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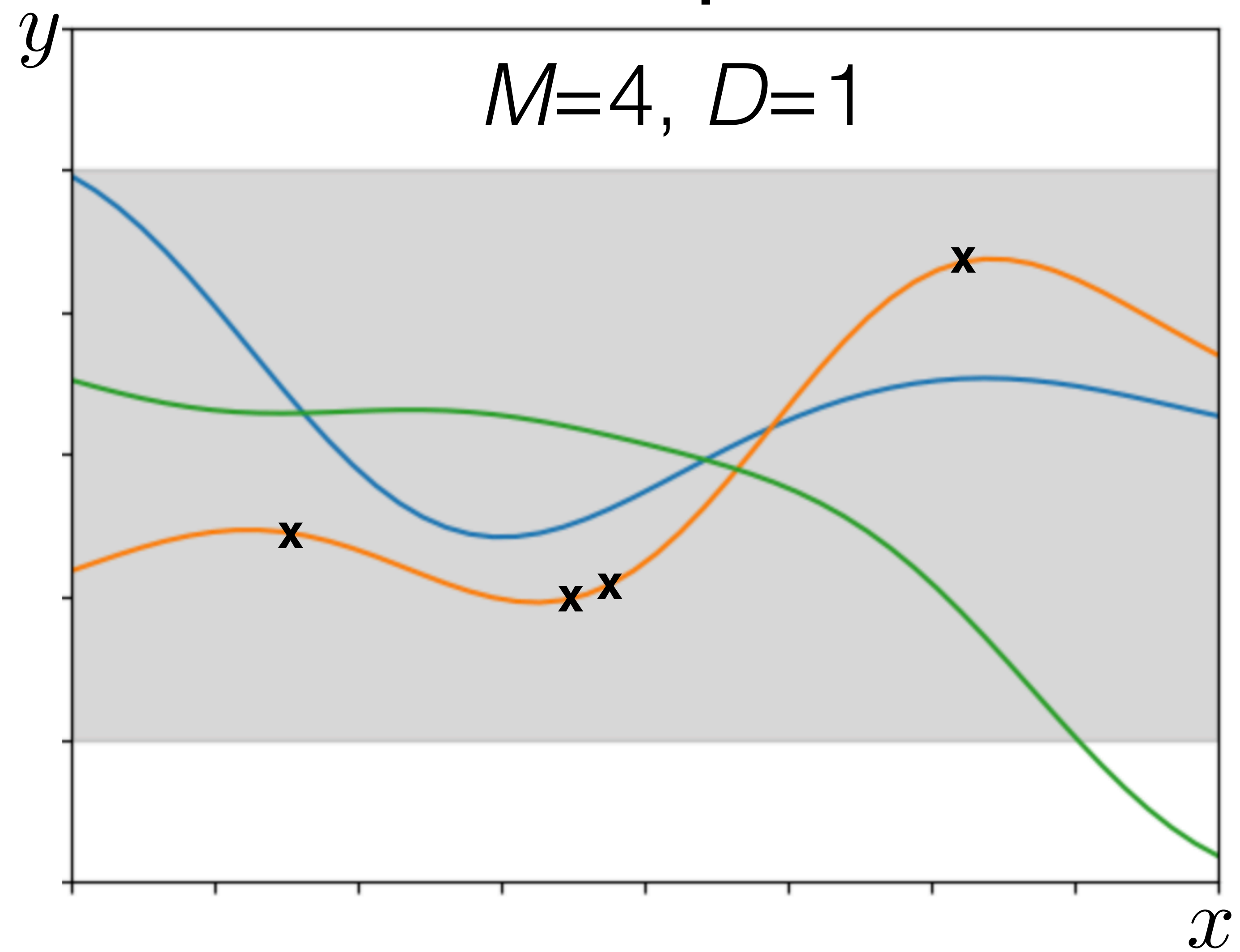

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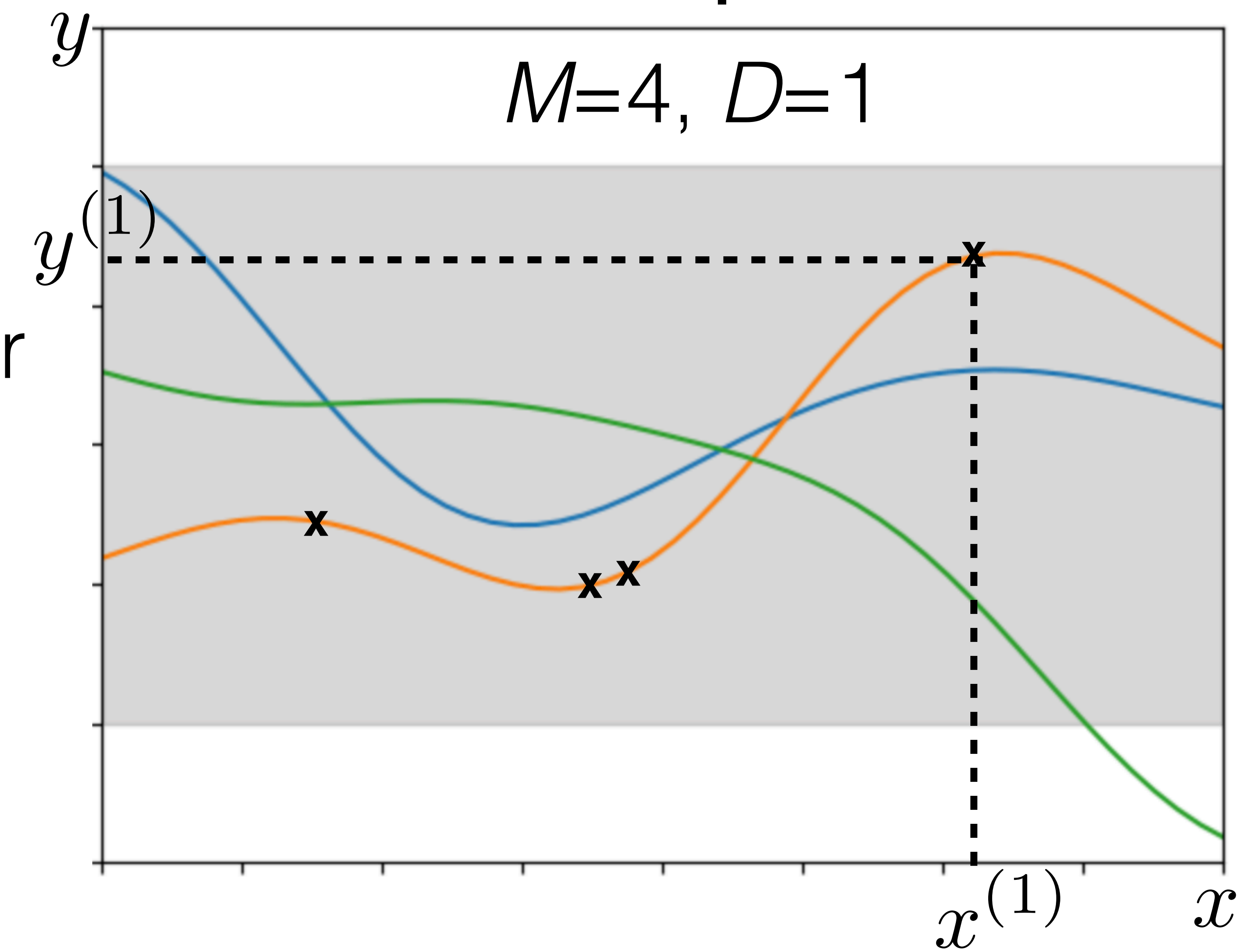

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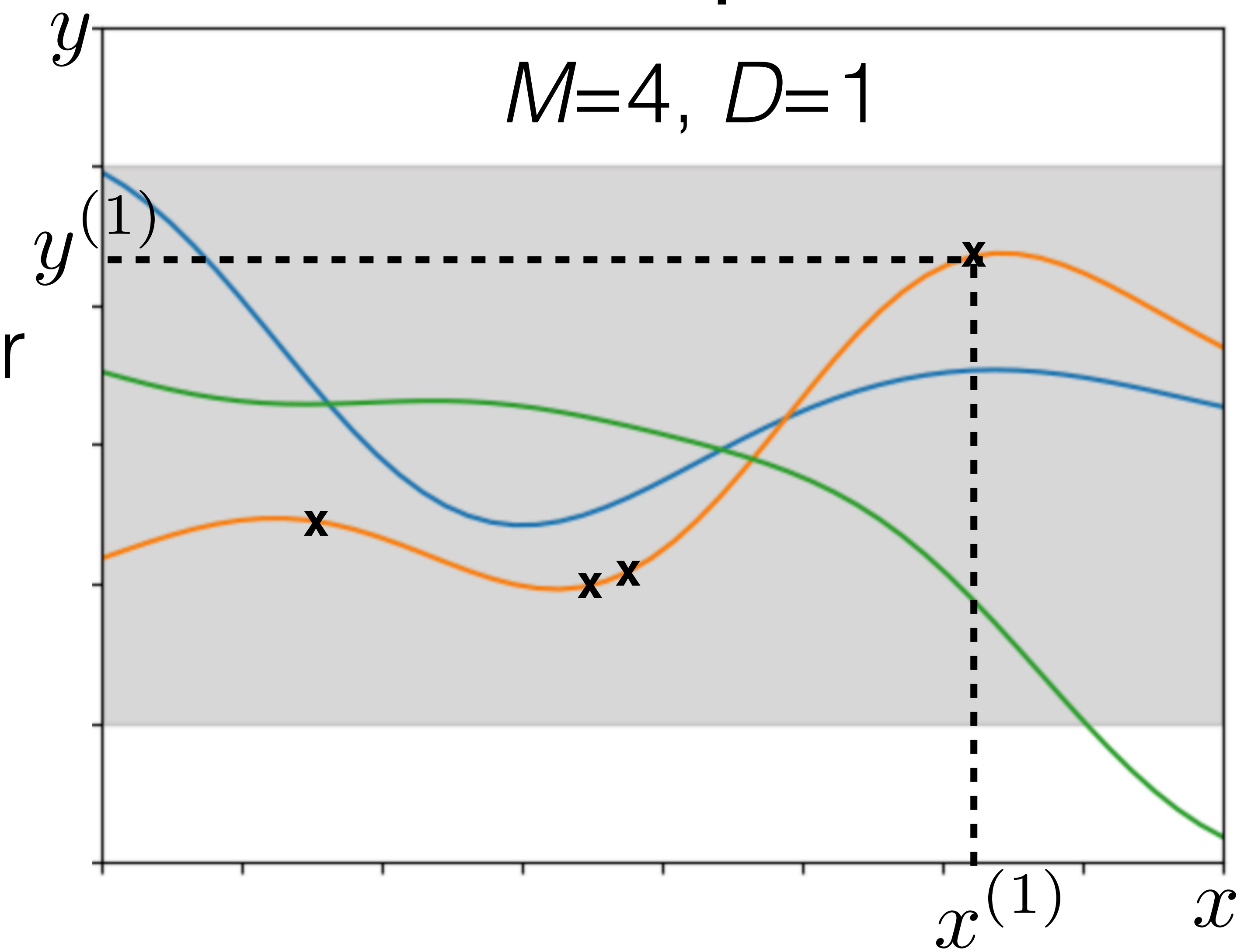

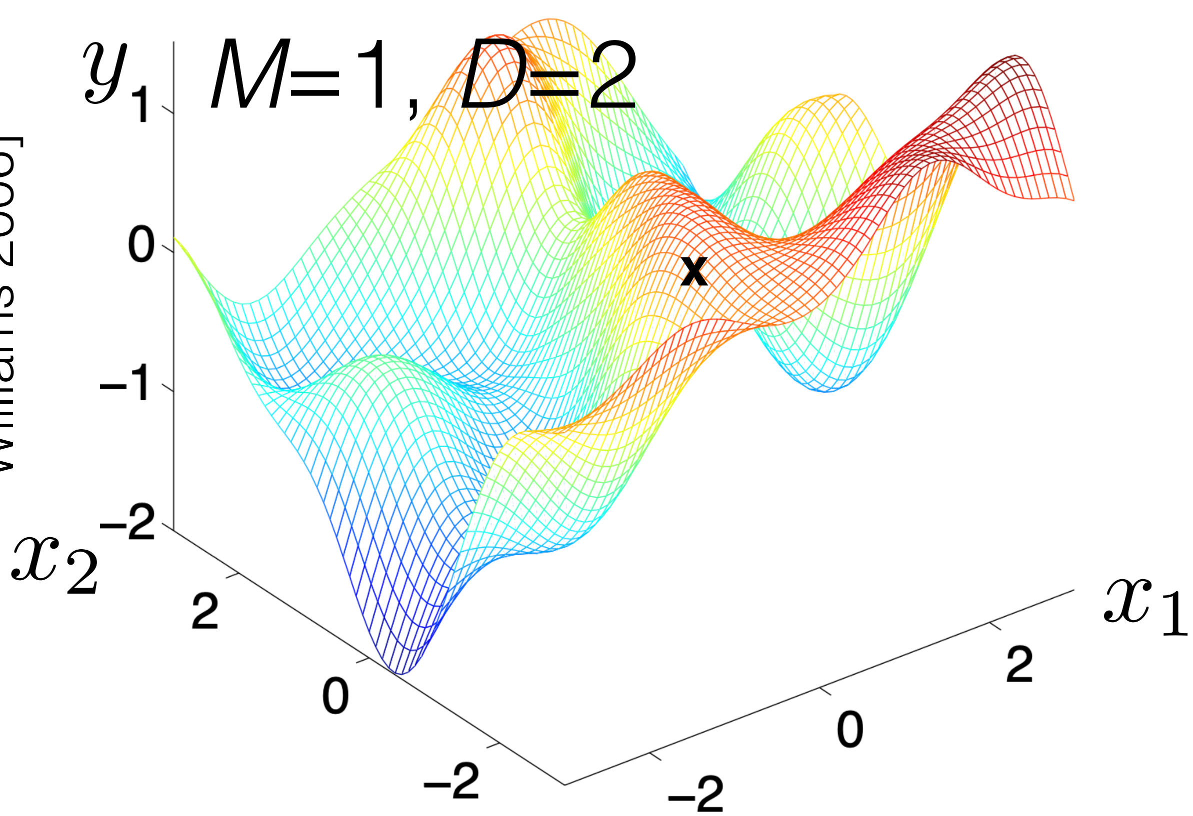

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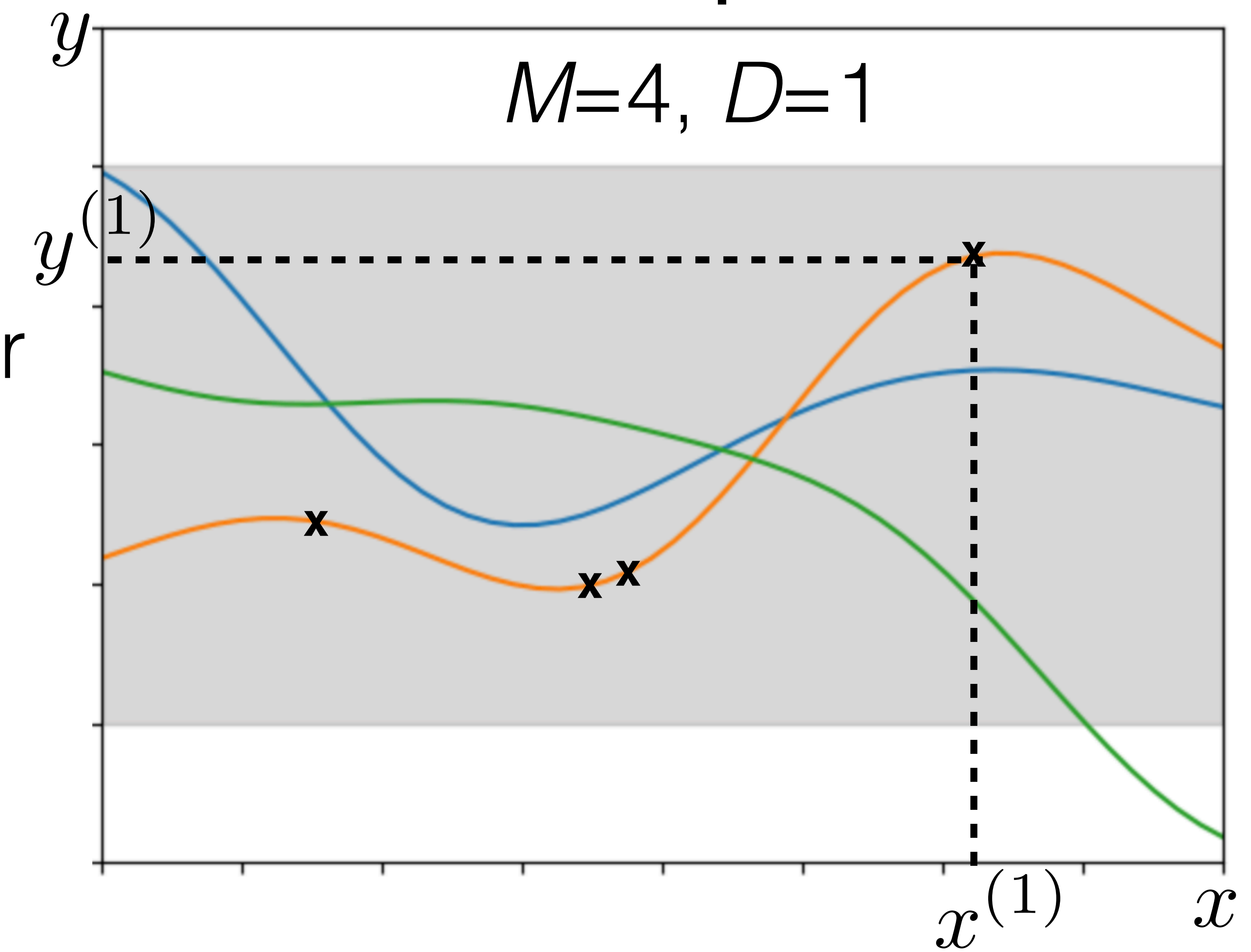

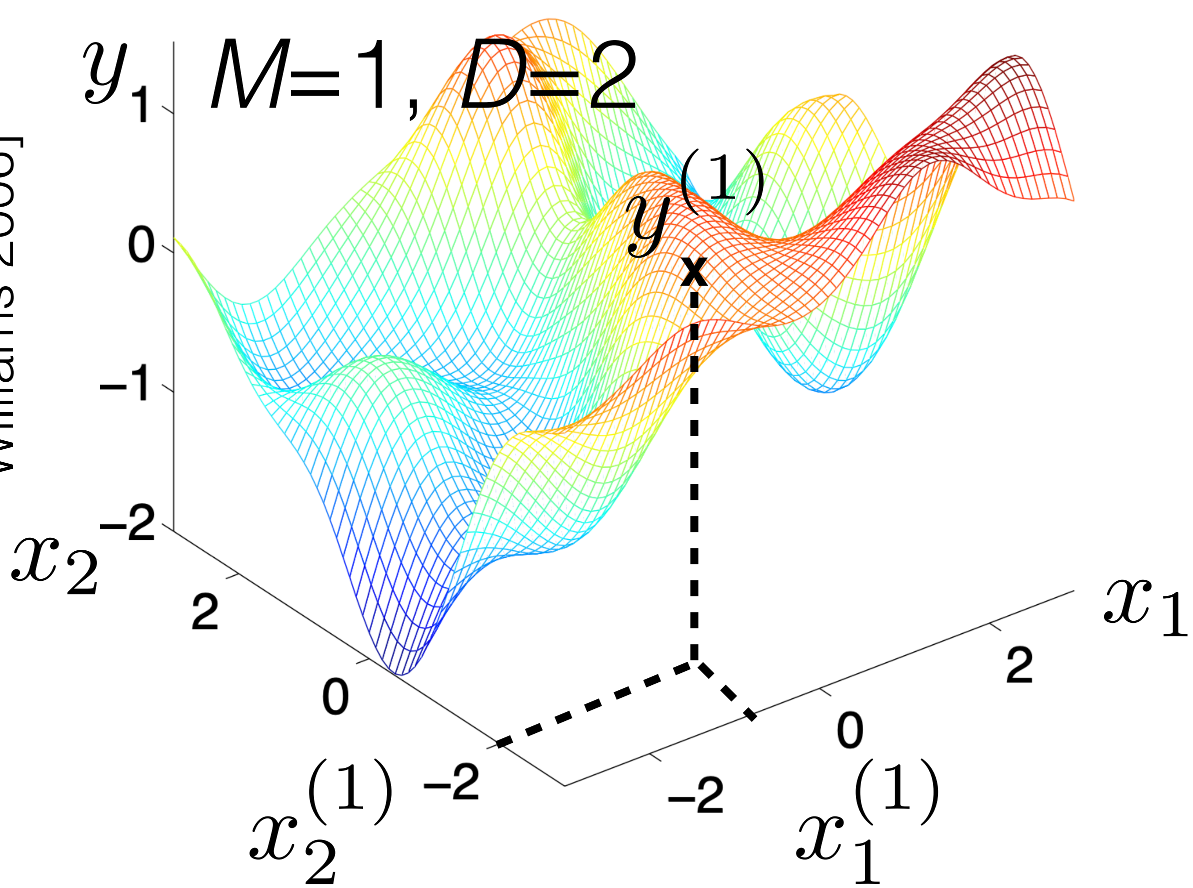

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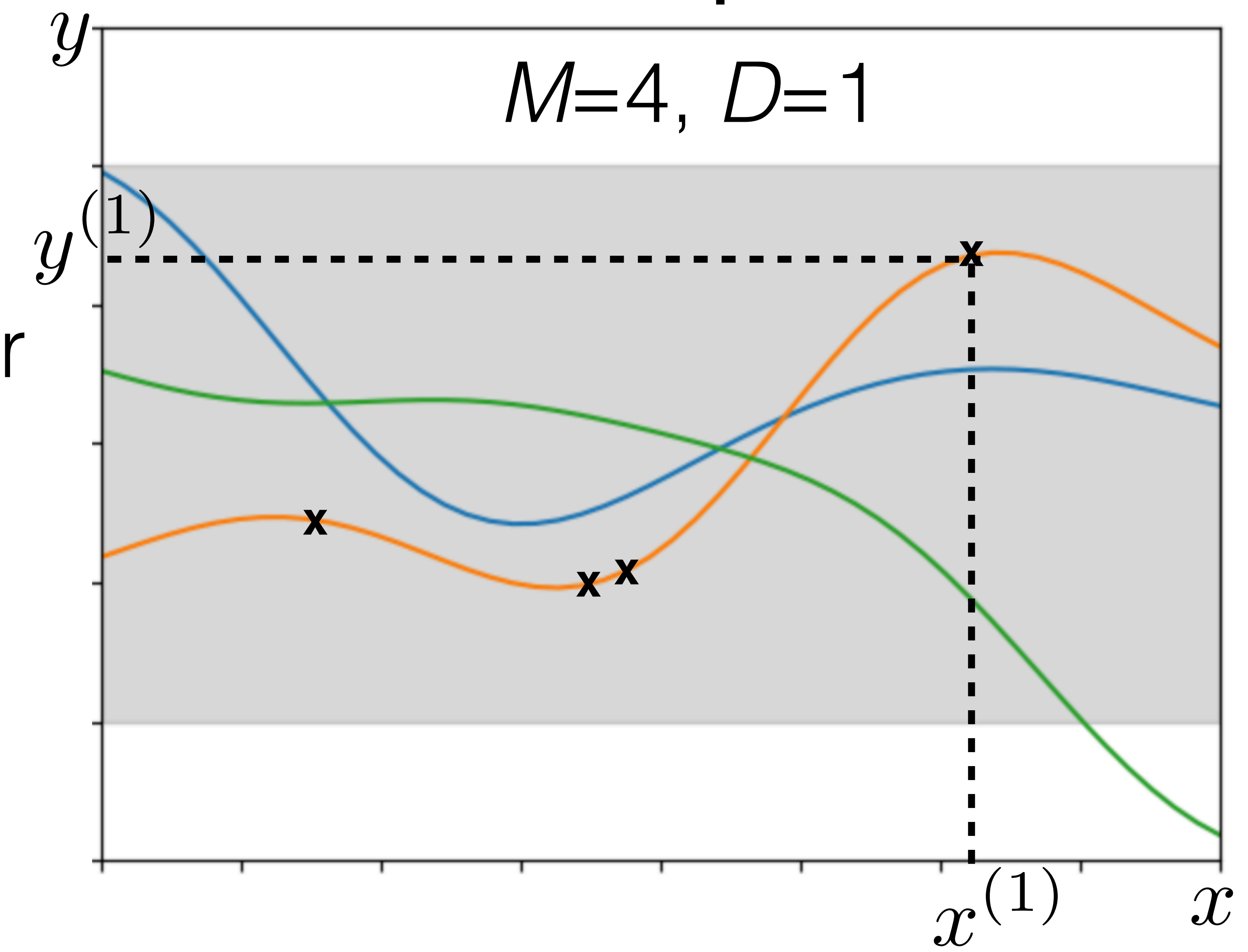

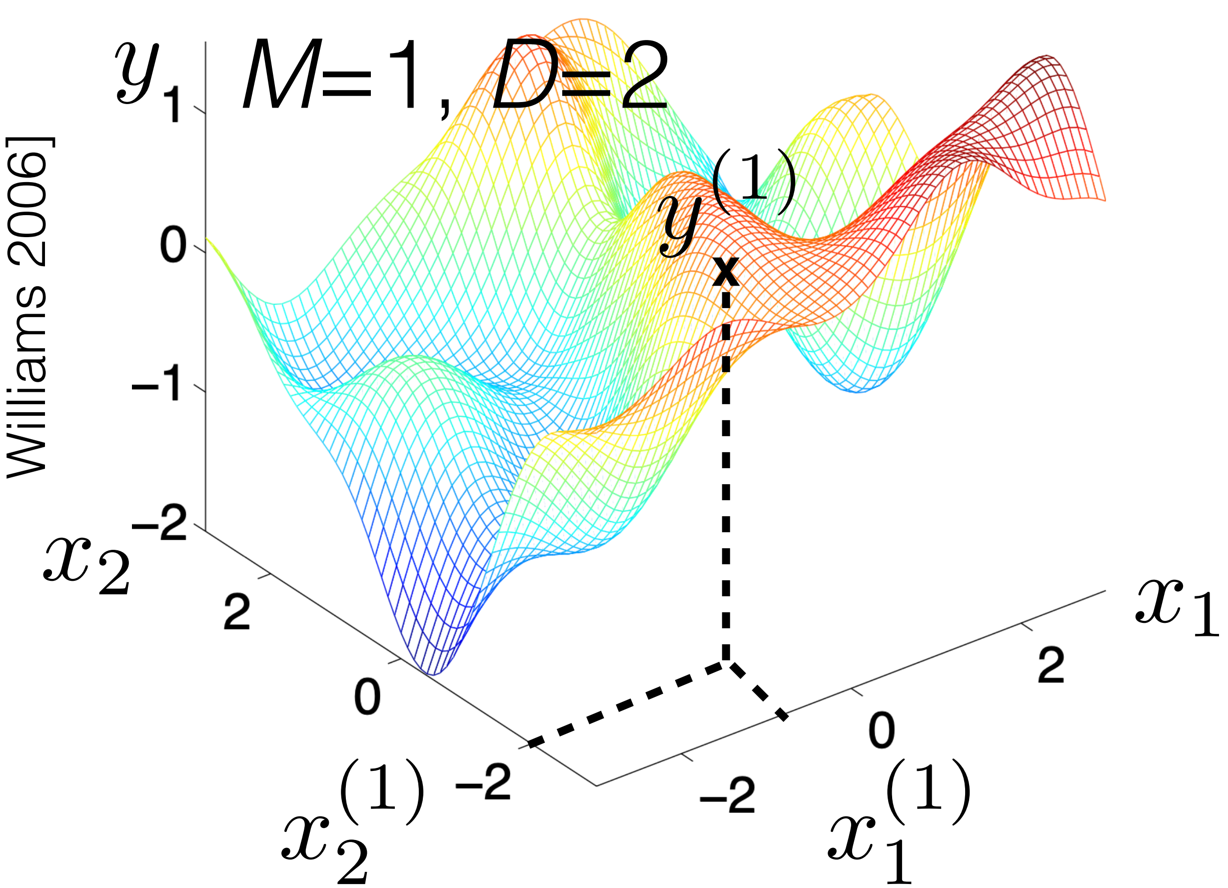

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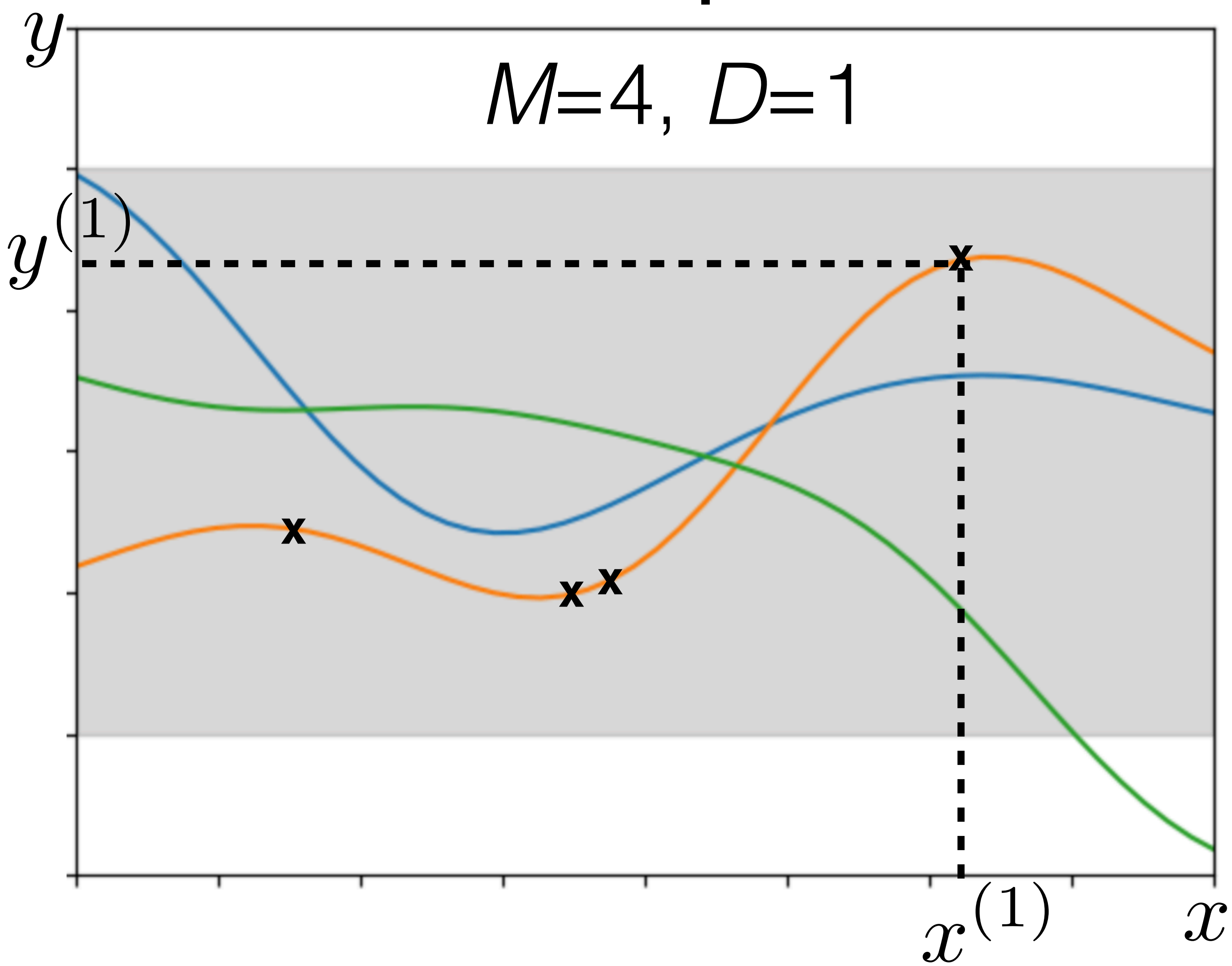

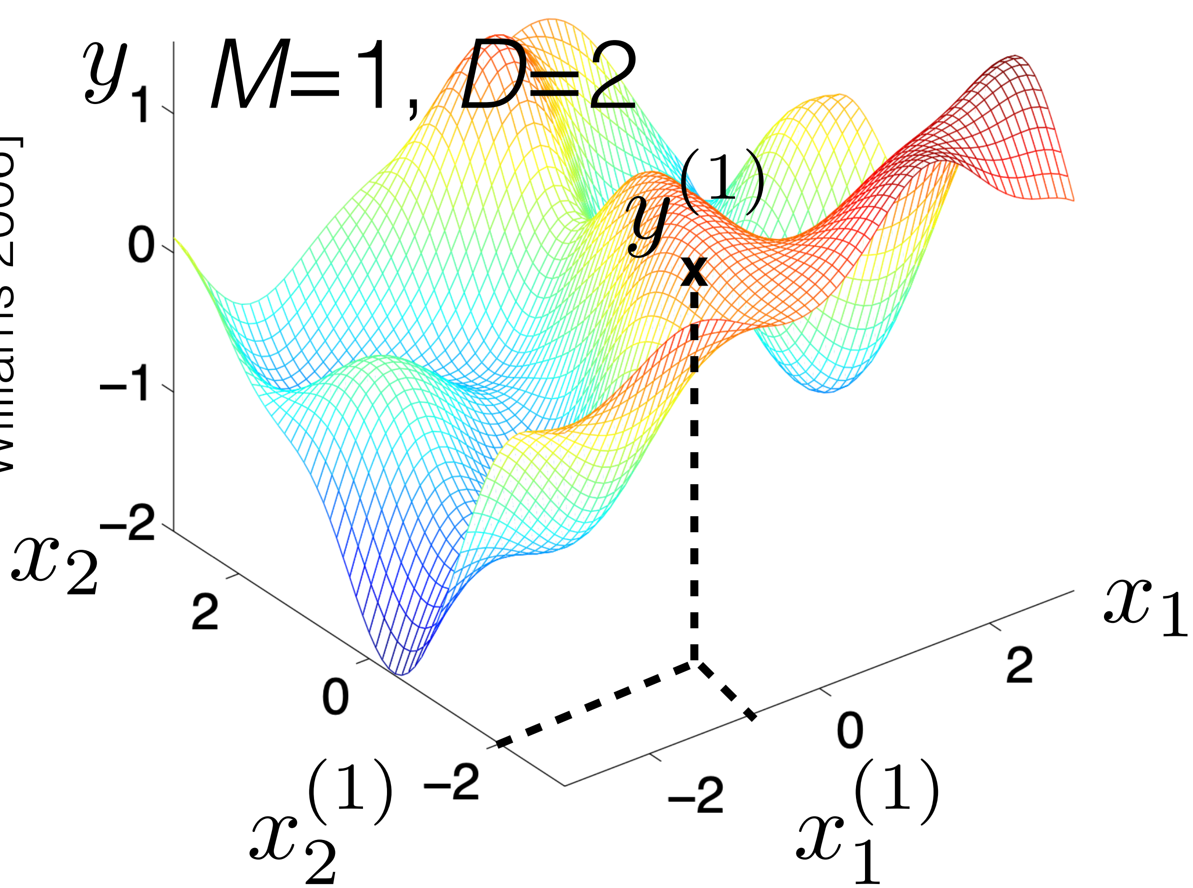

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- Note: all of our real-life examples from the start had number of inputs $D > 1$
- $D = 1$ is much easier to visualize, but might not be representative

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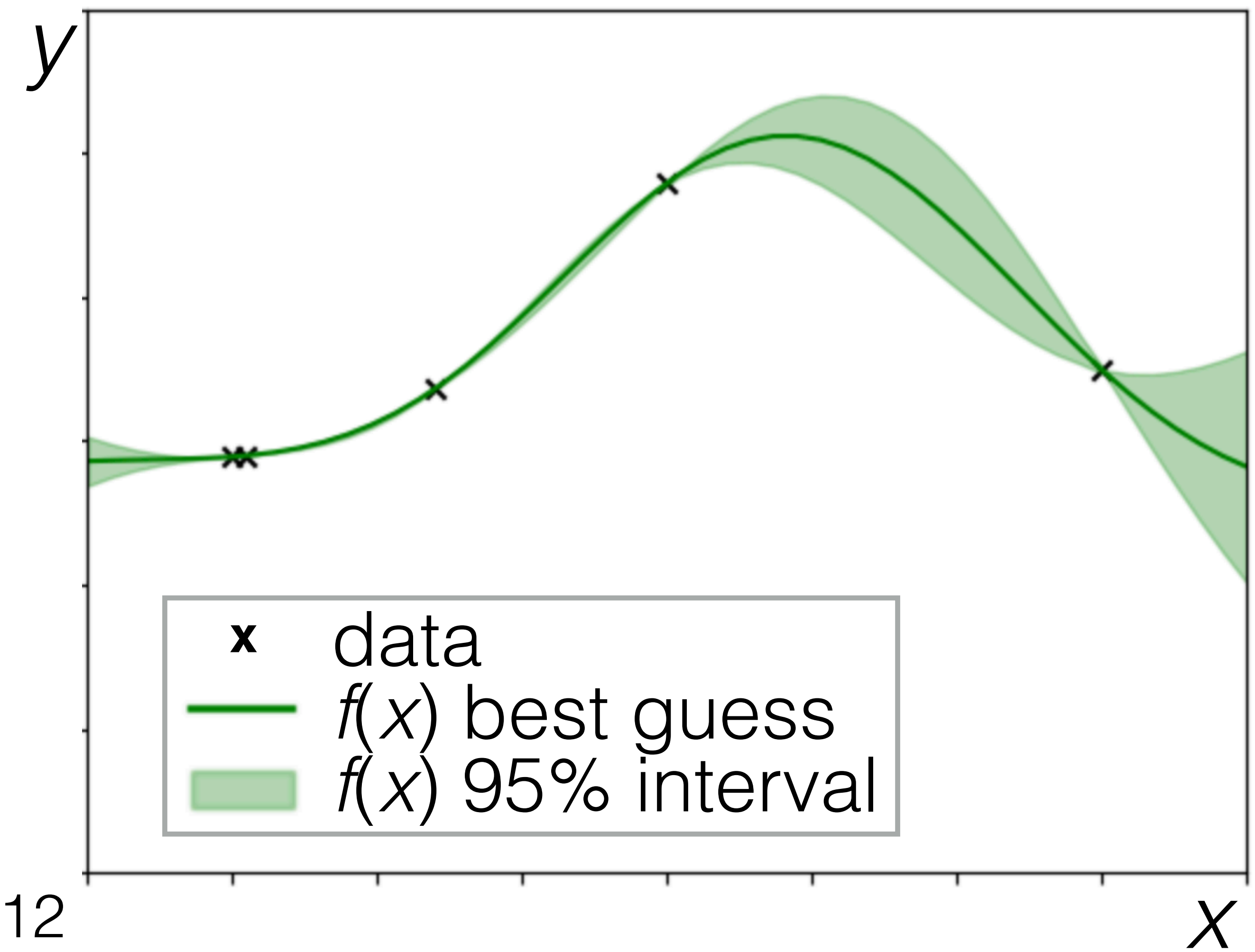

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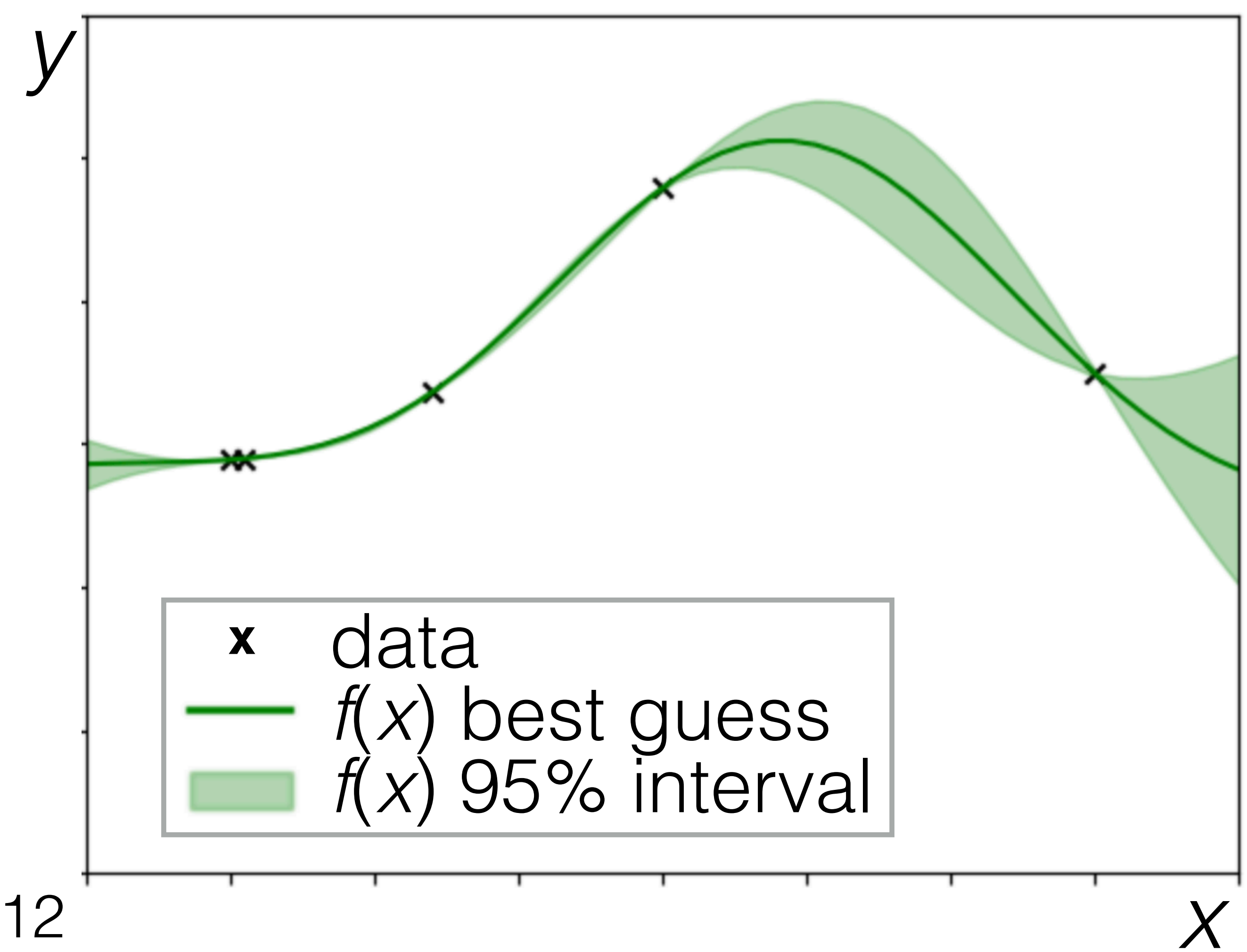

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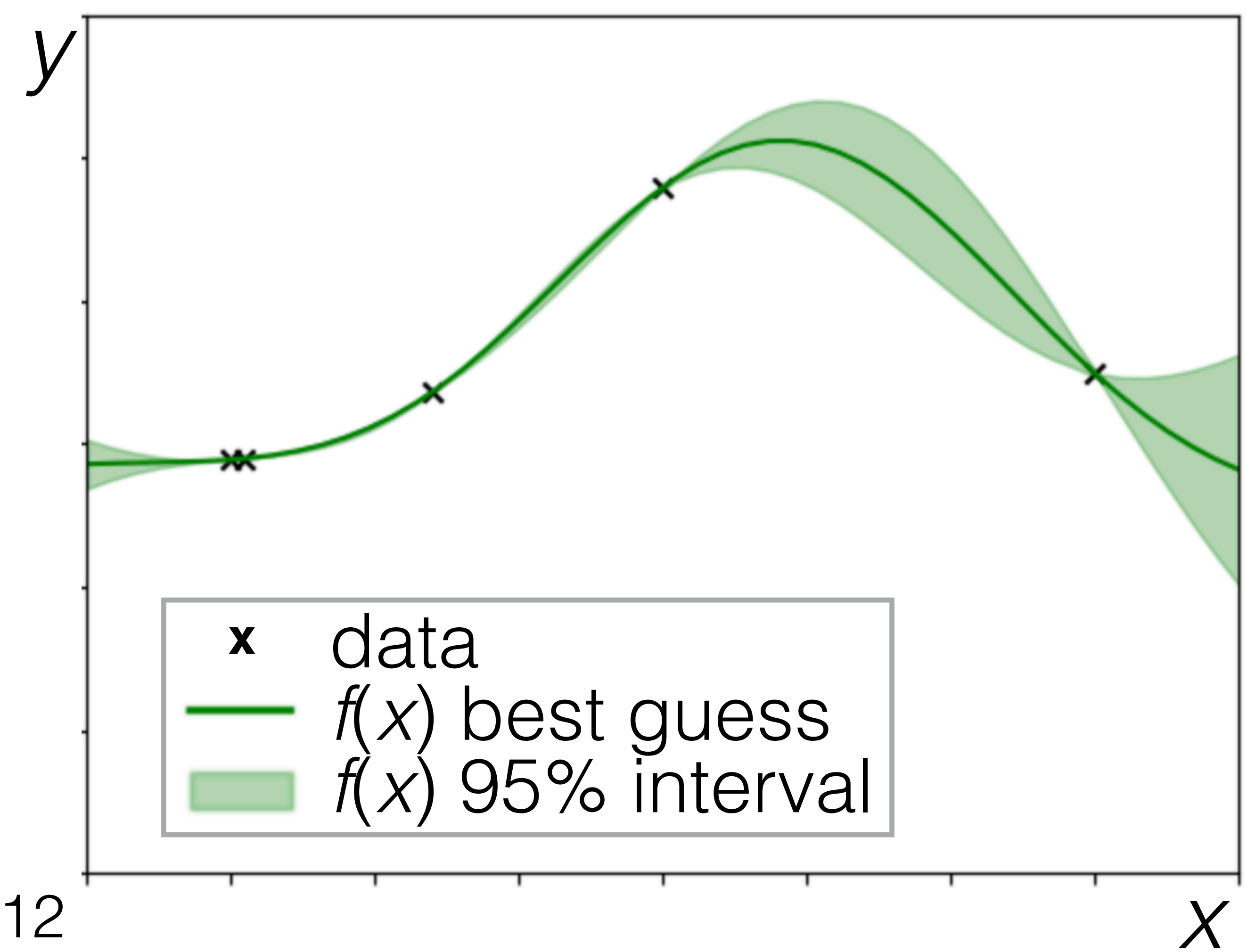

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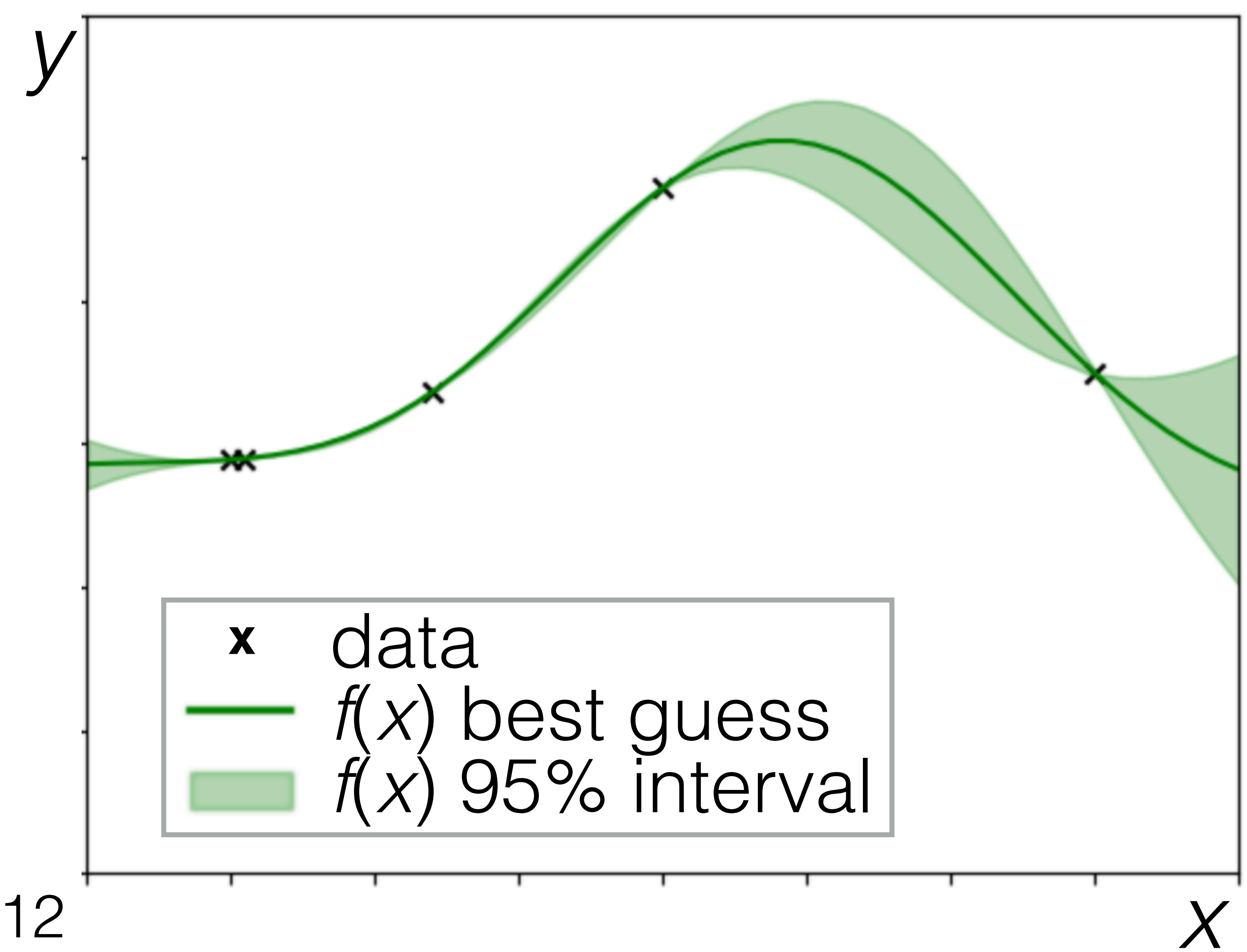

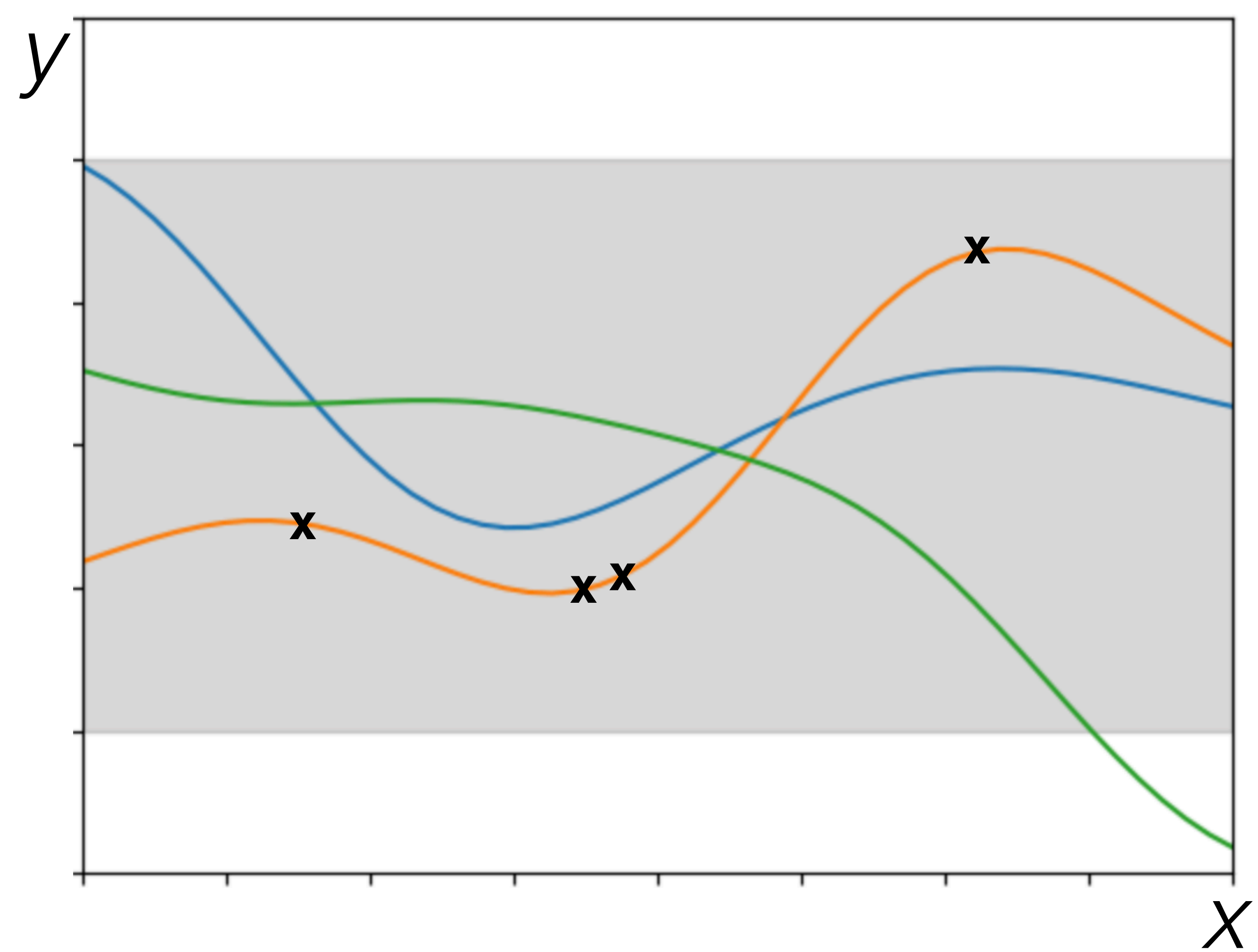

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Inference about unknowns given data

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    *M*x*M*

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- We’ll infer *f*(*X’*) given our simulated data; recall we’re using $k(x,x') = \sigma^2 \exp(-\frac{1}{2}(x-x')^2), \sigma = 1$

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[demo]

Closer look at the uncertainty interval

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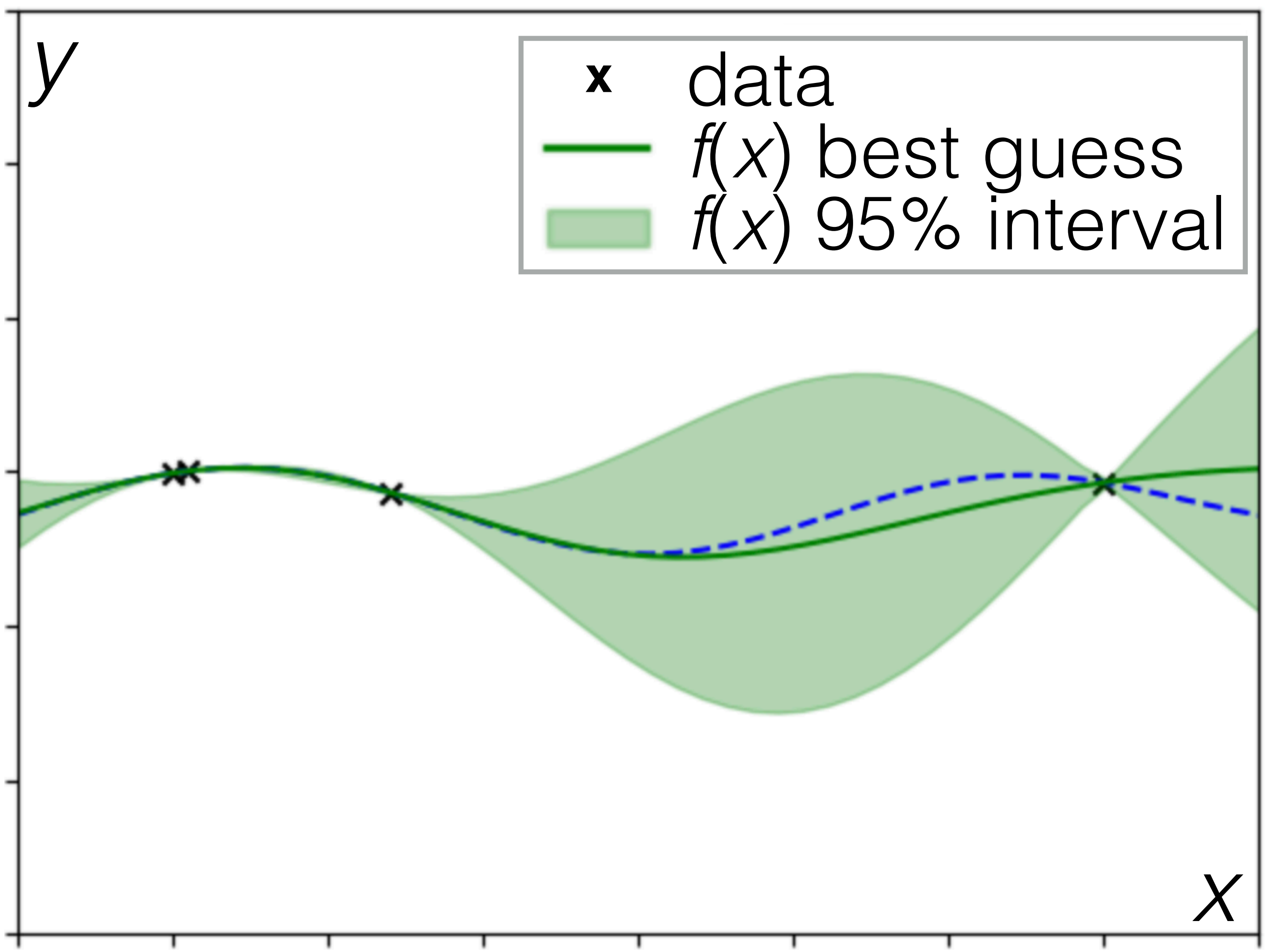

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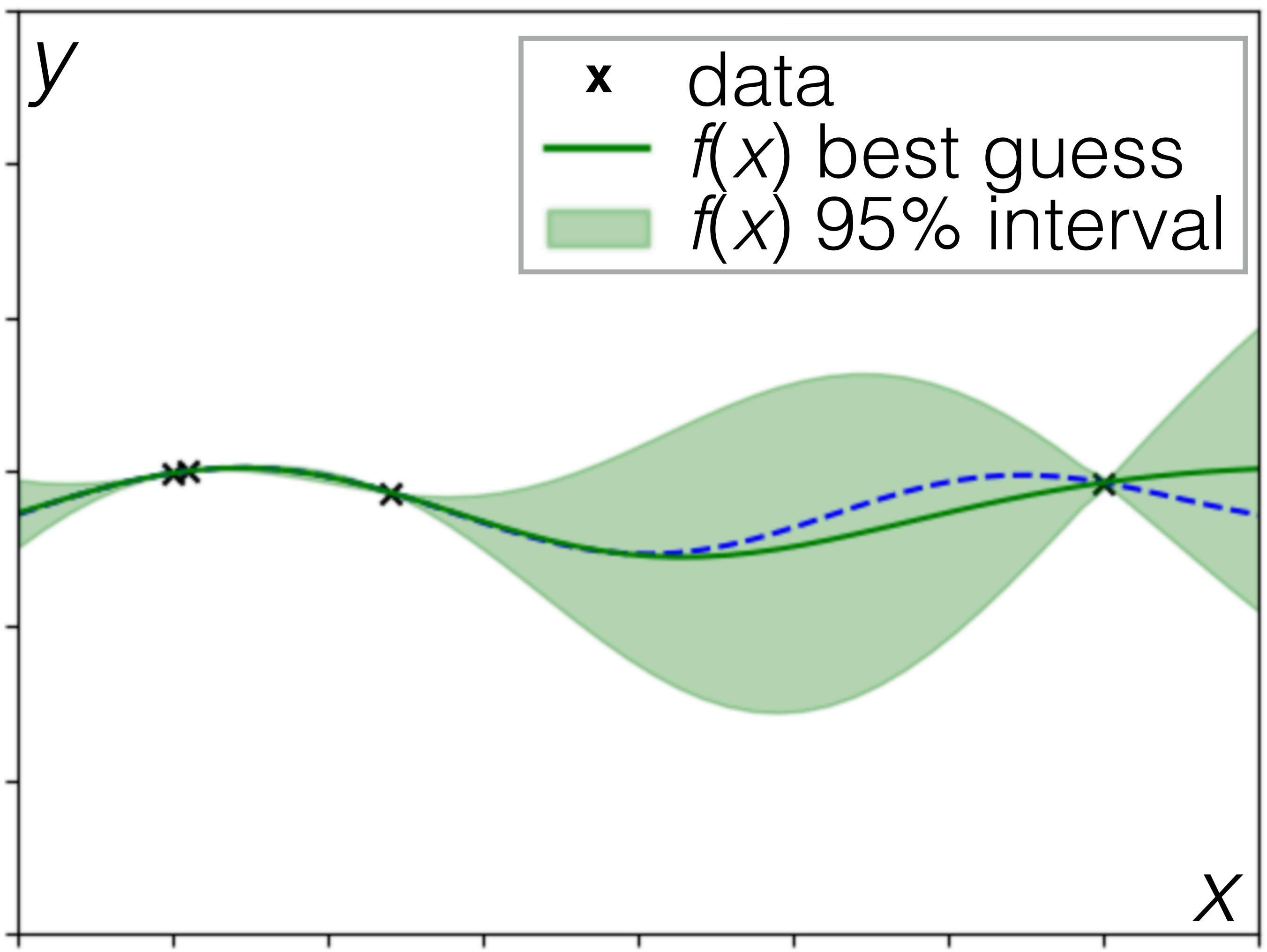

- Under GP, $f(x')|f(X), X, x'$ at a point *x'* is marginally Gaussian

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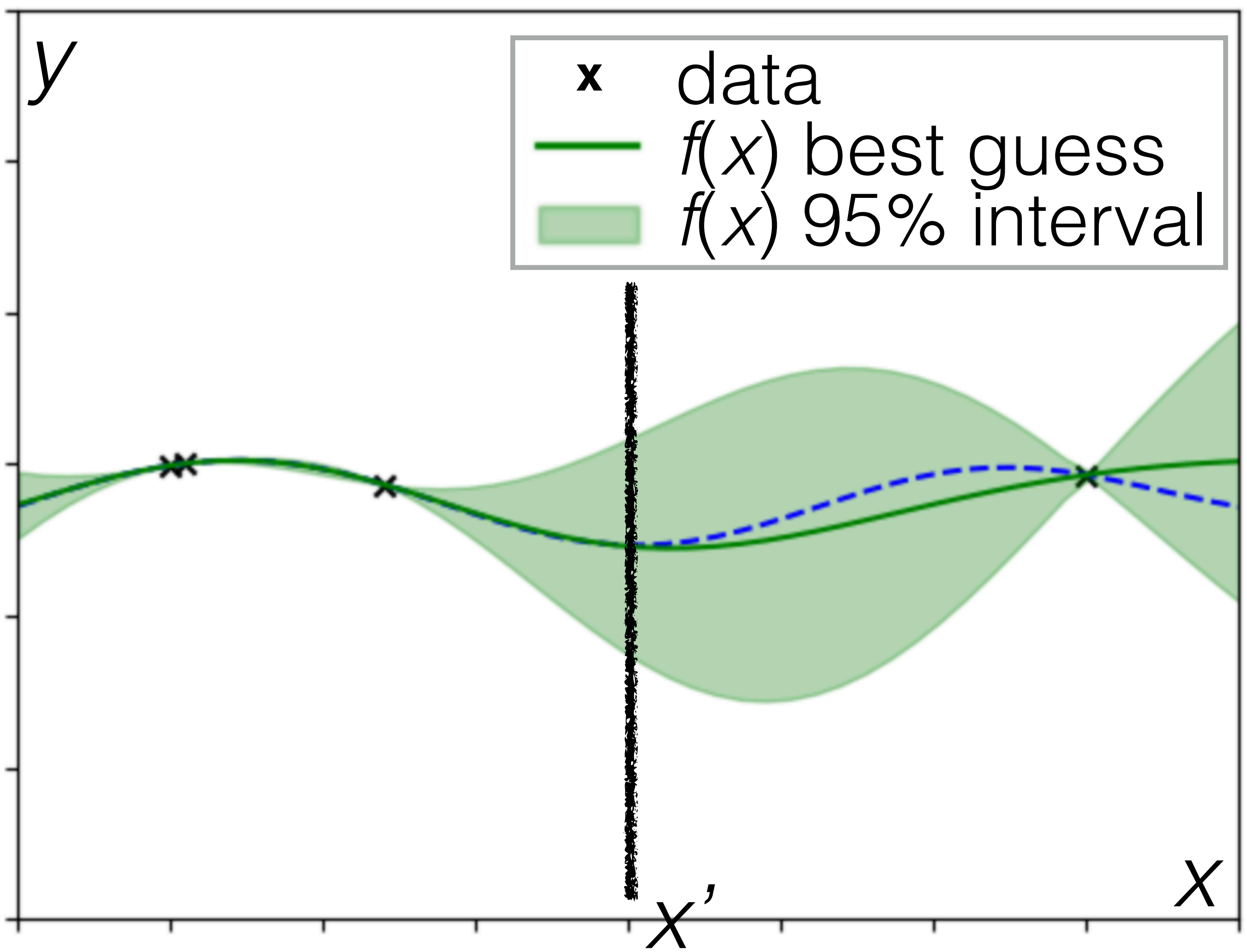

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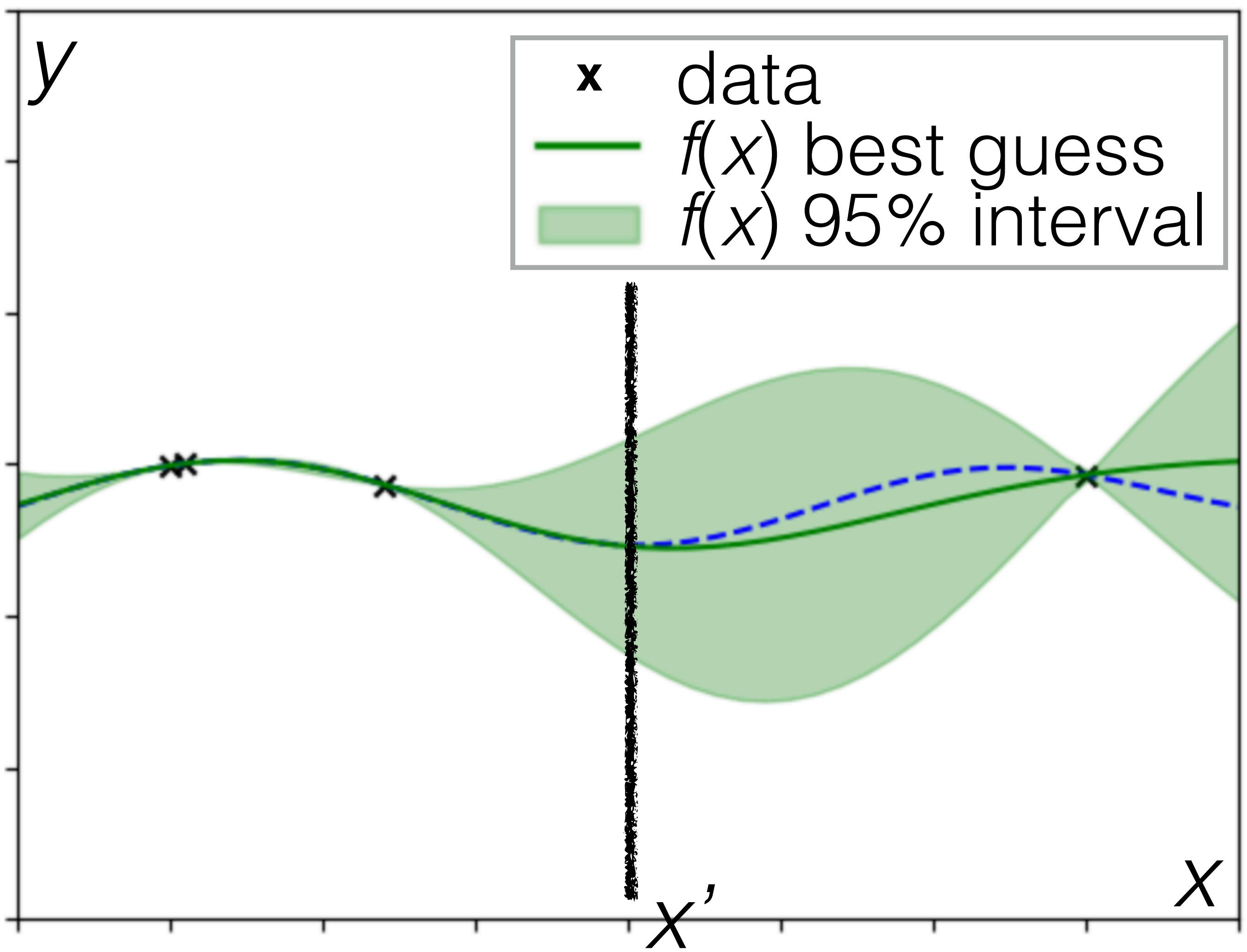

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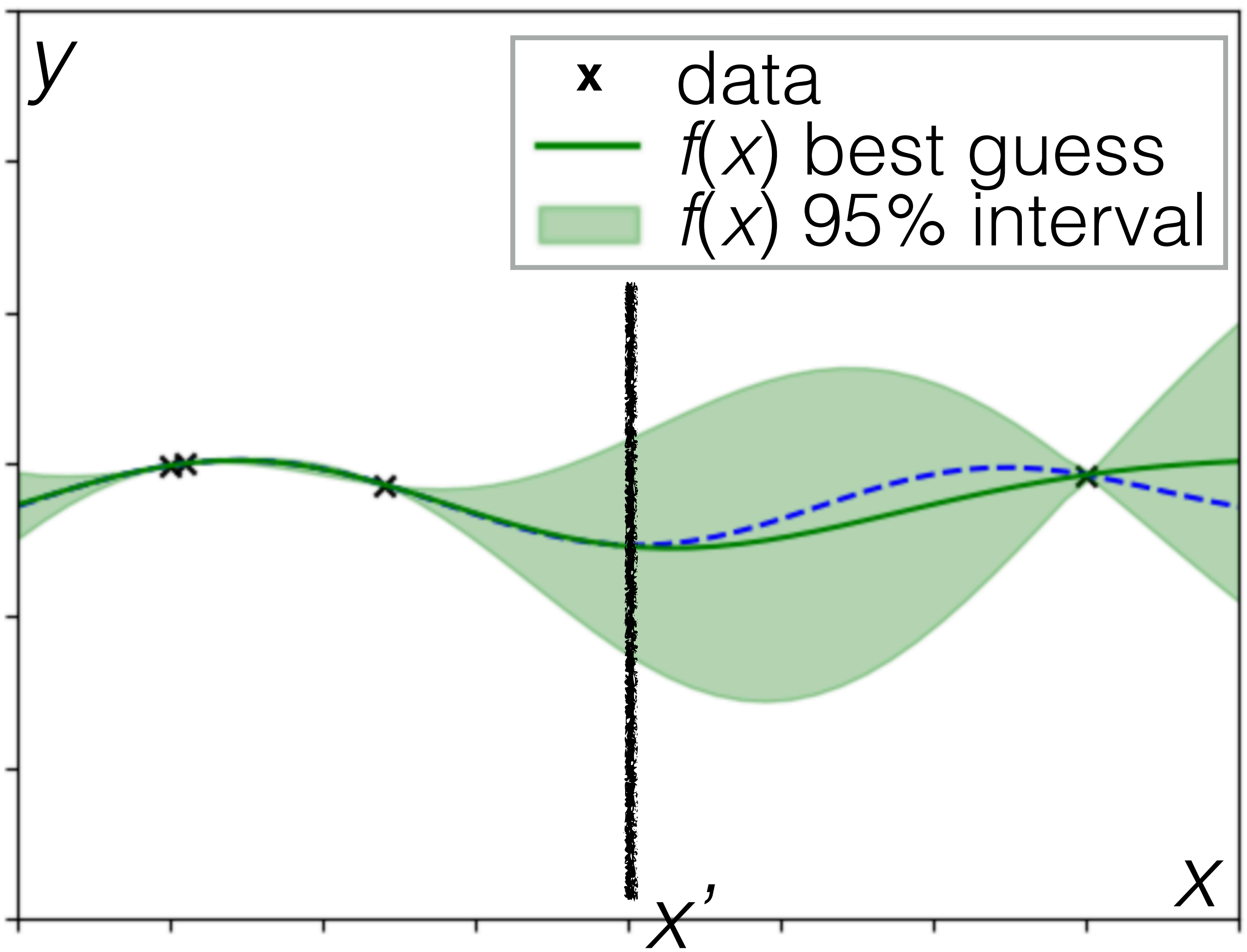

- Under GP, $f(x')|f(X), X, x'$ at a point *x'* is marginally Gaussian
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- The green interval at that point: mean +/- 2 std devs

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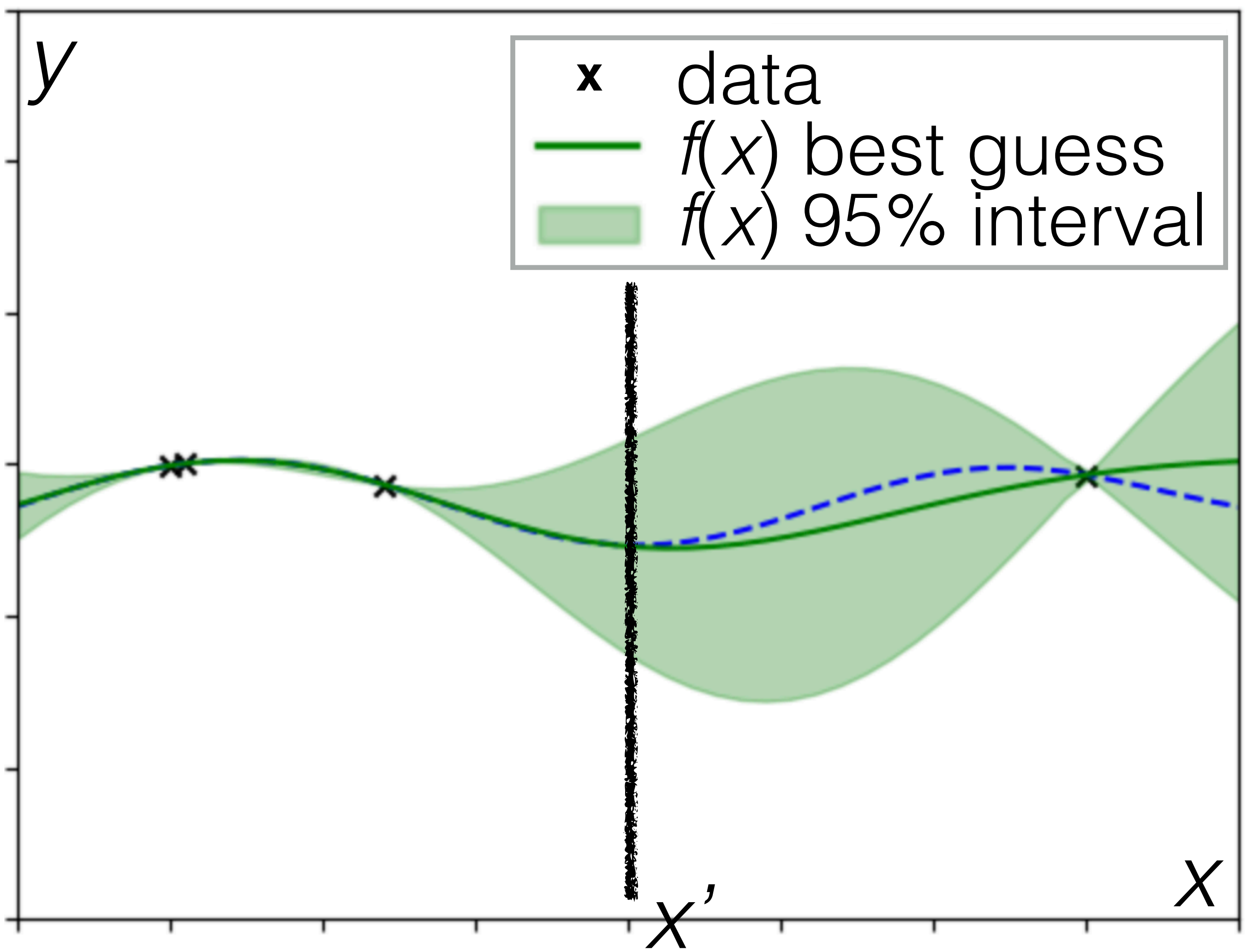

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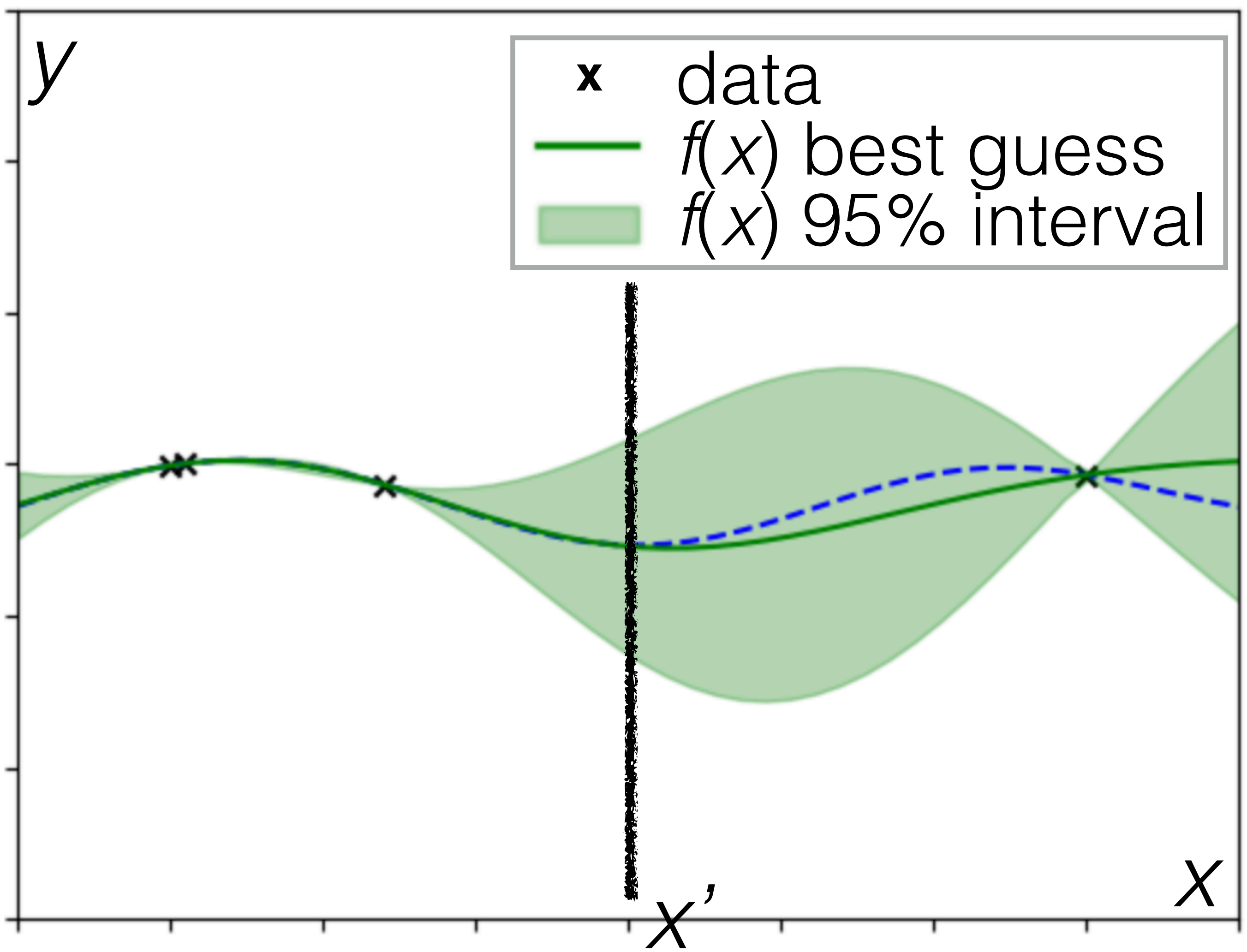

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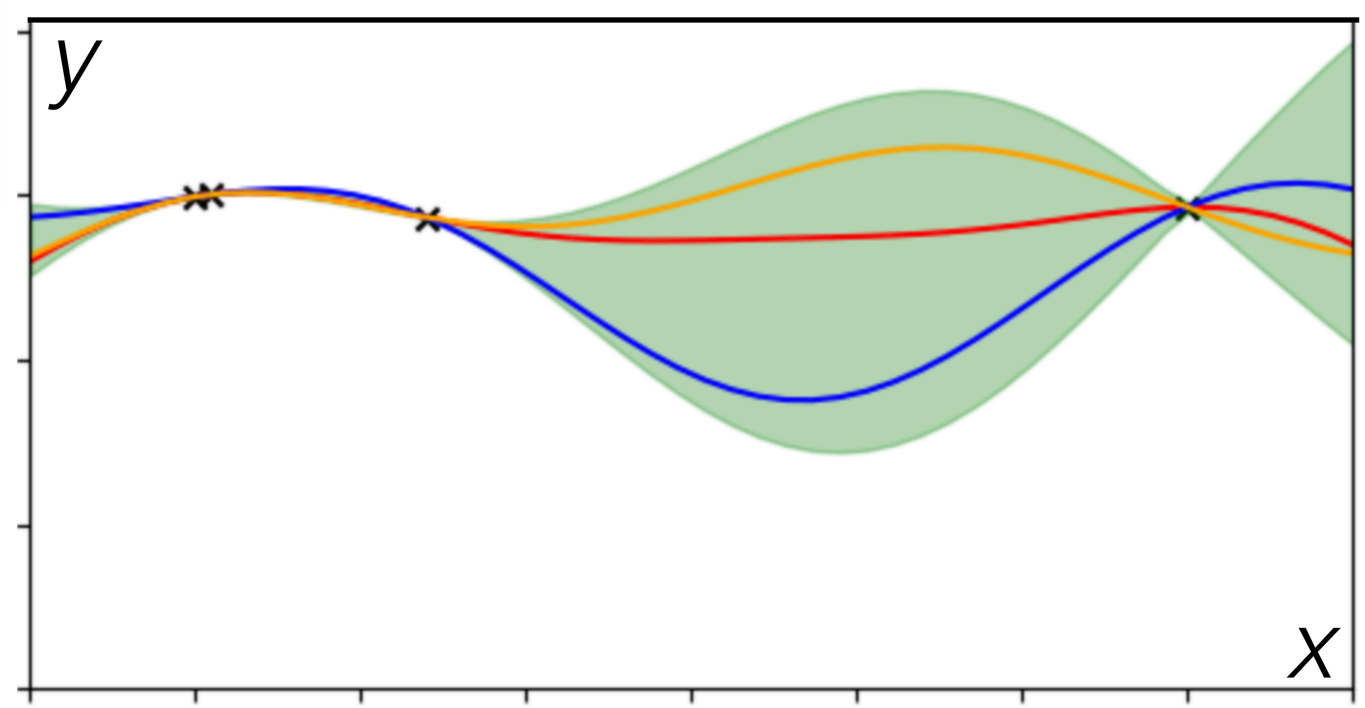

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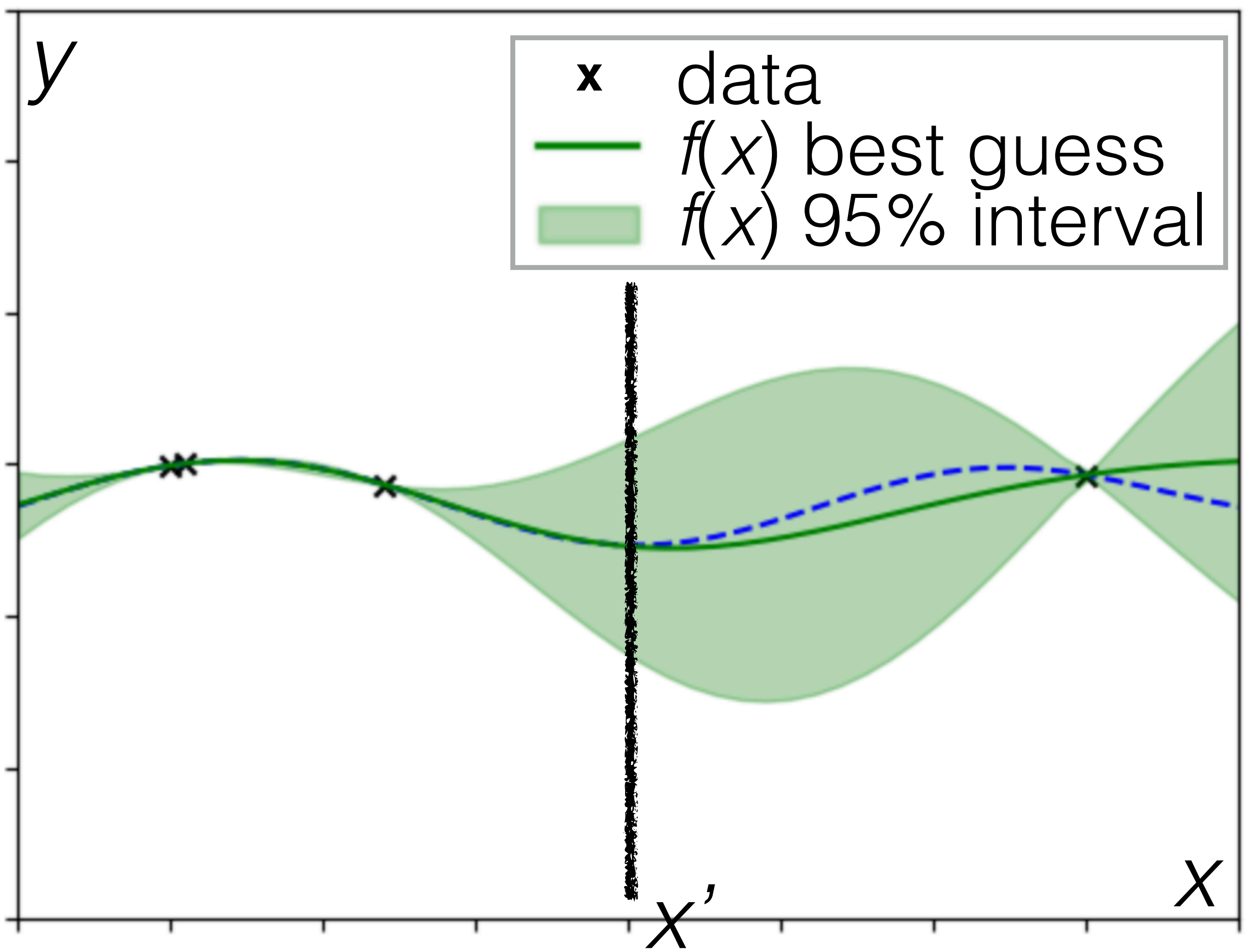

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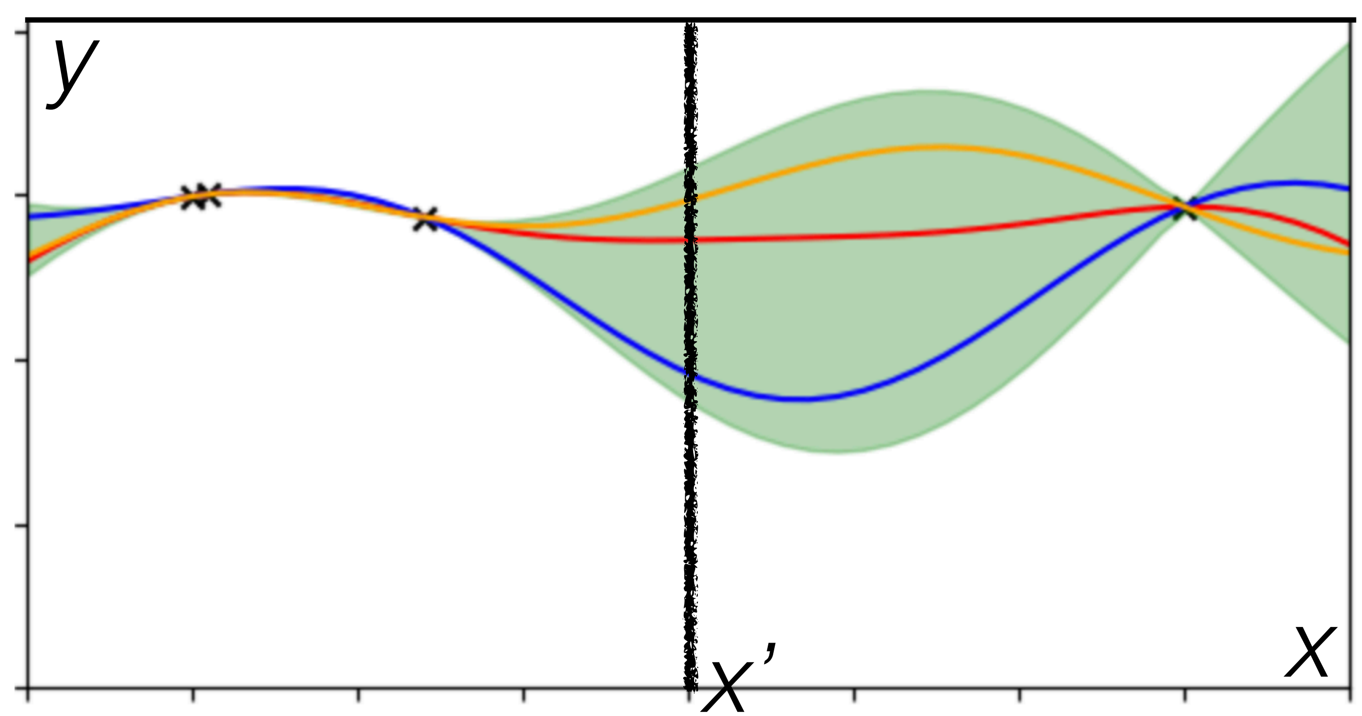

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