Machine Learning and Nonparametric Bayes

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"Wikipedia phenomenon"

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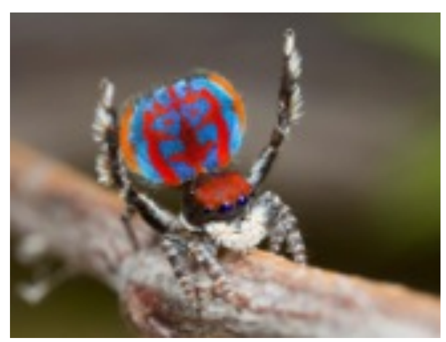
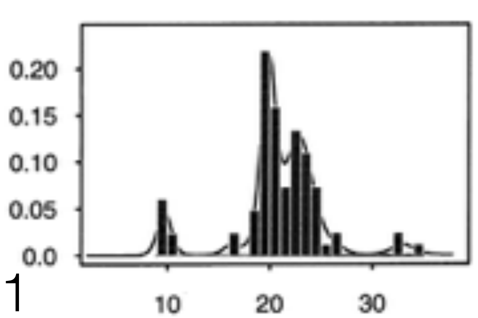


[Time Mag]



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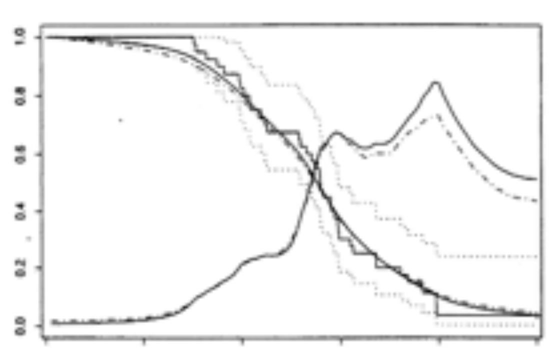

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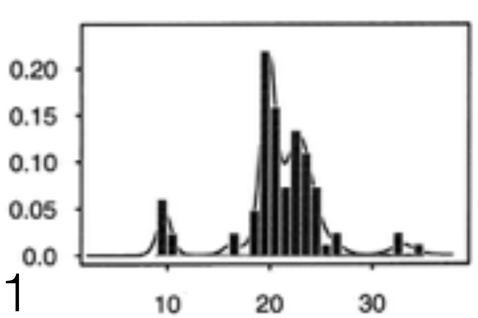

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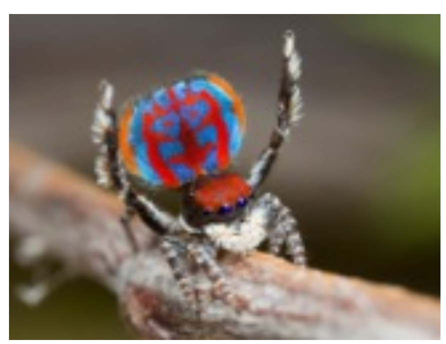
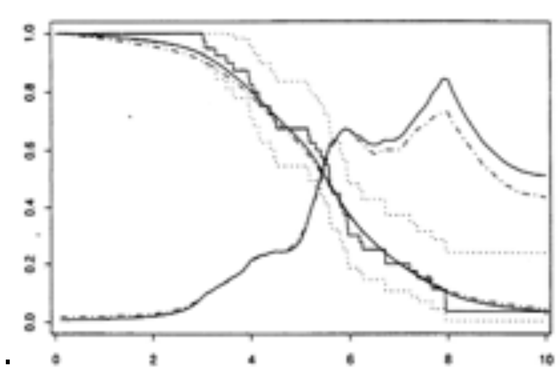


[Time Mag]

[Fox et al 2014]



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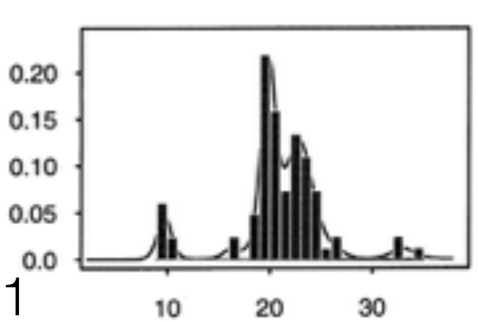

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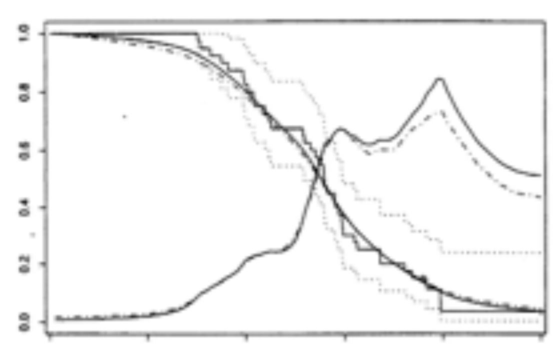

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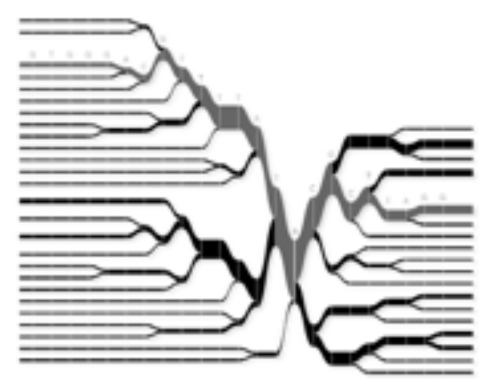
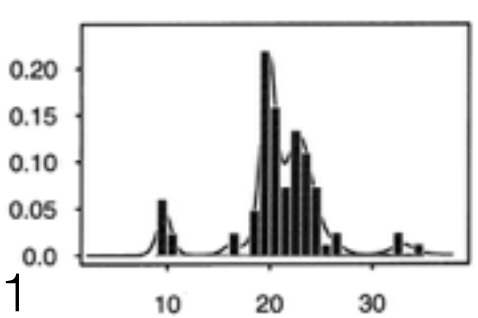


[Ewens, 1972; Hartl, Clark 2003]



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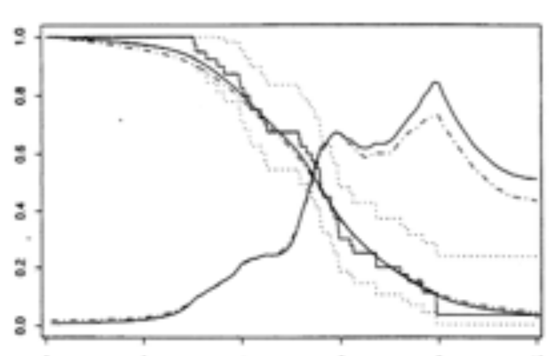

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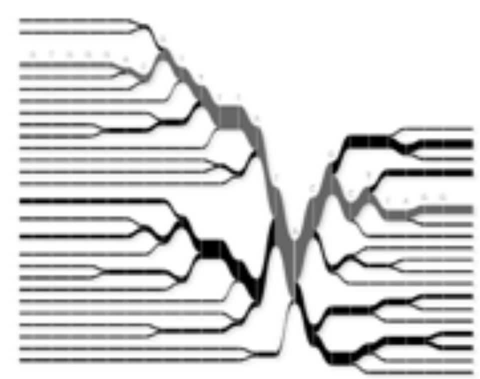

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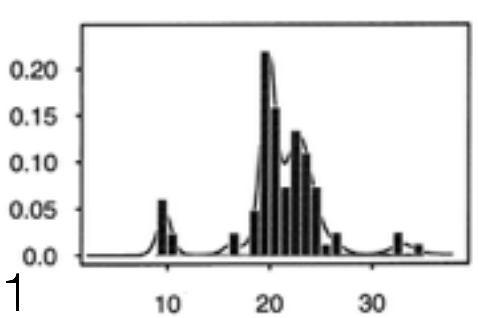

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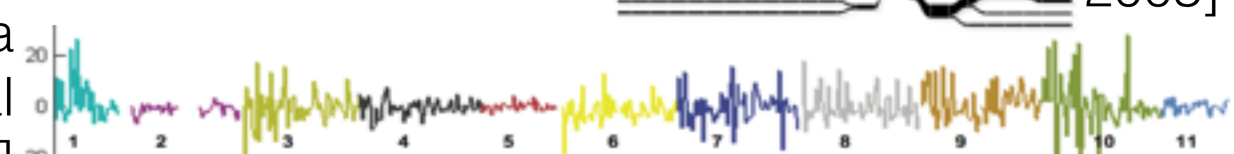

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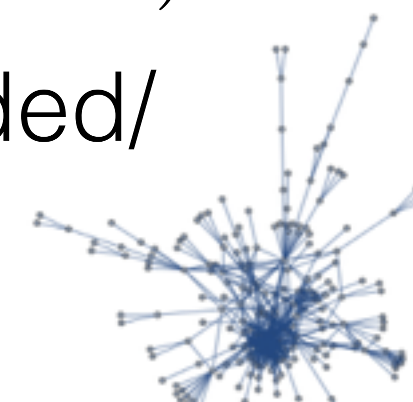

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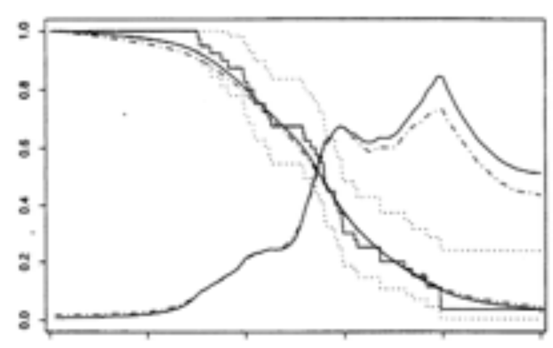

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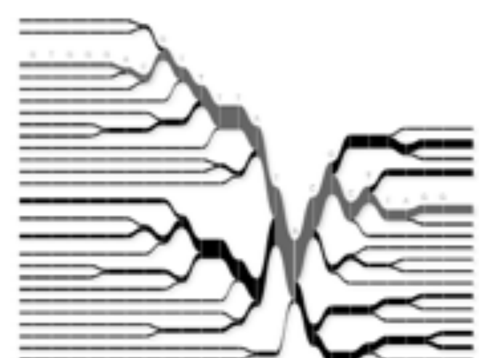

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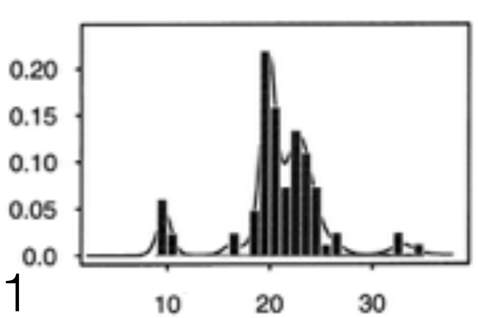

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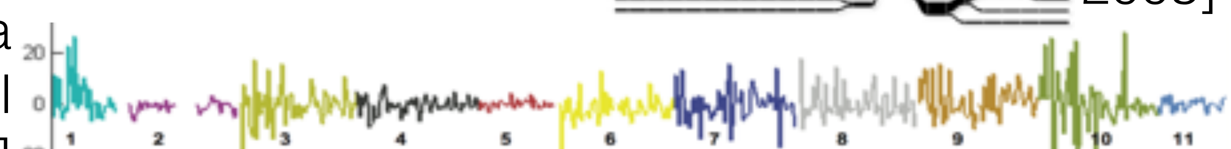

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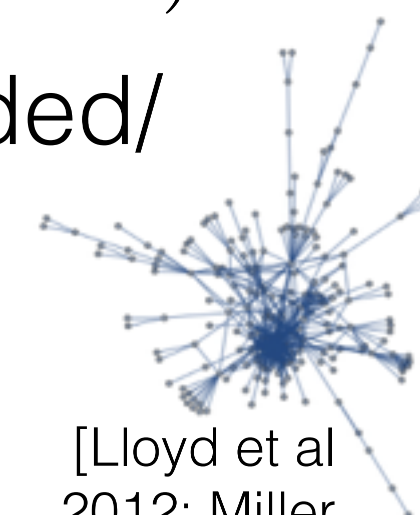
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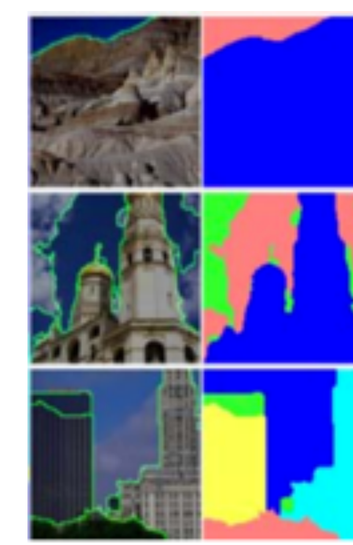
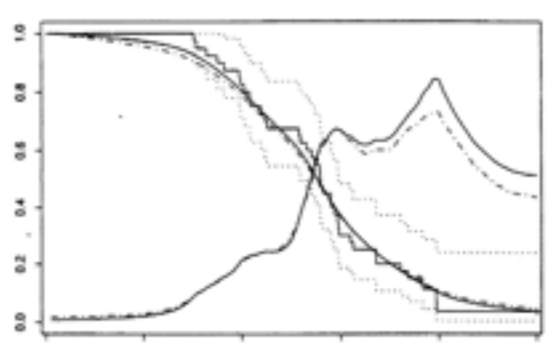

[Sudderth, Jordan 2009]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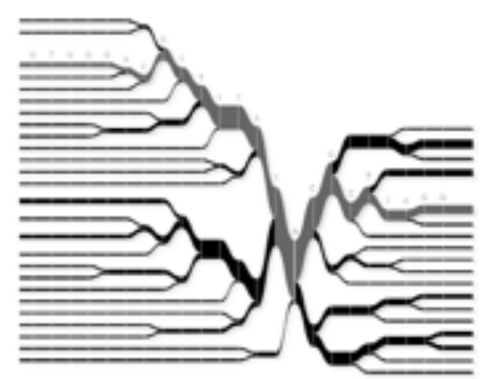
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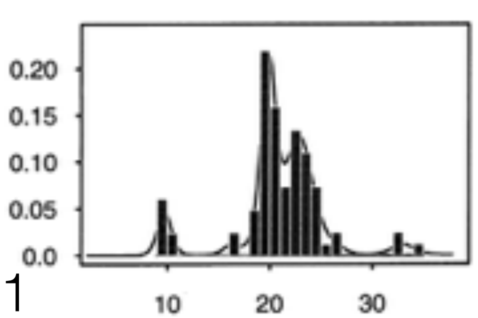

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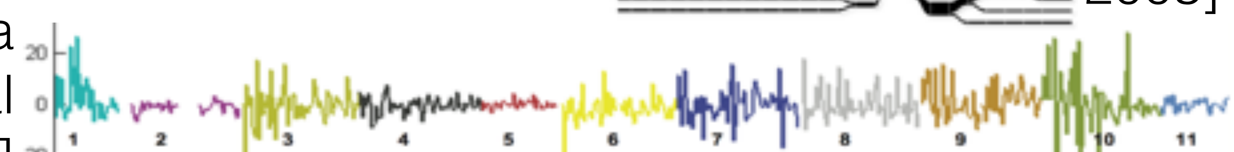
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[Hewitt, Savage 1955; Aldous 1983]

Outline

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- Dirichlet process

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  - Background for intuition

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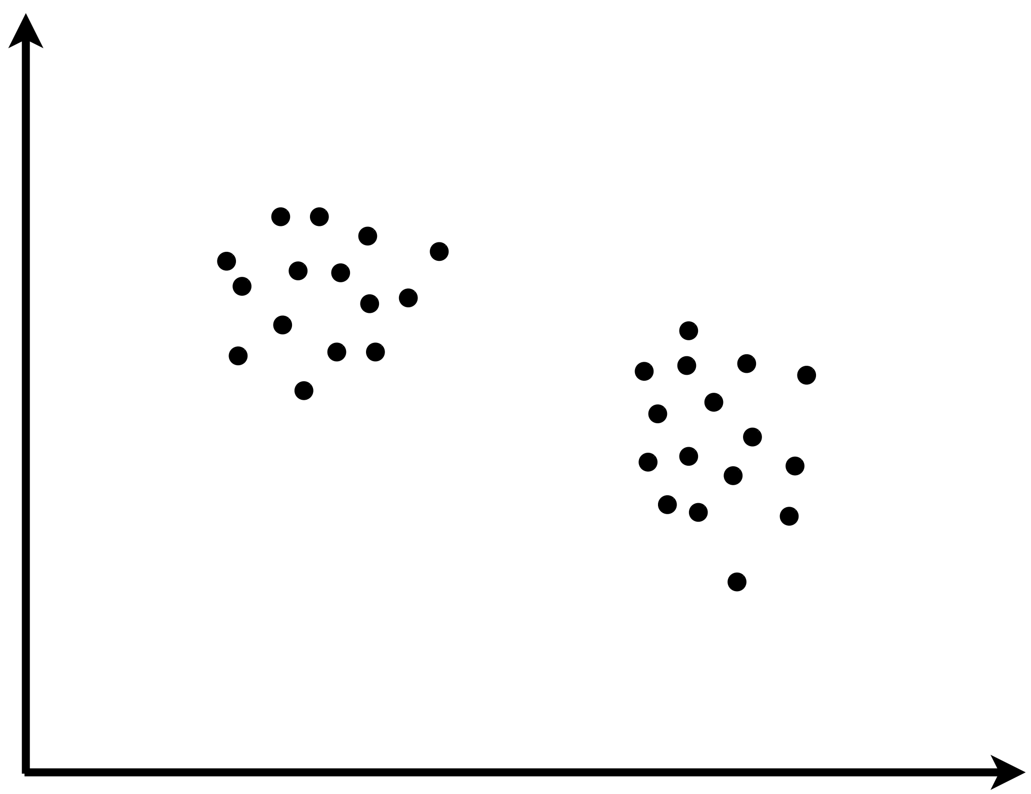
- Venture further into the wild world of Nonparametric Bayesian statistics

Generative model



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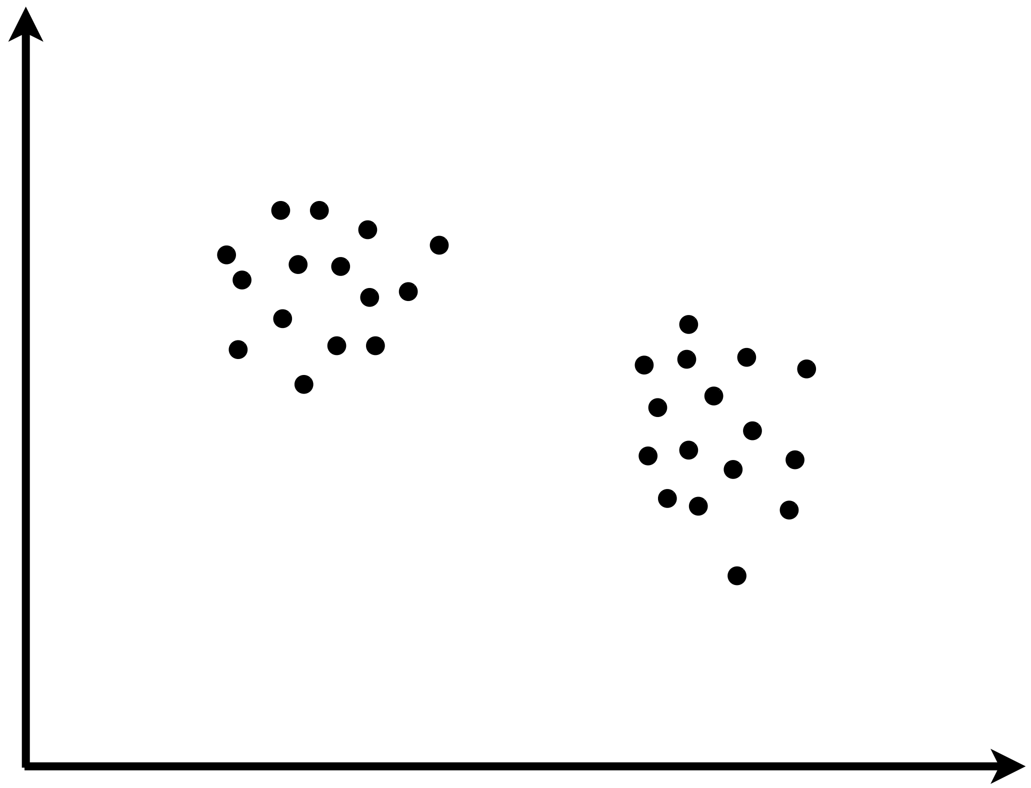

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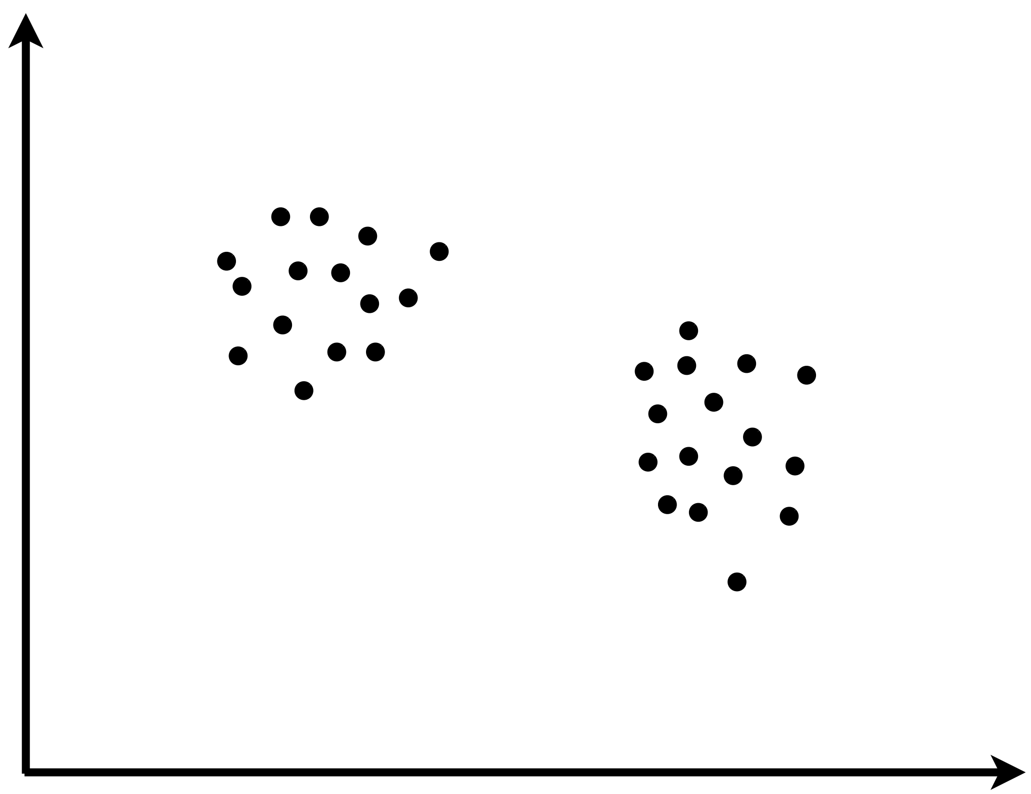

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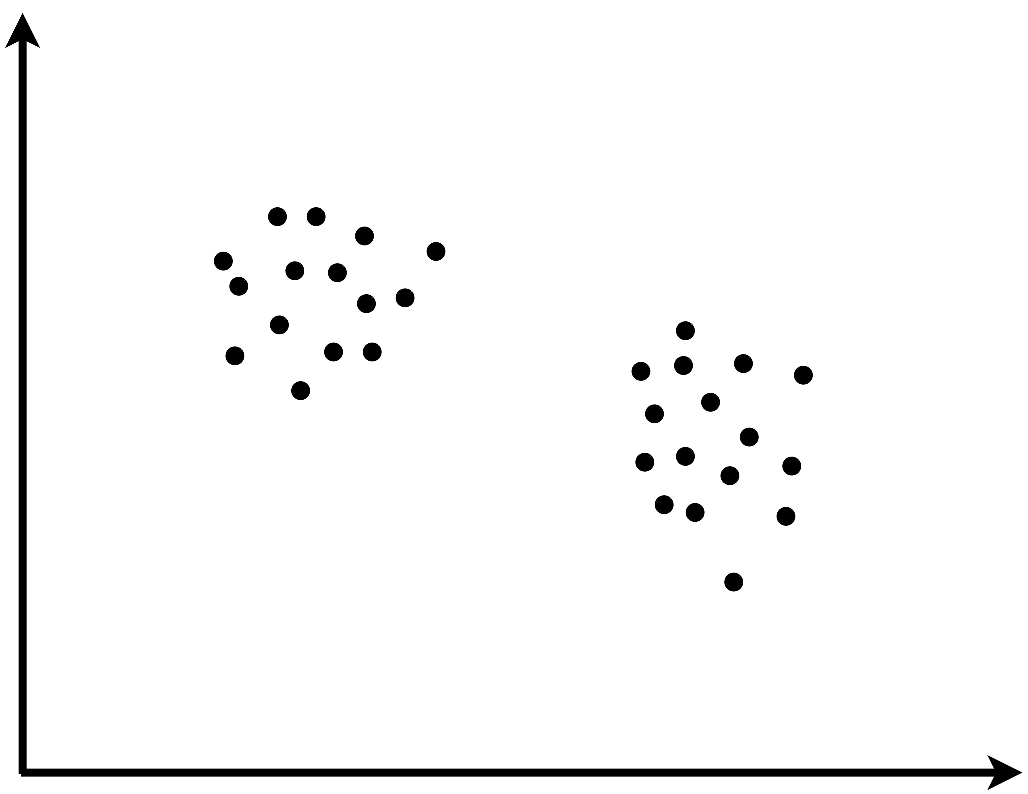

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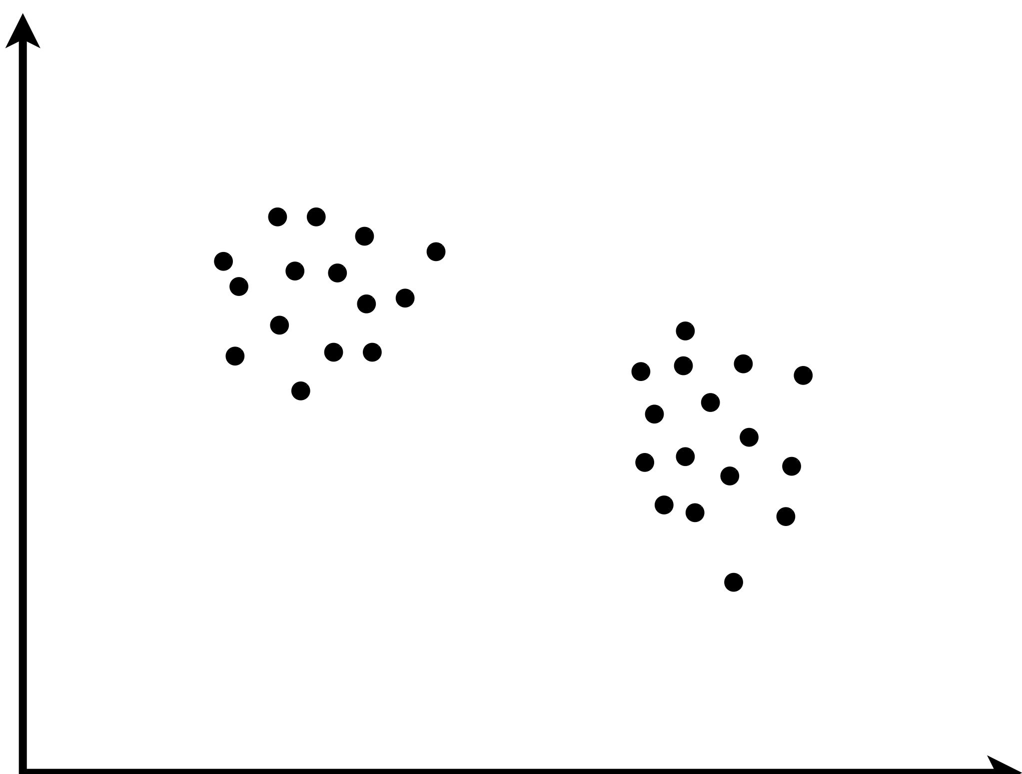

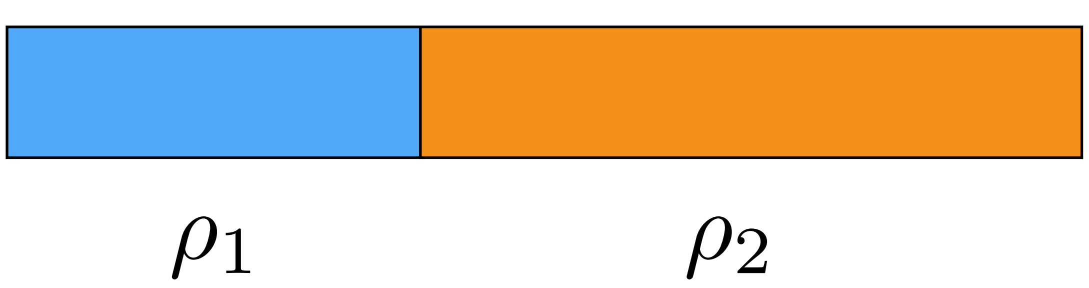

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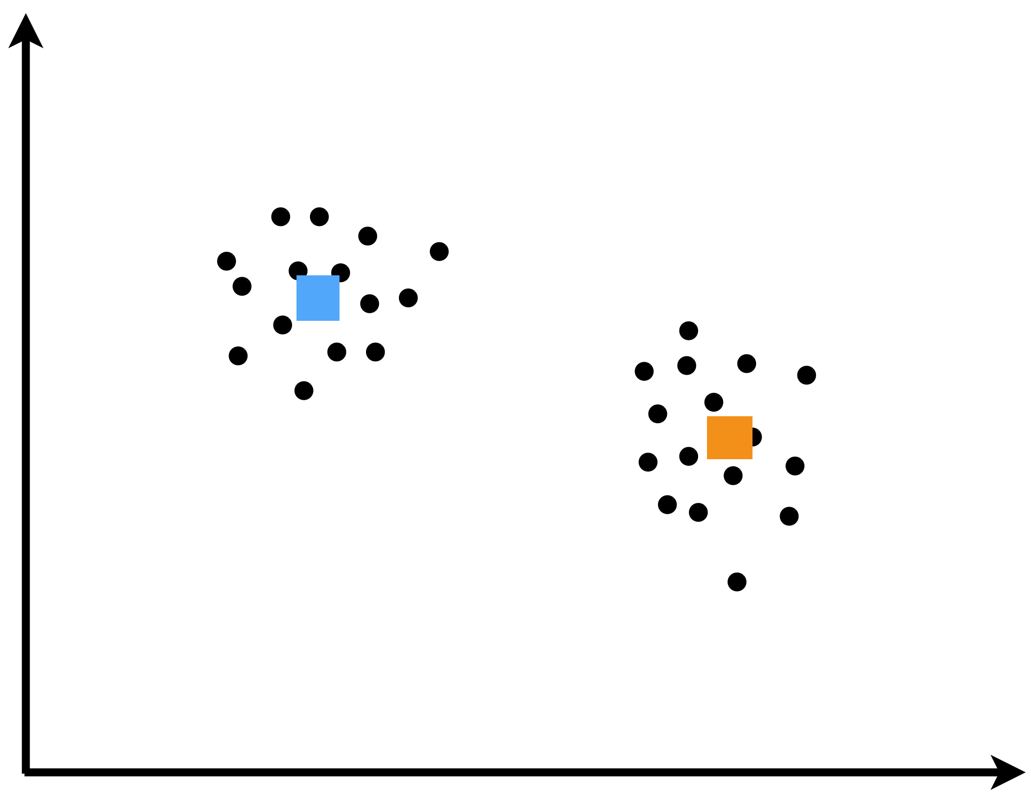

- Finite Gaussian mixture model (*K*=2 clusters)

$$z_n \overset{iid}{\sim} \text{Categorical}(\rho_1, \rho_2)$$
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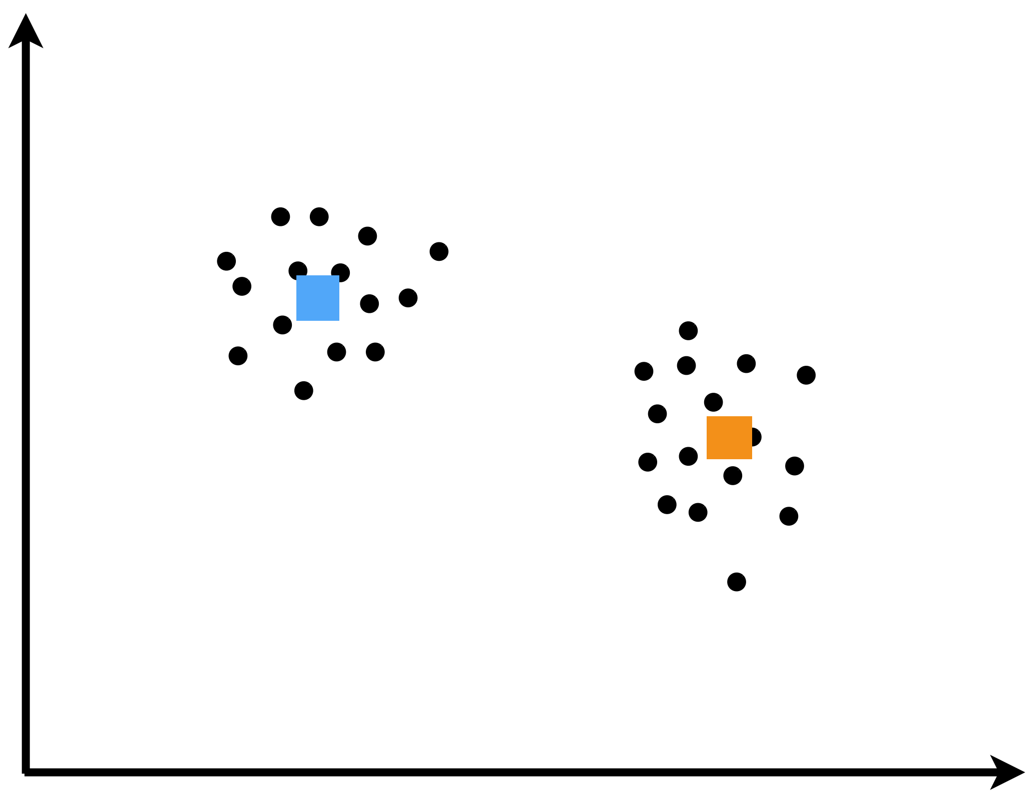

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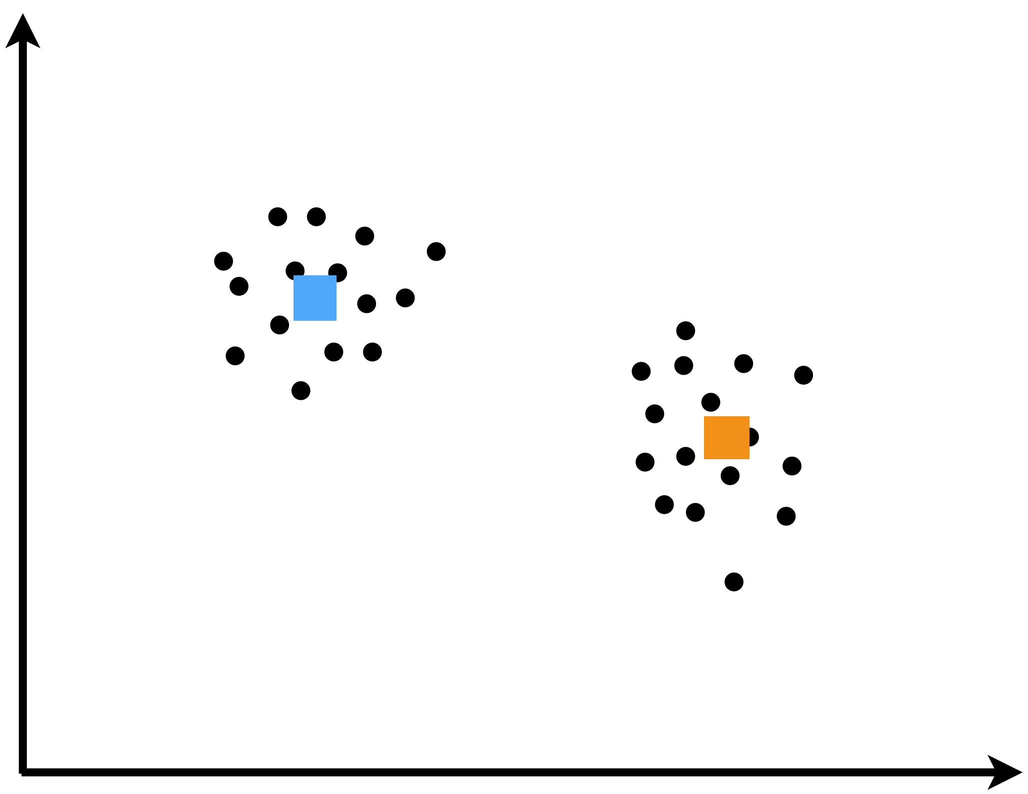

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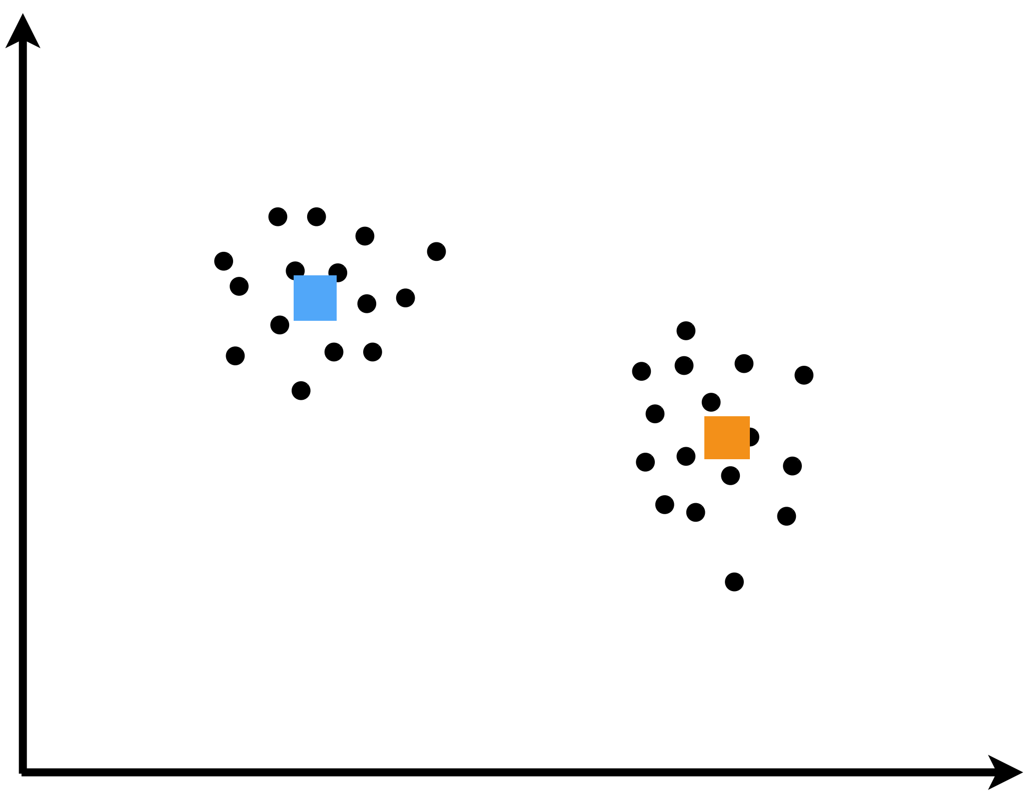

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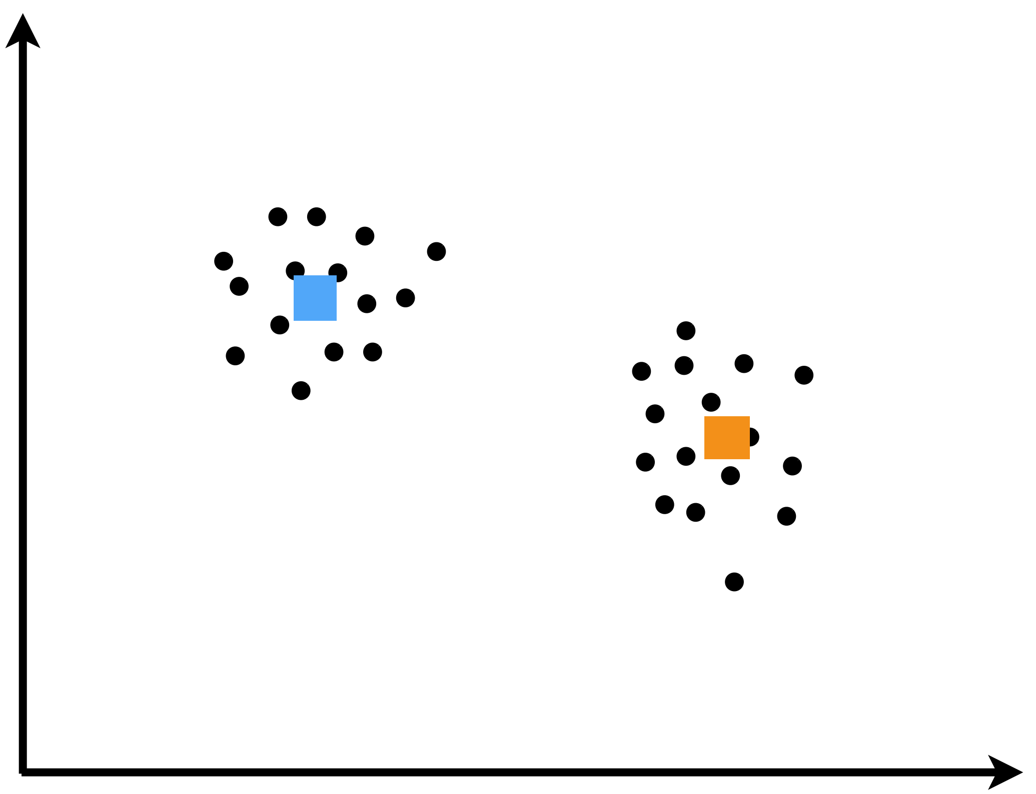

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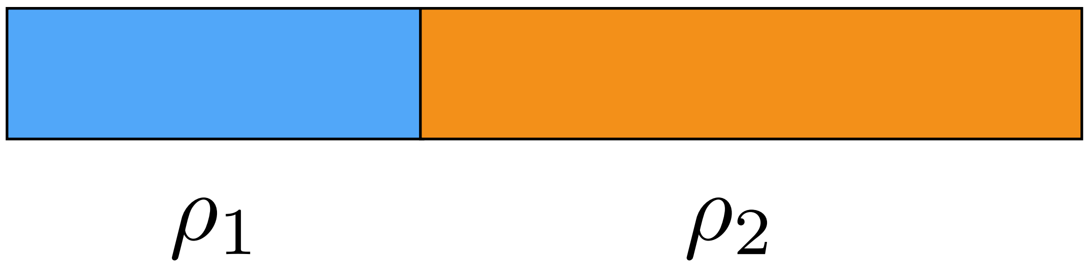
$$\rho_1 \sim \text{Beta}(a_1, a_2)$$
$$\rho_2 = 1 - \rho_1$$



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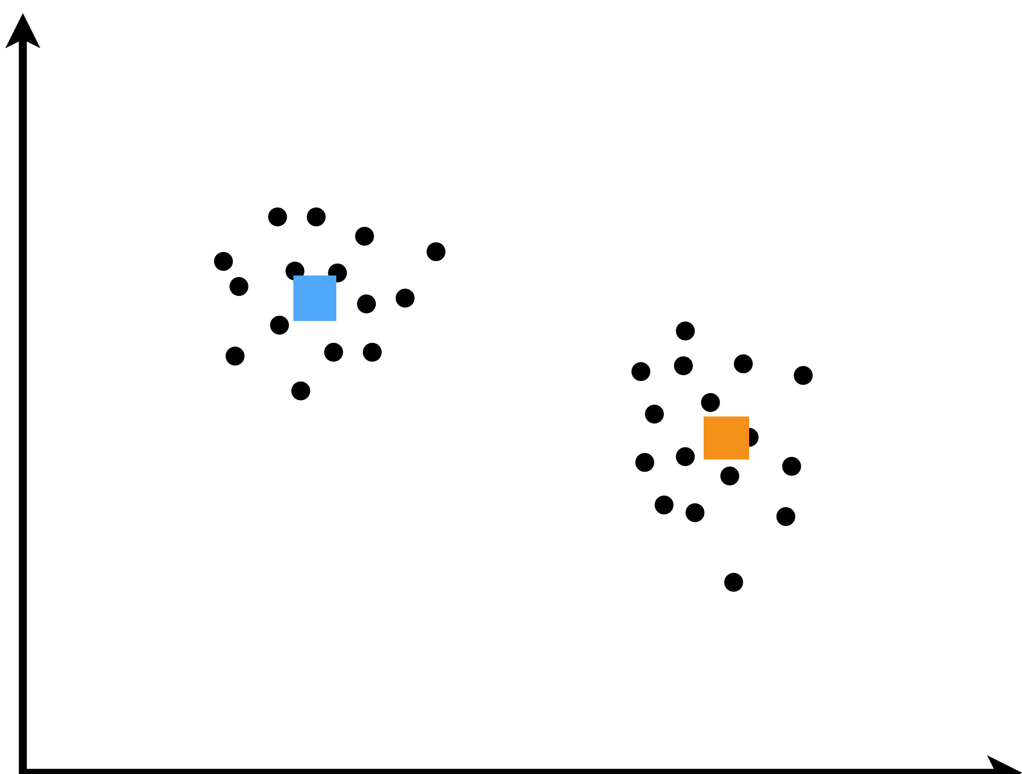

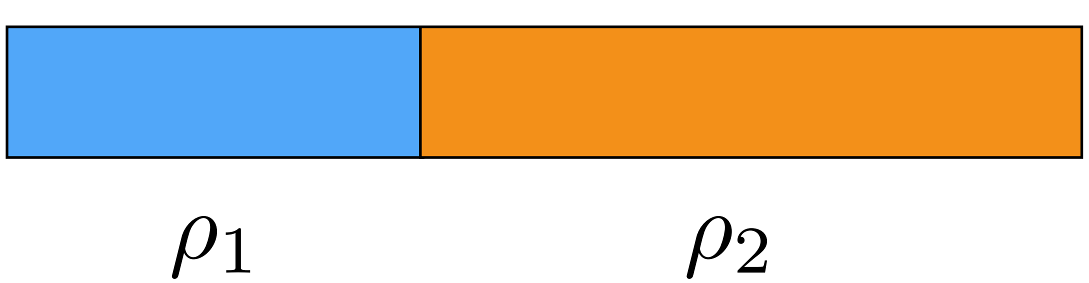

- Finite Gaussian mixture model (K=2 clusters)

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$$x_n \overset{indep}{\sim} \mathcal{N}(\mu_{z_n}, \Sigma)$$

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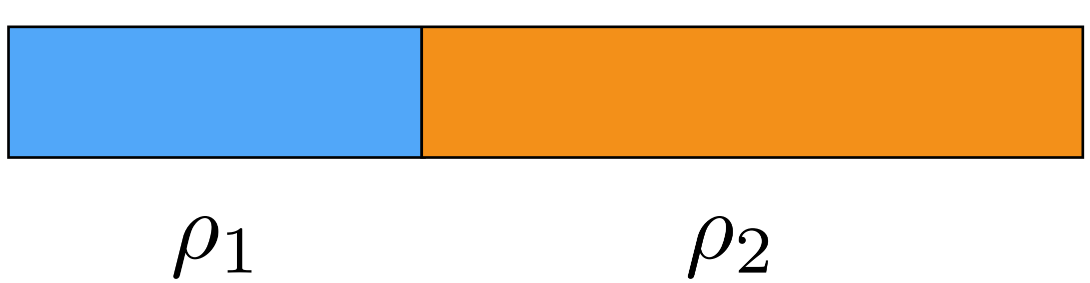

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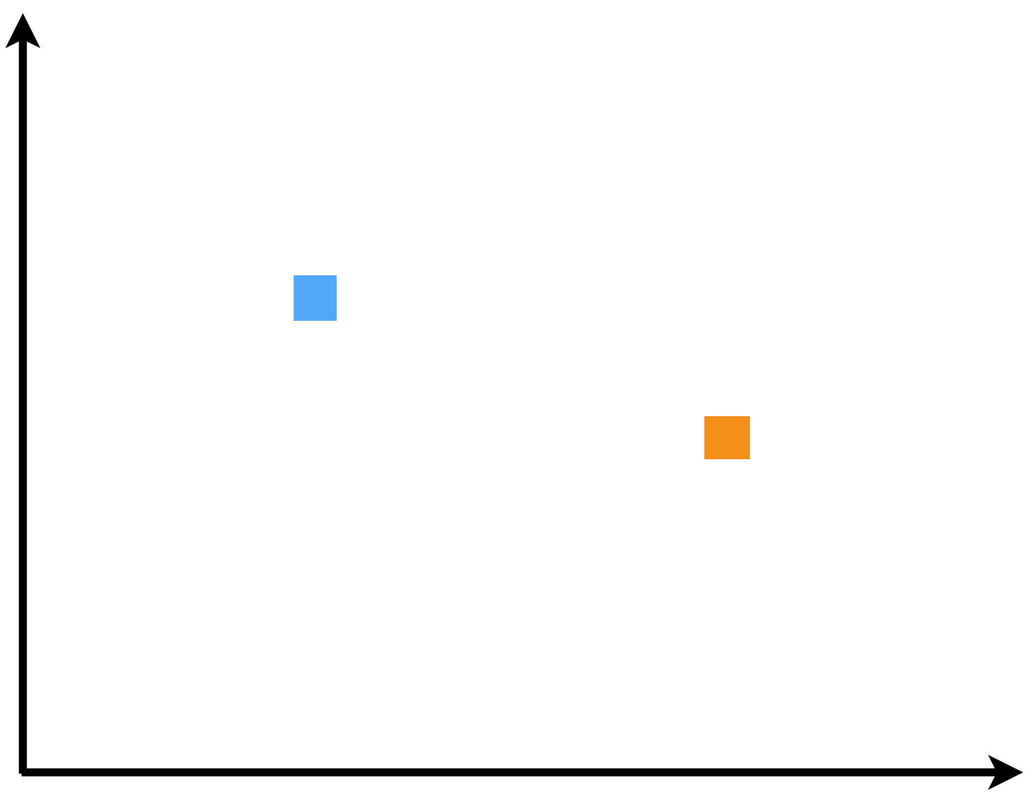

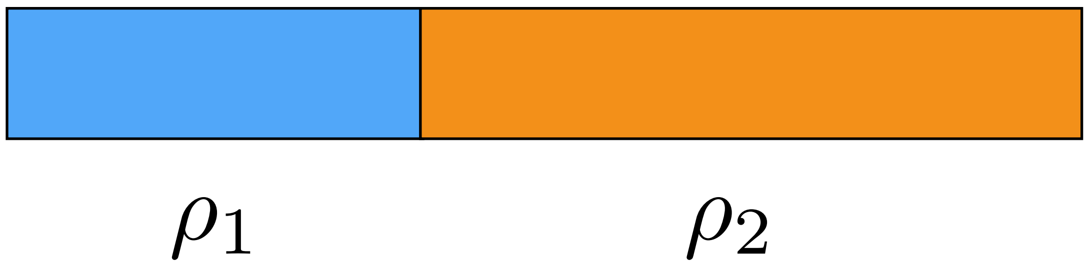

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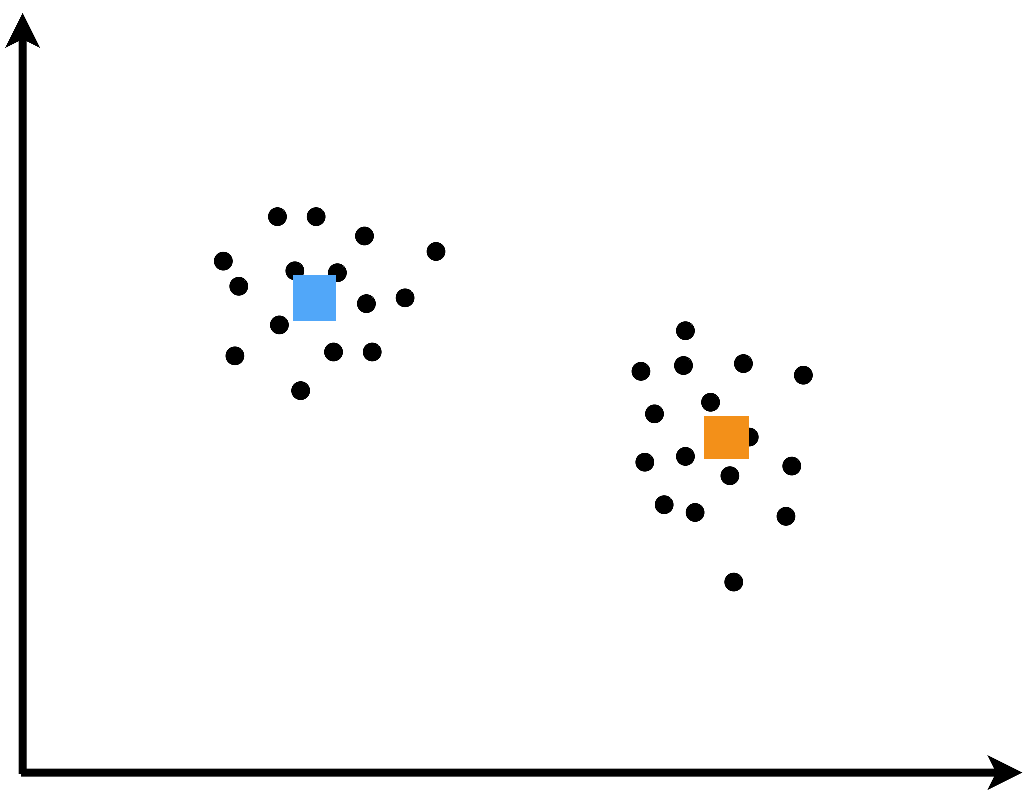

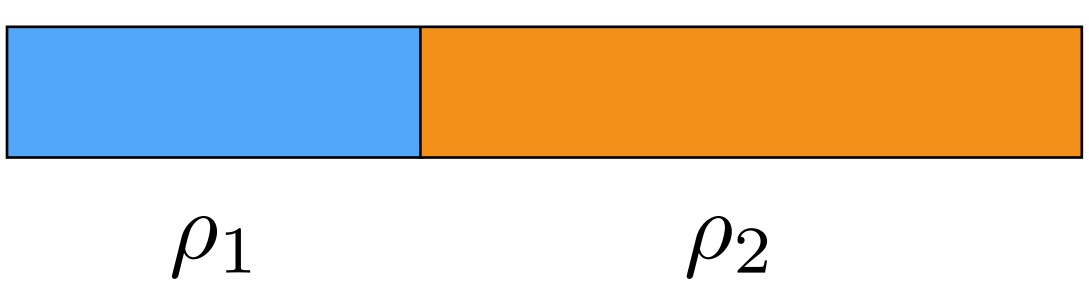

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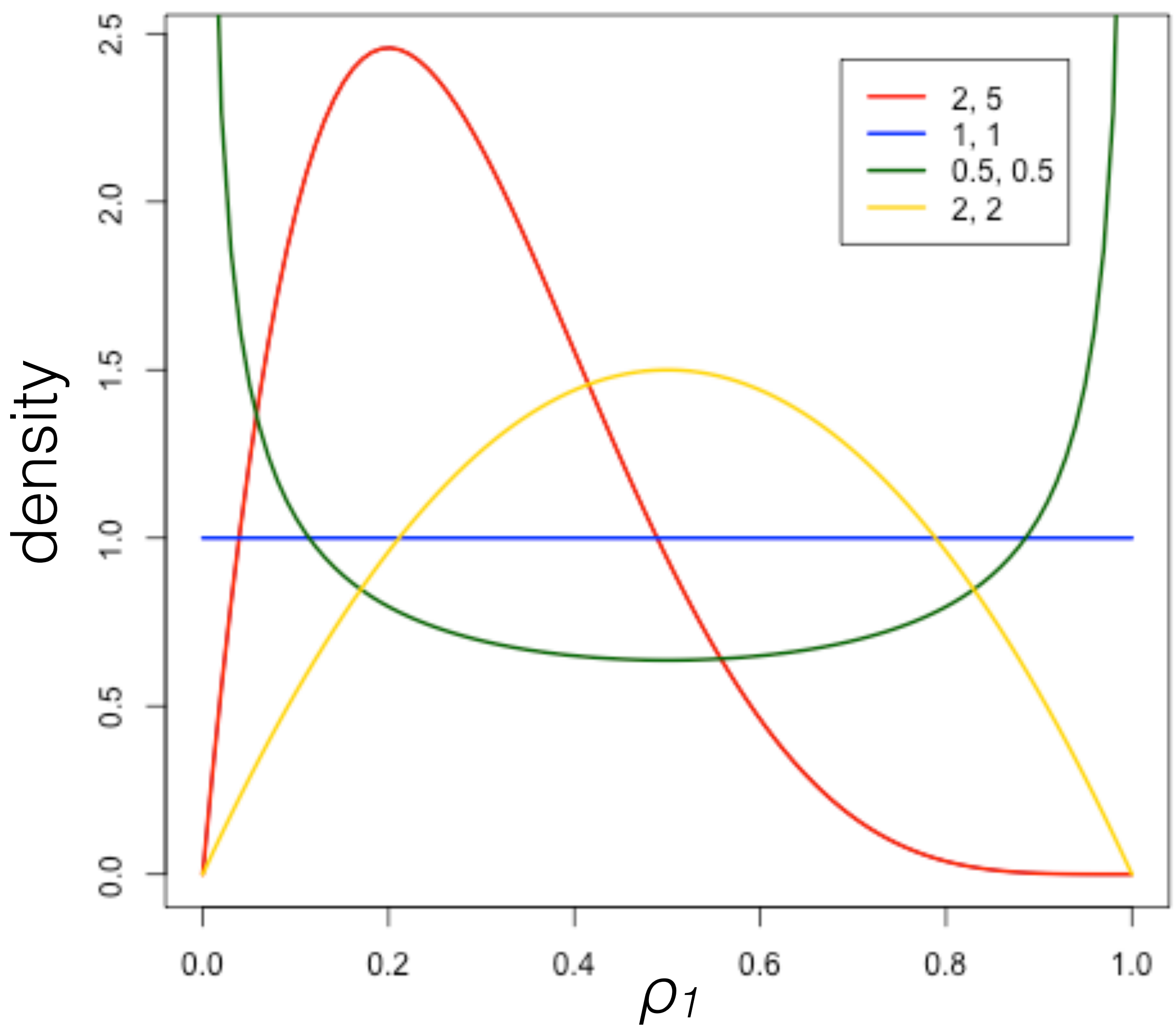

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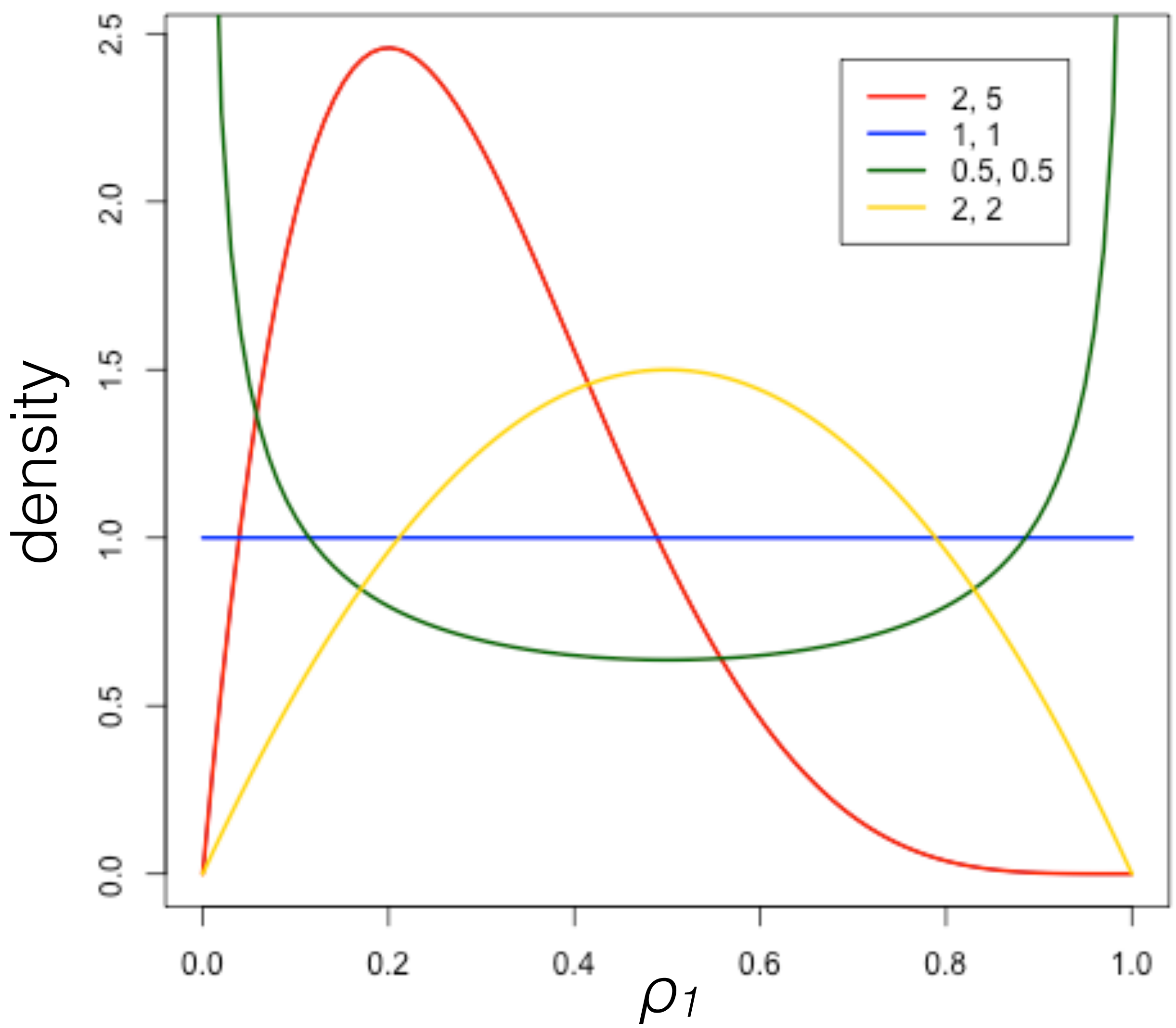

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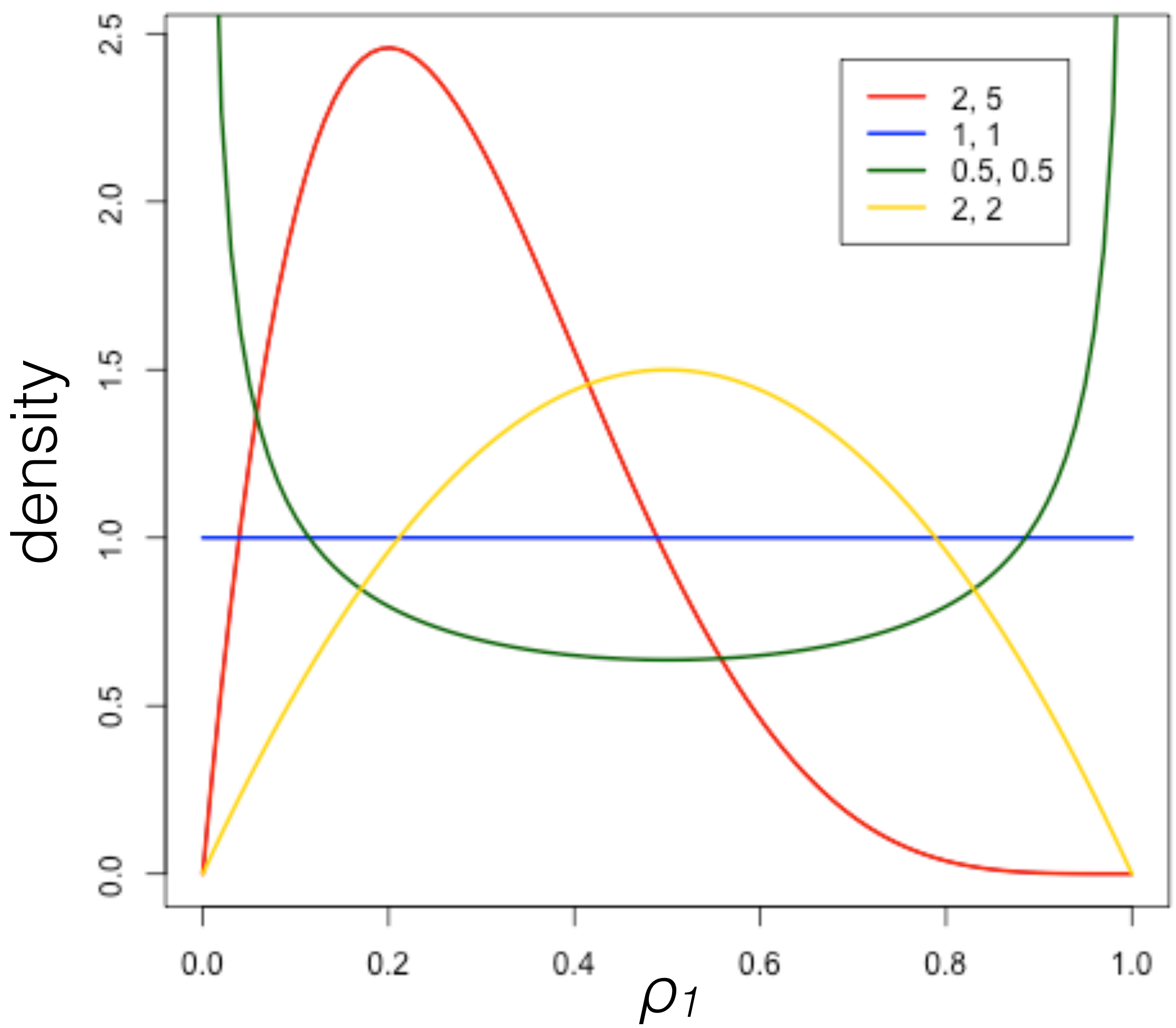

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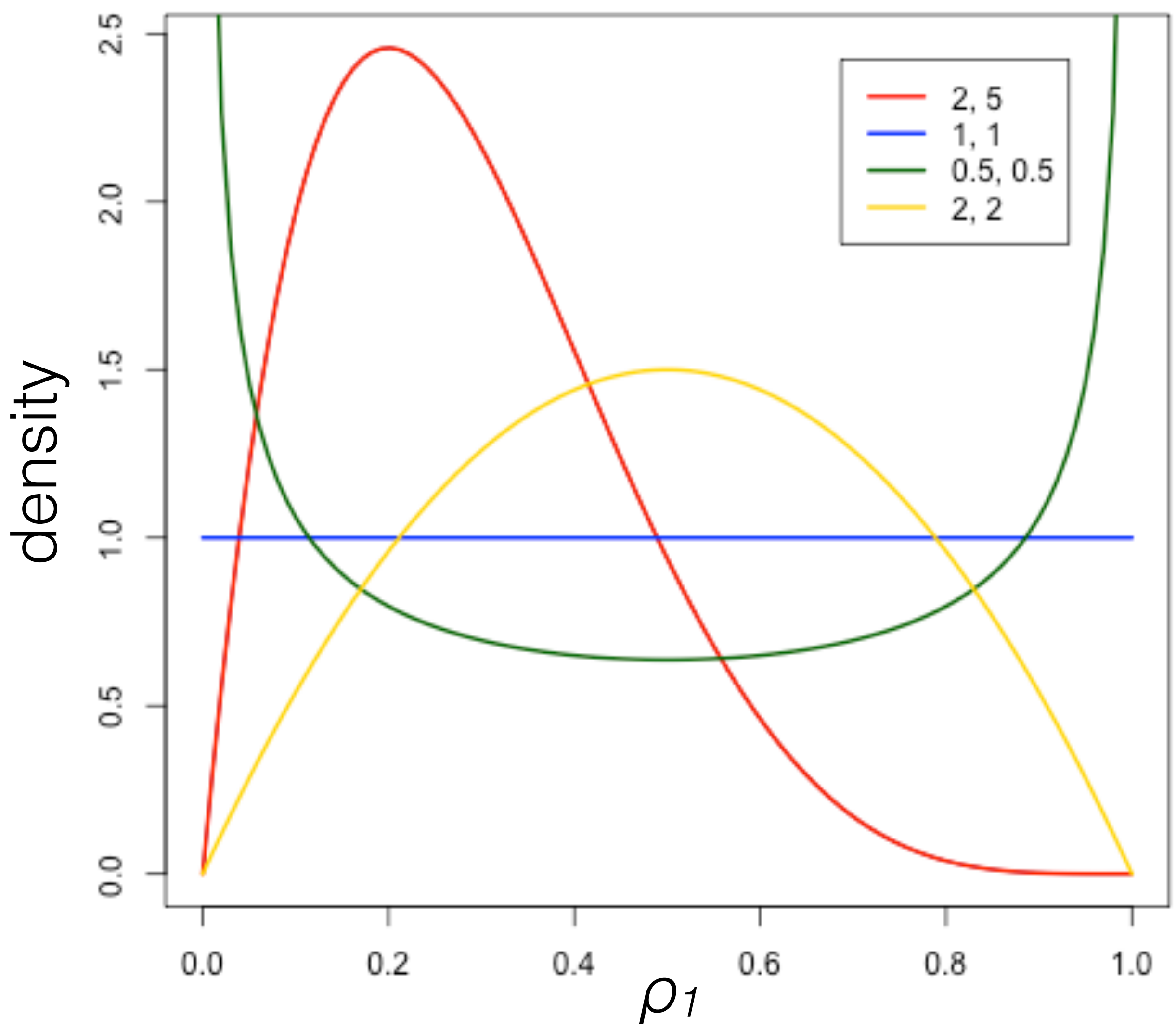

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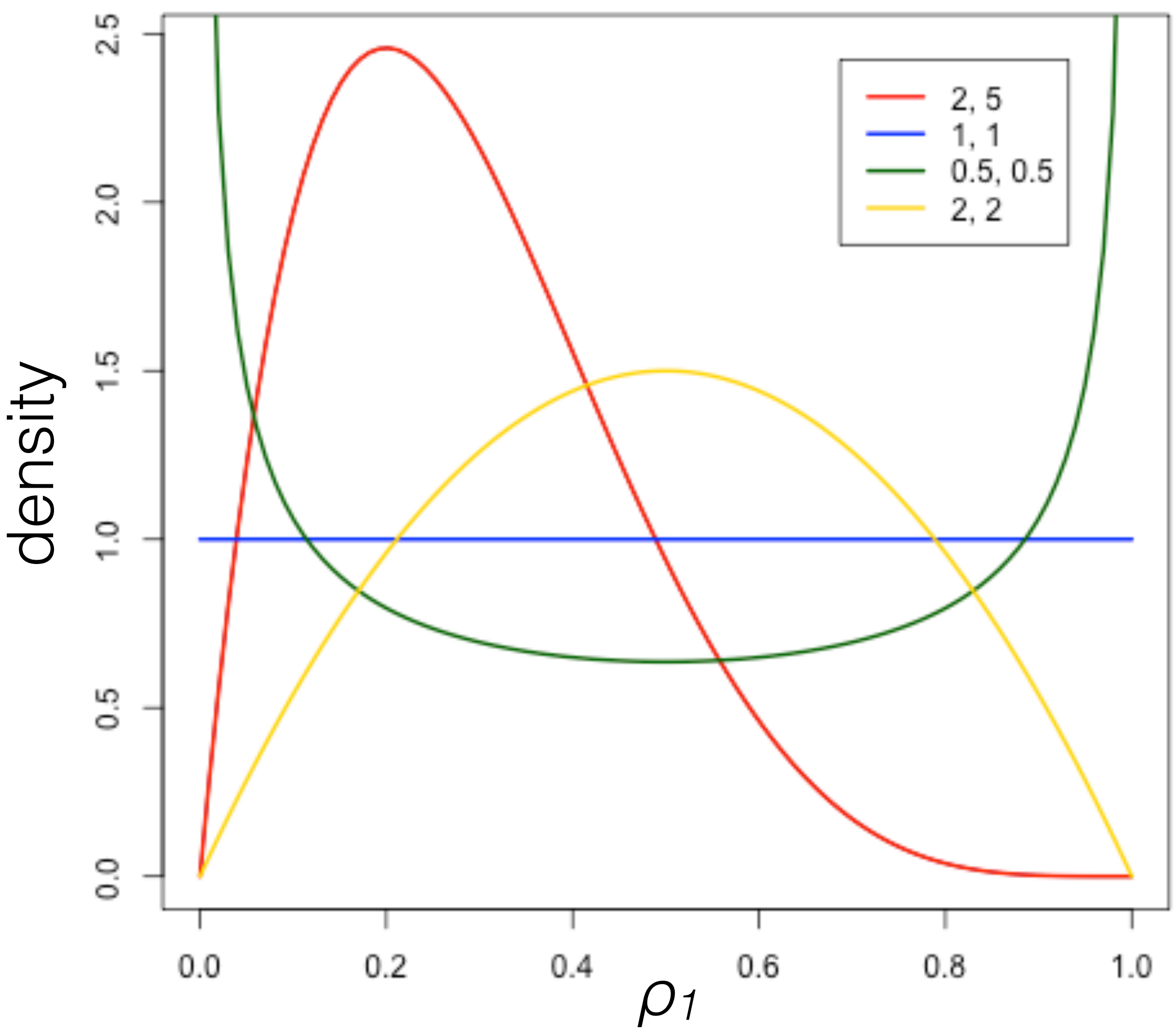

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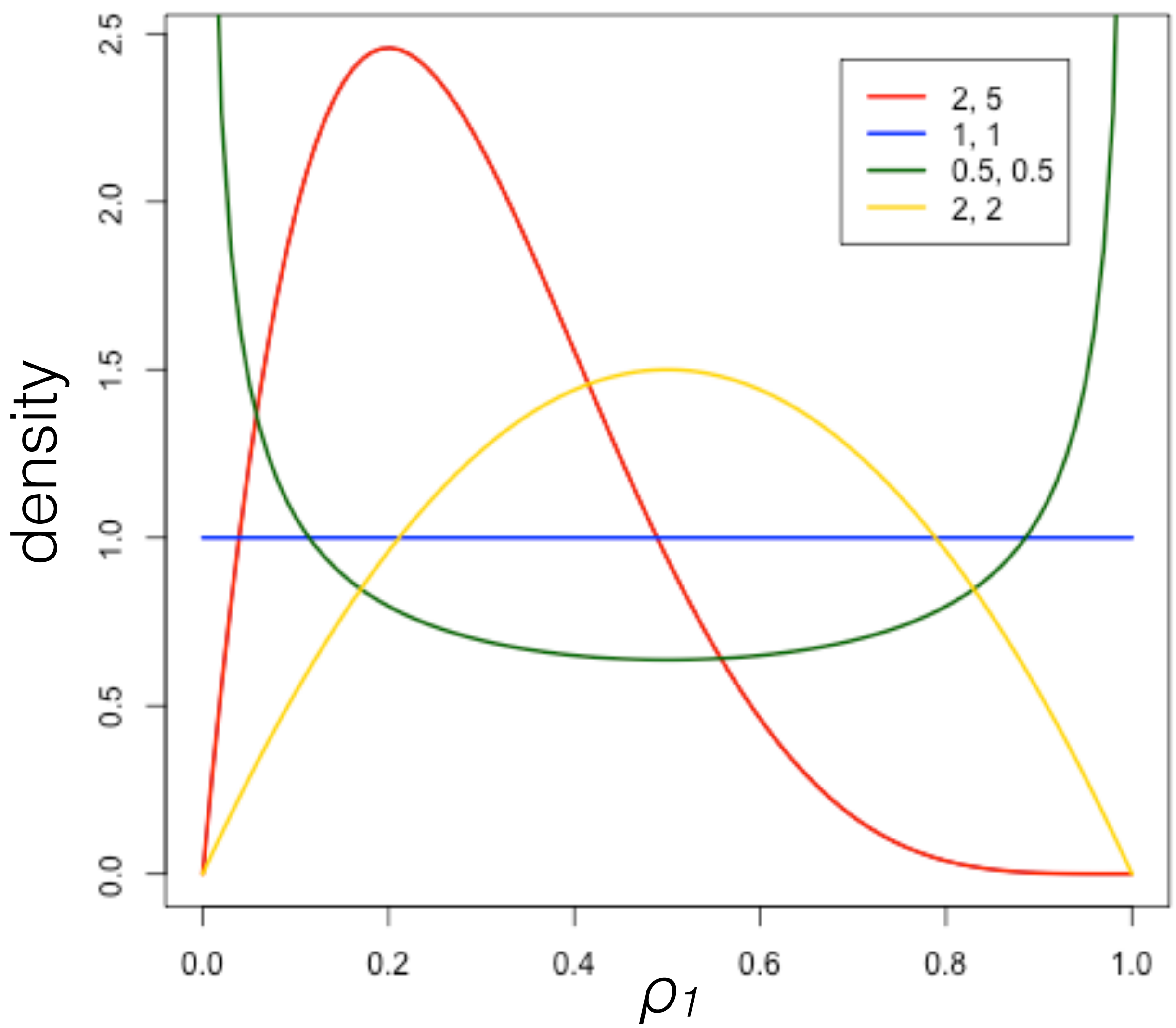

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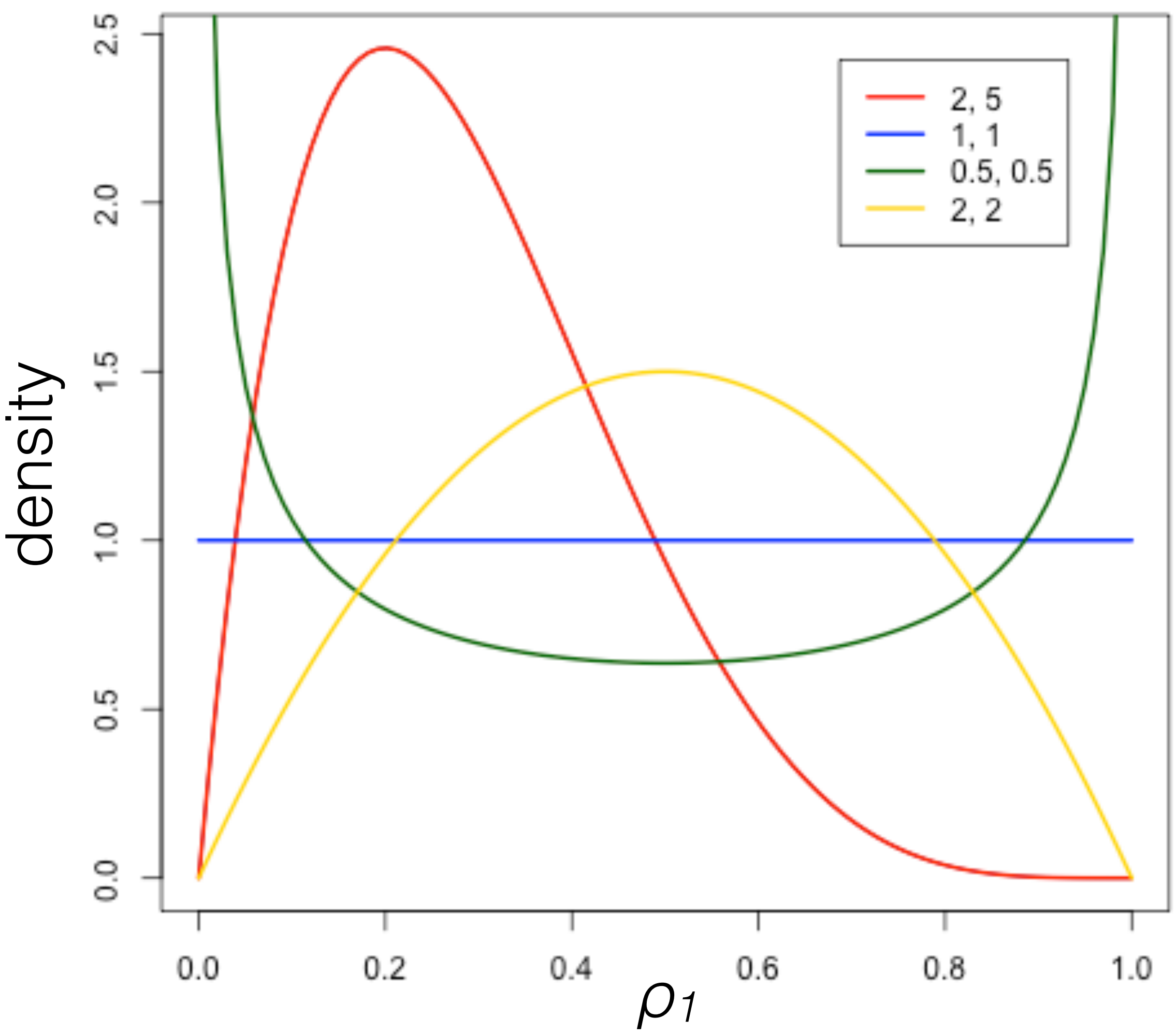

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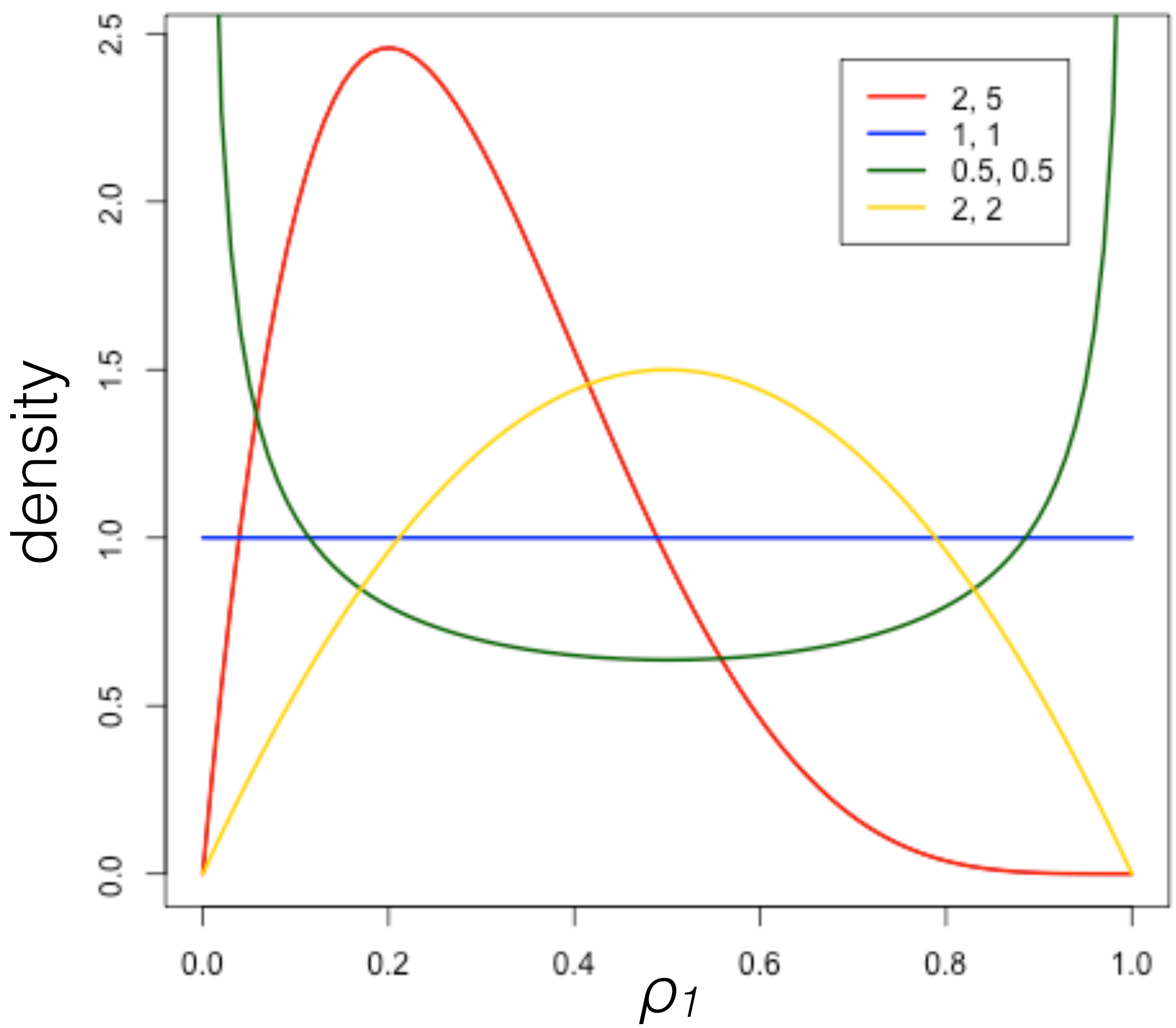

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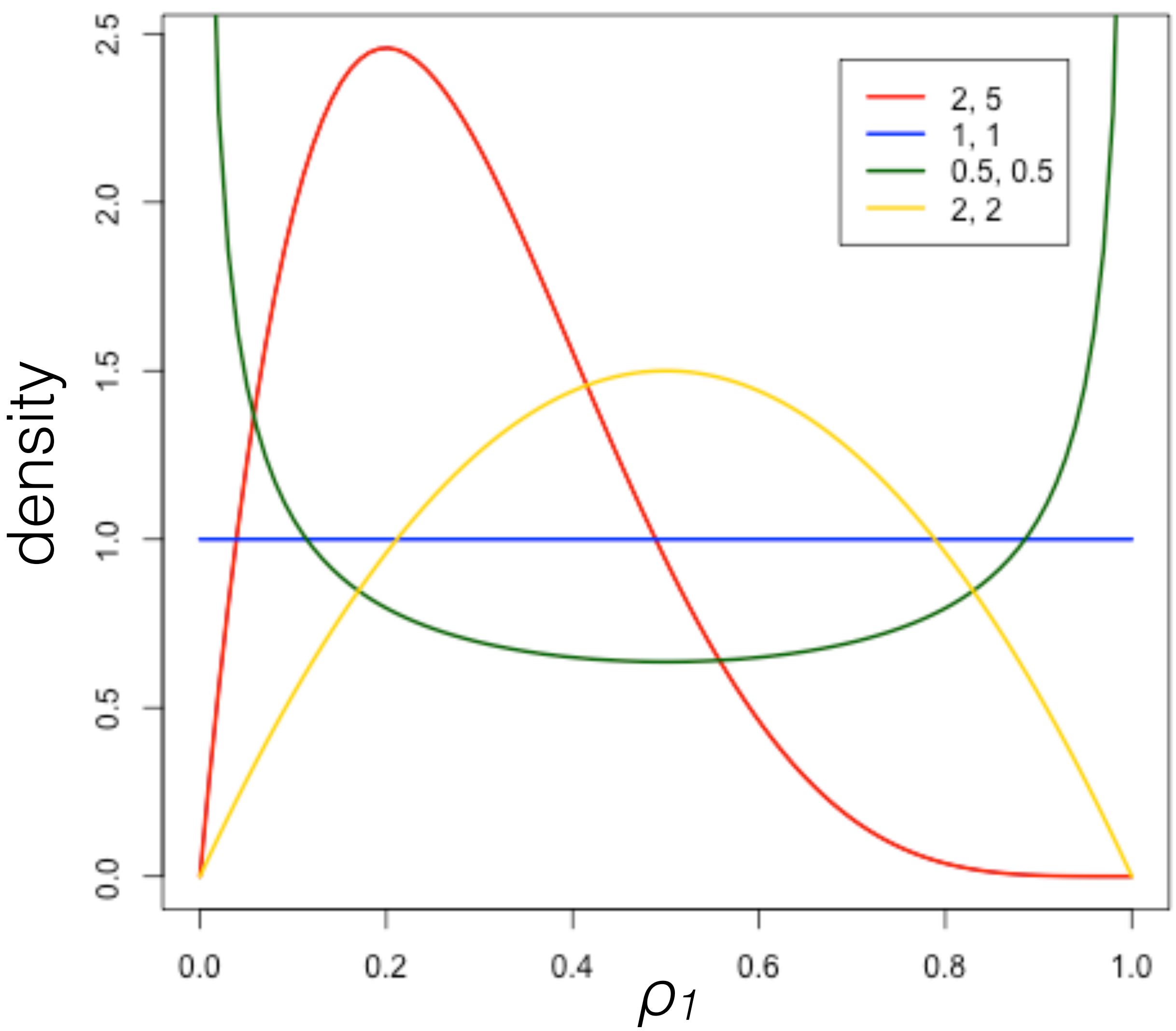

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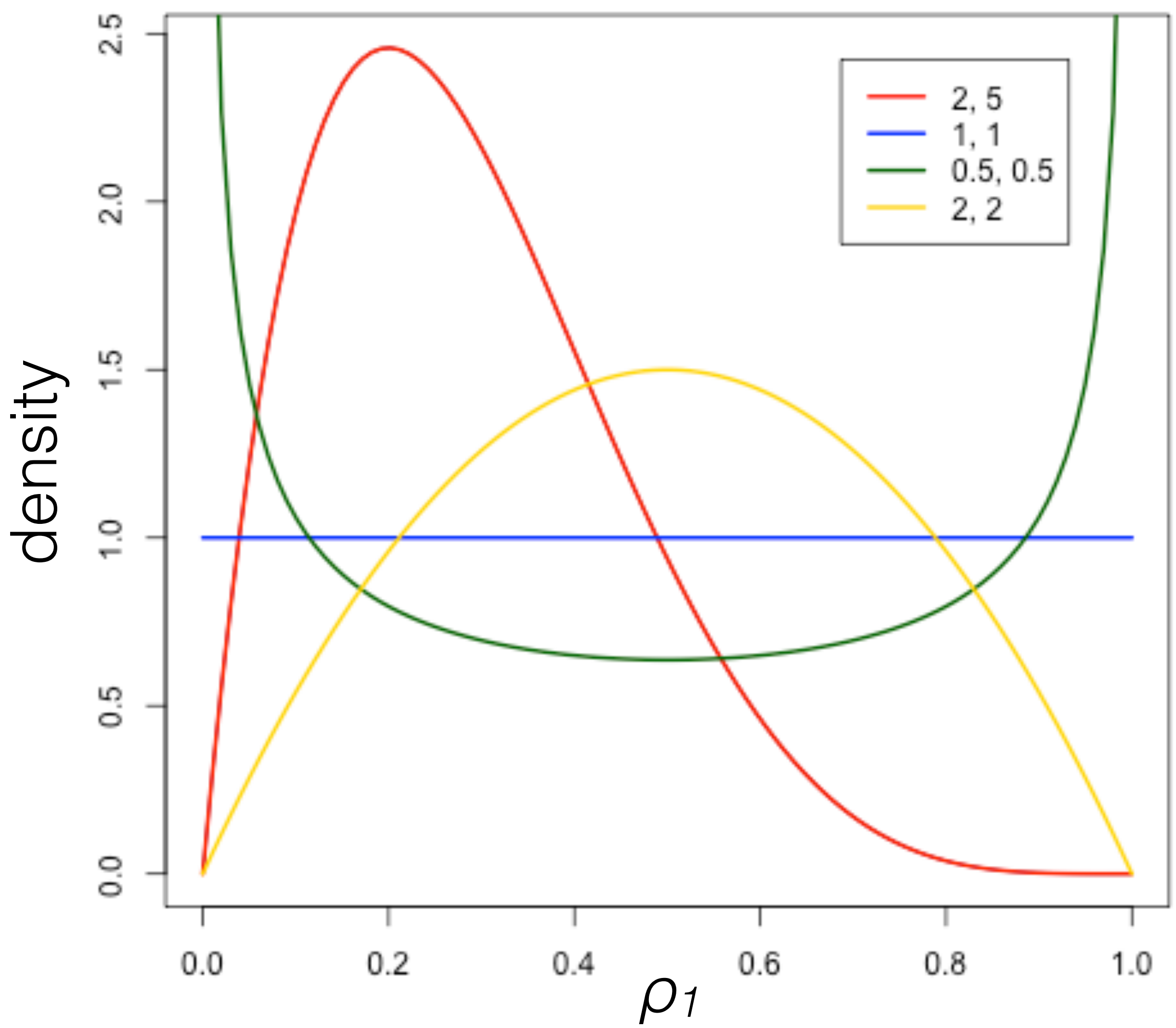

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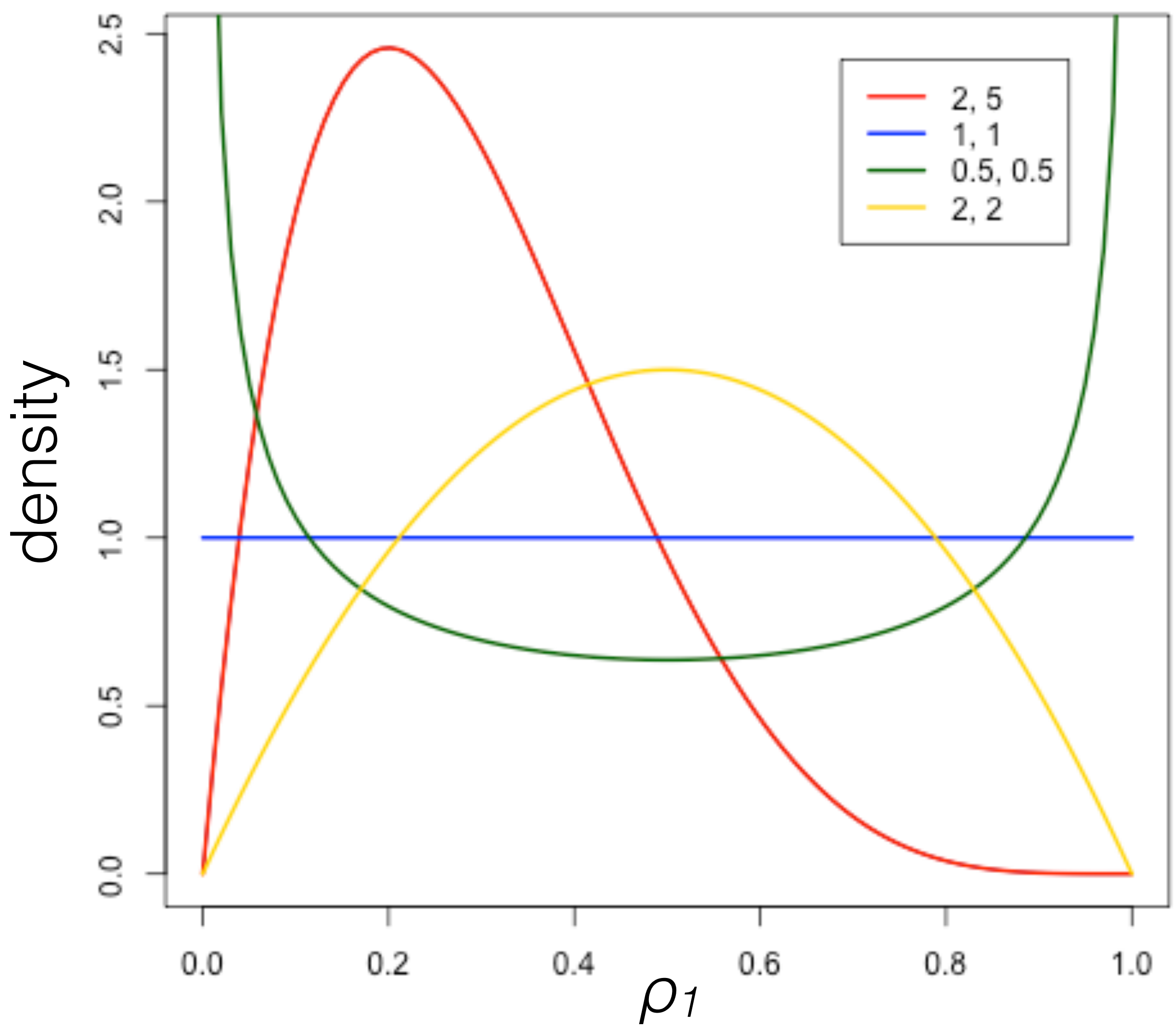

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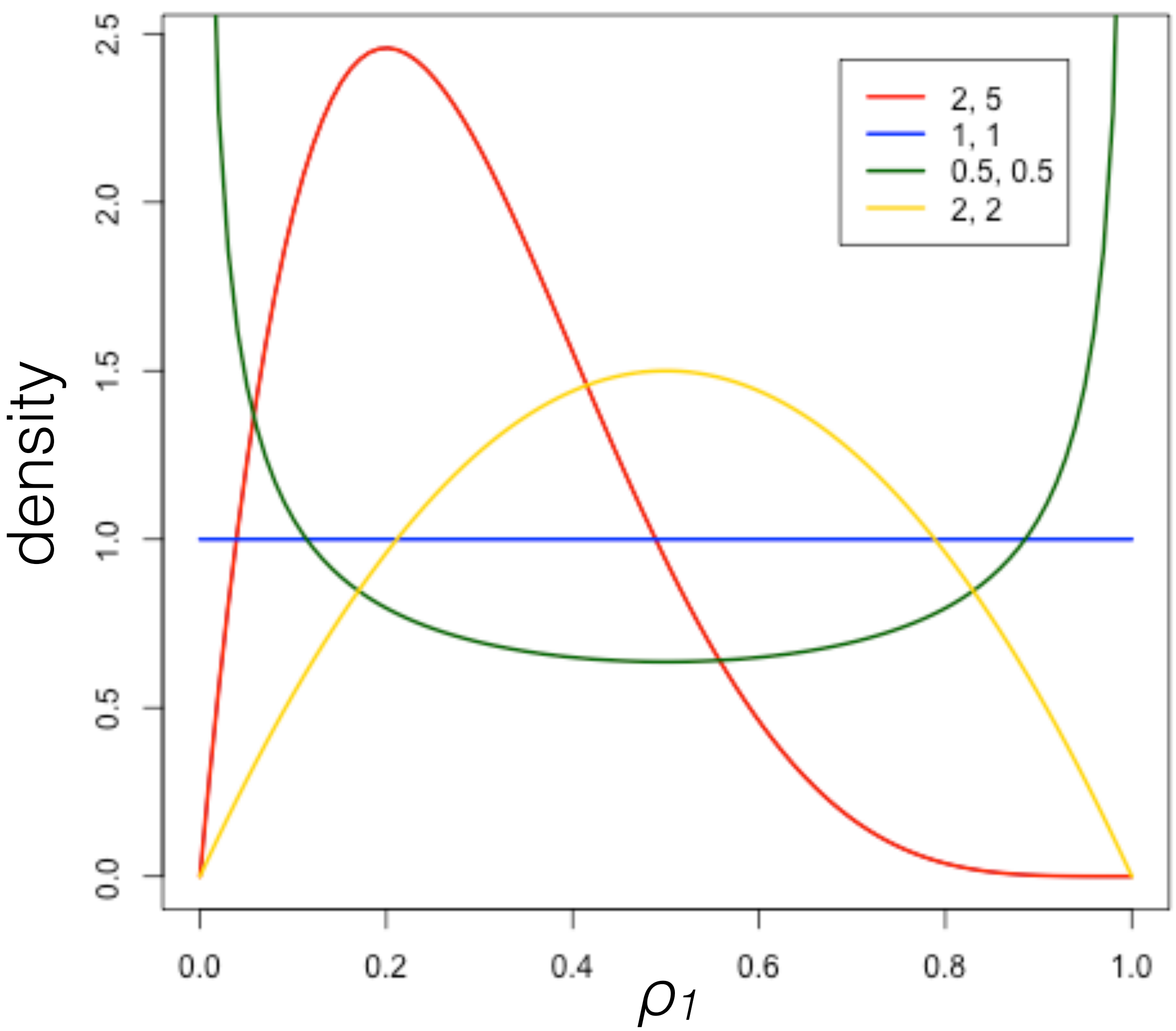

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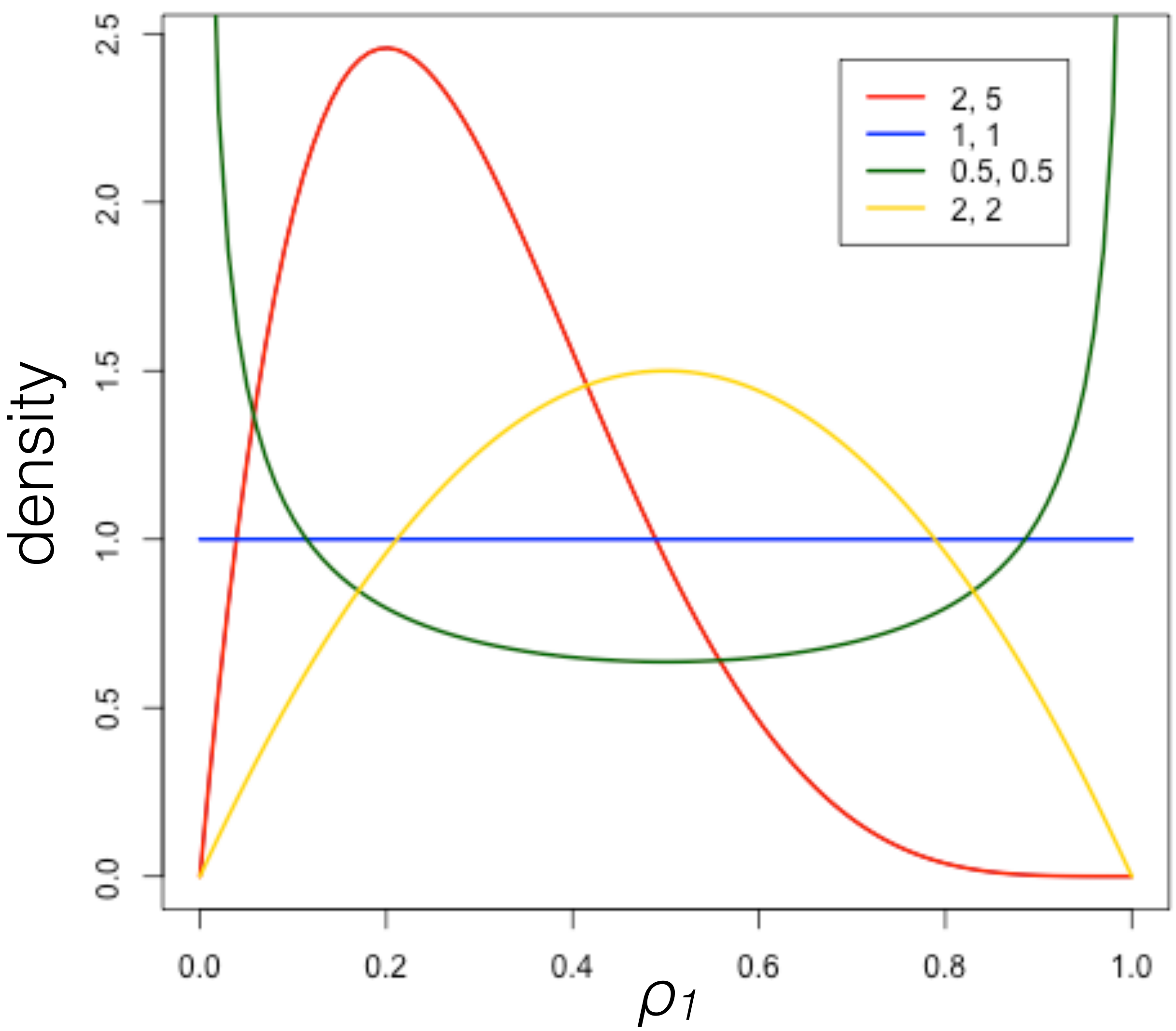

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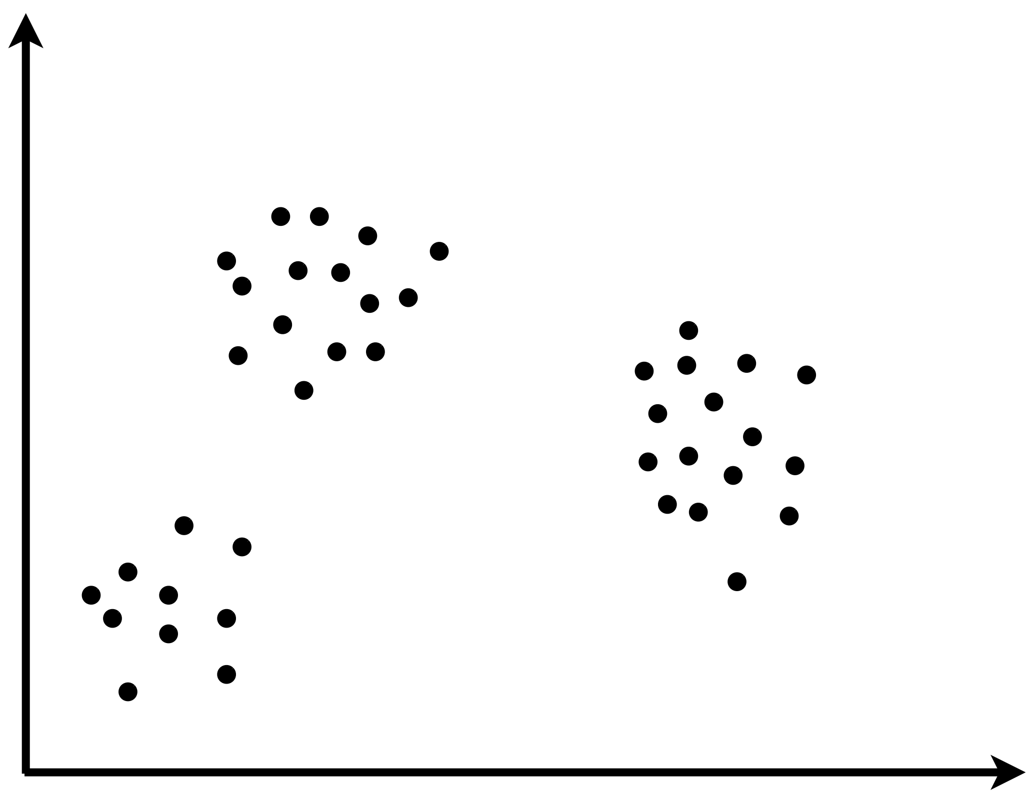

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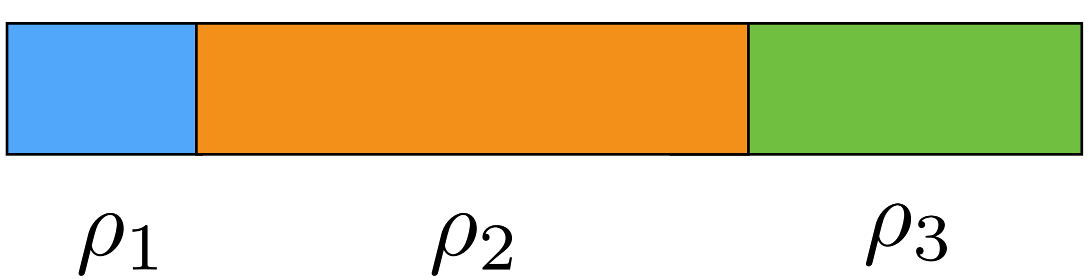

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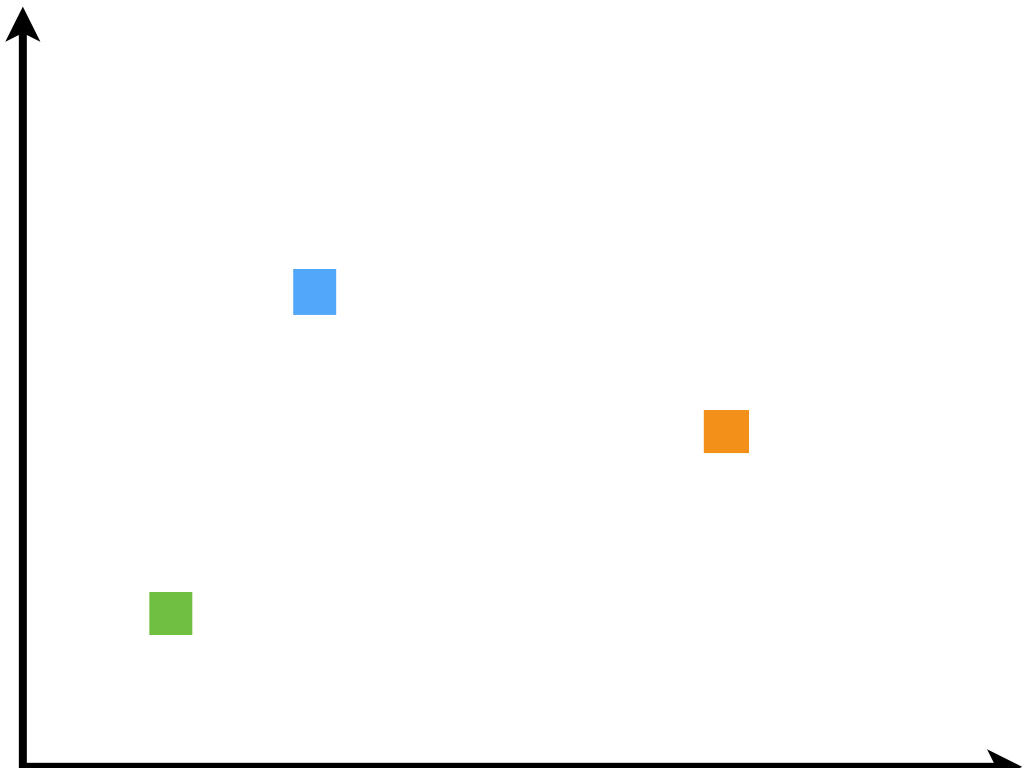

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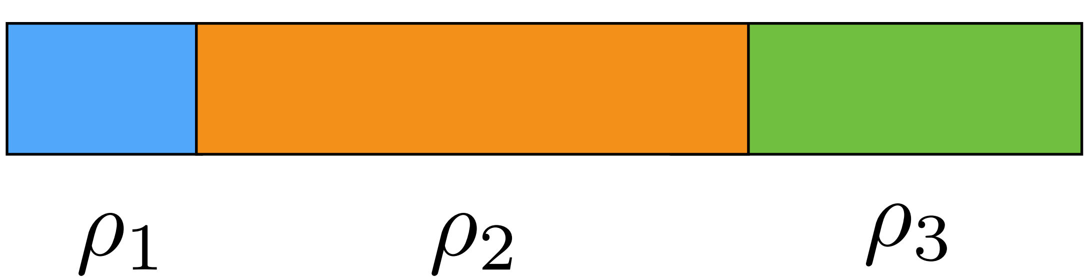

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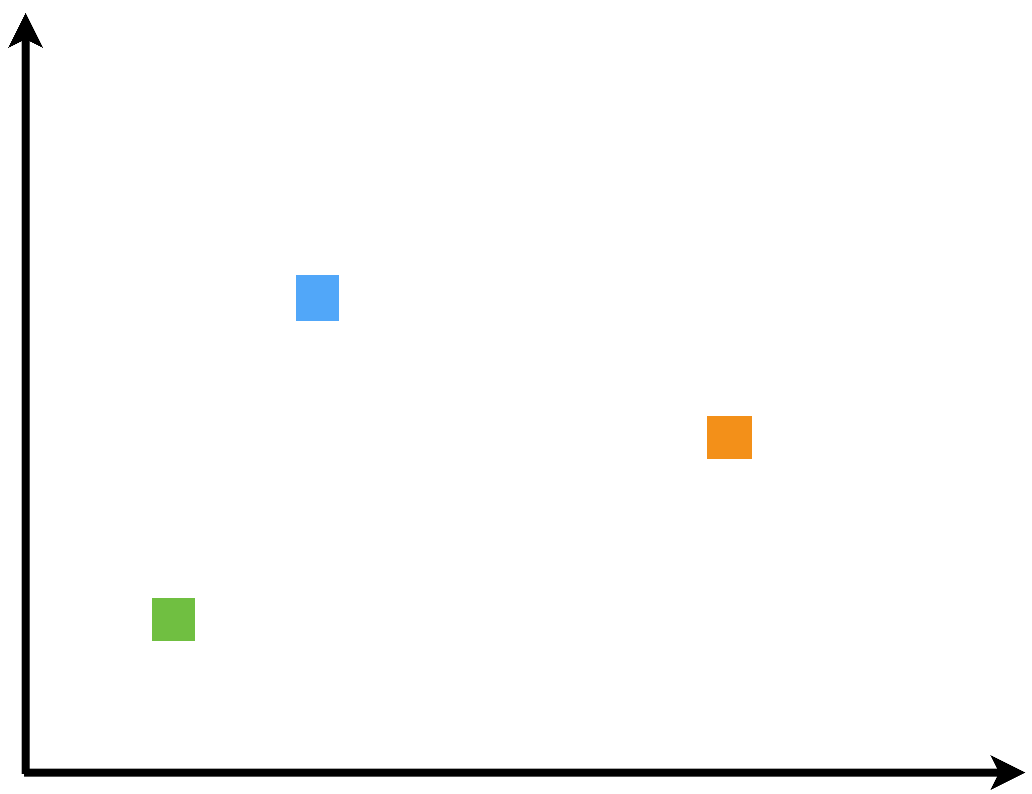

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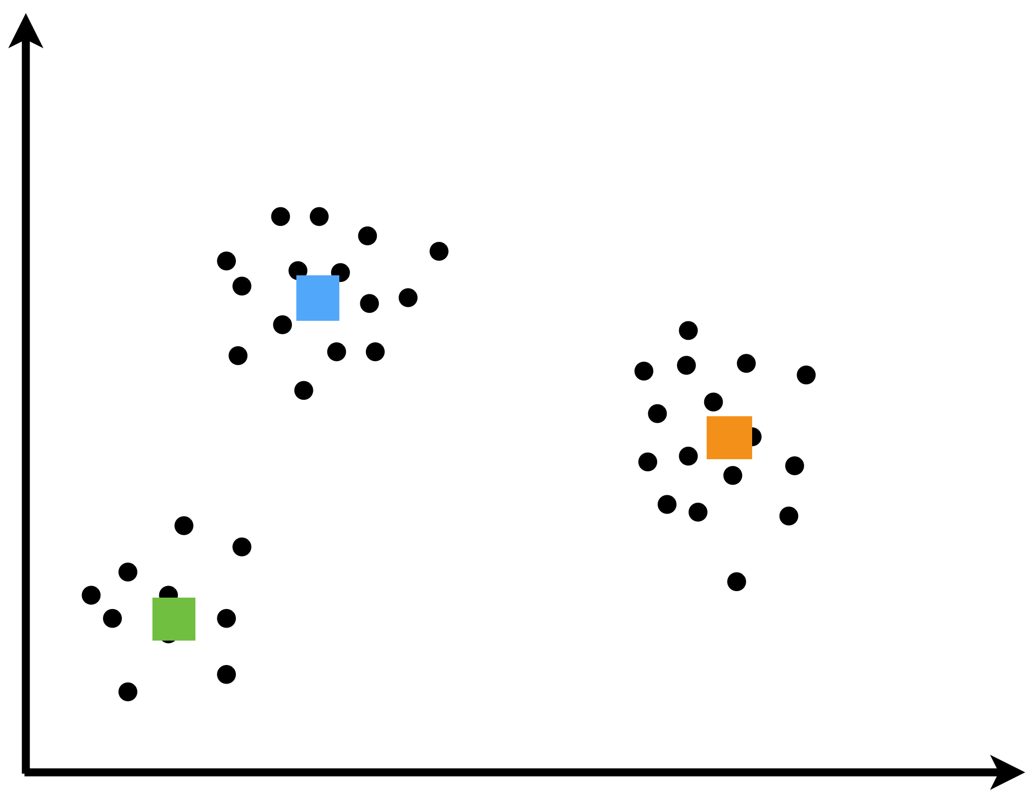
$$\mu_k \overset{iid}{\sim} \mathcal{N}(\mu_0, \Sigma_0)$$

$$z_n \overset{iid}{\sim} \text{Categorical}(\rho_{1:K})$$

ρ_1 ρ_2 ρ_3

Generative model

$$\mathbb{P}(\text{parameters}|\text{data}) \propto \mathbb{P}(\text{data}|\text{parameters})\mathbb{P}(\text{parameters})$$



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$$x_n \overset{indep}{\sim} \mathcal{N}(\mu_{z_n}, \Sigma)$$

Dirichlet distribution review

$$\text{Dirichlet}(\rho_{1:K}|a_{1:K}) = \frac{\Gamma(\sum_{k=1}^{K} a_k)}{\prod_{k=1}^{K} \Gamma(a_k)} \prod_{k=1}^{K} \rho_k^{a_k - 1} \qquad a_k > 0$$

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$$a_k > 0$$

$$\rho_k \in (0, 1)$$

$$\sum_k \rho_k = 1$$

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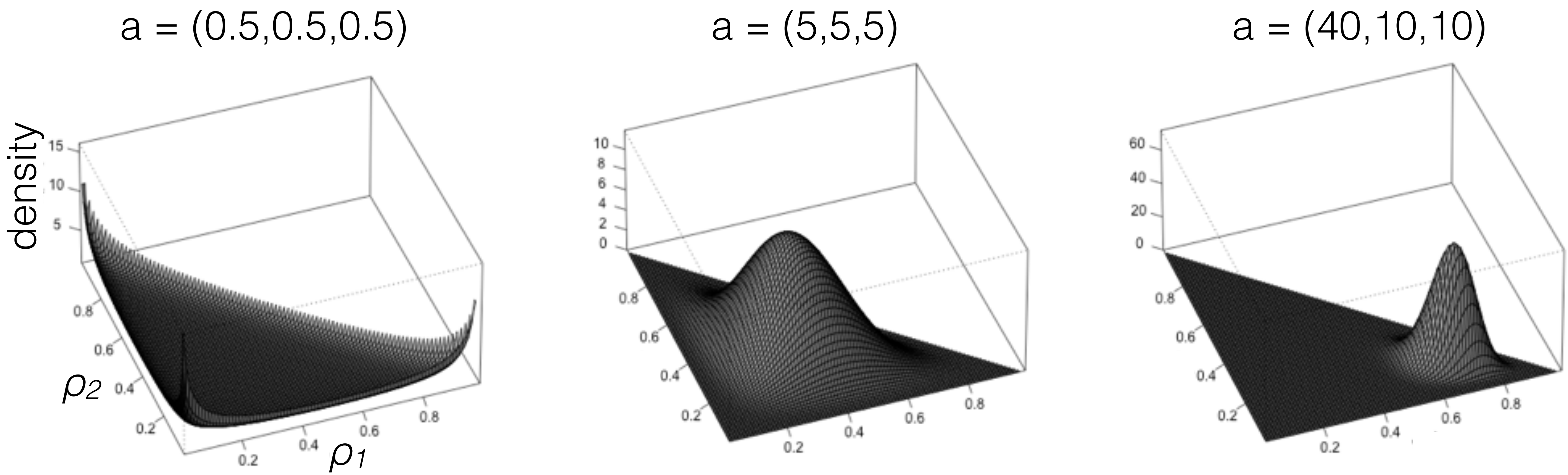

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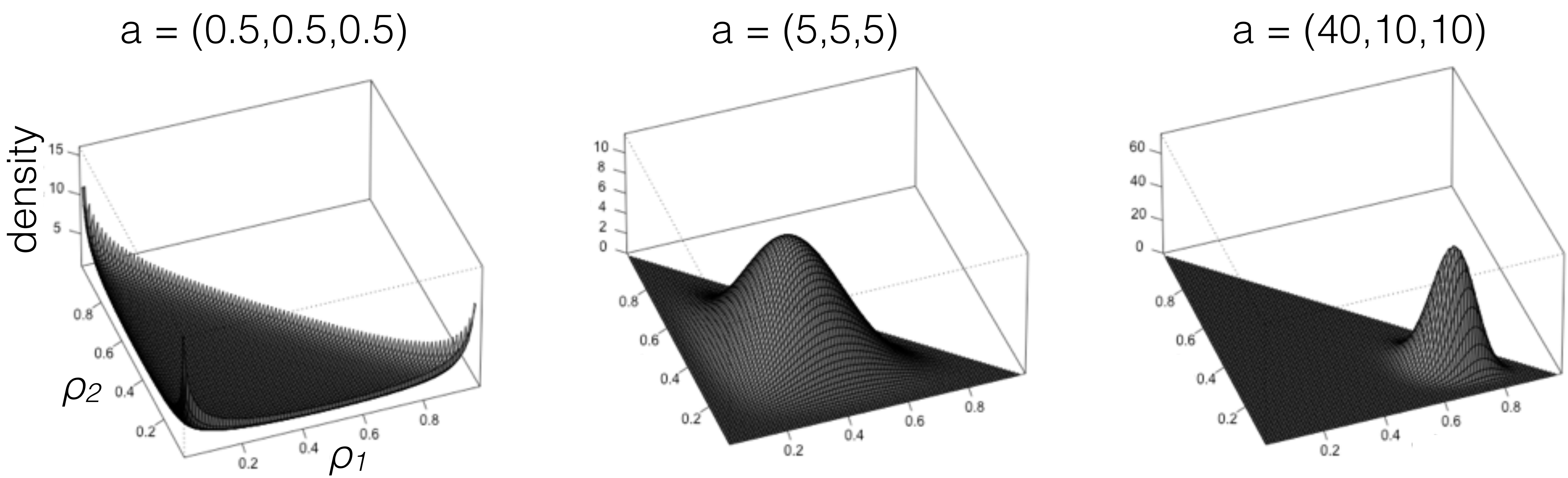

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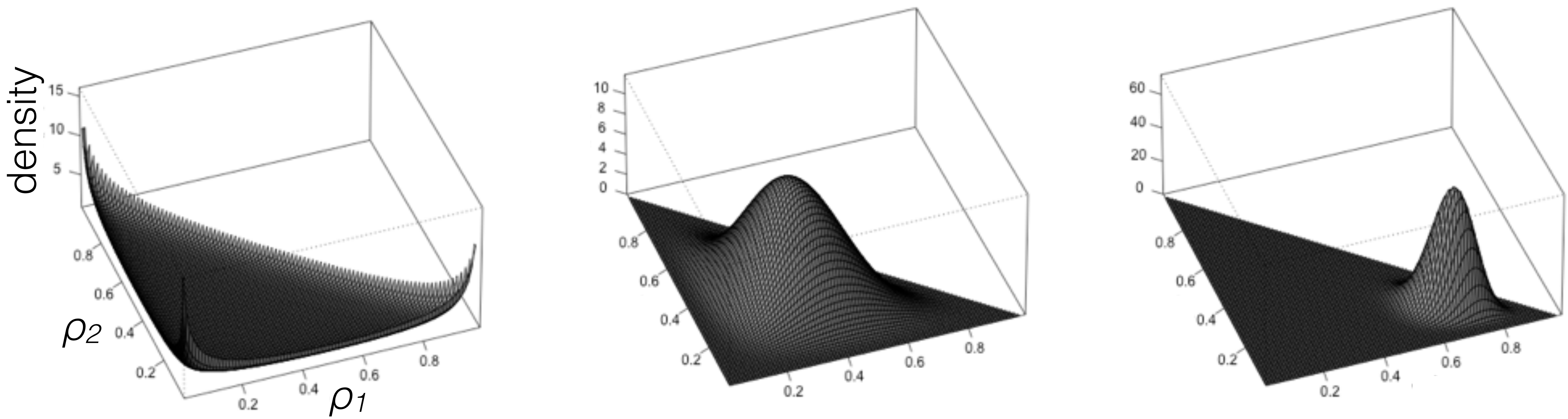
a = (0.5,0.5,0.5) a = (5,5,5) a = (40,10,10)



- What happens? $a = a_k = 1$ $\quad a = a_k \to 0$

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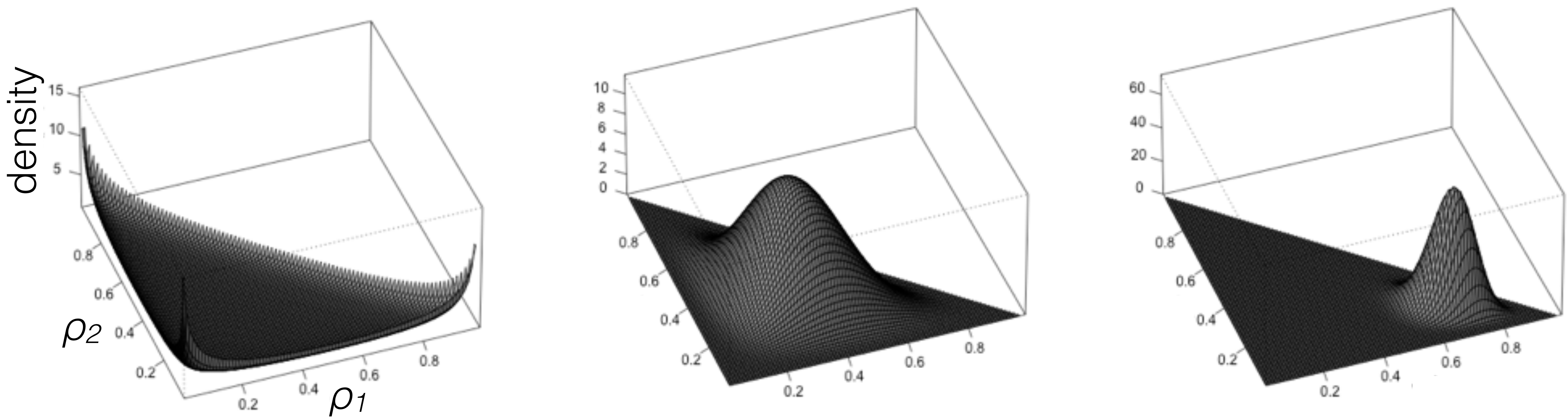

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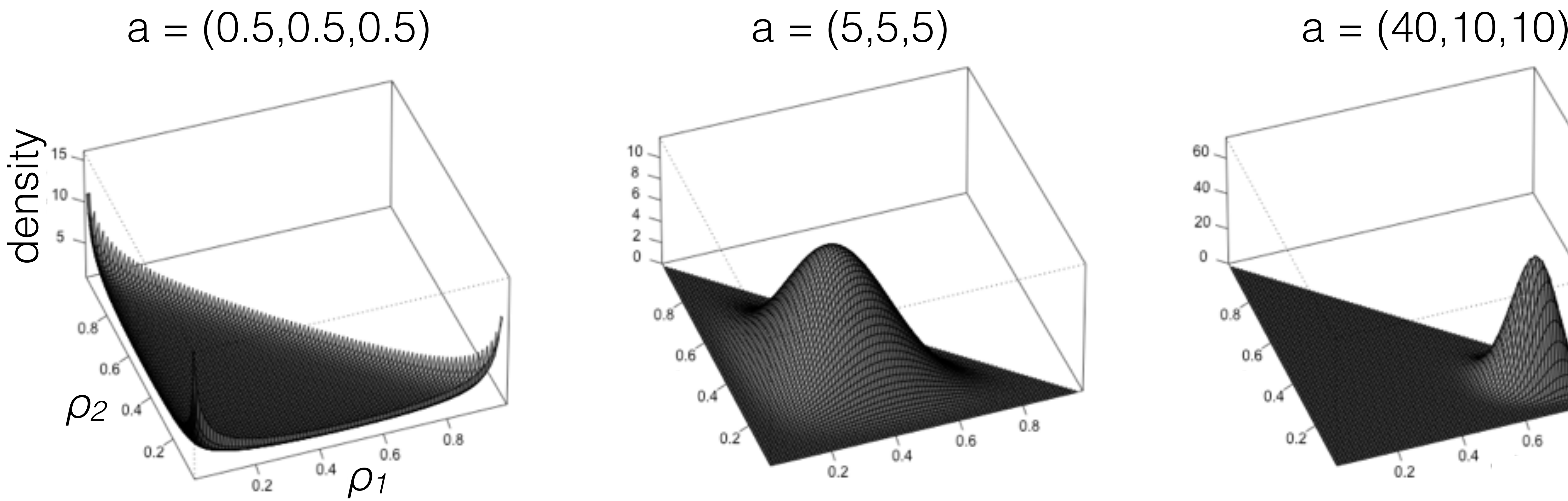

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[demo]

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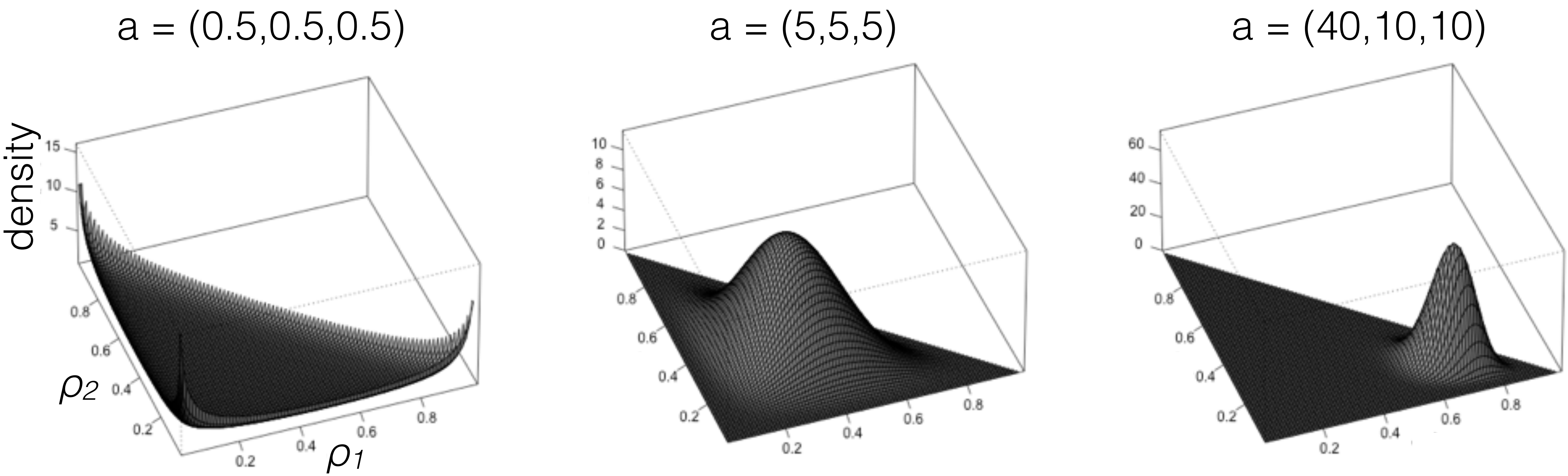

- What happens? $a = a_k = 1$ $\quad a = a_k \to 0$ $\quad a = a_k \to \infty$
- Dirichlet is conjugate to Categorical

[demo]

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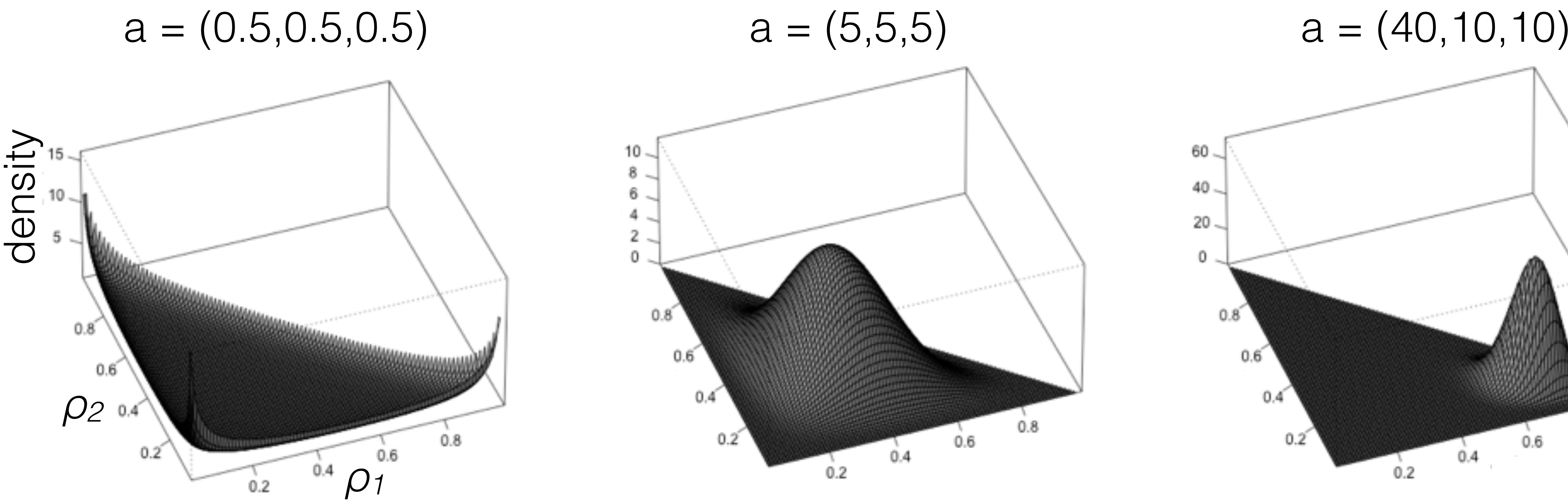

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  [demo]
- Dirichlet is conjugate to Categorical
  $\rho_{1:K} \sim \text{Dirichlet}(a_{1:K}), z \sim \text{Cat}(\rho_{1:K})$

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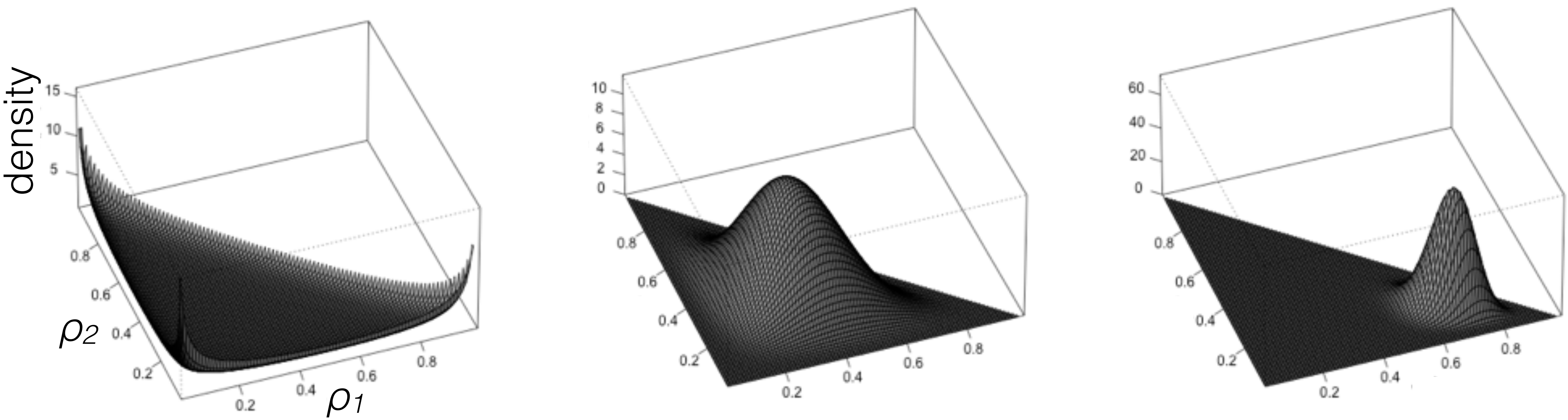

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[demo]

- Dirichlet is conjugate to Categorical

$$\rho_{1:K} \sim \mathrm{Dirichlet}(a_{1:K}), z \sim \mathrm{Cat}(\rho_{1:K})$$

$$\rho_{1:K}|z \stackrel{d}{=} \mathrm{Dirichlet}(a'_{1:K}), a'_k = a_k + \mathbf{1}\{z = k\}$$

What if $K > N$?

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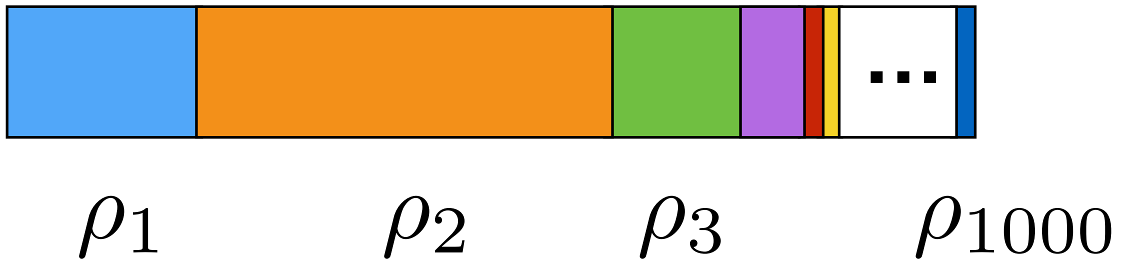
ρ_1 ρ_2 ρ_3 ρ_{1000}

What if $K > N$?

- e.g. species sampling, topic modeling, groups on a social network, etc.



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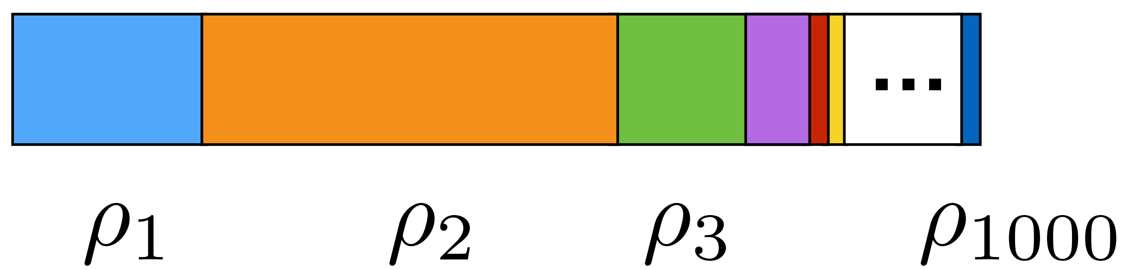

- Components: number of latent groups

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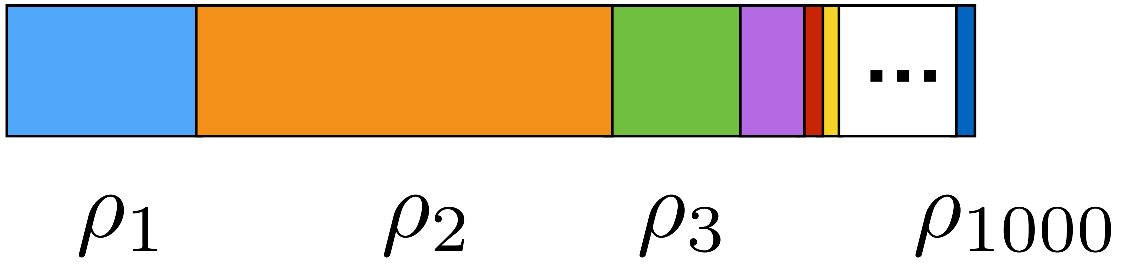

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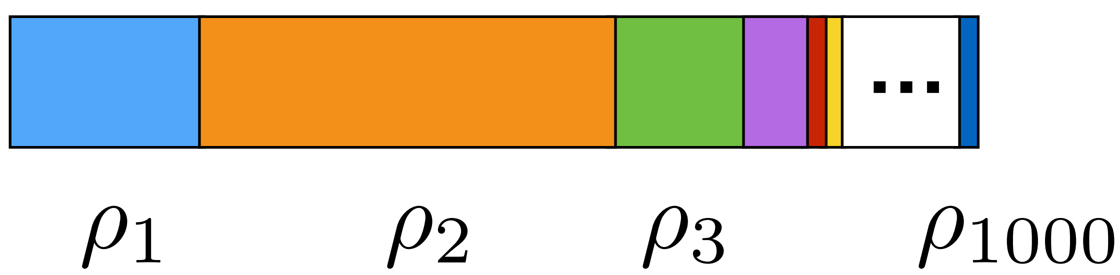

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- [demo 1, demo 2]

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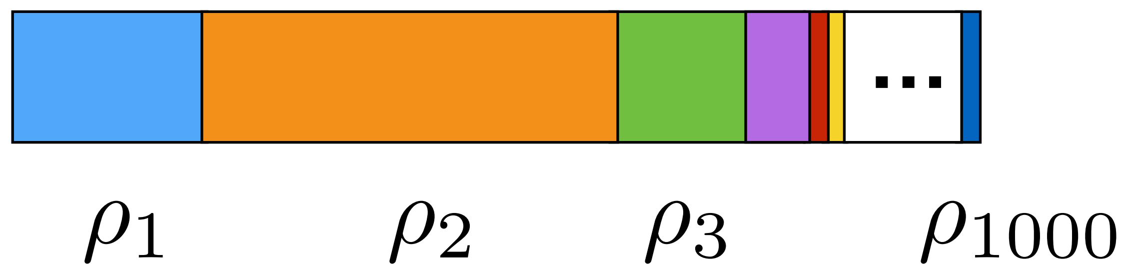

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- [demo 1, demo 2]
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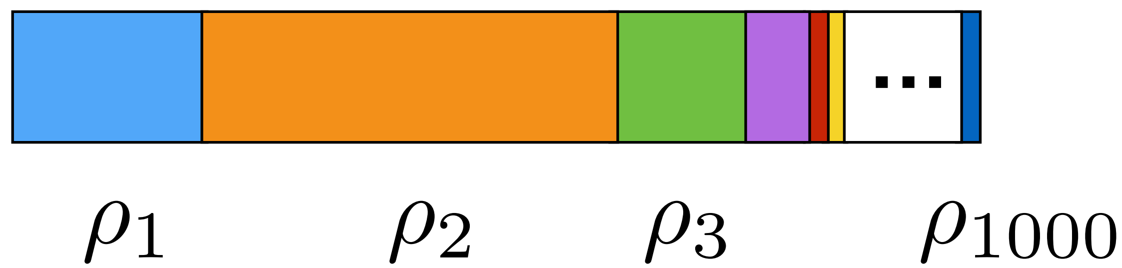

- Components: number of latent groups
- Clusters: number of components represented in the data
- [demo 1, demo 2]
- Number of clusters for N data points is $< K$ and random
- Number of clusters grows with N

- Here, difficult to choose finite K in advance (contrast with small K): don't know K, difficult to infer, streaming data

Choosing $K = \infty$

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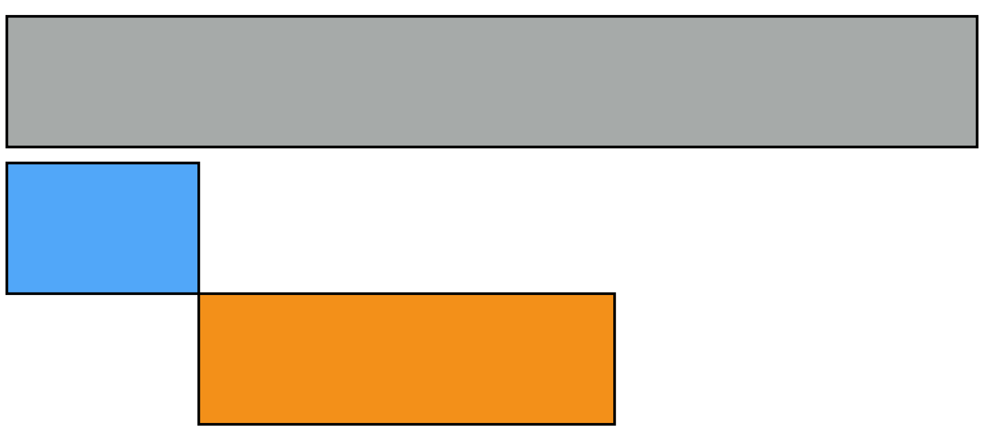

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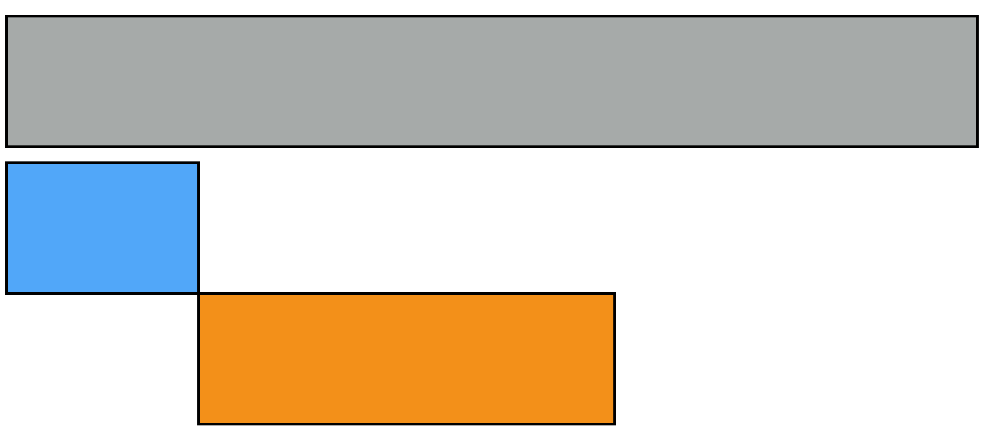

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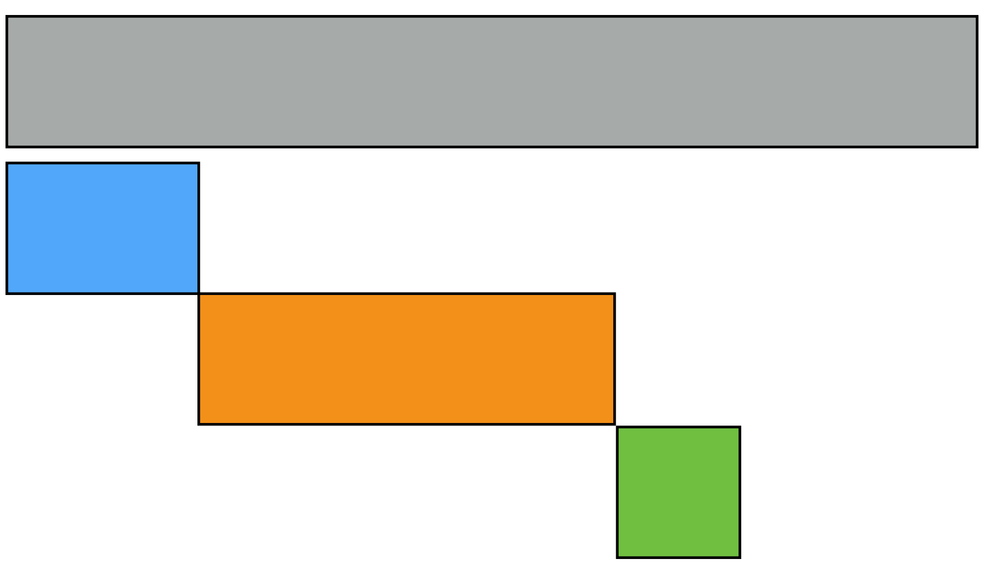

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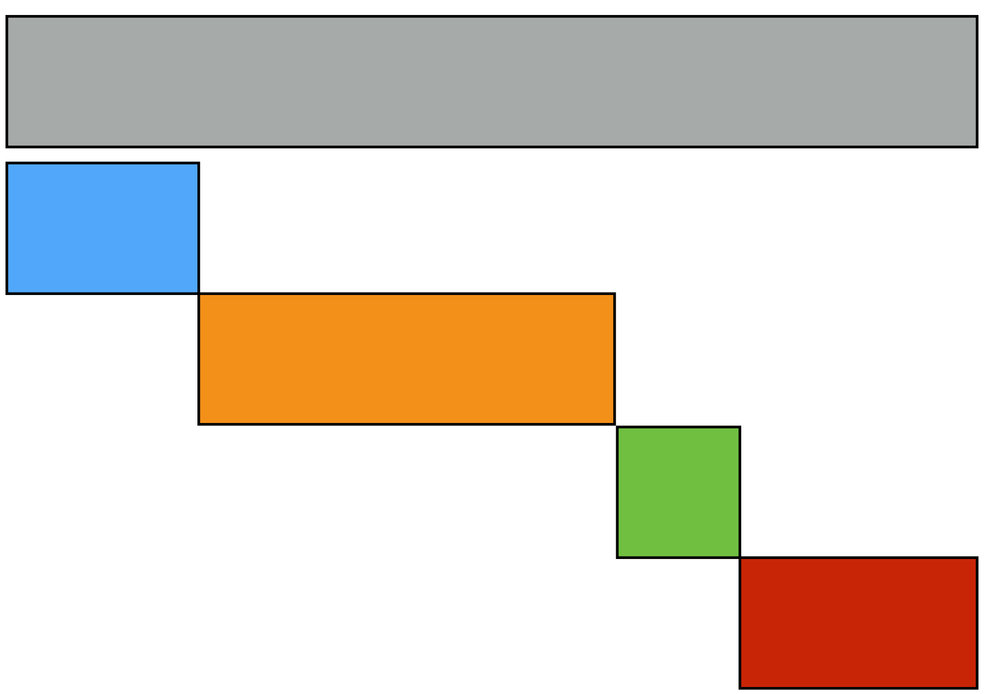

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$$\rho_4 = 1 - \sum_{k=1}^{3} \rho_k$$

Choosing $K = \infty$

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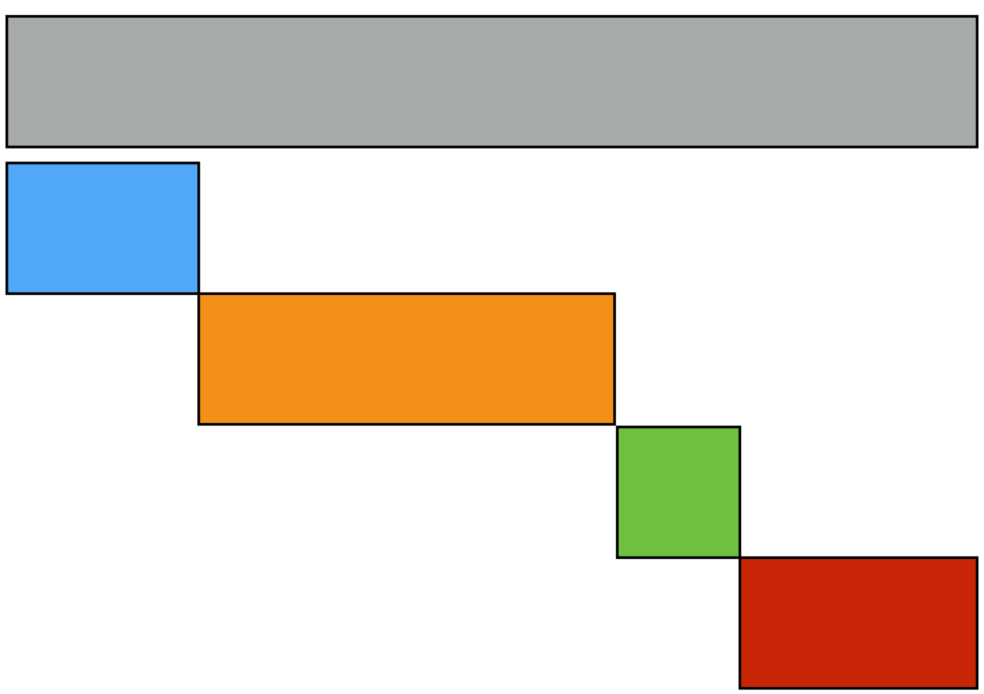

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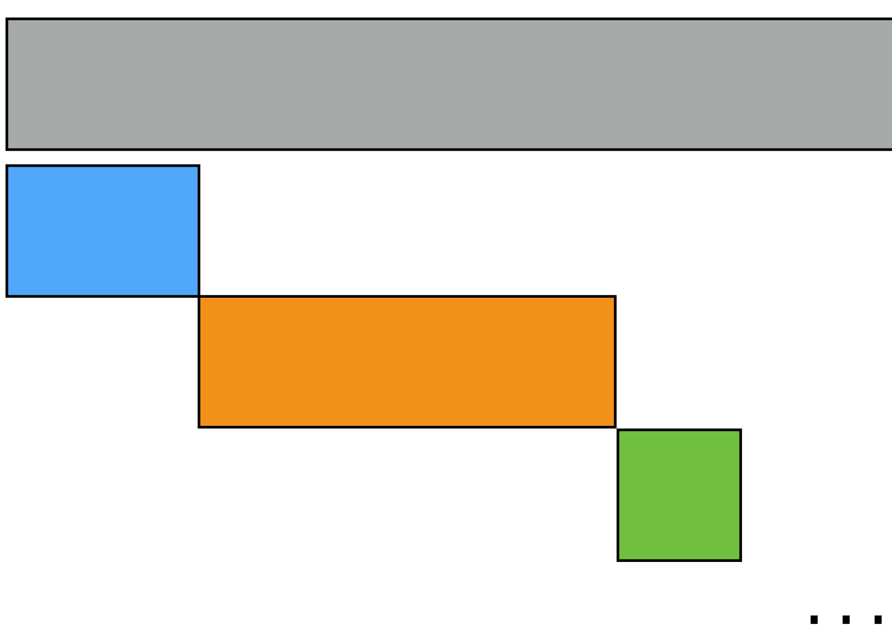

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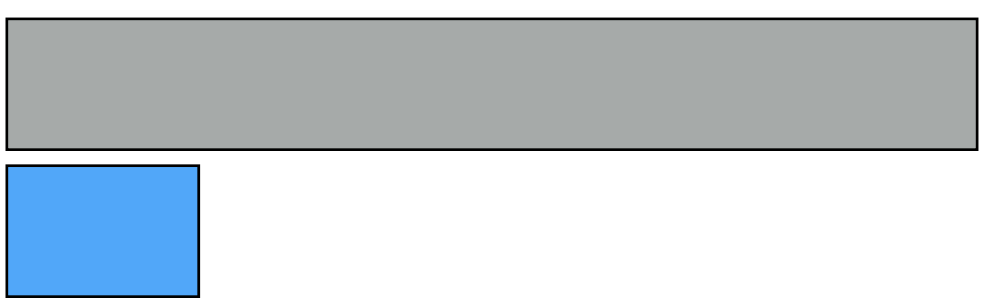

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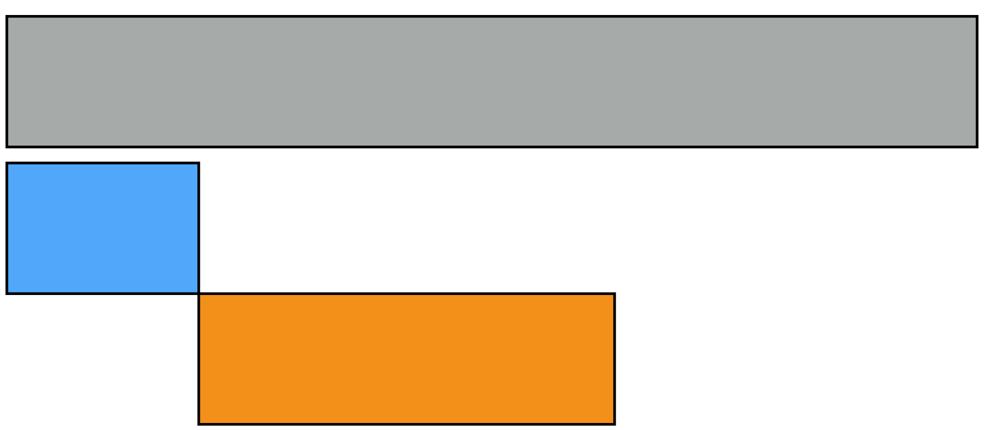

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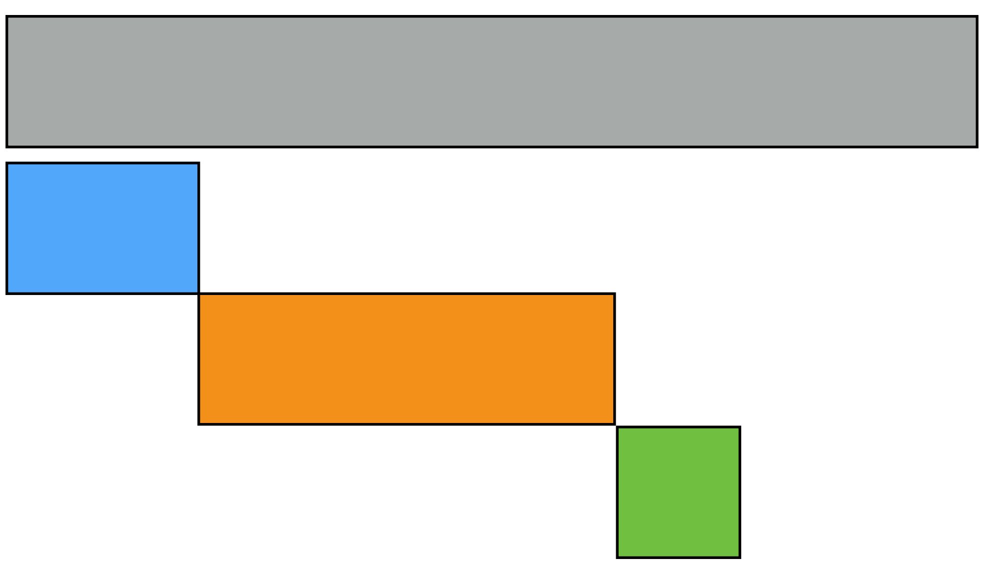

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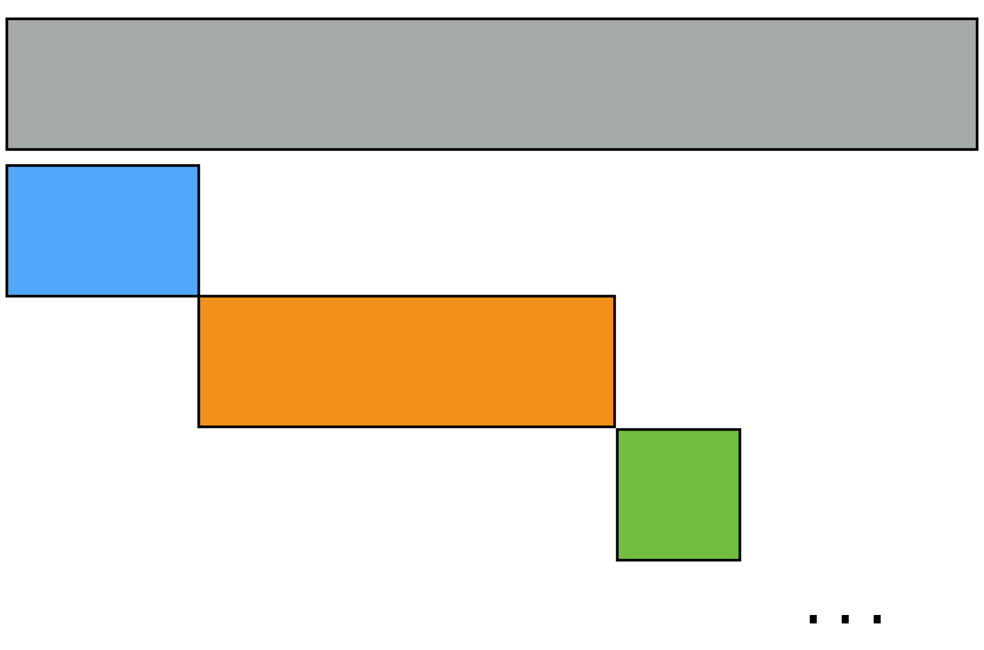

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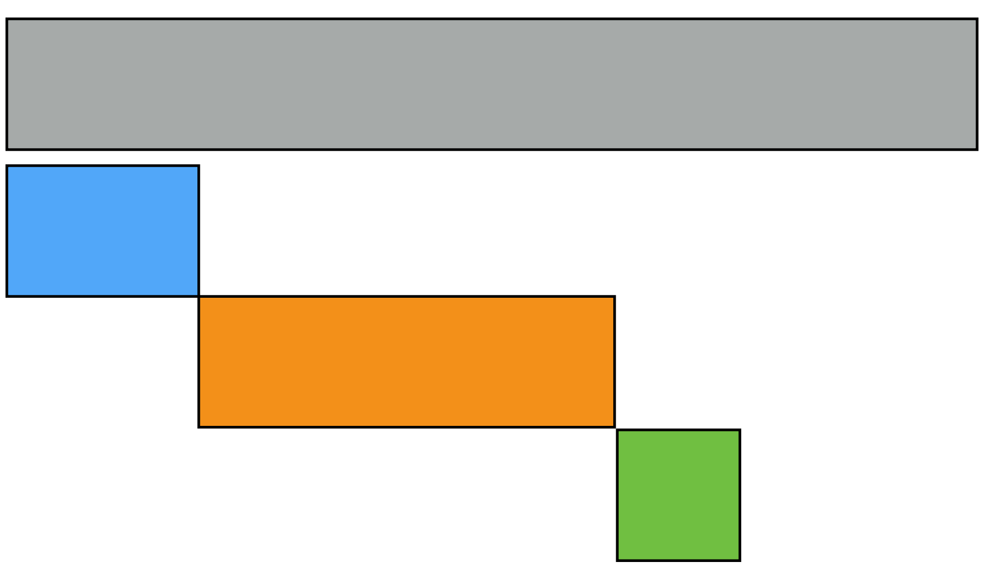

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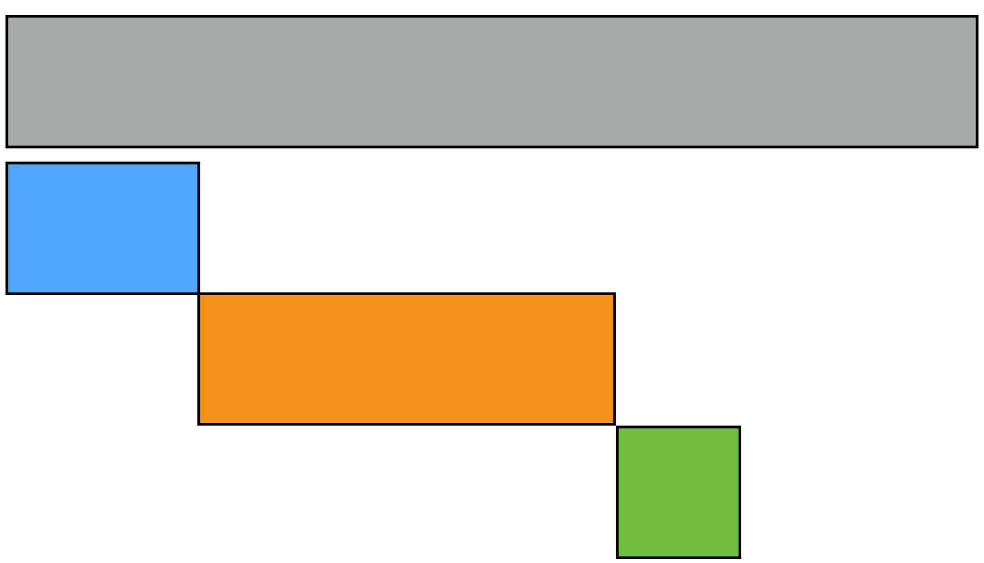

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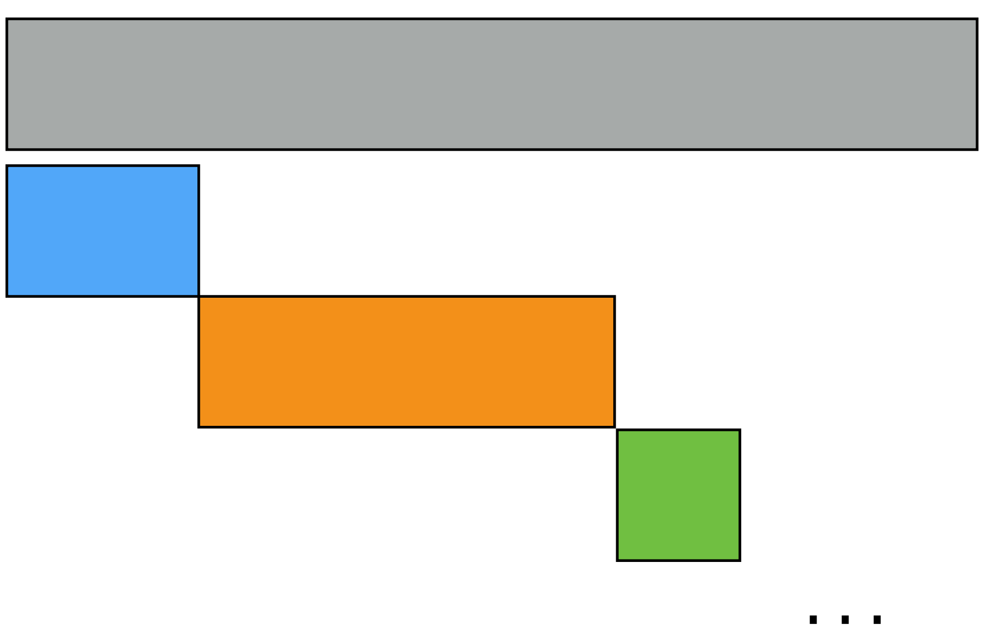

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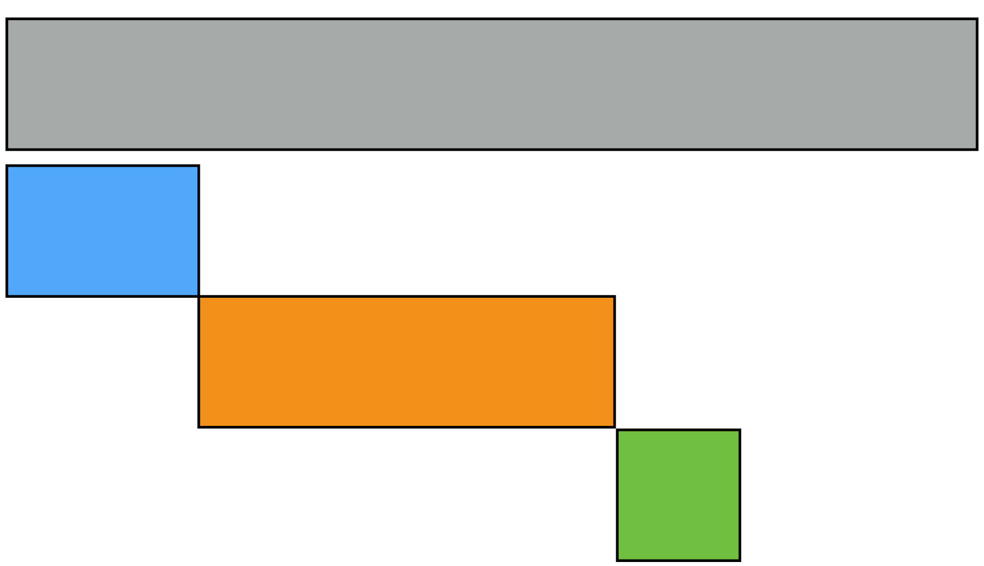

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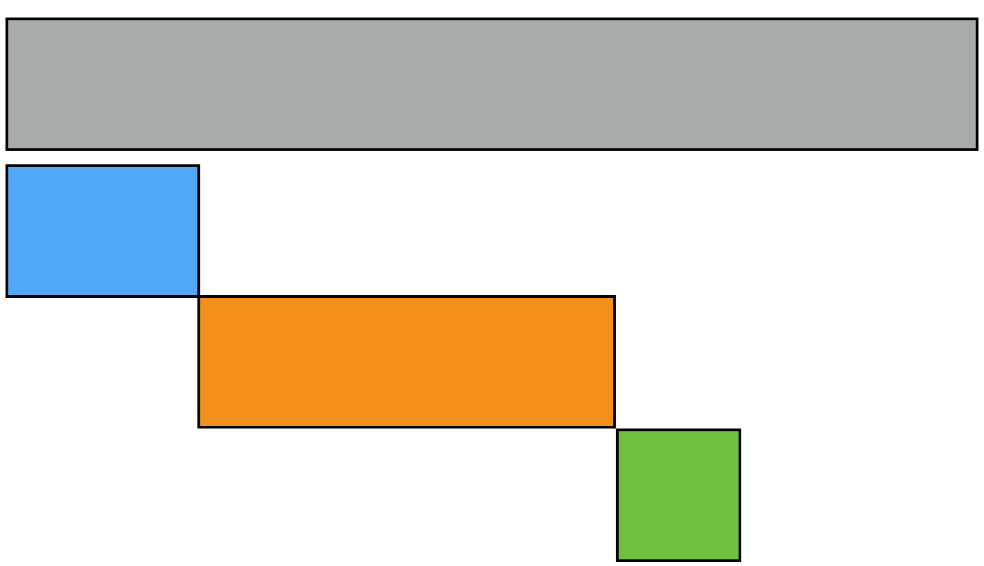

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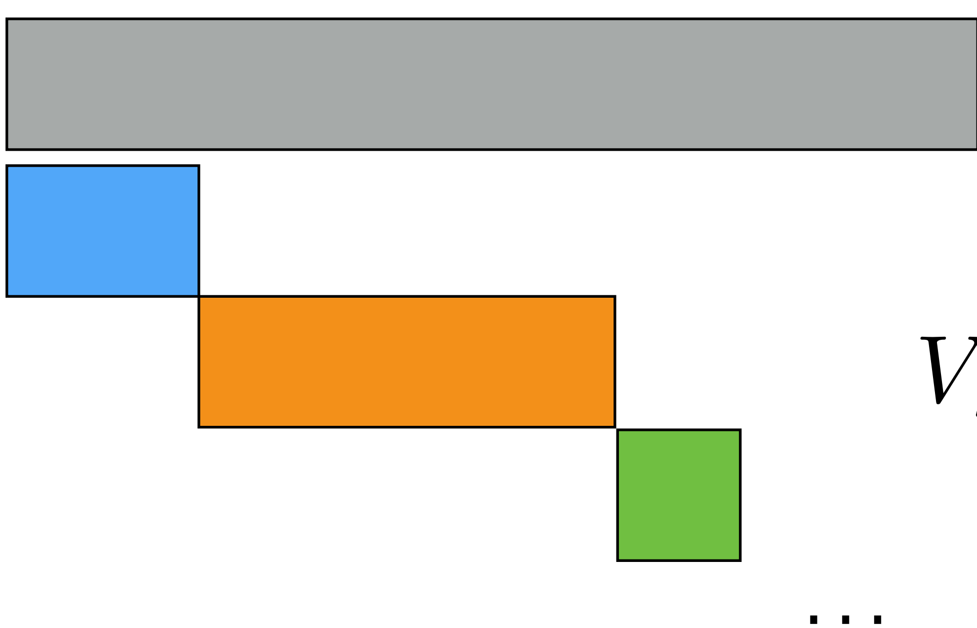

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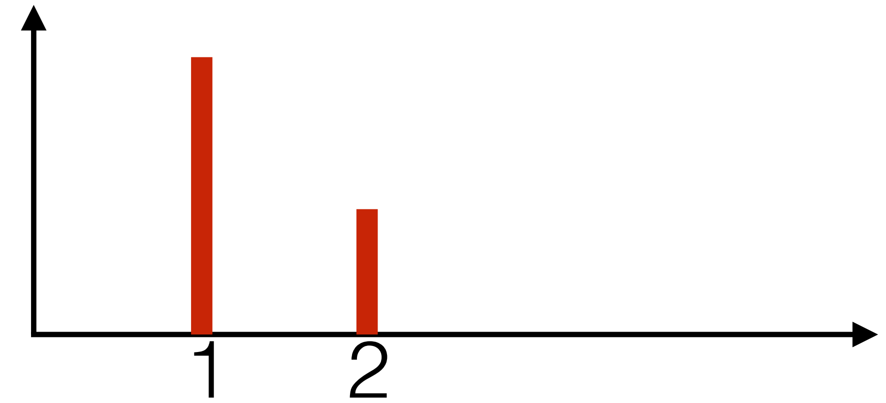

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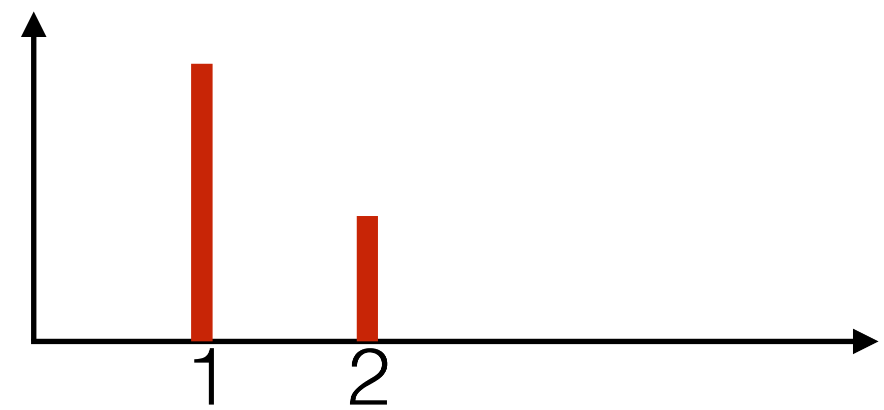

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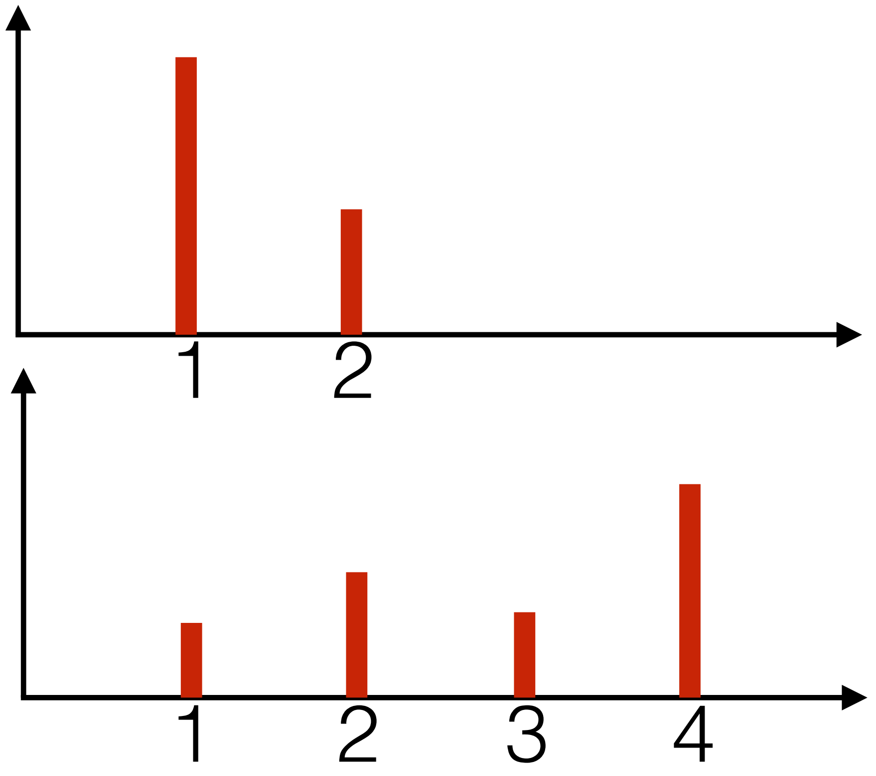

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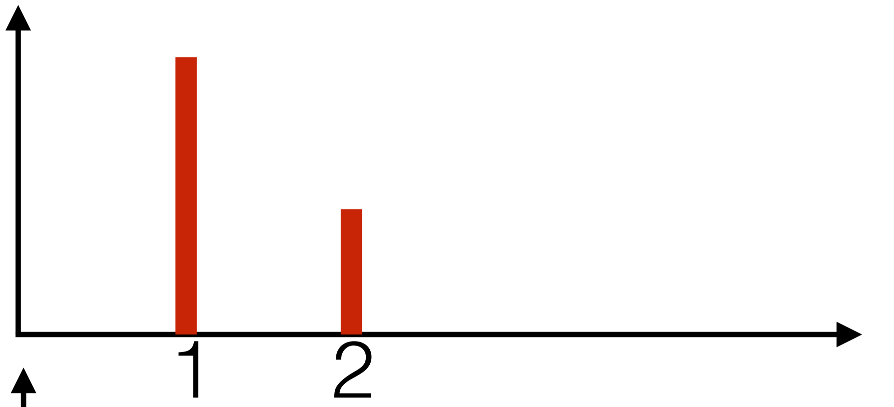

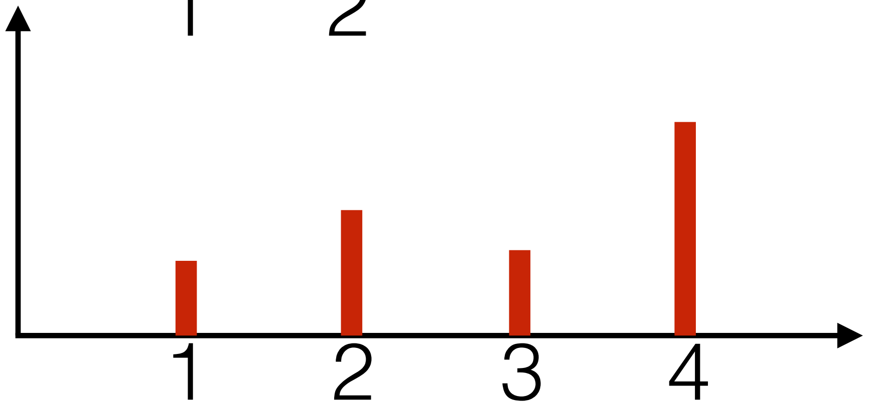

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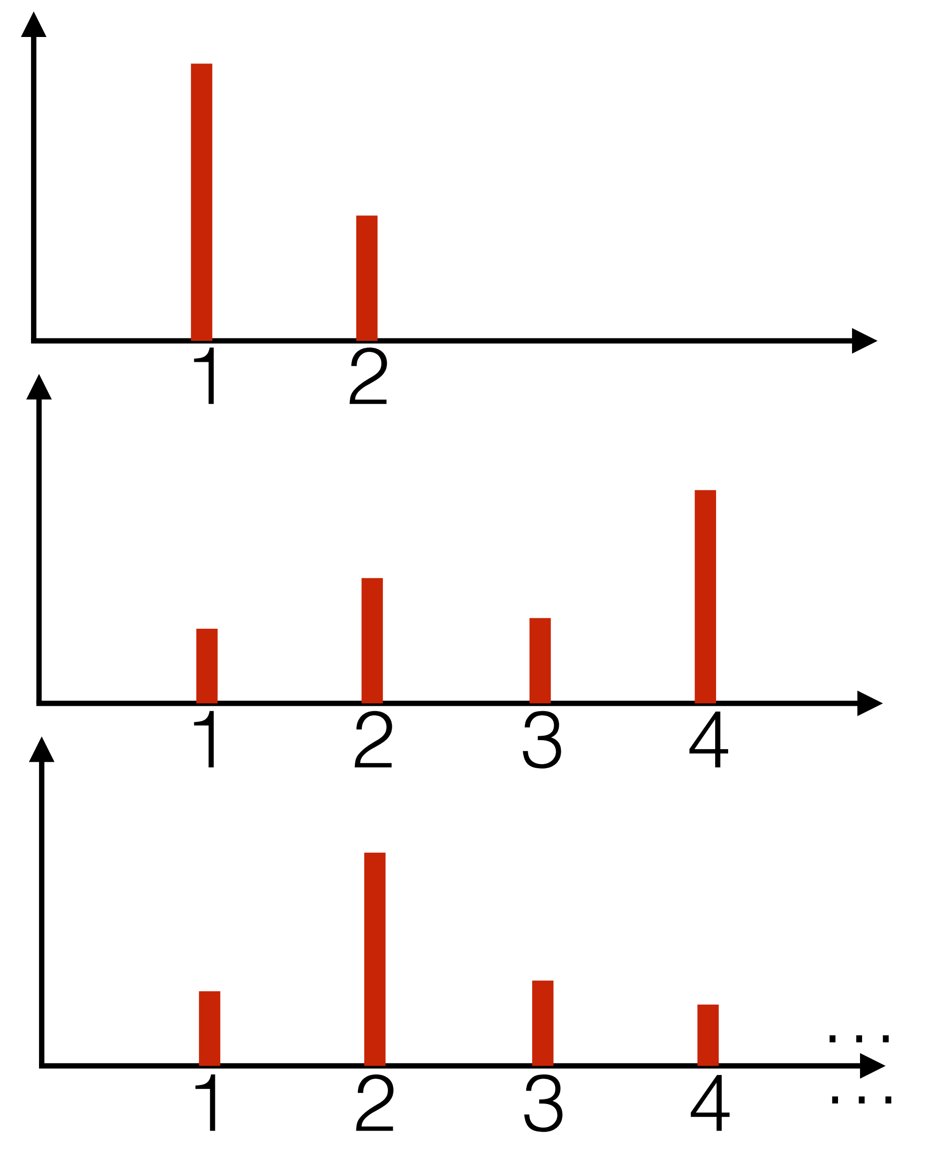

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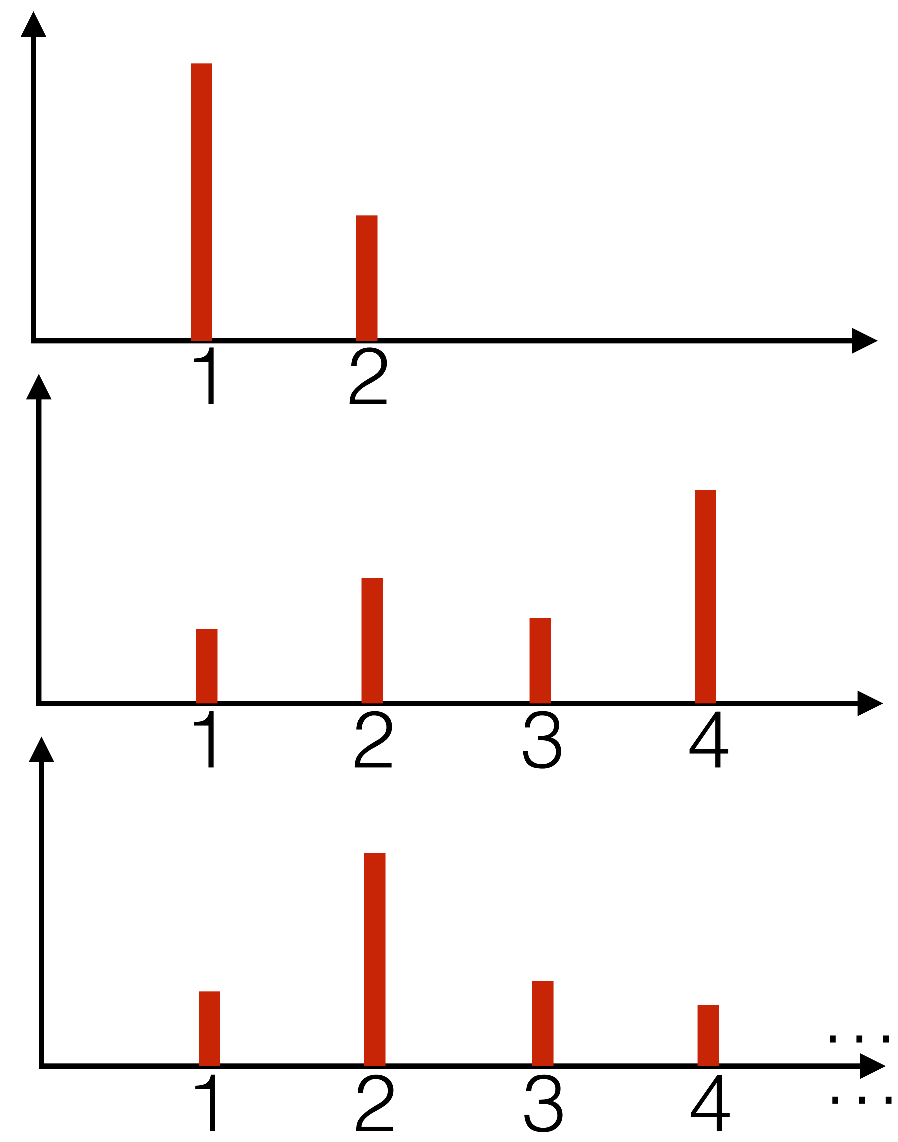
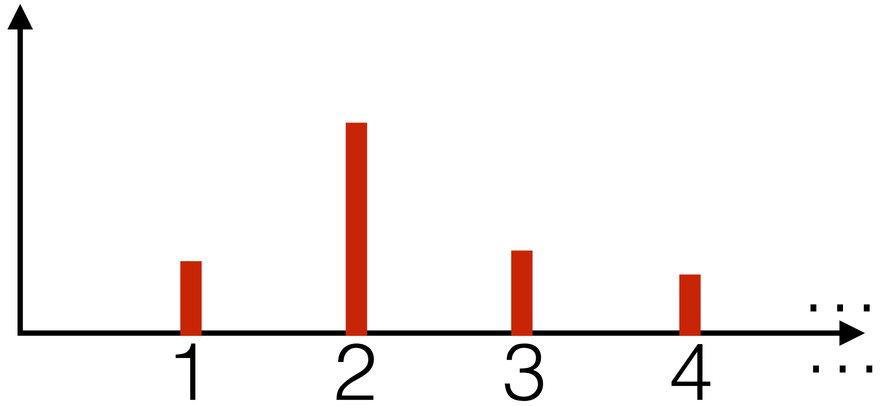


- Infinity of parameters: components
- Growing number of parameters: clusters

Exercises



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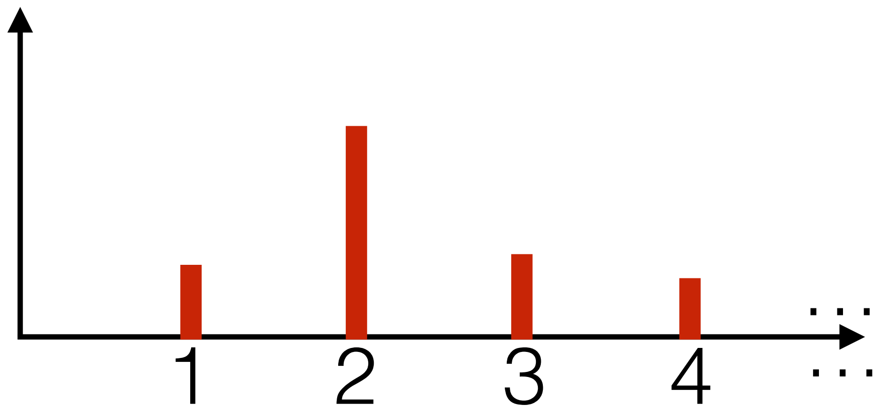

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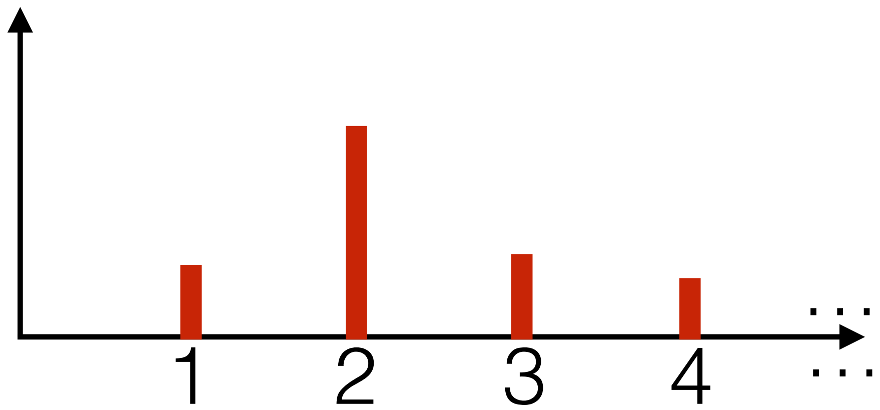

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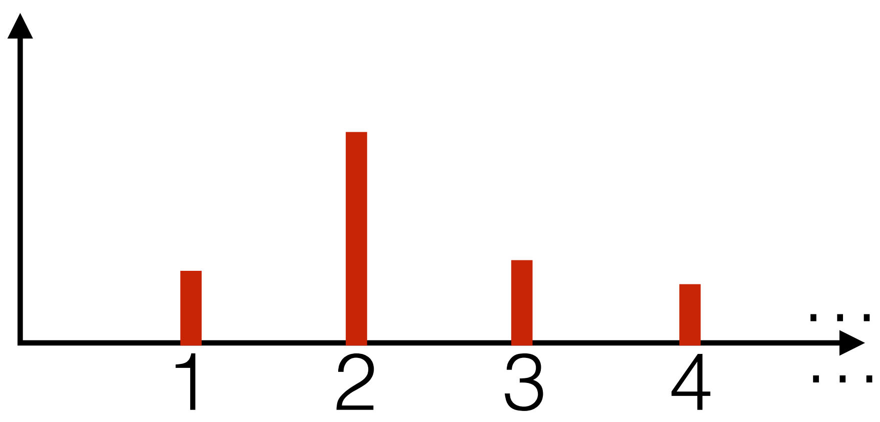

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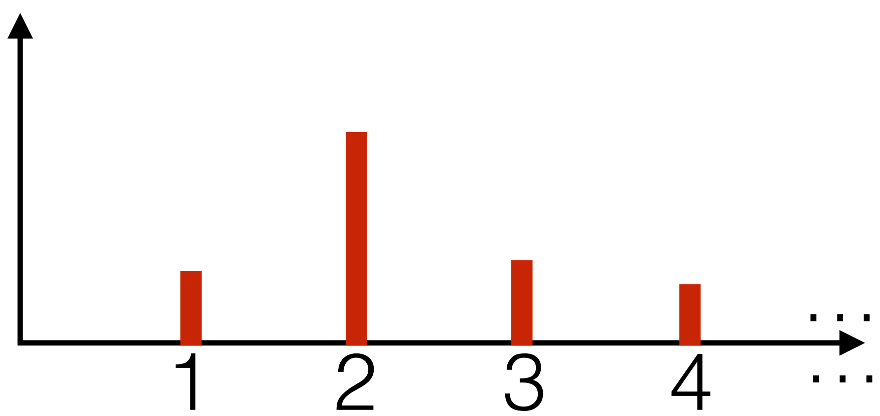

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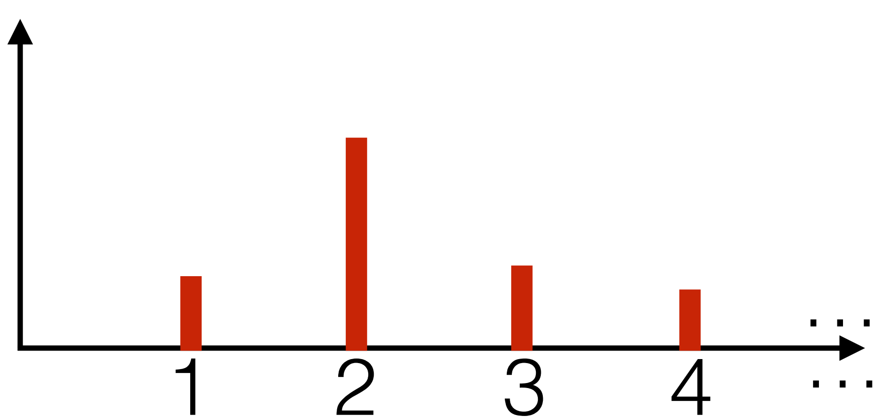

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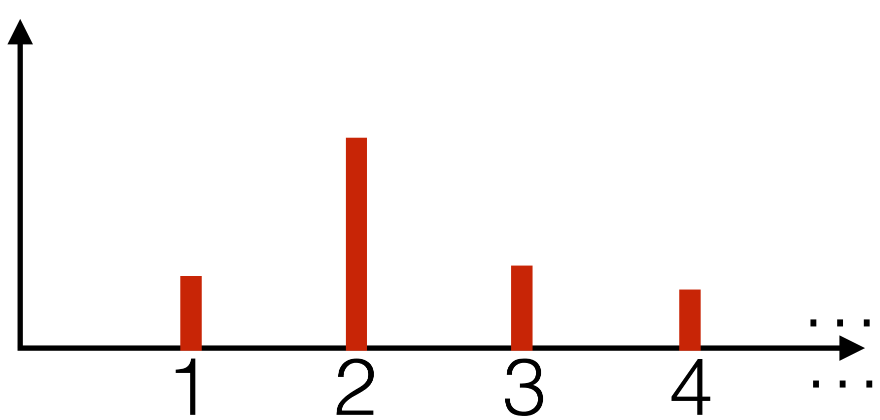

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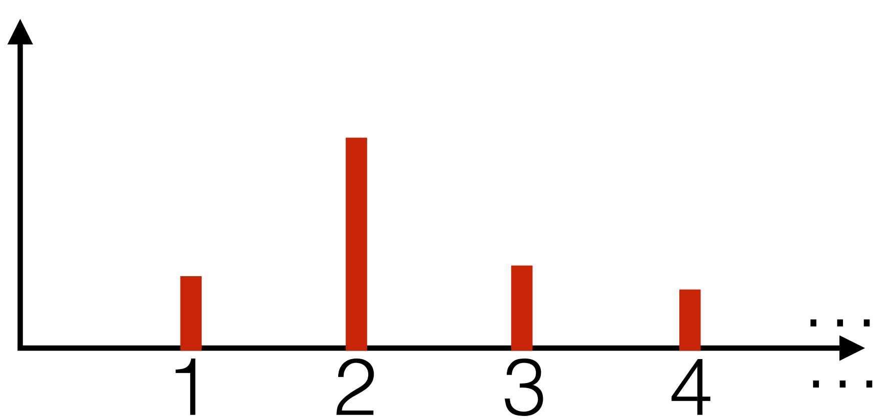

- Compare the number of clusters as *N* changes in the GEM case with the growth in the *K*=1000 case
- How does the growth in *N* change when you change α?

References

A full reference list is provided at the end of the “Part II” slides.