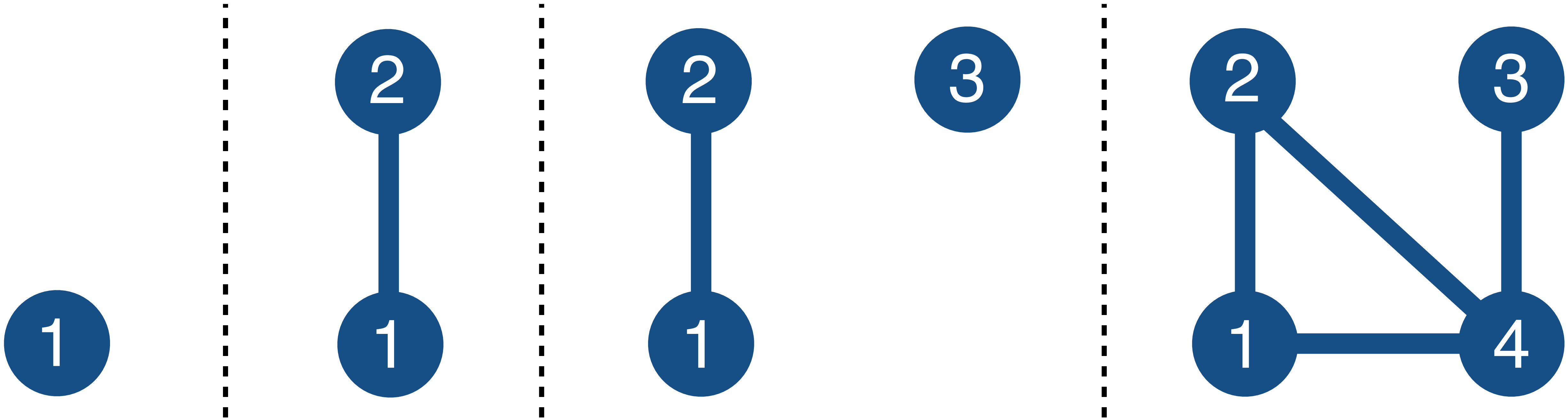

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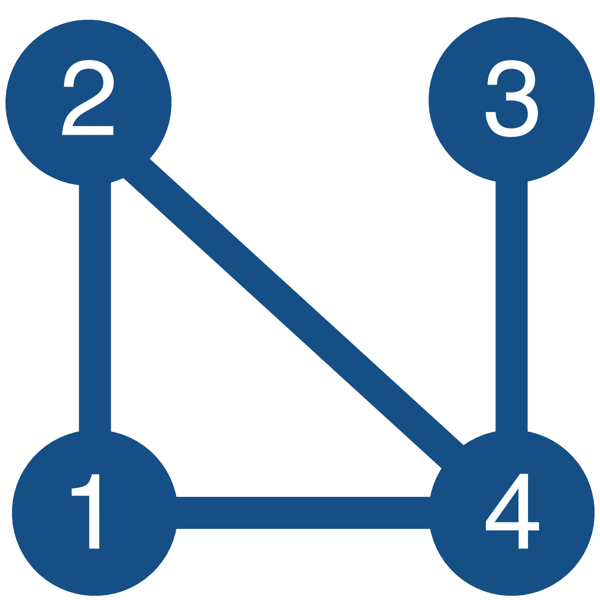
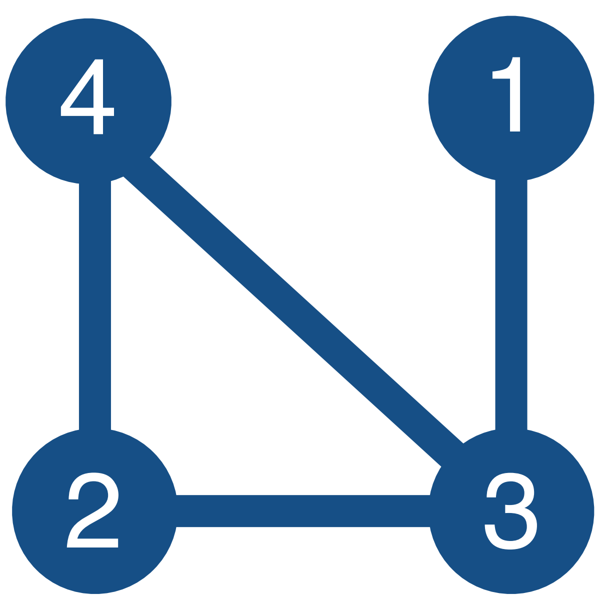
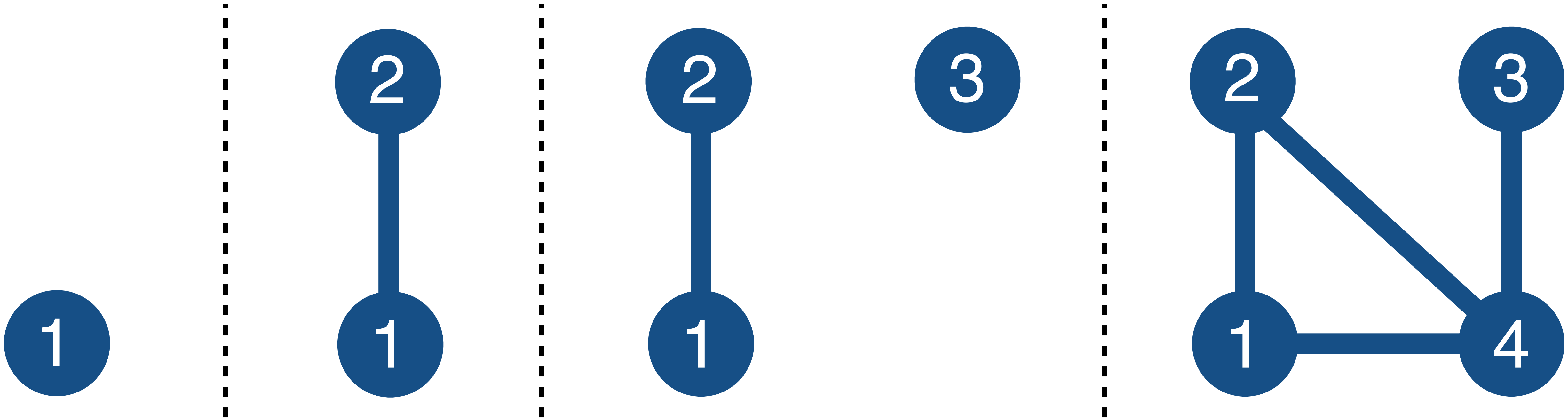
$p($ 

$) = p($ 

$)$

[Hoover 1979, Aldous 1981]

The Old Way: Node exchangeability



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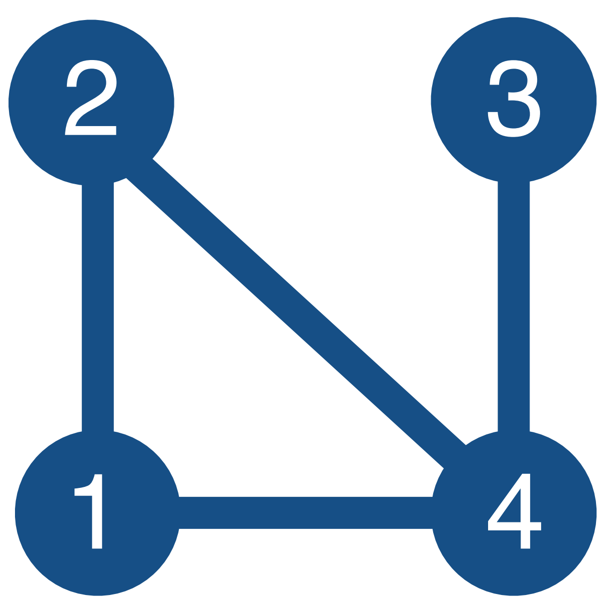
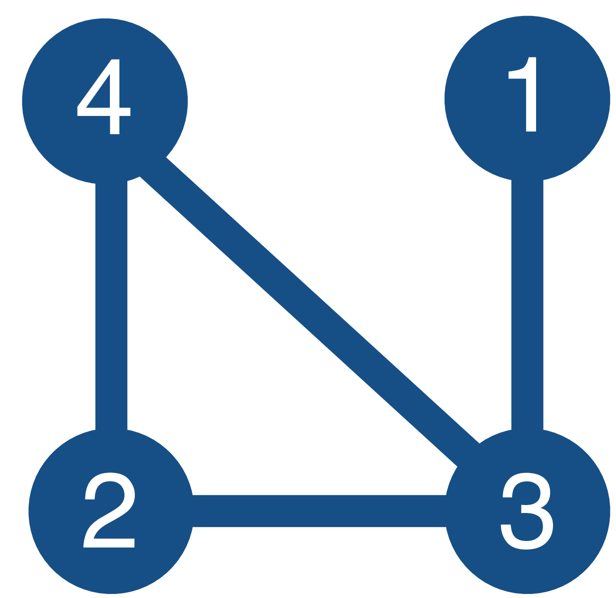
$p($ 

$) = p($ 

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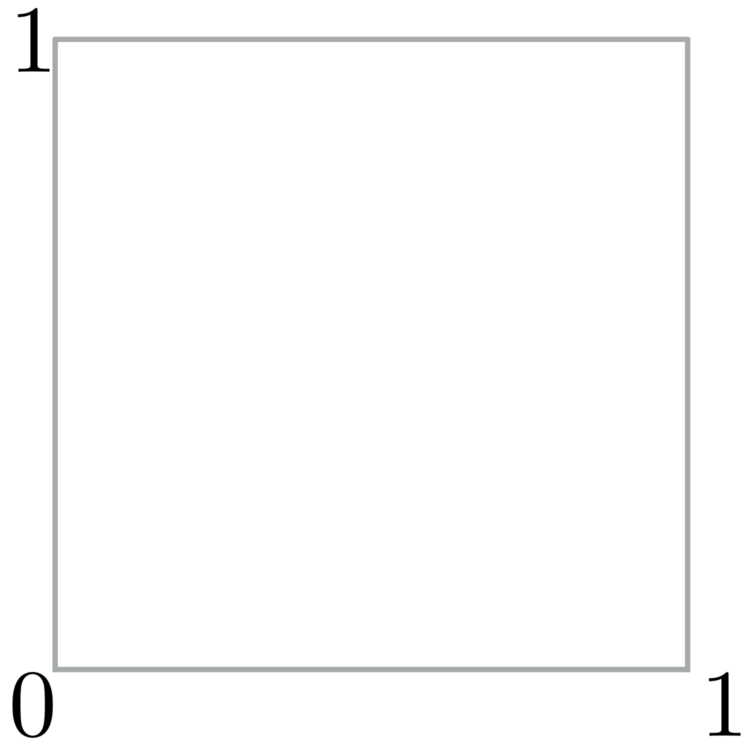
[Hoover 1979, Aldous 1981]

Aldous-Hoover



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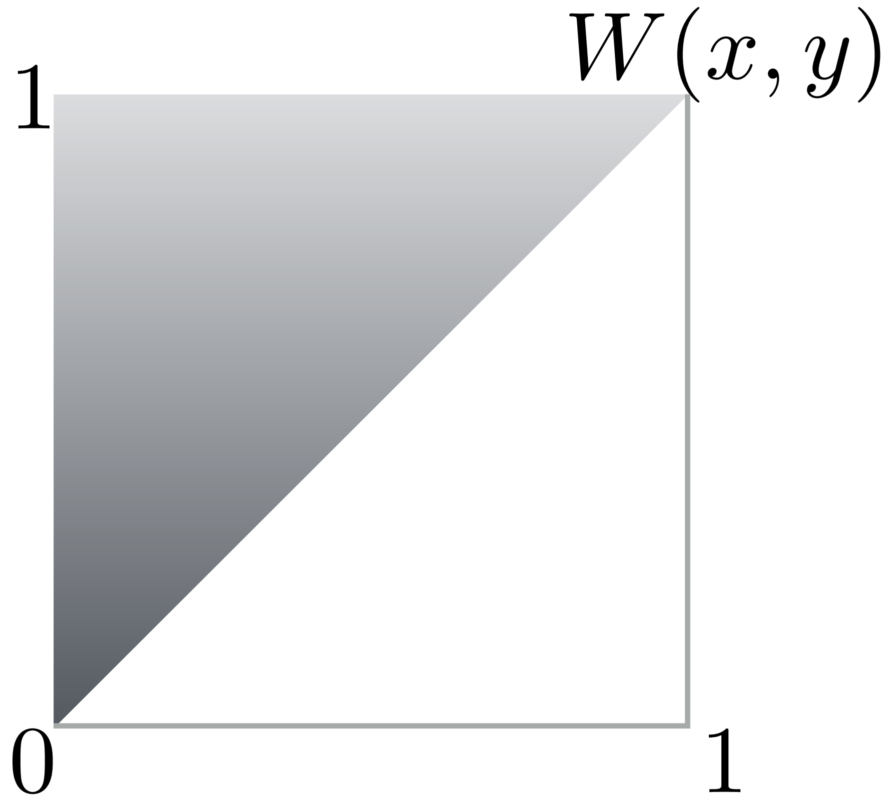

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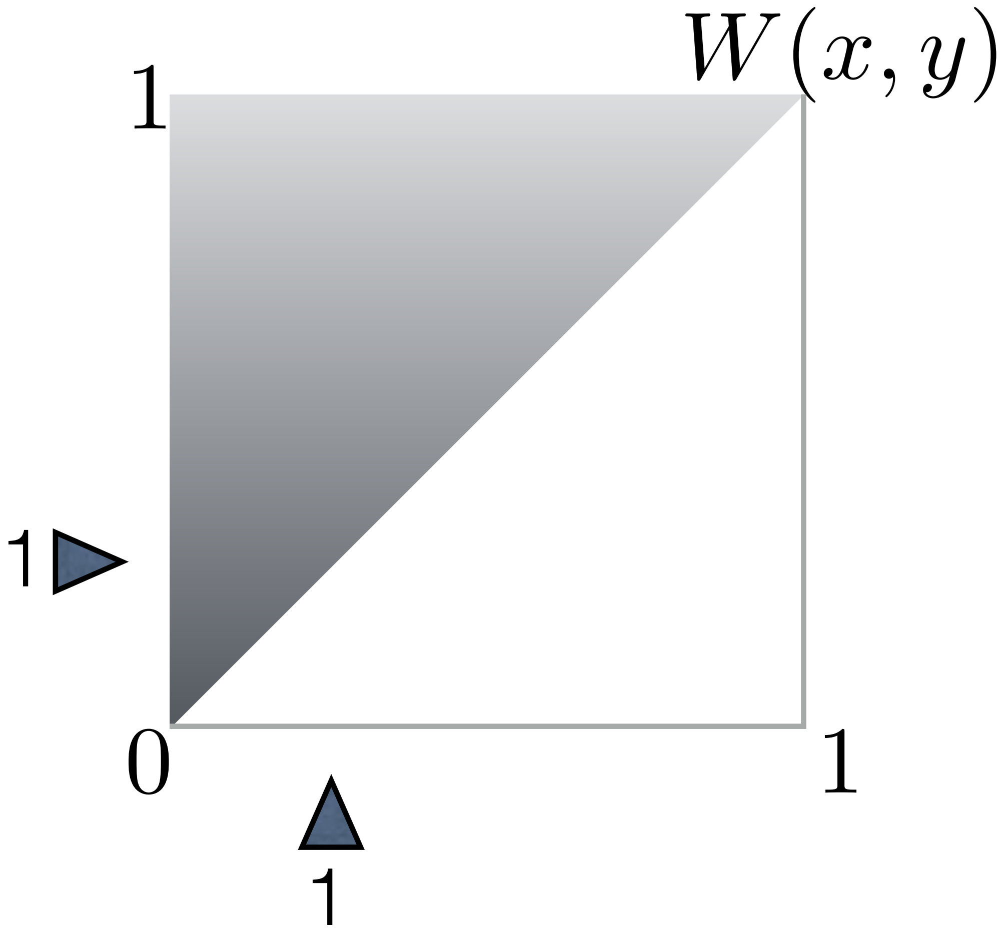

[Hoover 1979, Aldous 1981]

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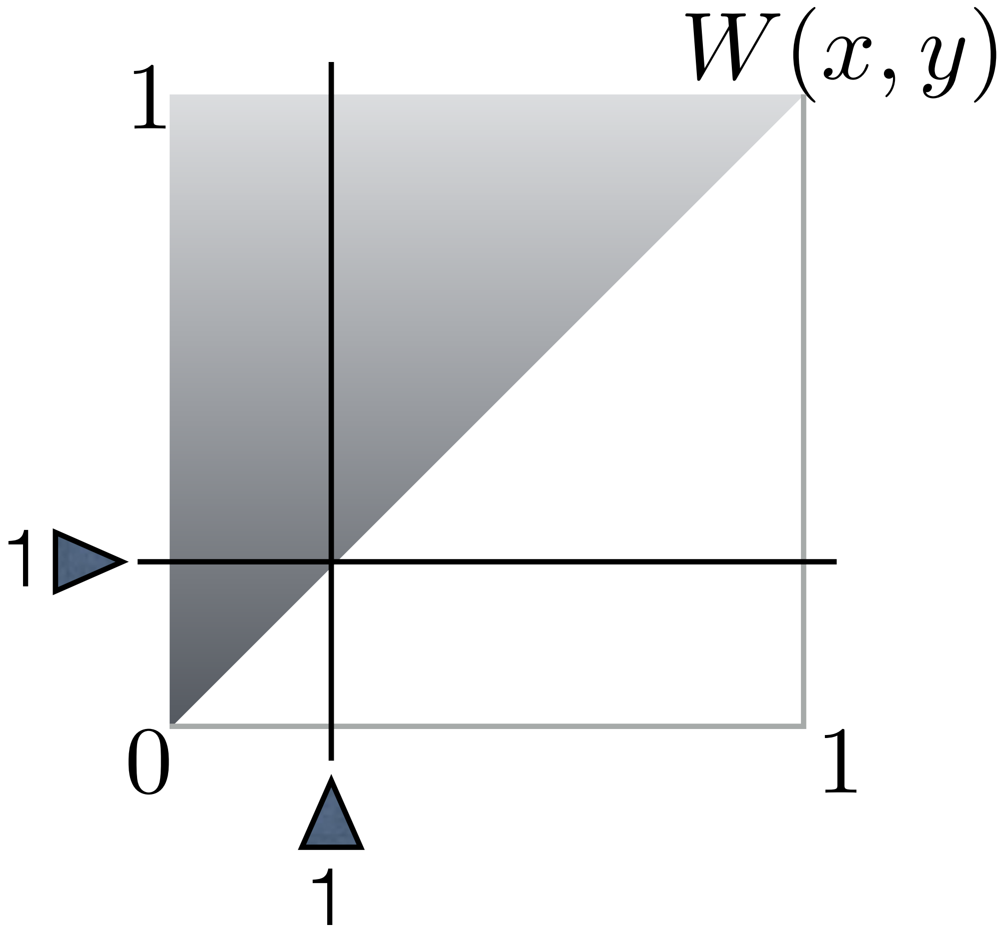

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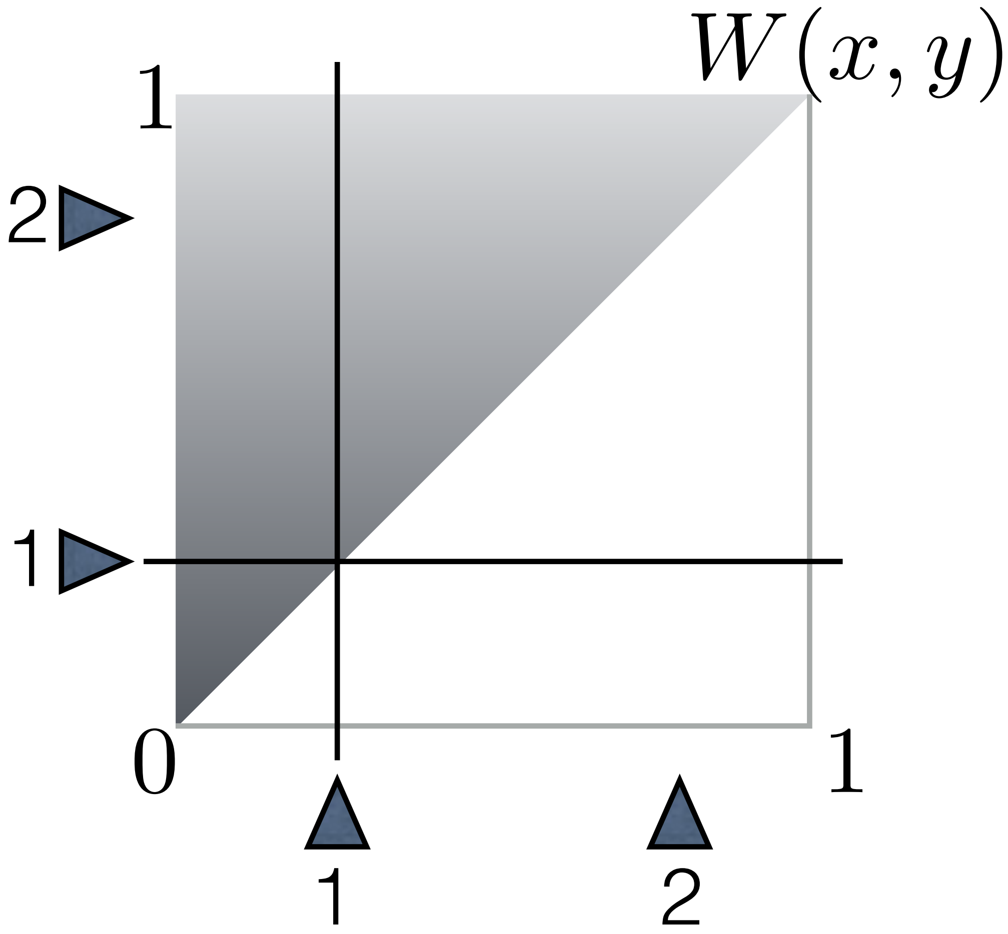

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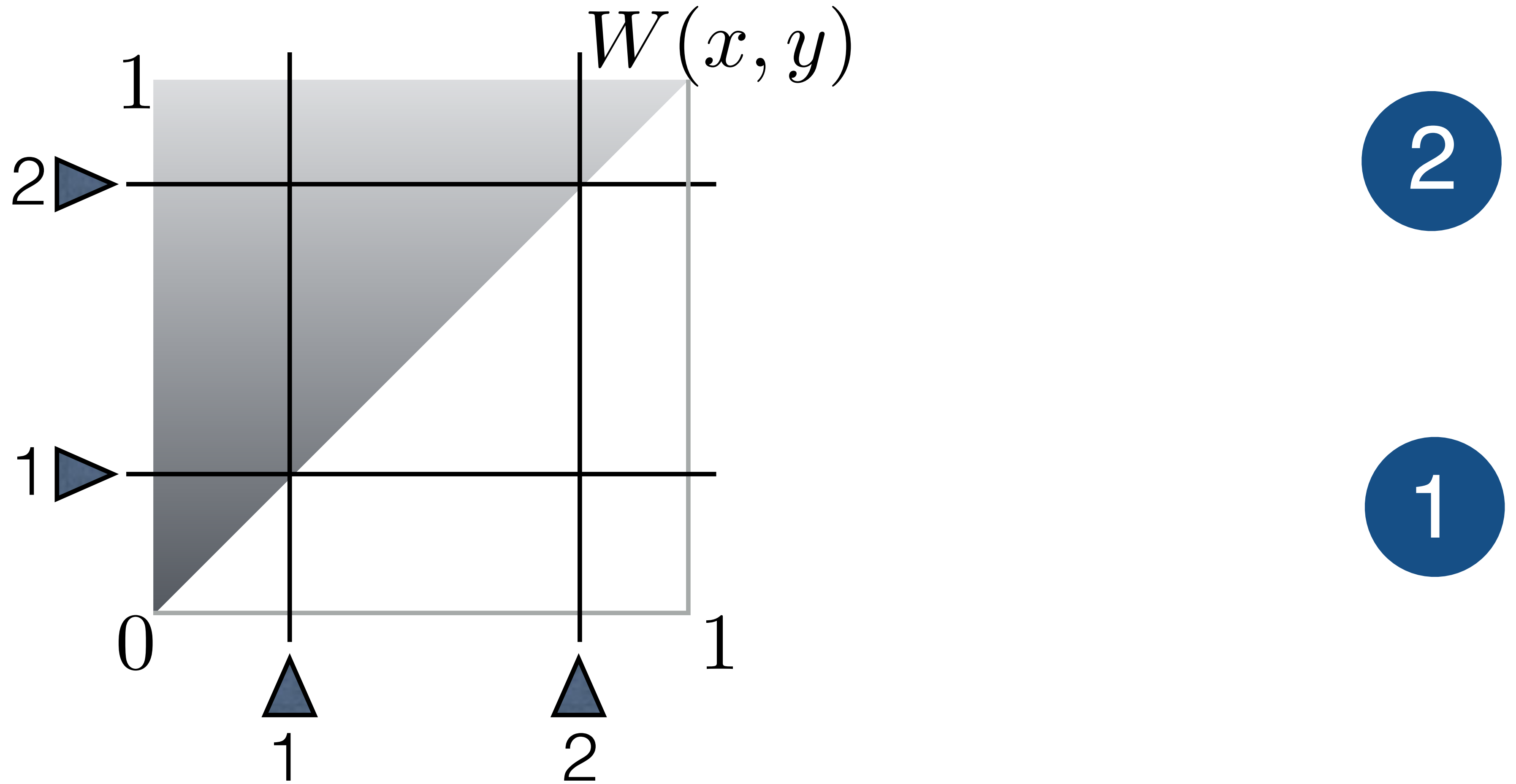

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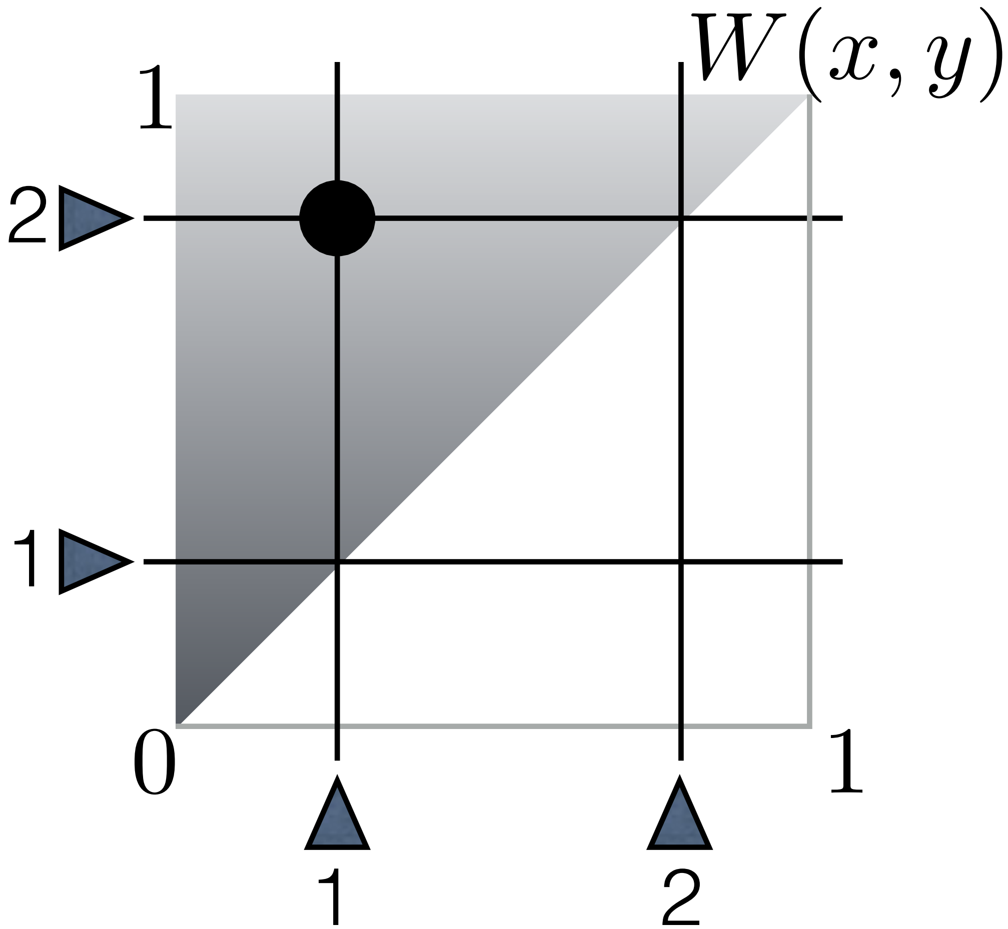

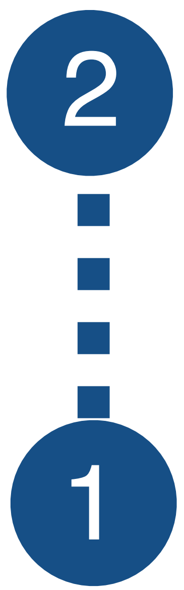

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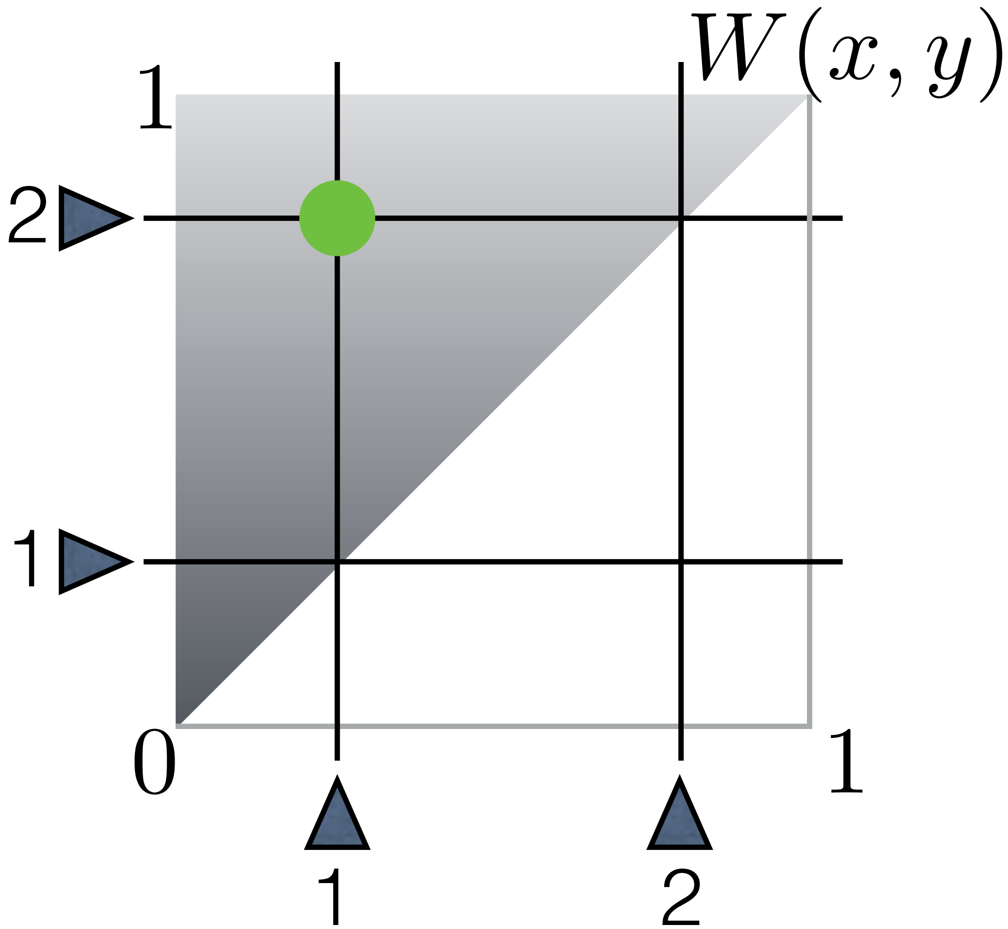

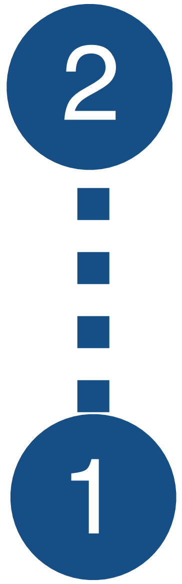

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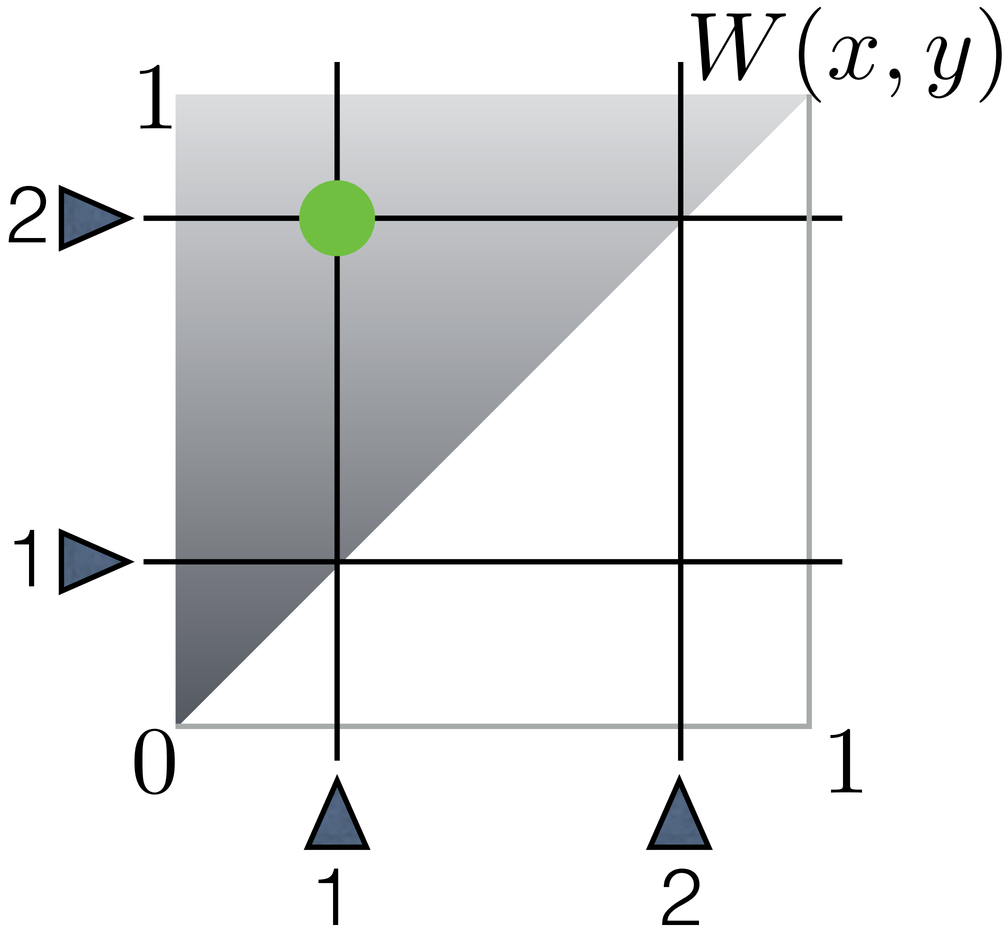

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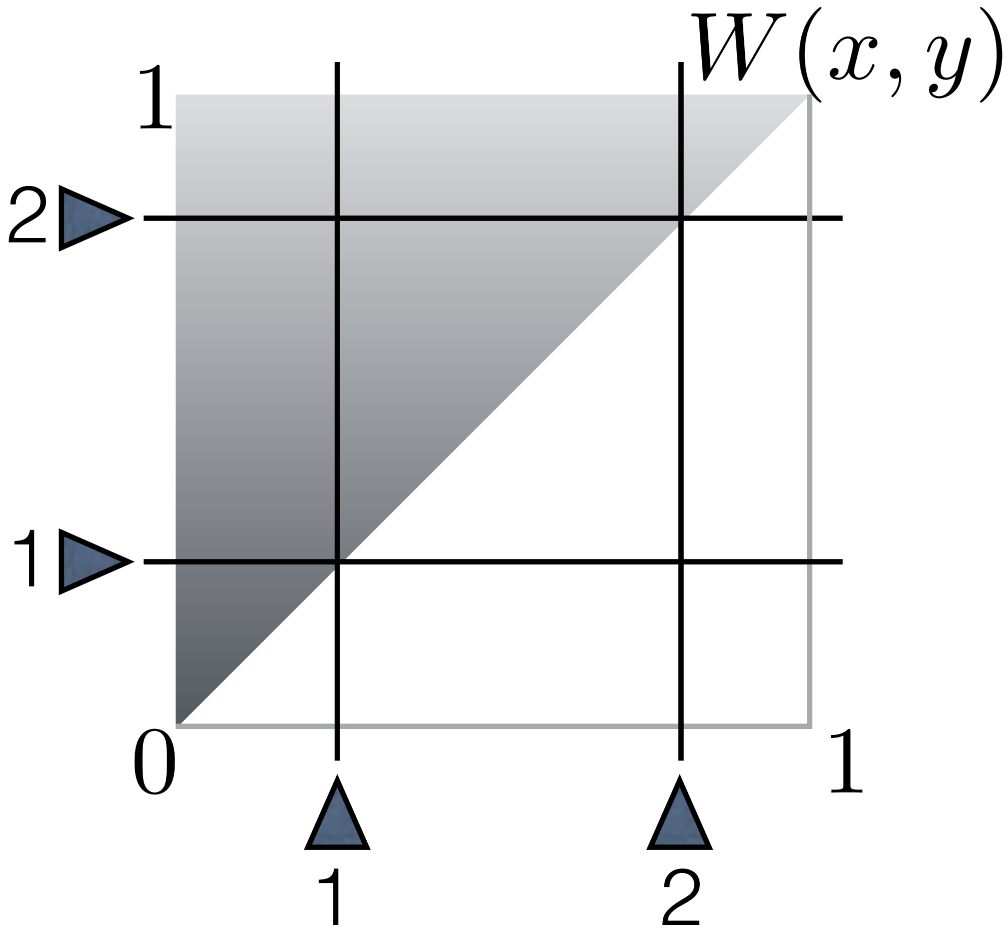

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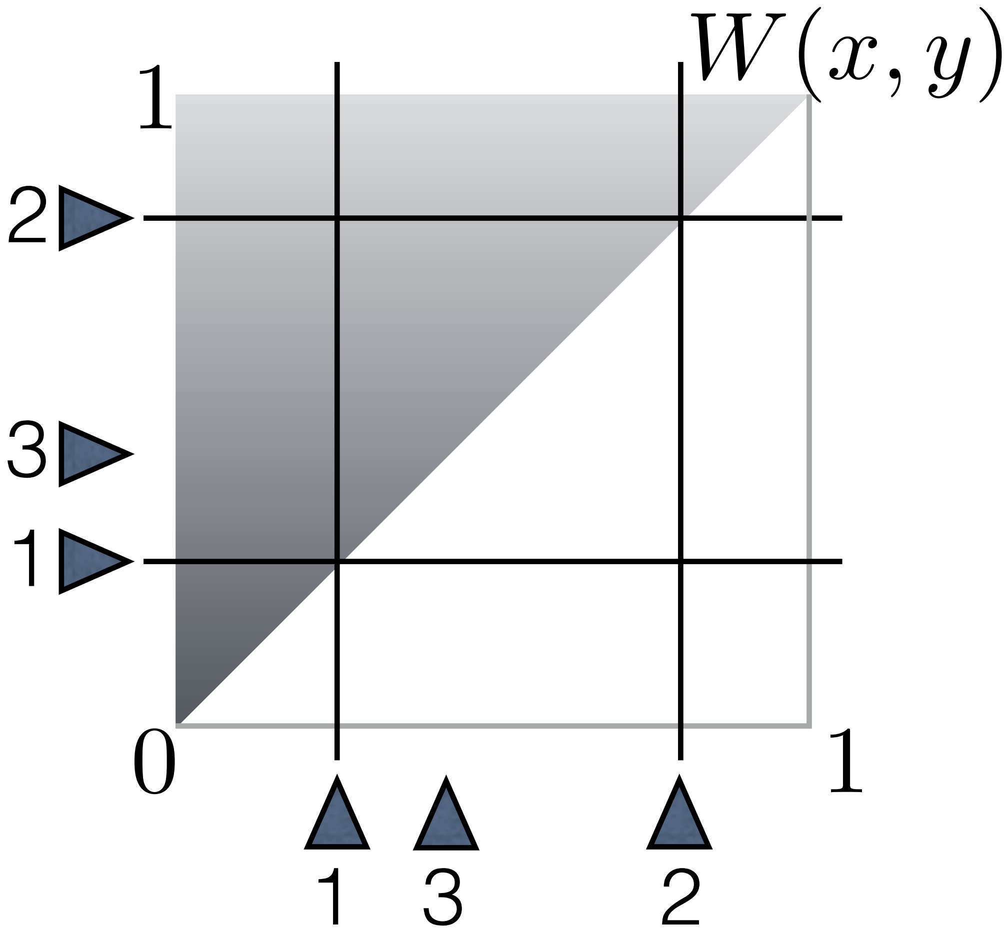

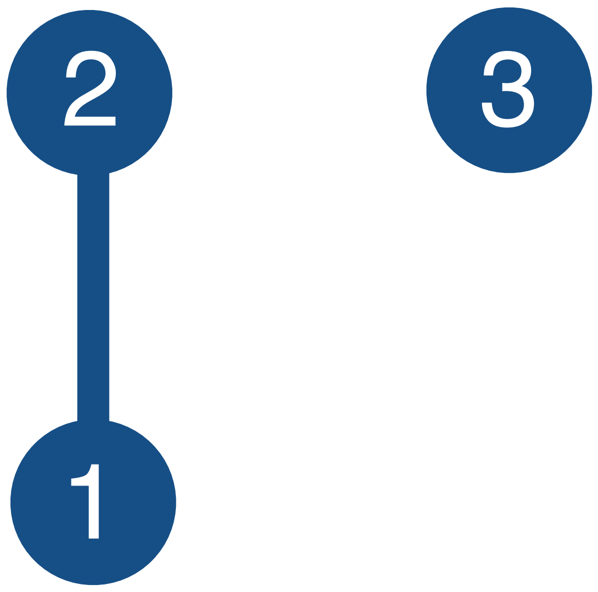

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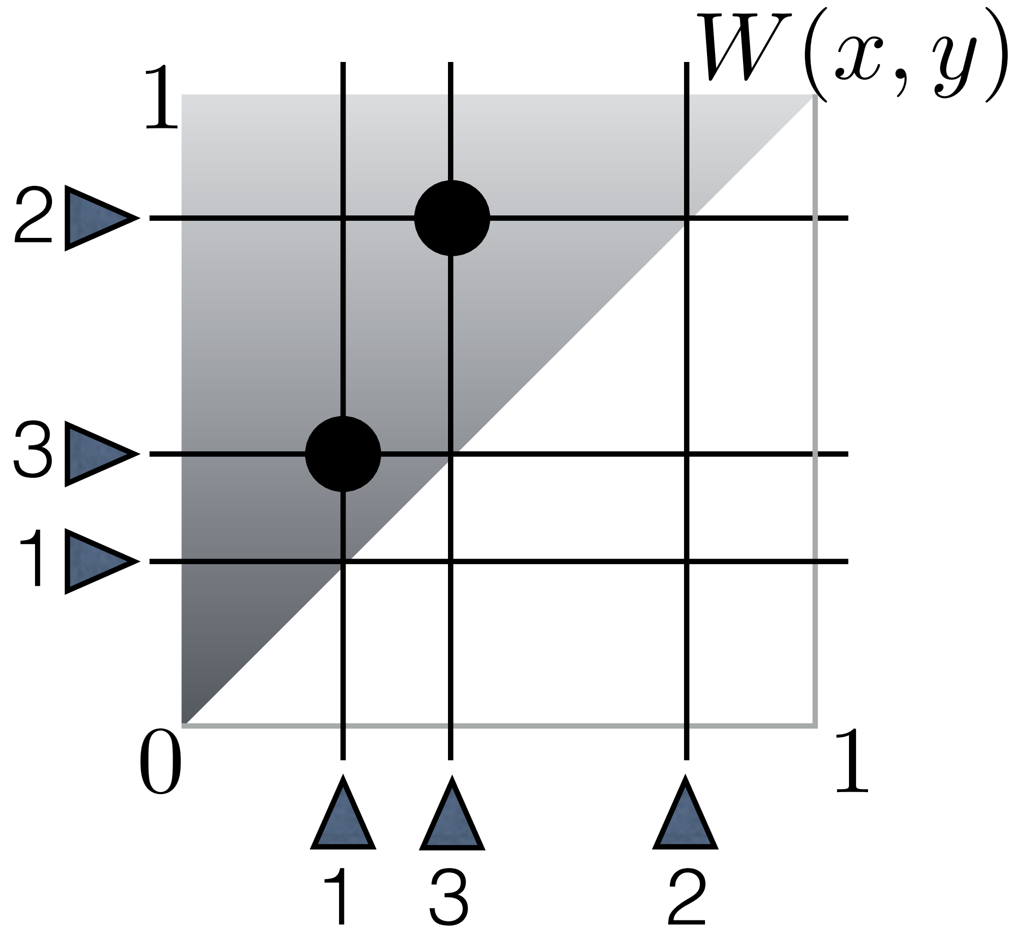

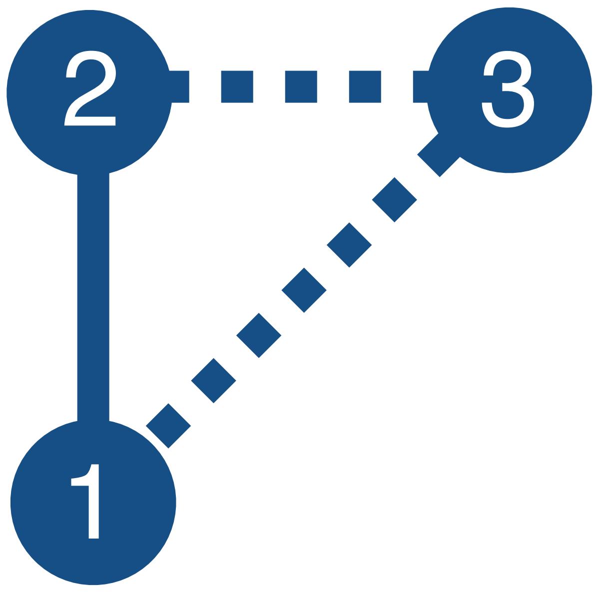

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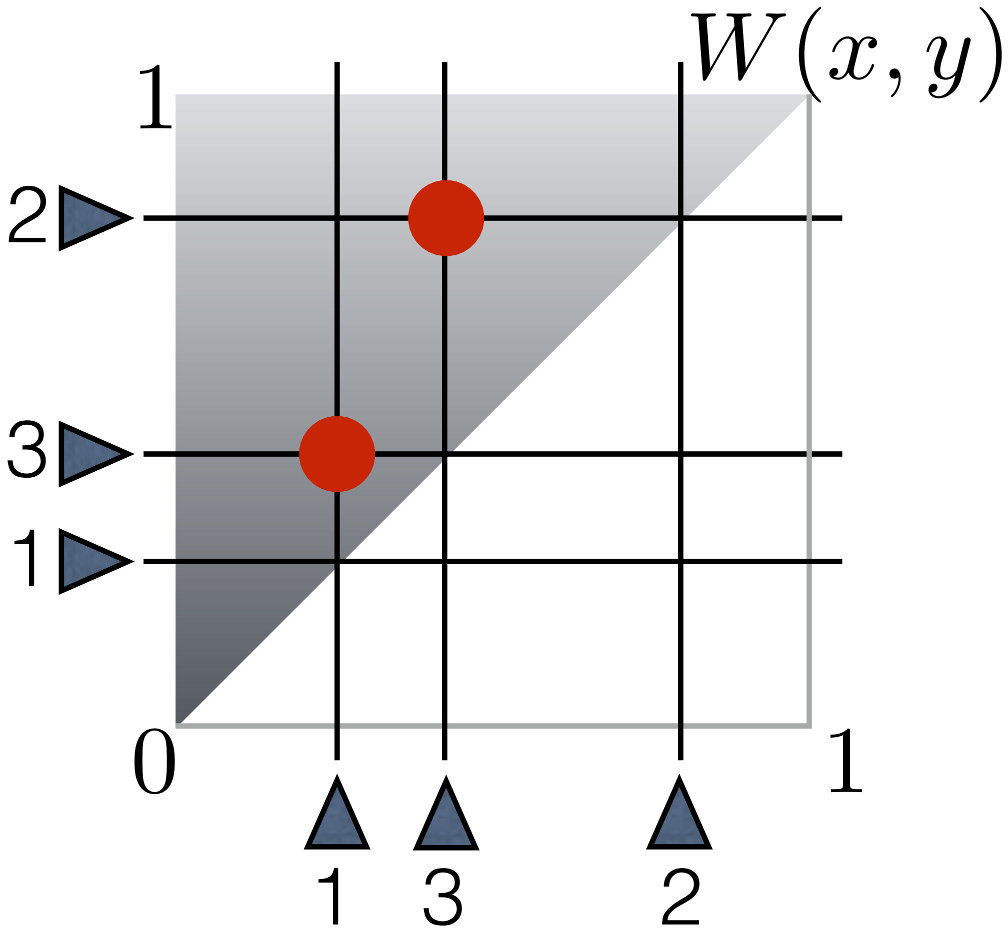

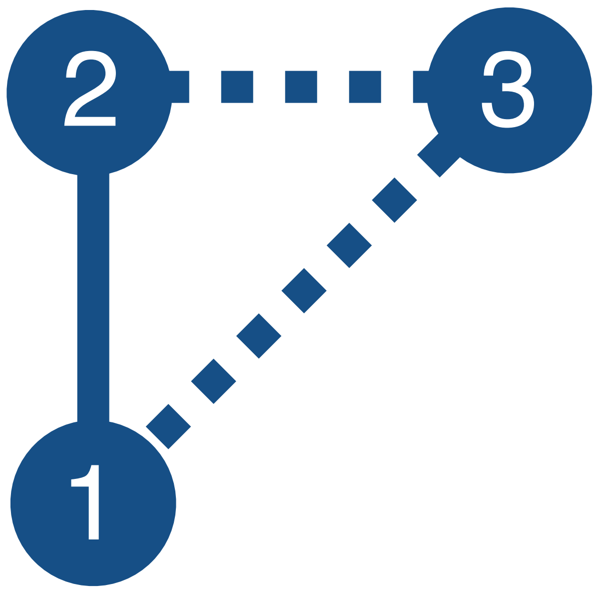

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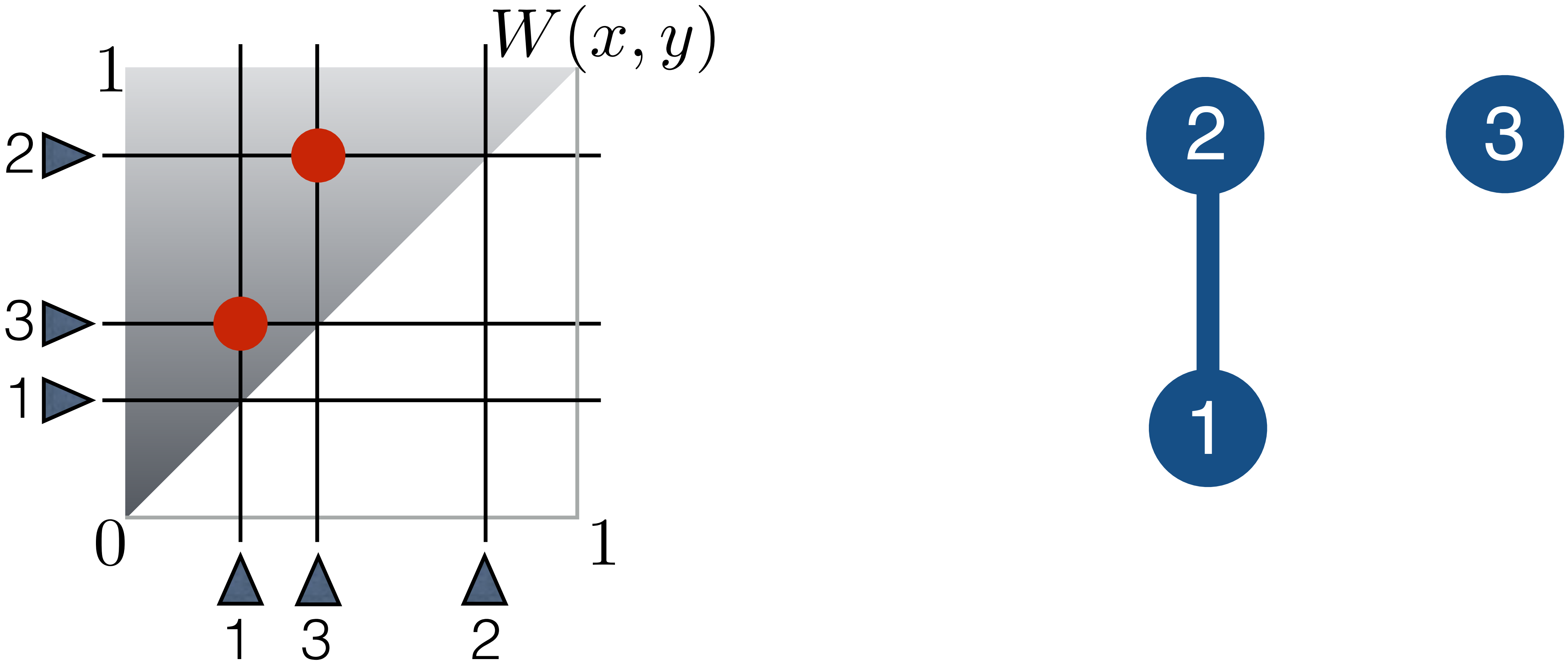

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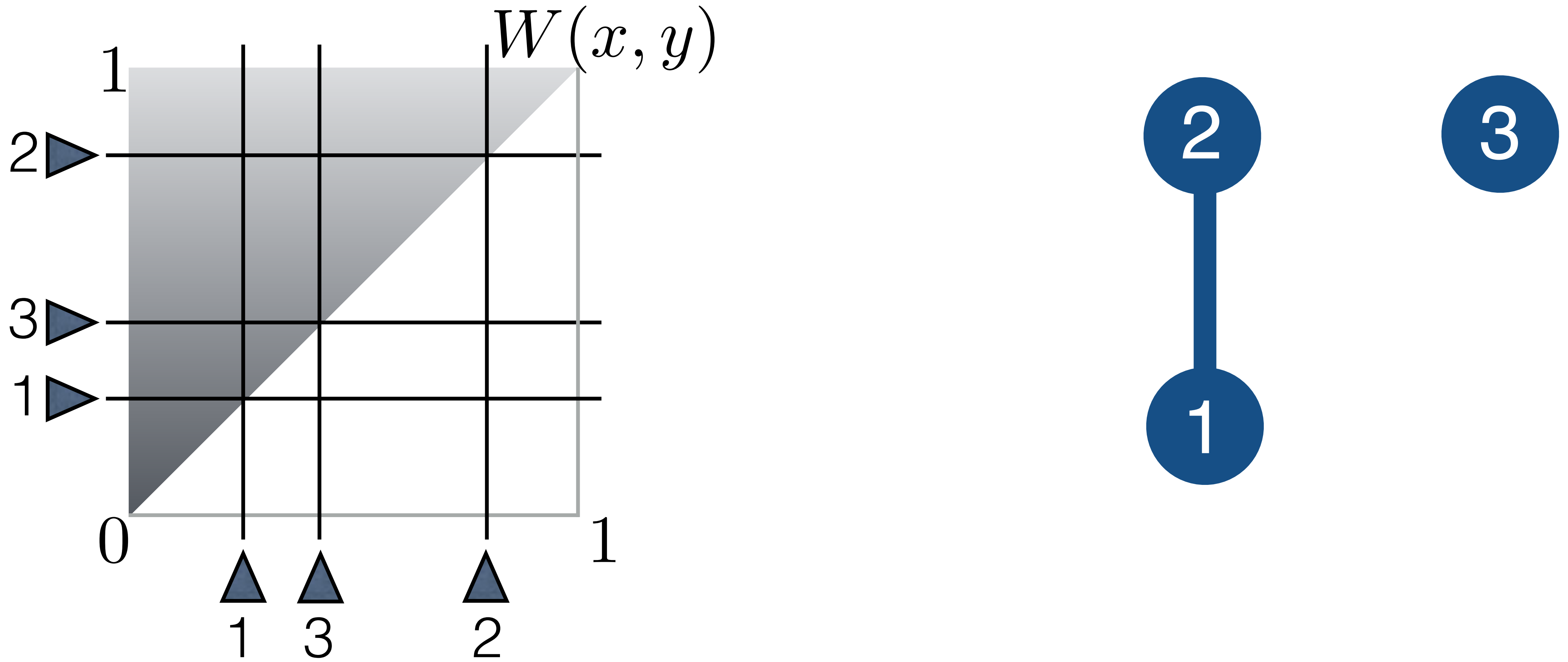

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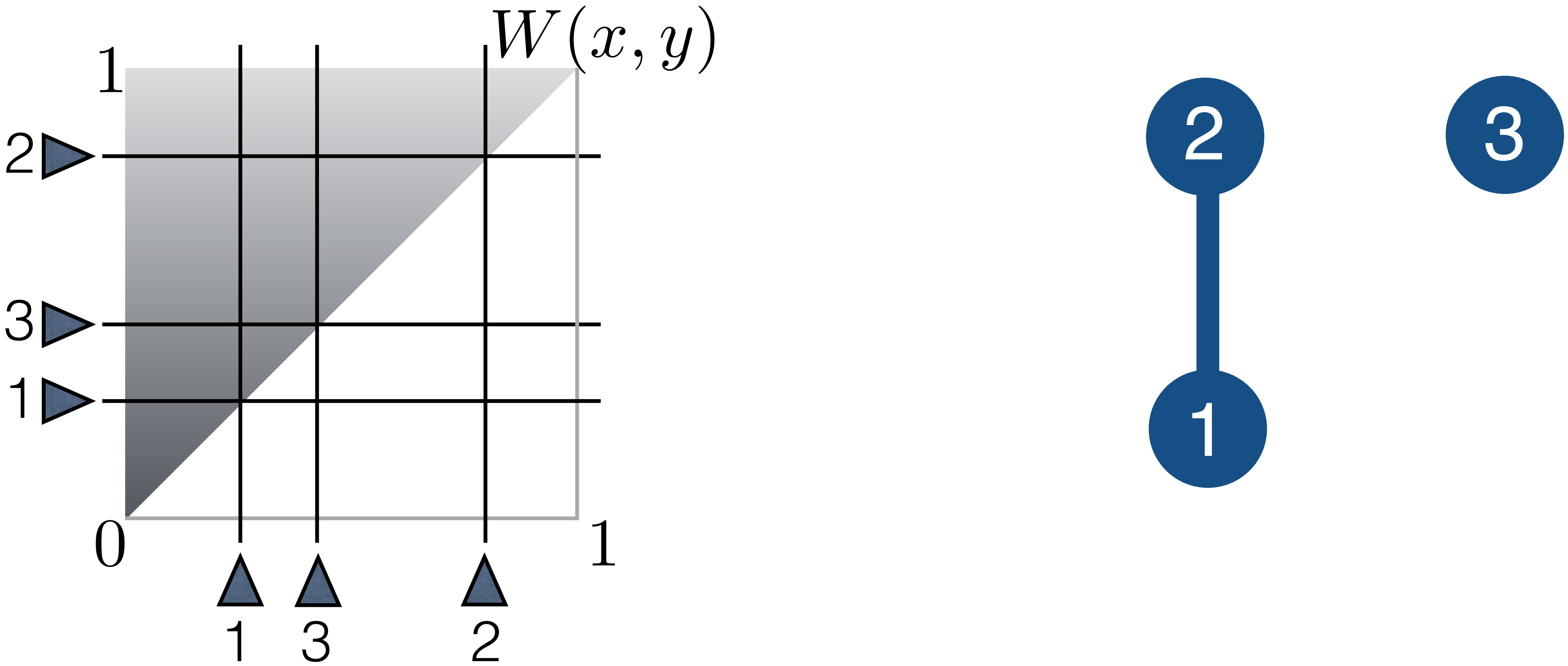

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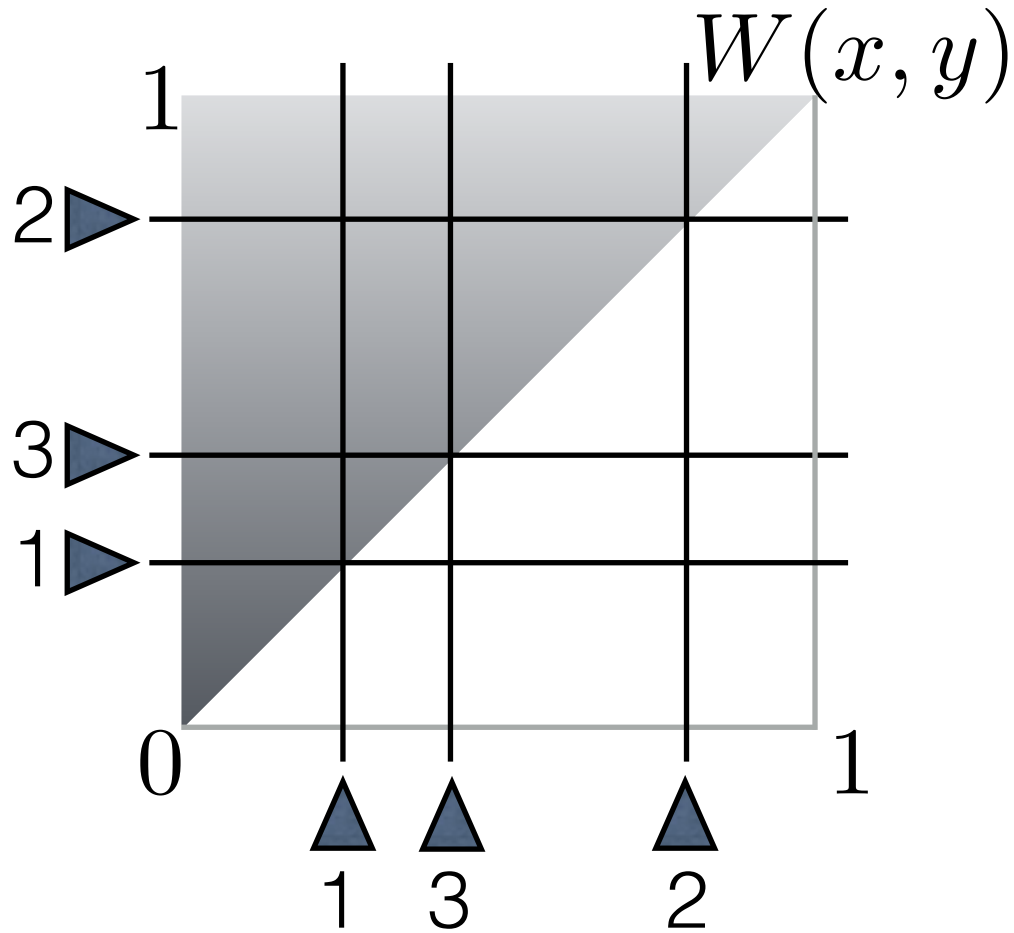

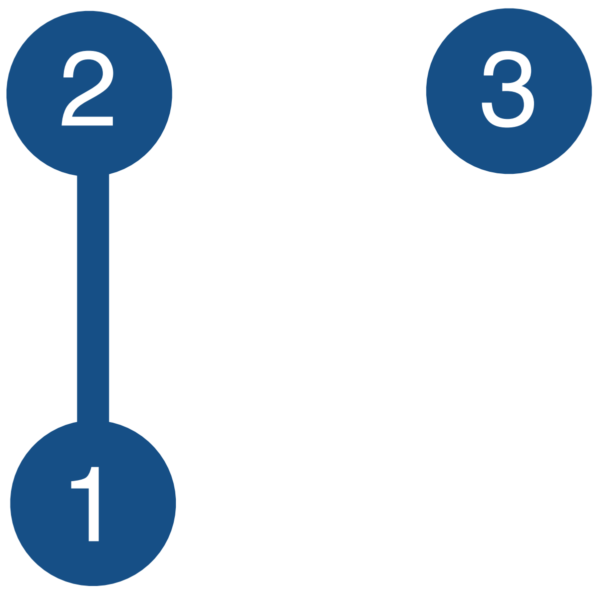

Thm (AH). Every node-exch. graph seq. has a *graphon* rep.

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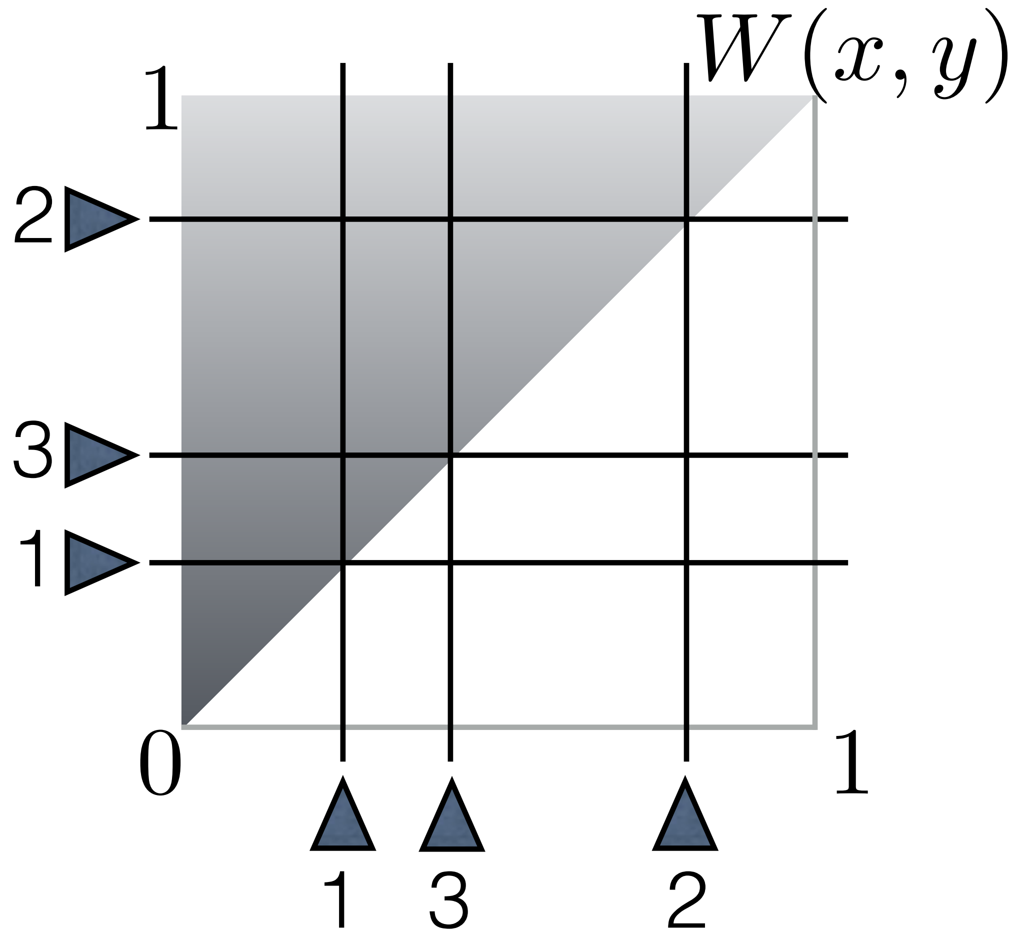

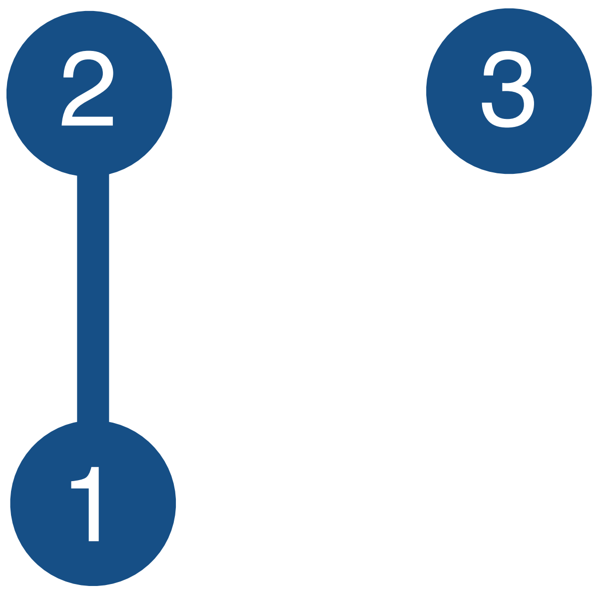

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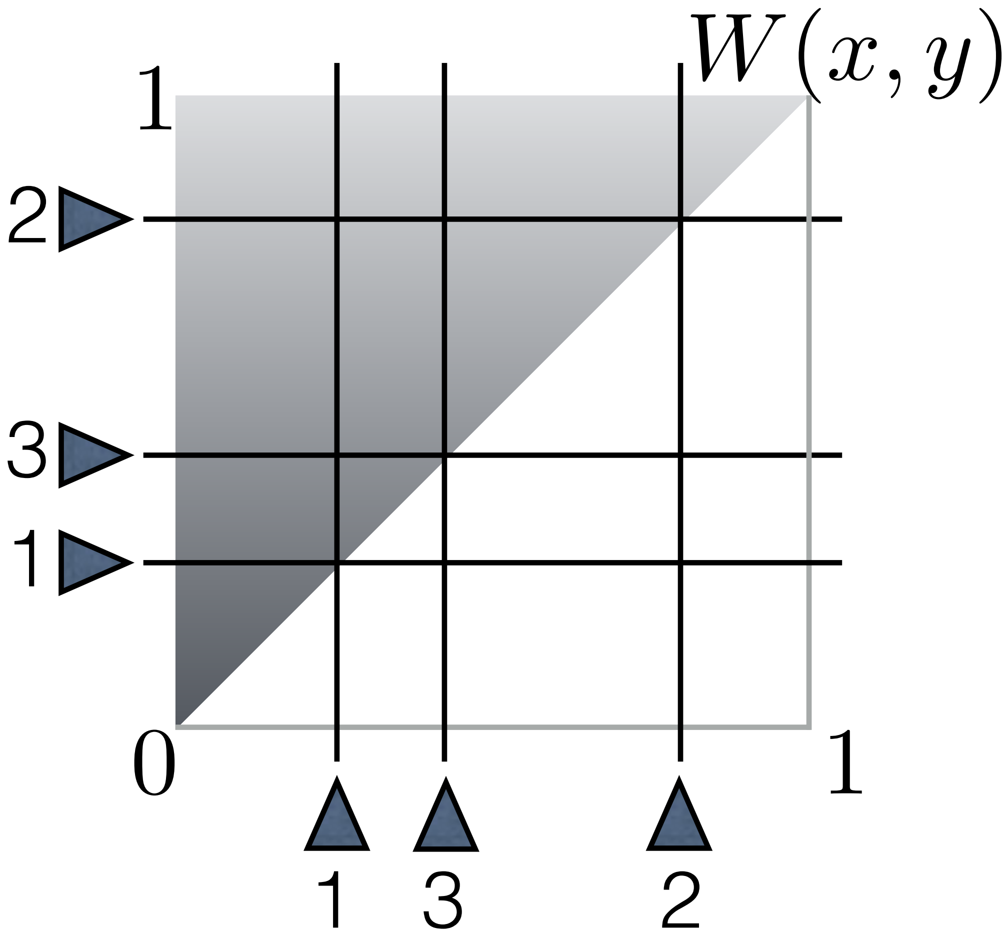

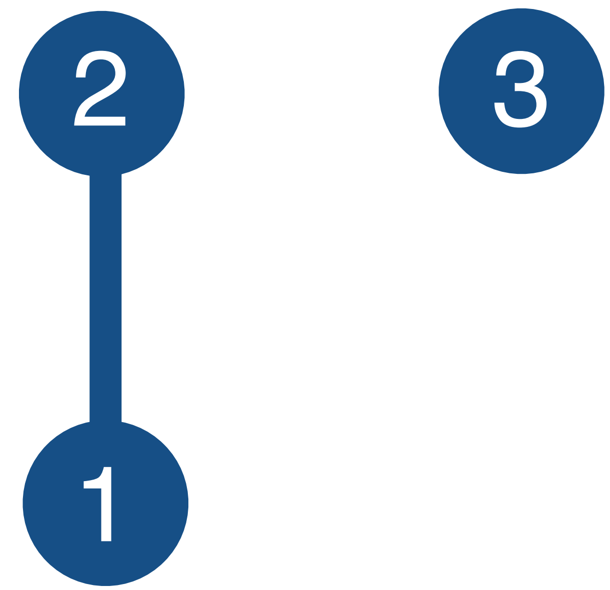

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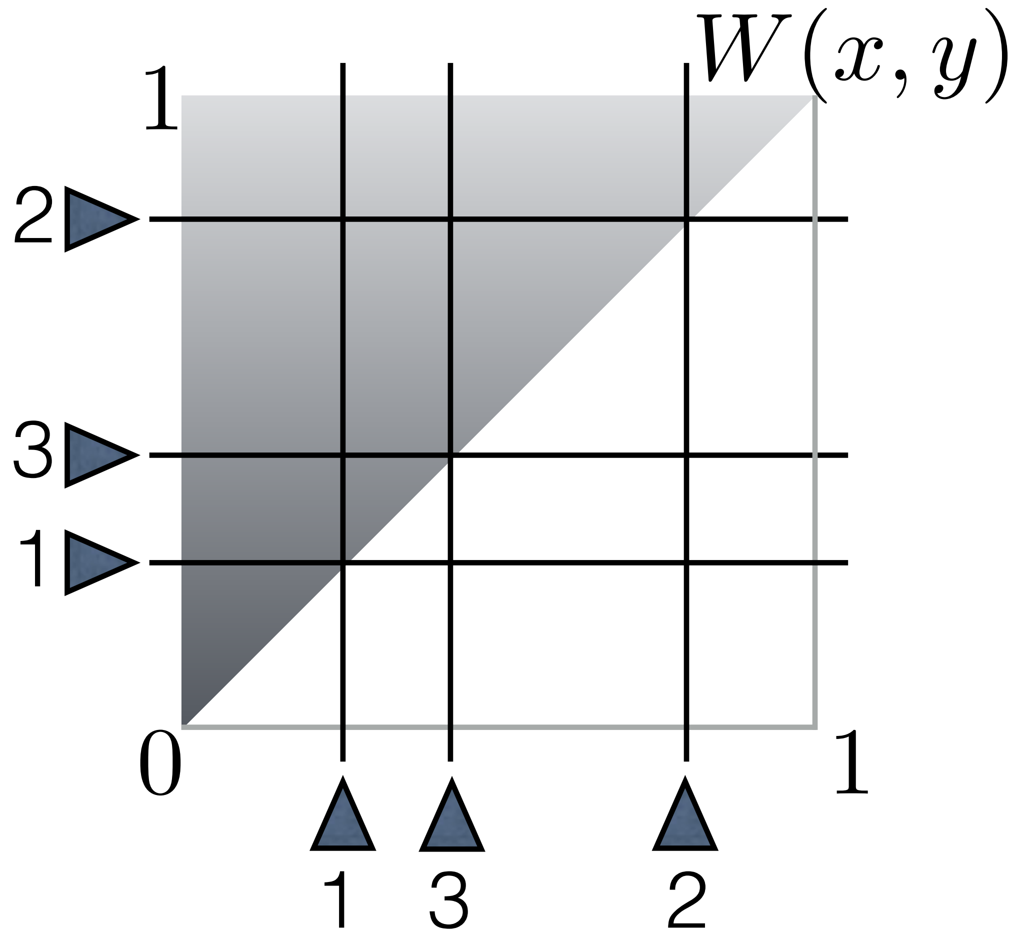

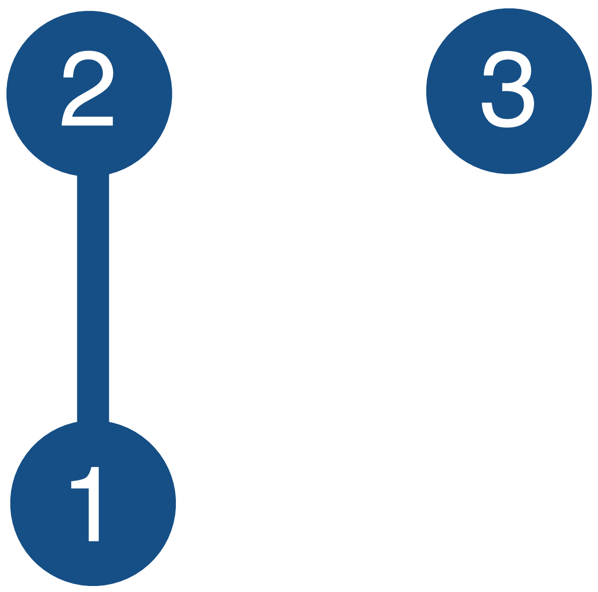

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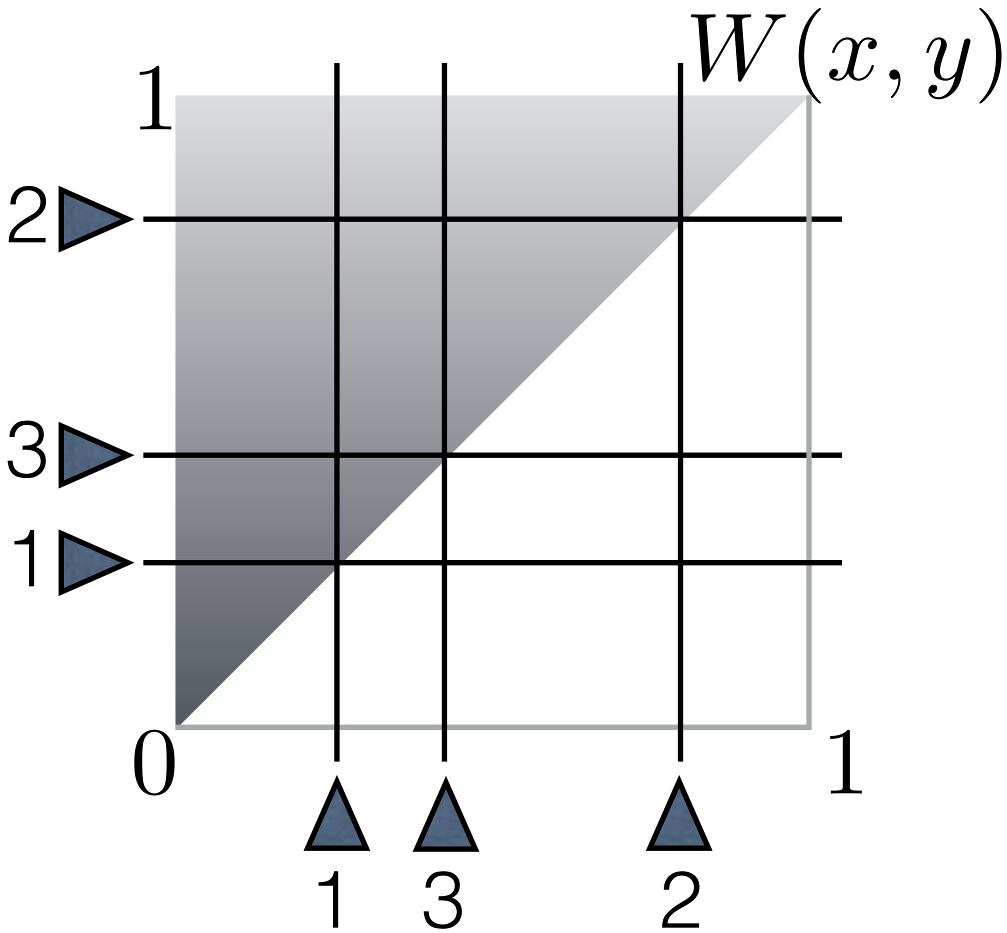

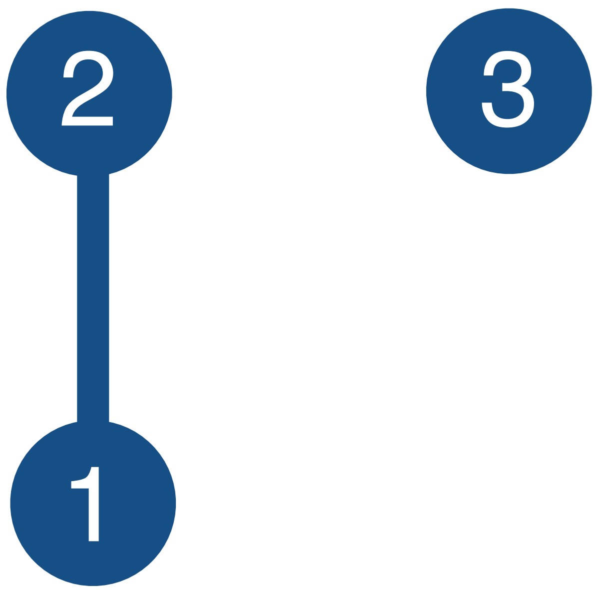

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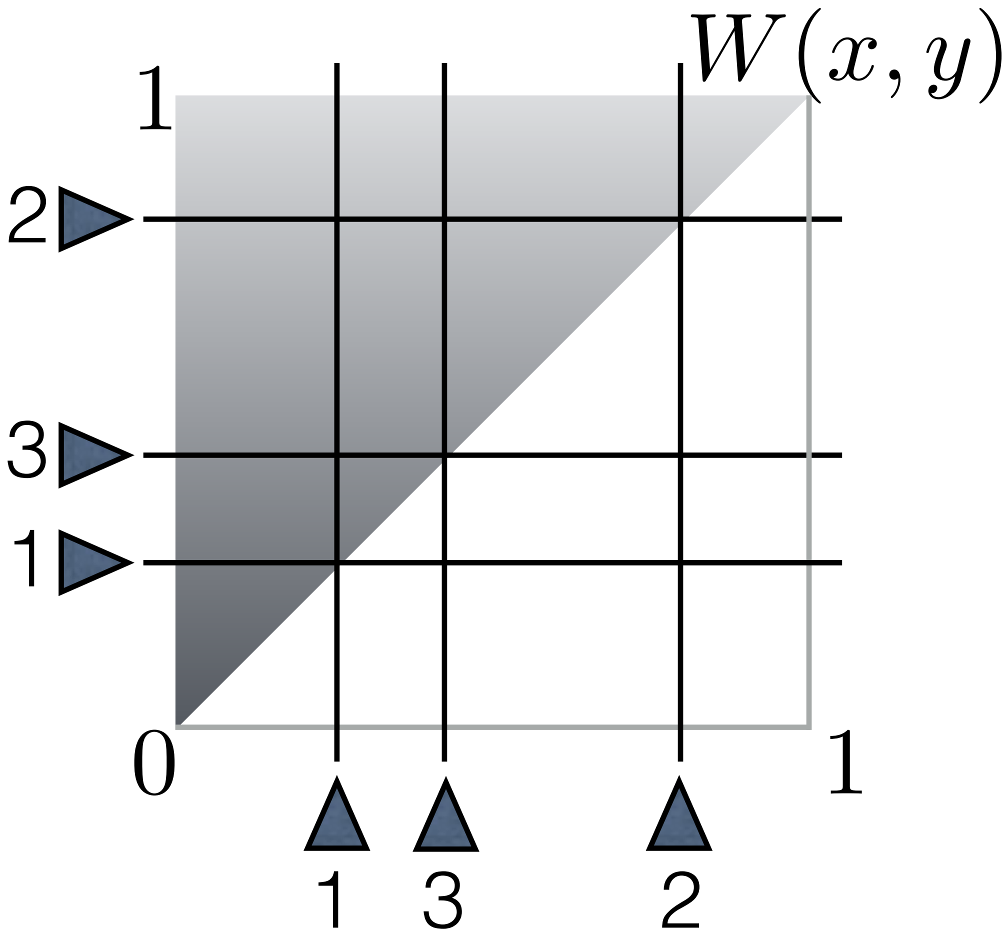

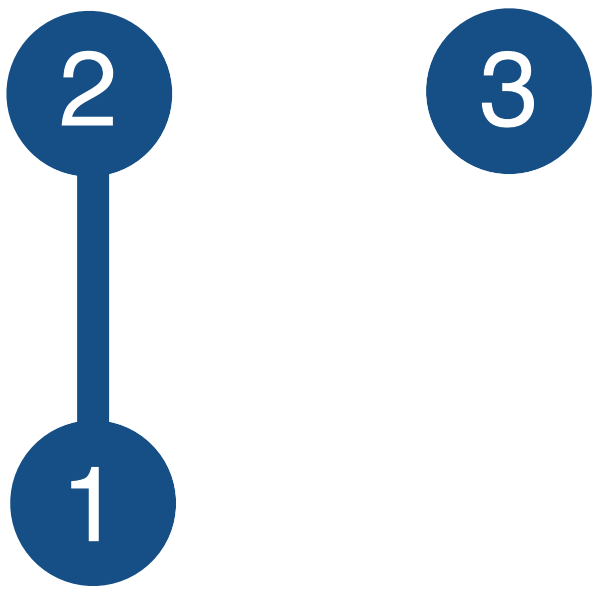

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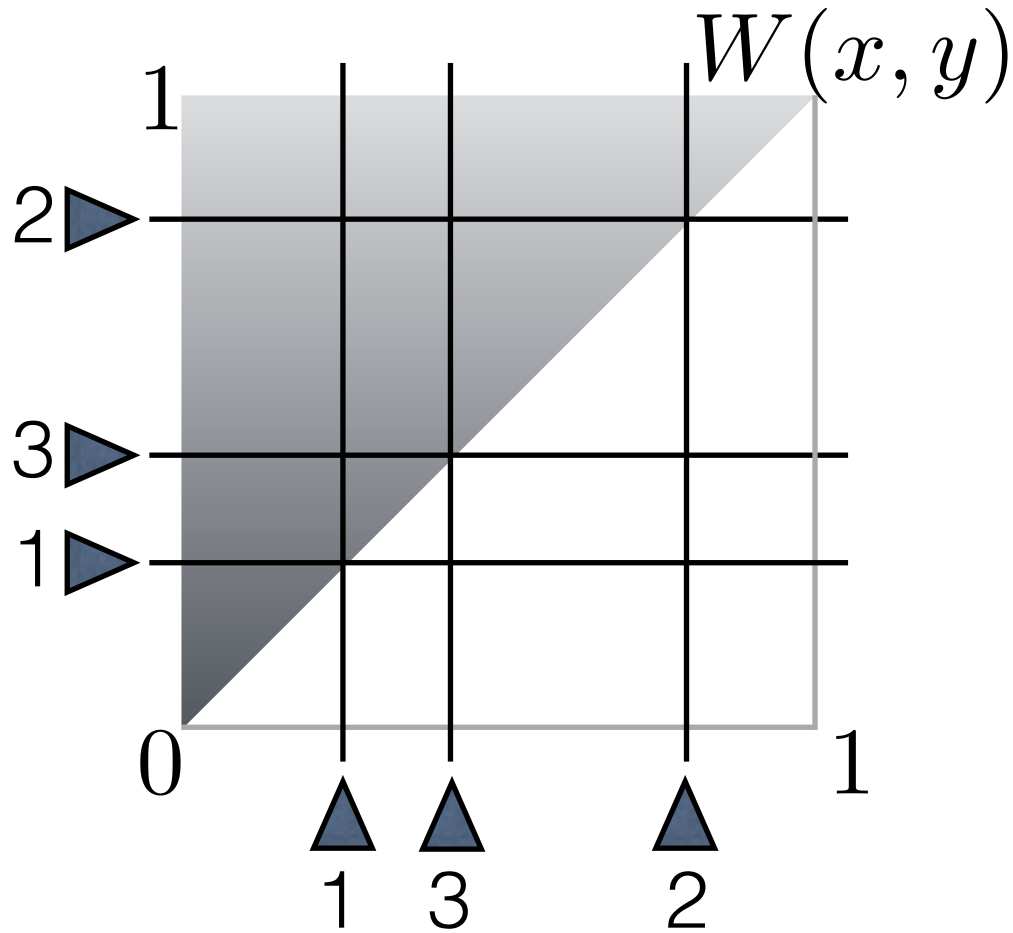

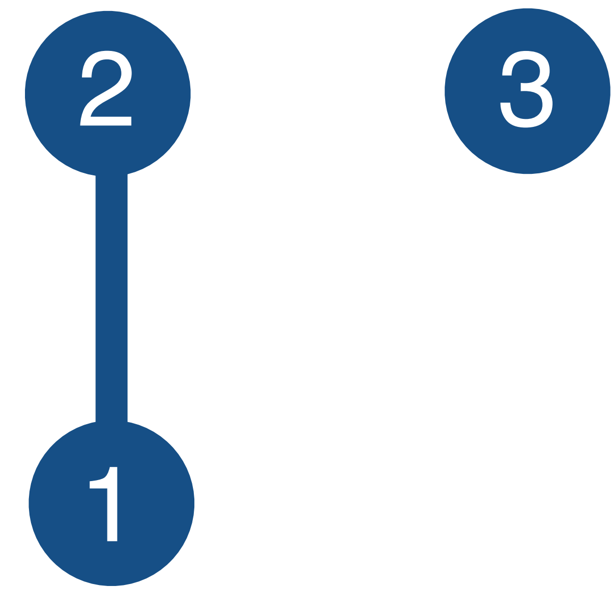

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Intuition: To a given node, all other nodes look the same.

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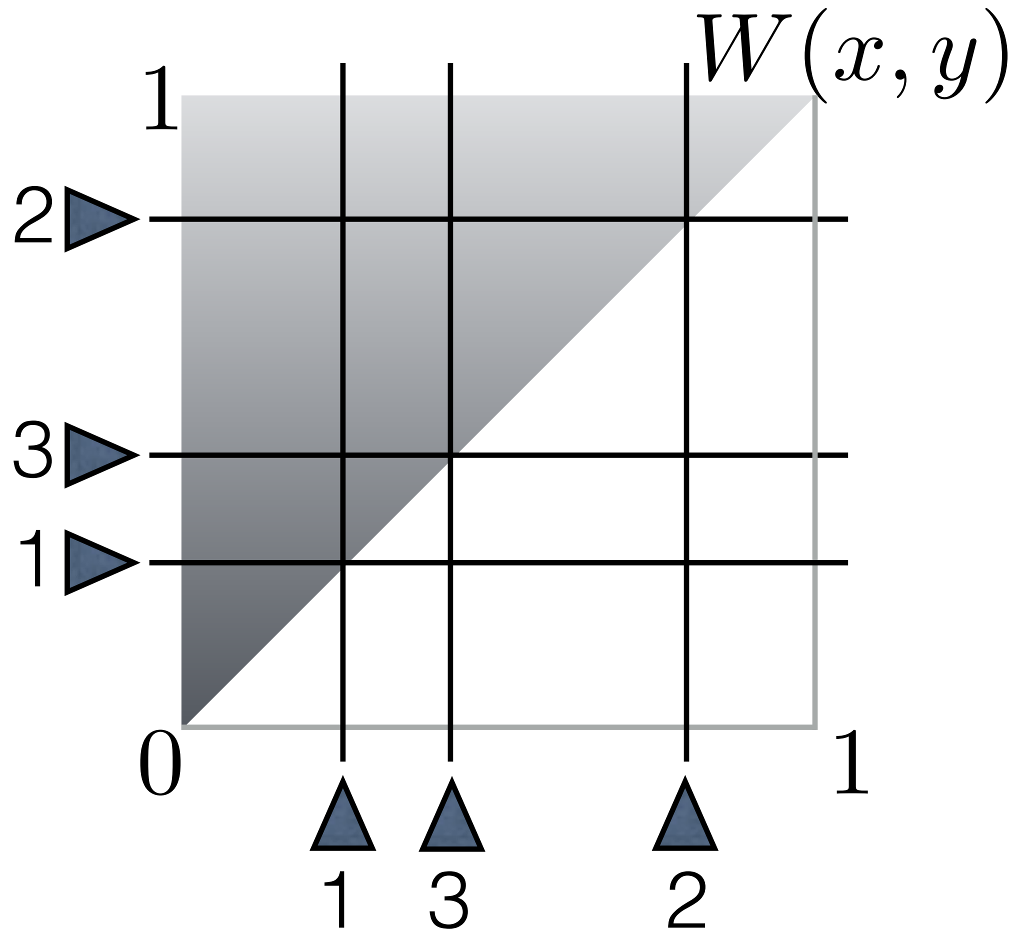

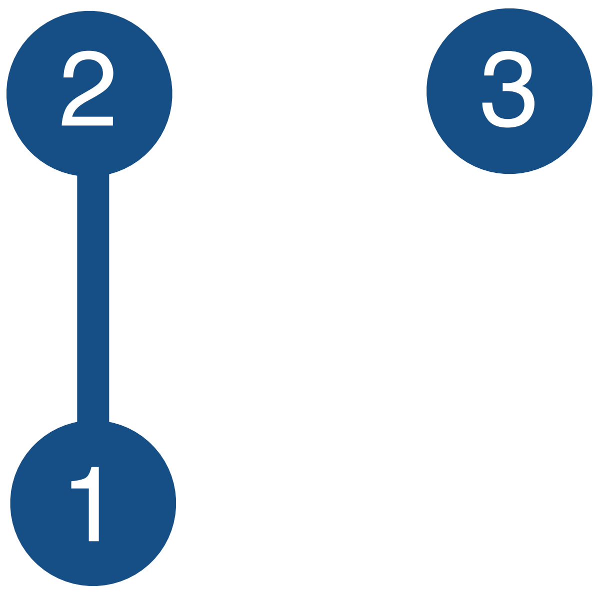

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Peter Orbanz Monday 11:30am

[Hoover 1979, Aldous 1981, Lloyd et al 2012, Orbanz, Roy 2015]

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A: **Many** ideas

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[Caron, Fox 2017; Veitch, Roy 2015, 2016; Borgs, Chayes, Cohn, Holden 2016; Broderick, Cai 2015; Cai, Broderick 2015a,b; Crane, Dempsey 2015a,b, 2016a; Cai, Campbell, Broderick 2016, Williamson 2016]

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[Caron, Fox 2017; Veitch, Roy 2015, 2016; Borgs, Chayes, Cohn, Holden 2016; Broderick, Cai 2015; Cai, Broderick 2015a,b; Crane, Dempsey 2015a,b, 2016a; Cai, Campbell, Broderick 2016, Williamson 2016]

- Graphexes, at this workshop:
  - Chayes, Monday 1:45pm
  - Borgs, Monday 2:30pm

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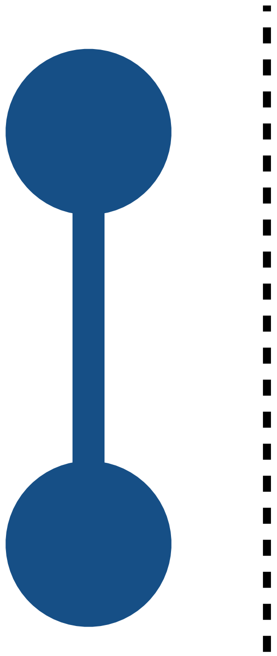
- Global sparsity and local density, at this workshop:
  - Williamson, Wedn 1:45pm
- Idea: exchange the edges instead of nodes
  - Our work + Don’t miss independent graphs work by Crane & Dempsey!

A New Way: Edges



G_1

 [Broderick, Cai 2015; Cai, Broderick 2015a,b; Crane, Dempsey 2015a,b,16a; Cai, Campbell, Broderick 2016]

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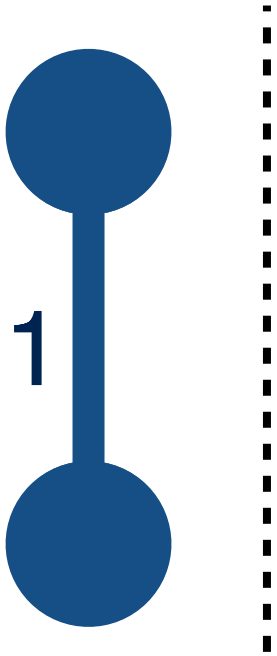

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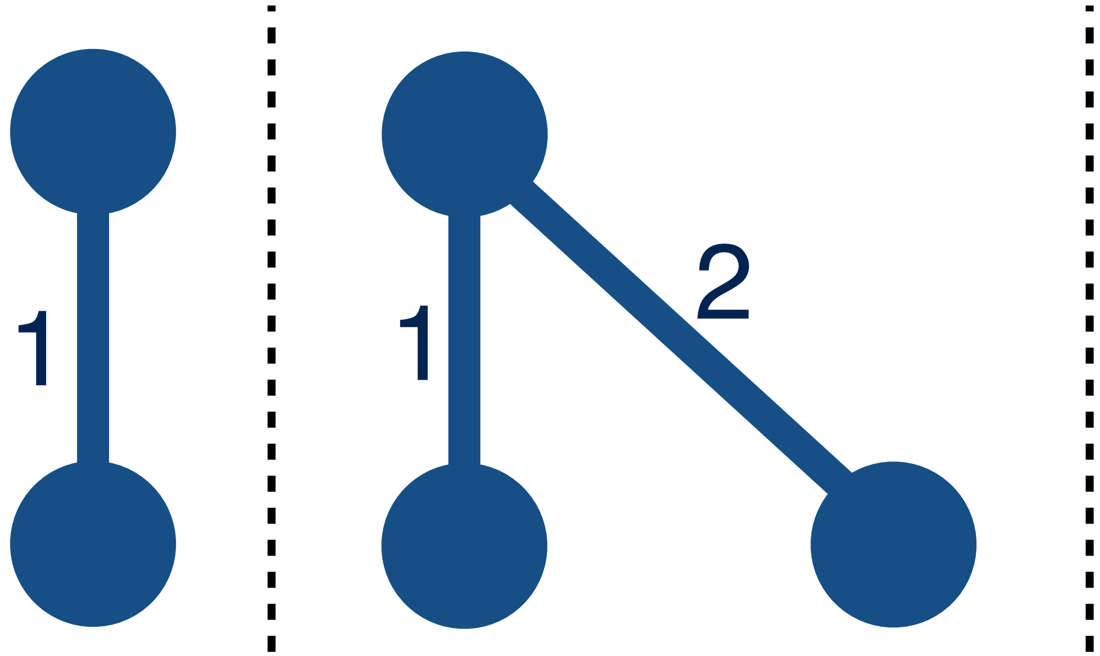

G_1 G_2

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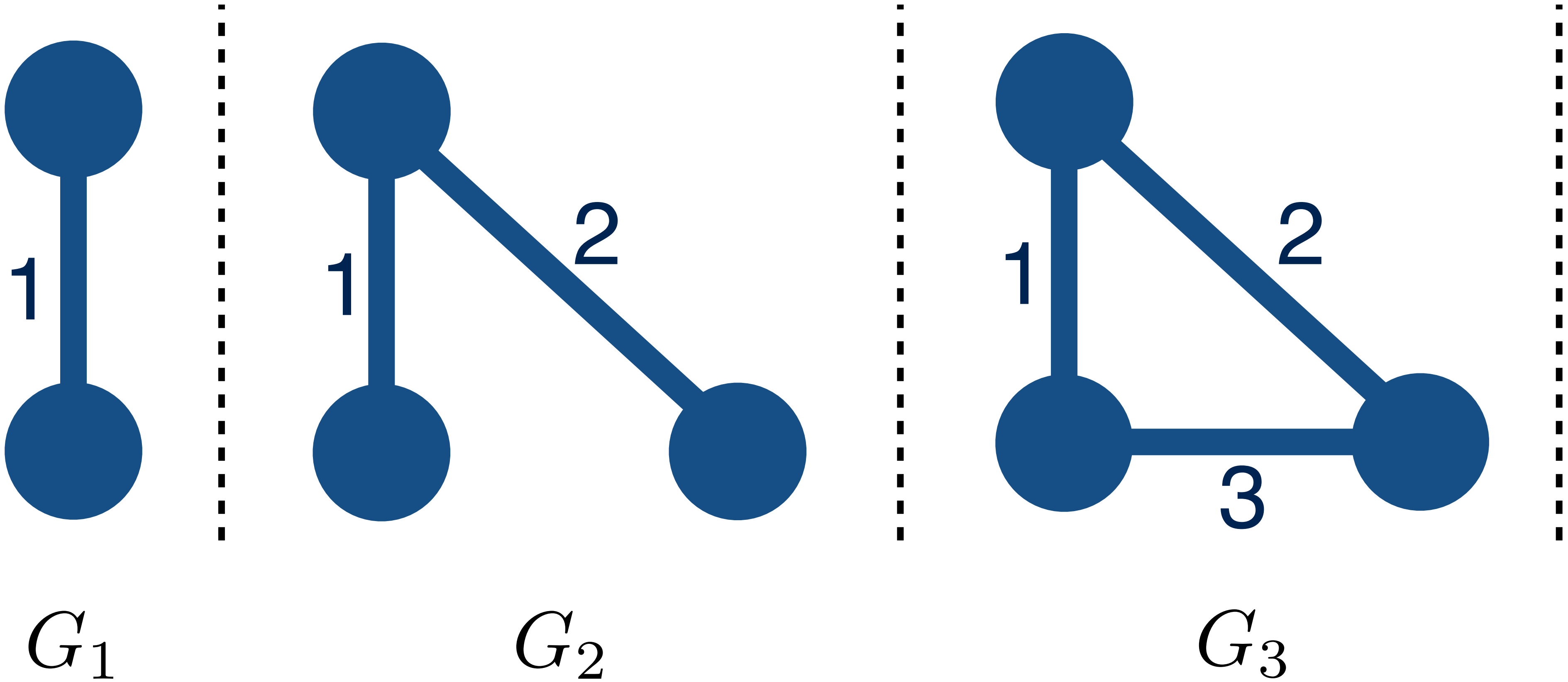

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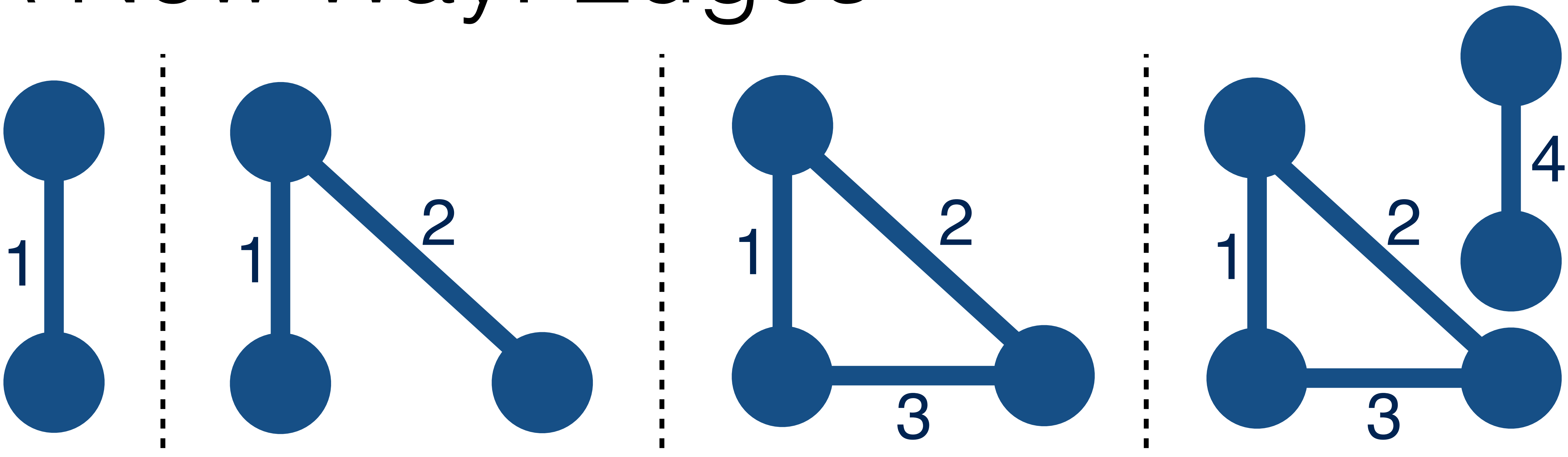

G_1 G_2 G_3 G_4

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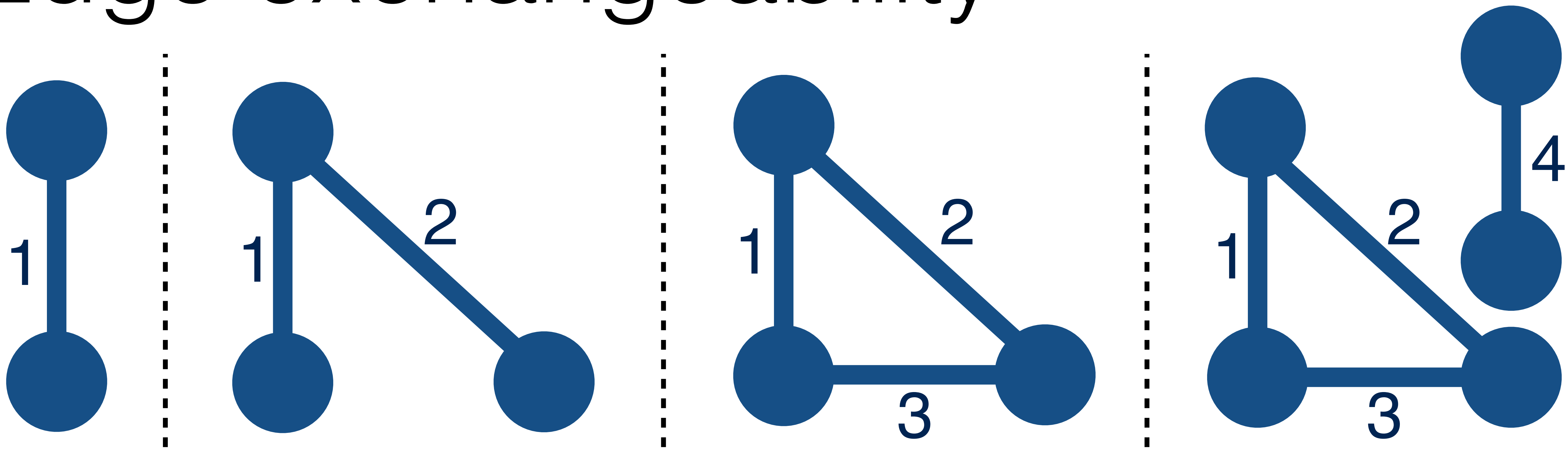
Edge exchangeability



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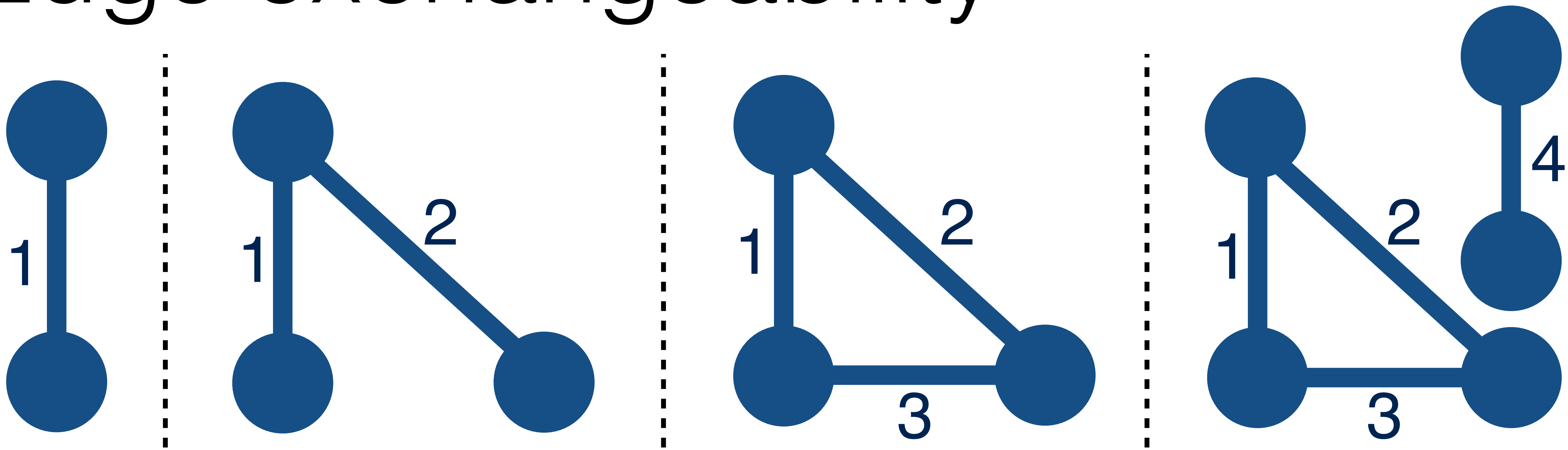
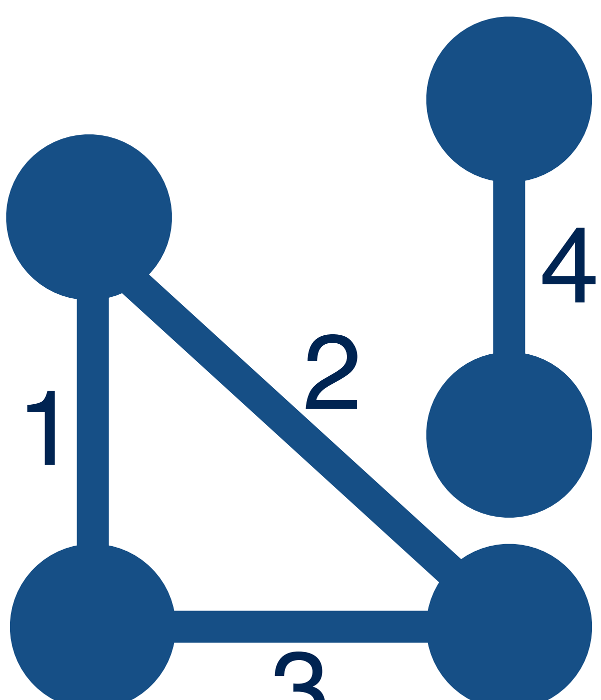


G_1 G_2 G_3 G_4



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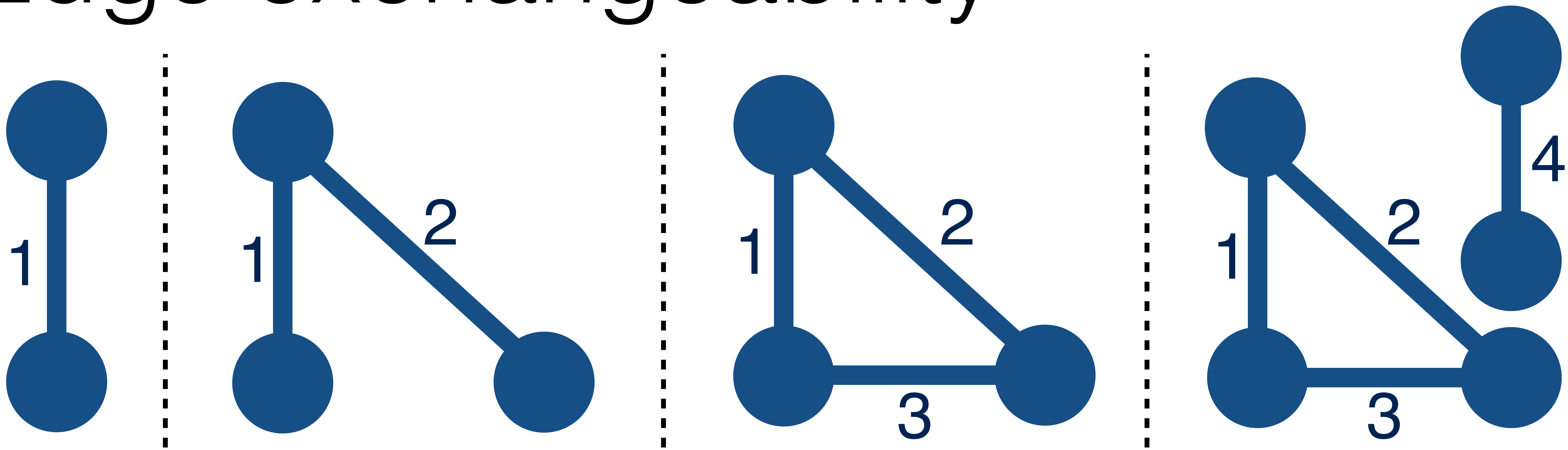

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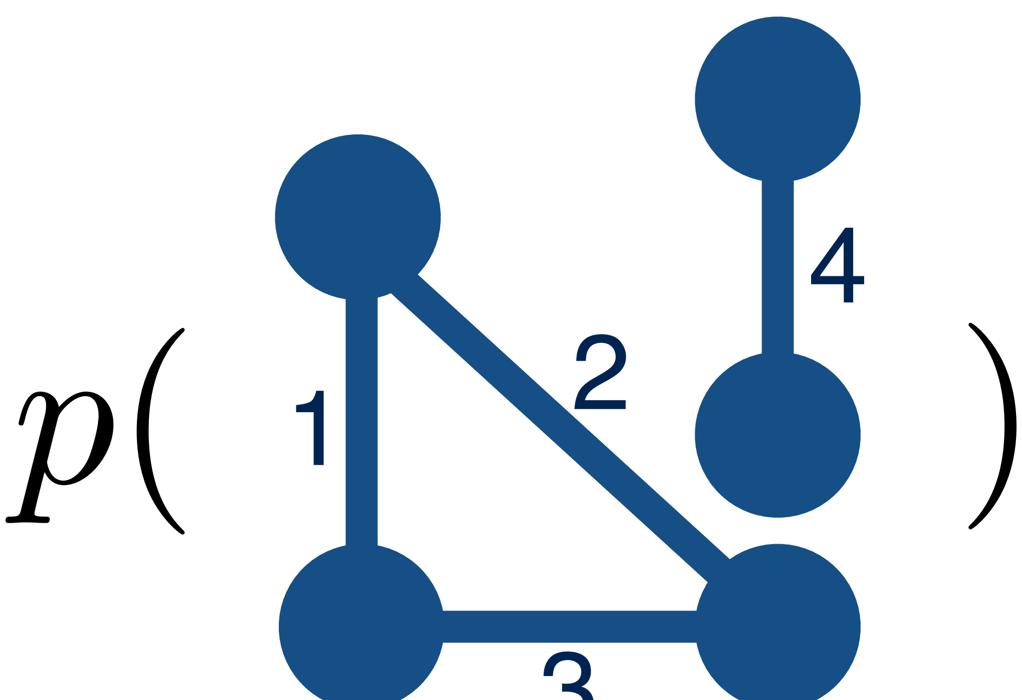

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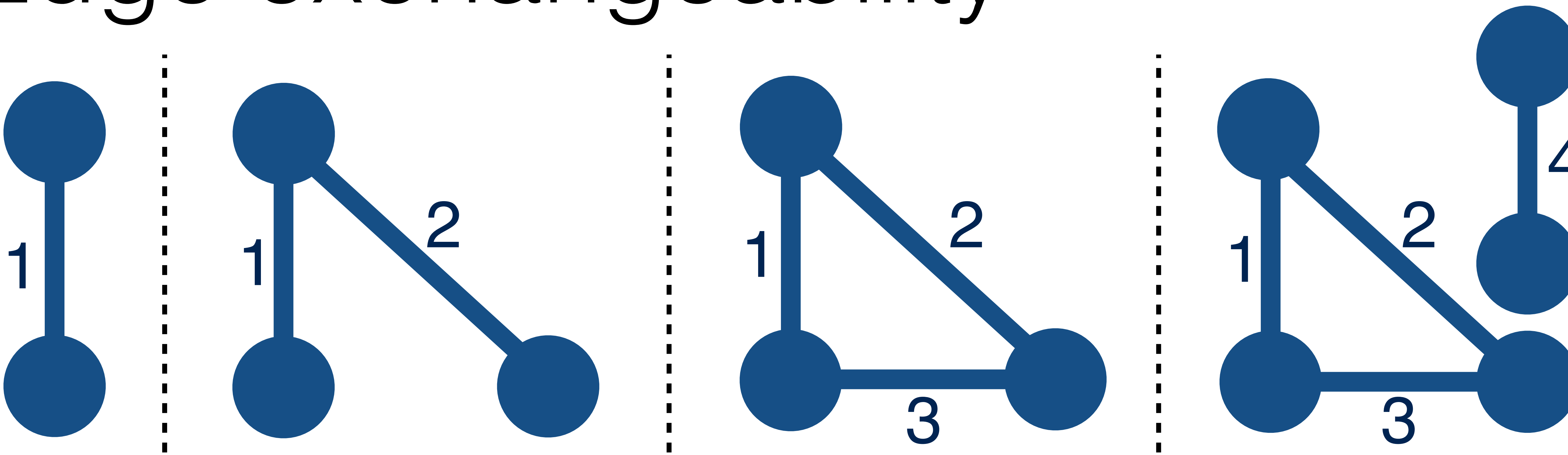

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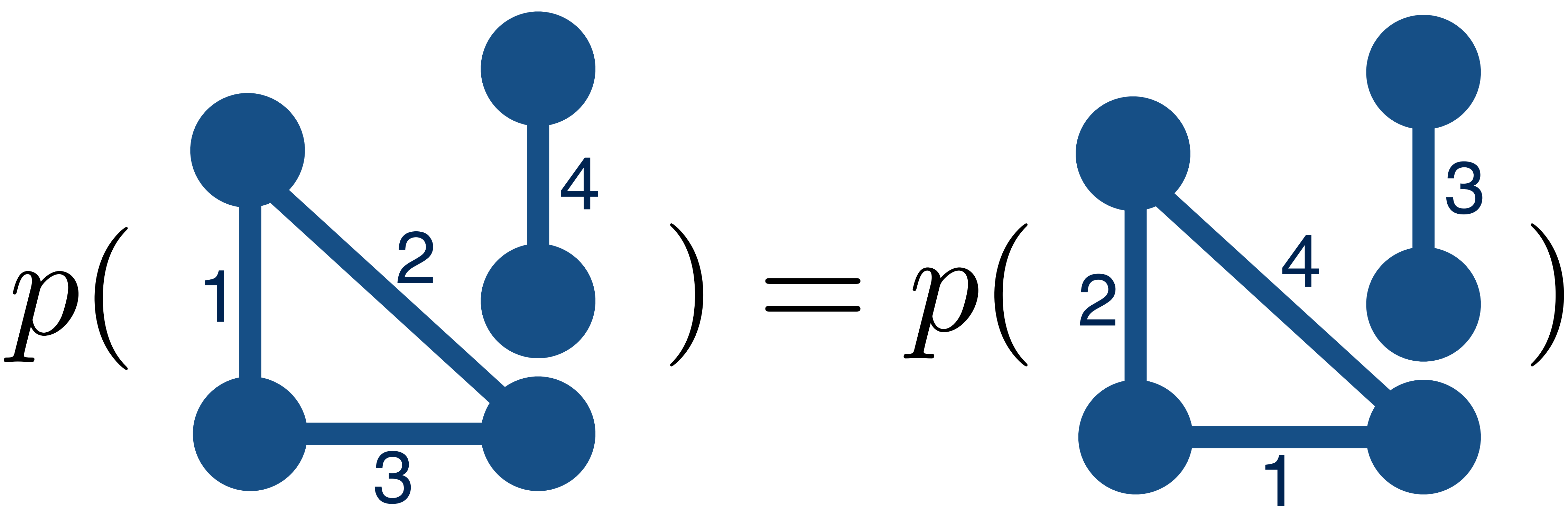

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Edge exchangeability

Thm. A wide class of edge-exchangeable graph models yields sparse graph sequences

Thm. A paintbox-style characterization for edge-exchangeable graph sequences

 [Broderick, Cai 2015; Cai, Broderick 2015a,b; Crane, Dempsey 2015a,b,16a; Cai, Campbell, Broderick 2016]

Roadmap

- Example problem: clustering
- Example NPBayes model: Dirichlet process (DP)
- De Finetti for clustering: Kingman Paintbox
- De Finetti for networks/graphs

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  - Why NPBayes? **Learn more from more data**
  - What does an infinite/growing number of parameters really mean (in NPBayes)? **Components vs clusters; latent vs realized**
  - Why is NPBayes challenging but practical? **Infinite dimensional parameter but finitely many realized (in practice, e.g., can integrate out or truncate the infinity)**
  - How does thinking about exchangeability help us use NPBayes in practice?

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Roadmap [http://www.tamarabroderick.com/tutorials.html]

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