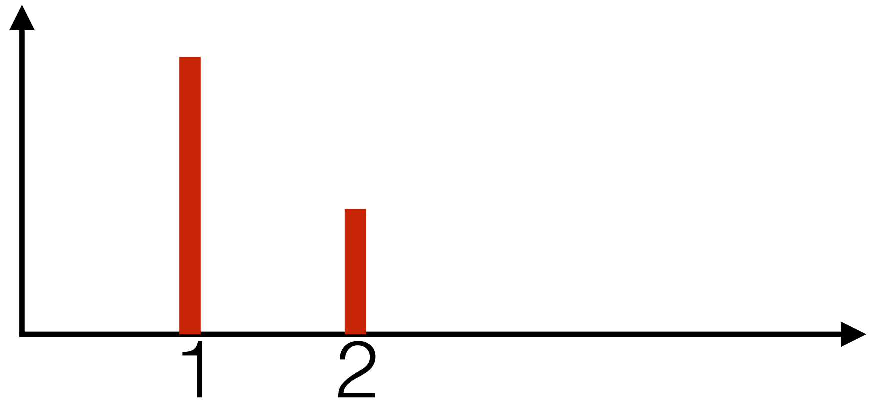
Nonparametric Bayes: Part II

Tamara Broderick
ITT Career Development Assistant Professor
Electrical Engineering & Computer Science
MIT

Distributions

Distributions

- Beta → random distribution over $1, 2$



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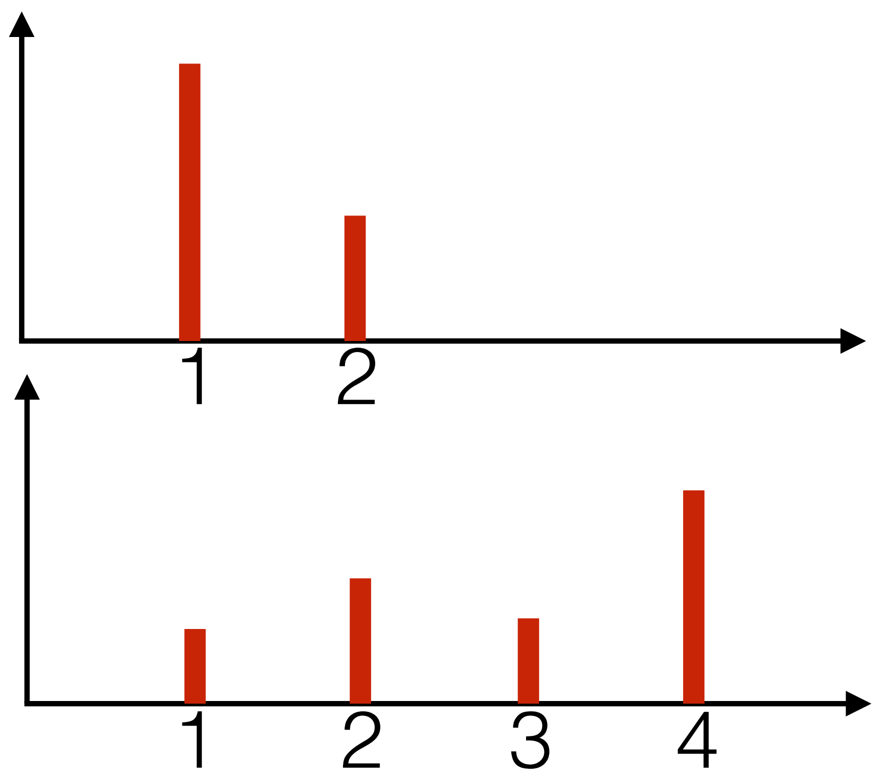

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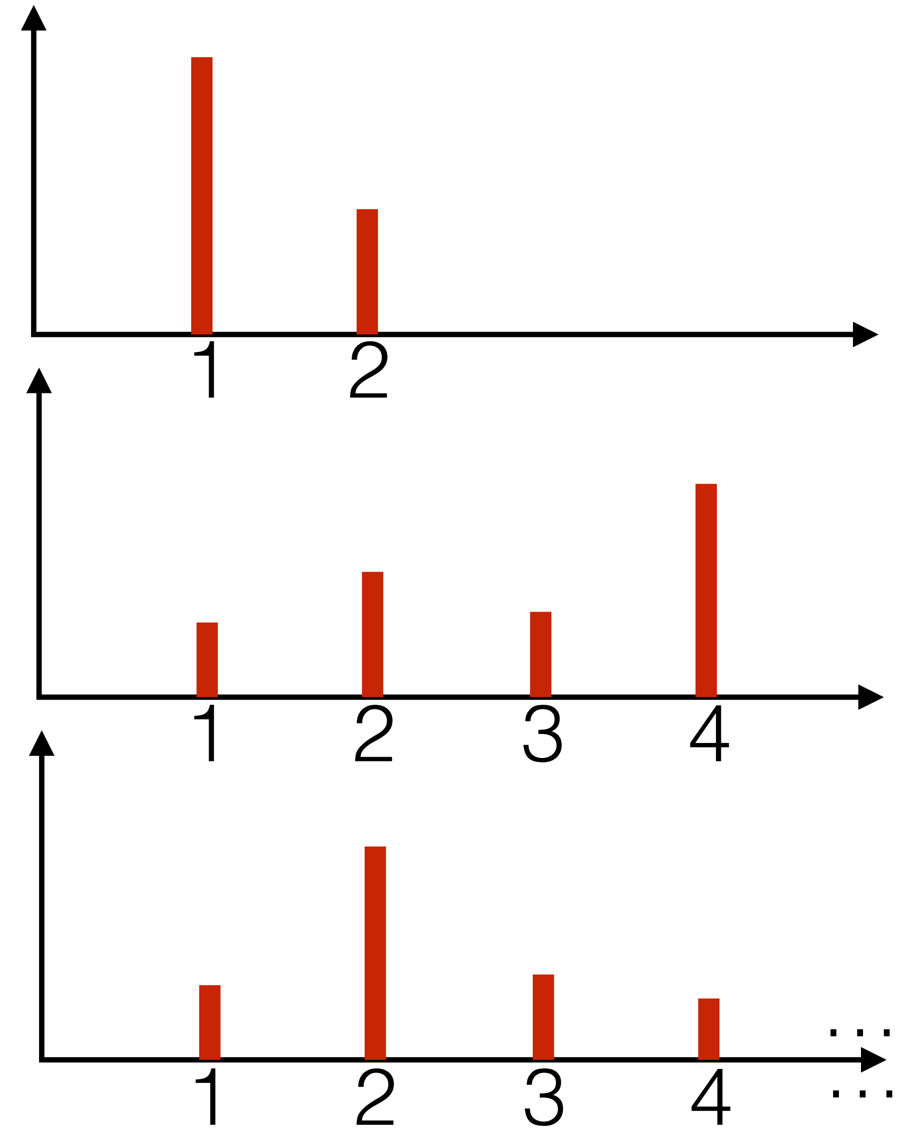

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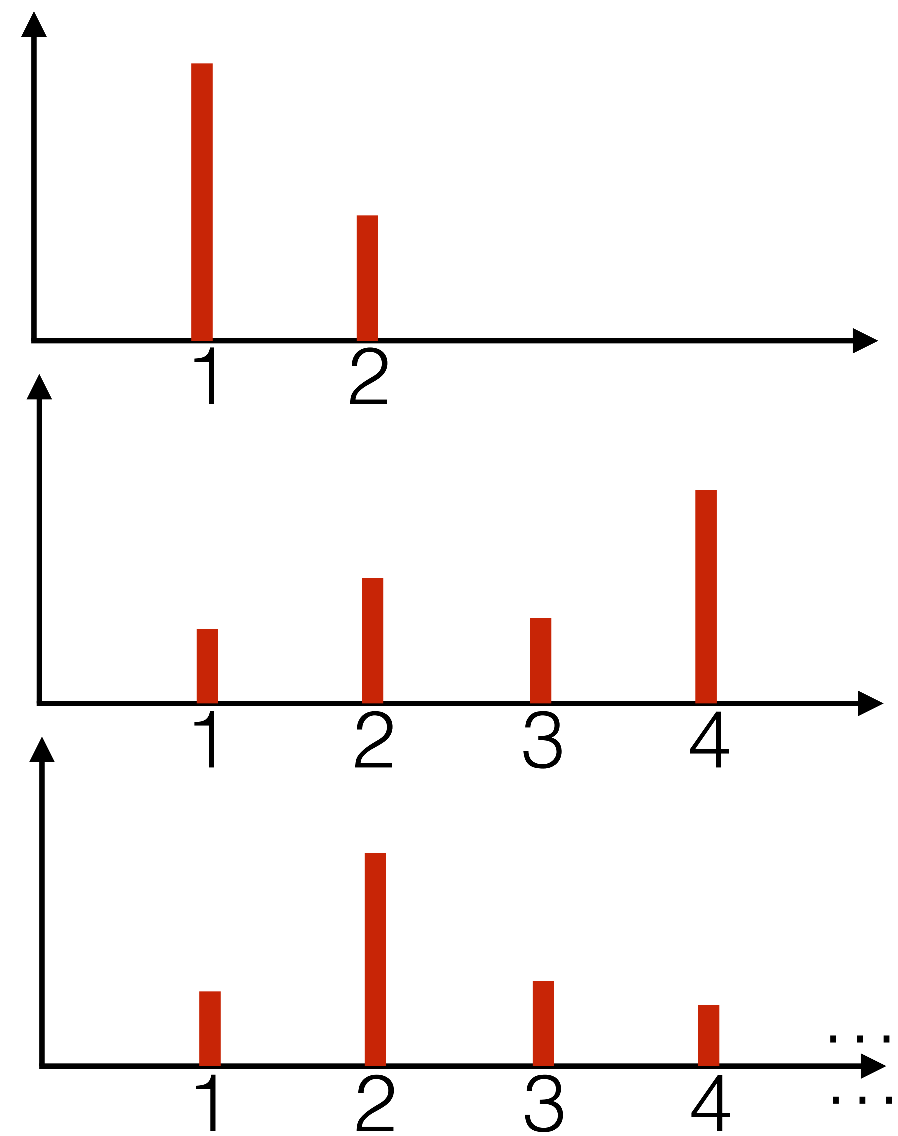

- Infinity of parameters: components
- Growing number of parameters: clusters

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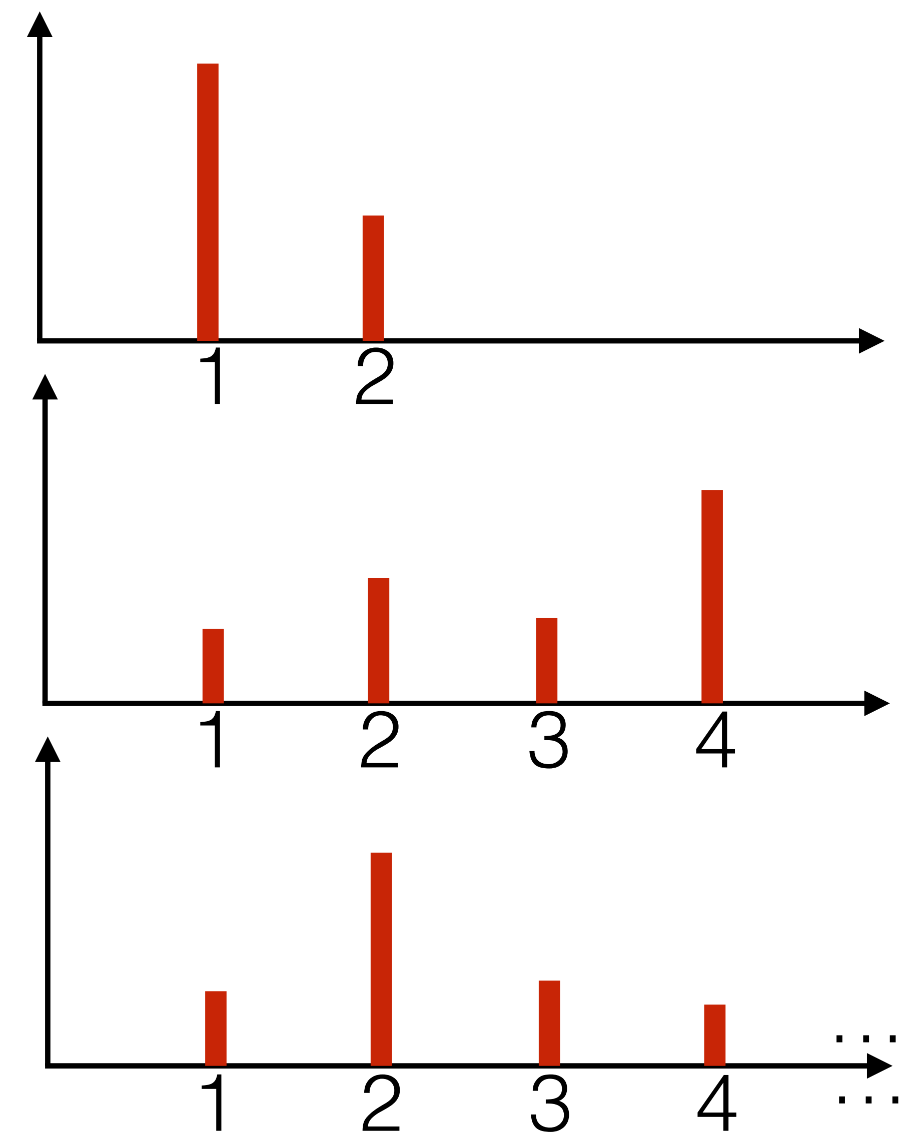

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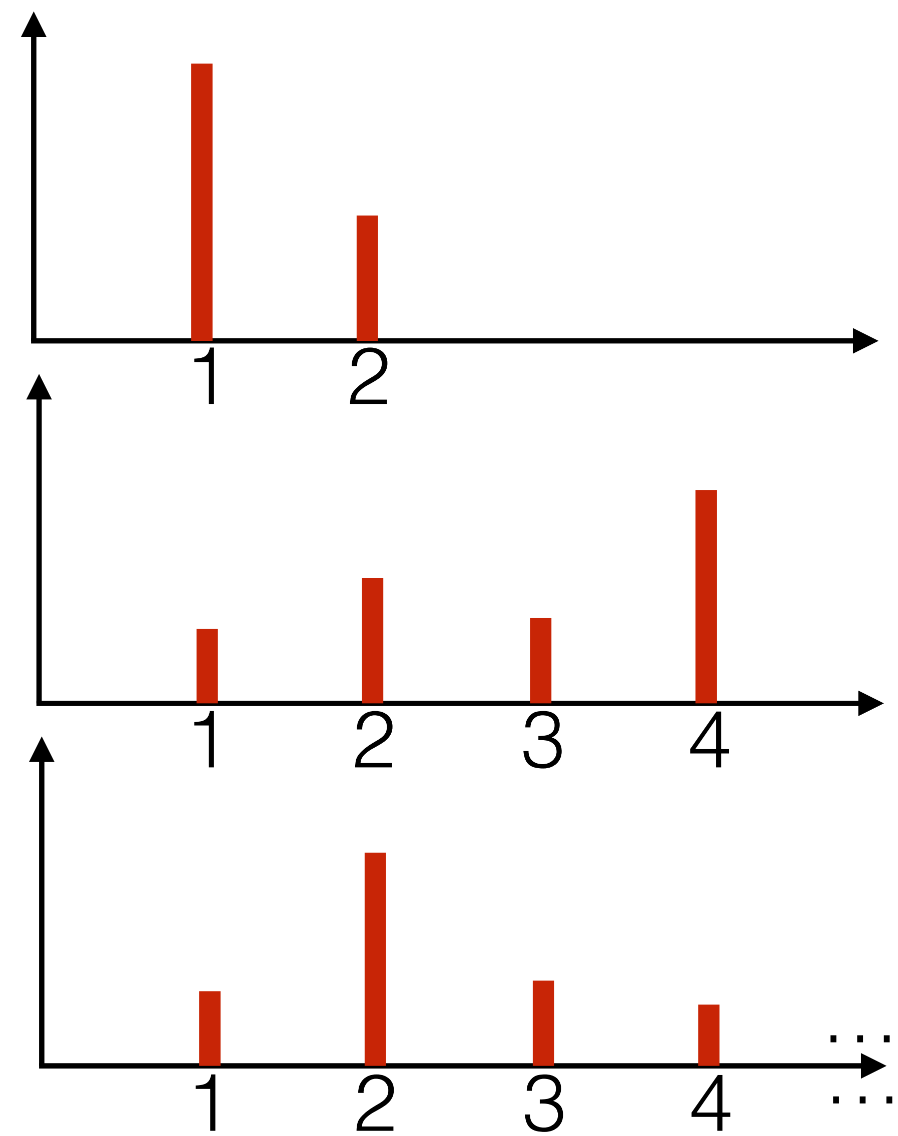

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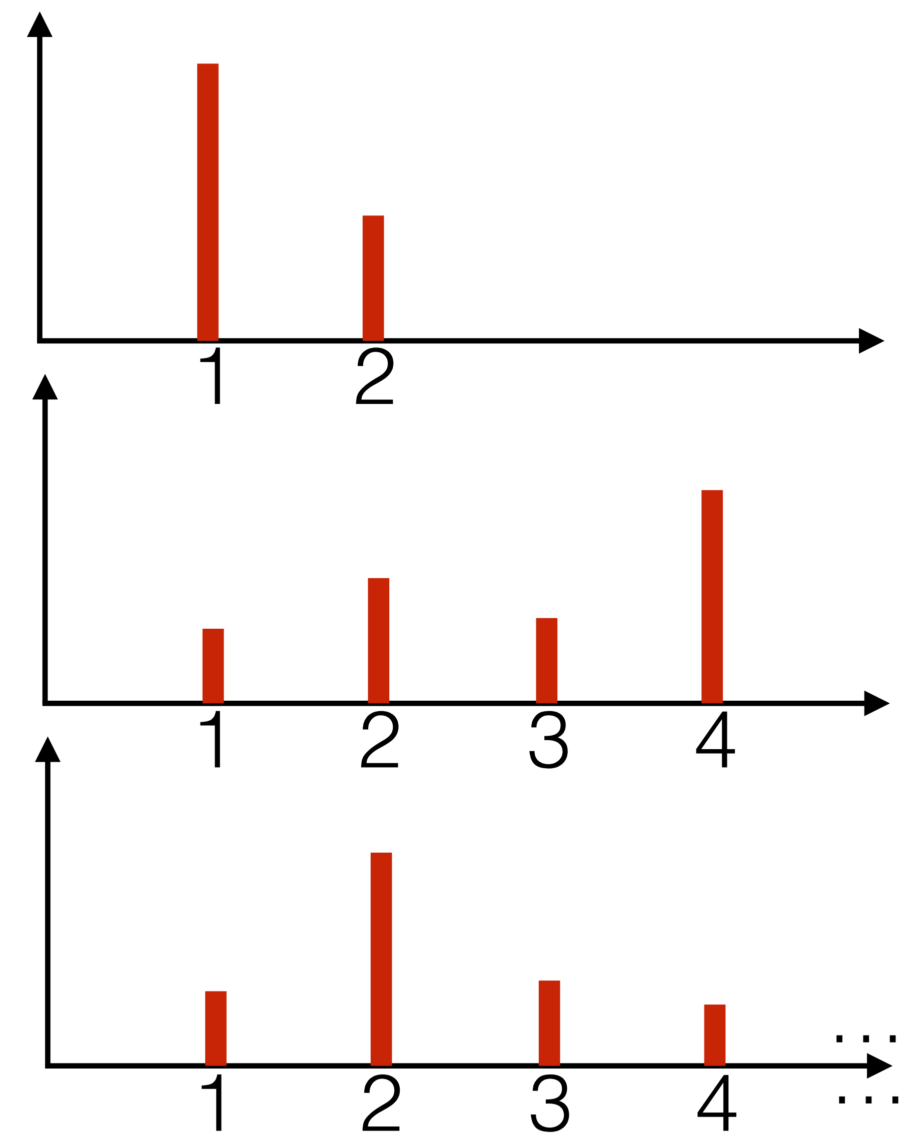

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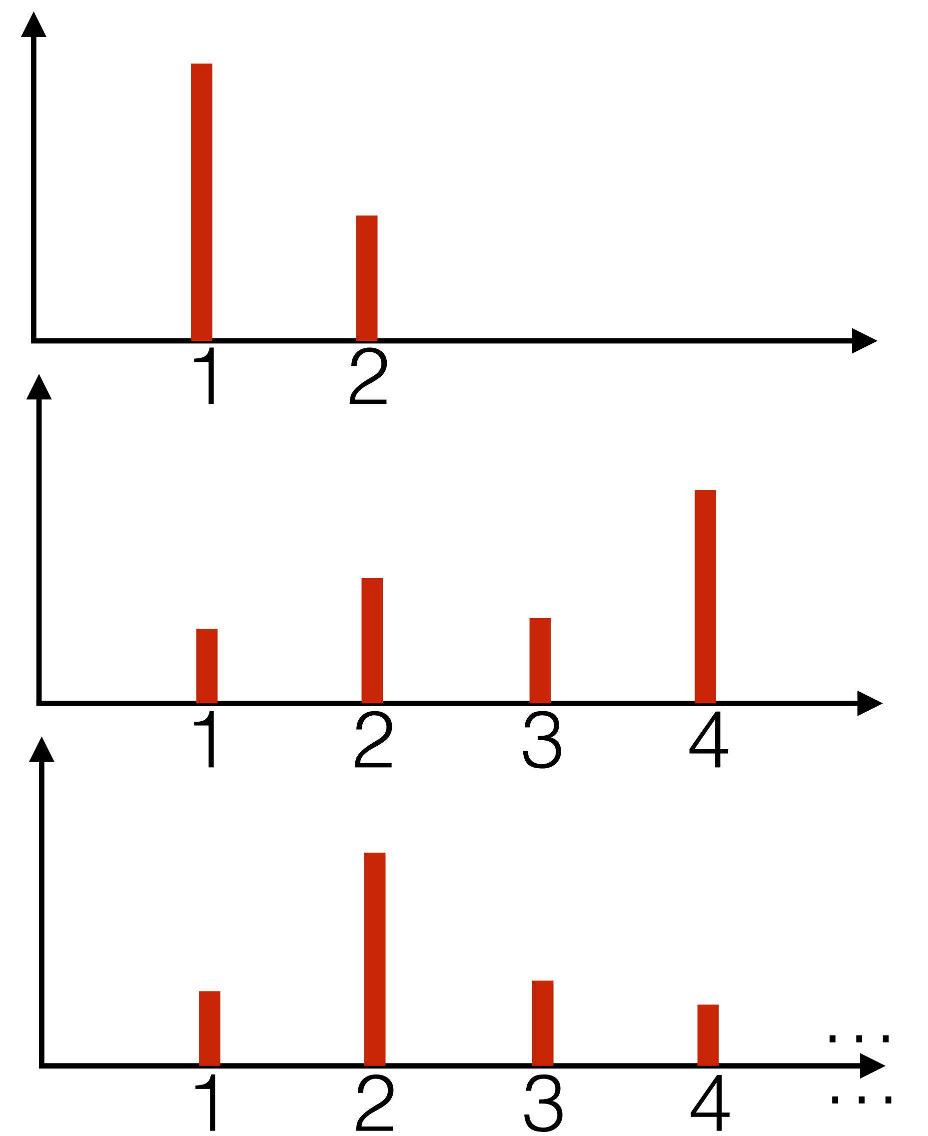

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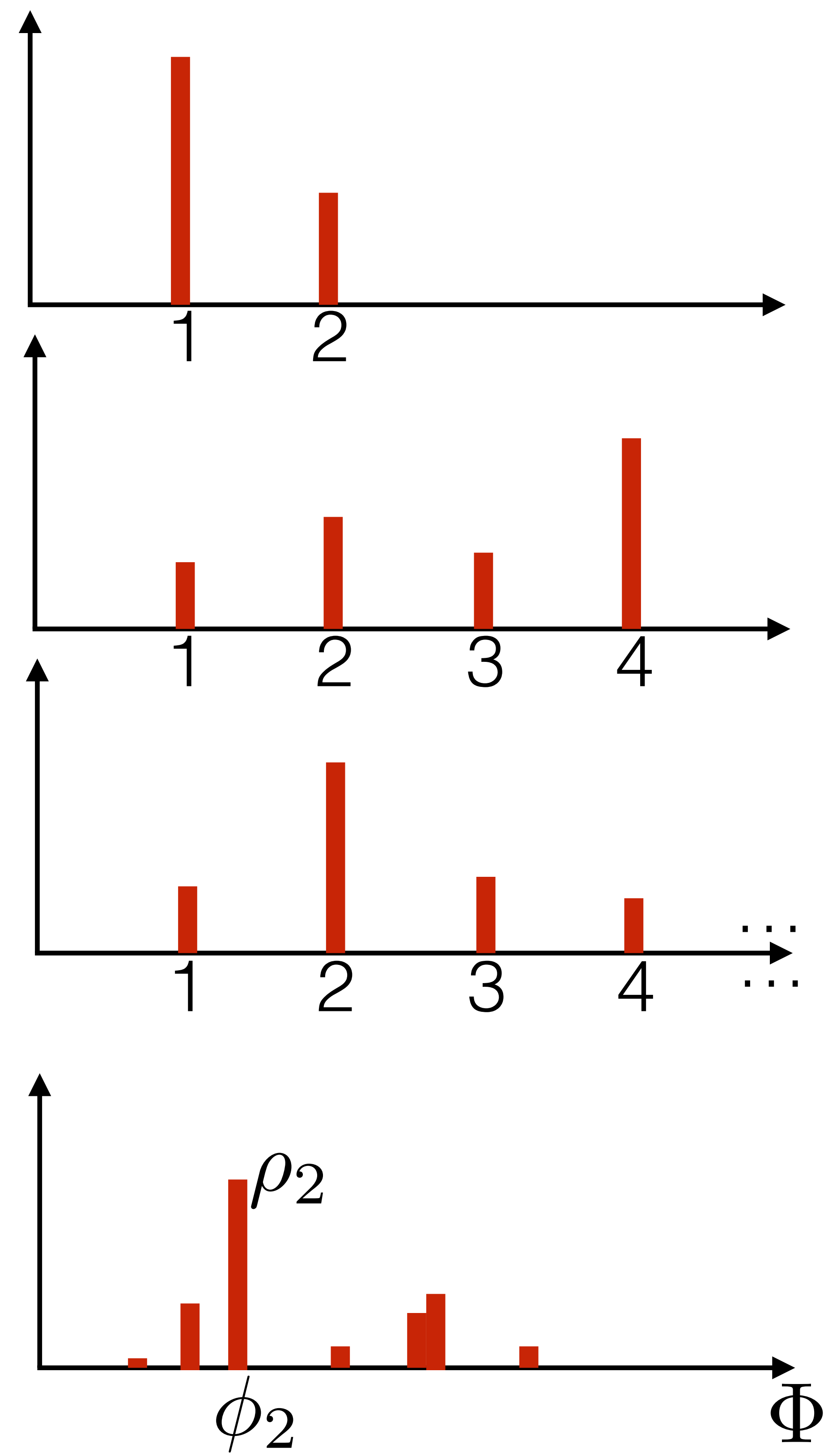

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- **Dirichlet process →** random distribution over Φ:
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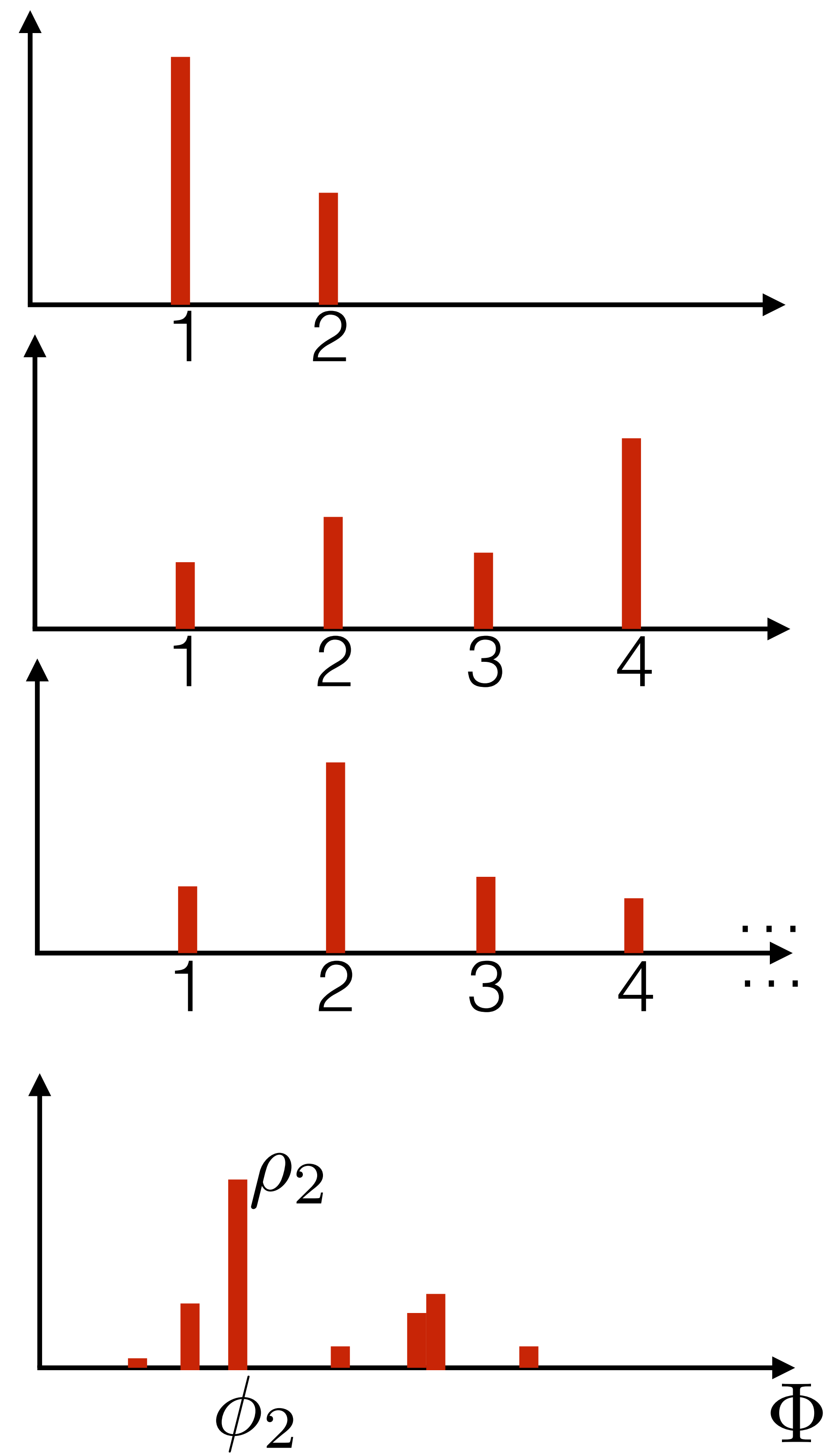

[Ferguson 1973]

Dirichlet process mixture model

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- Gaussian mixture model

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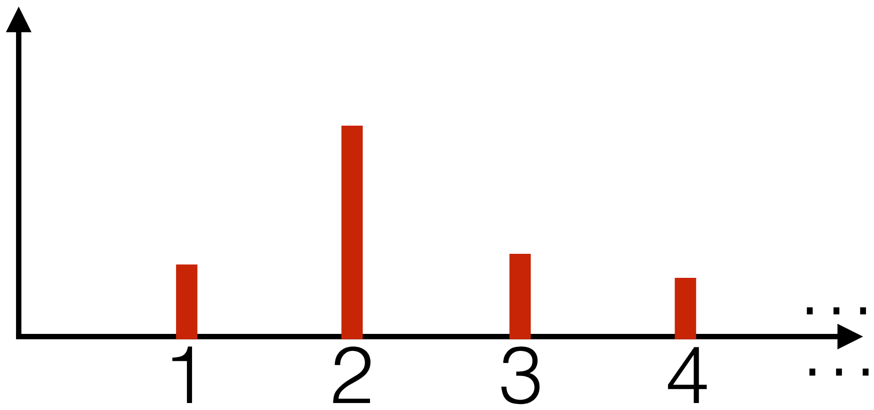

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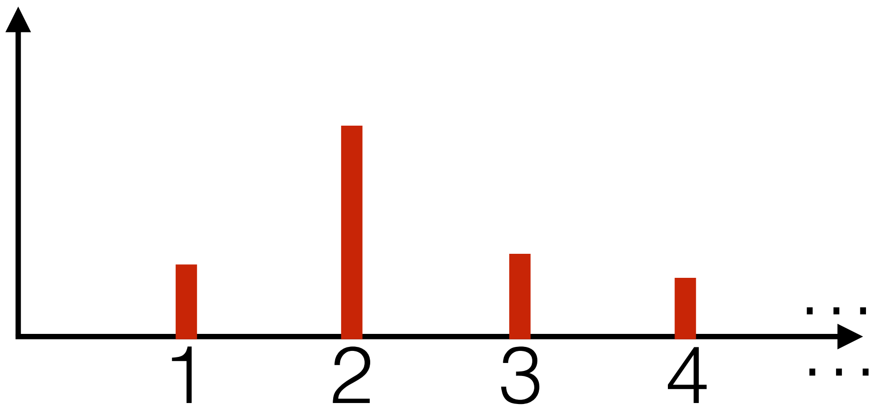

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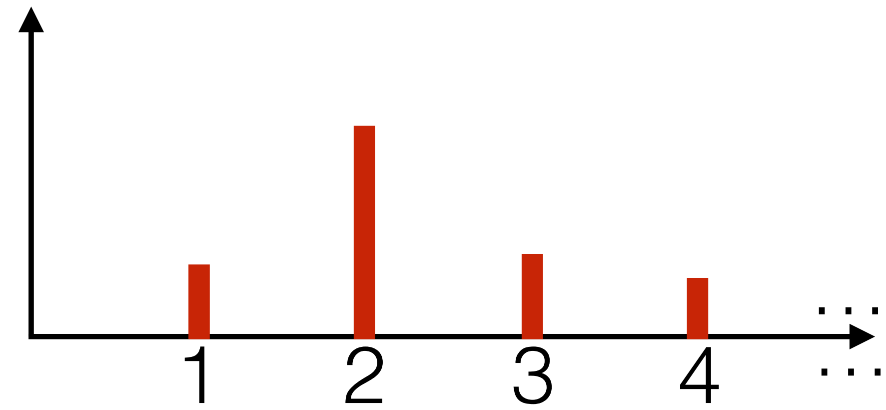

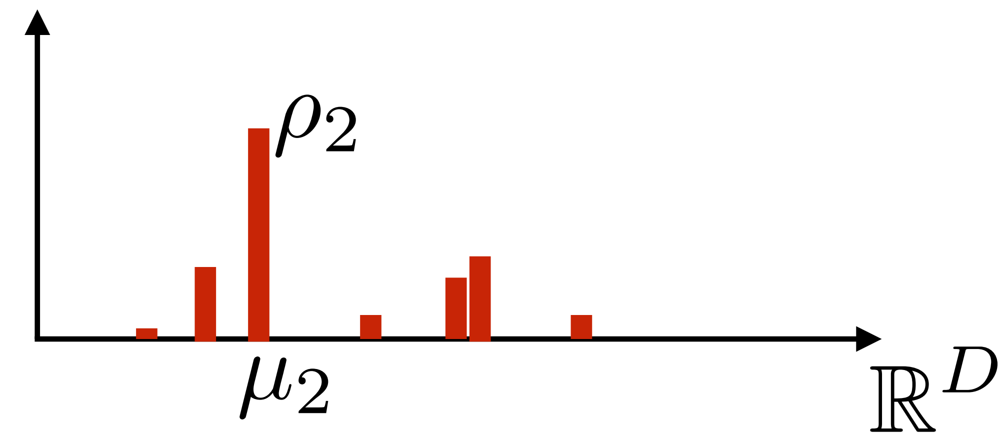

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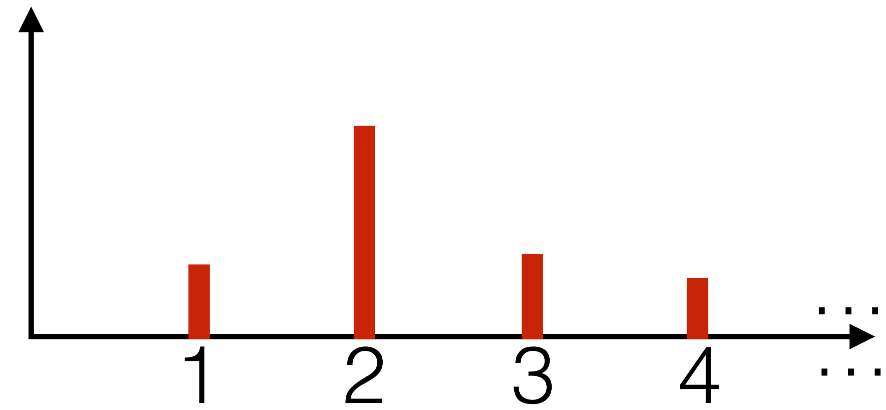

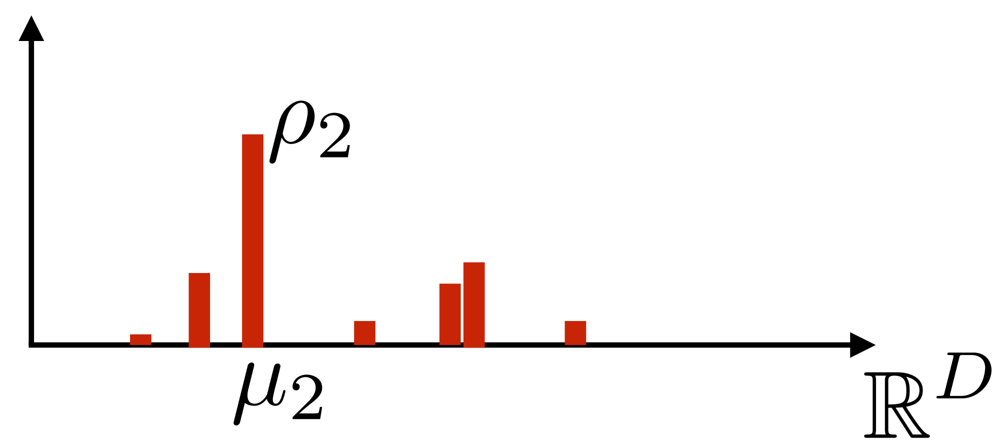

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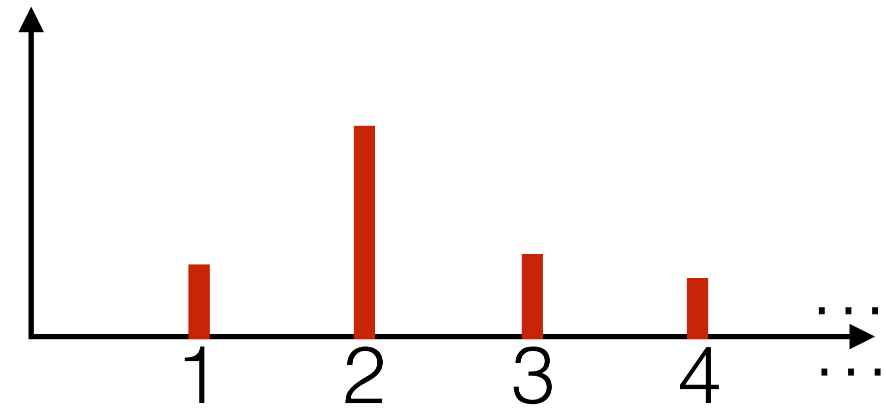

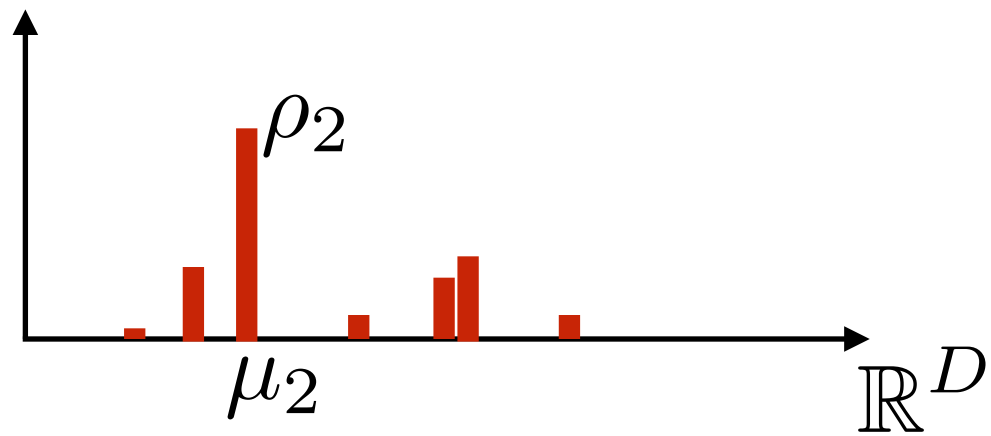

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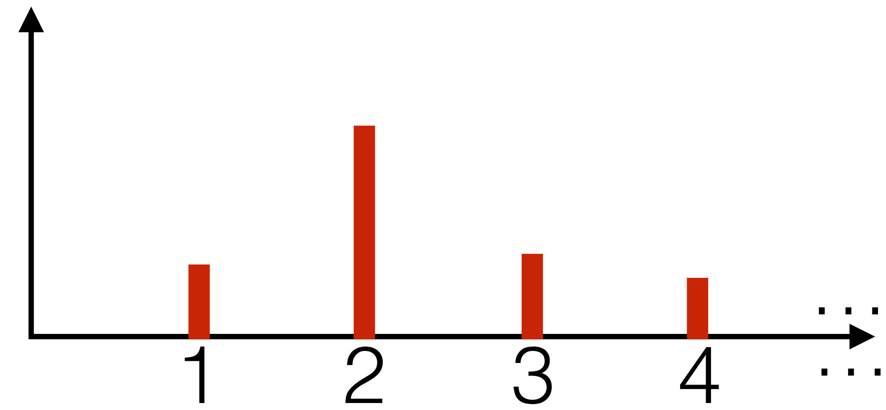

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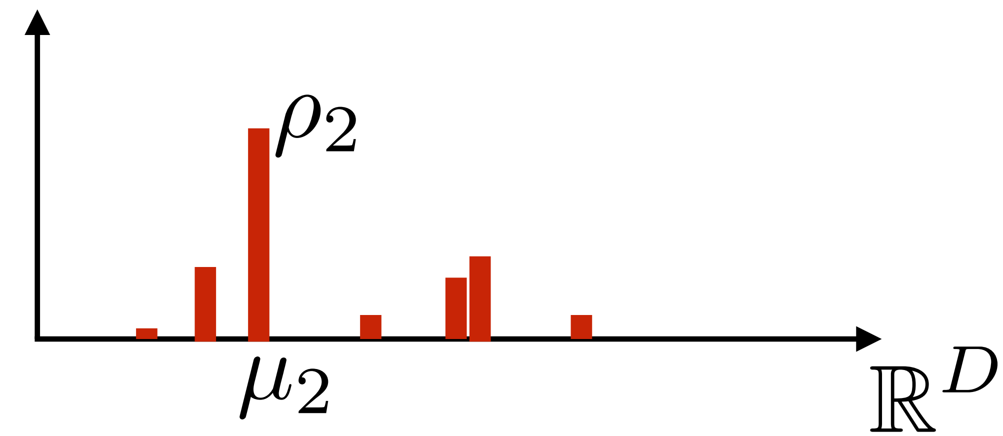

$$z_n \stackrel{iid}{\sim} \mathrm{Categorical}(\rho)$$

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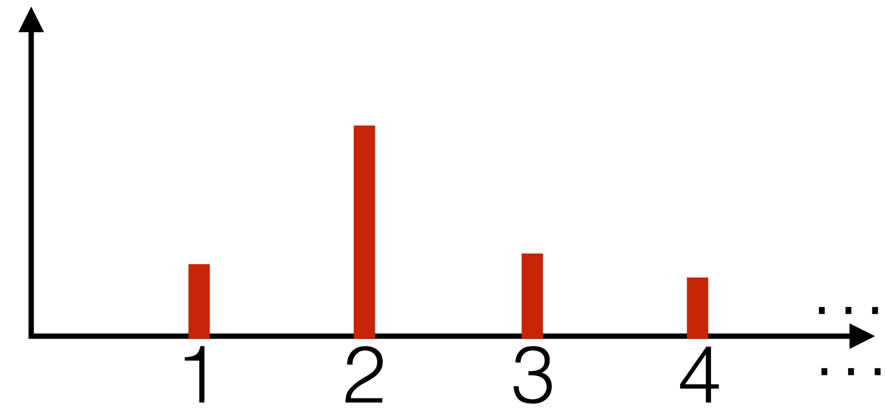

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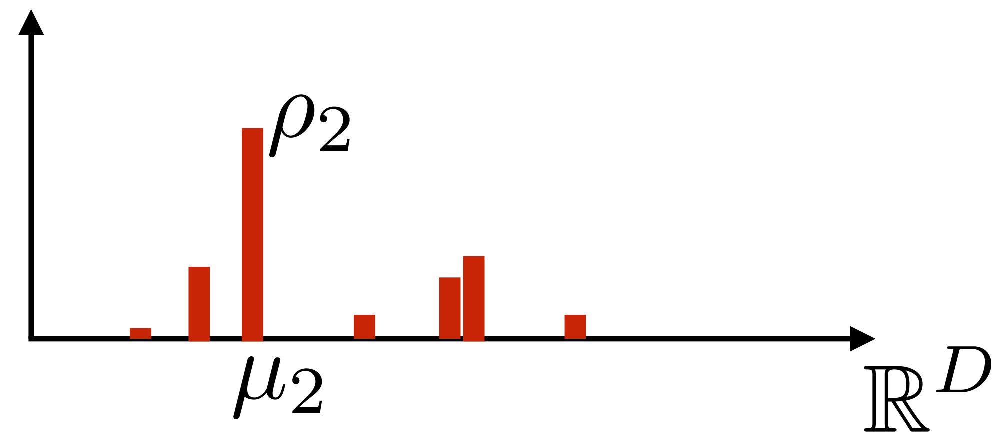

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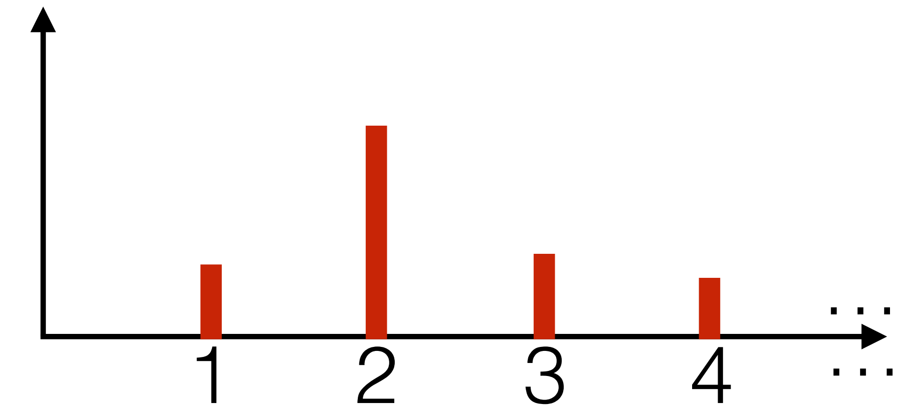

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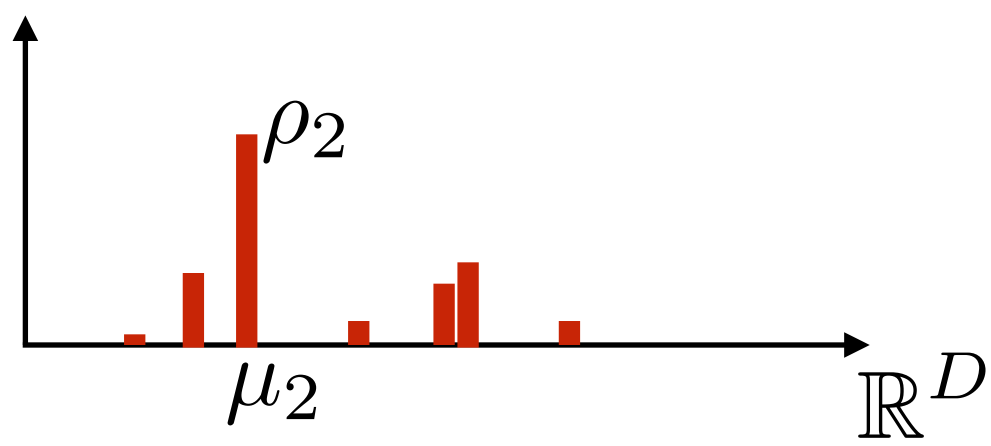

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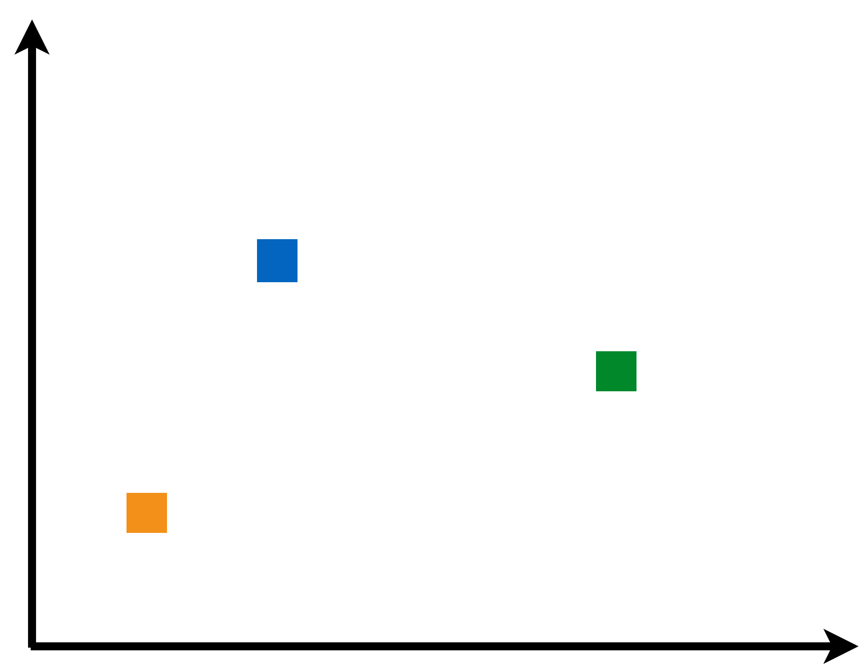

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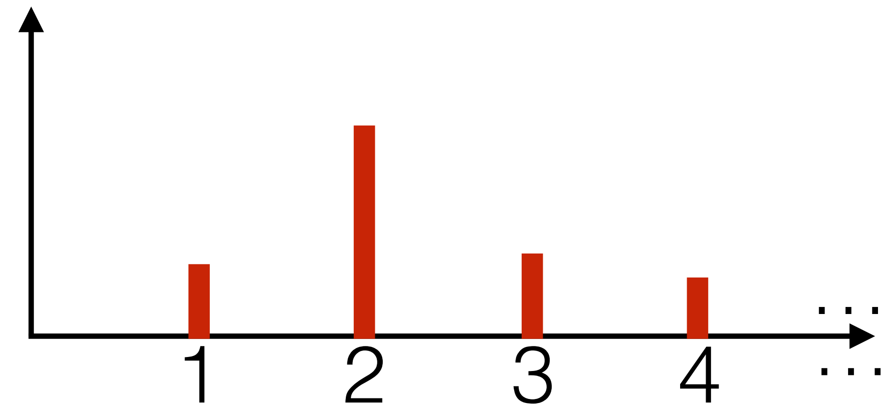

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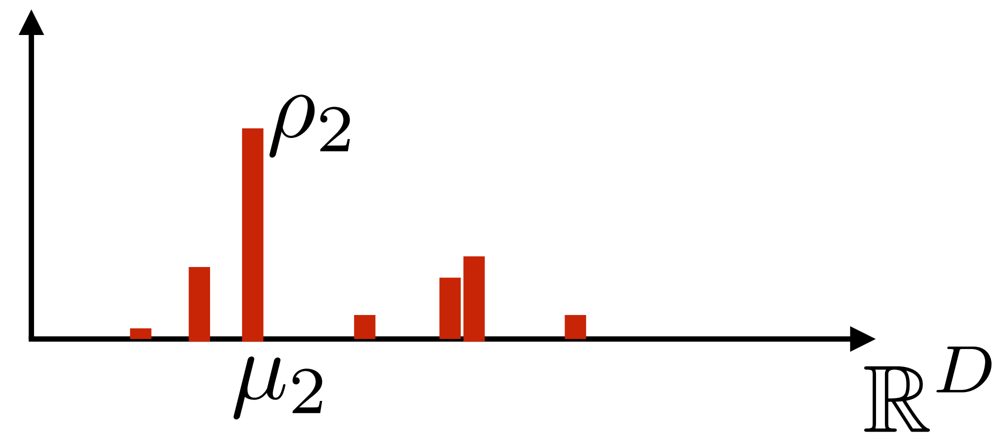

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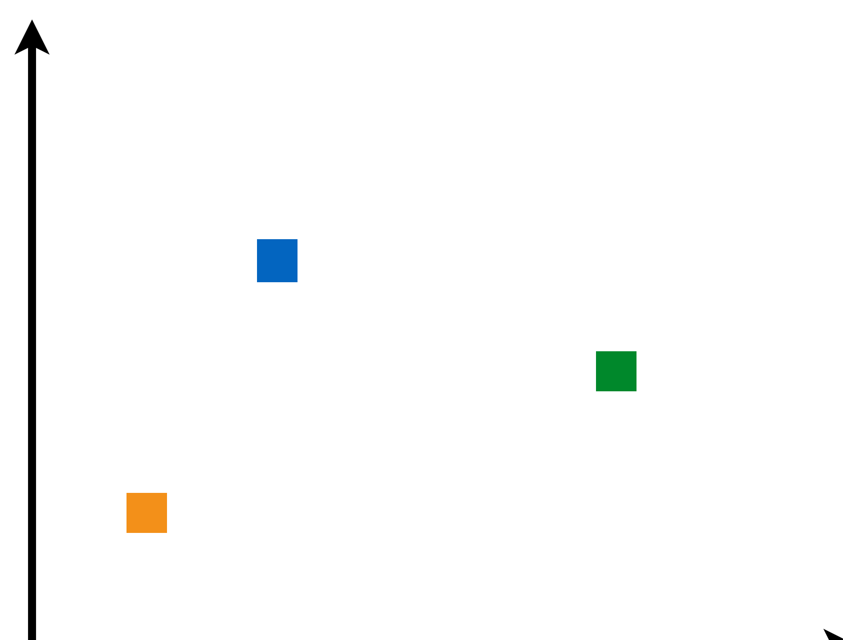

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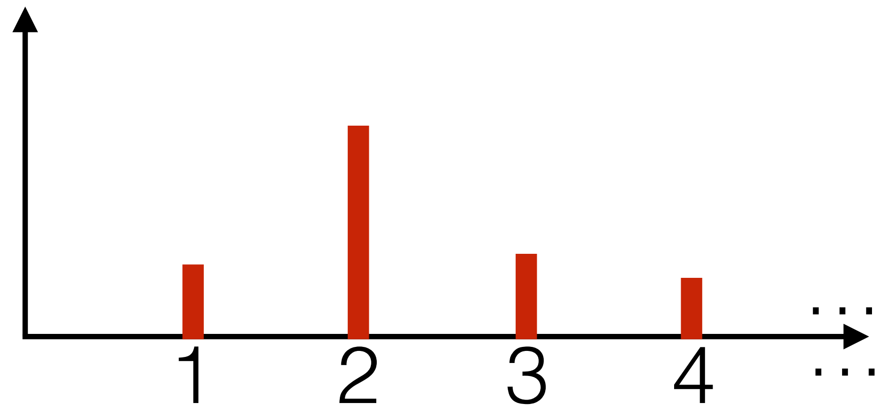

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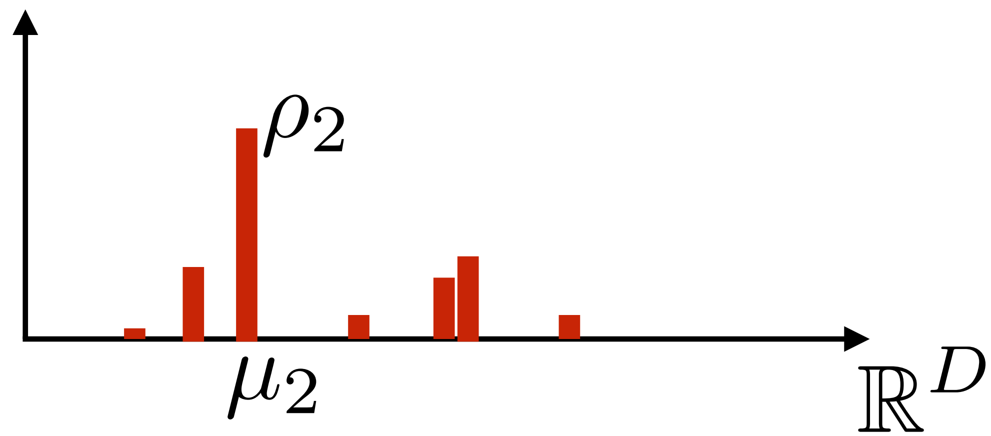

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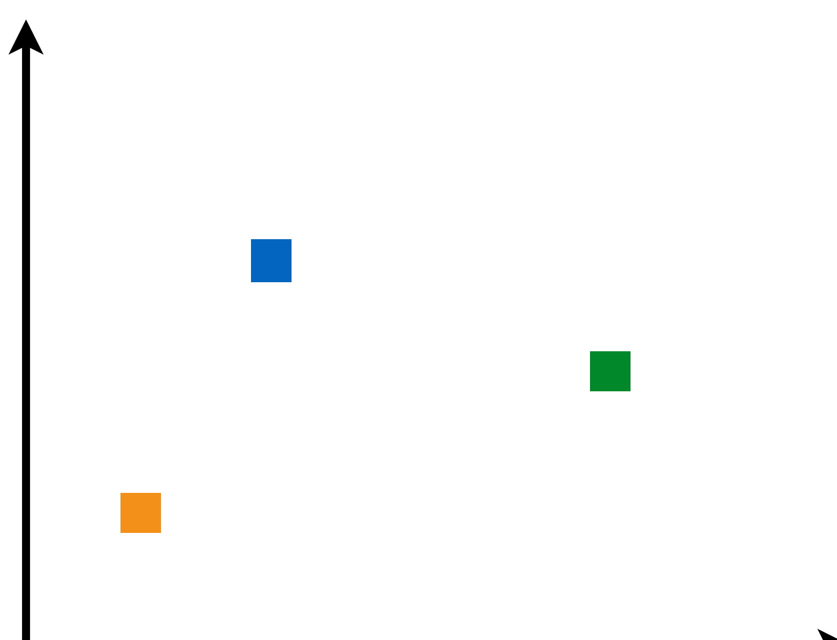

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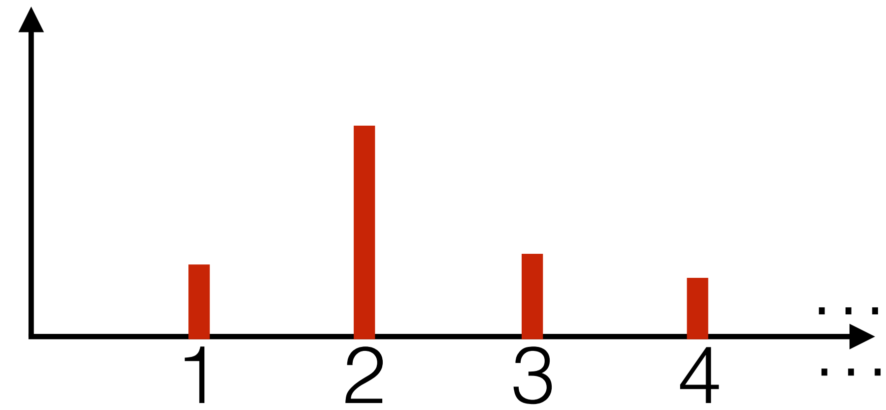

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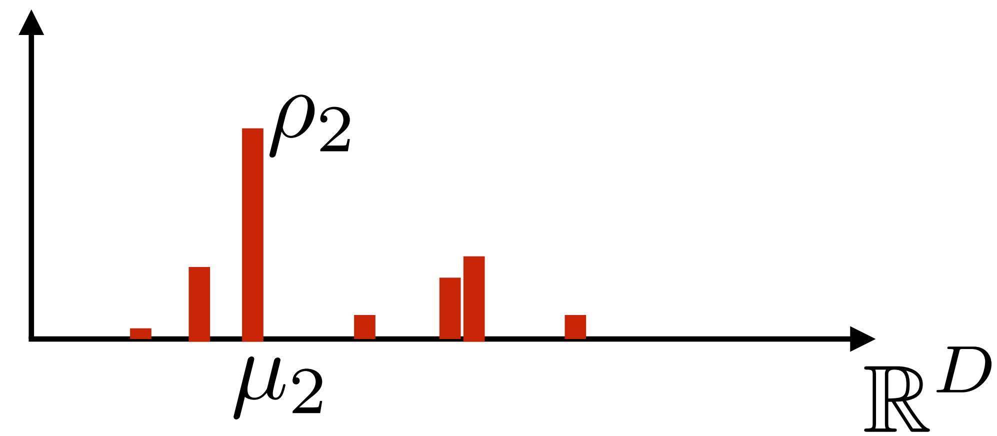

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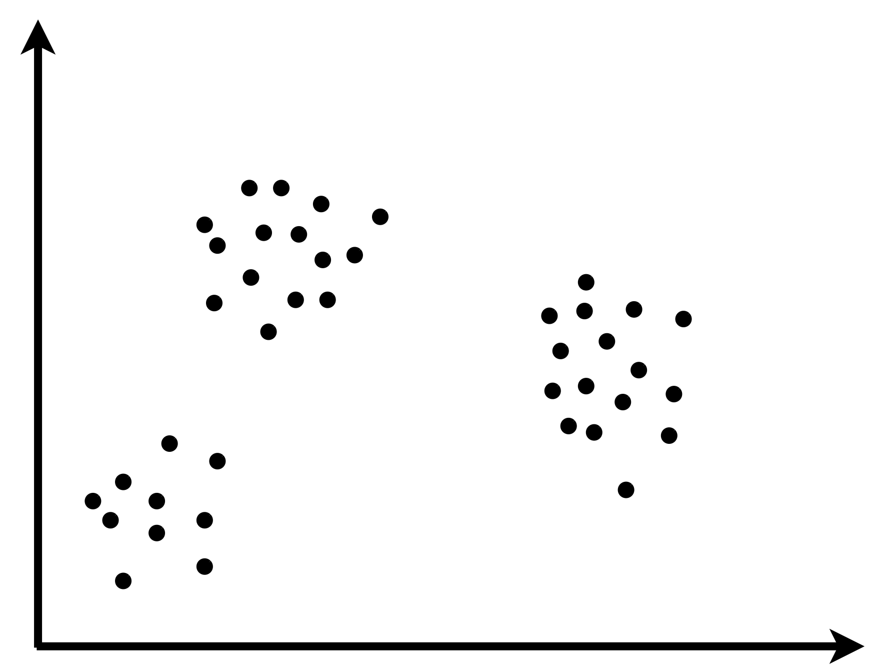

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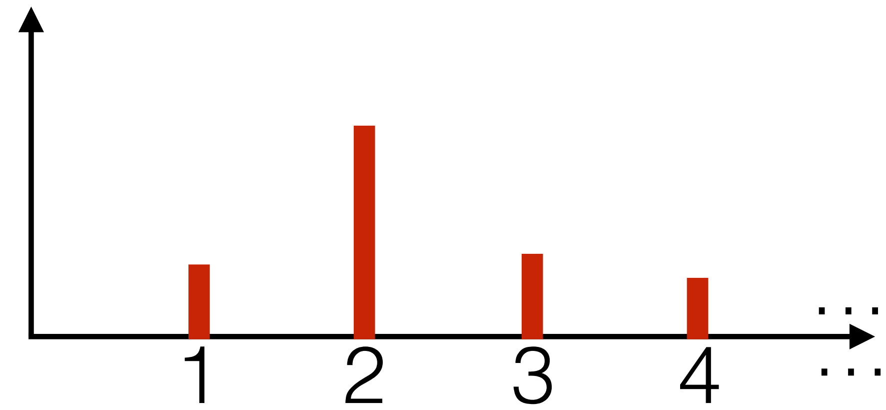

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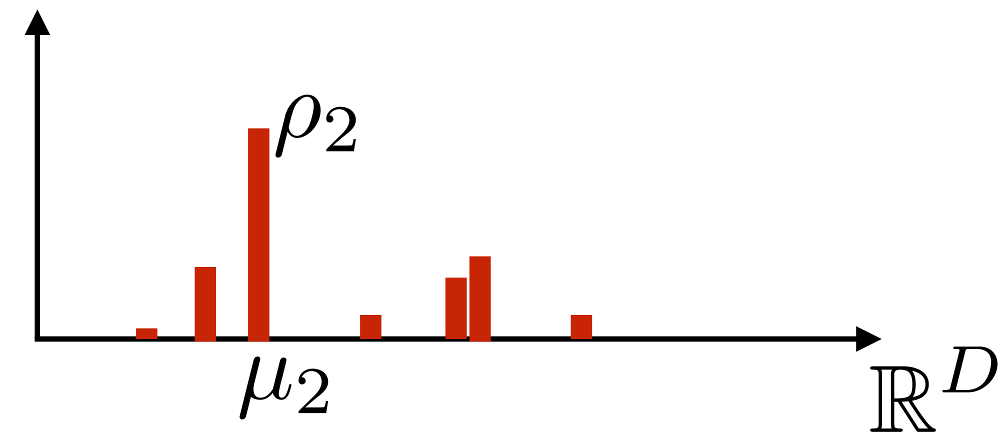

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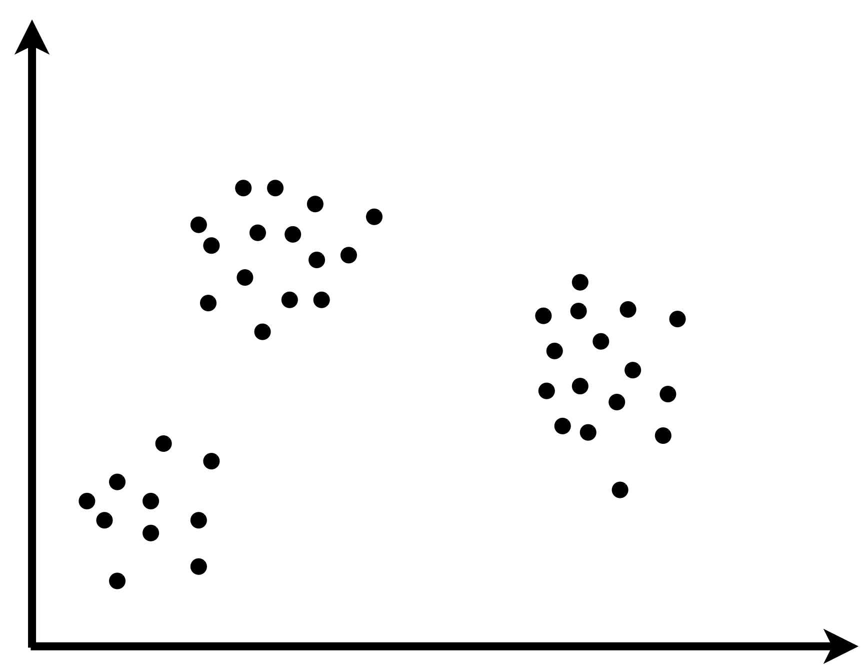

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[demo]

Dirichlet process mixture model

- More generally

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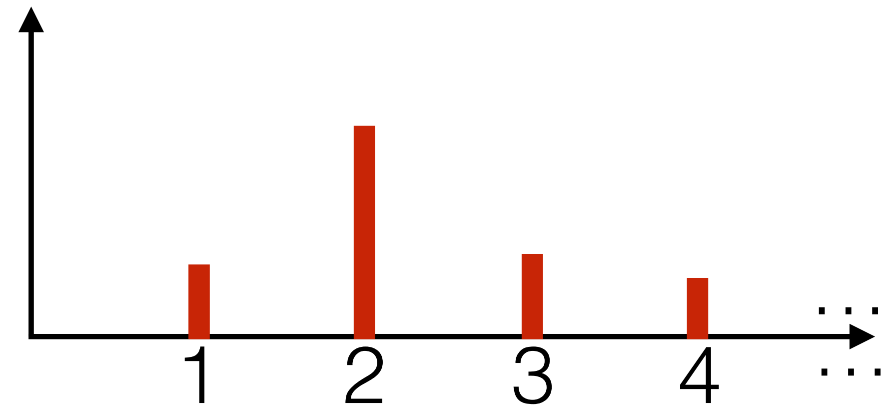

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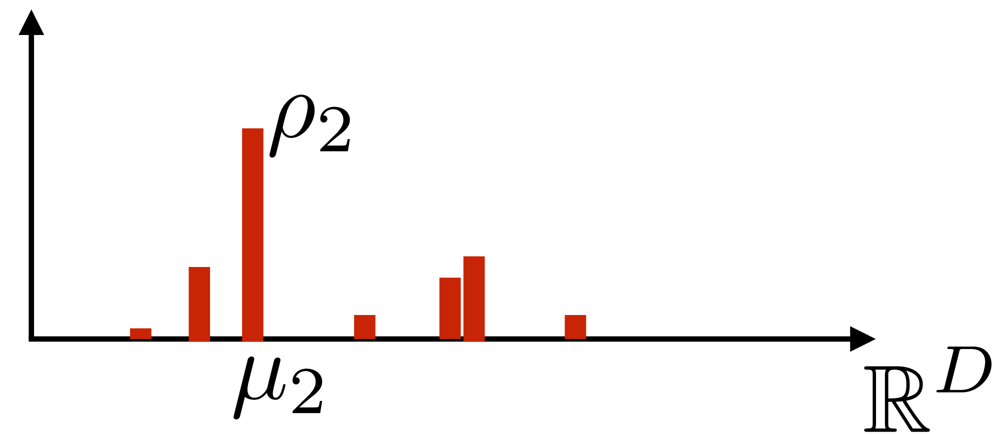

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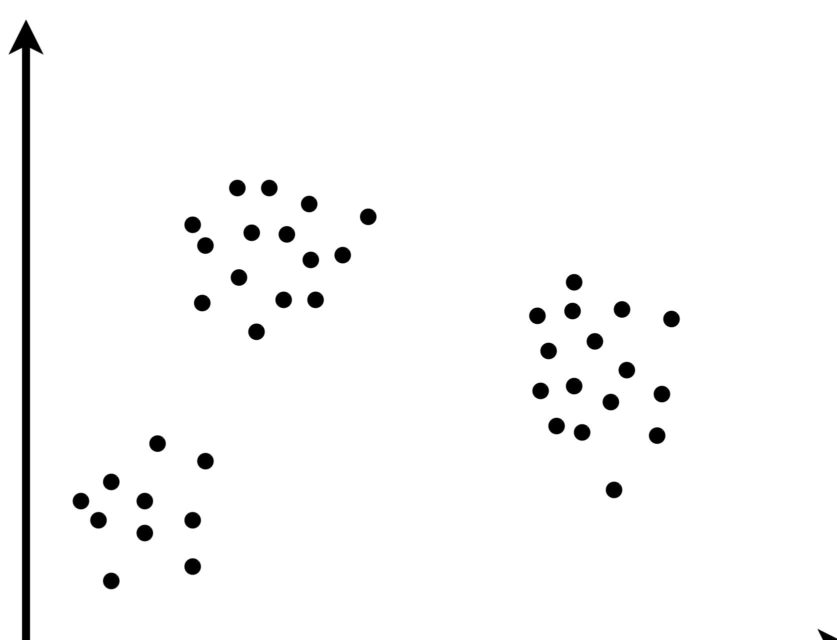

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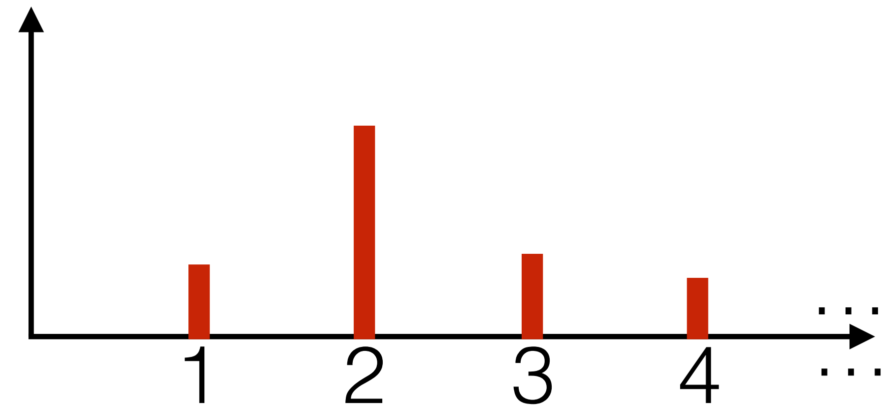

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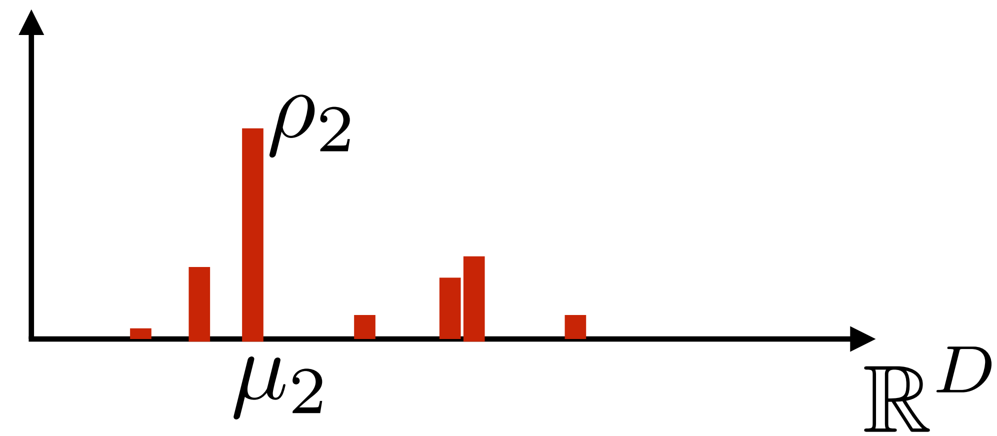

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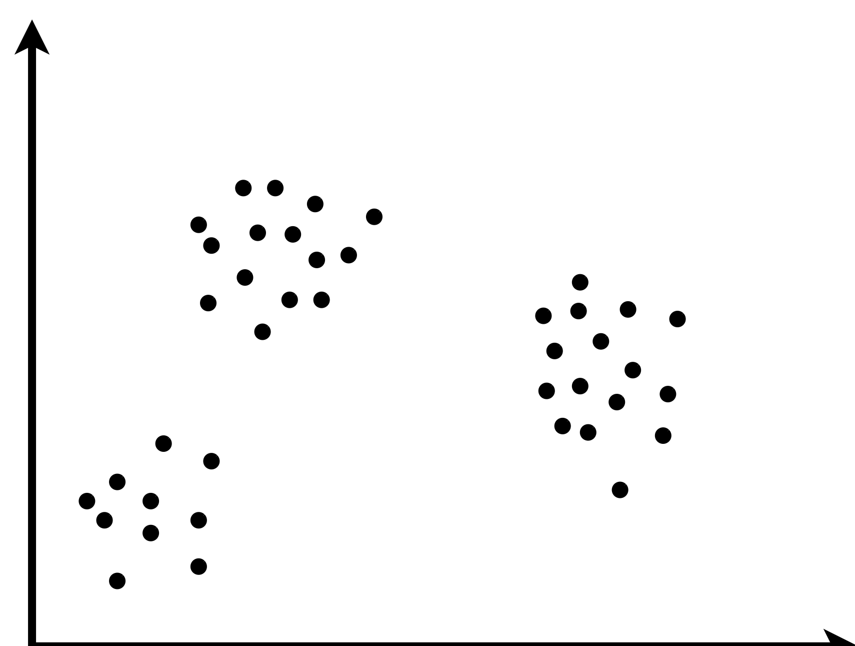

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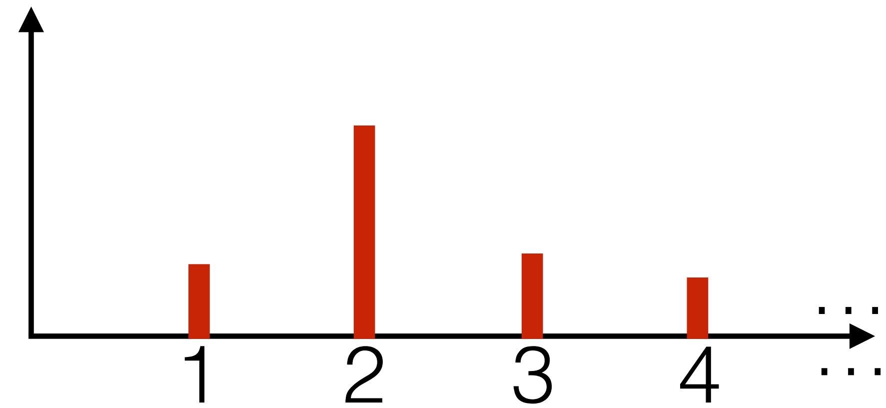

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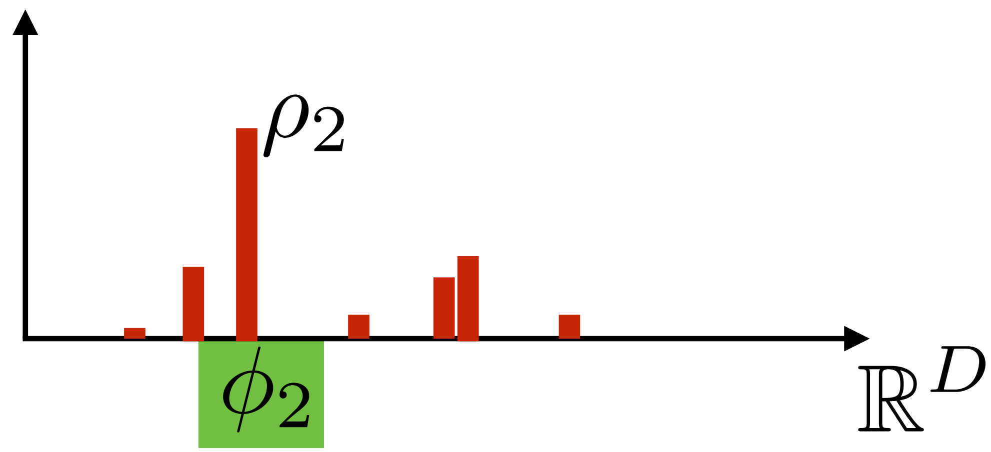

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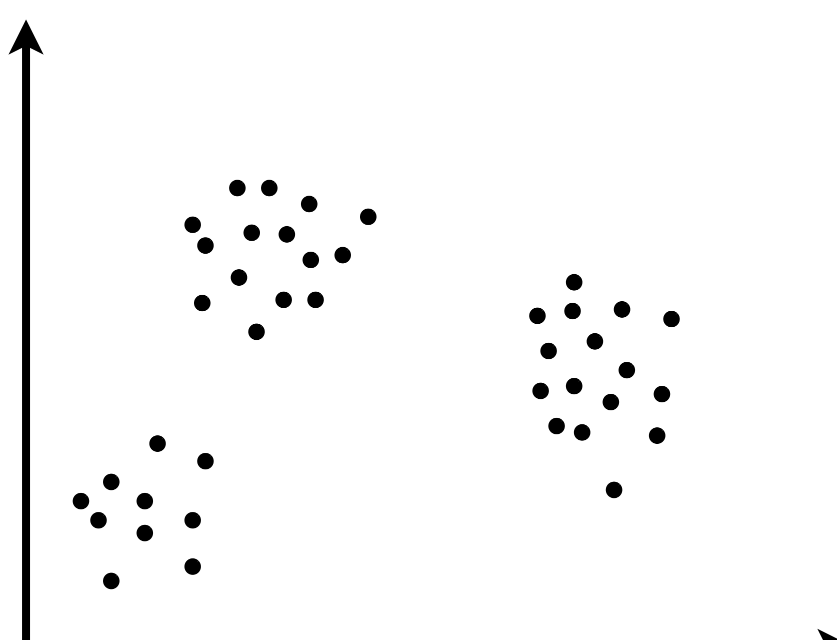
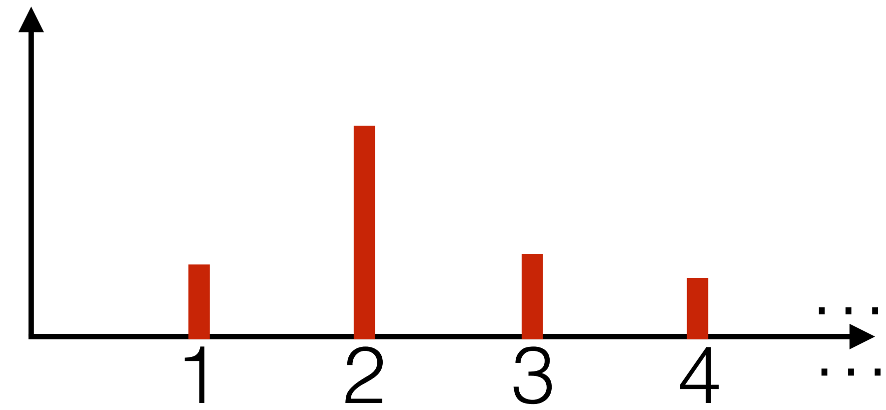


Dirichlet process mixture model

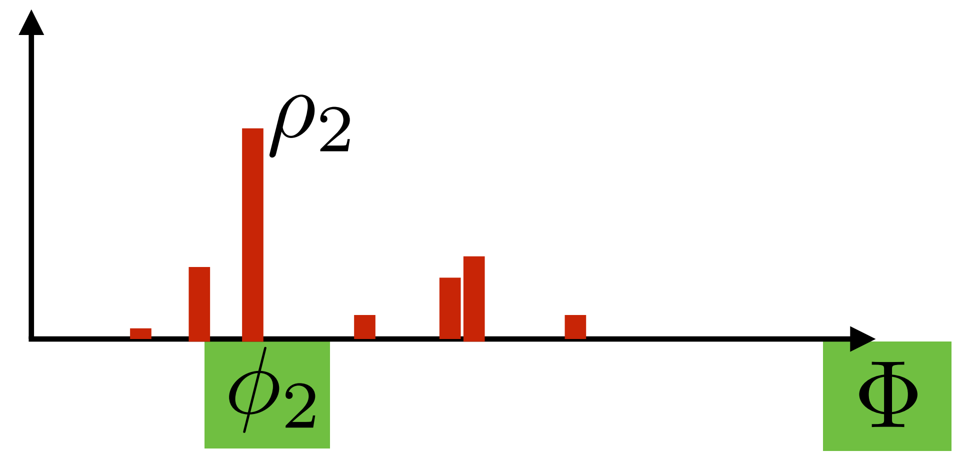
- More generally

$$\rho = (\rho_1, \rho_2, \ldots) \sim \text{GEM}(\alpha)$$

$$\phi_k \stackrel{iid}{\sim} G_0 \qquad k = 1, 2, \ldots$$



- i.e. $G = \sum_{k=1}^{\infty} \rho_k \delta_{\mu_k} \stackrel{d}{=} \text{DP}(\alpha, \mathcal{N}(\mu_0, \Sigma_0))$



$$z_n \stackrel{iid}{\sim} \text{Categorical}(\rho)$$

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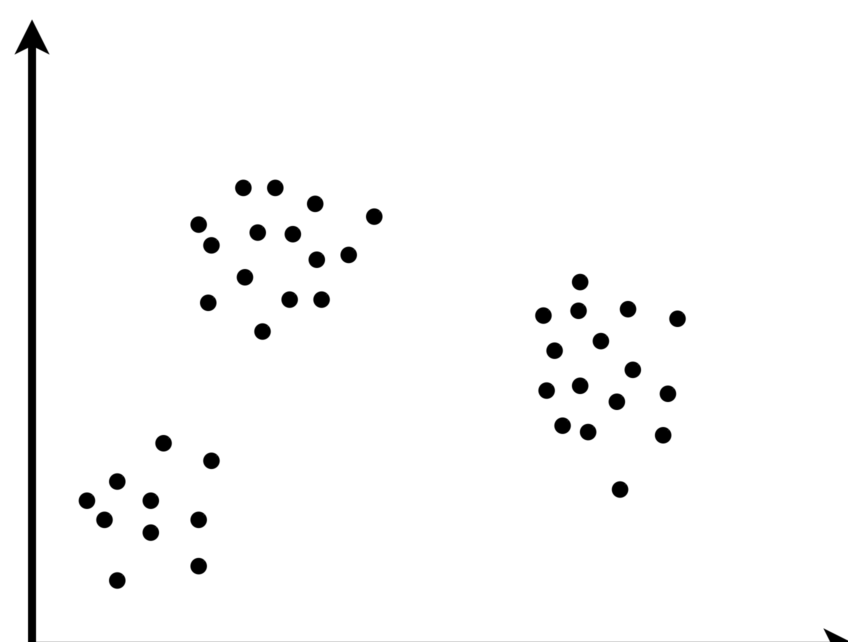

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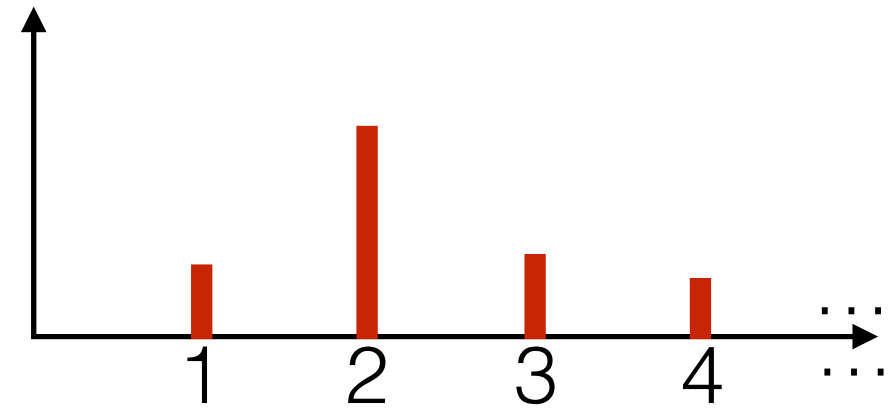

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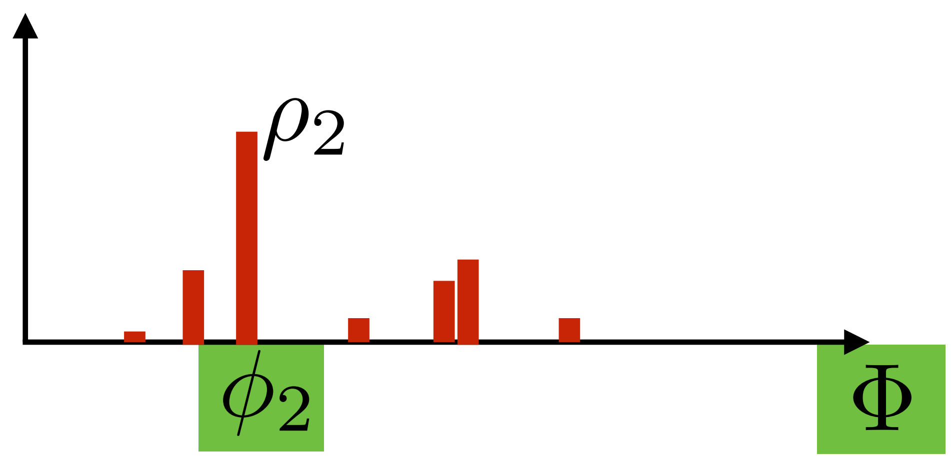

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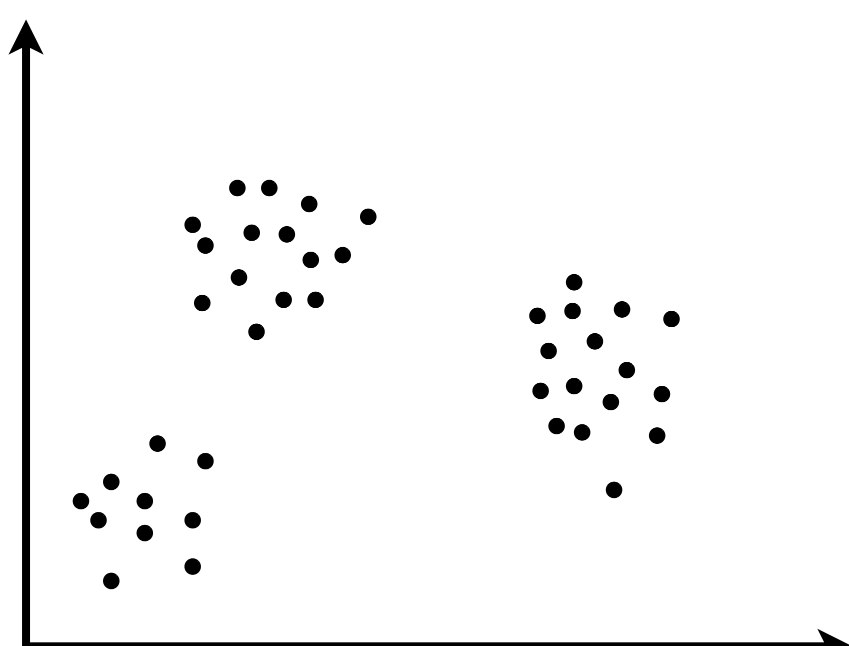

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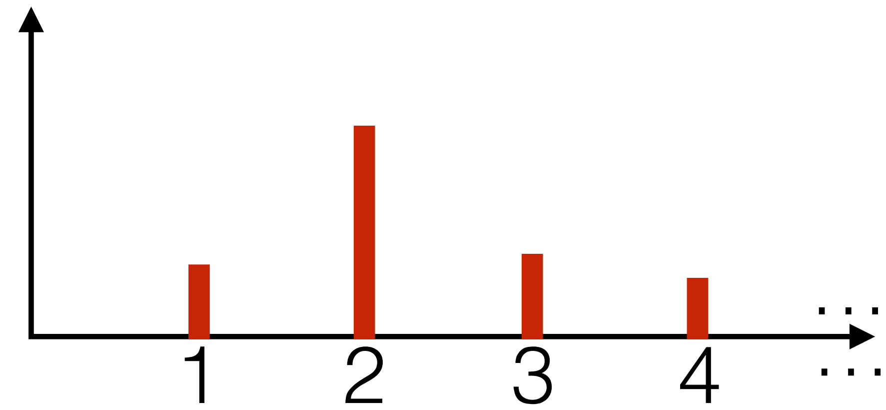

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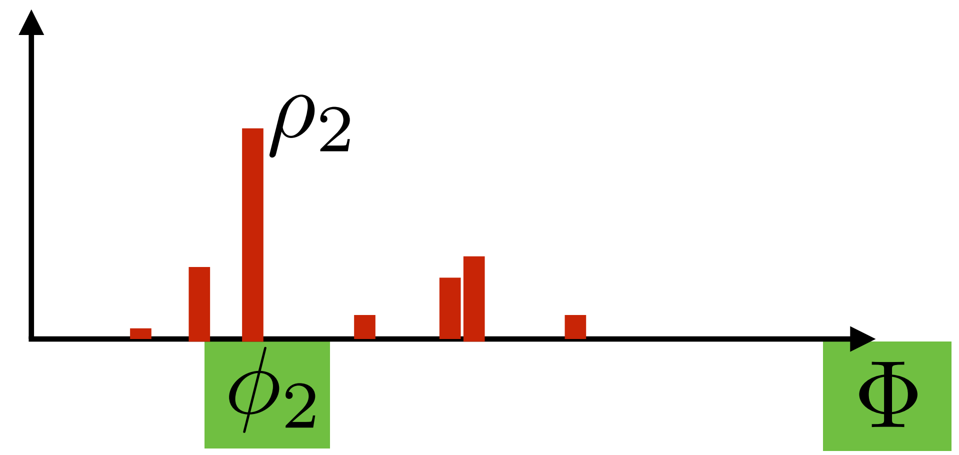

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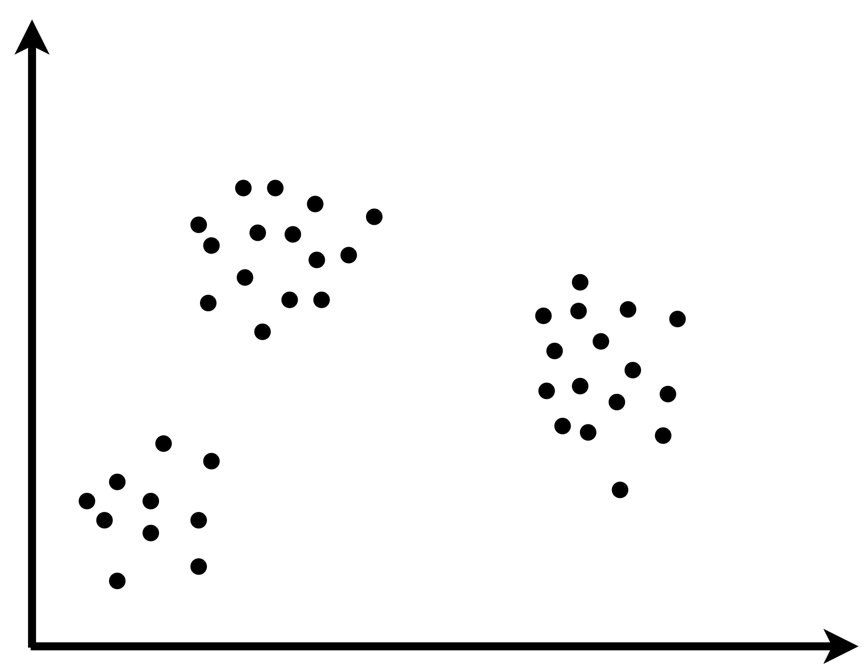

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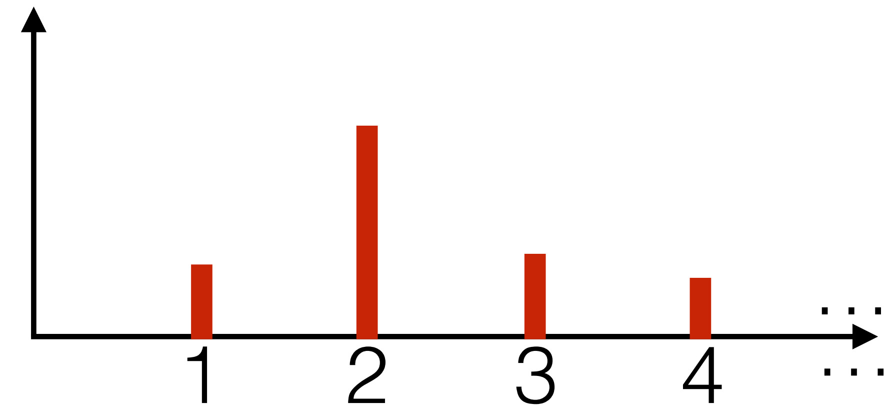
Dirichlet process mixture model



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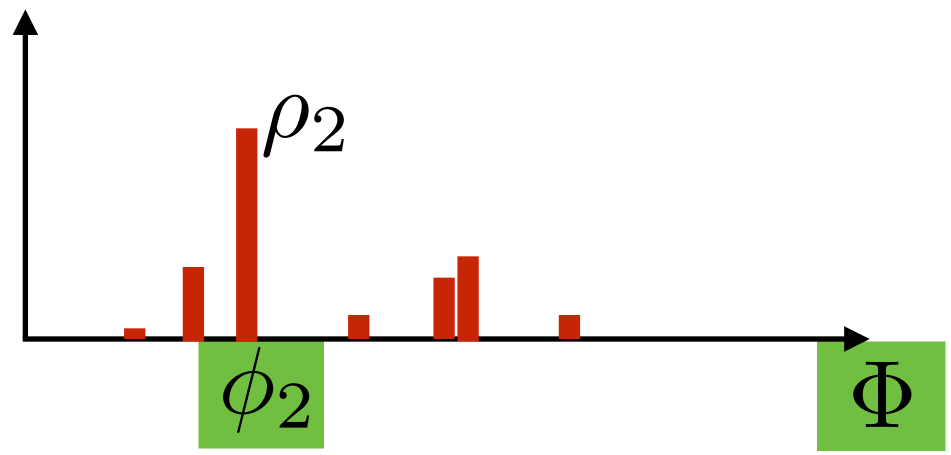

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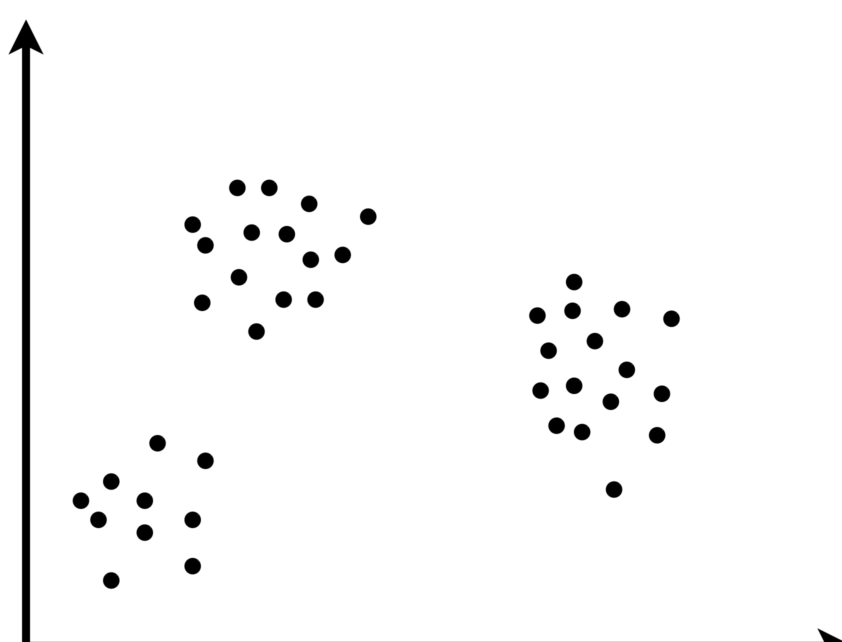

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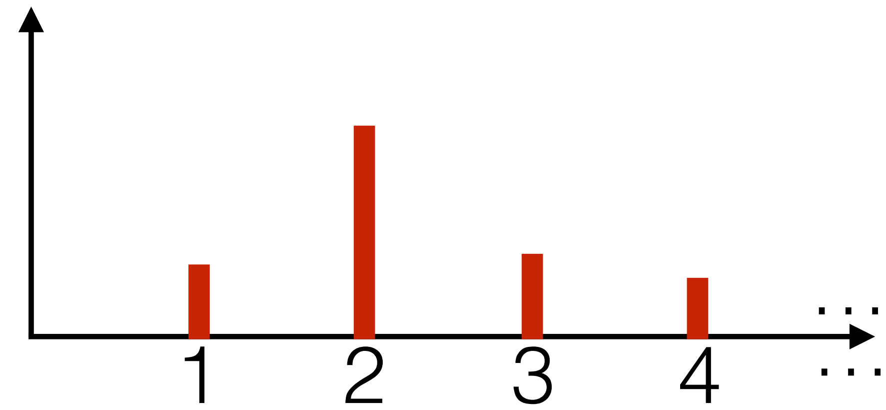

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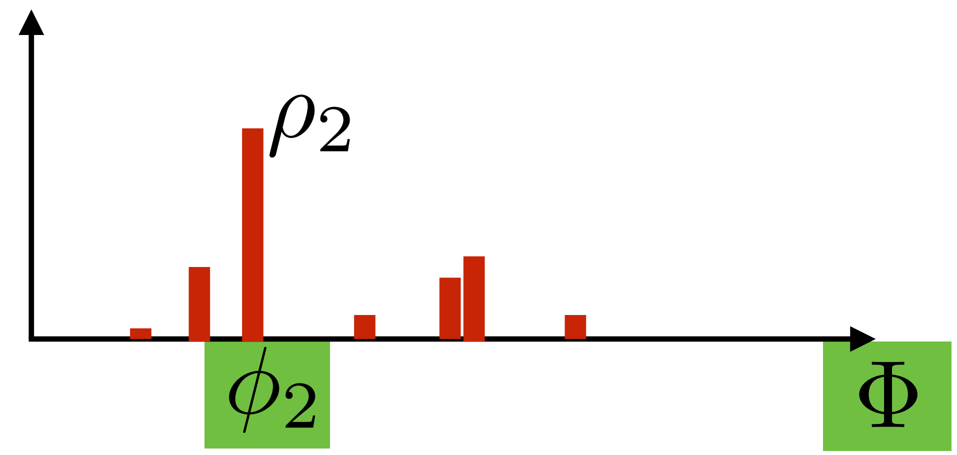

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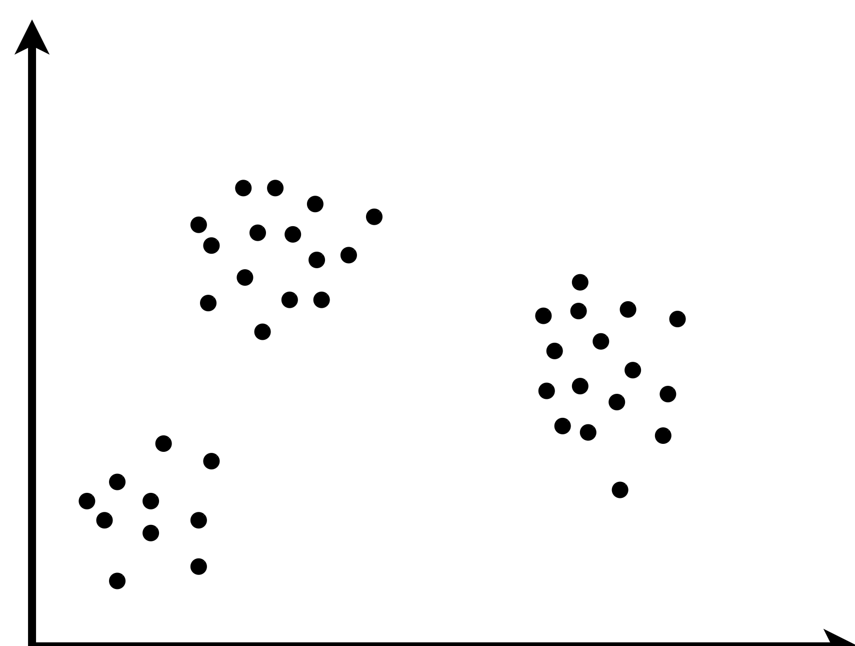

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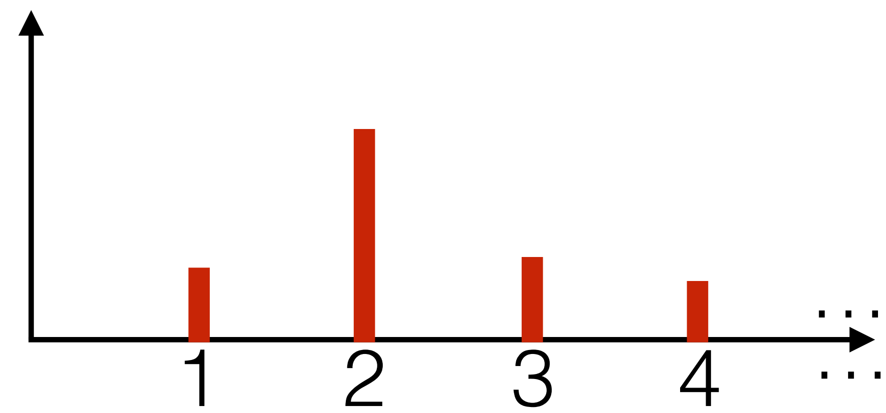

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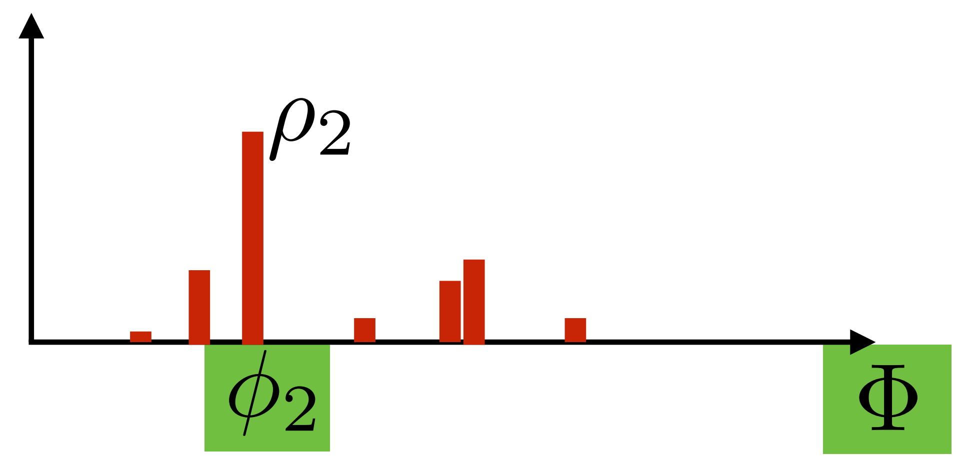

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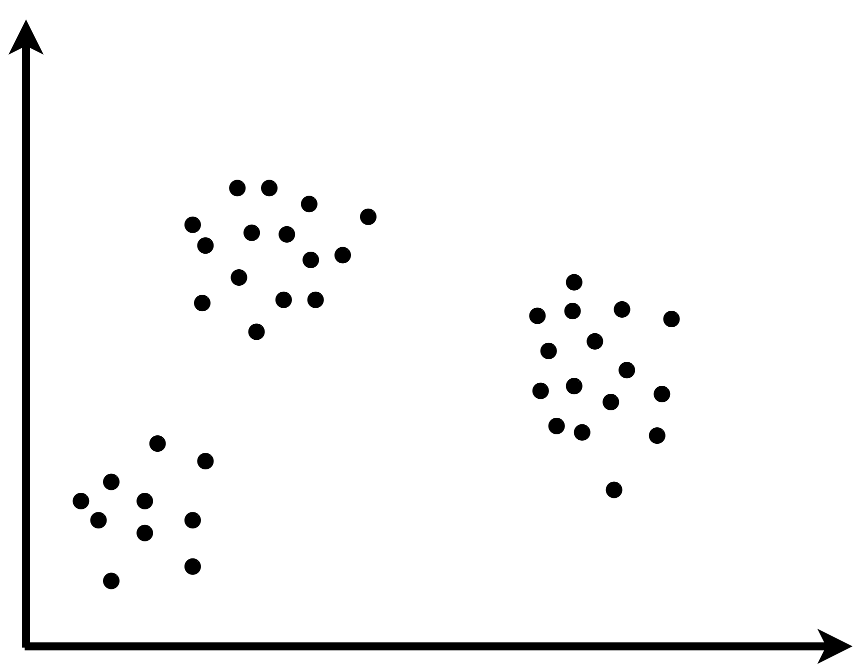

$$x_n \overset{indep}{\sim} F(\theta_n)$$

Dirichlet process mixture model

- More generally

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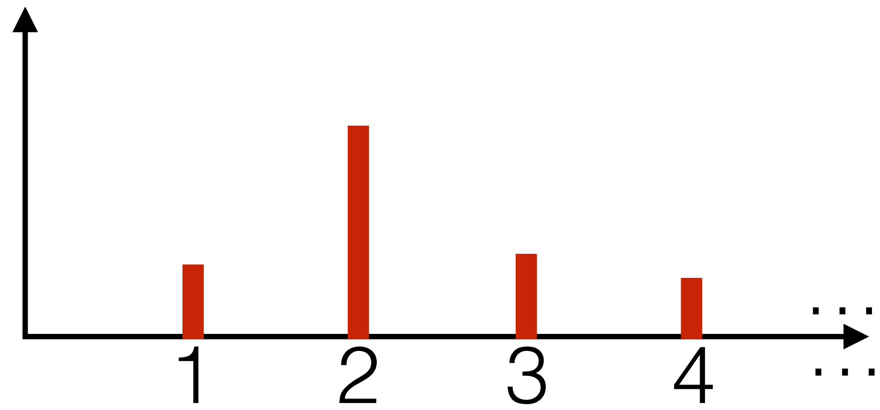

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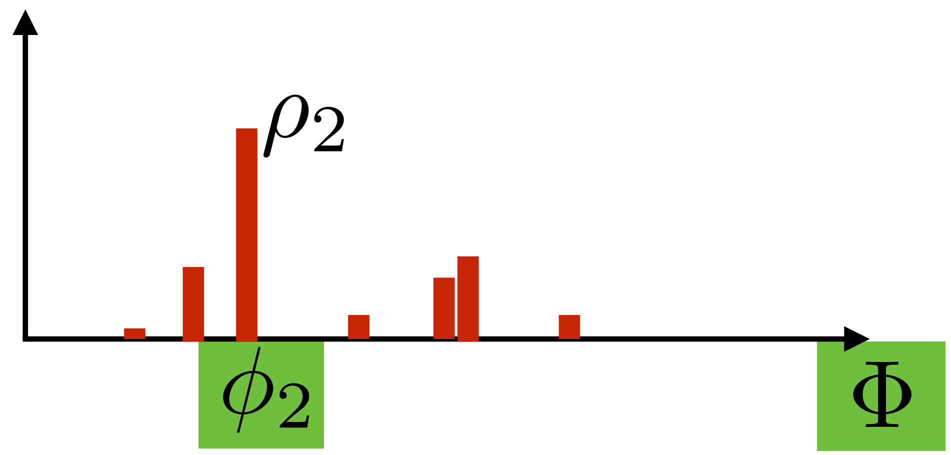

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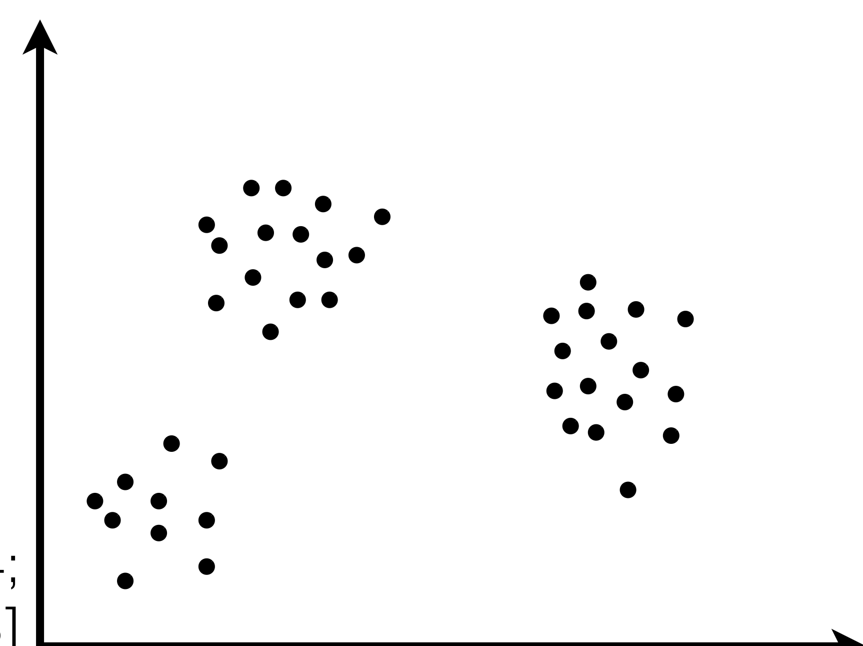
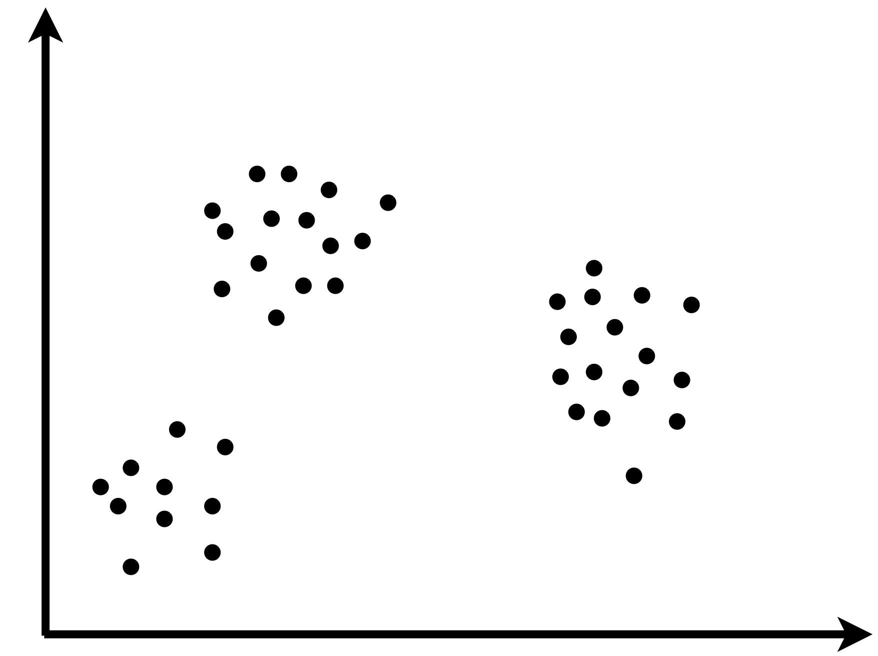


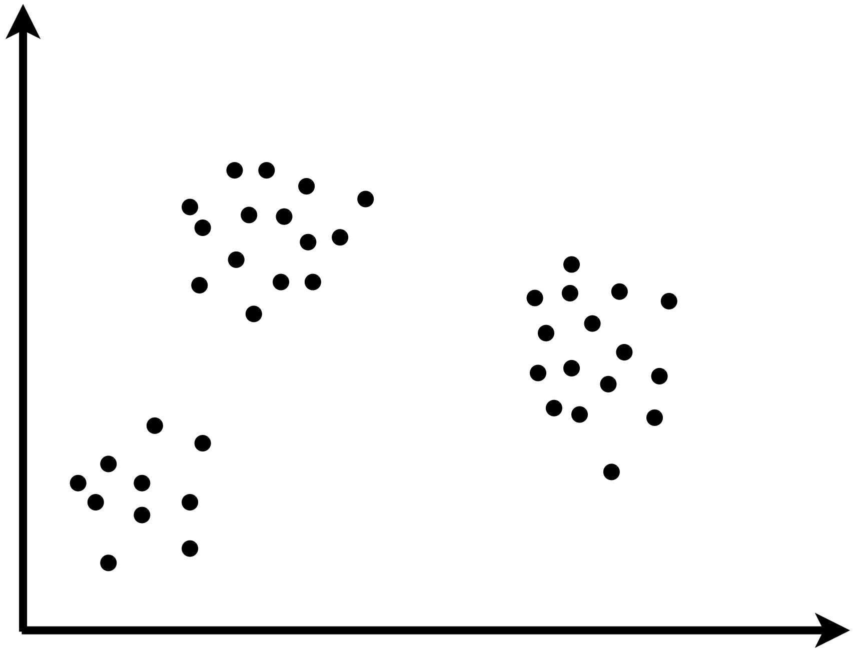
[Antoniak 1974; Ferguson 1983; West, Müller, Escobar 1994; Escobar, West 1995; MacEachern, Müller 1998]

DP or not DP, that is the question



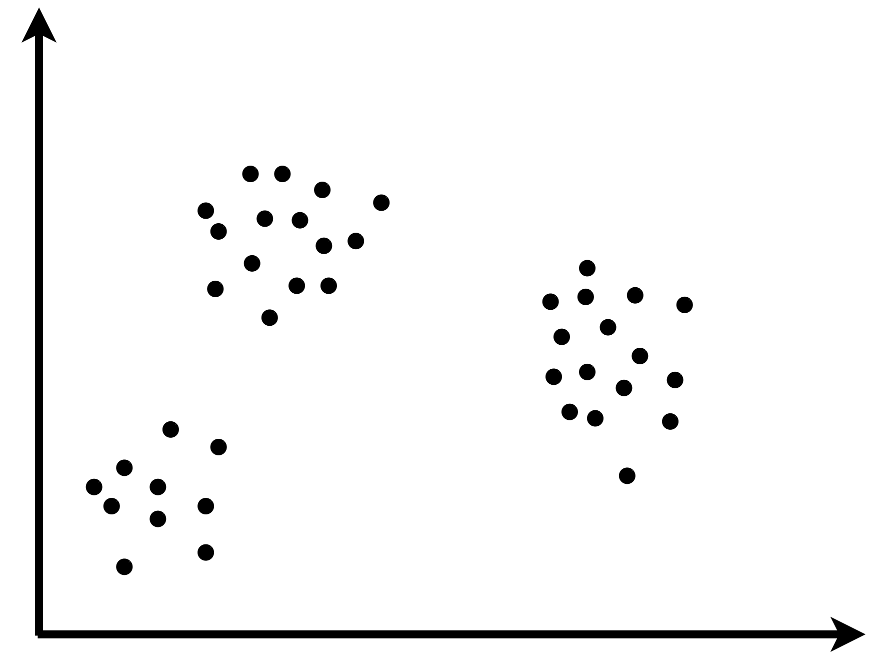
DP or not DP, that is the question

- GEM: 



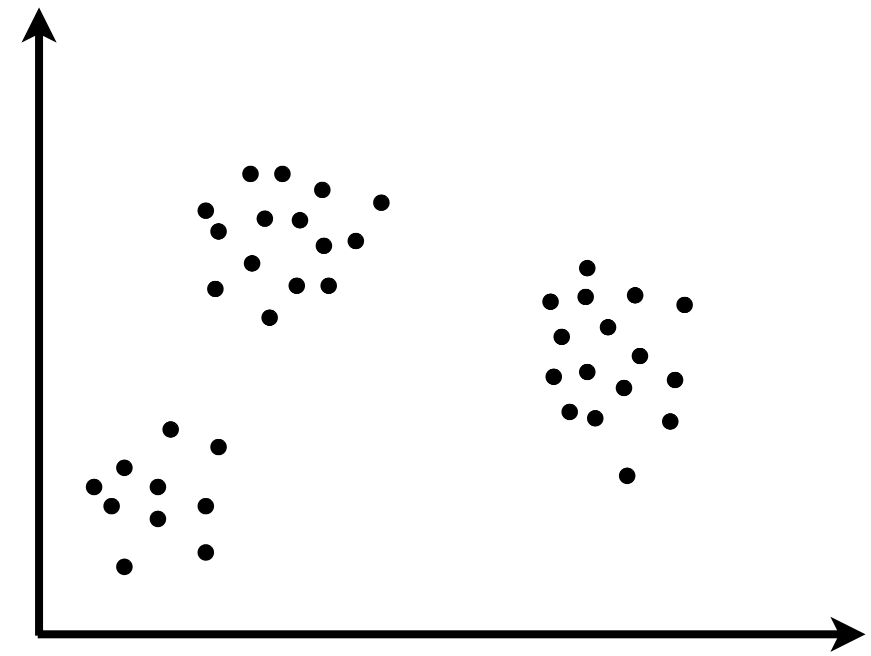
DP or not DP, that is the question

- GEM: 
- Compare to:



DP or not DP, that is the question

- GEM: 
- Compare to:
  - Finite (small *K*) mixture model



DP or not DP, that is the question

- GEM: 

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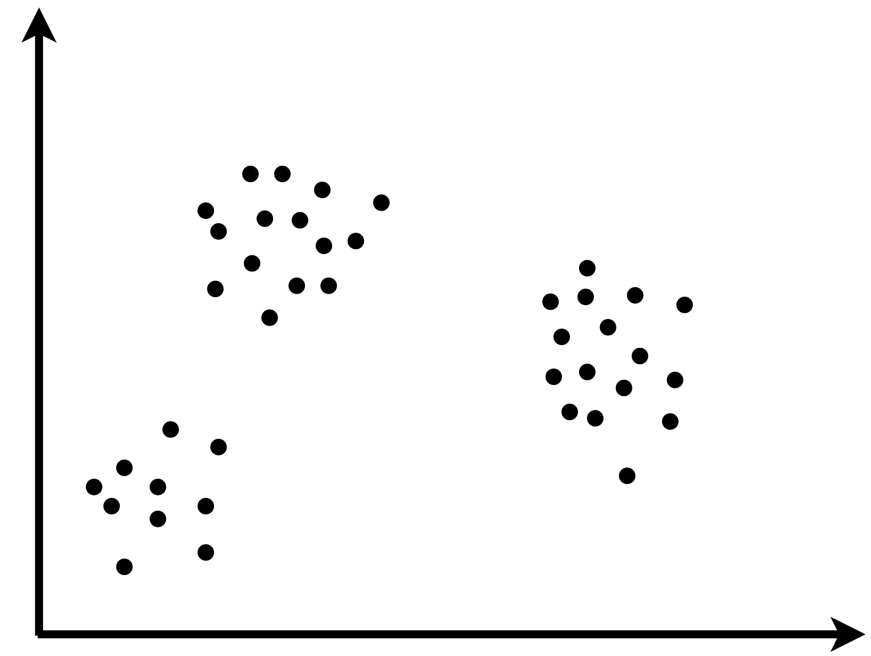
  - Finite (large *K*) mixture model



DP or not DP, that is the question

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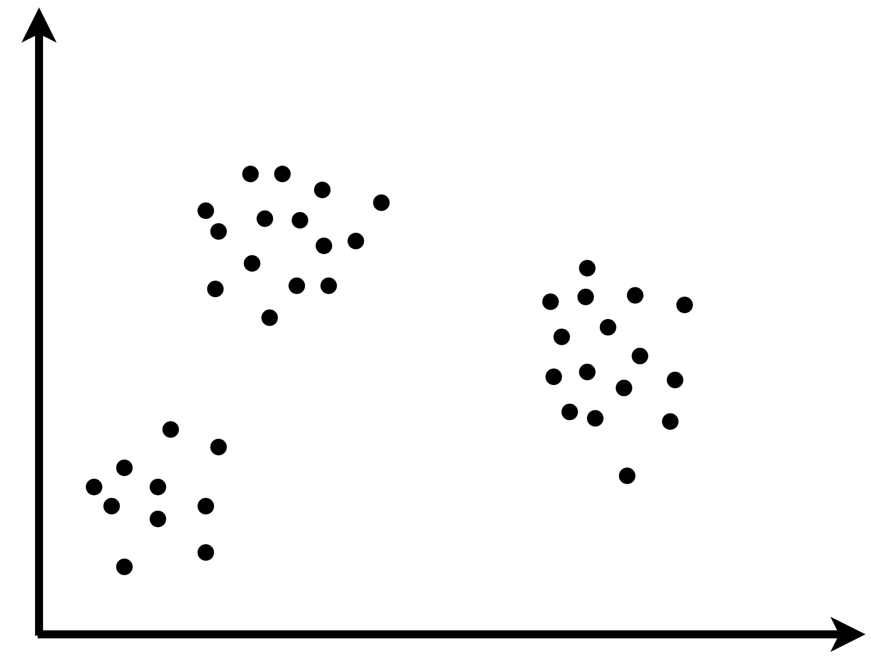
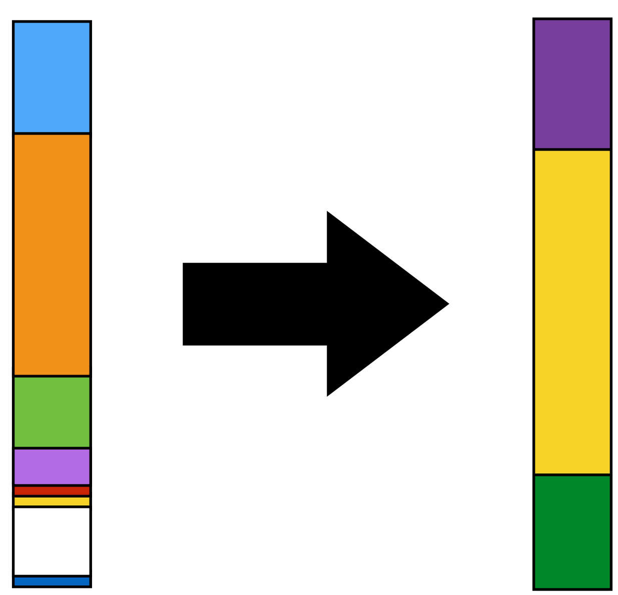


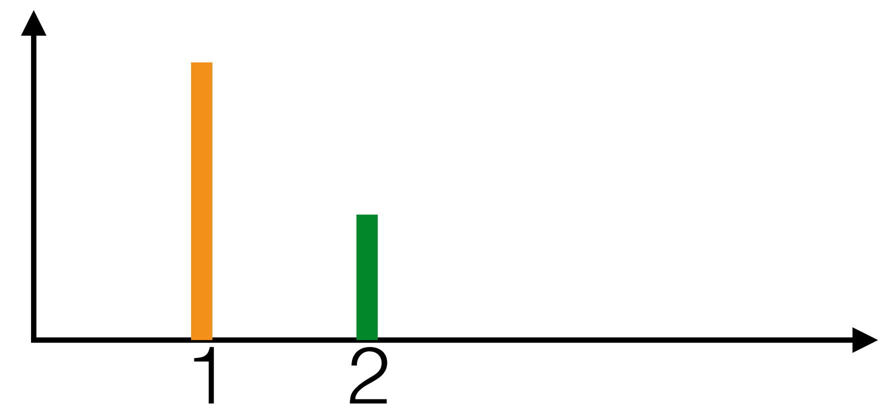
  - Finite (large *K*) mixture model

  - Time series



Marginal cluster assignments

$$\rho_1 \sim \mathrm{Beta}(a_1, a_2), z_n \stackrel{iid}{\sim} \mathrm{Cat}(\rho_1, \rho_2)$$



Marginal cluster assignments

- Integrate out the frequencies

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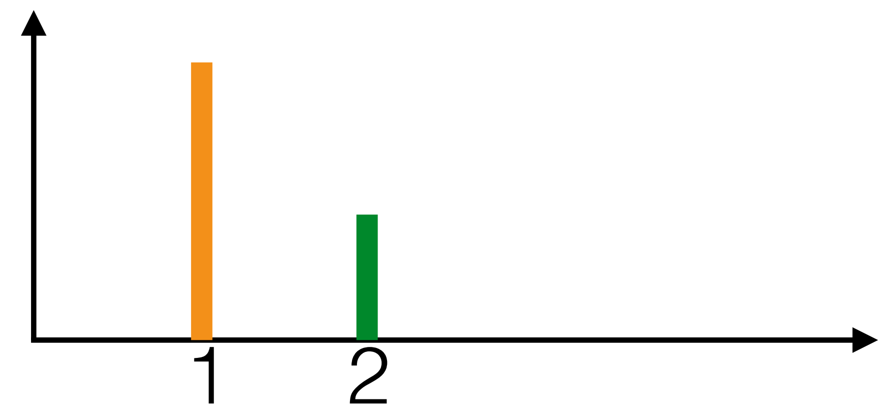

Marginal cluster assignments

- Integrate out the frequencies

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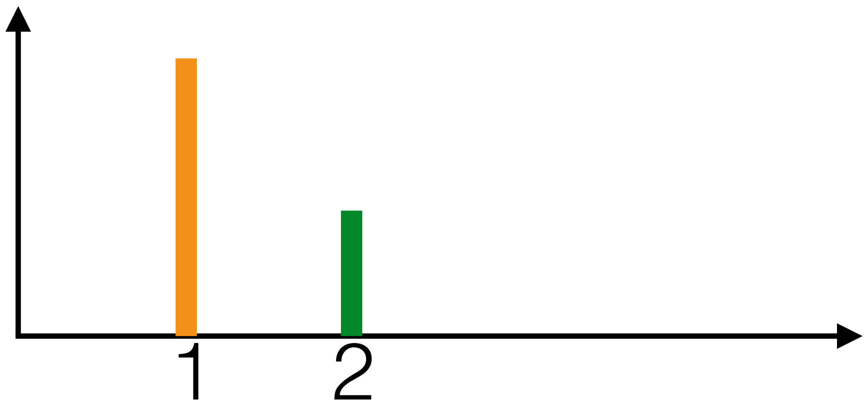
$$p(z_n = 1 | z_1, \ldots, z_{n-1})$$



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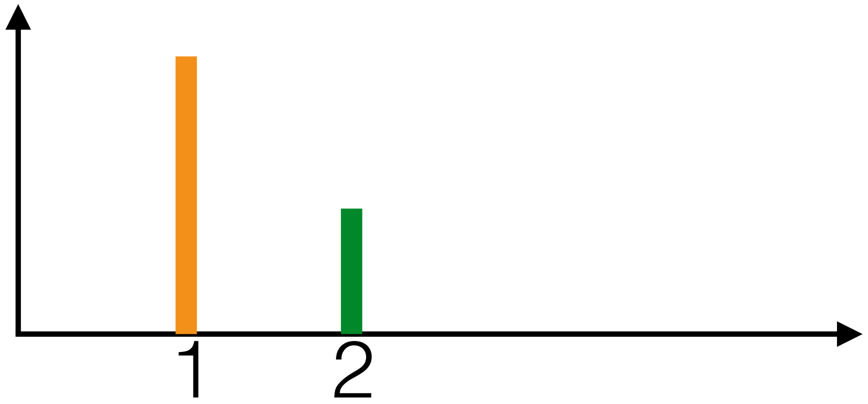
$$p(z_n = 1 | z_1, \ldots, z_{n-1})$$
$$= \int p(z_n = 1, \rho_1 | z_1, \ldots, z_{n-1}) d\rho_1$$



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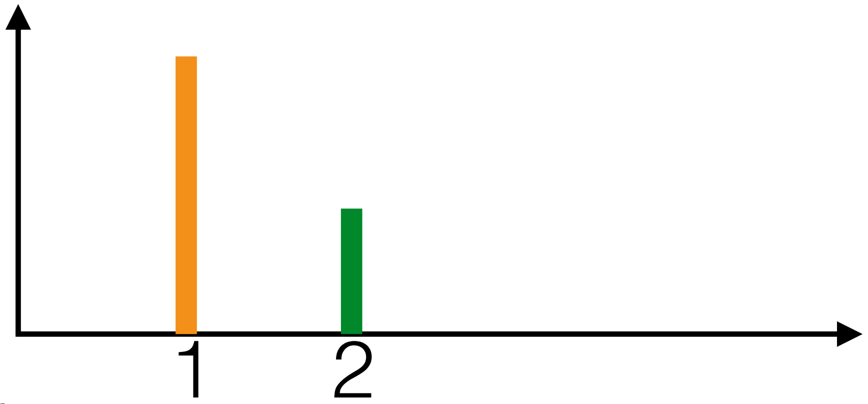

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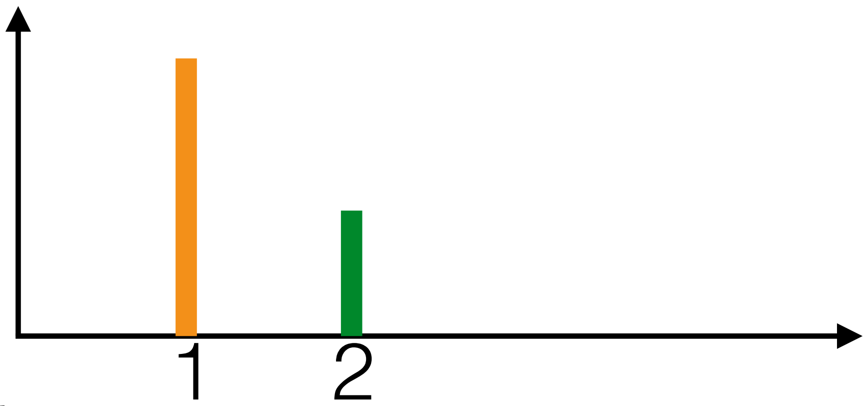

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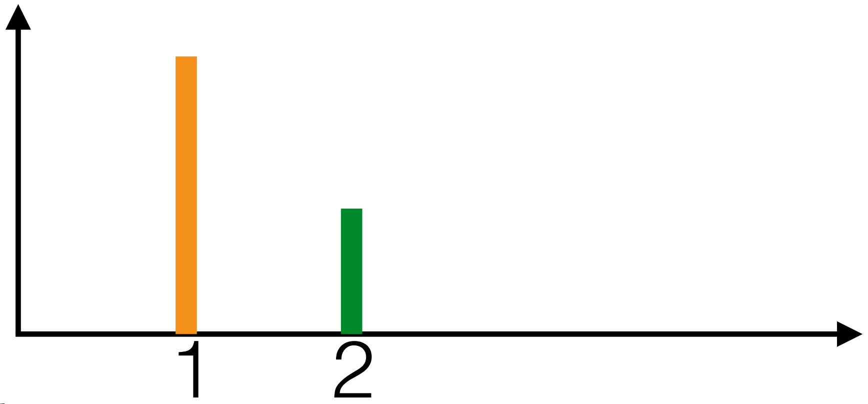
- Integrate out the frequencies



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$$= \int p(z_n = 1 | \rho_1) p(\rho_1 | z_1, \ldots, z_{n-1}) d\rho_1$$

$$= \int \rho_1$$

Marginal cluster assignments

- Integrate out the frequencies

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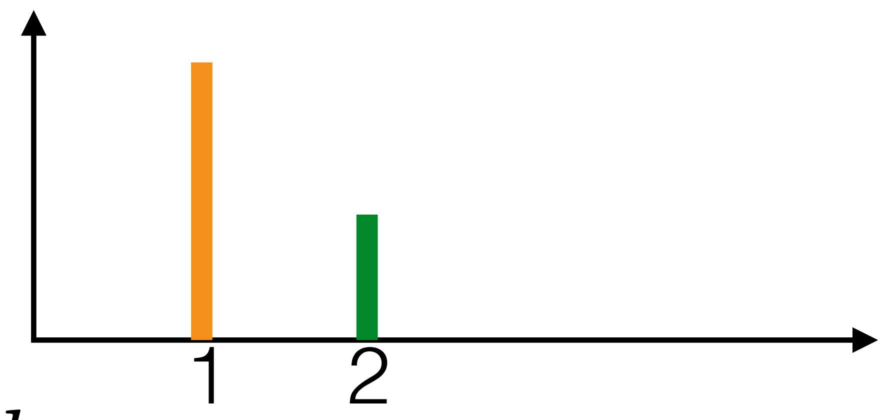

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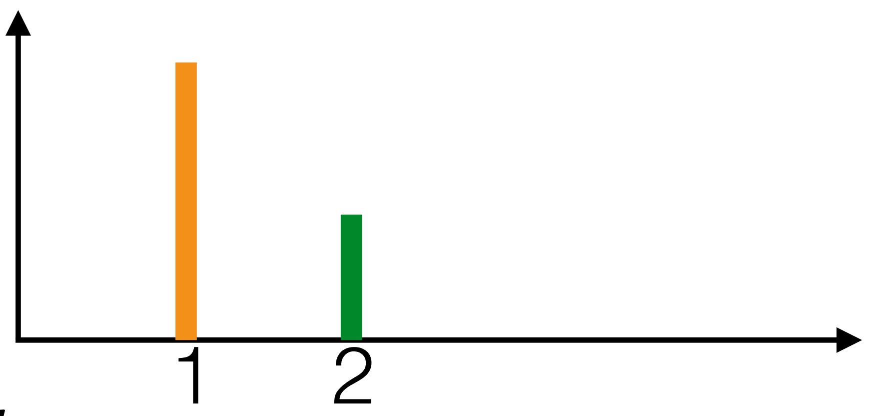

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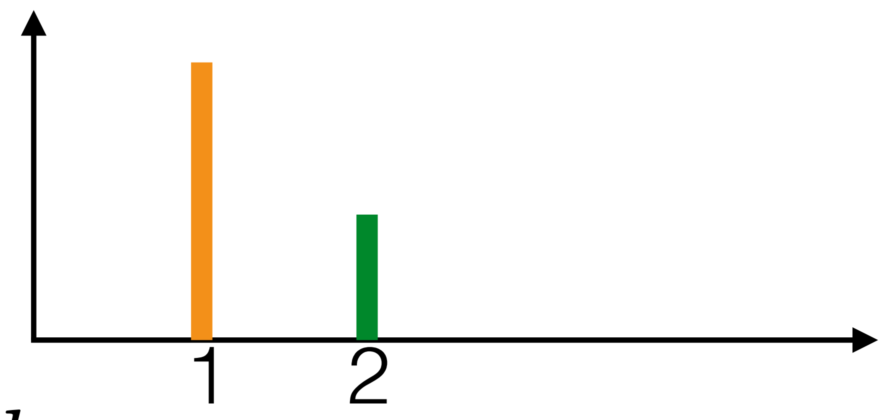

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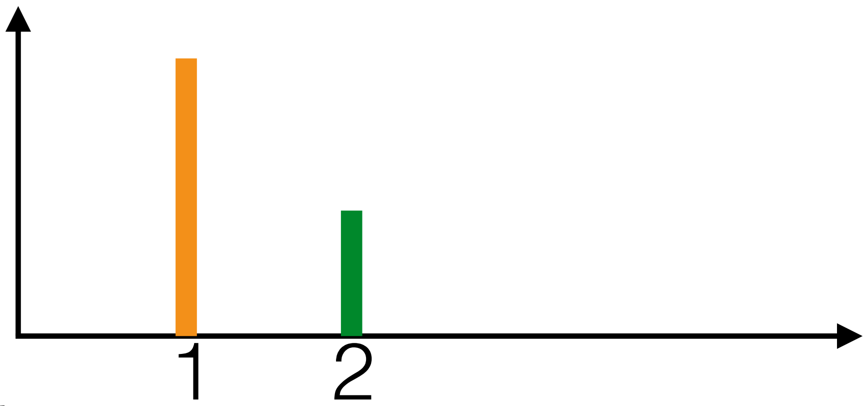

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$$a_{1,n} := a_1 + \sum_{m=1}^{n-1} \mathbf{1}\{z_m = 1\}, a_{2,n} = a_2 + \sum_{m=1}^{n-1} \mathbf{1}\{z_m = 2\}$$

Marginal cluster assignments

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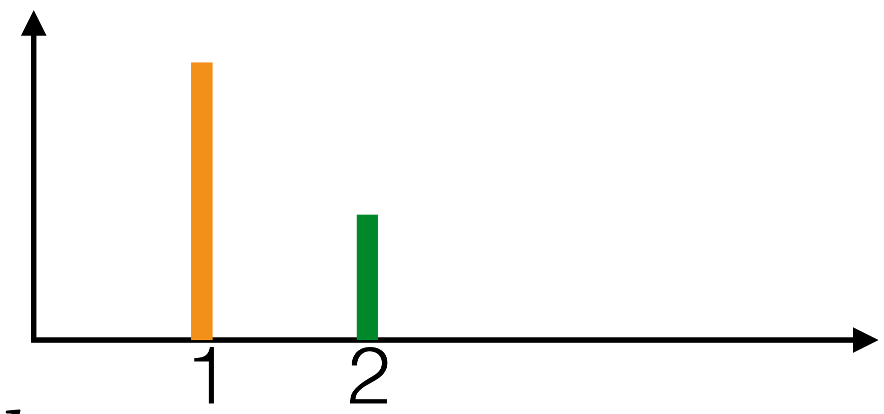

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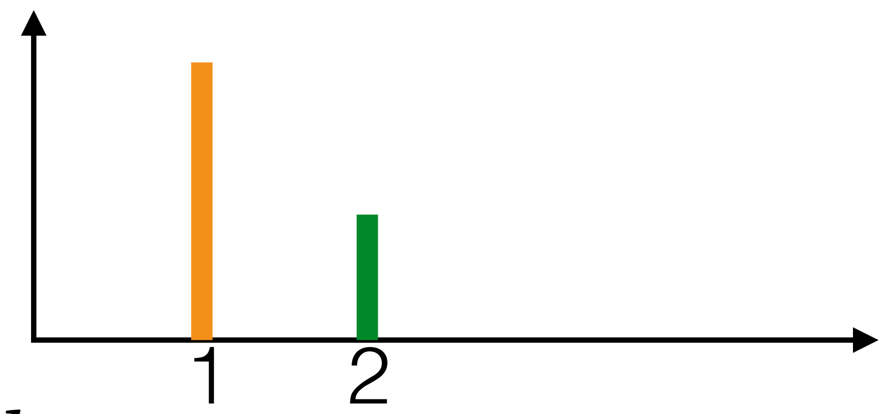
$$= \int \rho_1 \frac{\Gamma(a_{1,n} + a_{2,n})}{\Gamma(a_{1,n})\Gamma(a_{2,n})} \rho_1^{a_{1,n}-1} (1 - \rho_1)^{a_{2,n}-1} d\rho_1$$

Marginal cluster assignments

- Integrate out the frequencies



$$\rho_1 \sim \mathrm{Beta}(a_1, a_2),\, z_n \stackrel{iid}{\sim} \mathrm{Cat}(\rho_1, \rho_2)$$

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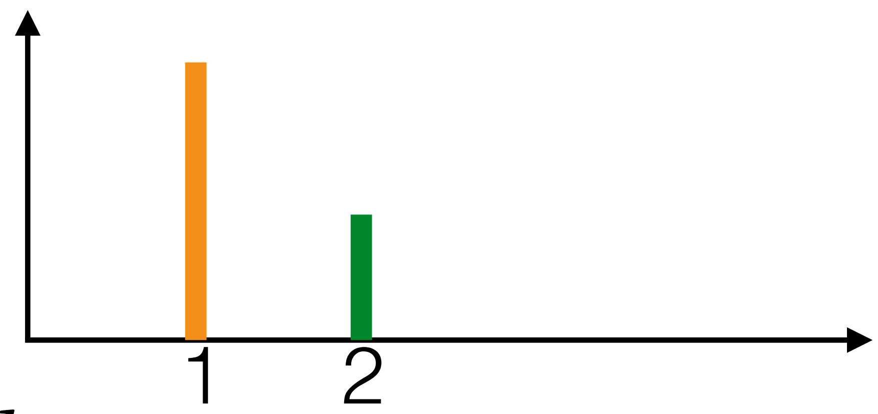
$$= \frac{\Gamma(a_{1,n} + a_{2,n})}{\Gamma(a_{1,n})\Gamma(a_{2,n})} \frac{\Gamma(a_{1,n} + 1)\Gamma(a_{2,n})}{\Gamma(a_{1,n} + a_{2,n} + 1)}$$

Marginal cluster assignments

- Integrate out the frequencies



$$\rho_1 \sim \text{Beta}(a_1, a_2),\, z_n \overset{iid}{\sim} \text{Cat}(\rho_1, \rho_2)$$

$$p(z_n = 1 | z_1, \ldots, z_{n-1})$$

$$= \int p(z_n = 1 | \rho_1) p(\rho_1 | z_1, \ldots, z_{n-1}) d\rho_1$$

$$= \int \rho_1 \text{Beta}(\rho_1 | a_{1,n}, a_{2,n}) d\rho_1$$

$$a_{1,n} := a_1 + \sum_{m=1}^{n-1} \mathbf{1}\{z_m = 1\},\, a_{2,n} = a_2 + \sum_{m=1}^{n-1} \mathbf{1}\{z_m = 2\}$$

$$= \int \rho_1 \frac{\Gamma(a_{1,n} + a_{2,n})}{\Gamma(a_{1,n})\Gamma(a_{2,n})} \rho_1^{a_{1,n}-1} (1 - \rho_1)^{a_{2,n}-1} d\rho_1$$

$$= \frac{\Gamma(a_{1,n} + a_{2,n})}{\Gamma(a_{1,n})\Gamma(a_{2,n})} \frac{\Gamma(a_{1,n} + 1)\Gamma(a_{2,n})}{\Gamma(a_{1,n} + a_{2,n} + 1)}$$

Recall
$\Gamma(x+1) = x\Gamma(x)$

Marginal cluster assignments

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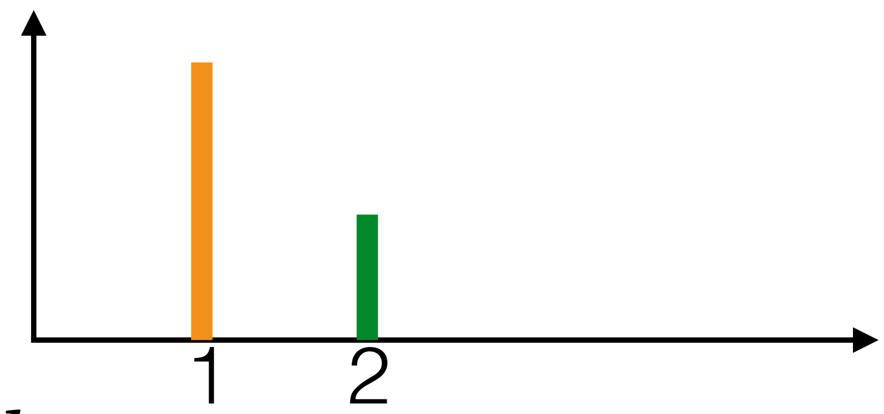

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$$= \frac{\Gamma(a_{1,n} + a_{2,n})}{\Gamma(a_{1,n})\Gamma(a_{2,n})} \frac{\Gamma(a_{1,n}+1)\Gamma(a_{2,n})}{\Gamma(a_{1,n} + a_{2,n} + 1)}$$
$$= \frac{a_{1,n}}{a_{1,n} + a_{2,n}}$$

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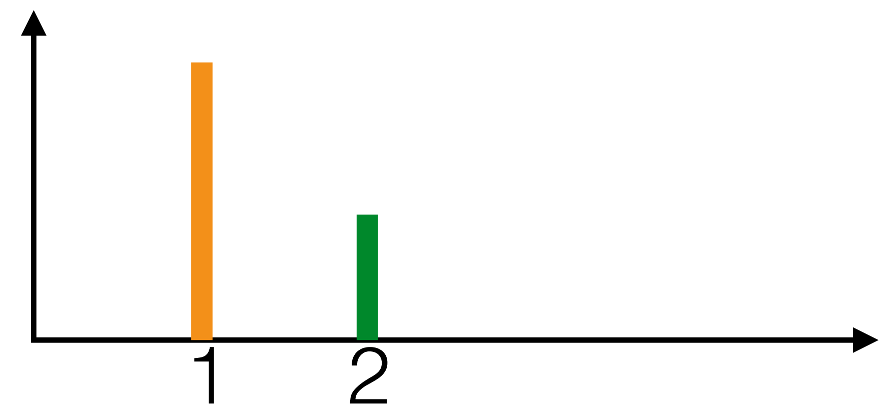

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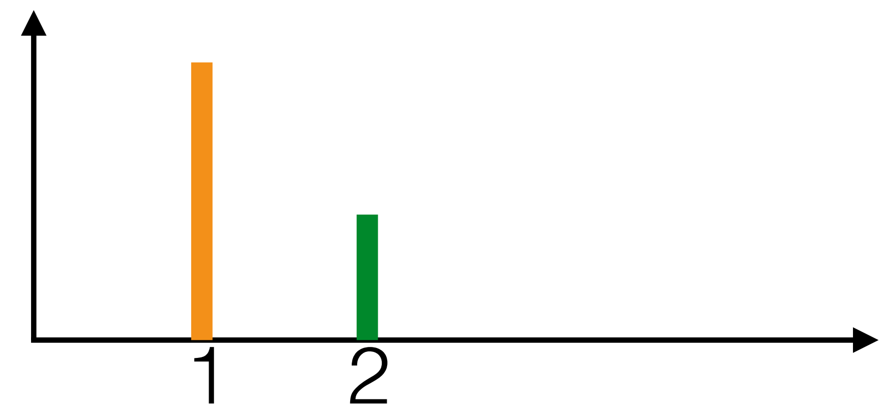

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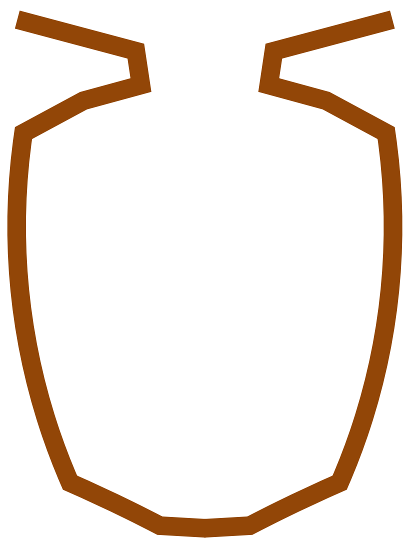

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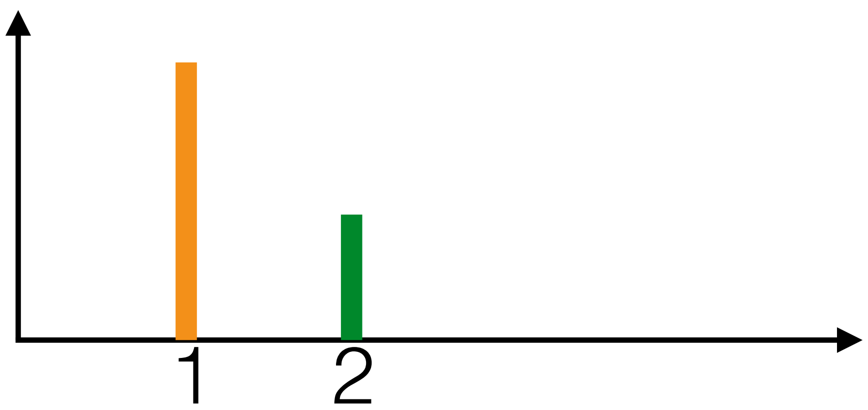

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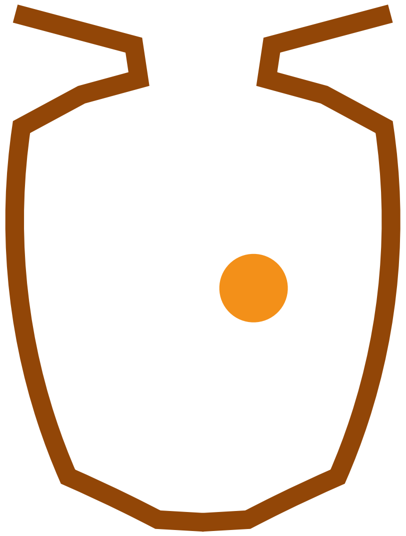

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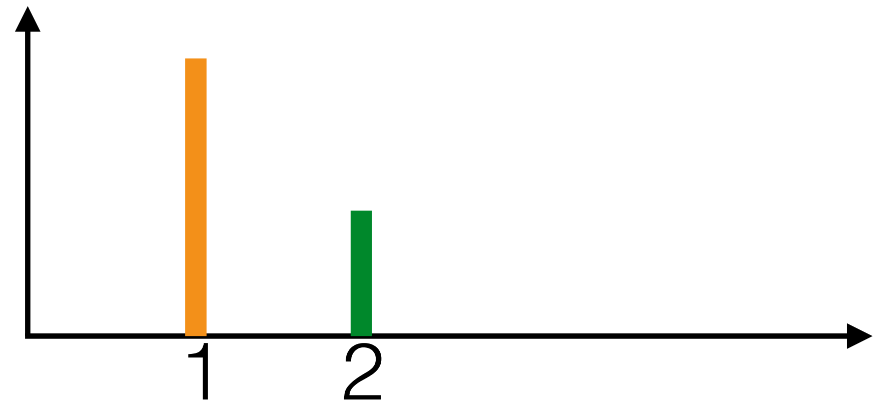

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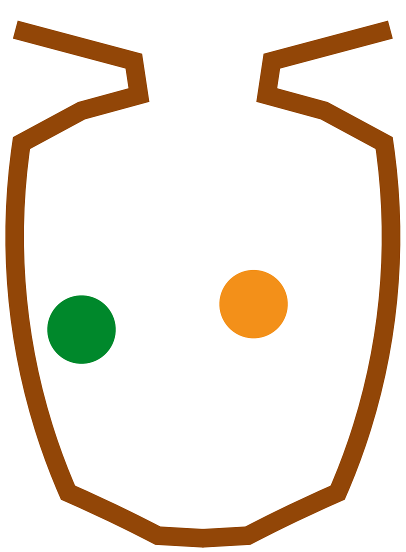

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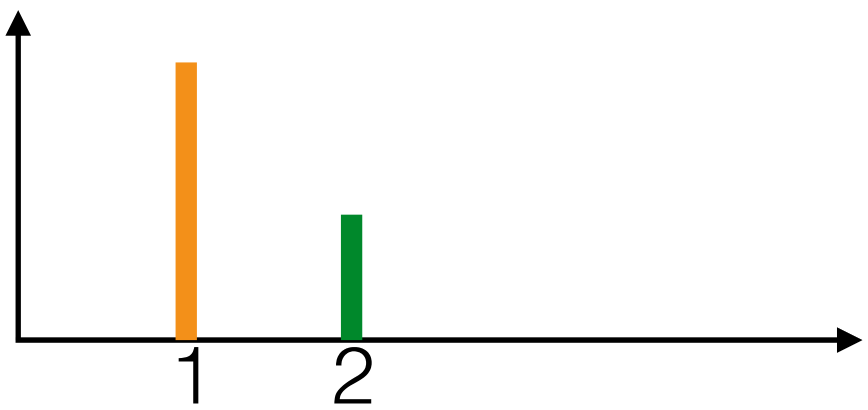

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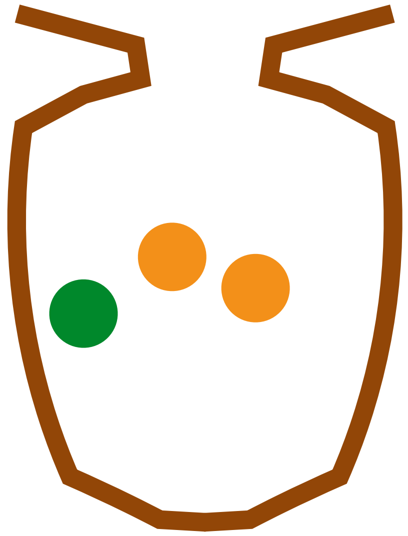

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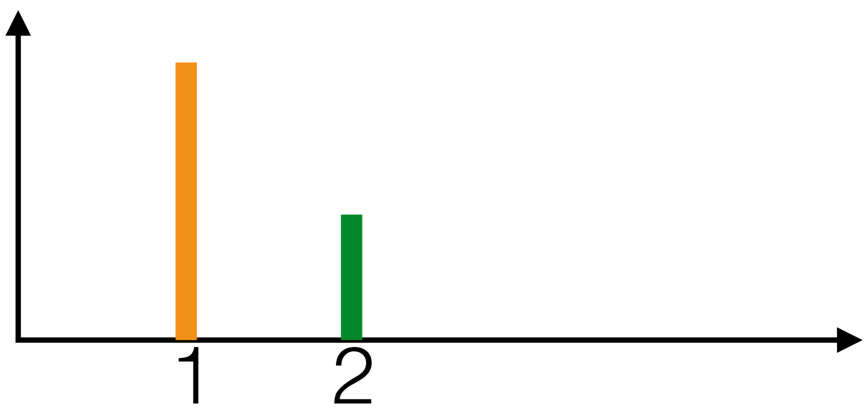

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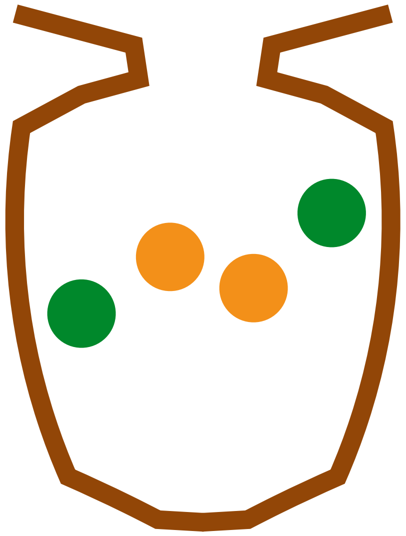

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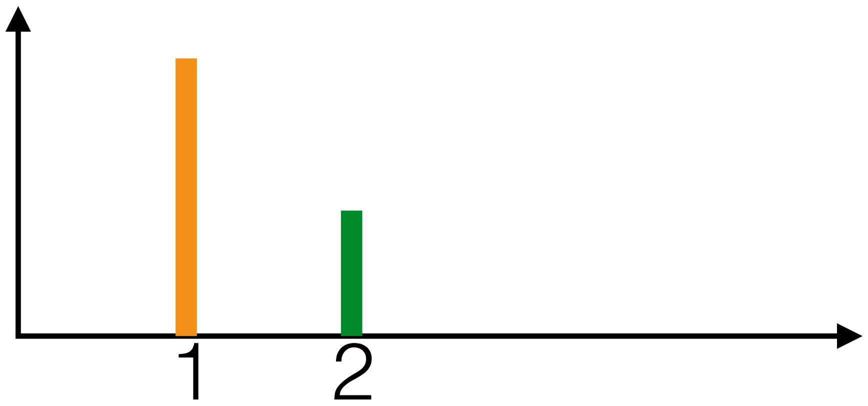

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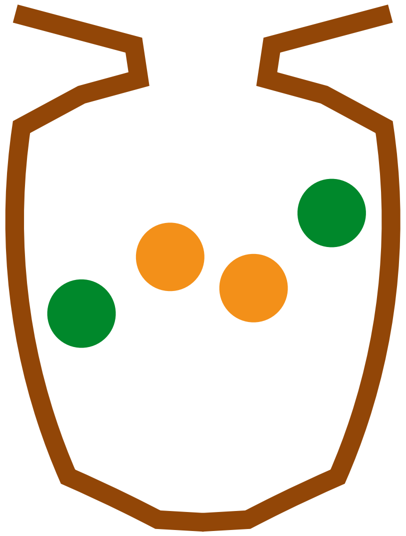

$$\lim_{n \to \infty} \frac{\#\ \text{orange}}{\#\ \text{total}}$$

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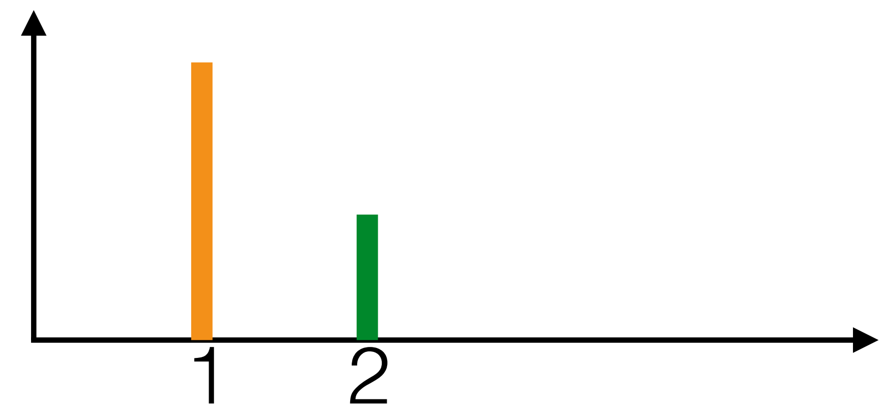

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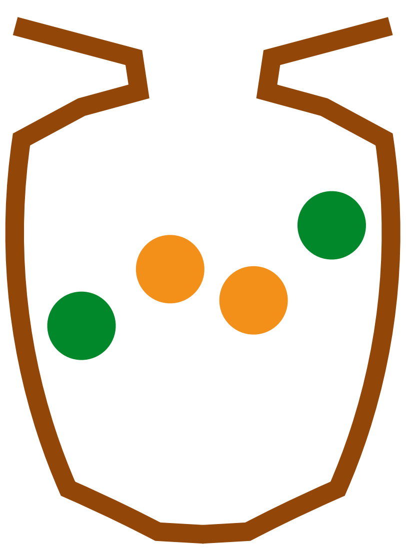

$$\lim_{n \to \infty} \frac{\#\ \text{orange}}{\#\ \text{total}} = \rho_{\text{orange}}$$

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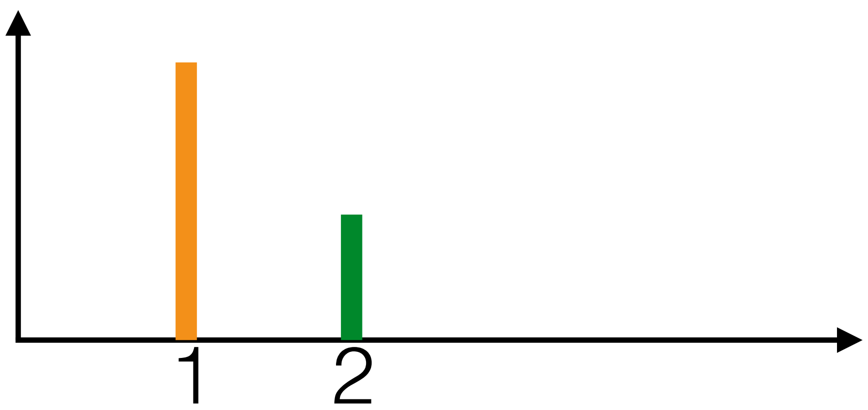

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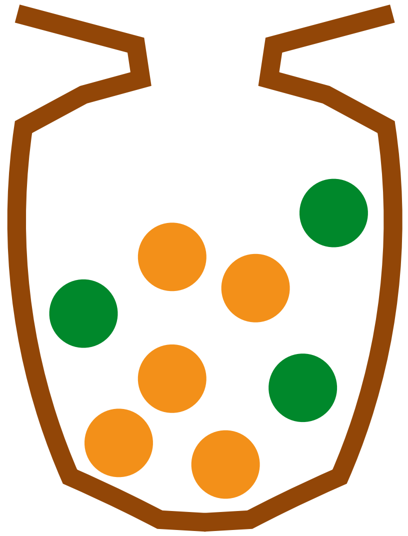

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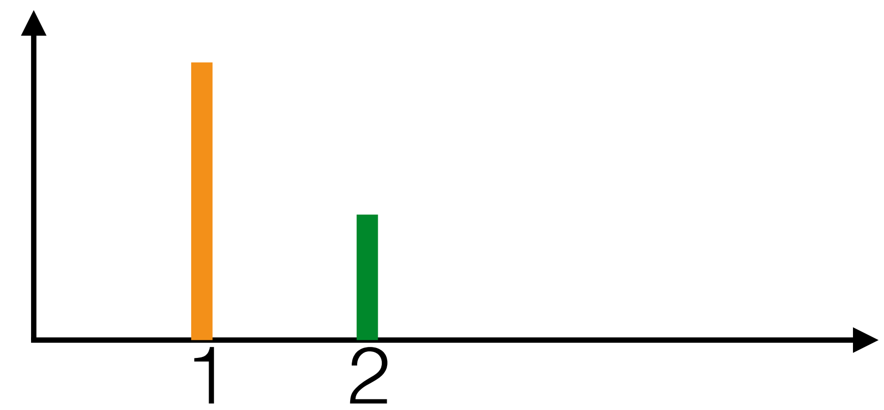

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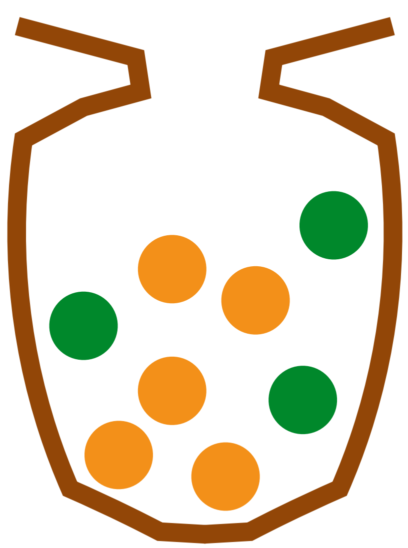

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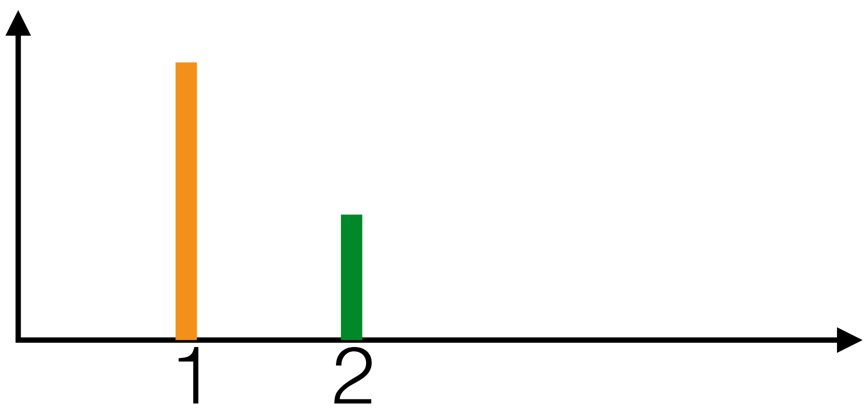

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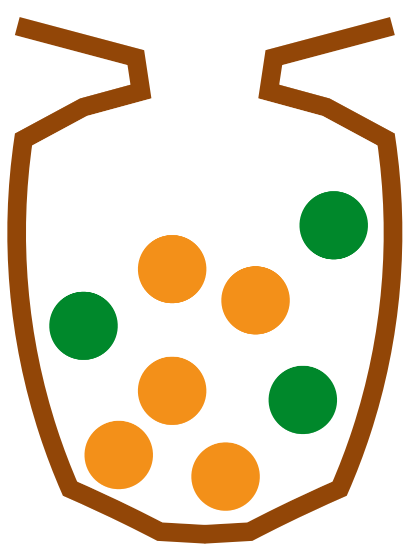

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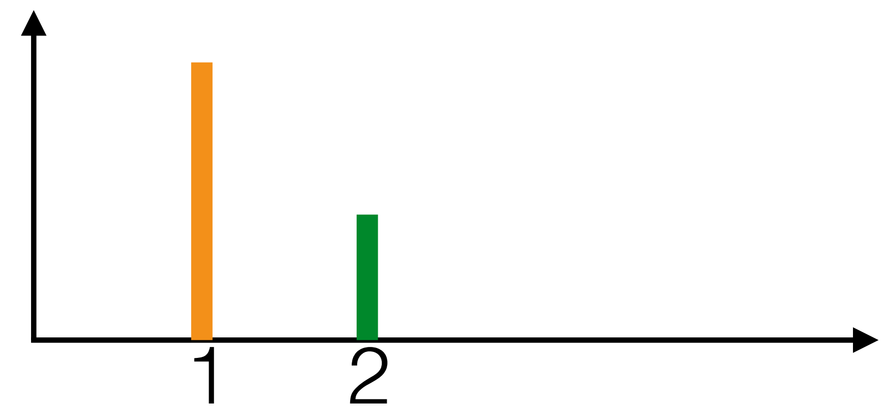

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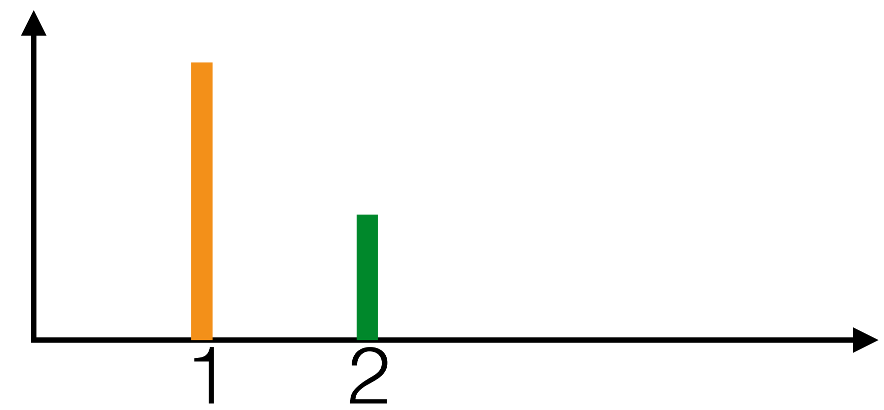

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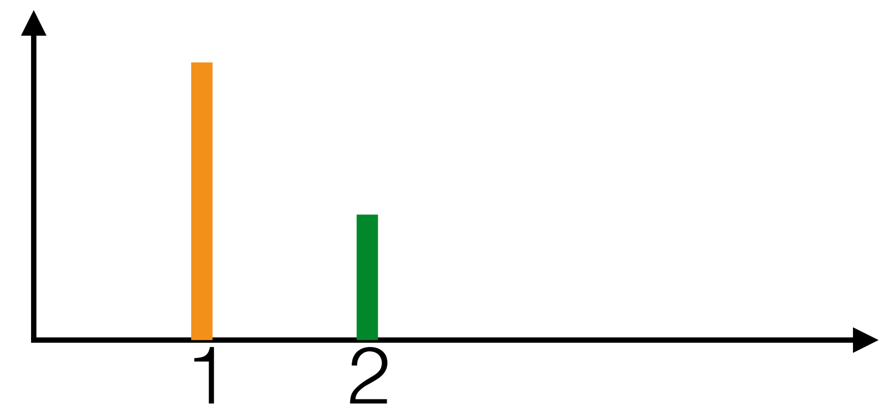

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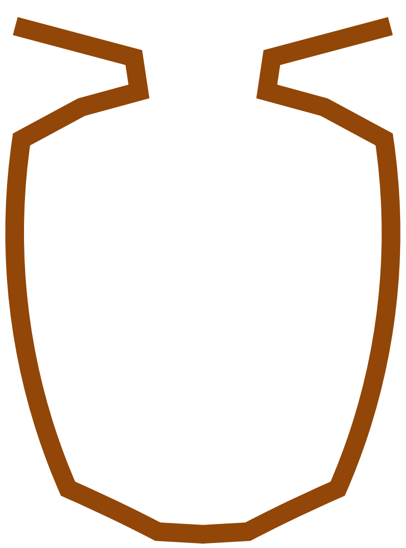

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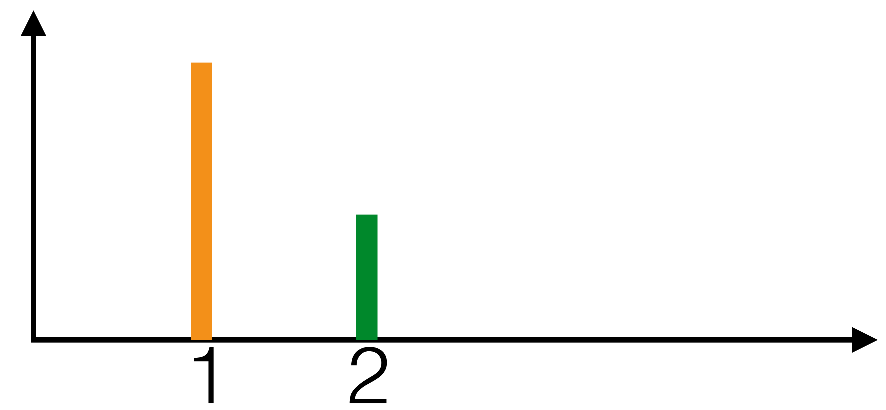

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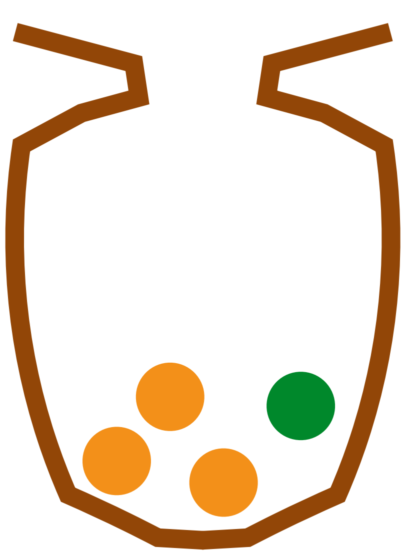

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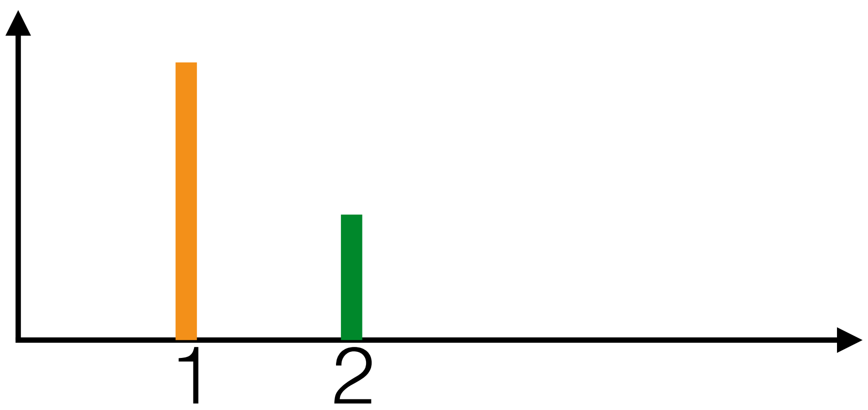

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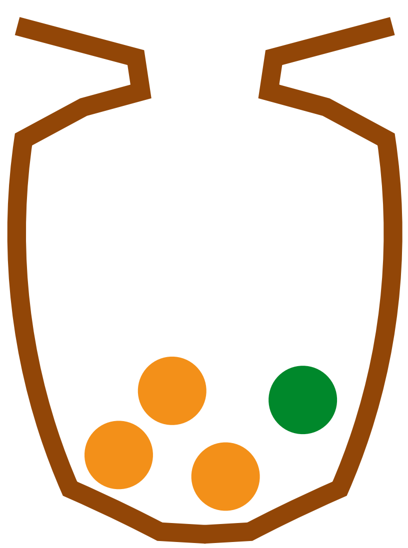
- Pólya urn
  - Choose any ball with equal probability



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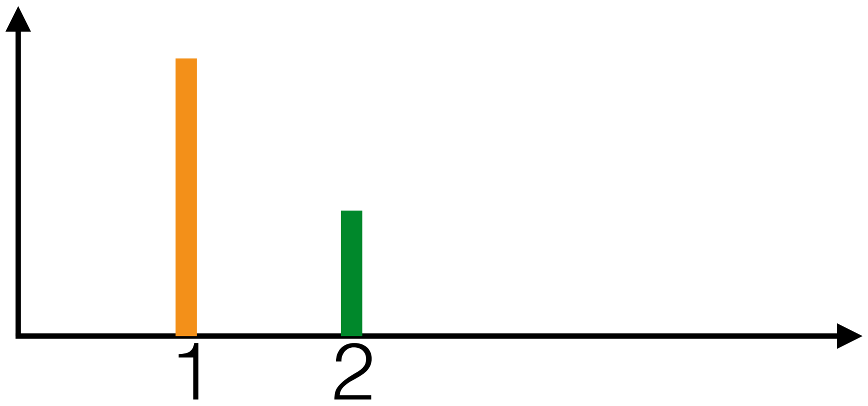

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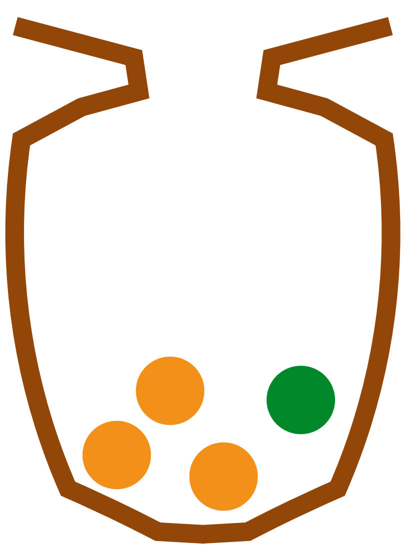
- Pólya urn
  - Choose any ball with equal probability
  - Replace and add ball of same color



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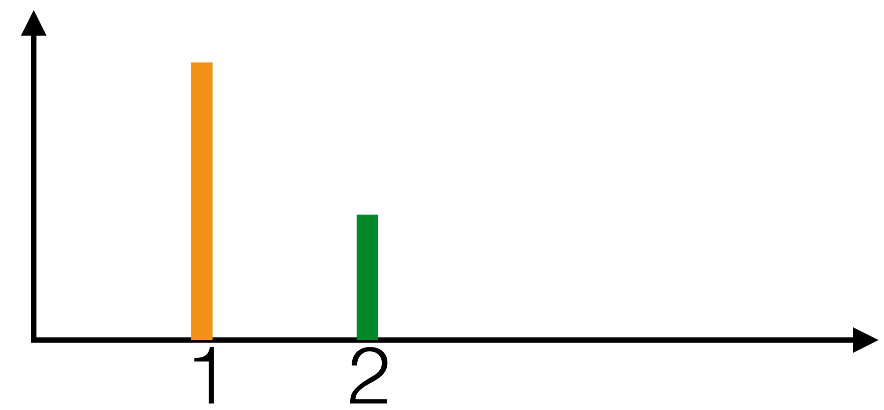

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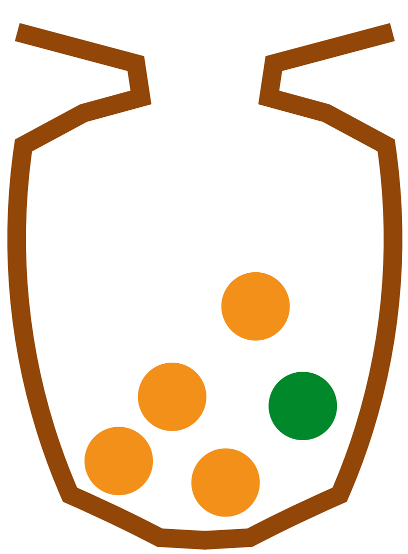
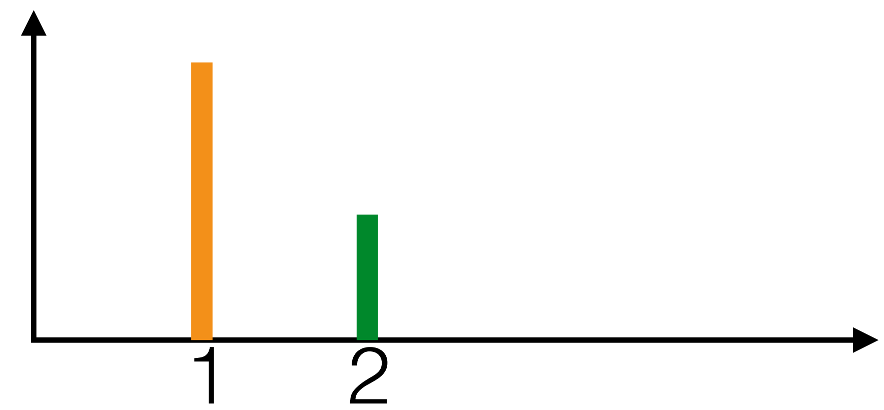


Marginal cluster assignments

- Integrate out the frequencies

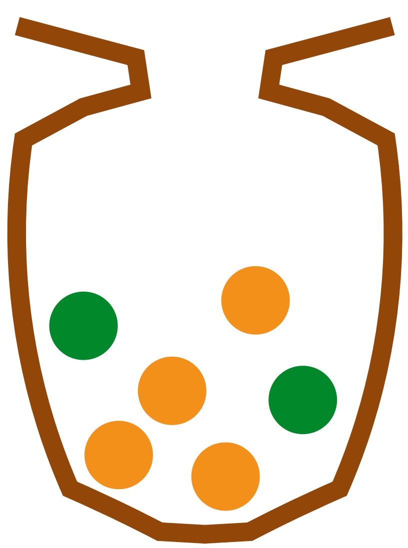
$$\rho_1 \sim \text{Beta}(a_1, a_2),\, z_n \overset{iid}{\sim} \text{Cat}(\rho_1, \rho_2)$$

$$p(z_n = 1 | z_1, \ldots, z_{n-1}) = \frac{a_{1,n}}{a_{1,n} + a_{2,n}}$$



$$a_{1,n} := a_1 + \sum_{m=1}^{n-1} \mathbf{1}\{z_m = 1\},\, a_{2,n} = a_2 + \sum_{m=1}^{n-1} \mathbf{1}\{z_m = 2\}$$

- Pólya urn
  - Choose any ball with equal probability
  - Replace and add ball of same color



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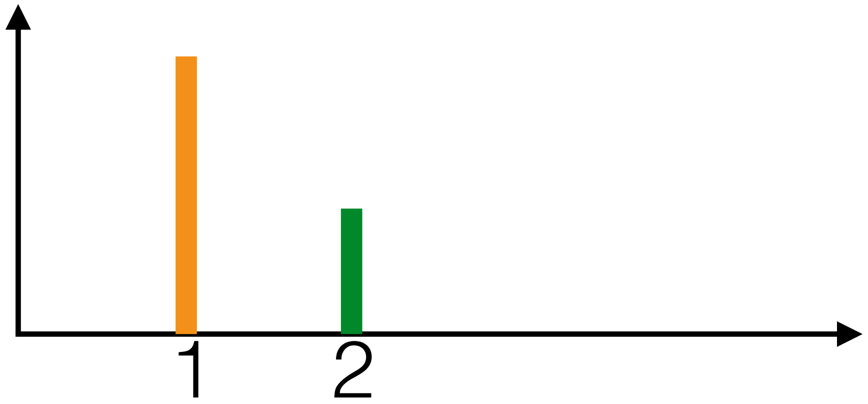

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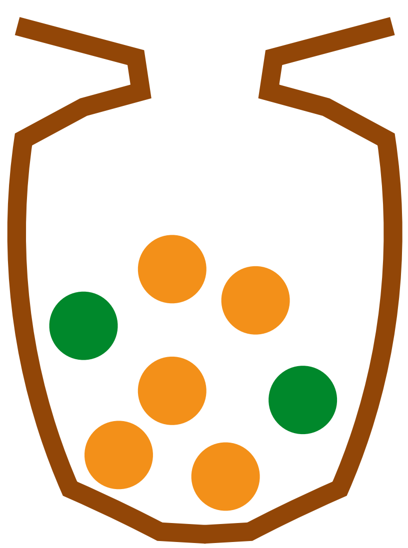

Marginal cluster assignments

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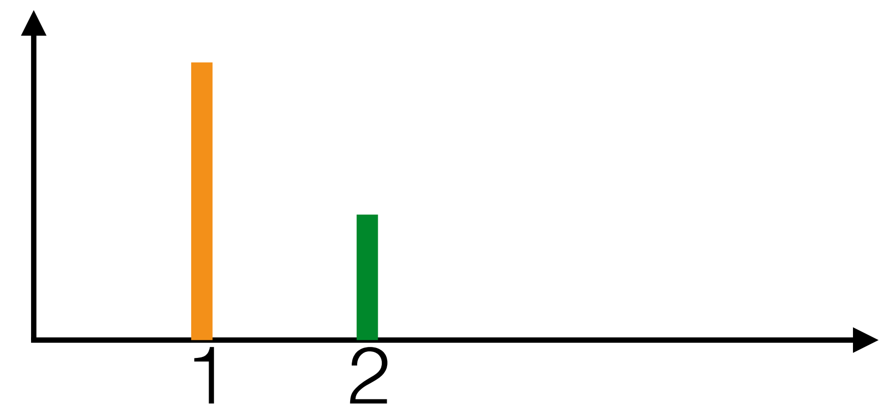

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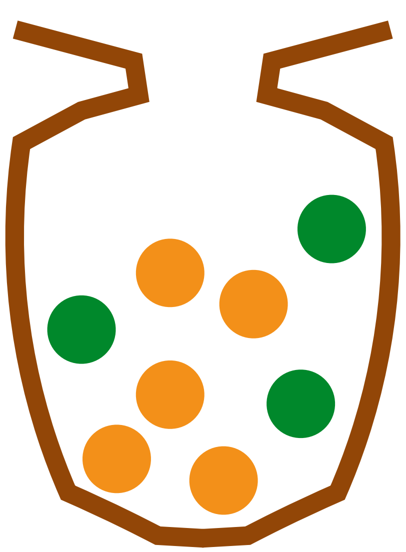

Marginal cluster assignments

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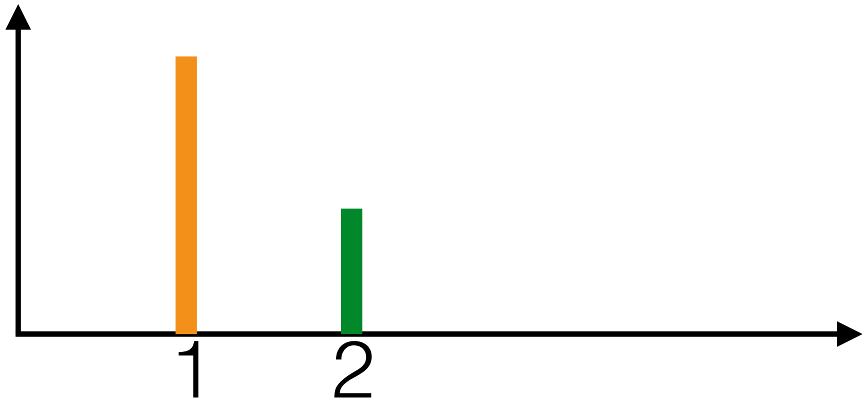

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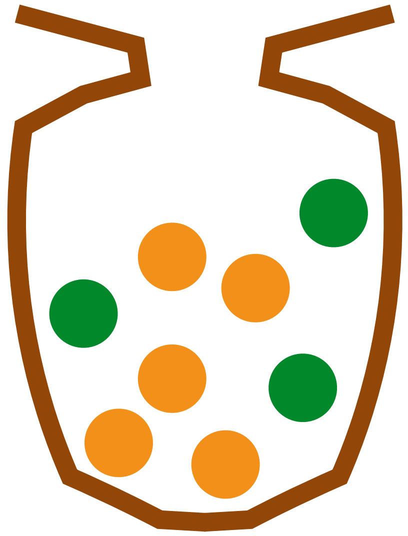

$$\lim_{n \to \infty} \frac{\#\ \text{orange}}{\#\ \text{total}}$$

Marginal cluster assignments

- Integrate out the frequencies

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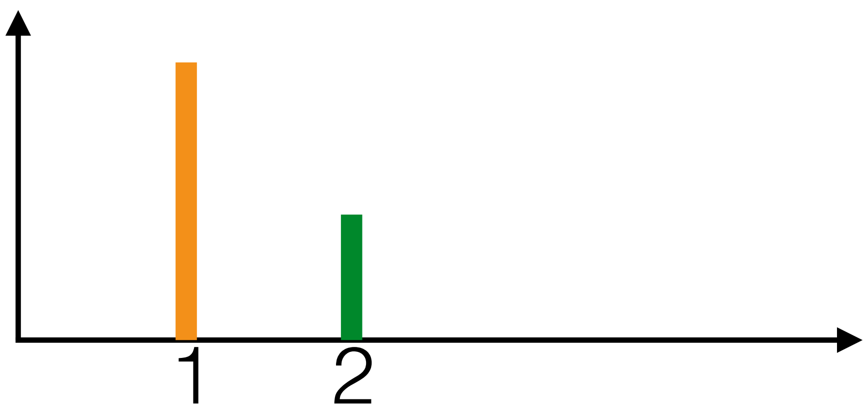

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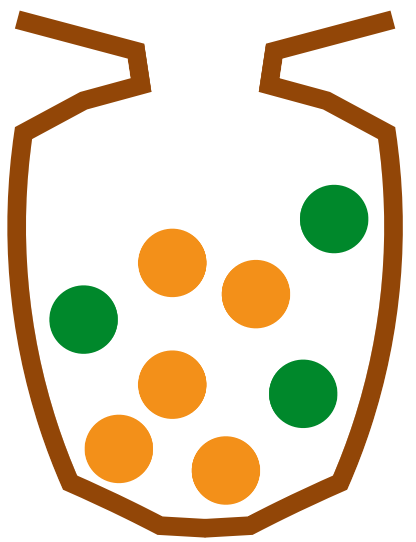
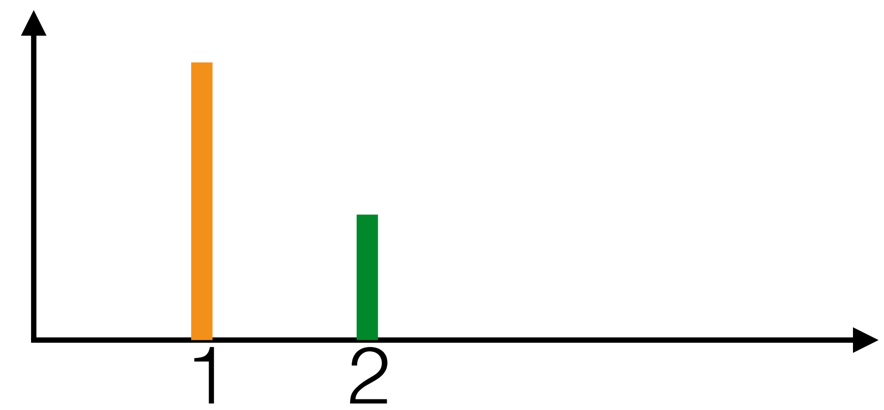


$$\lim_{n \to \infty} \frac{\#\ \text{orange}}{\#\ \text{total}} = \rho_{\text{orange}}$$

Marginal cluster assignments

- Integrate out the frequencies



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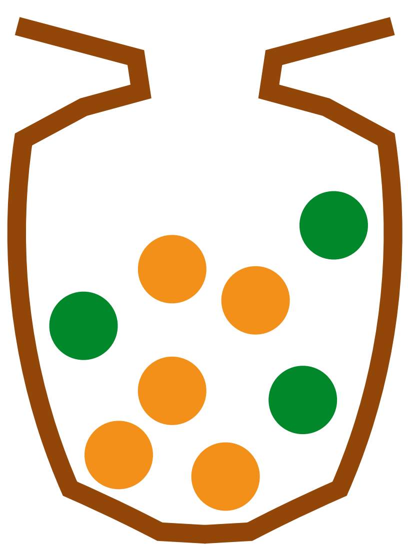
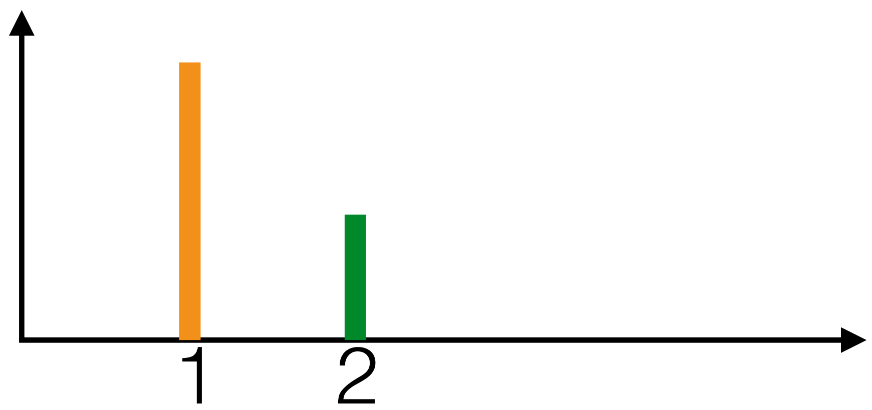


$$\lim_{n \to \infty} \frac{\#\ \text{orange}}{\#\ \text{total}} = \rho_{\text{orange}} \stackrel{d}{=} \mathrm{Beta}(a_{\text{orange}}, a_{\text{green}})$$

Marginal cluster assignments

- Integrate out the frequencies



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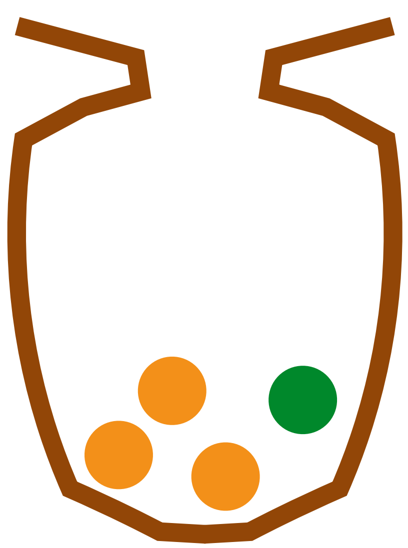

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Marginal cluster assignments

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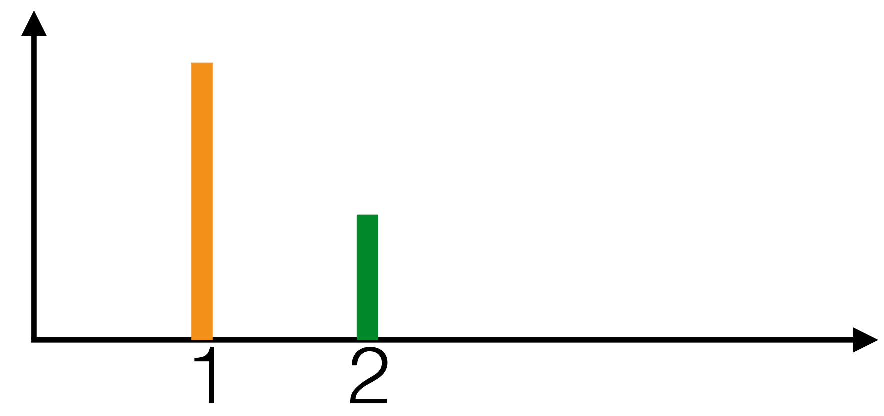

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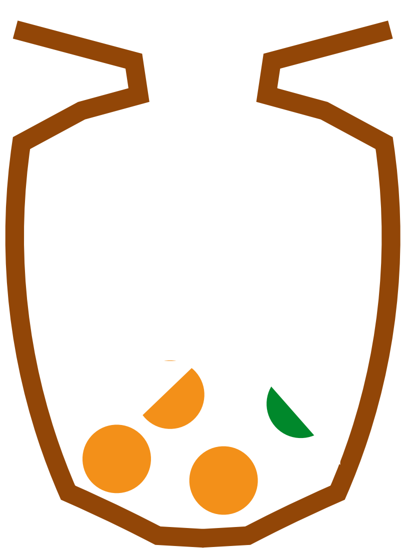

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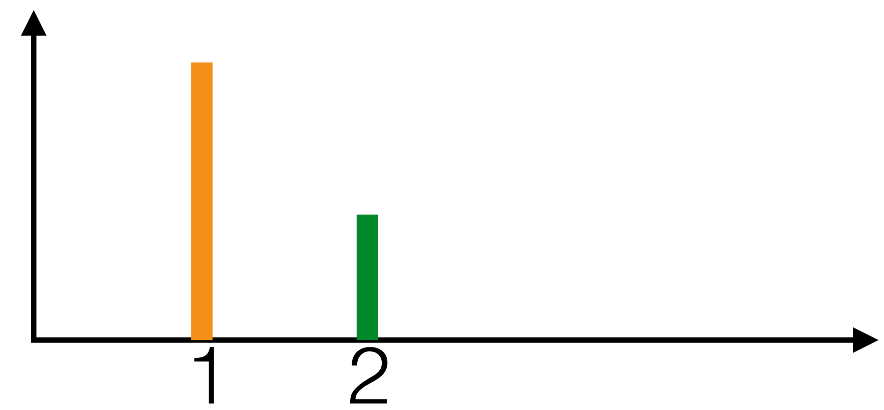

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    - Choose any ball with prob proportional to its mass
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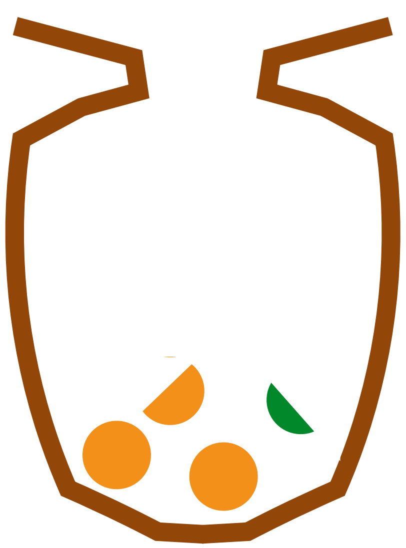

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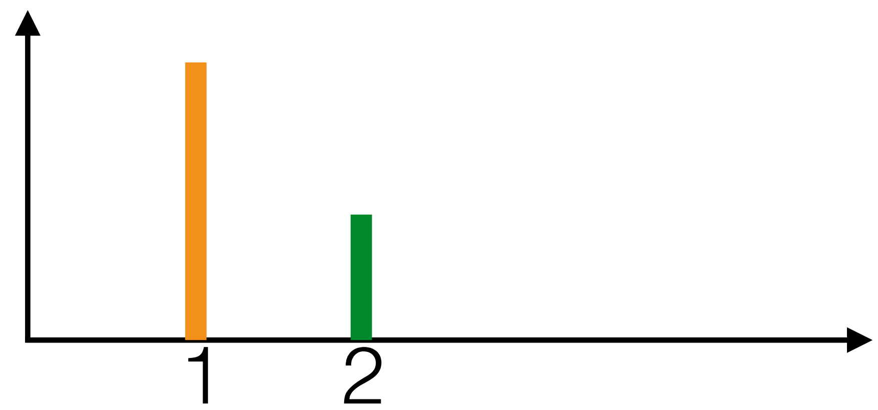

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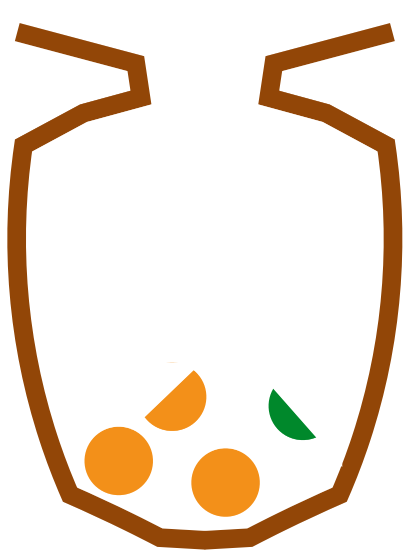

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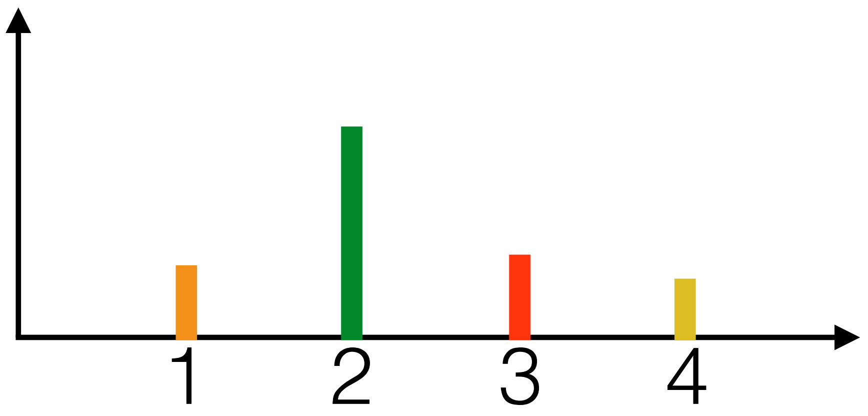
$$\text{PolyaUrn}(a_{\text{orange}}, a_{\text{green}})$$

Marginal cluster assignments

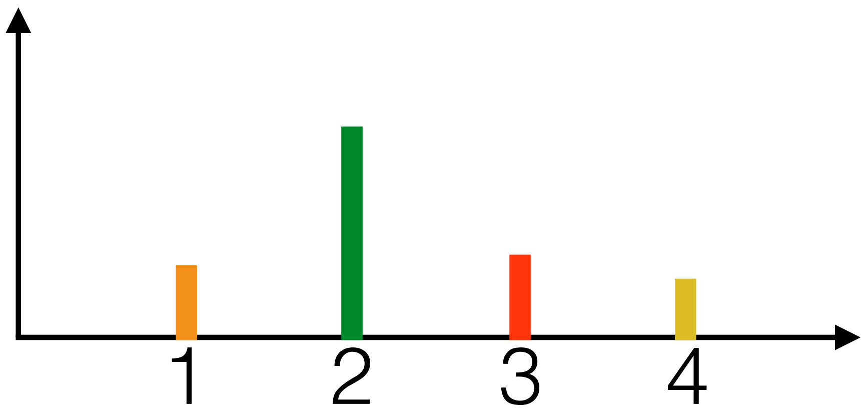
- Integrate out the frequencies



Marginal cluster assignments

- Integrate out the frequencies

$$\rho_{1:K} \sim \text{Dirichlet}(a_{1:K}),\, z_n \overset{iid}{\sim} \text{Cat}(\rho_{1:K})$$



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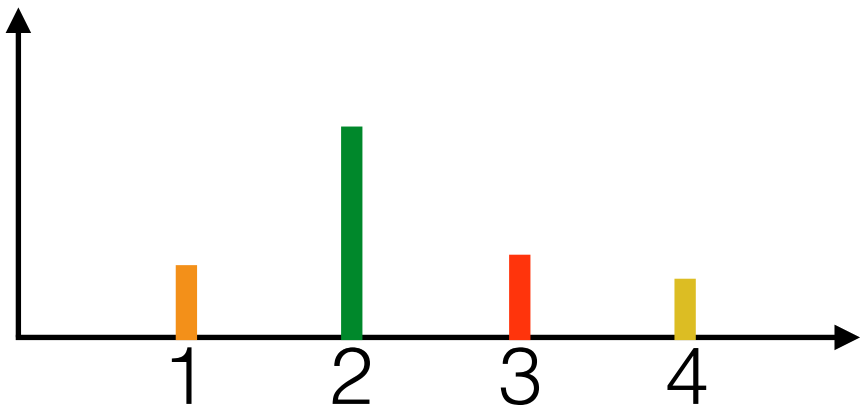

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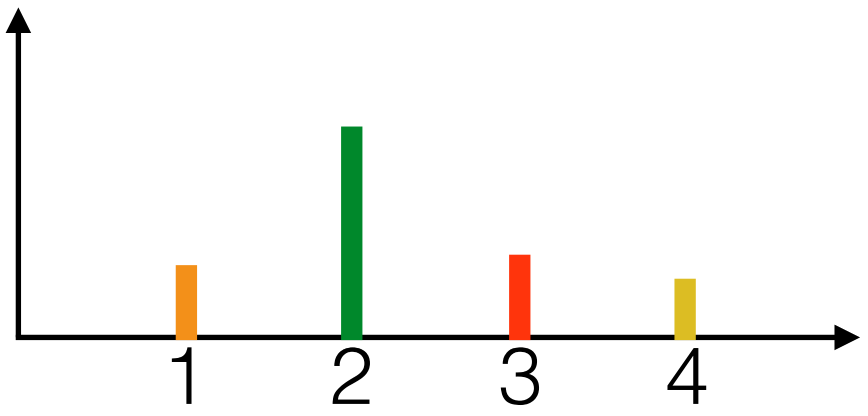

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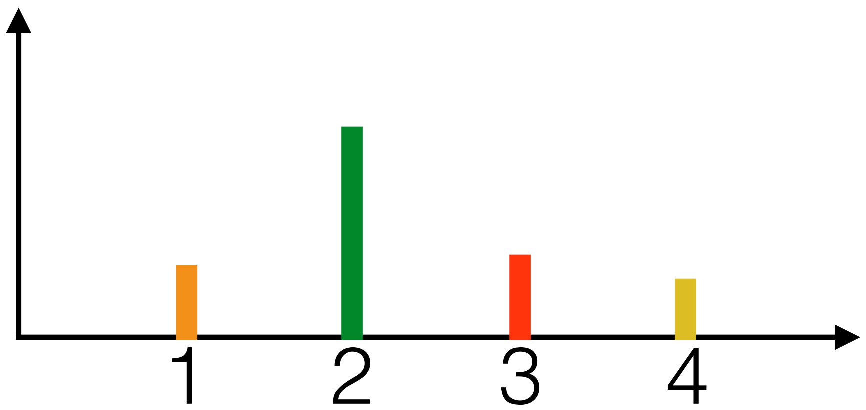

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- multivariate Pólya urn

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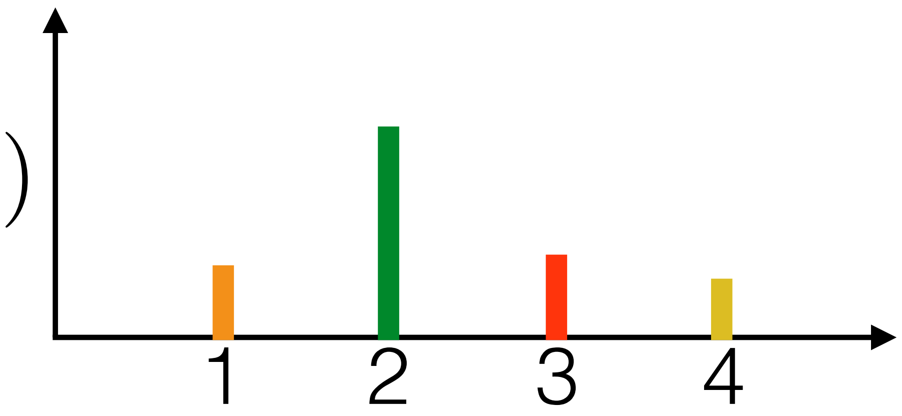

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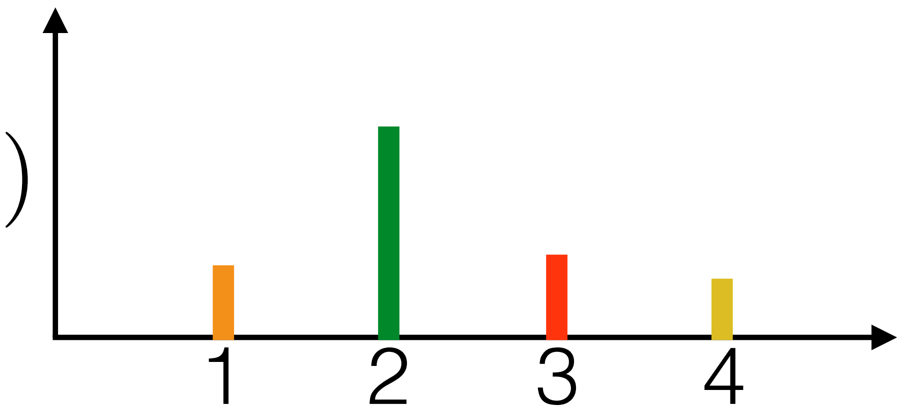

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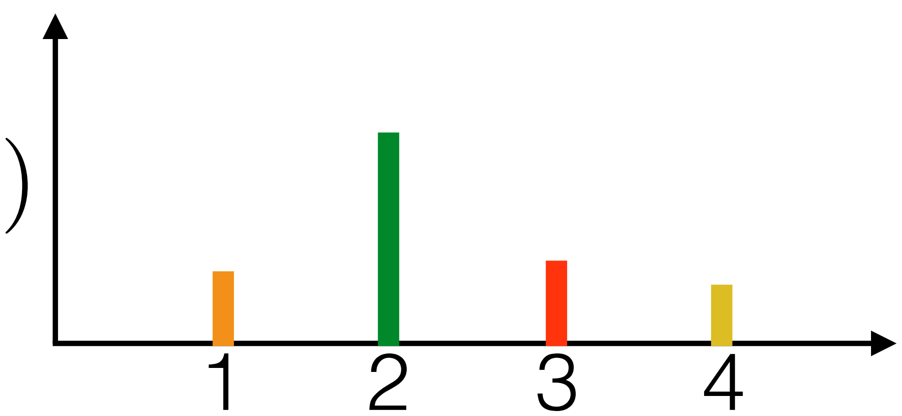

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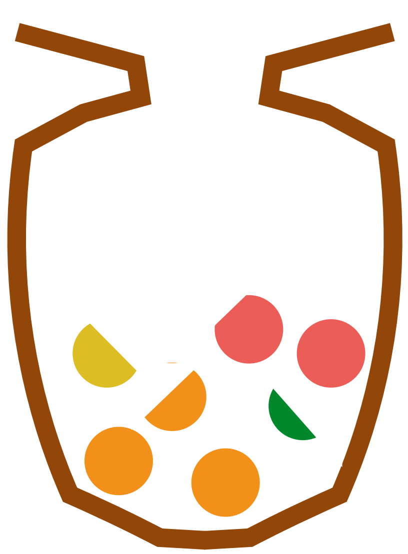

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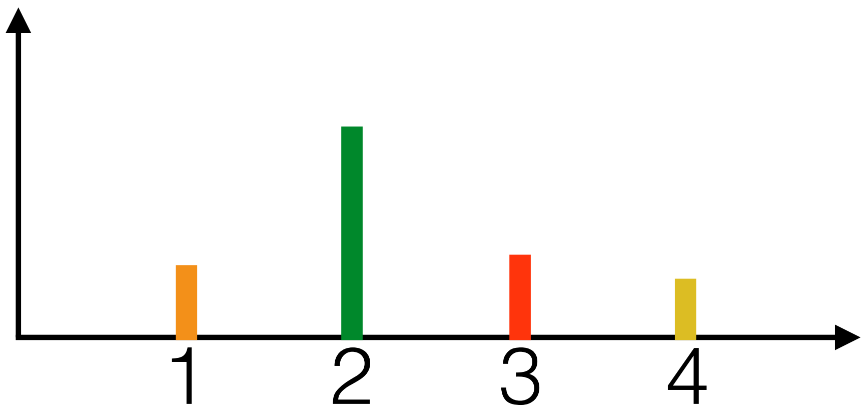

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$$\lim_{n \to \infty} \frac{(\#\ \text{orange}, \#\ \text{green}, \#\ \text{red}, \#\ \text{yellow})}{\#\ \text{total}}$$

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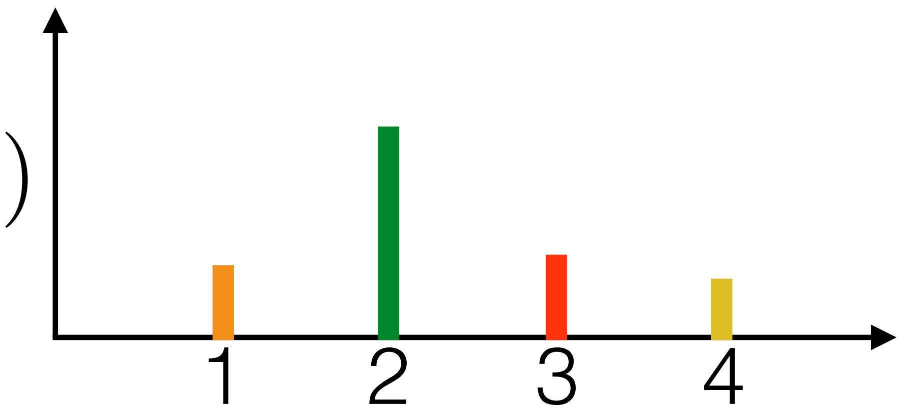

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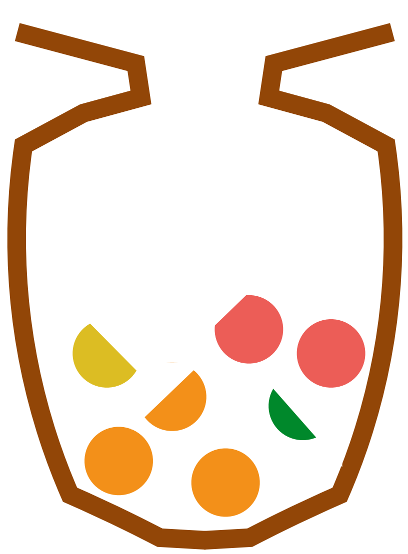

$$\lim_{n\to\infty} \frac{(\#\text{ orange},\, \#\text{ green},\, \#\text{ red},\, \#\text{ yellow})}{\#\text{ total}}$$

$$\to (\rho_{\text{orange}}, \rho_{\text{green}}, \rho_{\text{red}}, \rho_{\text{yellow}})$$

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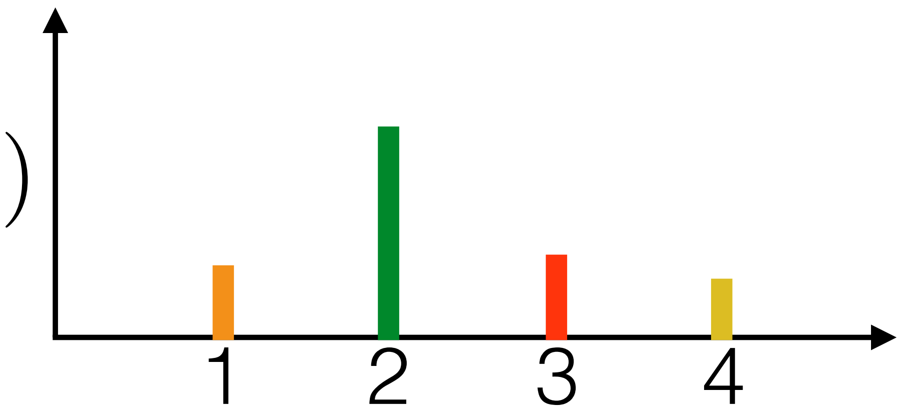

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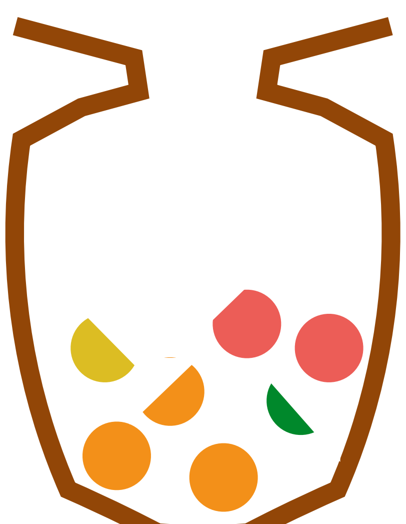

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$$\to (\rho_{\text{orange}}, \rho_{\text{green}}, \rho_{\text{red}}, \rho_{\text{yellow}})$$

$$\stackrel{d}{=} \text{Dirichlet}(a_{\text{orange}}, a_{\text{green}}, a_{\text{red}}, a_{\text{yellow}})$$

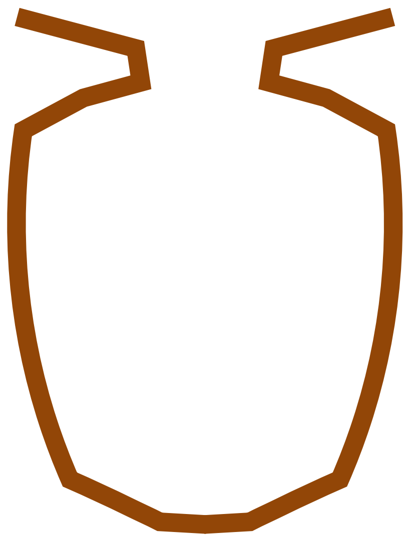
Marginal cluster assignments

- Hoppe urn / Blackwell-MacQueen urn

[Blackwell, MacQueen 1973; Hoppe 1984]

Marginal cluster assignments

- Hoppe urn / Blackwell-MacQueen urn



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Marginal cluster assignments

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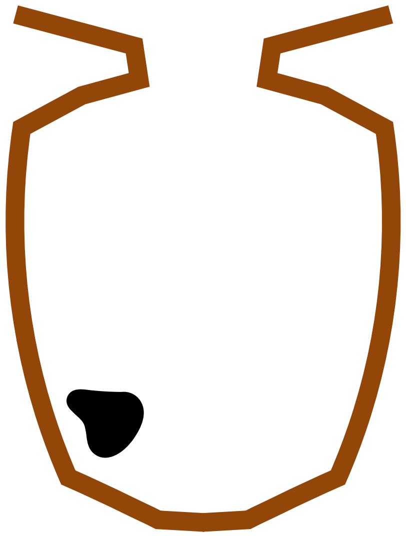
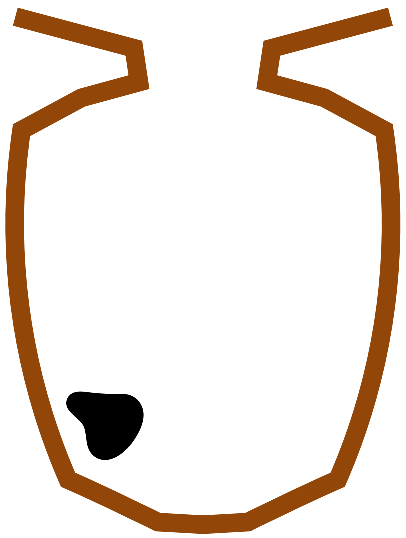


[Blackwell, MacQueen 1973; Hoppe 1984]

Marginal cluster assignments

- Hoppe urn / Blackwell-MacQueen urn

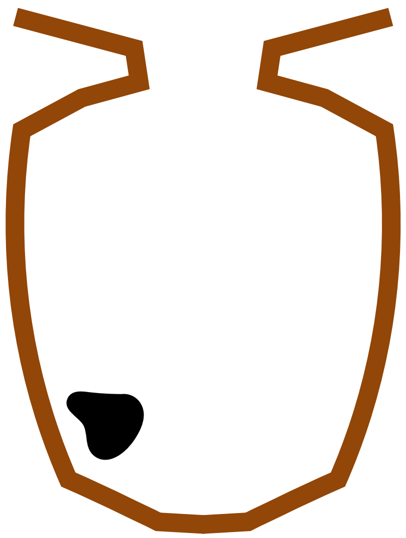


  - Choose ball with prob proportional to its mass

[Blackwell, MacQueen 1973; Hoppe 1984]

Marginal cluster assignments

- Hoppe urn / Blackwell-MacQueen urn



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Marginal cluster assignments

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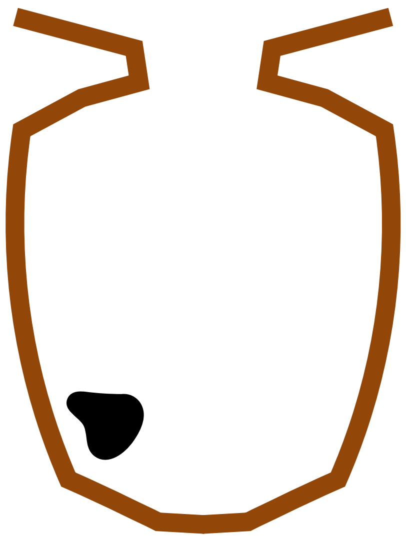

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Marginal cluster assignments

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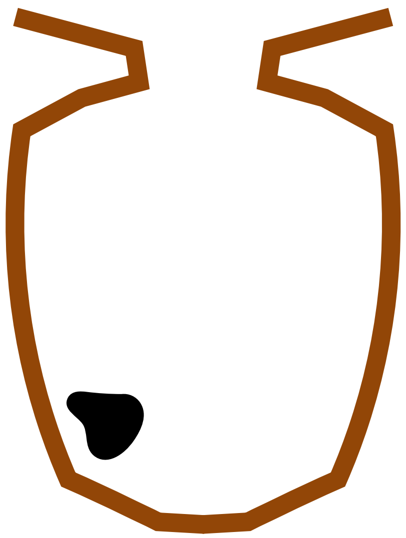

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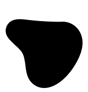
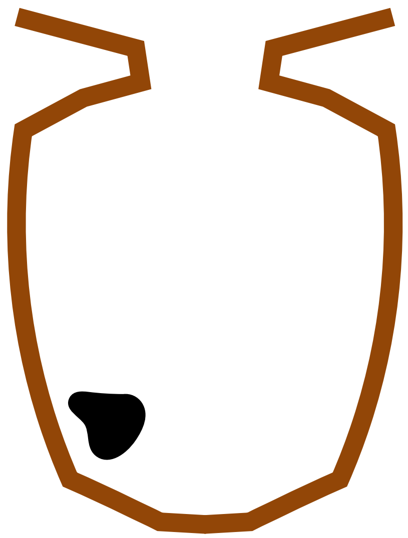
Step 0



[Blackwell, MacQueen 1973; Hoppe 1984]

Marginal cluster assignments

- Hoppe urn / Blackwell-MacQueen urn



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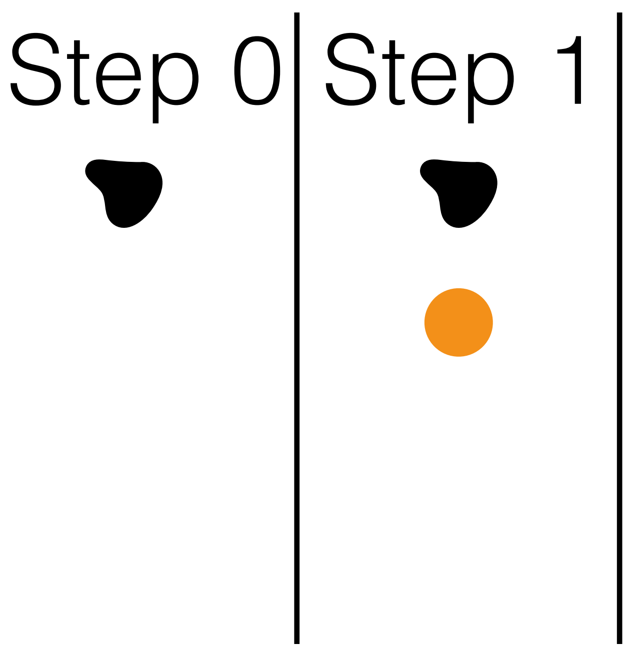

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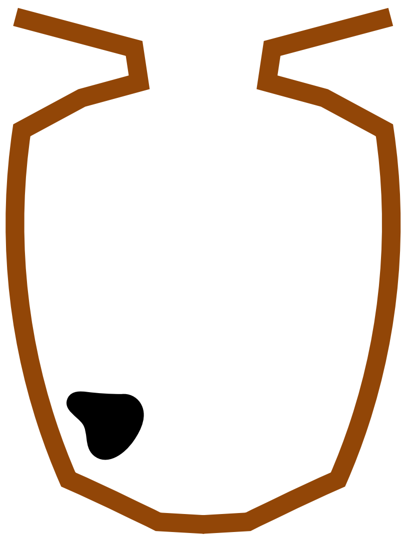
Marginal cluster assignments

- Hoppe urn / Blackwell-MacQueen urn



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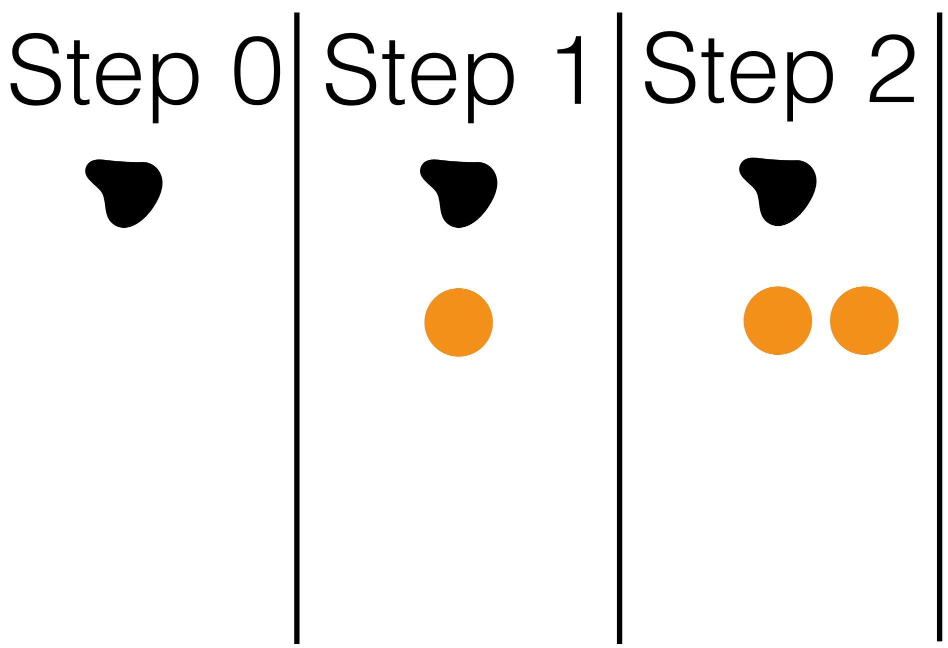

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Marginal cluster assignments

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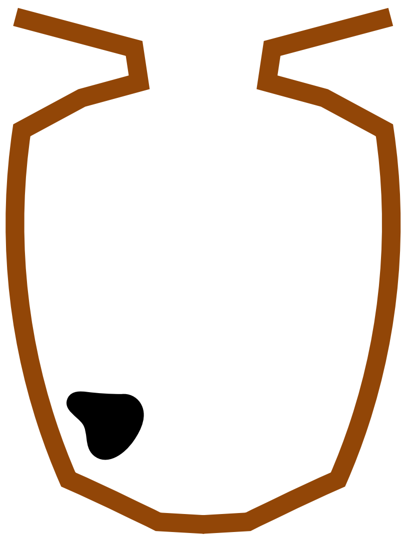

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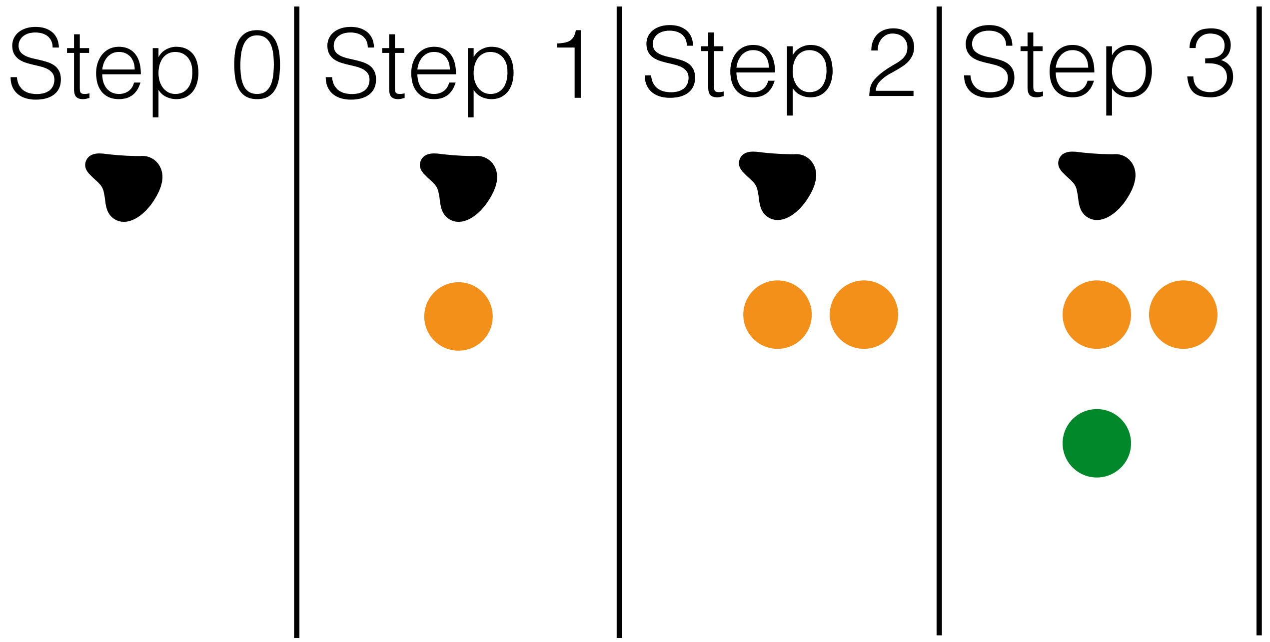

[Blackwell, MacQueen 1973; Hoppe 1984]

Marginal cluster assignments

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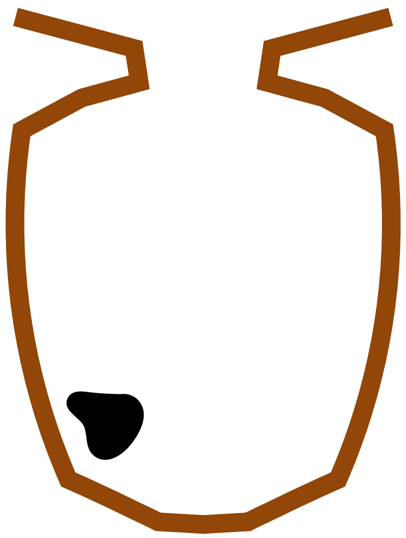

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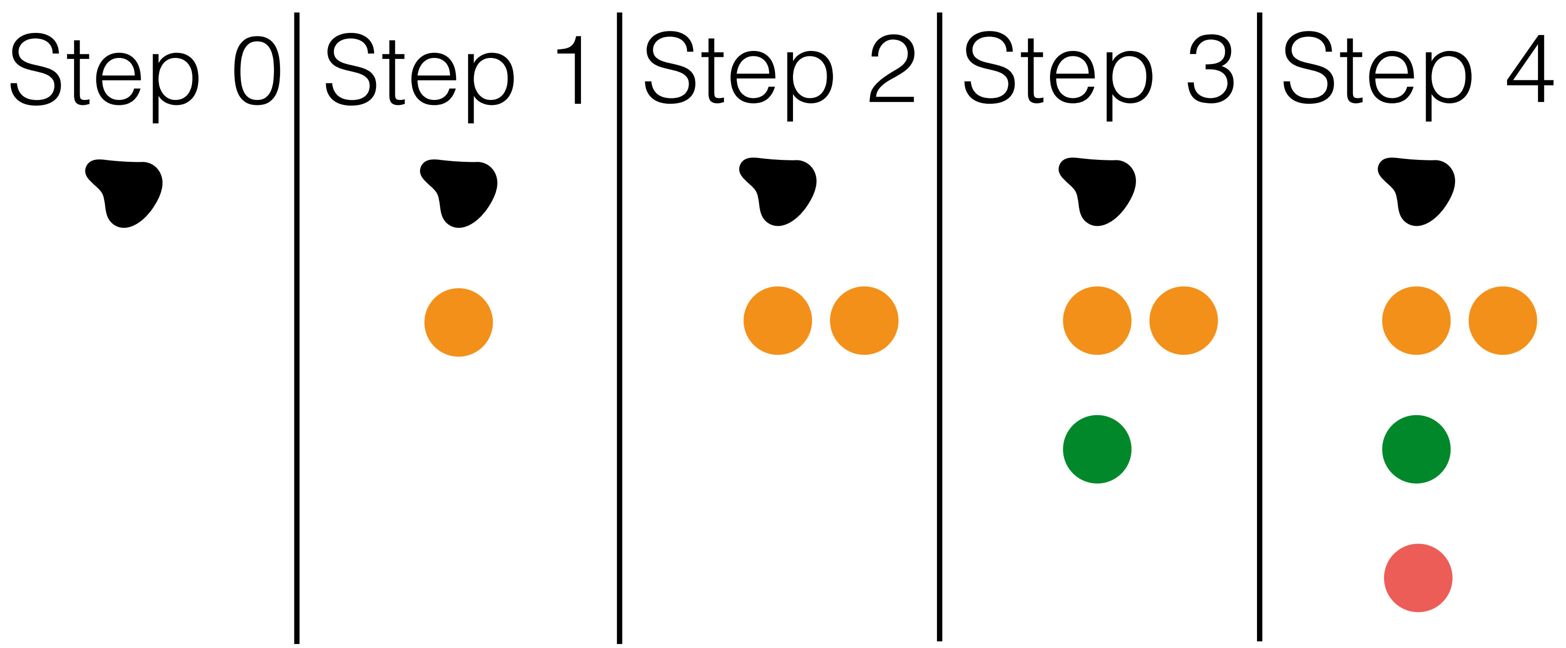

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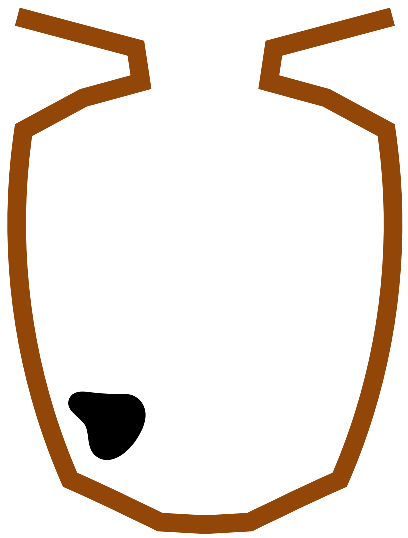
Marginal cluster assignments

- Hoppe urn / Blackwell-MacQueen urn



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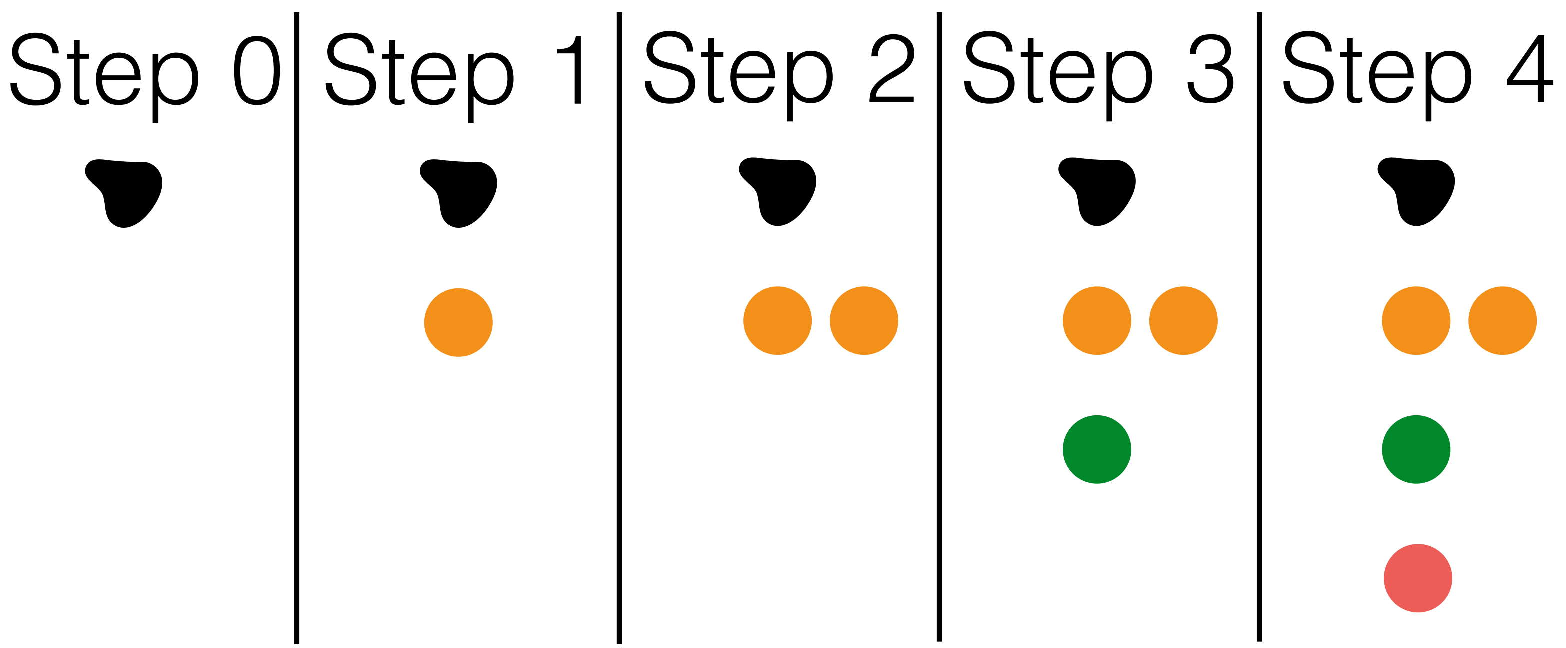
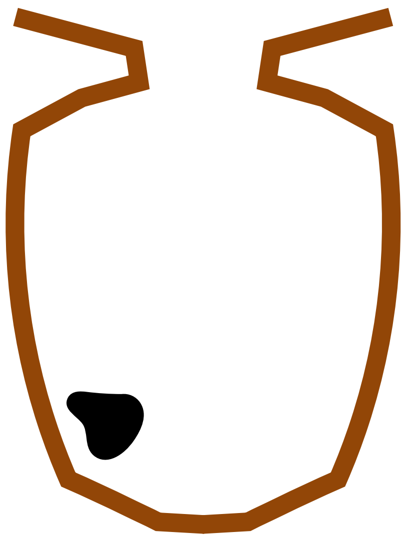


$$(\#\text{orange}, \#\text{other}) = \text{PolyaUrn}(1, \alpha)$$

Marginal cluster assignments

- Hoppe urn / Blackwell-MacQueen urn



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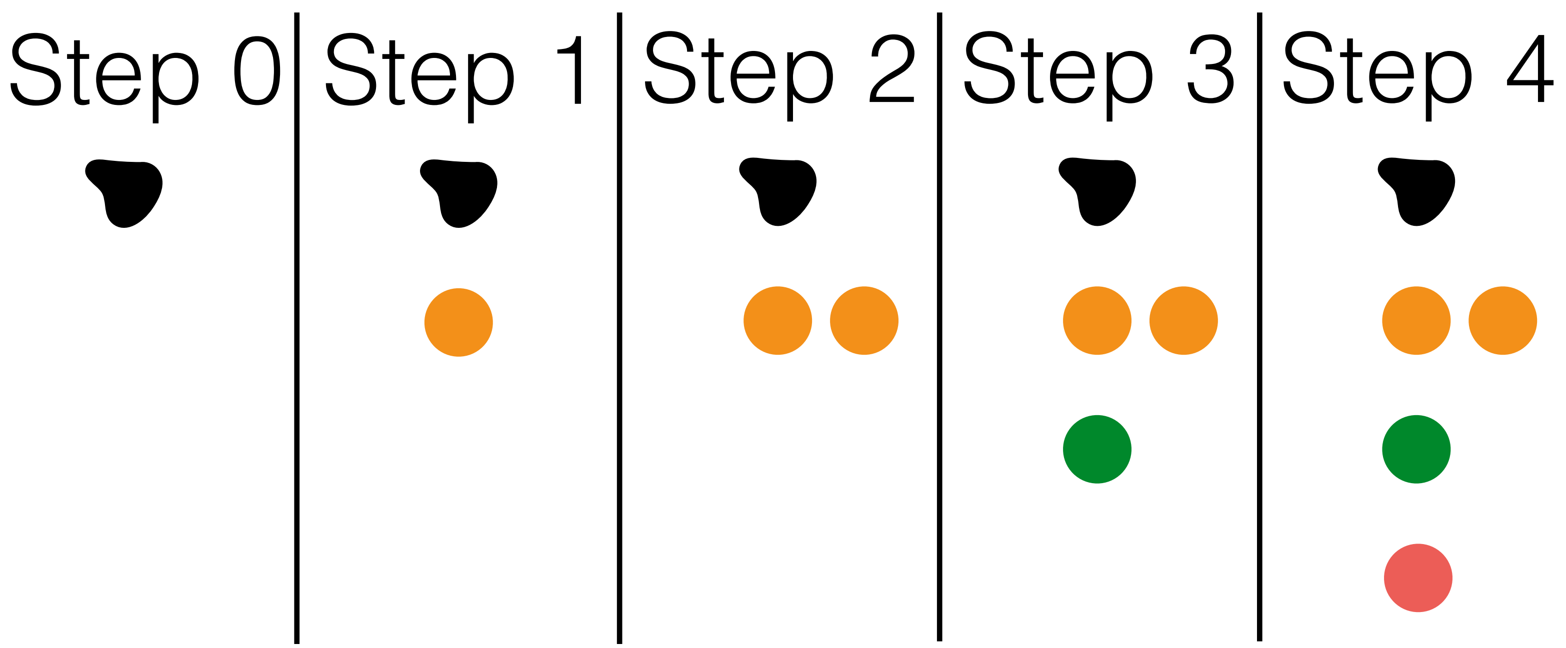
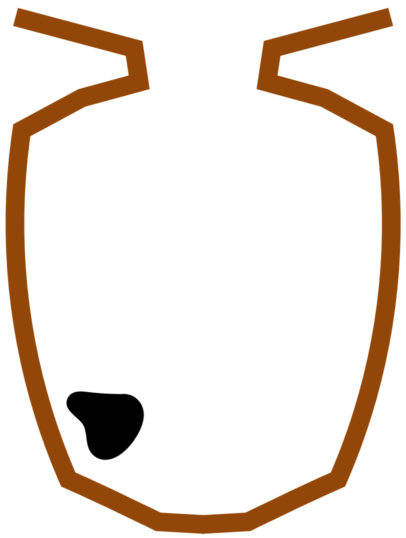


$(\#\text{orange}, \#\text{other}) = \text{PolyaUrn}(1, \alpha)$

- not orange: $(\#\text{green}, \#\text{other}) = \text{PolyaUrn}(1, \alpha)$

Marginal cluster assignments

- Hoppe urn / Blackwell-MacQueen urn



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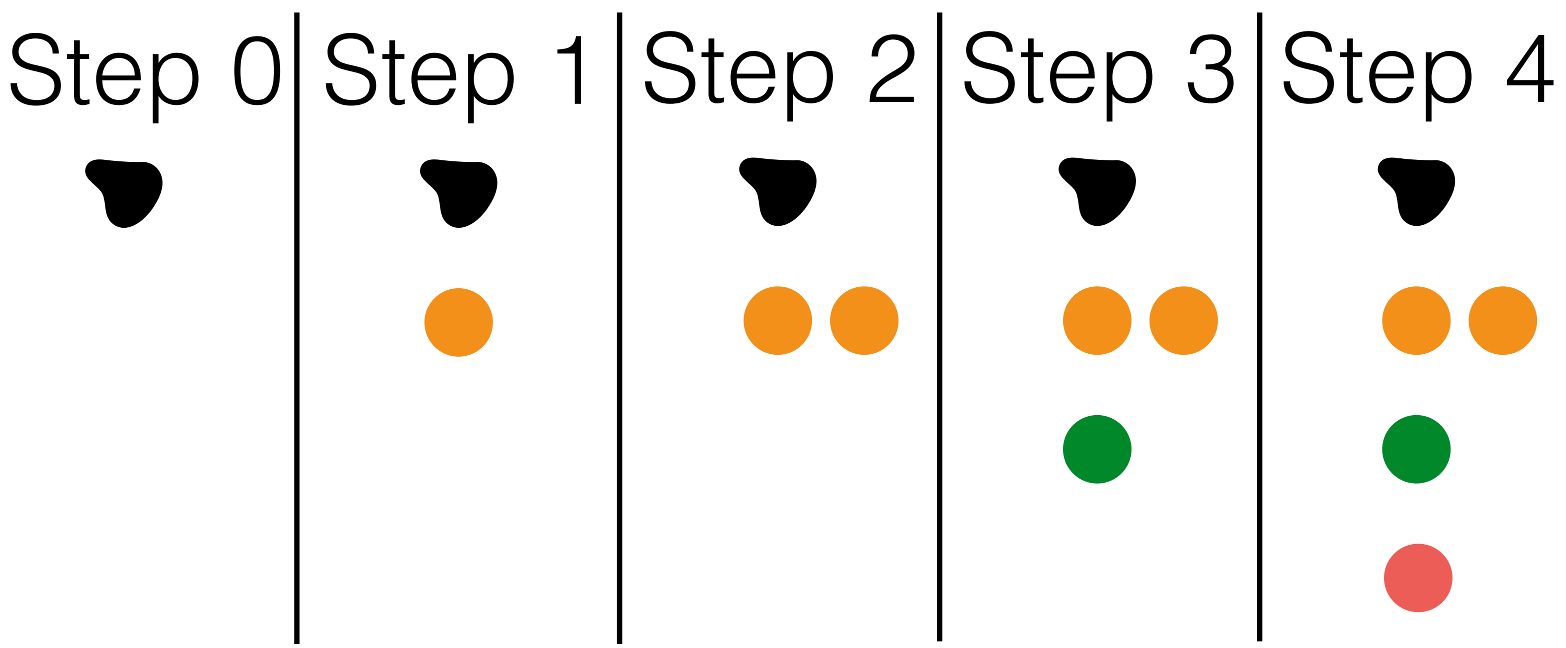
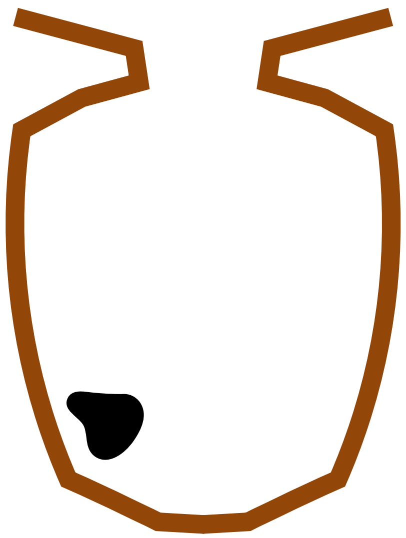


$$(\#\text{orange}, \#\text{other}) = \text{PolyaUrn}(1, \alpha)$$

- not orange: $(\#\text{green}, \#\text{other}) = \text{PolyaUrn}(1, \alpha)$
- not orange, green: $(\#\text{red}, \#\text{other}) = \text{PolyaUrn}(1, \alpha)$

Marginal cluster assignments

- Hoppe urn / Blackwell-MacQueen urn



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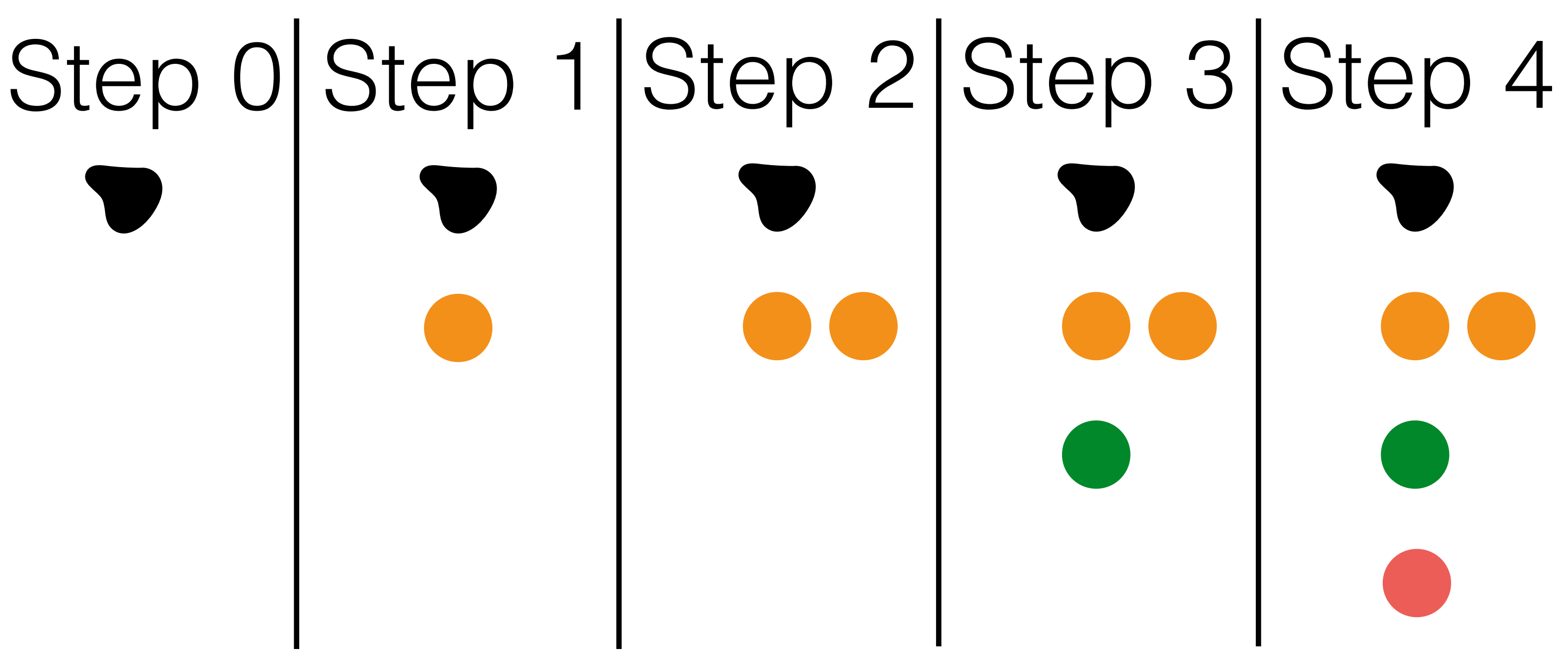

$$V_k \overset{iid}{\sim} \text{Beta}(1, \alpha)$$

$$(\#\text{orange}, \#\text{other}) = \text{PolyaUrn}(1, \alpha)$$

- not orange: $(\#\text{green}, \#\text{other}) = \text{PolyaUrn}(1, \alpha)$
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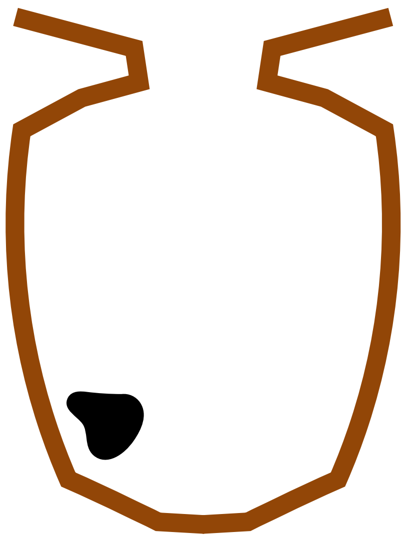
Marginal cluster assignments

- Hoppe urn / Blackwell-MacQueen urn



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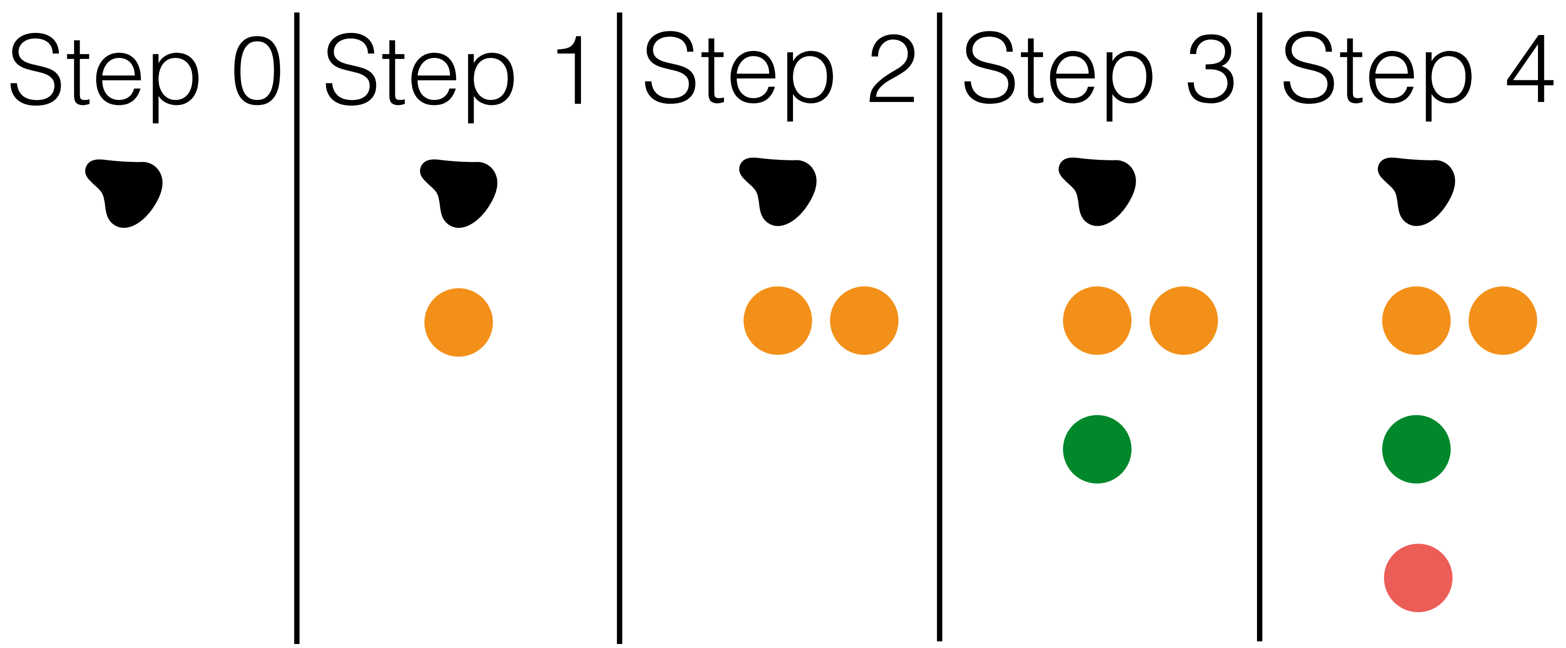

$$V_k \stackrel{iid}{\sim} \mathrm{Beta}(1, \alpha)$$

$$\rho_1 = V_1$$

$$(\#\mathrm{orange}, \#\mathrm{other}) = \mathrm{PolyaUrn}(1, \alpha)$$

- not orange: $(\#\mathrm{green}, \#\mathrm{other}) = \mathrm{PolyaUrn}(1, \alpha)$
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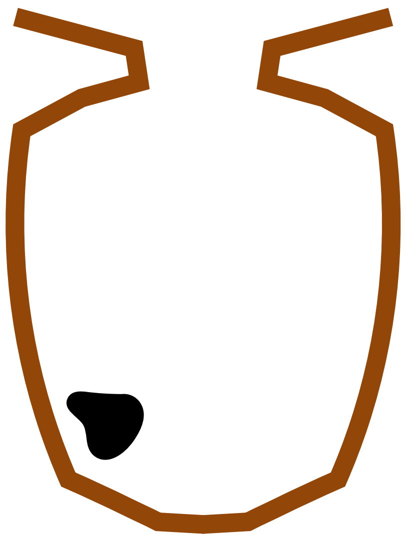
Marginal cluster assignments

- Hoppe urn / Blackwell-MacQueen urn



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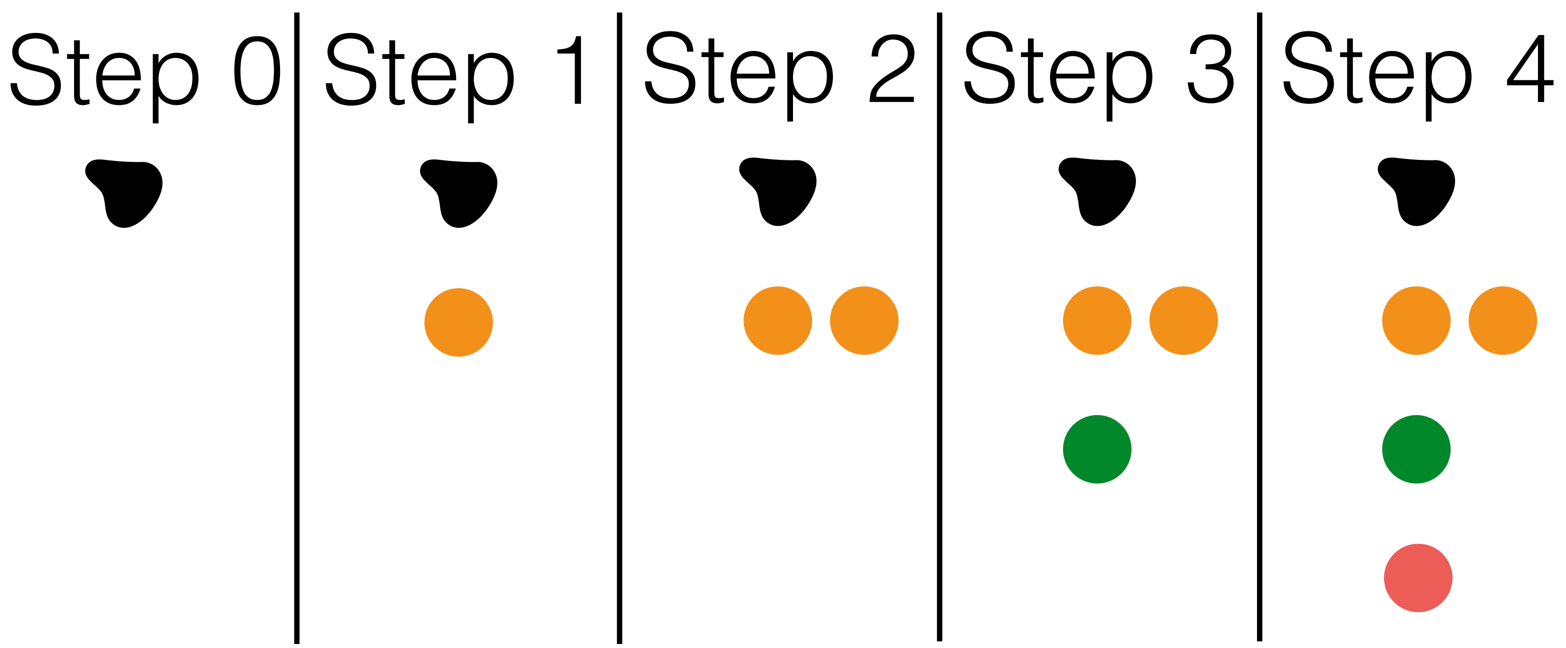

$$V_k \stackrel{iid}{\sim} \text{Beta}(1, \alpha)$$

$$\rho_1 = V_1$$

$$\rho_2 = (1 - V_1)V_2$$

$$(\#\text{orange}, \#\text{other}) = \text{PolyaUrn}(1, \alpha)$$

- not orange: $(\#\text{green}, \#\text{other}) = \text{PolyaUrn}(1, \alpha)$
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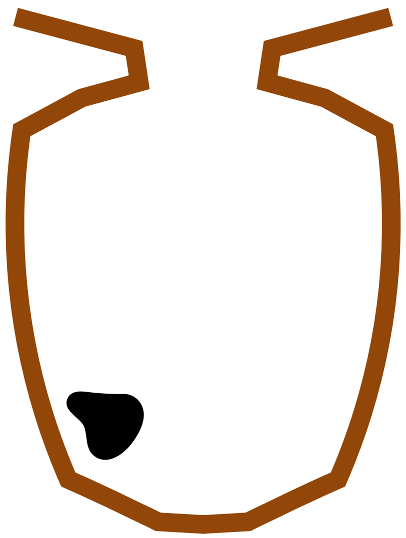
Marginal cluster assignments

- Hoppe urn / Blackwell-MacQueen urn



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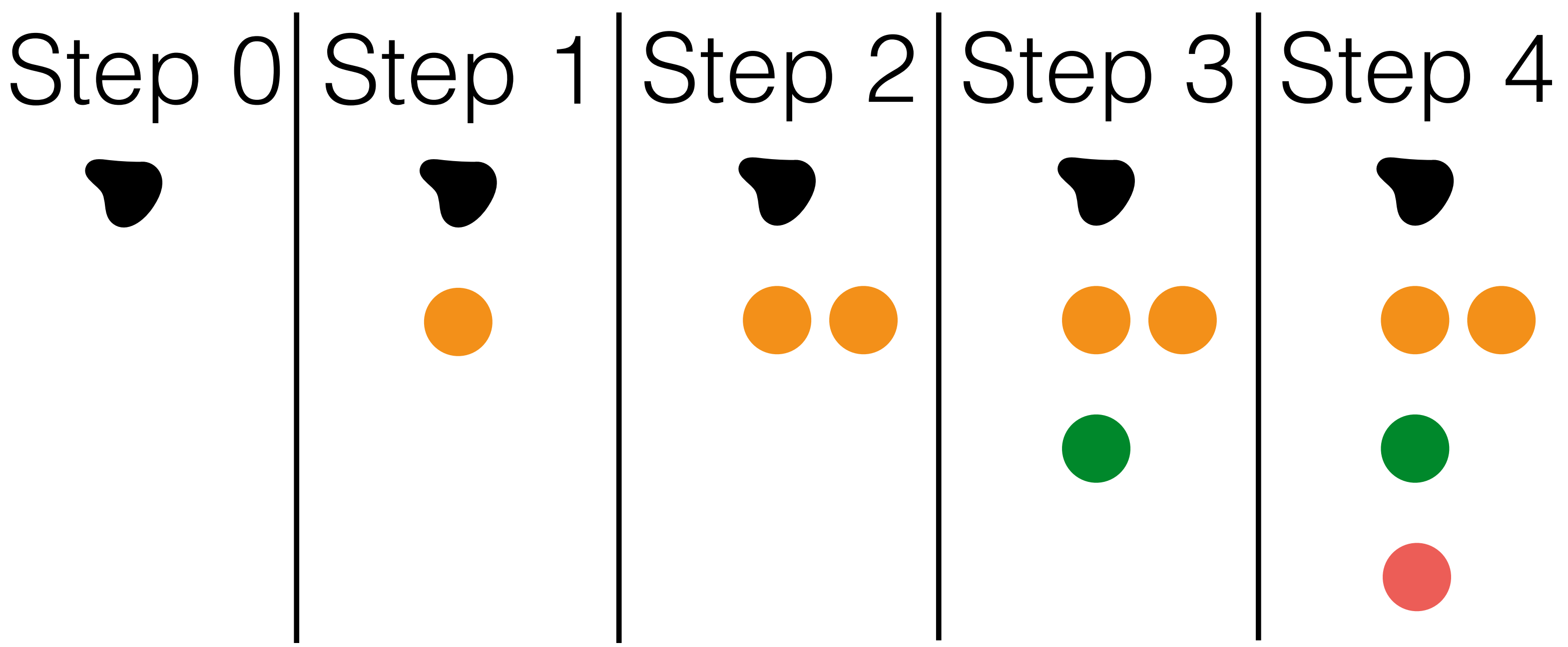

$$V_k \overset{iid}{\sim} \text{Beta}(1, \alpha)$$

$$\rho_1 = V_1$$

$$\rho_2 = (1 - V_1)V_2$$

$$\rho_3 = [\prod_{k=1}^{2}(1 - V_k)]V_3$$

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Chinese restaurant process

Chinese restaurant process

- Same thing we just did

Chinese restaurant process

- Same thing we just did
- Each customer walks into the restaurant

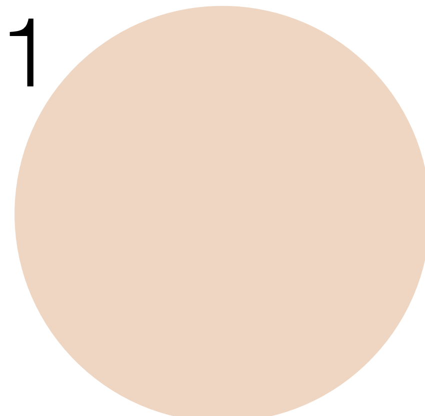
Chinese restaurant process

- Same thing we just did
- Each customer walks into the restaurant
    - Sits at existing table with prob proportional to # people there

Chinese restaurant process

- Same thing we just did
- Each customer walks into the restaurant
  - Sits at existing table with prob proportional to # people there
  - Forms new table with prob proportional to α

Chinese restaurant process



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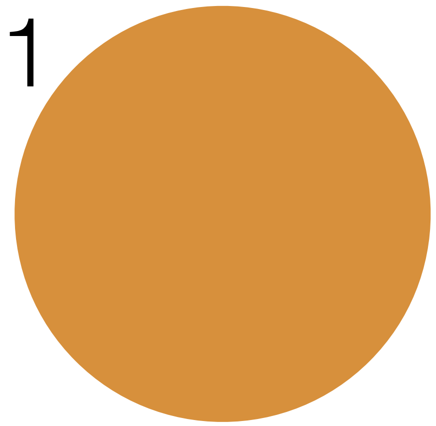
Chinese restaurant process



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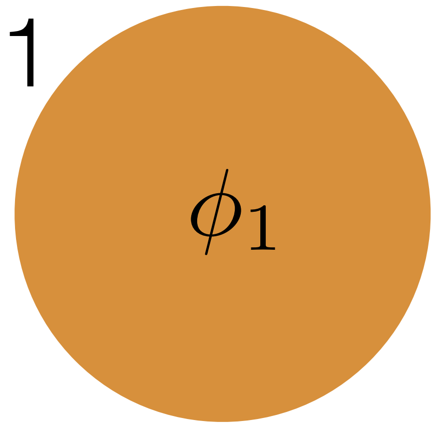
Chinese restaurant process



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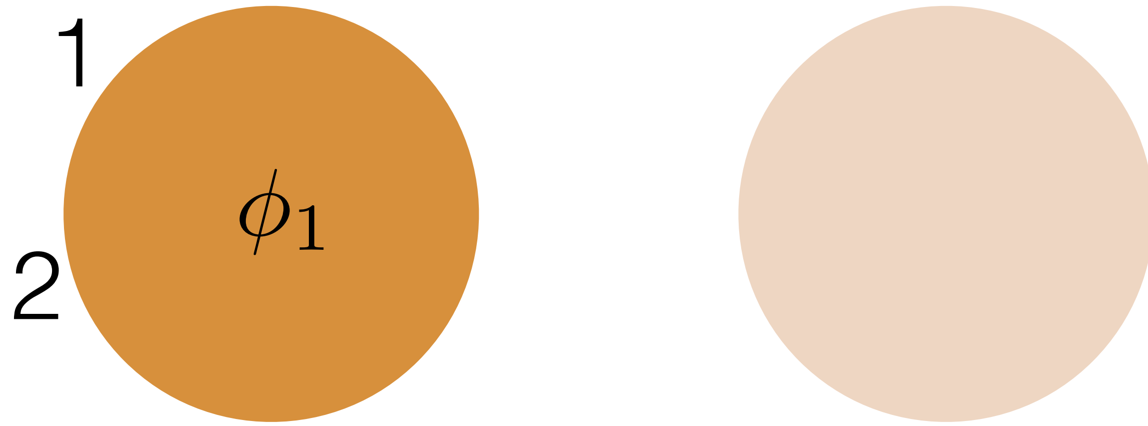
Chinese restaurant process



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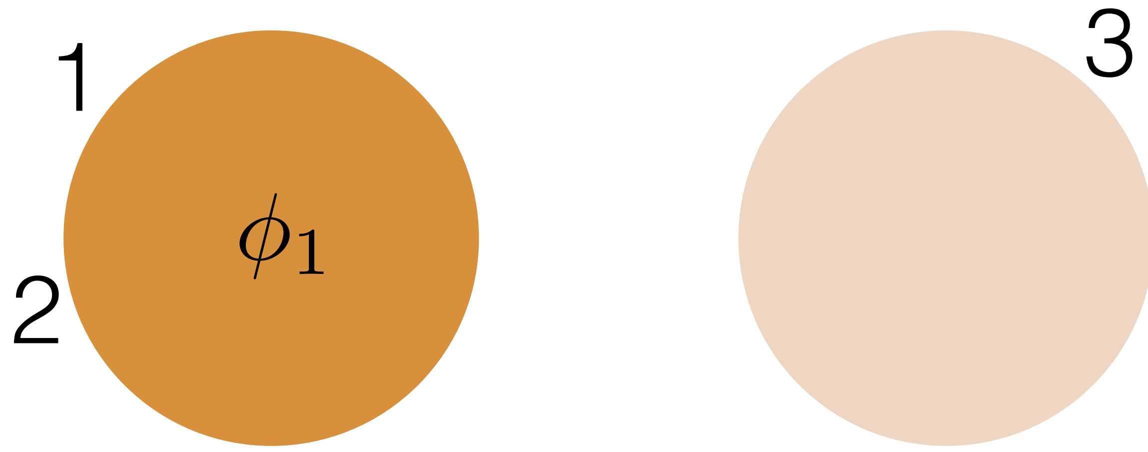

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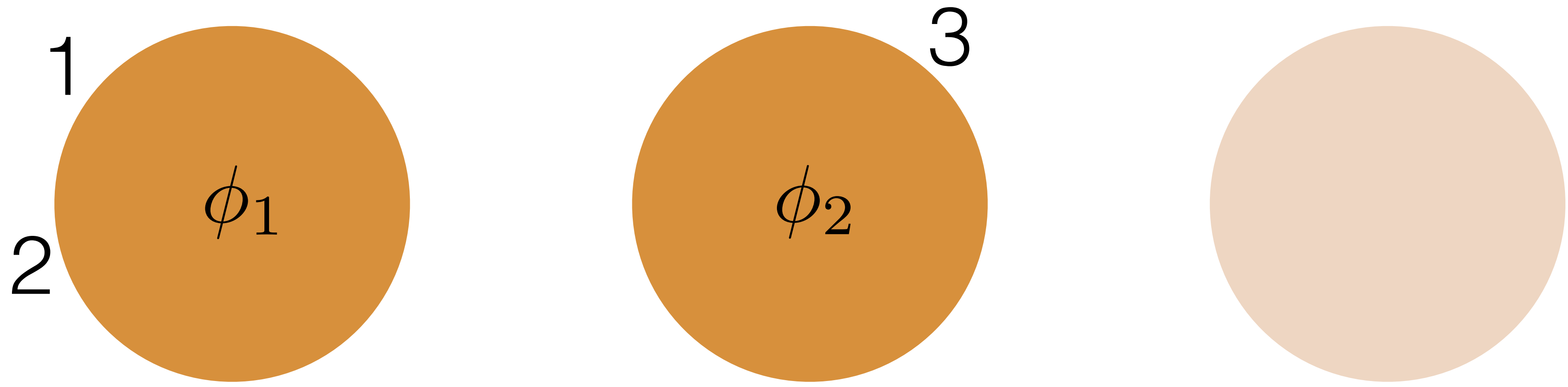

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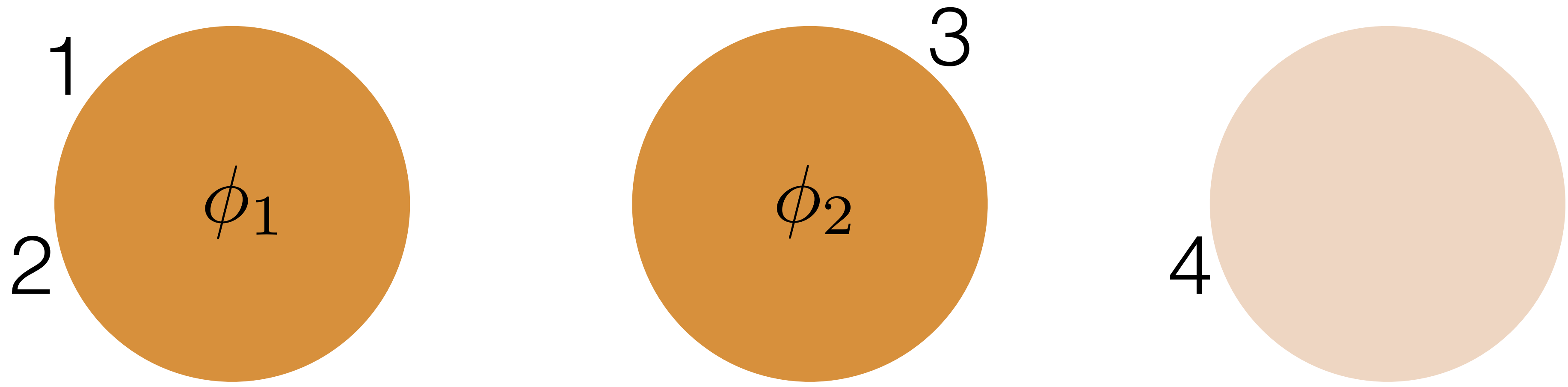
Chinese restaurant process



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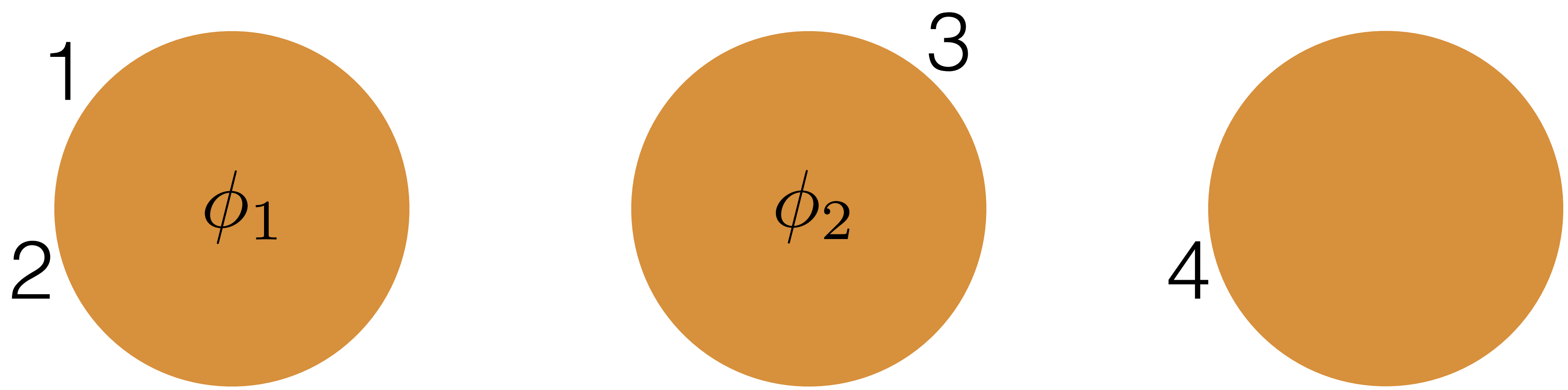
Chinese restaurant process



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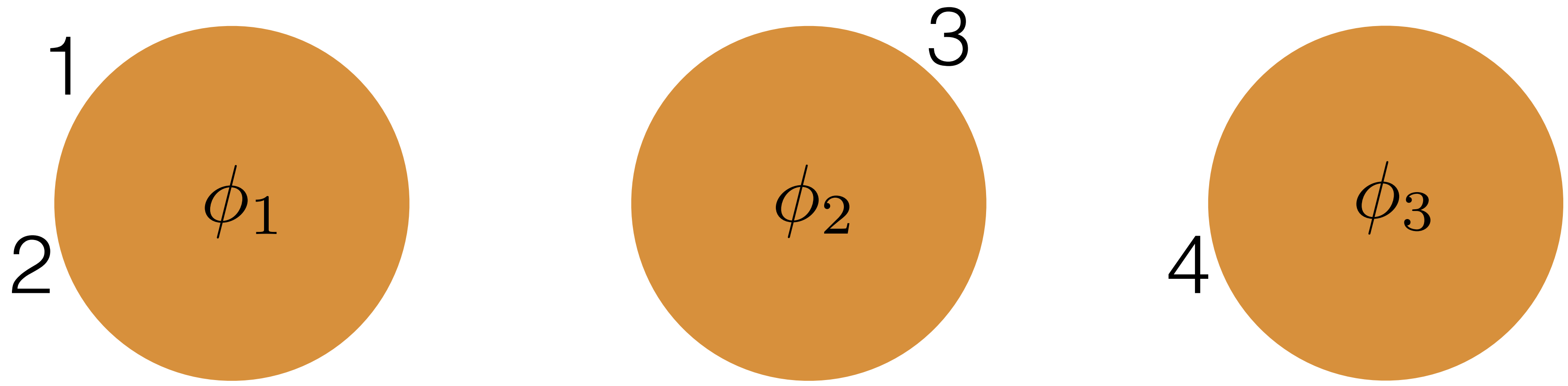
Chinese restaurant process



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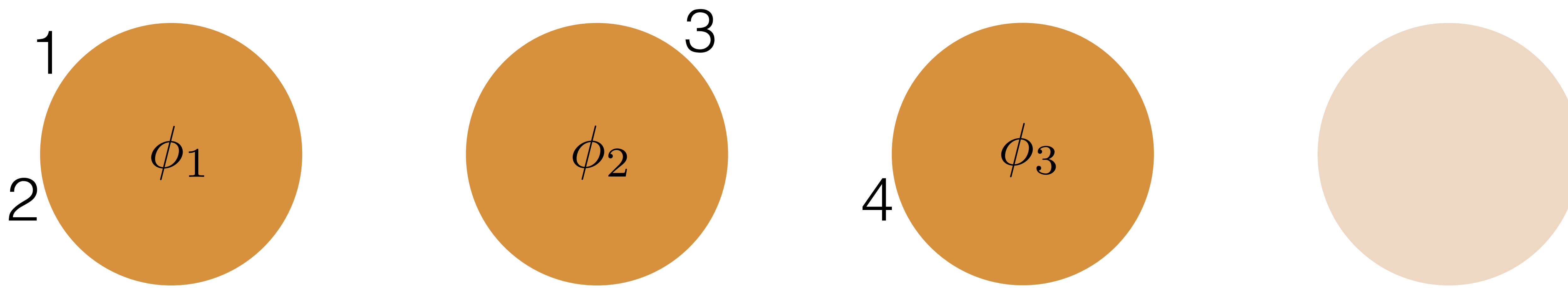

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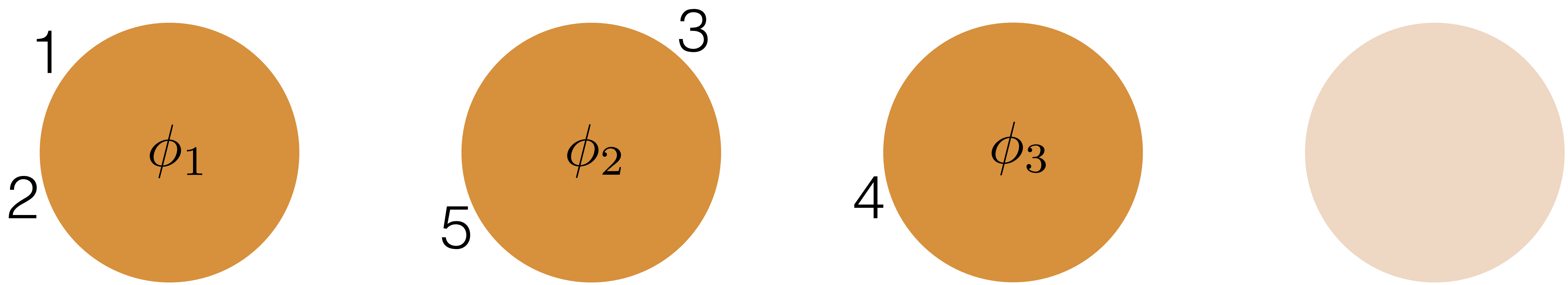

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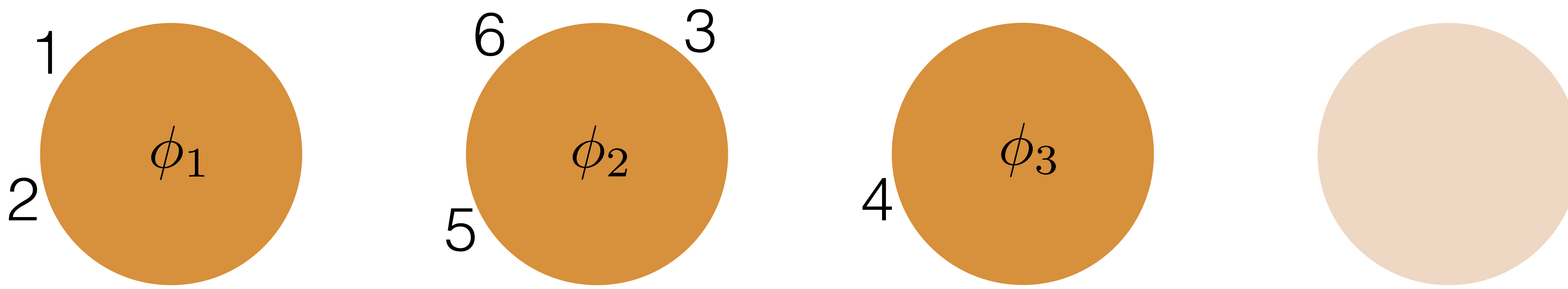

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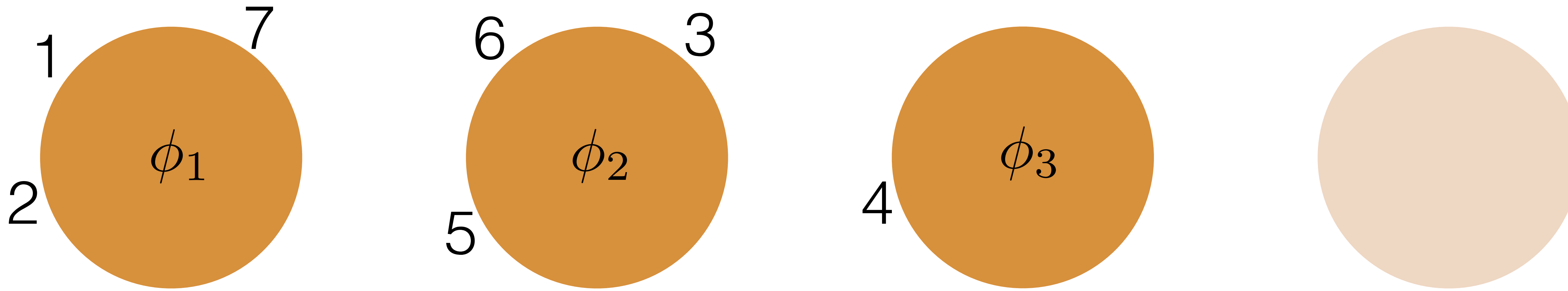

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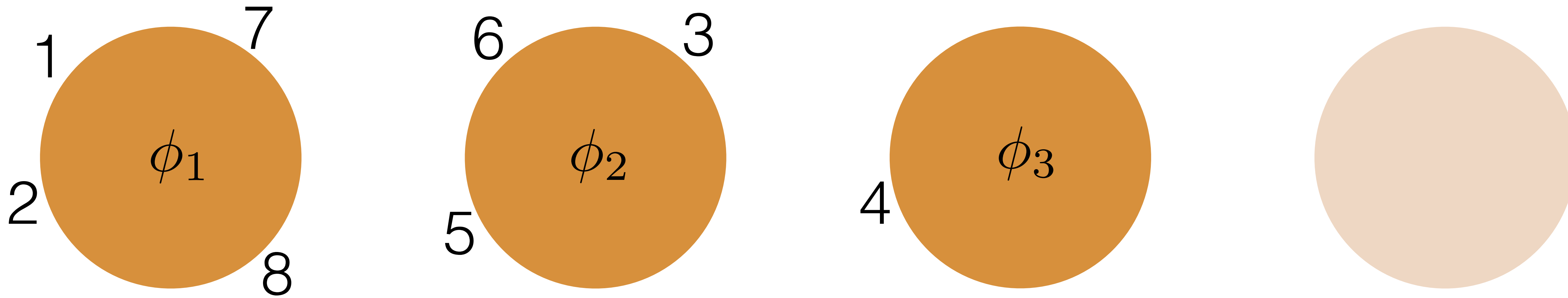
Chinese restaurant process



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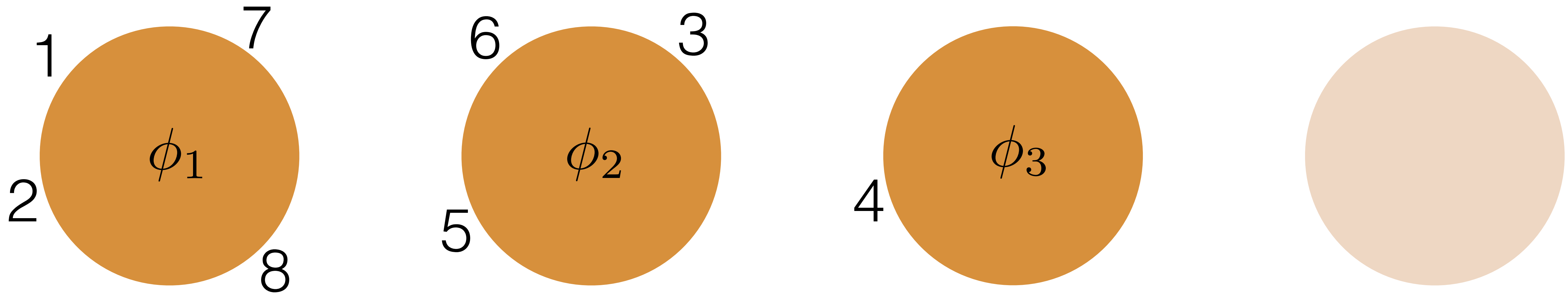
Chinese restaurant process



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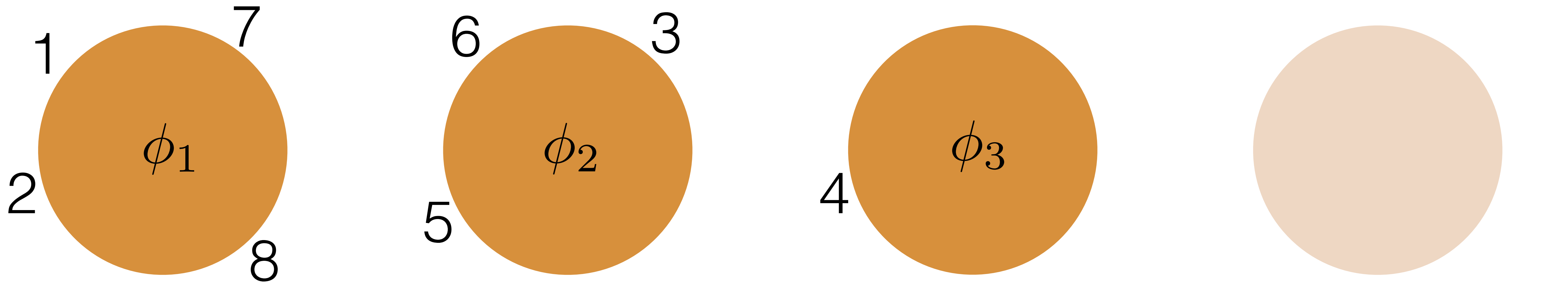
[Aldous 1983]

Chinese restaurant process



- Same thing we just did
- Each customer walks into the restaurant
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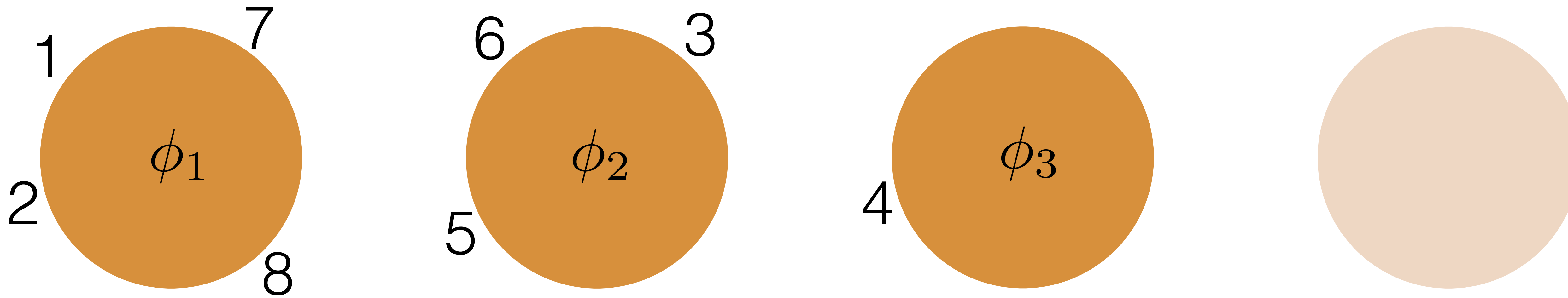
  - Forms new table with prob proportional to α
- Marginal for the Categorical likelihood with GEM prior

Chinese restaurant process



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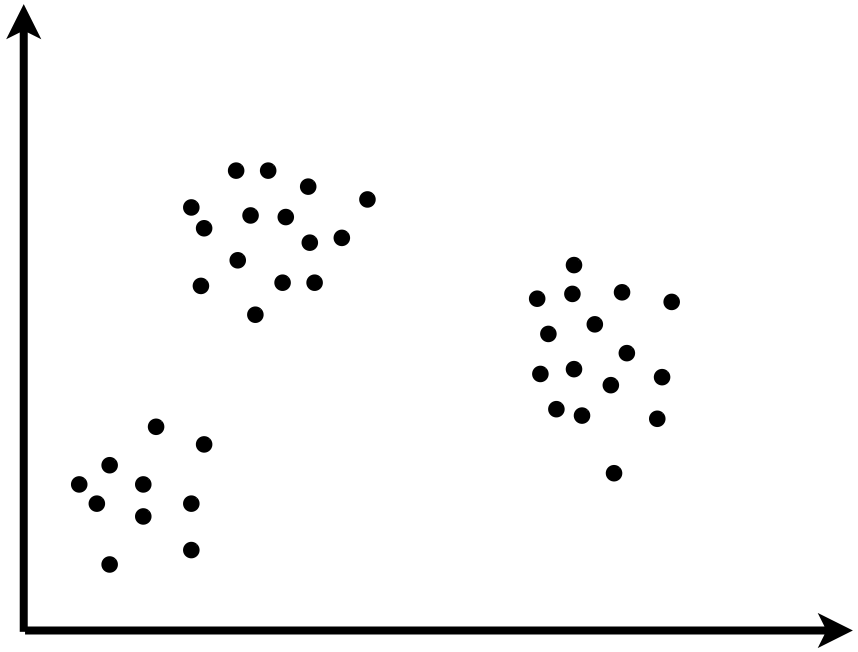
We've seen: Dirichlet process, Chinese restaurant process

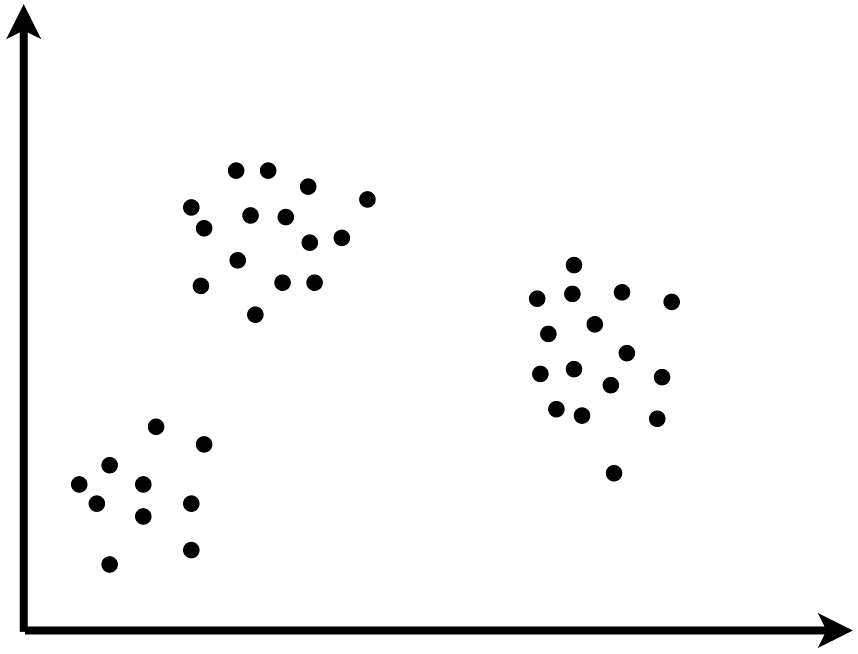
- Infinity of parameters (components)
- Growing number of parameters (clusters)

Inference



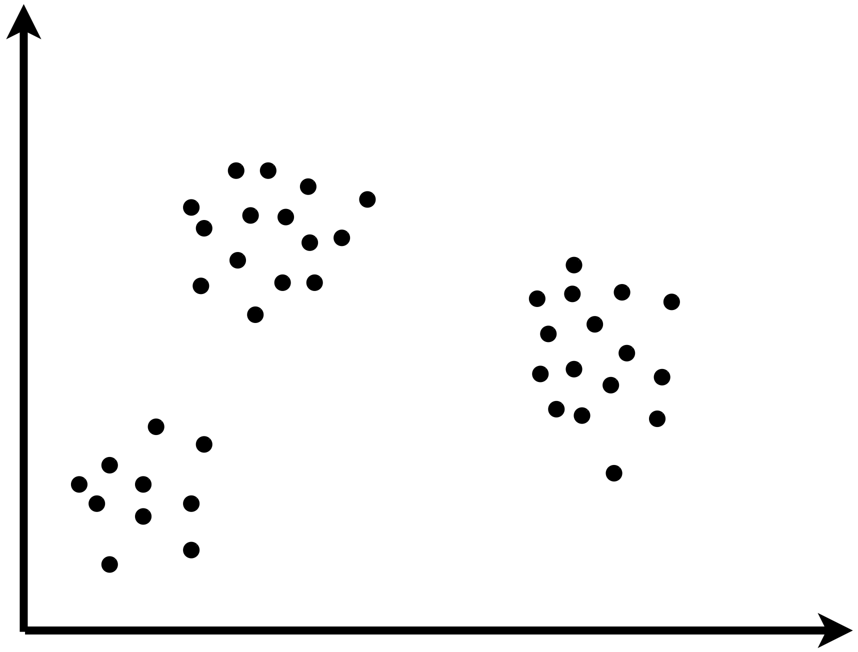
Inference

- DPMM



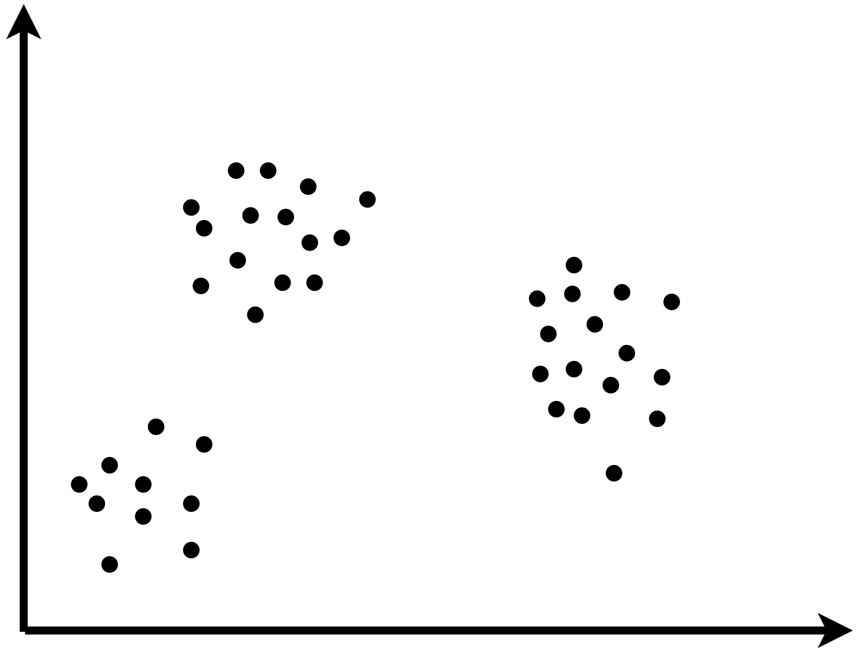
Inference

- DPMM; goal is a posterior over:



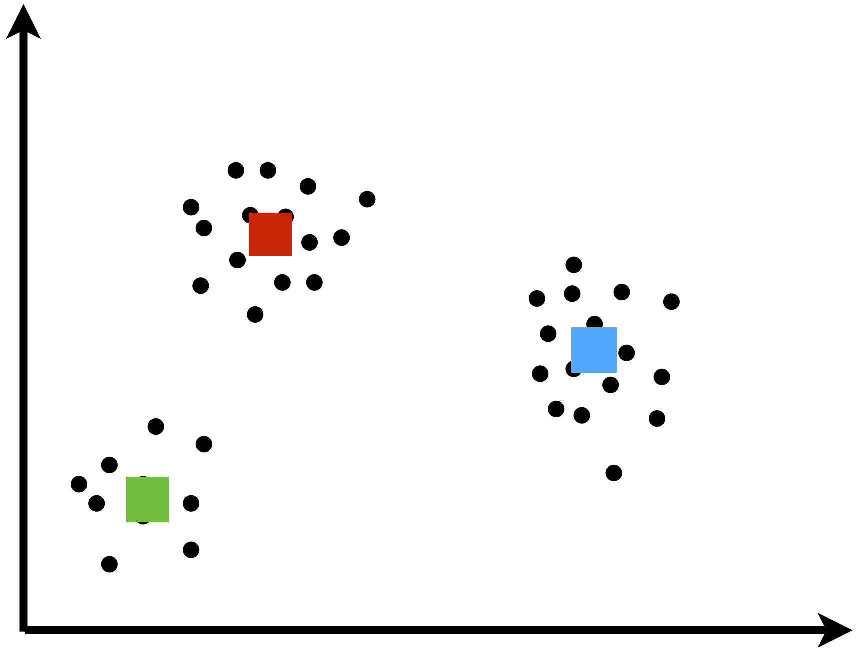
Inference

- DPMM; goal is a posterior over:
  - Cluster parameters (center, shape)



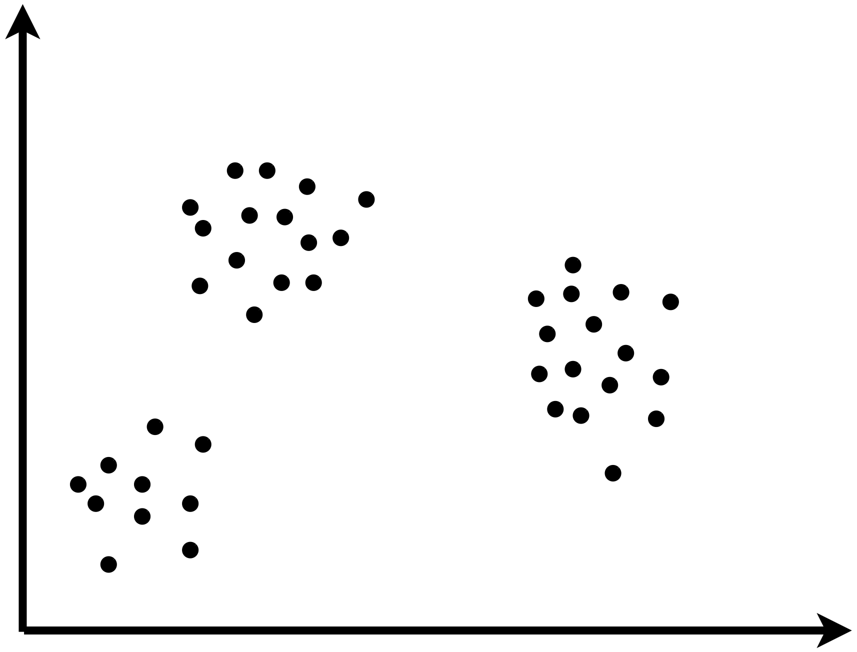
Inference

- DPMM; goal is a posterior over:
  - Cluster parameters (center, shape)



Inference

- DPMM; goal is a posterior over:
  - Cluster parameters (center, shape)
  - Assignments of data points to clusters



Inference

- DPMM; goal is a posterior over:
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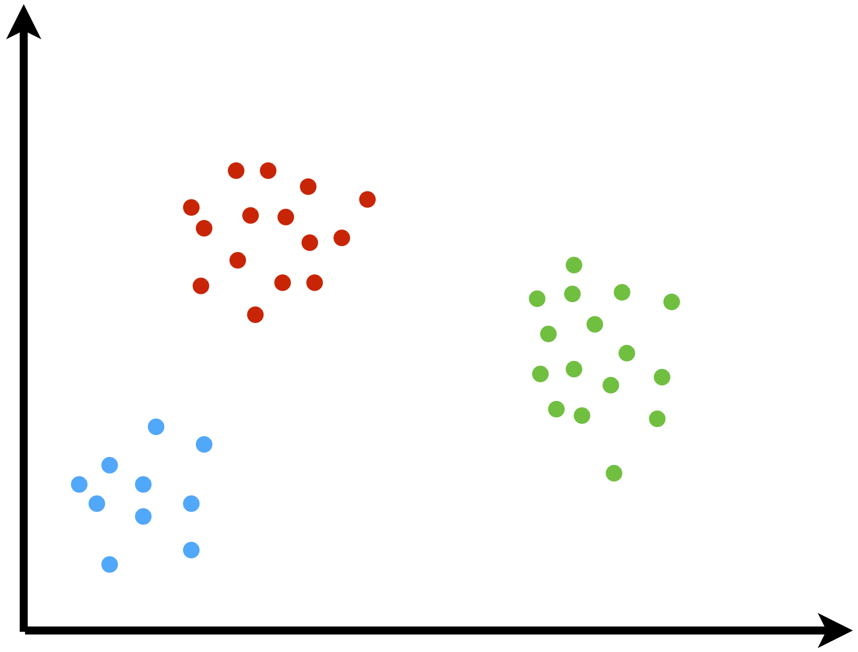

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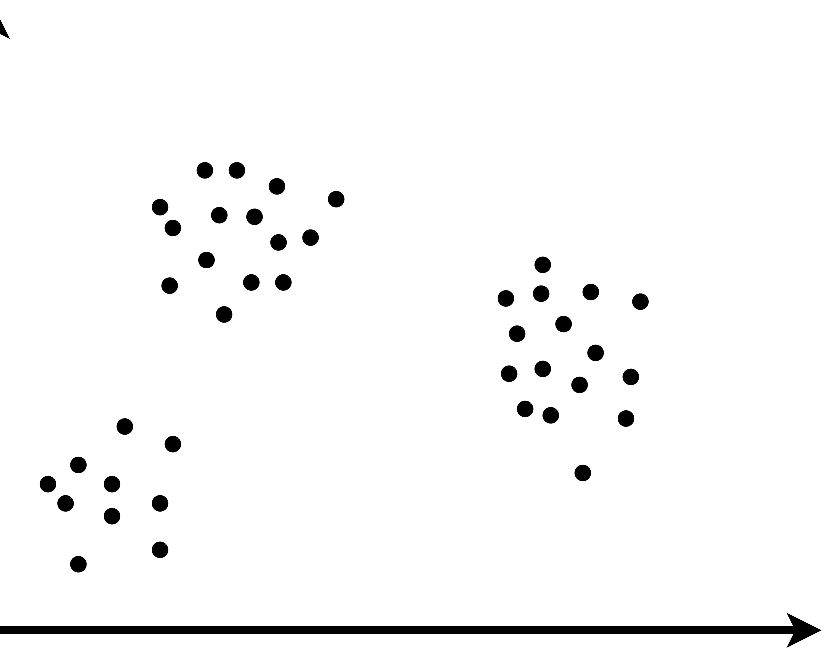

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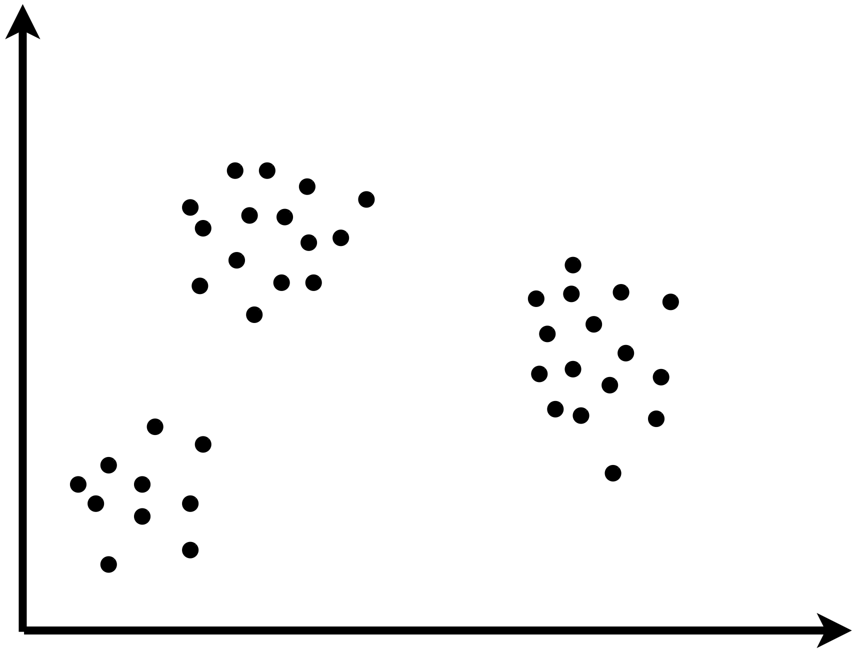
Inference



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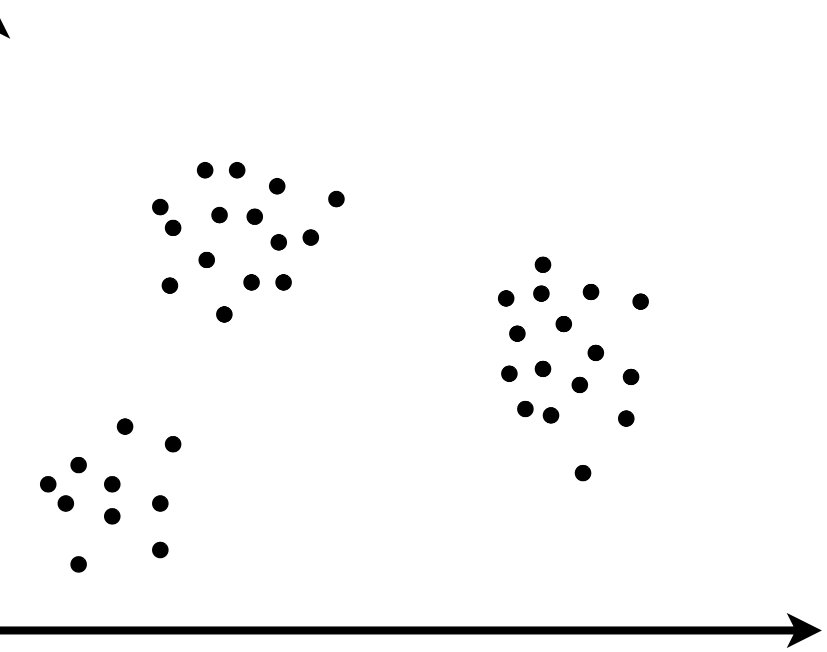
Inference



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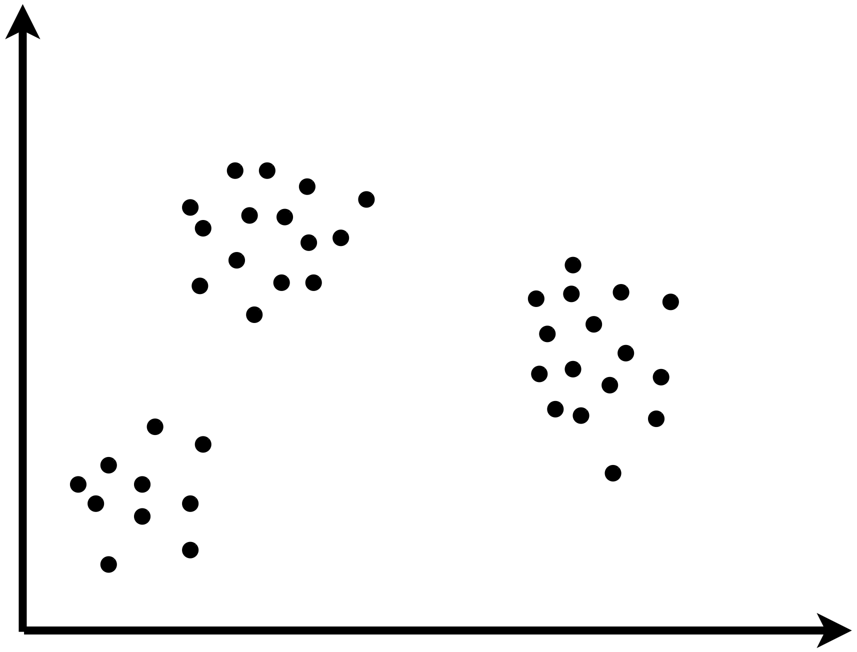

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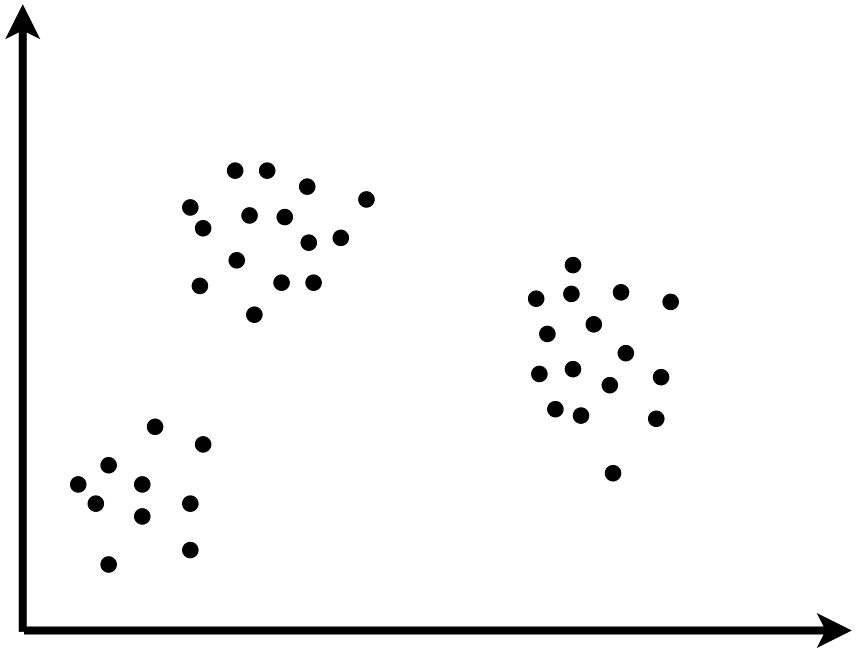

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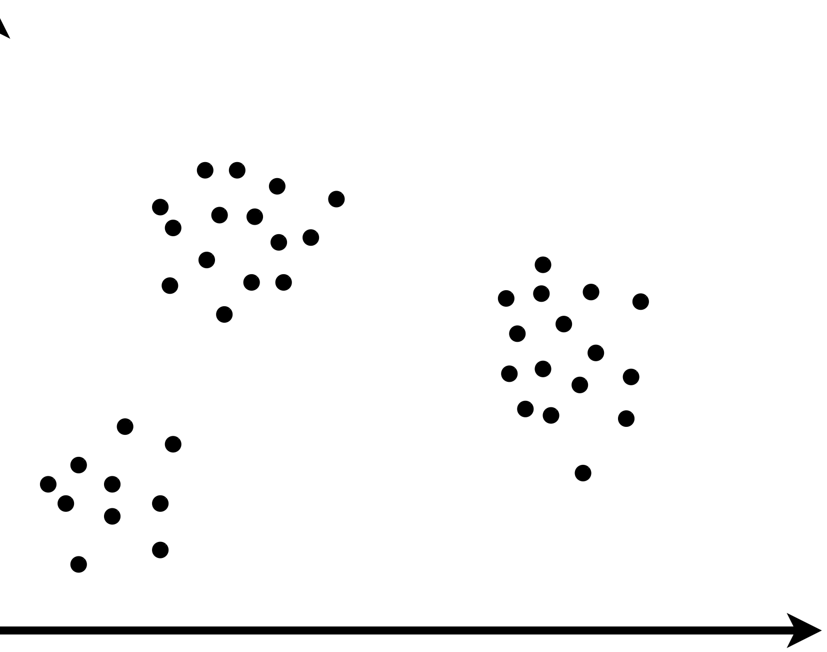

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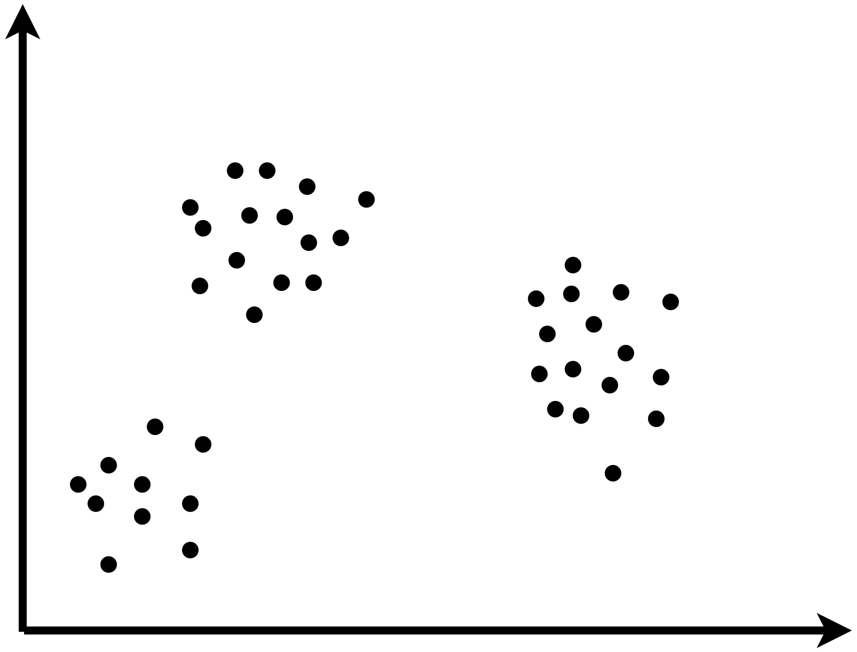
Inference



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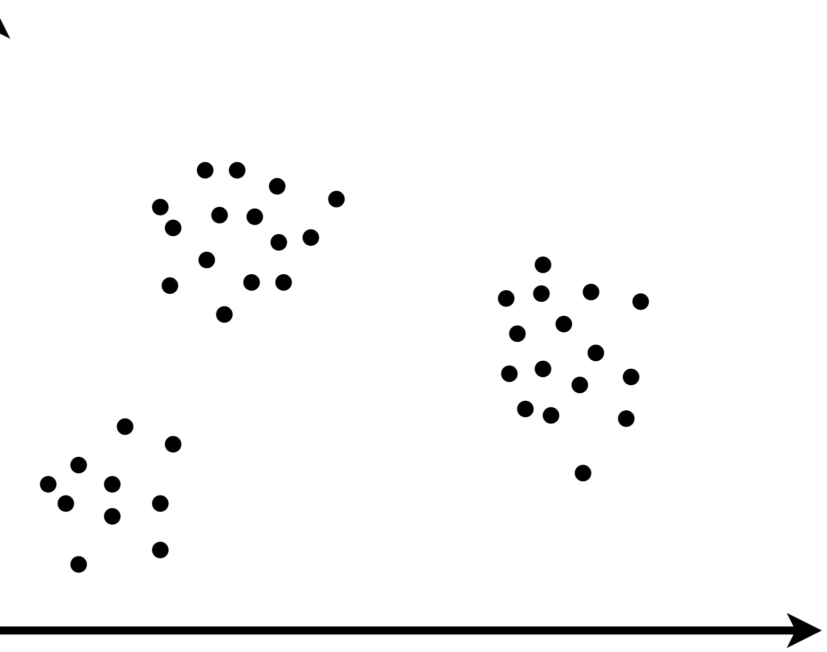
Inference



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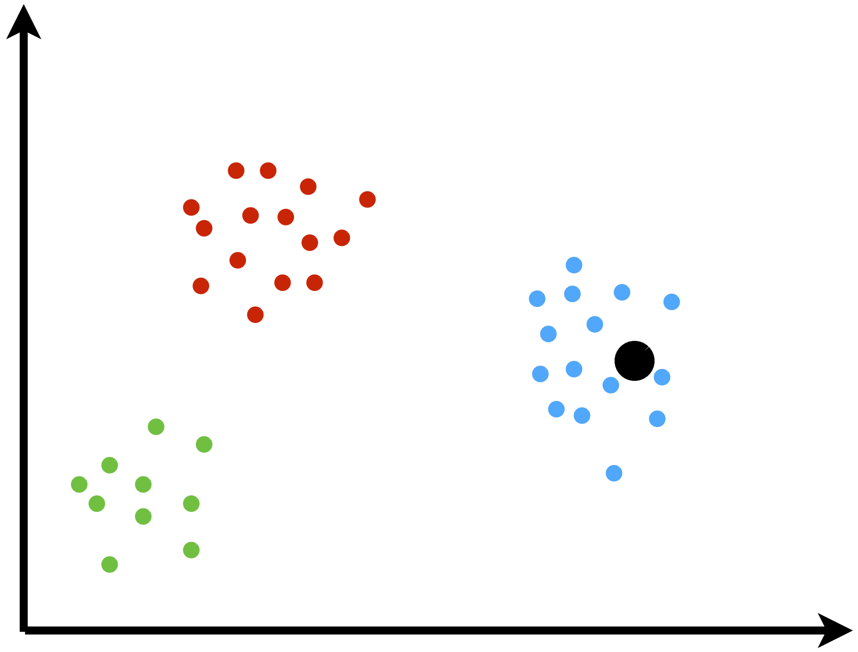
- Gibbs with the CRP: use exchangeability
  - Treat every data point like the final customer

Inference



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More Markov Chain Monte Carlo

More Markov Chain Monte Carlo

- Slice sampling

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  - auxiliary variable ➔ finite conditionals

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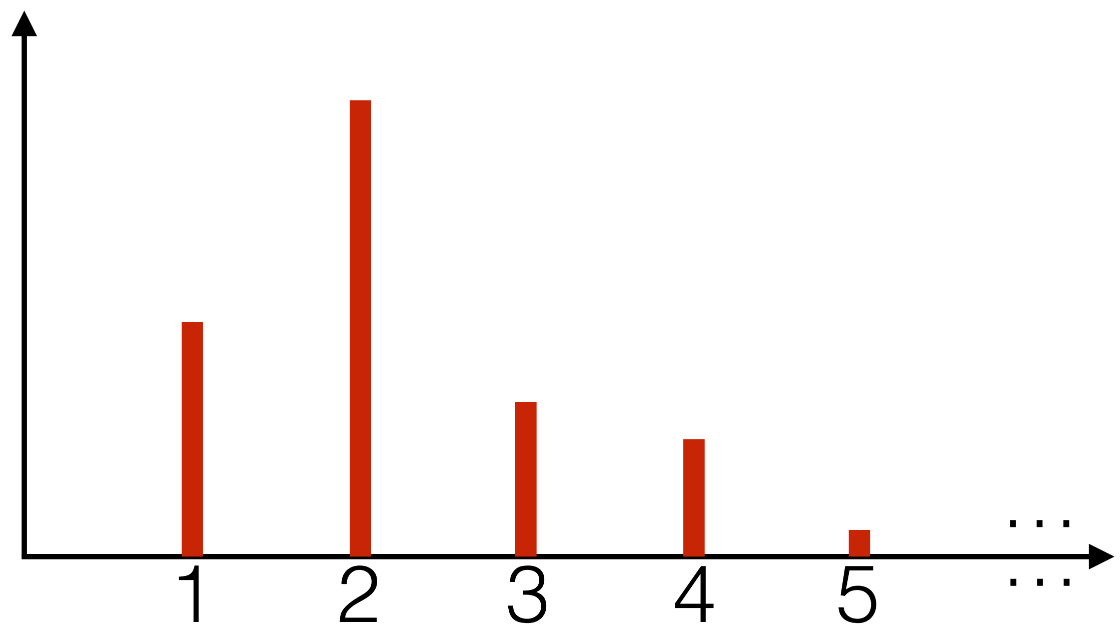

More Markov Chain Monte Carlo

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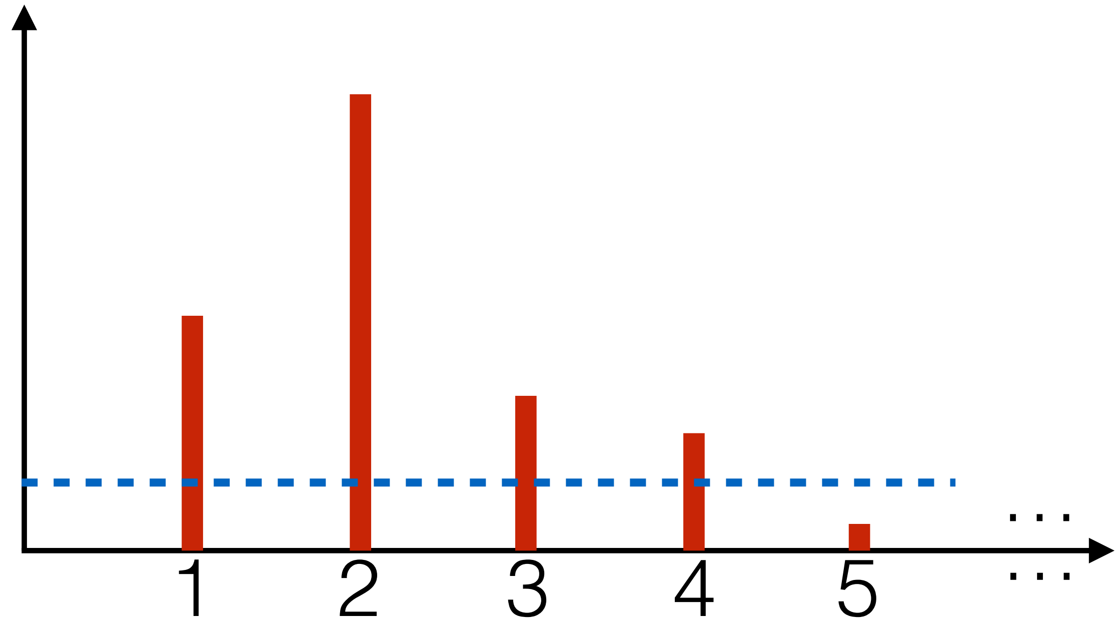

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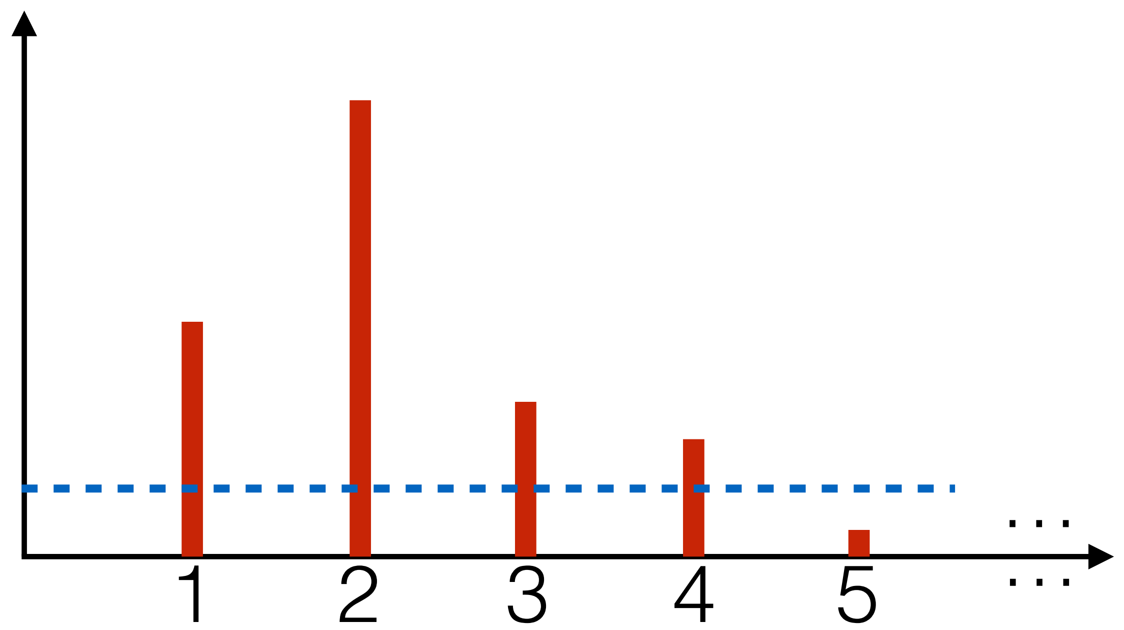

- Approximate with truncated distribution

[Ishwaran, James 2001; Campbell*, Huggins*, Broderick 2016]

More Markov Chain Monte Carlo

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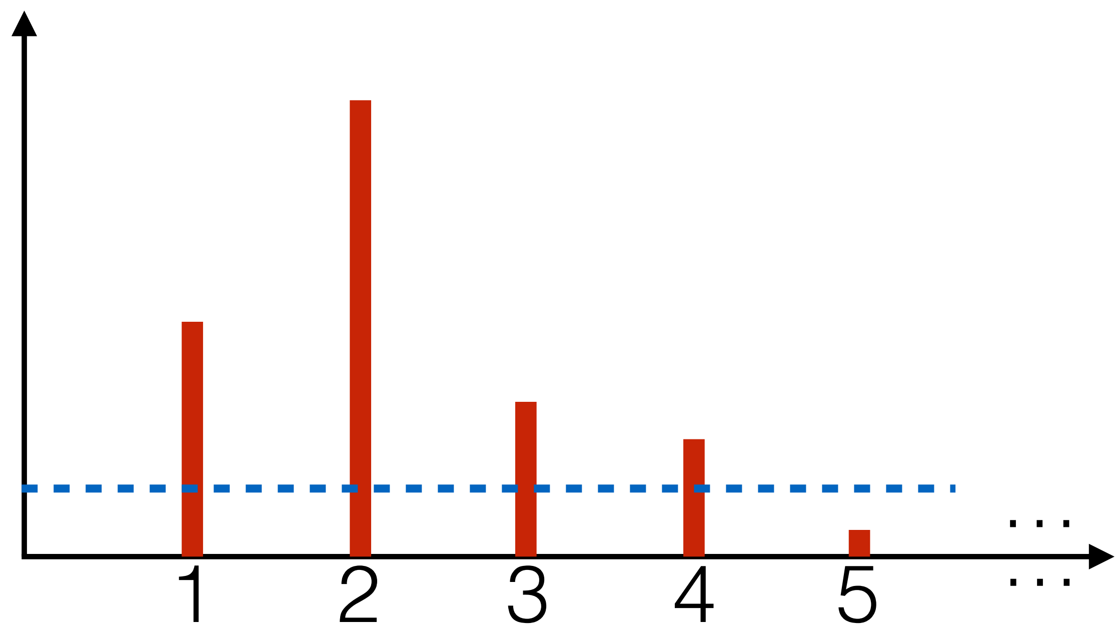

- Approximate with truncated distribution
  - E.g., Hamiltonian Monte Carlo

[Ishwaran, James 2001; Campbell*, Huggins*, Broderick 2016]

Variational Bayes

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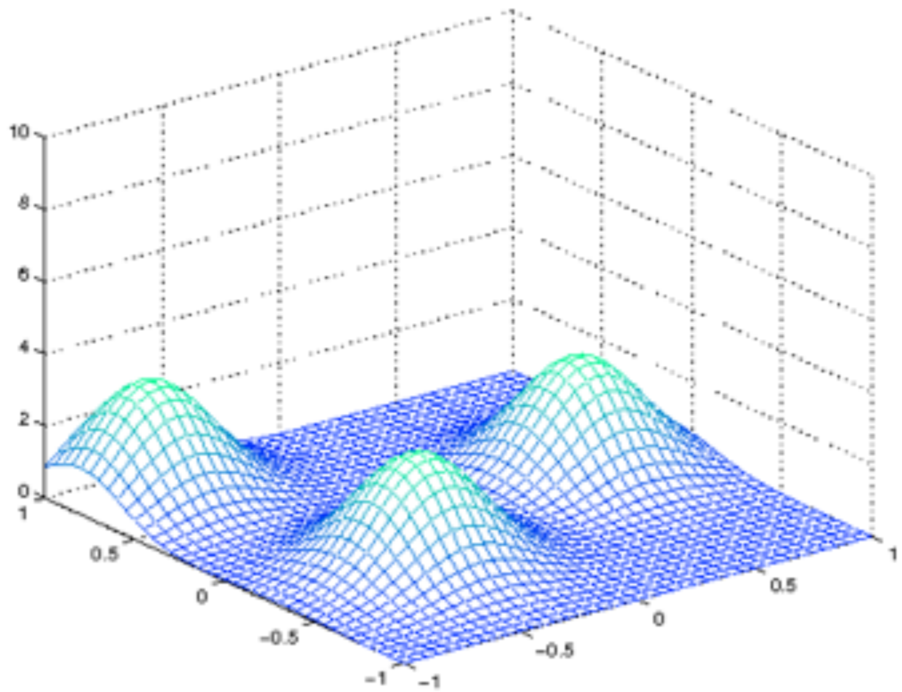
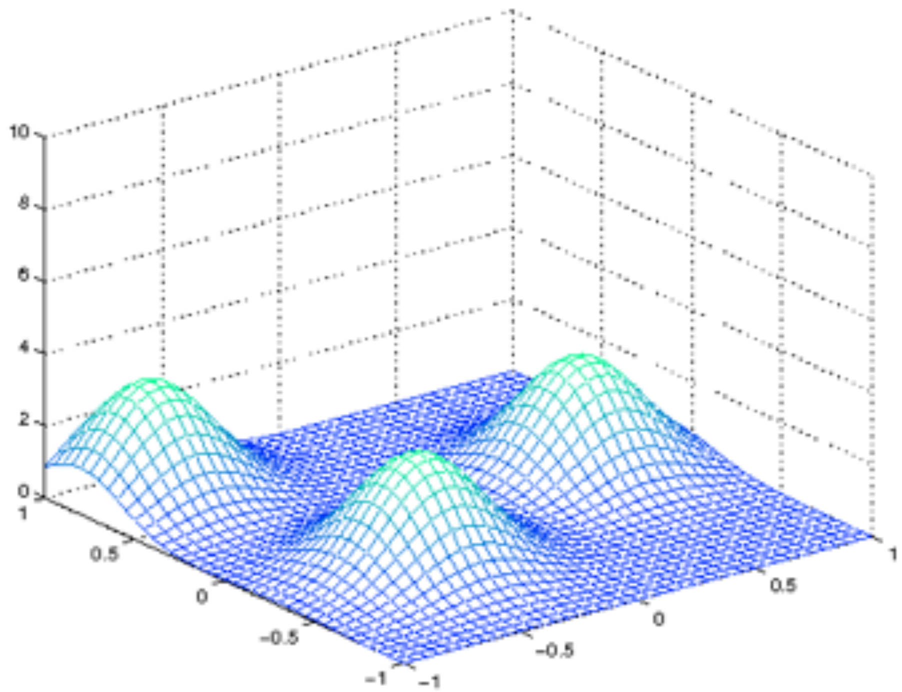


- Variational Bayes (VB)

Variational Bayes



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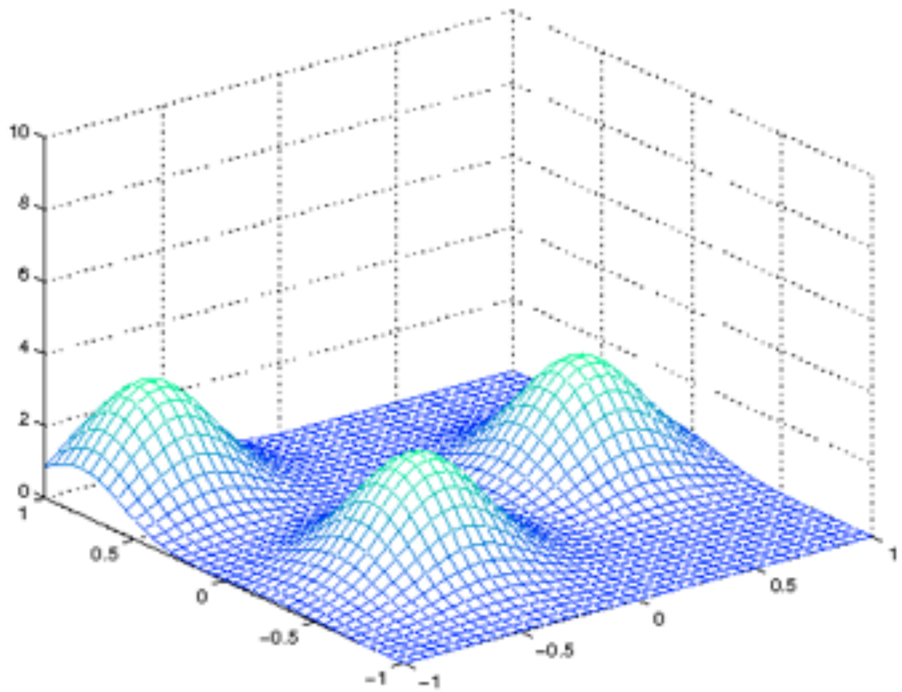

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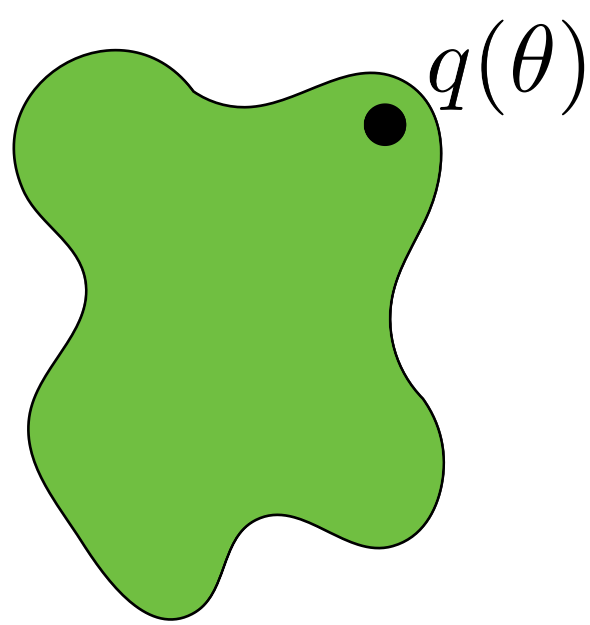

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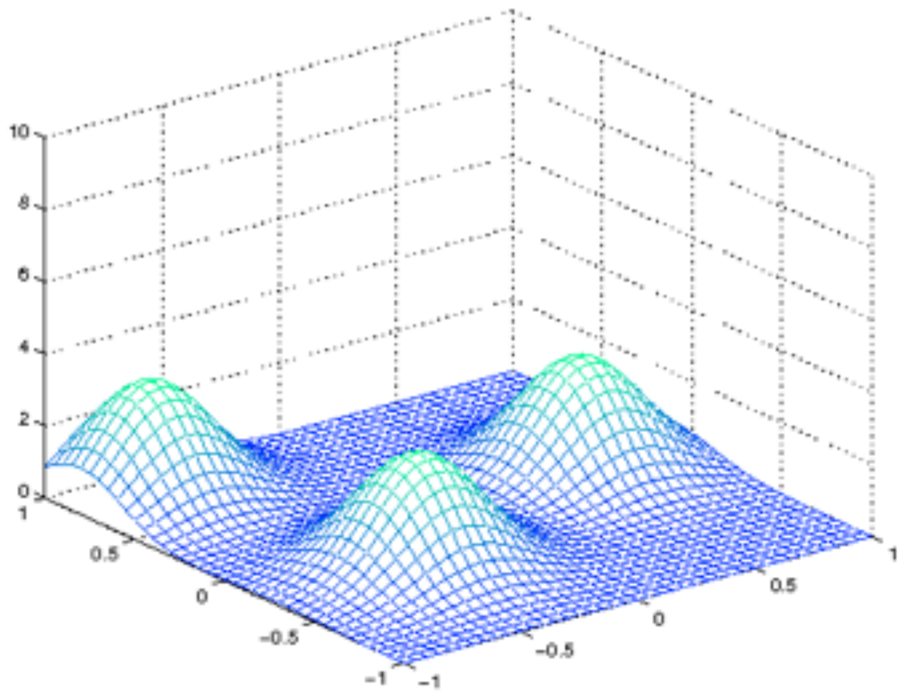

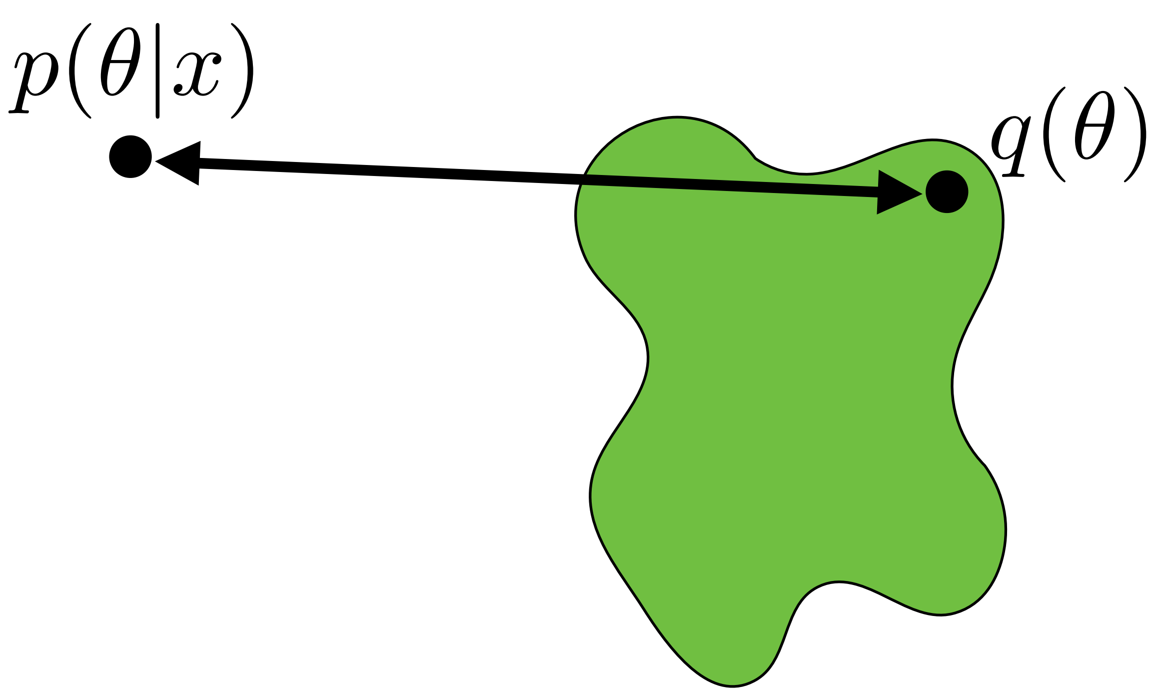

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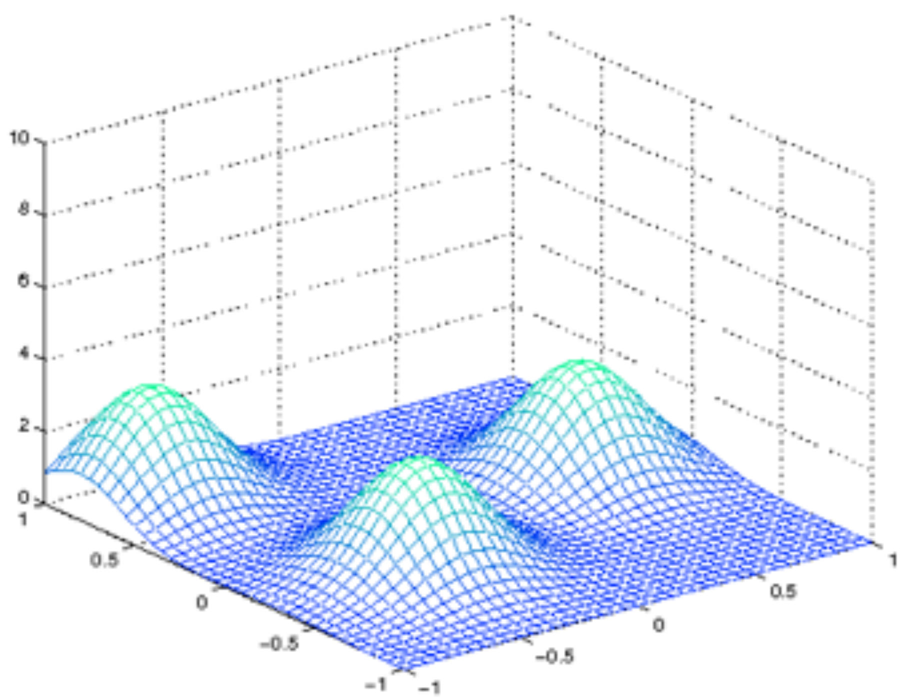

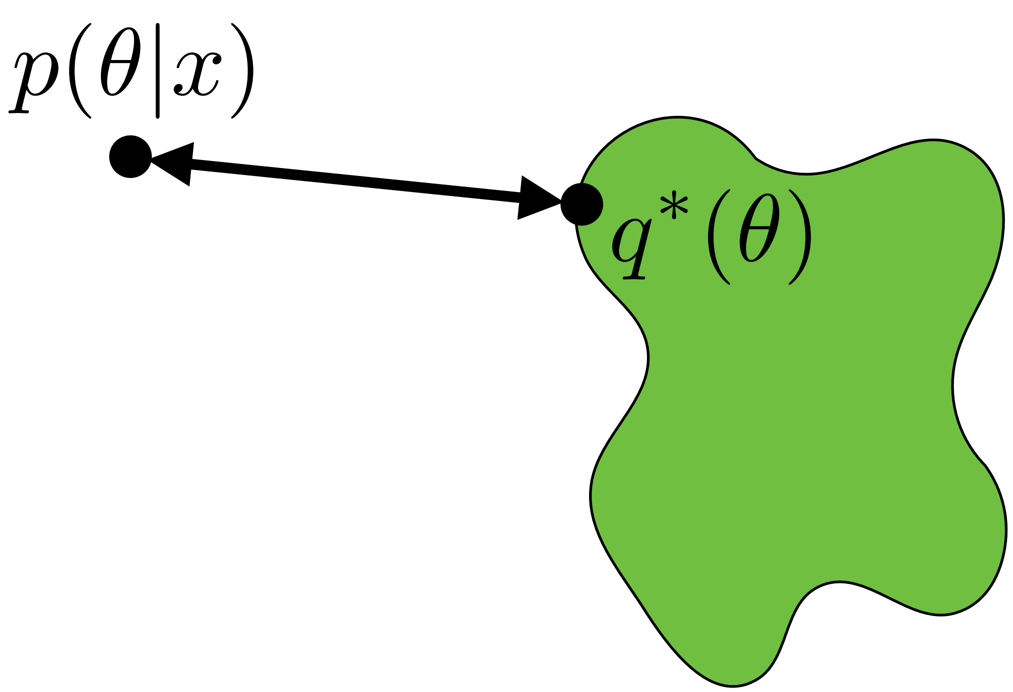

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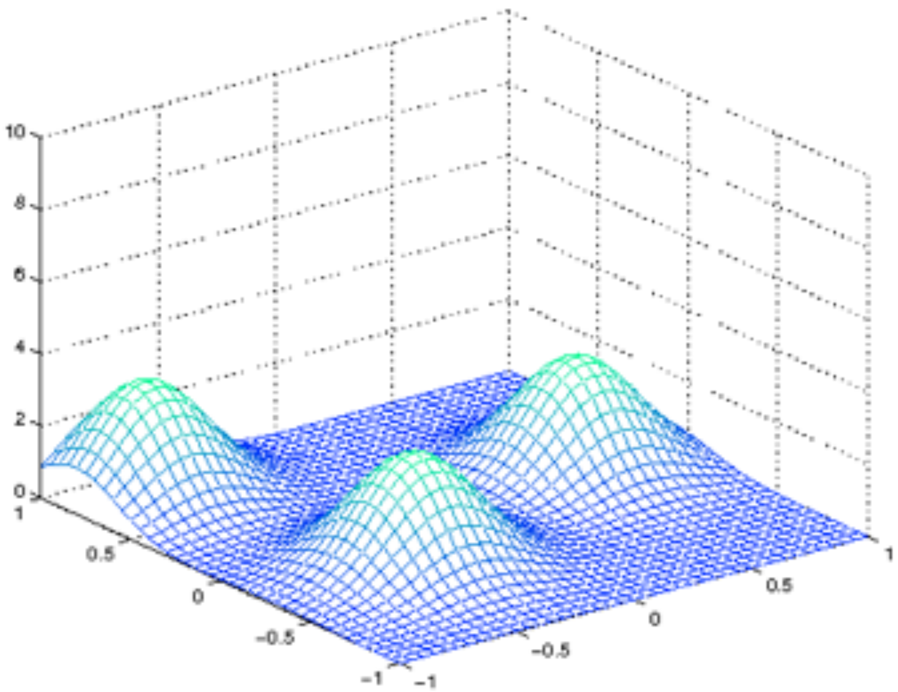

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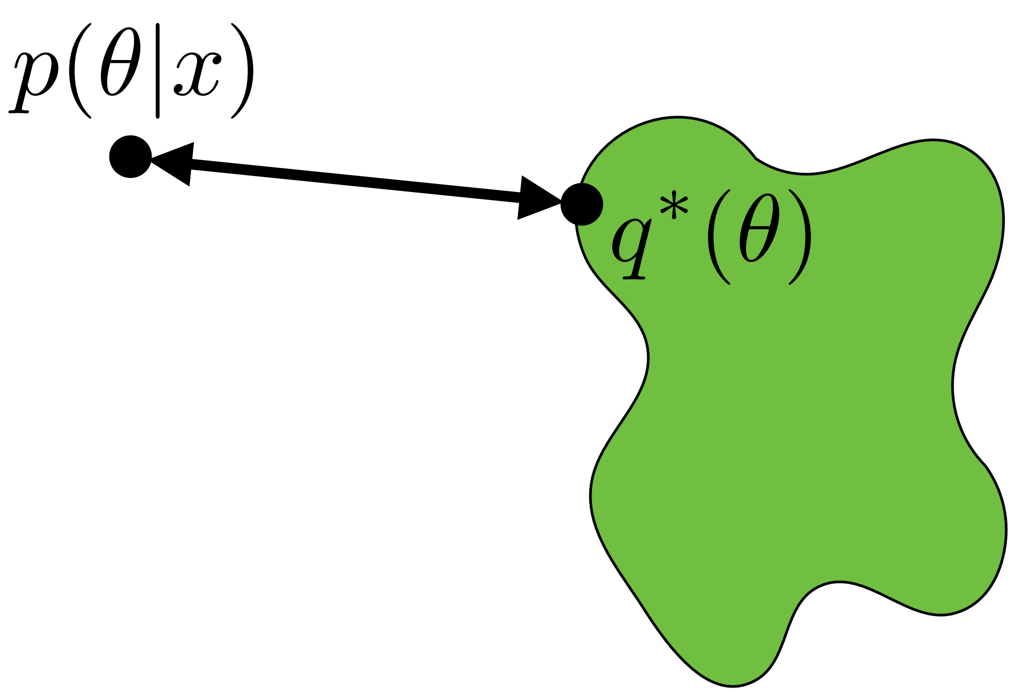

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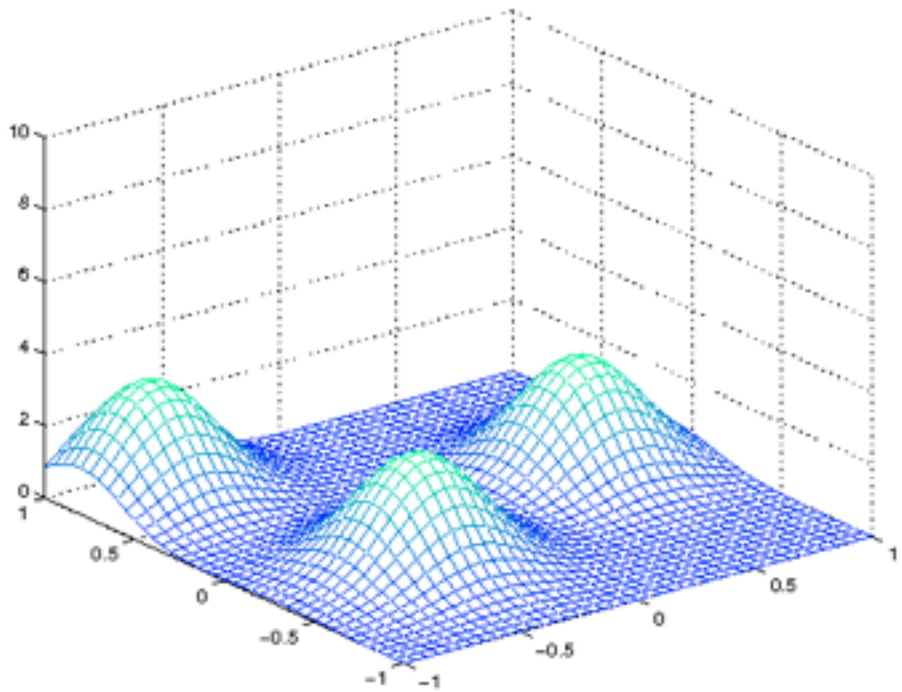

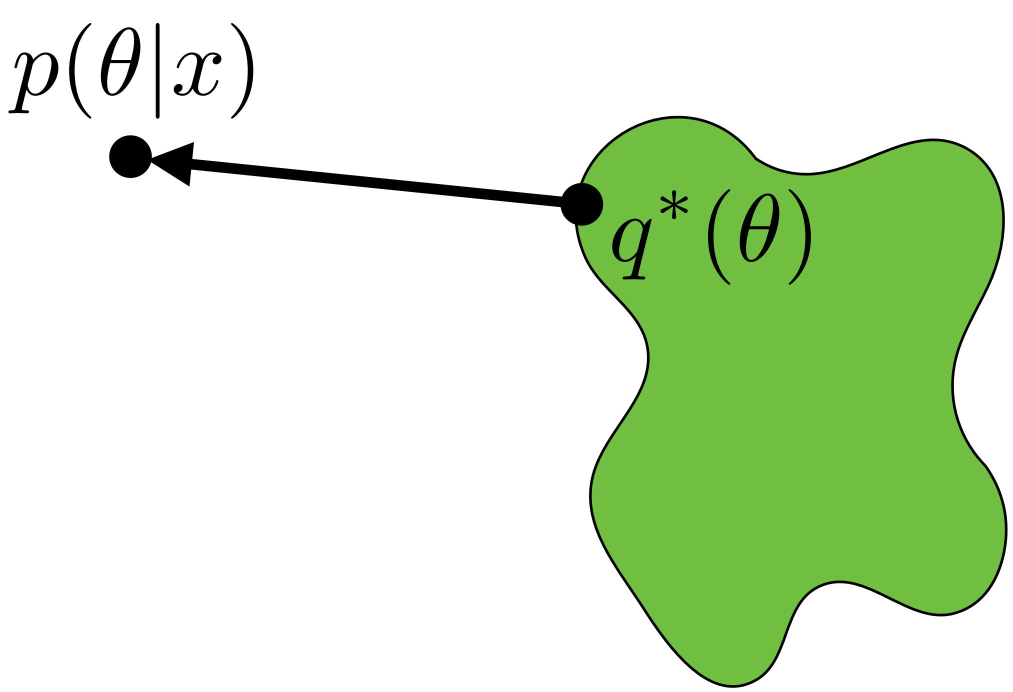

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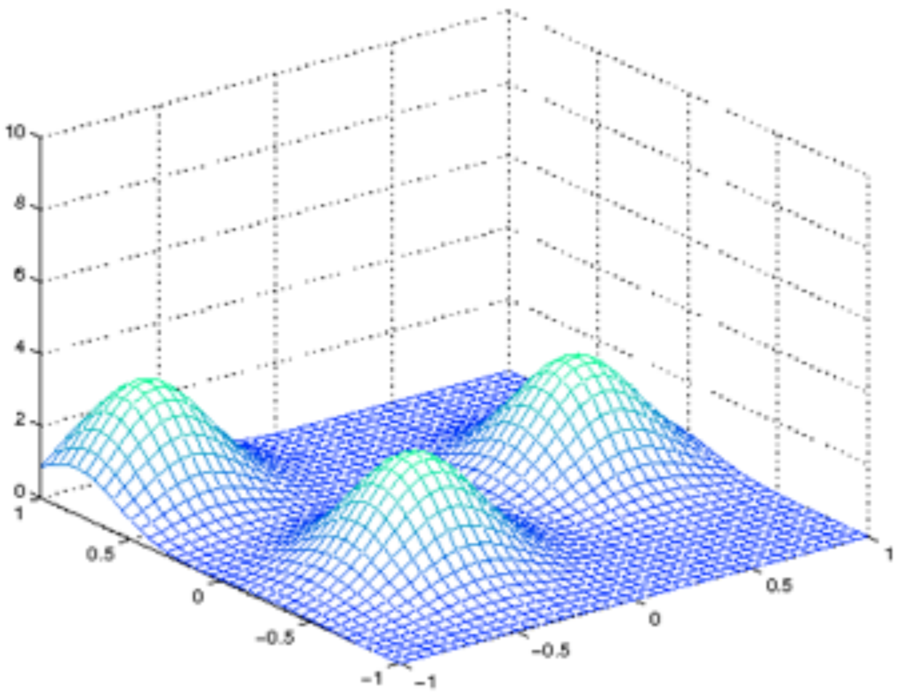

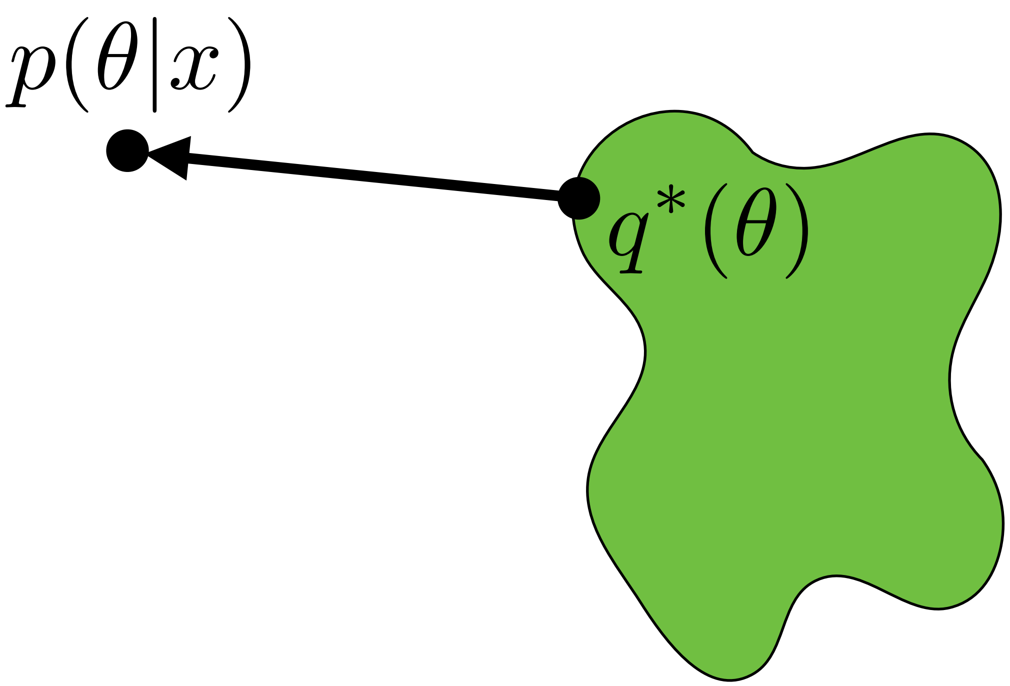

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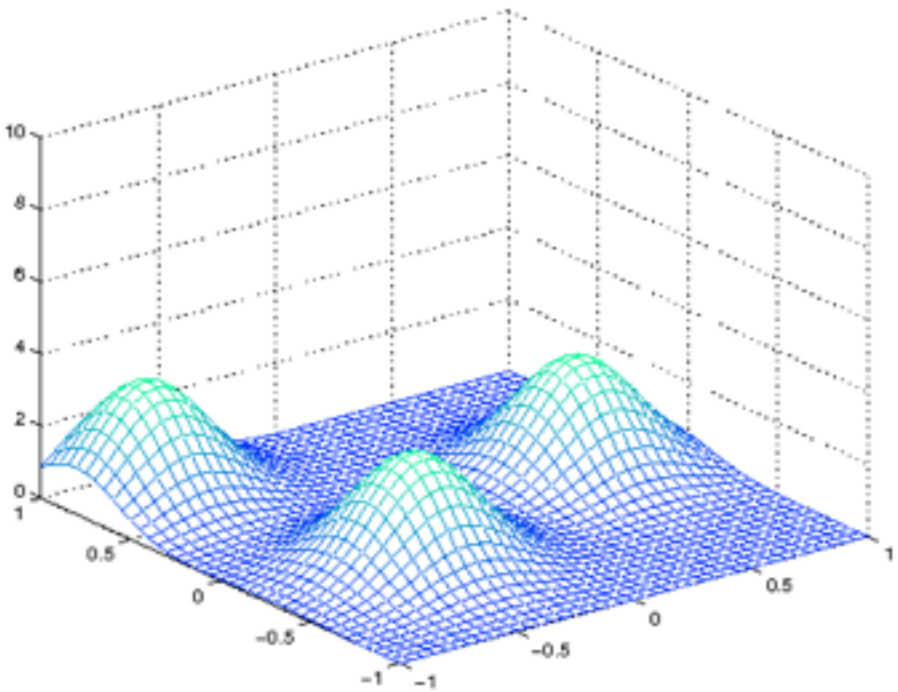

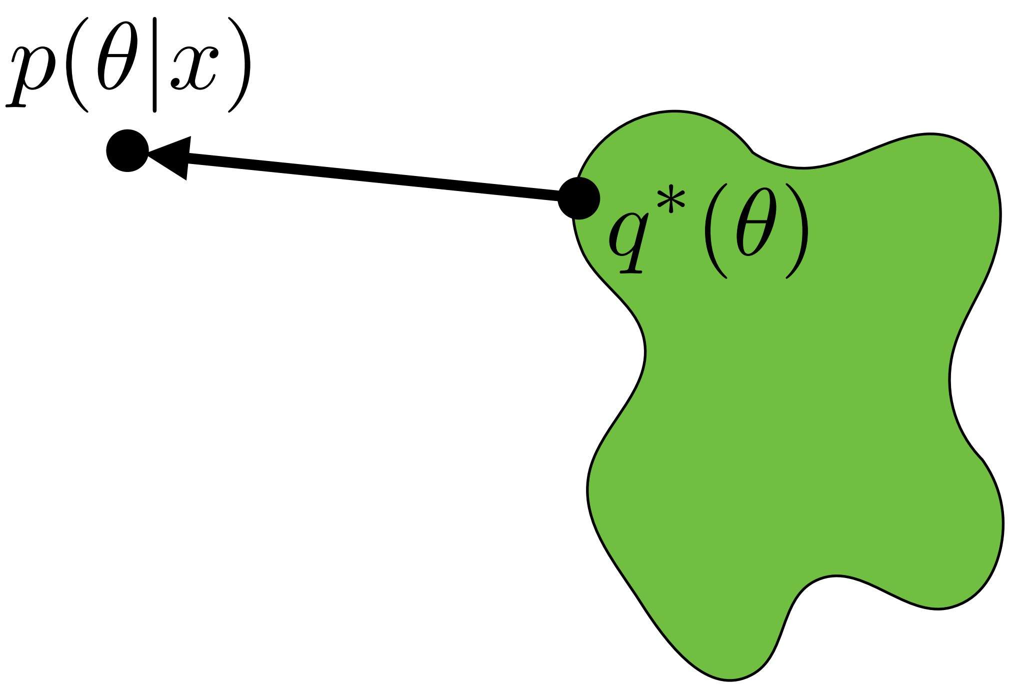

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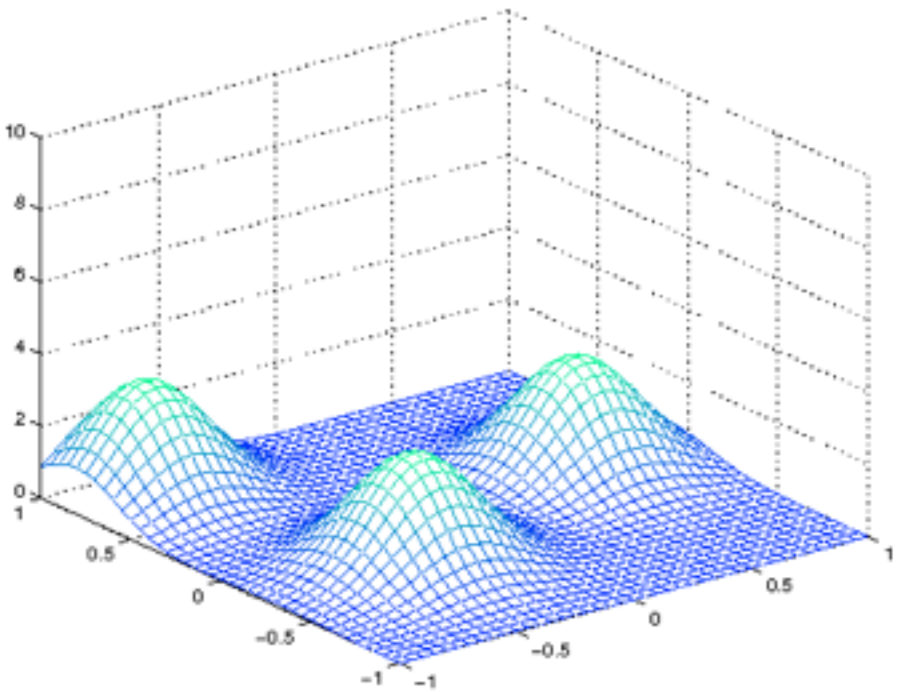

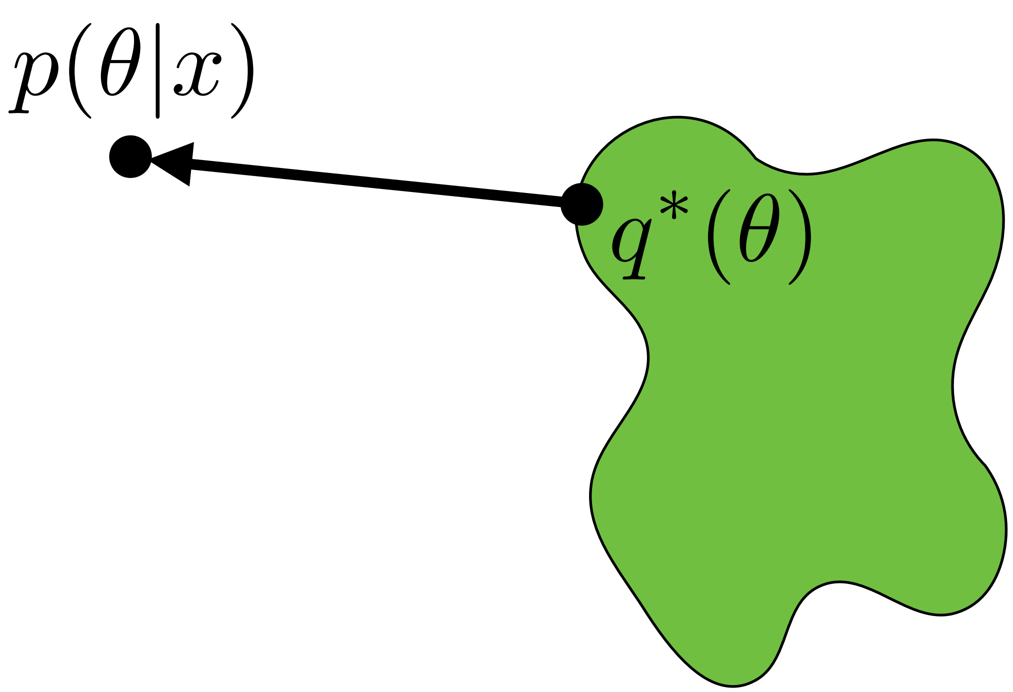

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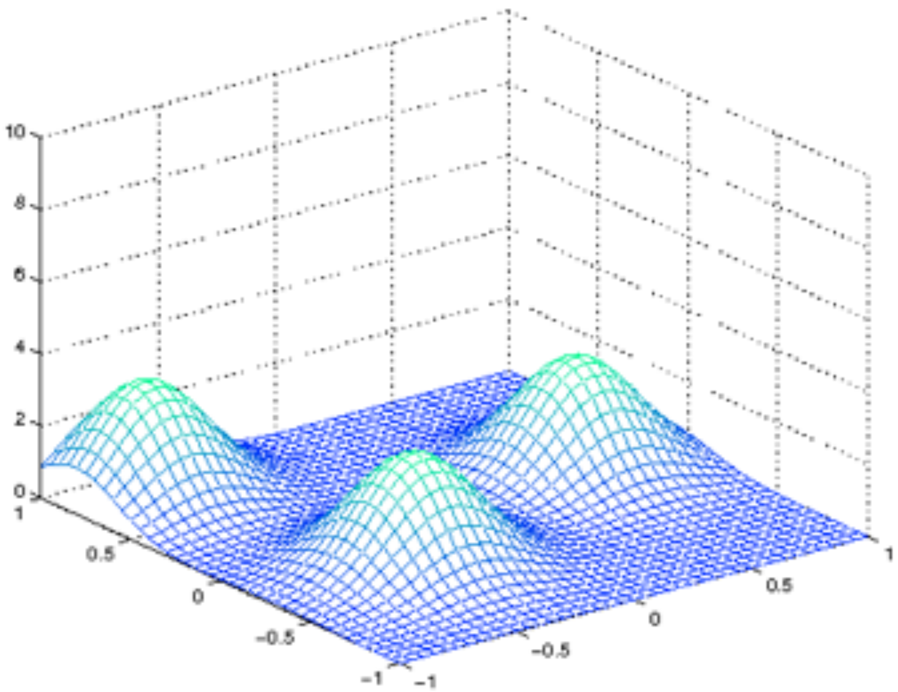
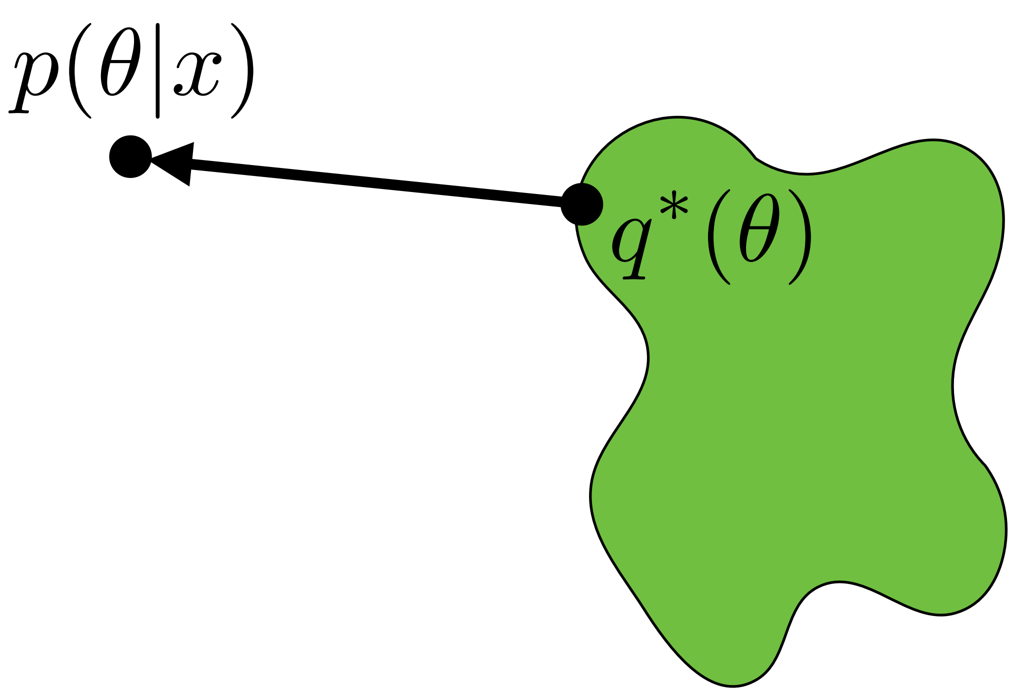
Variational Bayes





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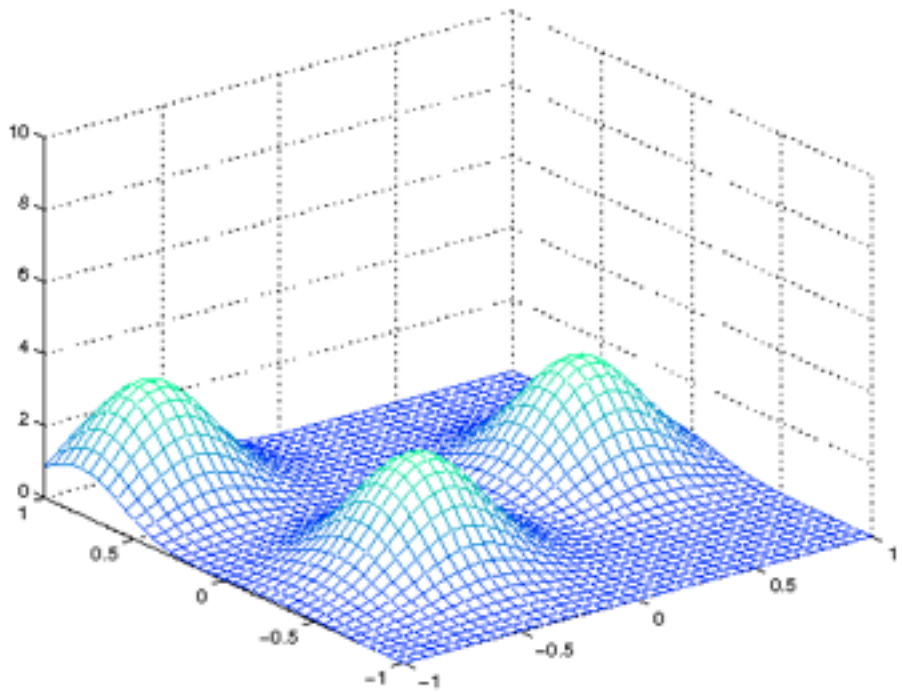
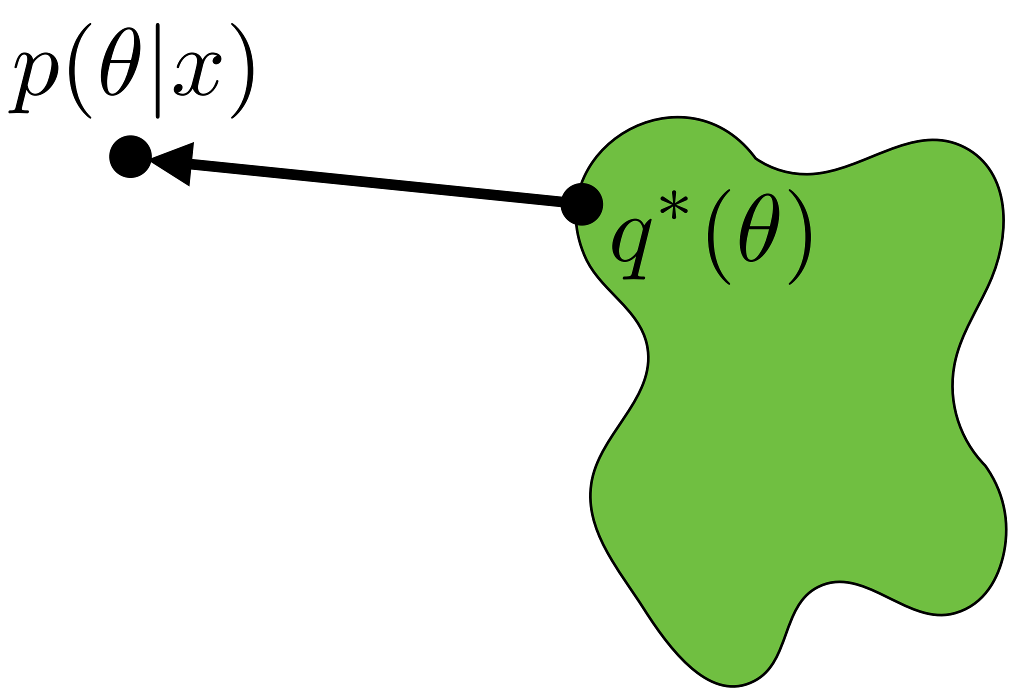
[Broderick, Boyd, Wibisono, Wilson, Jordan 2013]

Variational Bayes





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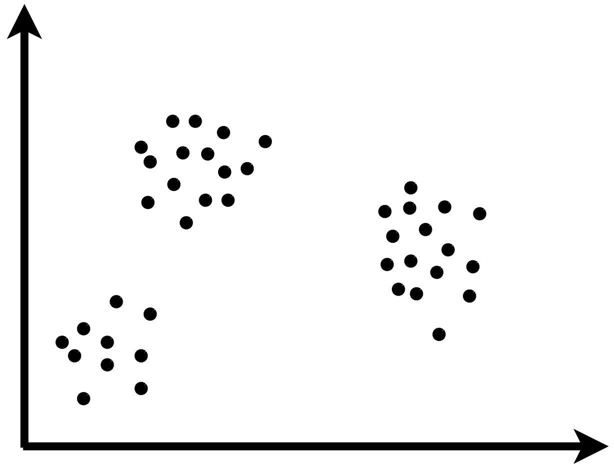
  - Linear response VB (LRVB) for accurate covariance

[Broderick, Boyd, Wibisono, Wilson, Jordan 2013; Giordano, Broderick, Jordan 2015]

Exercises



Exercises

- Code a DPMM simulator using a:
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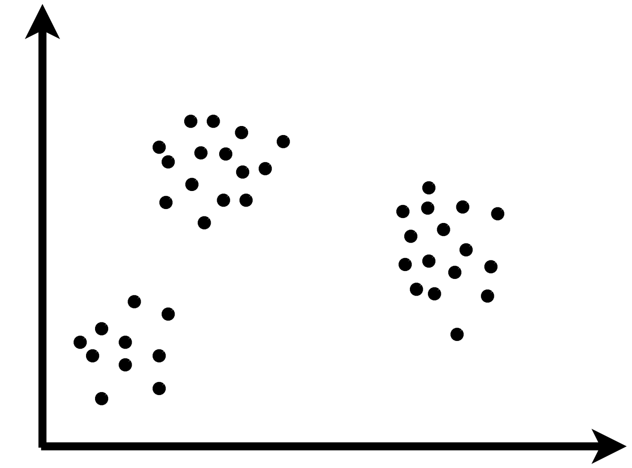

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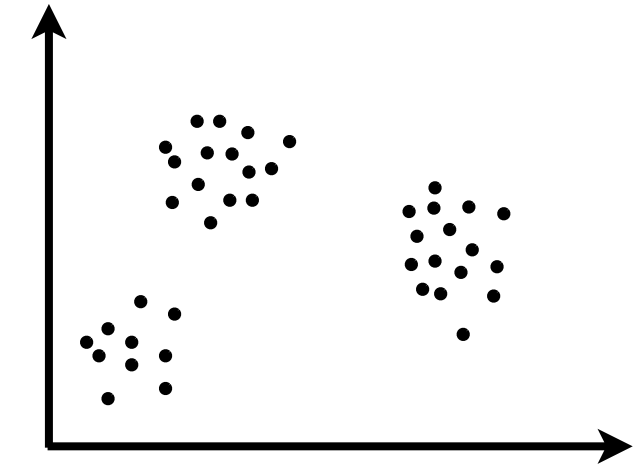

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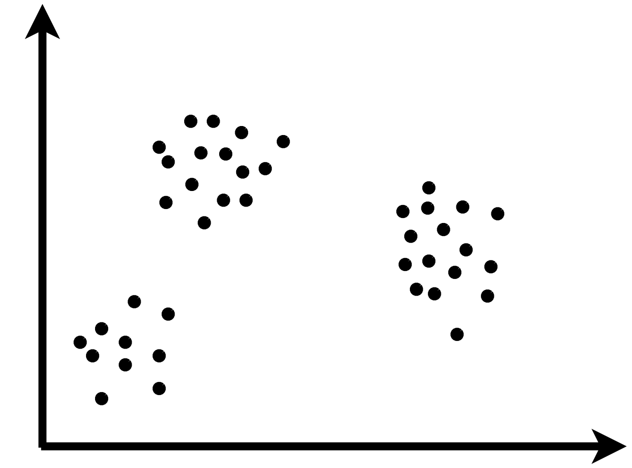

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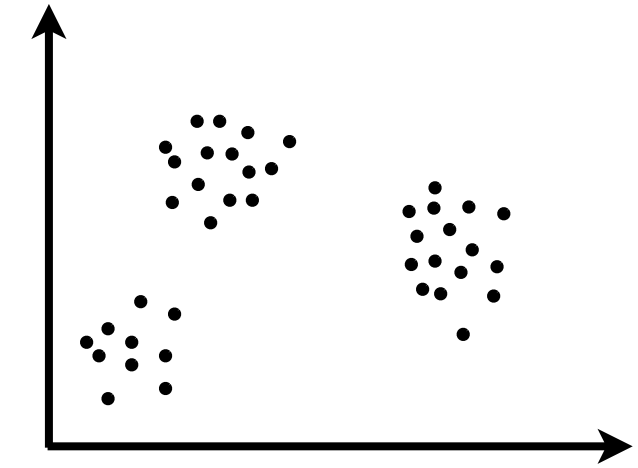

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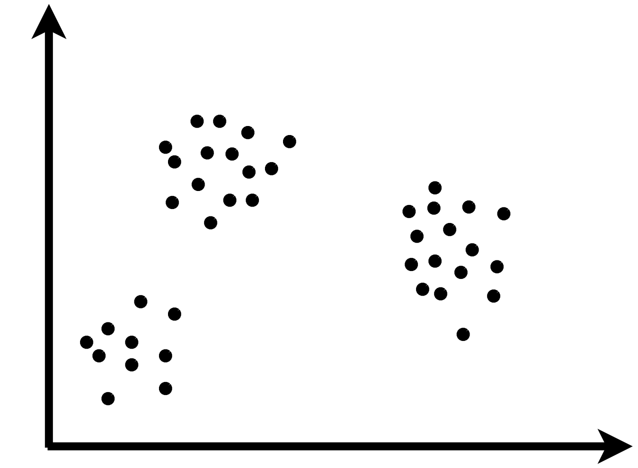

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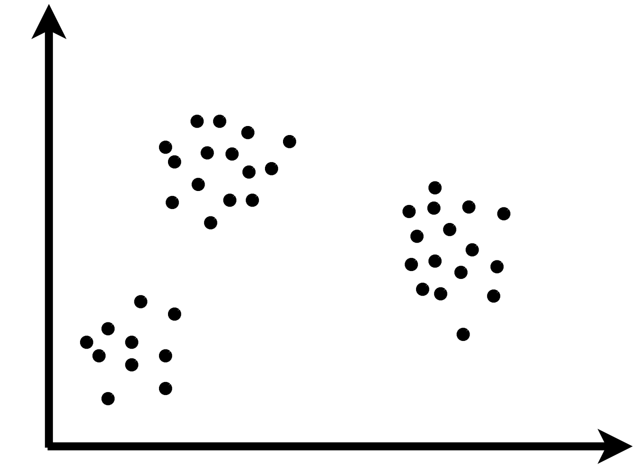

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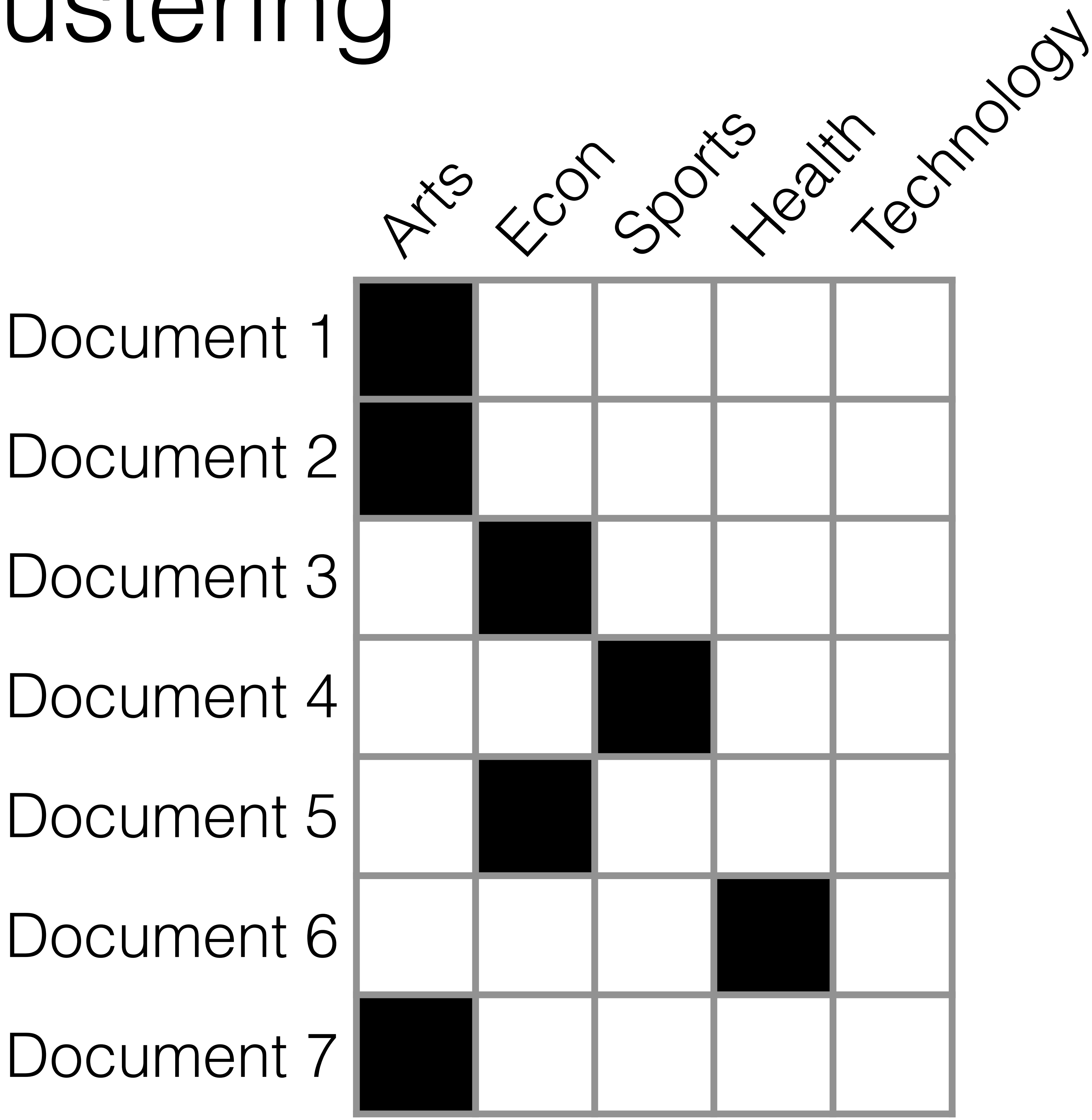
- Read [Blei, Jordan 2006] and code variational inference for the DPMM

Resources online

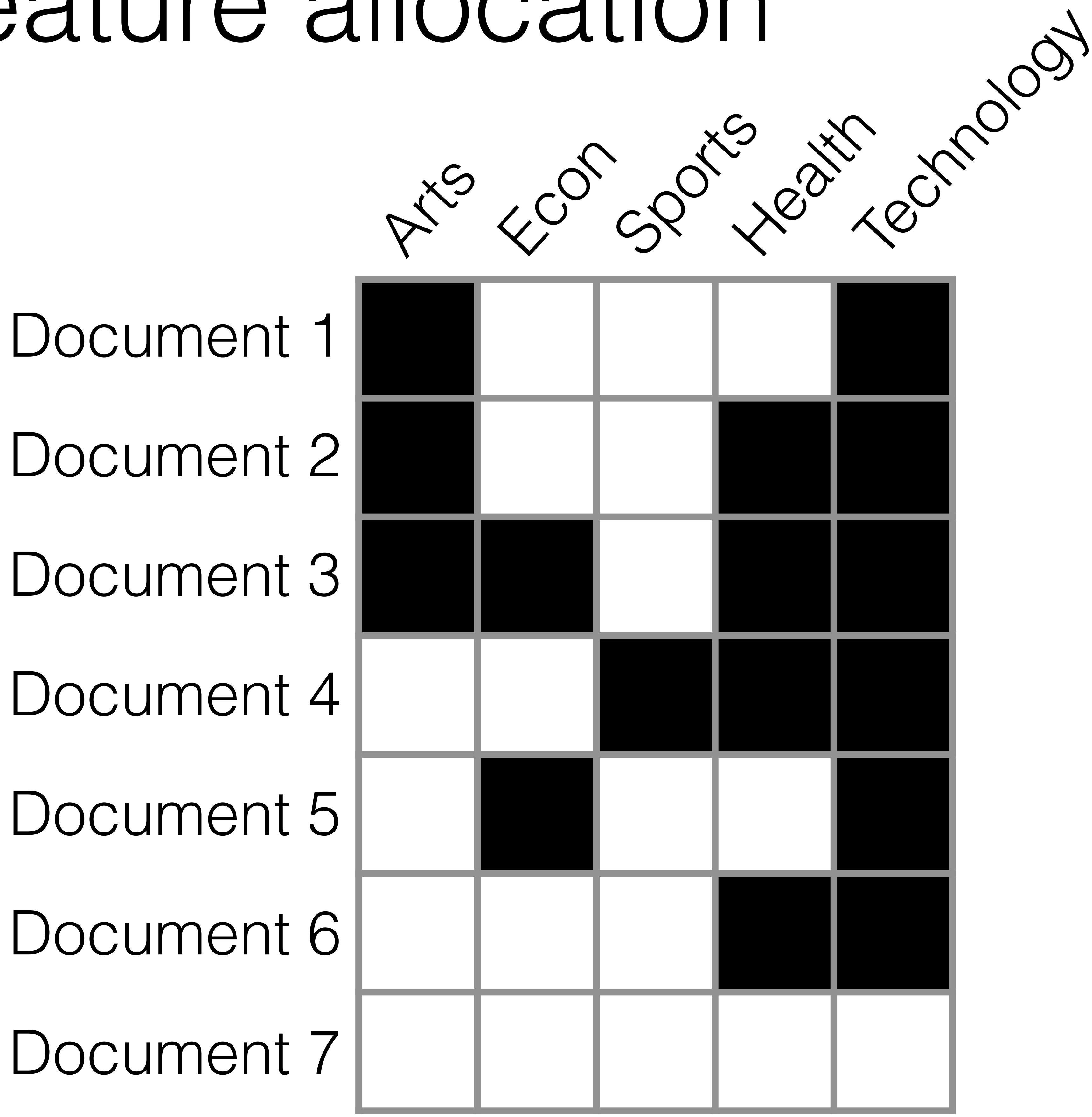
www.tamarabroderick.com/tutorials.html

[Tiia Monto, https://commons.wikimedia.org/wiki/File:Trail_and_mountain.jpg]

Clustering


Arts
Econ
Sports
Health
Technology
Document 1
Document 2
Document 3
Document 4
Document 5
Document 6
Document 7

Feature allocation



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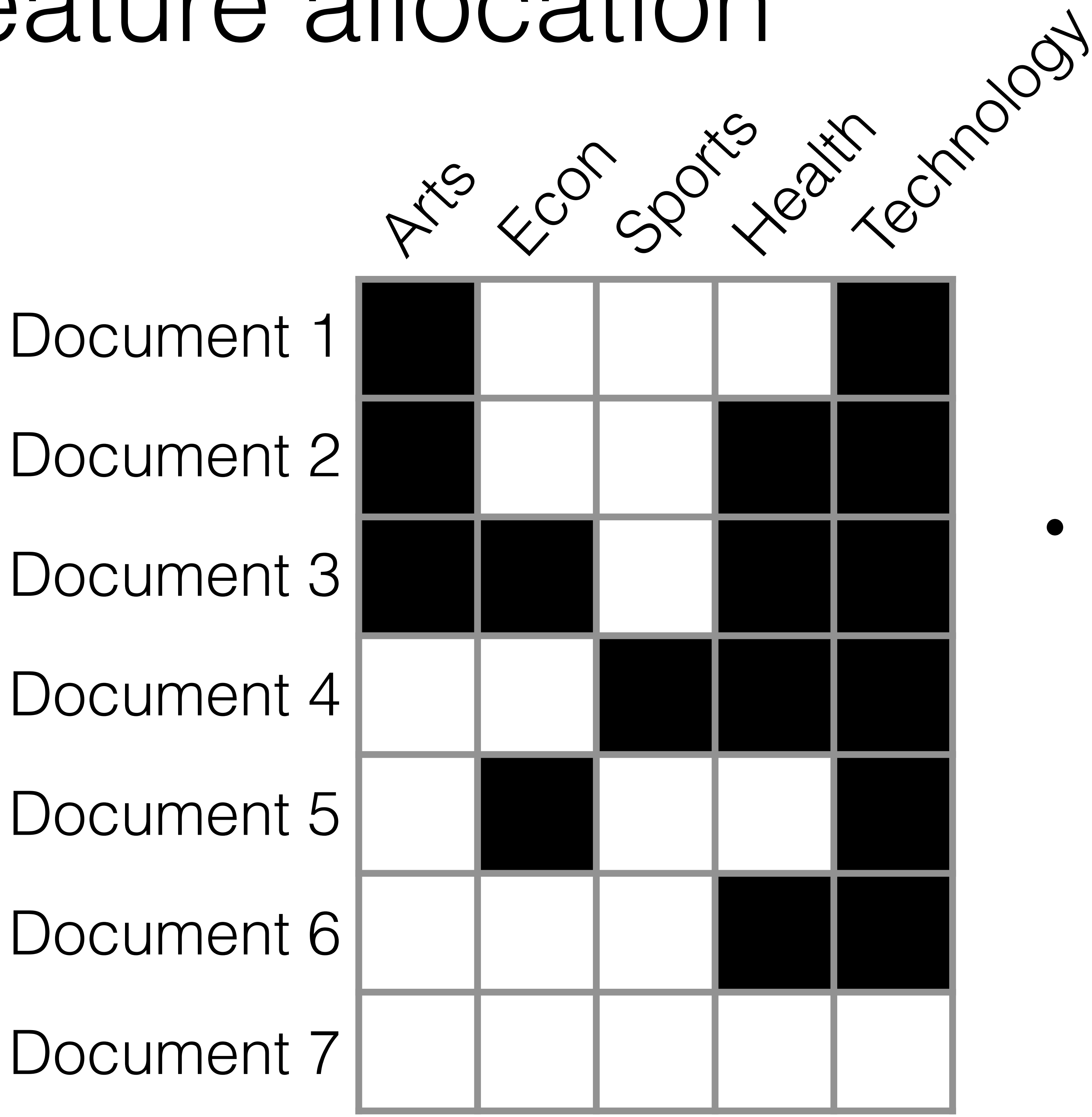
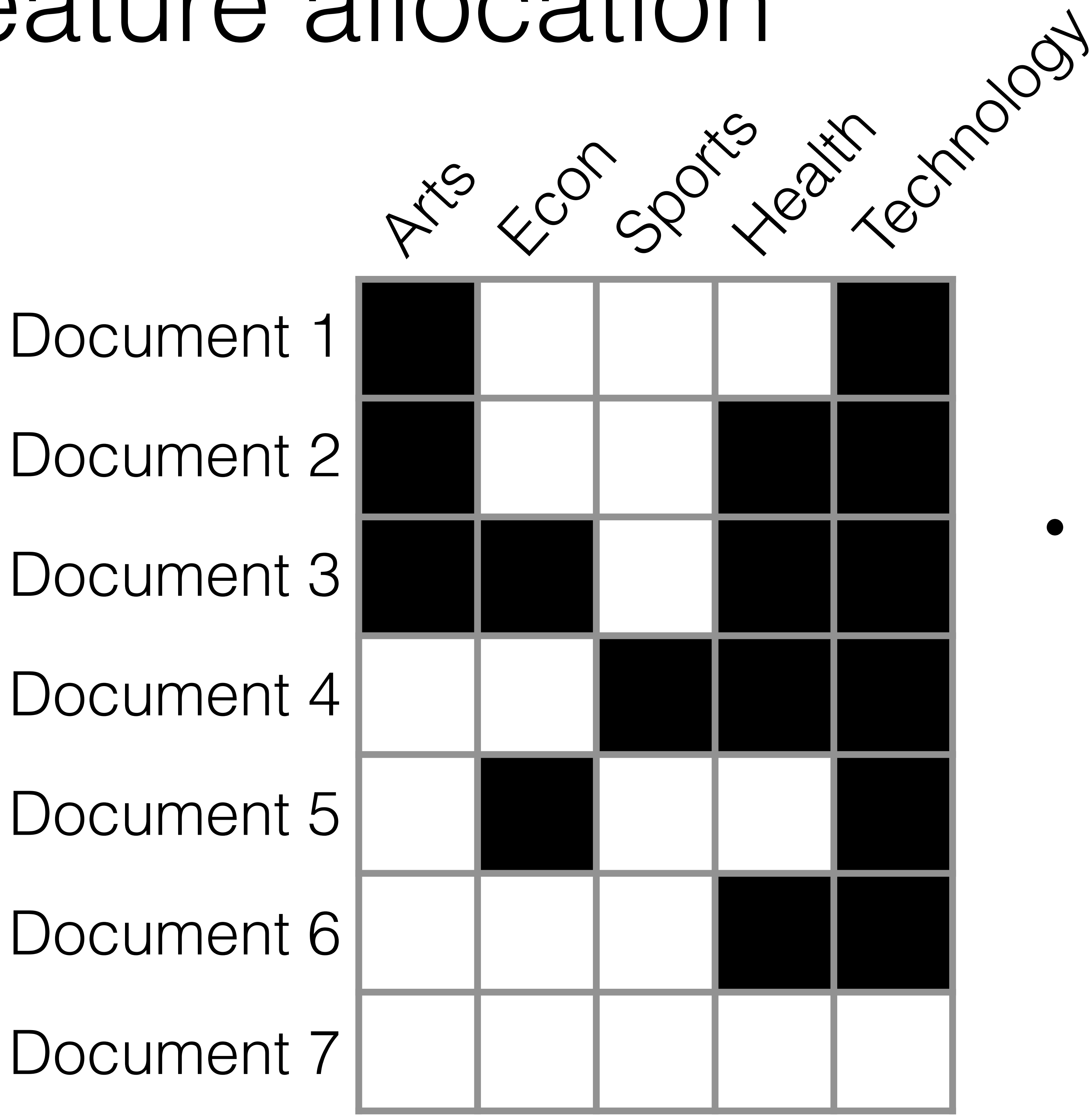


- Indian buffet process

Feature allocation



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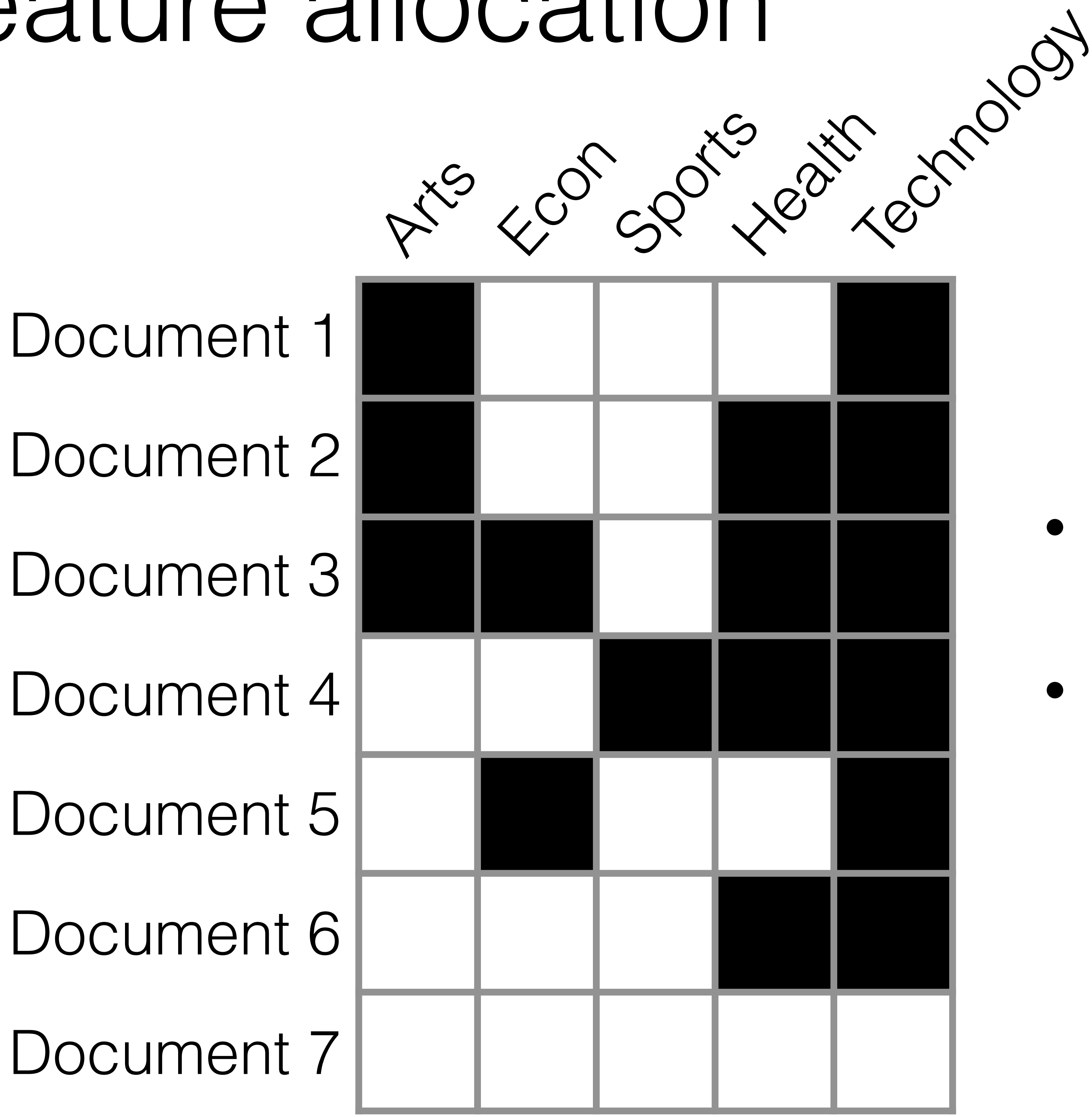

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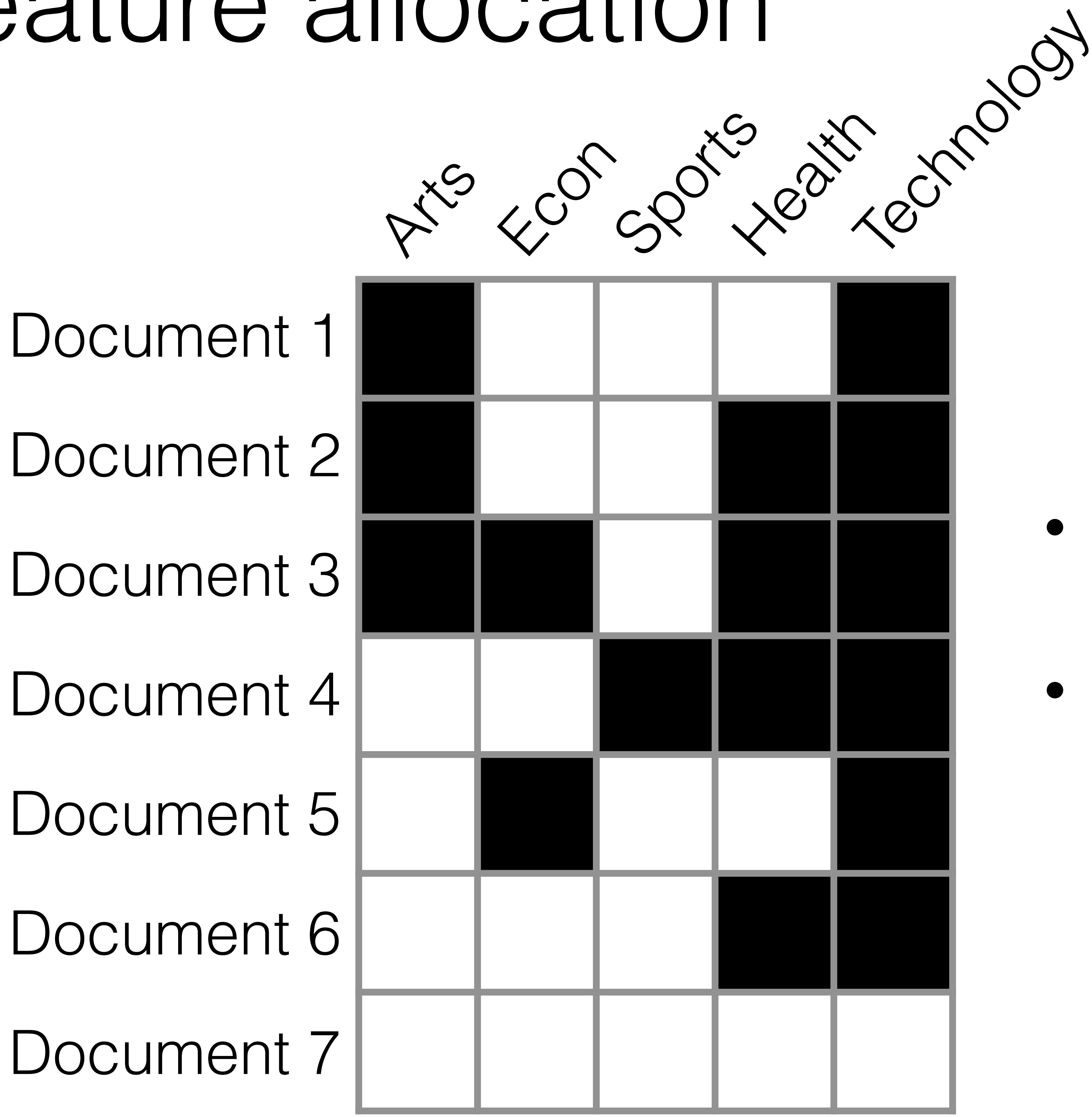

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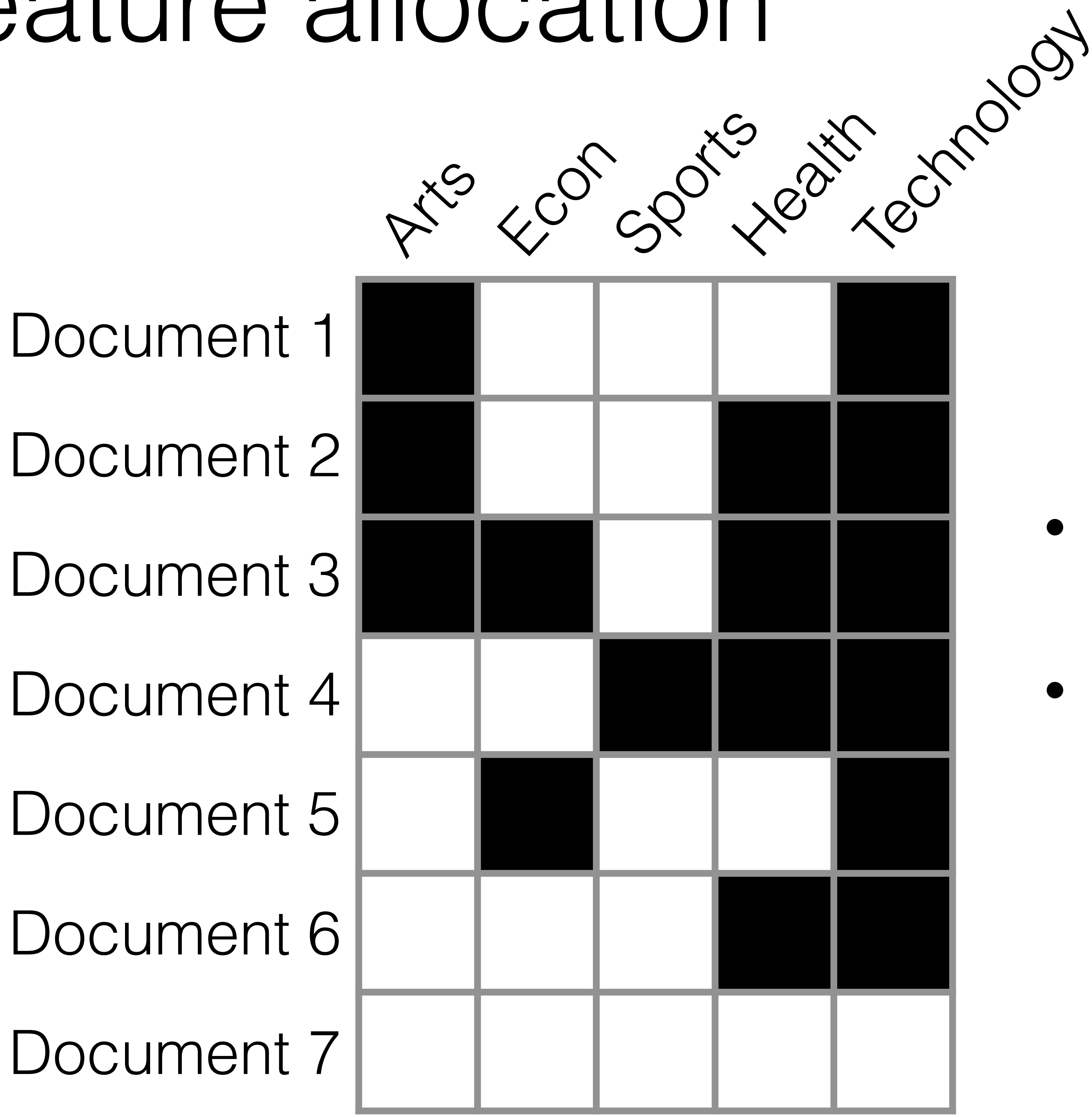

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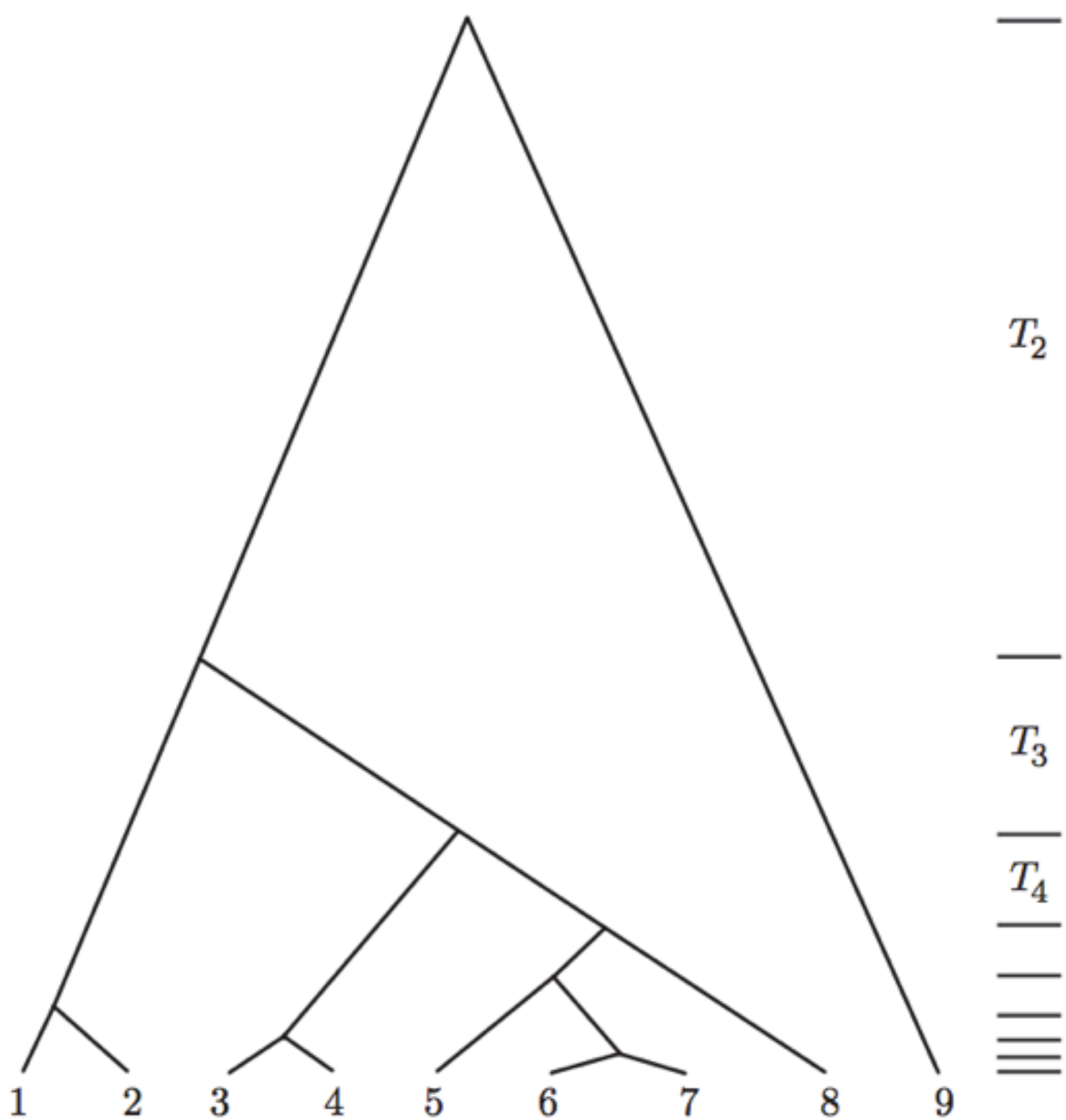
Genealogy, trees, beyond trees



[Wakeley 2008]

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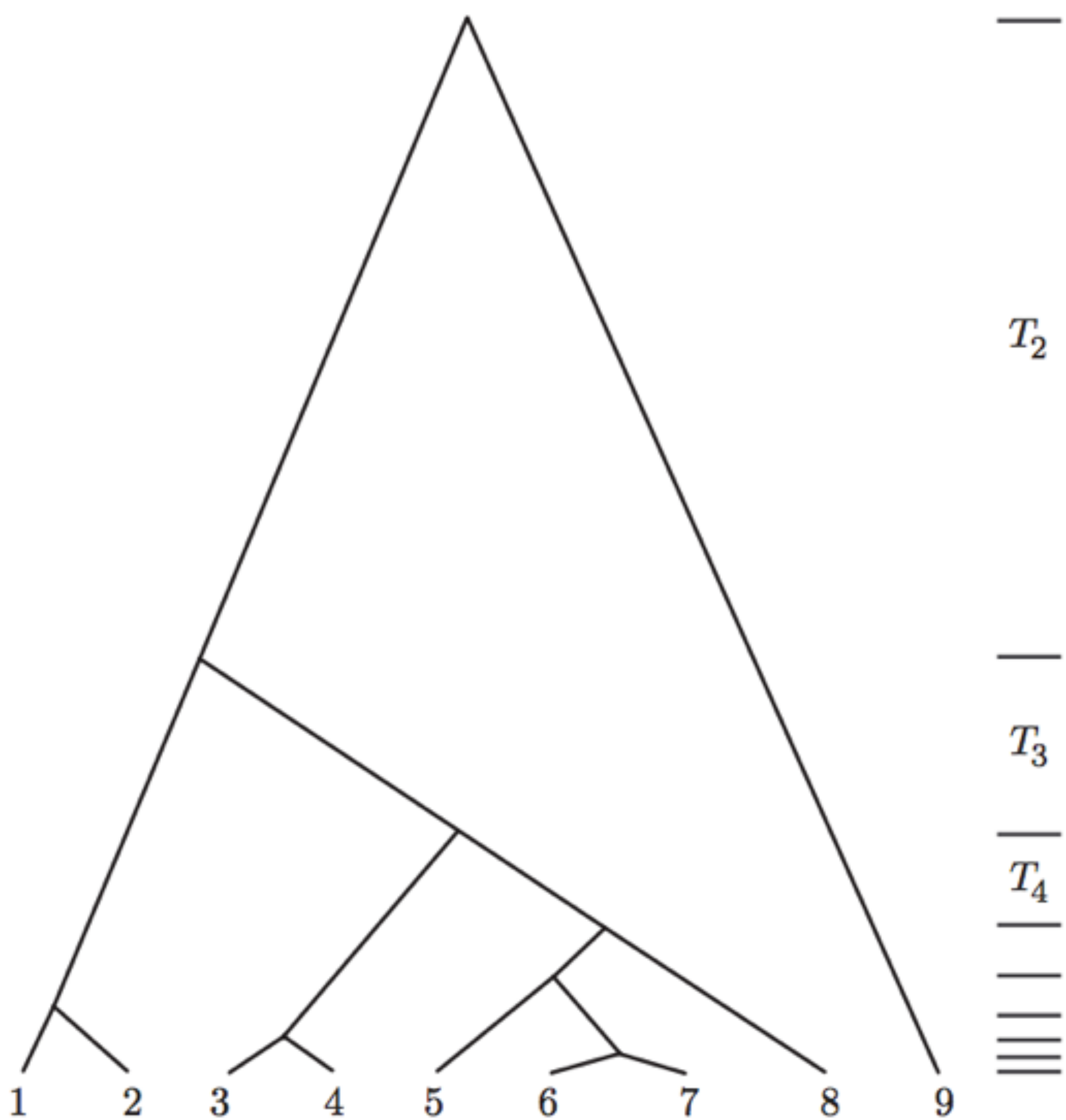

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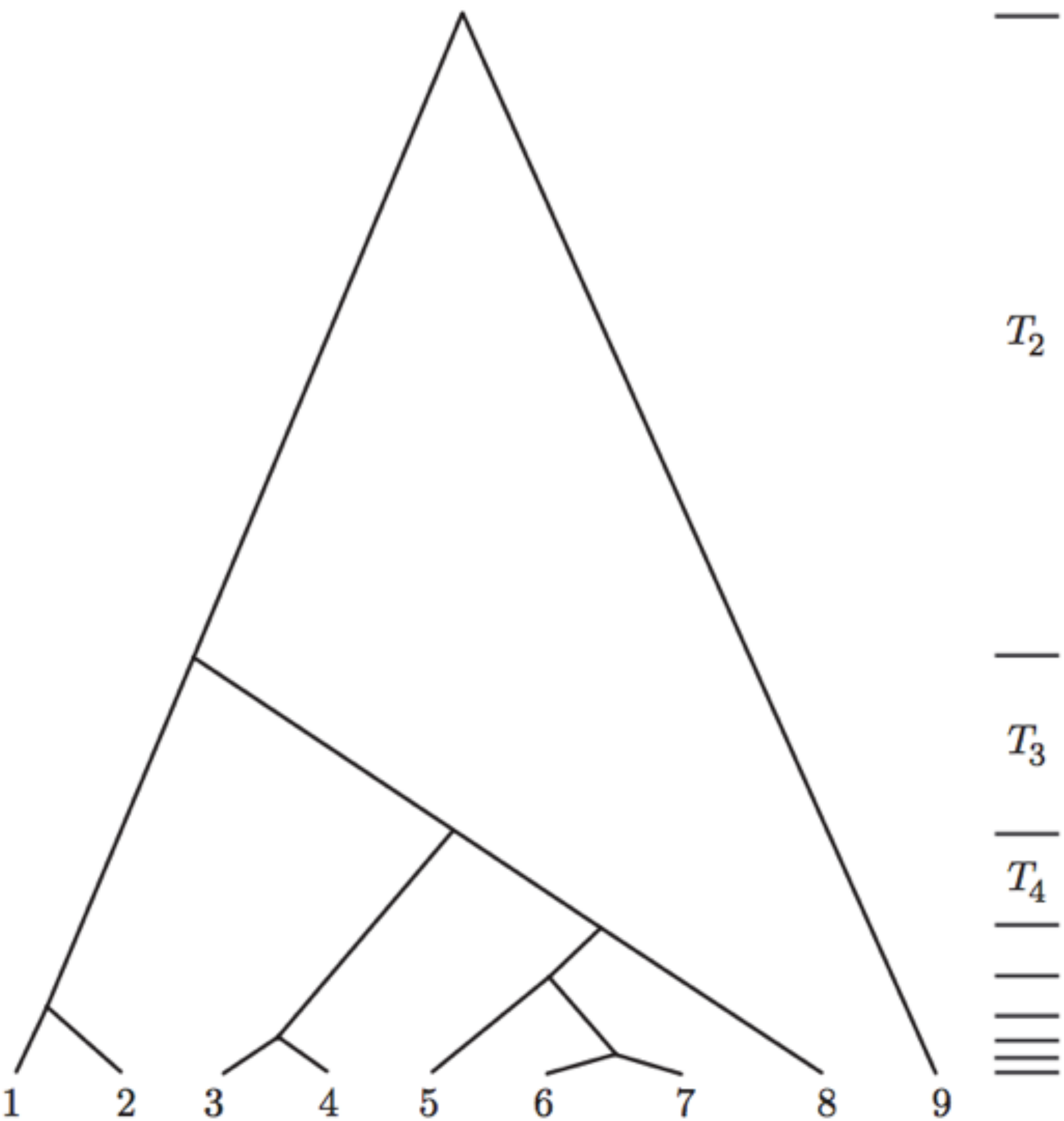

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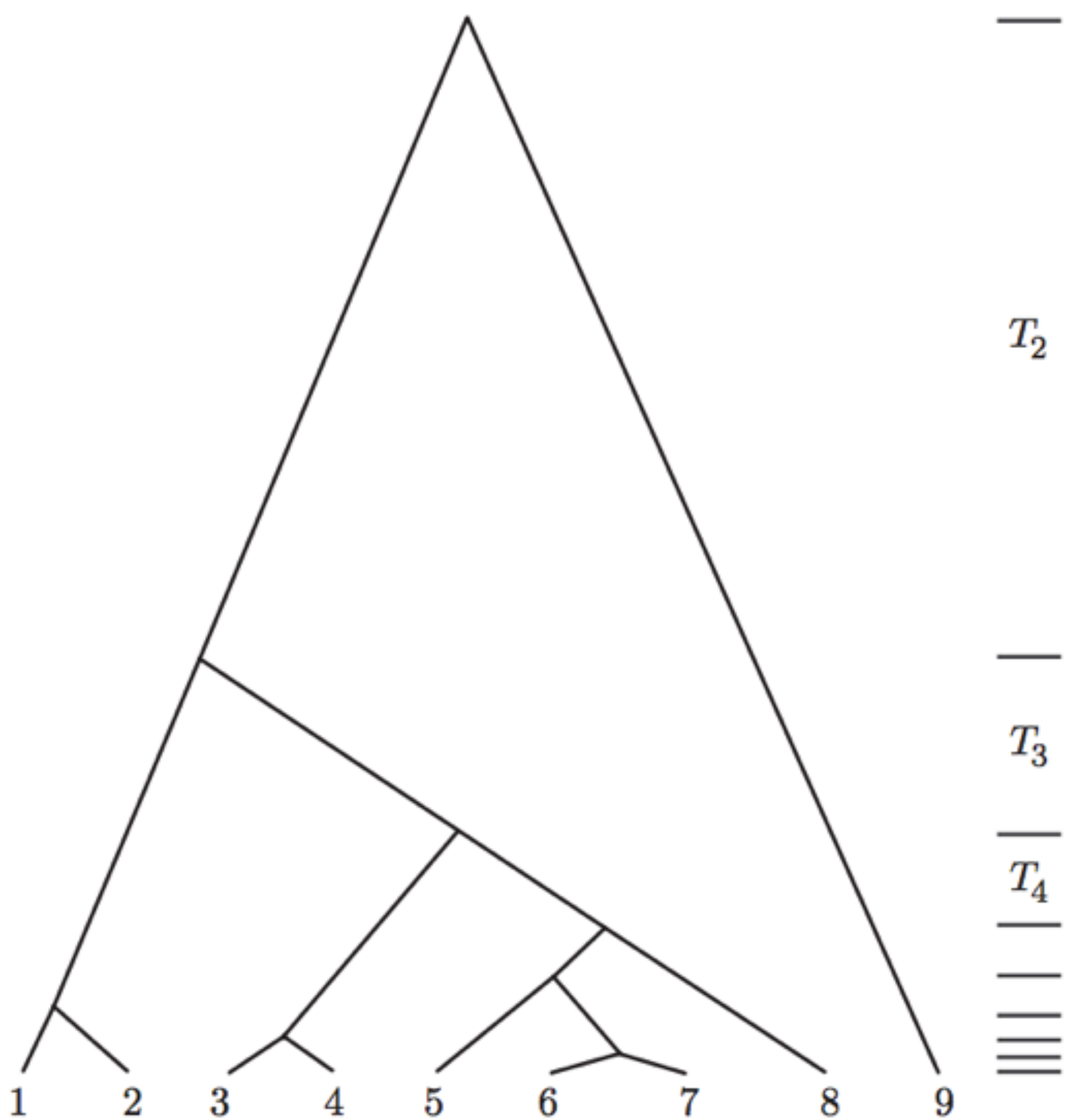

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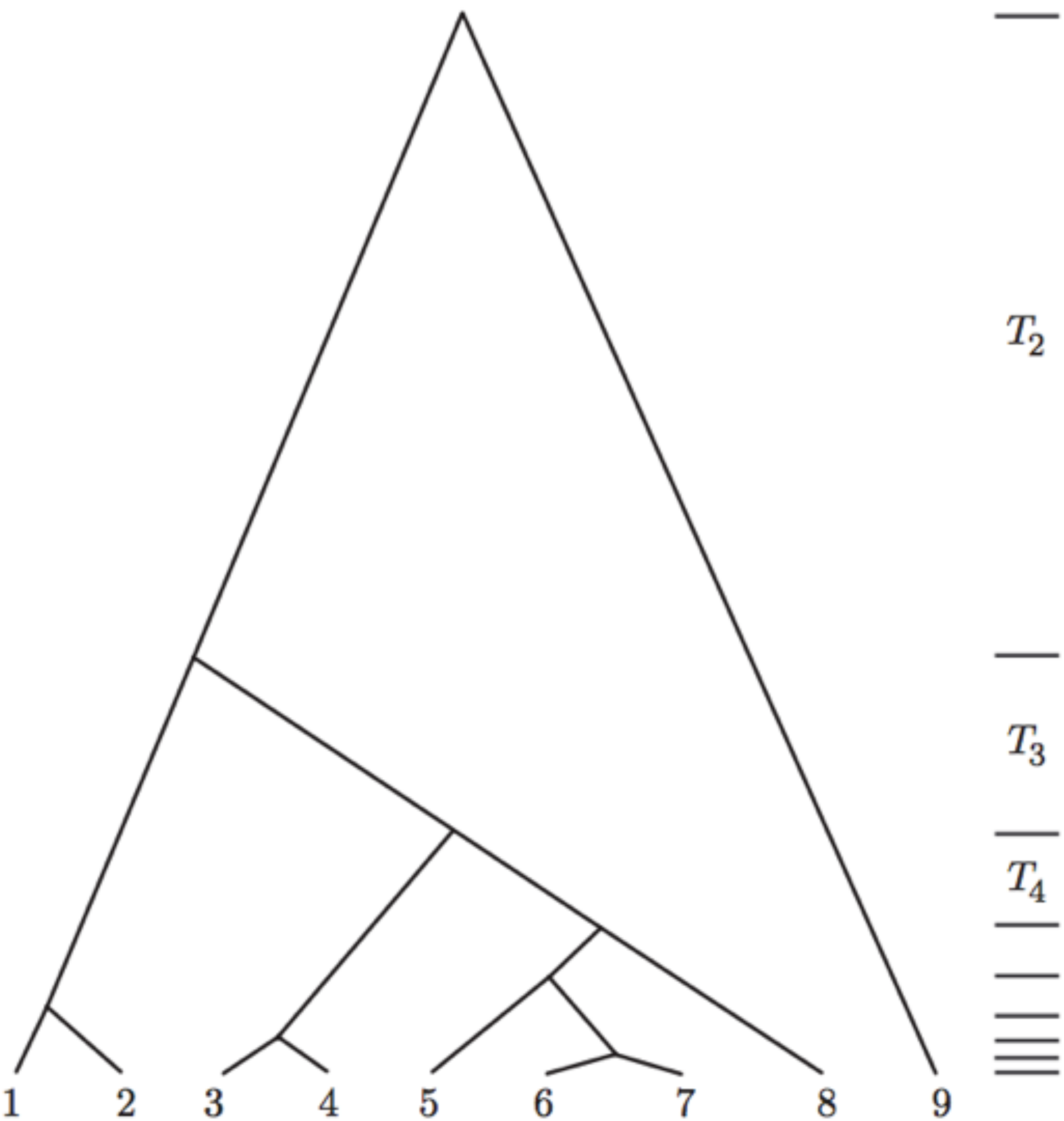

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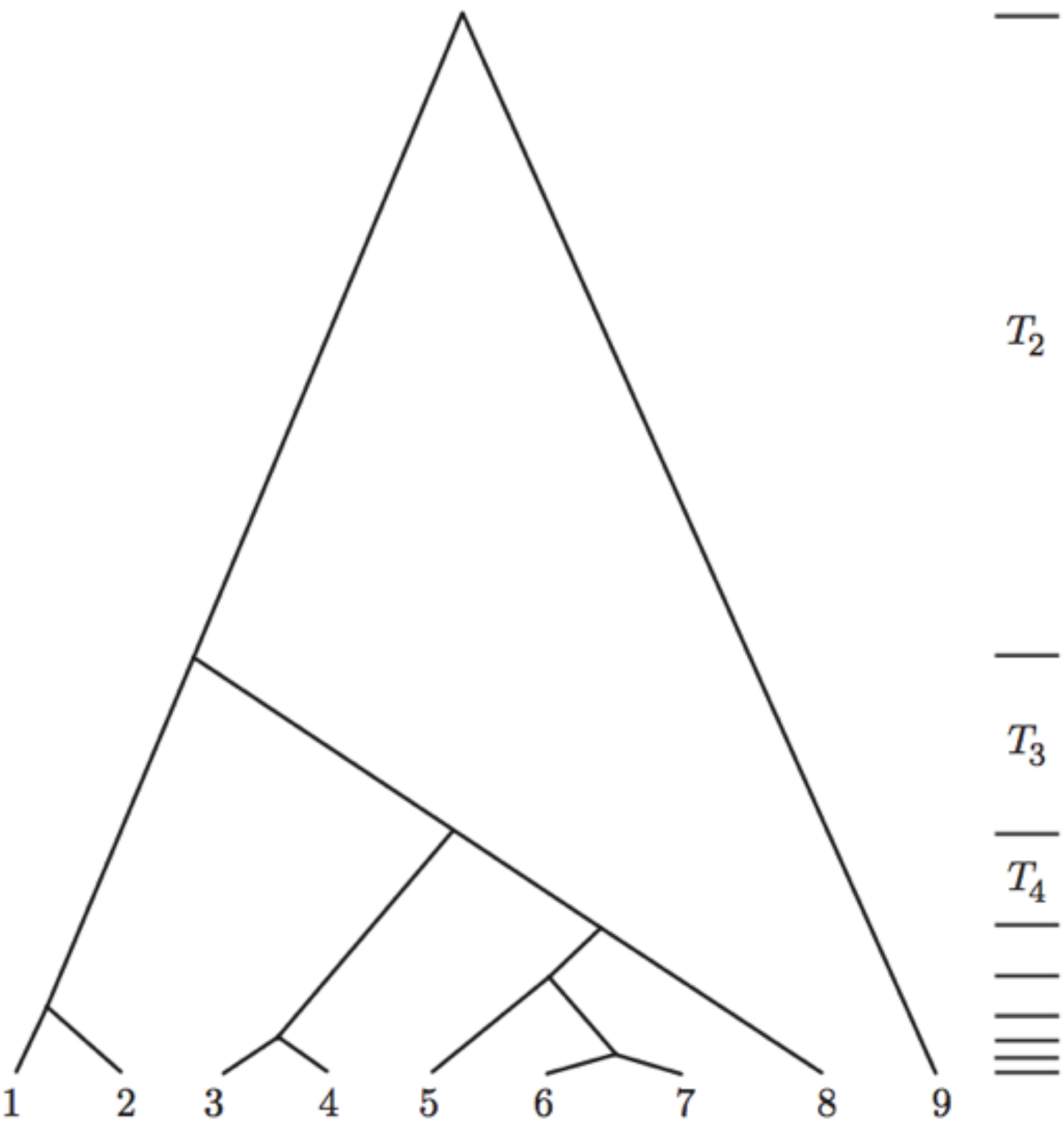

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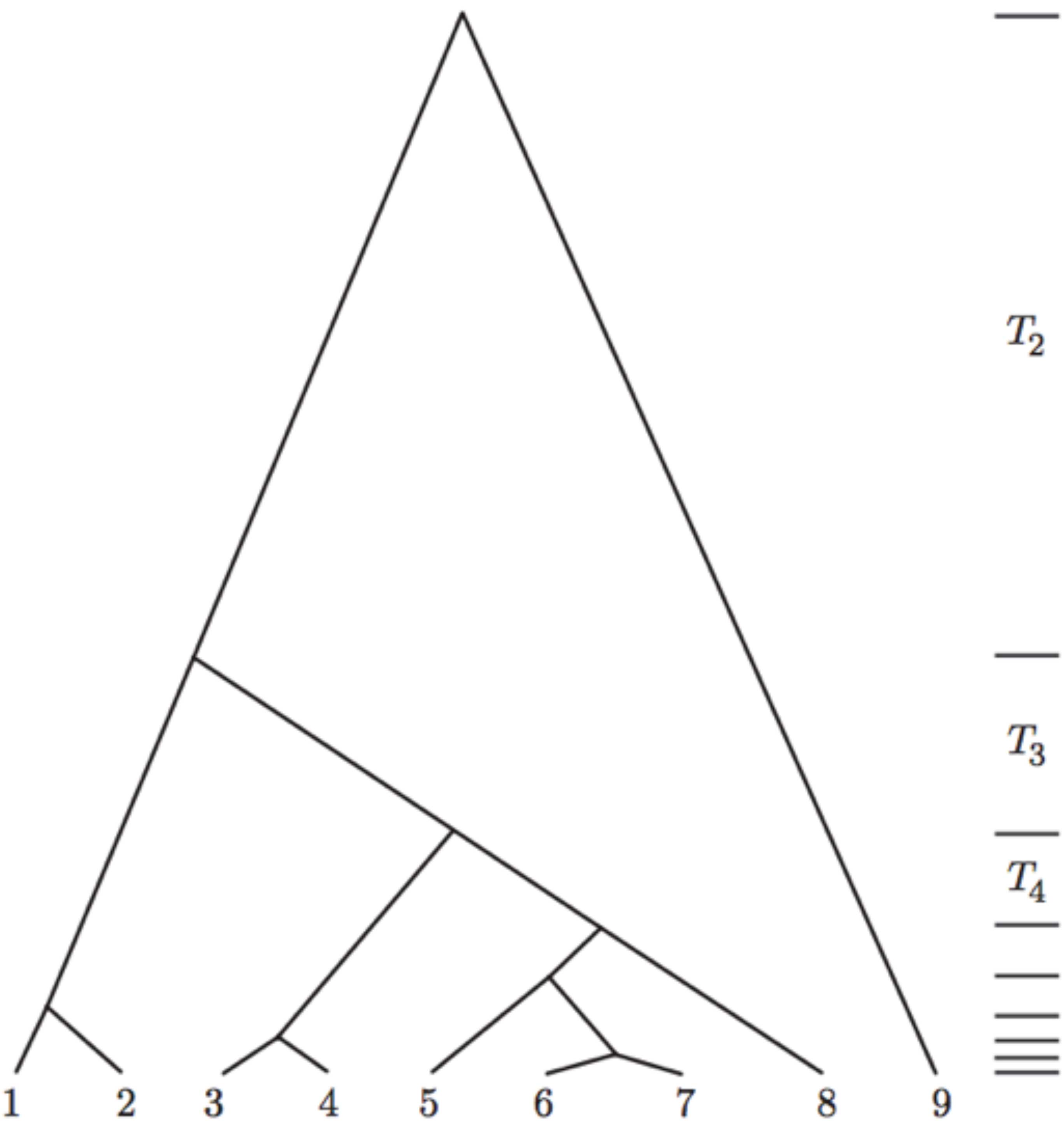

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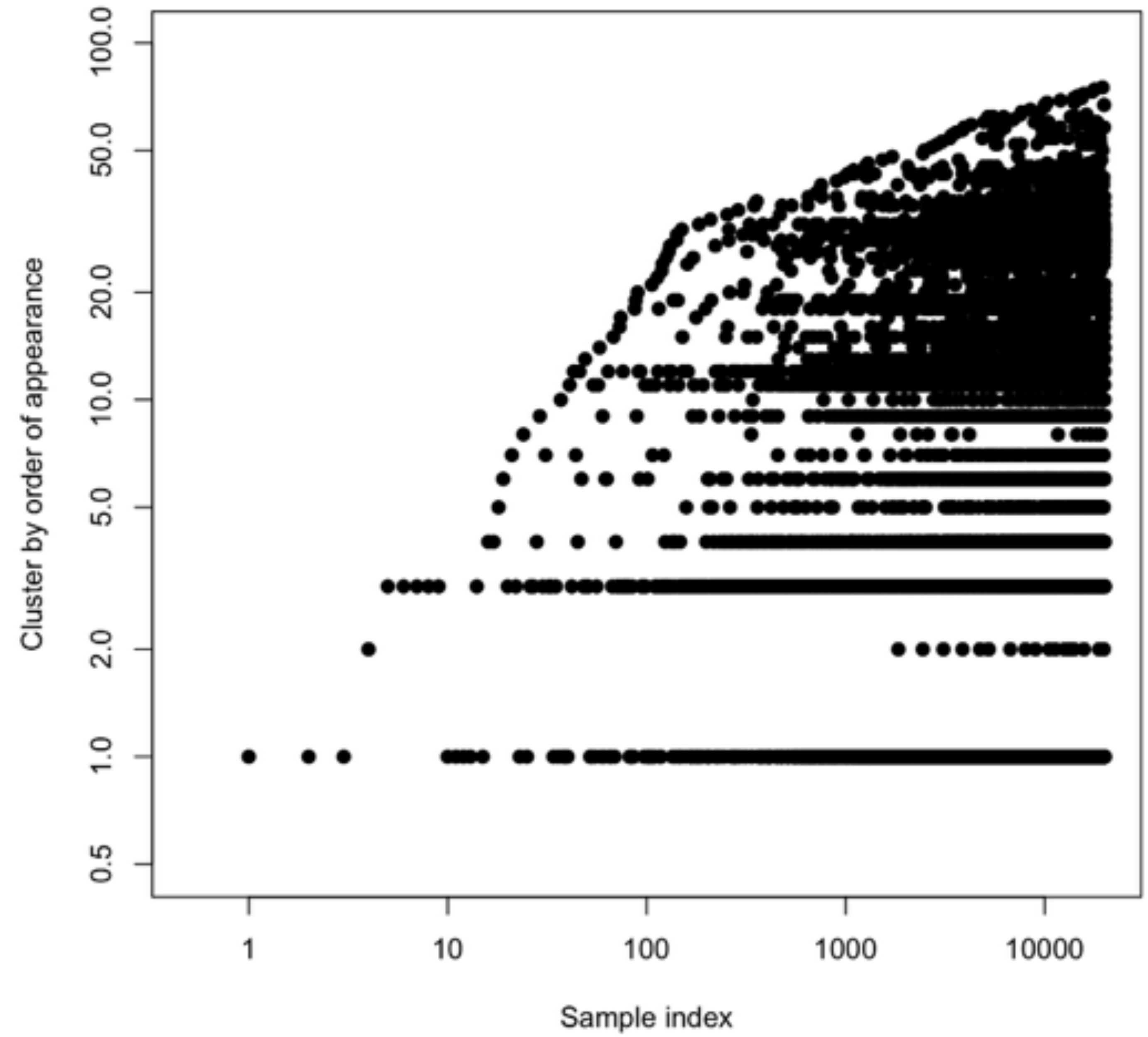
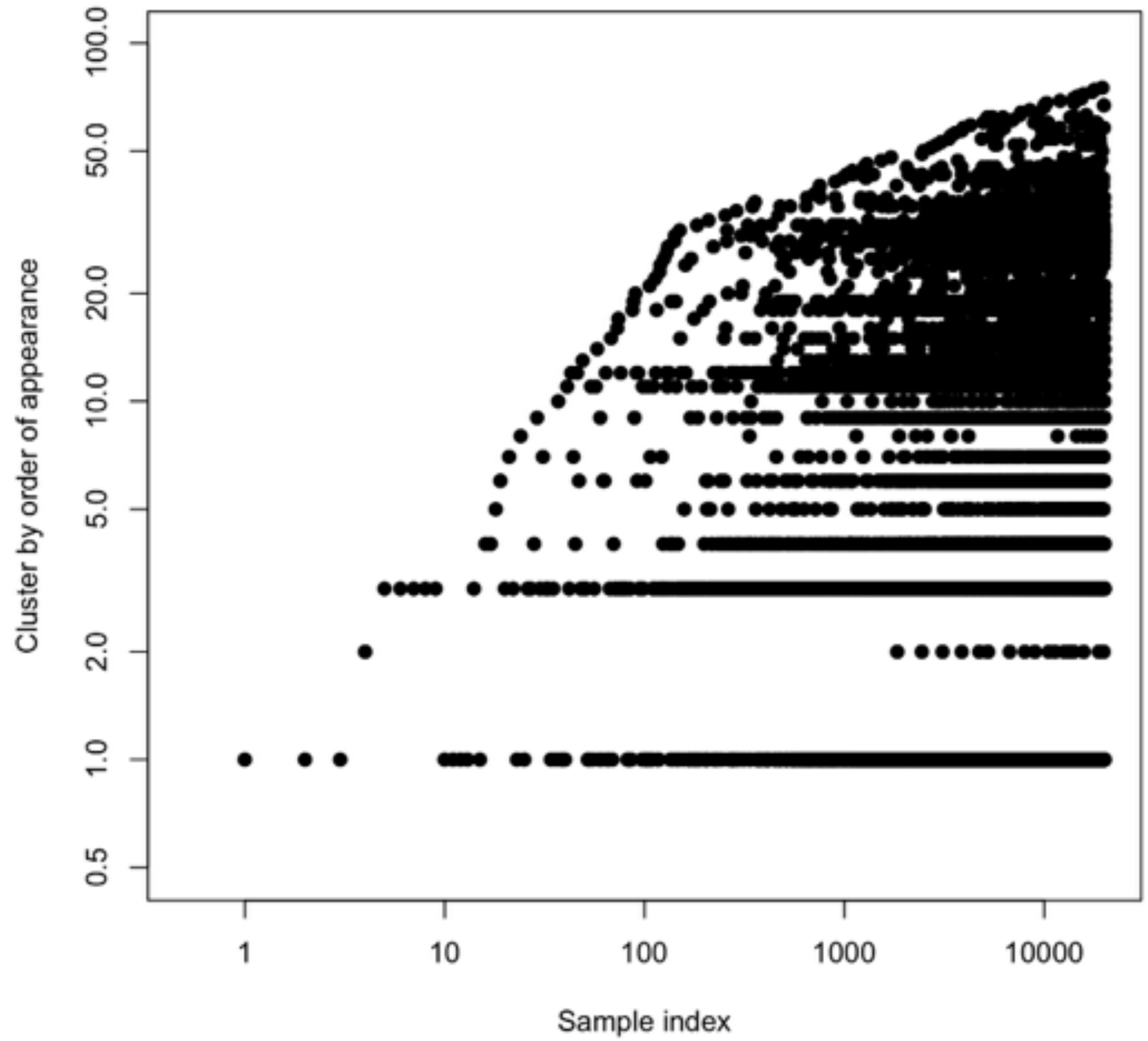
Power laws



Cluster by order of appearance
100.0
50.0
20.0
10.0
5.0
2.0
1.0
0.5
1
10
100
1000
10000
Sample index

Power laws

- K_N := # clusters occupied by N data points



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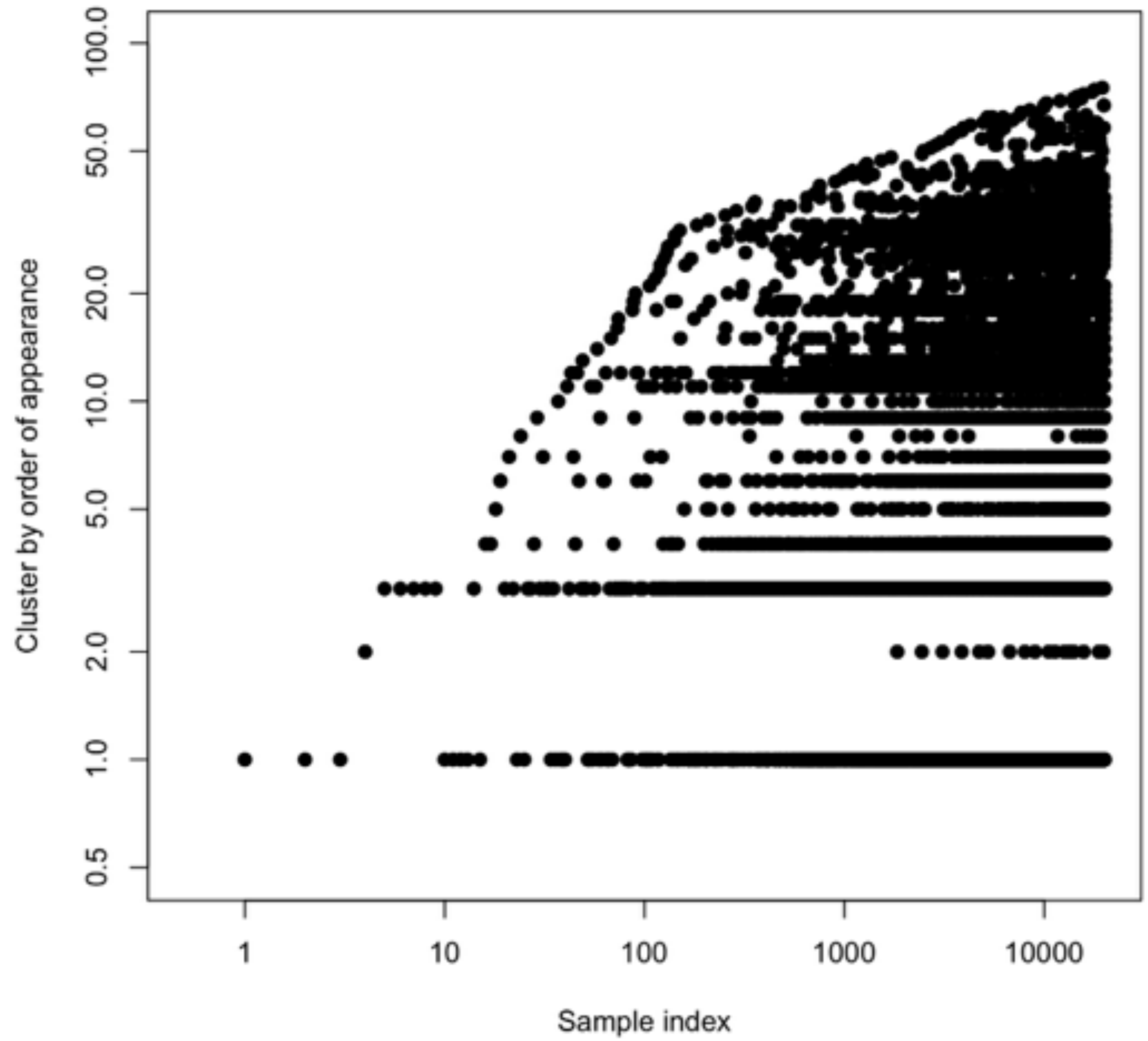

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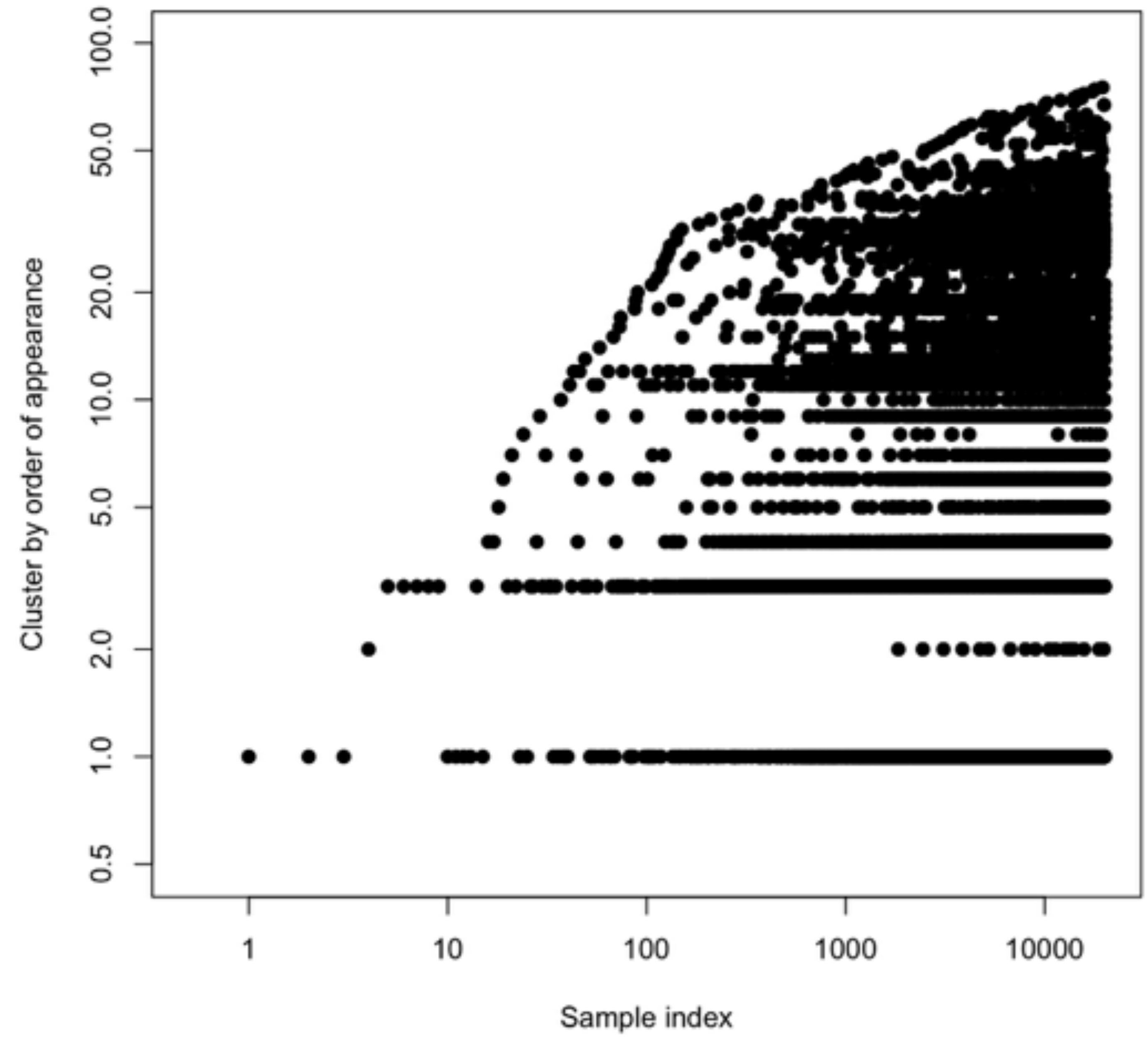

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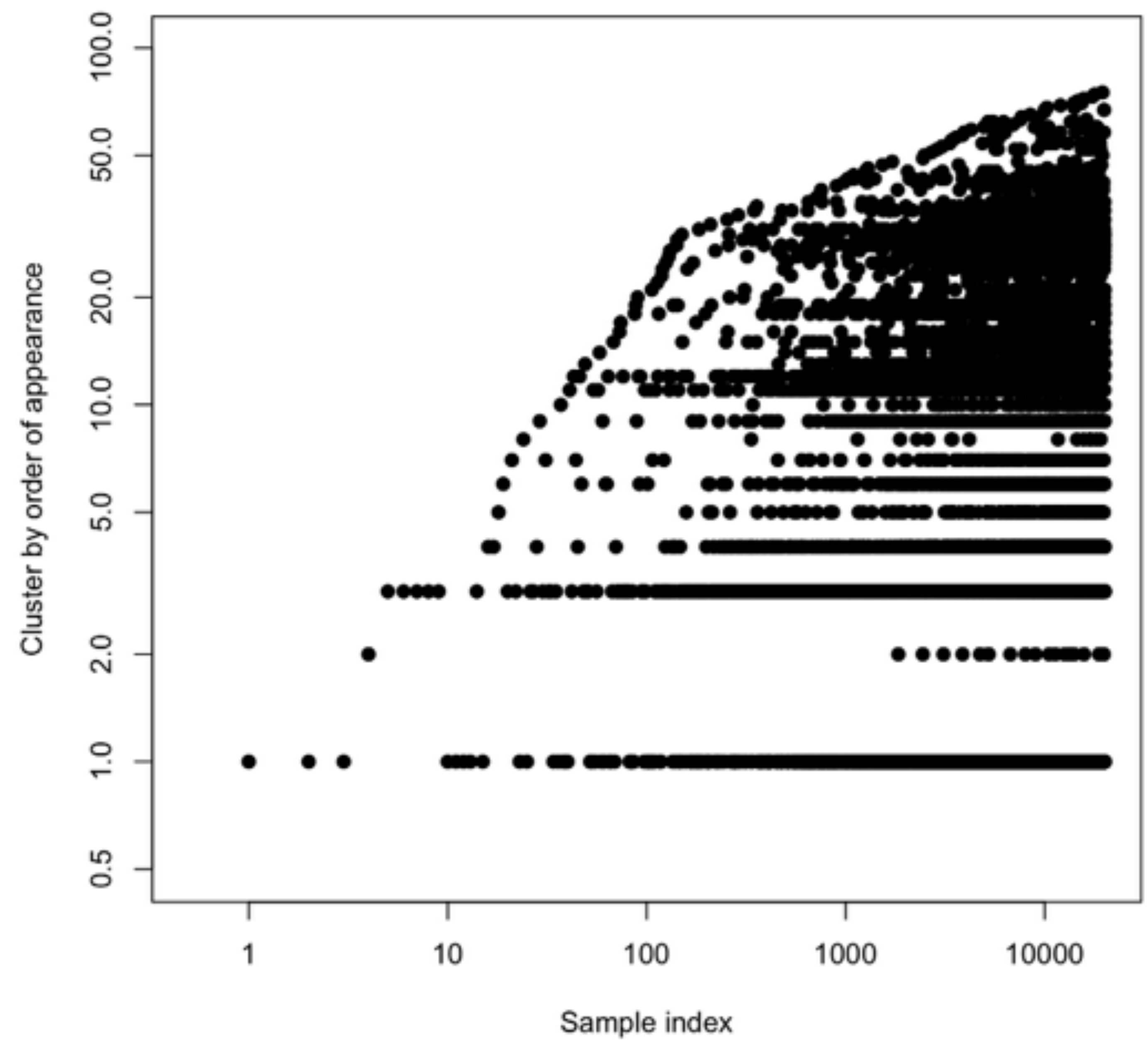

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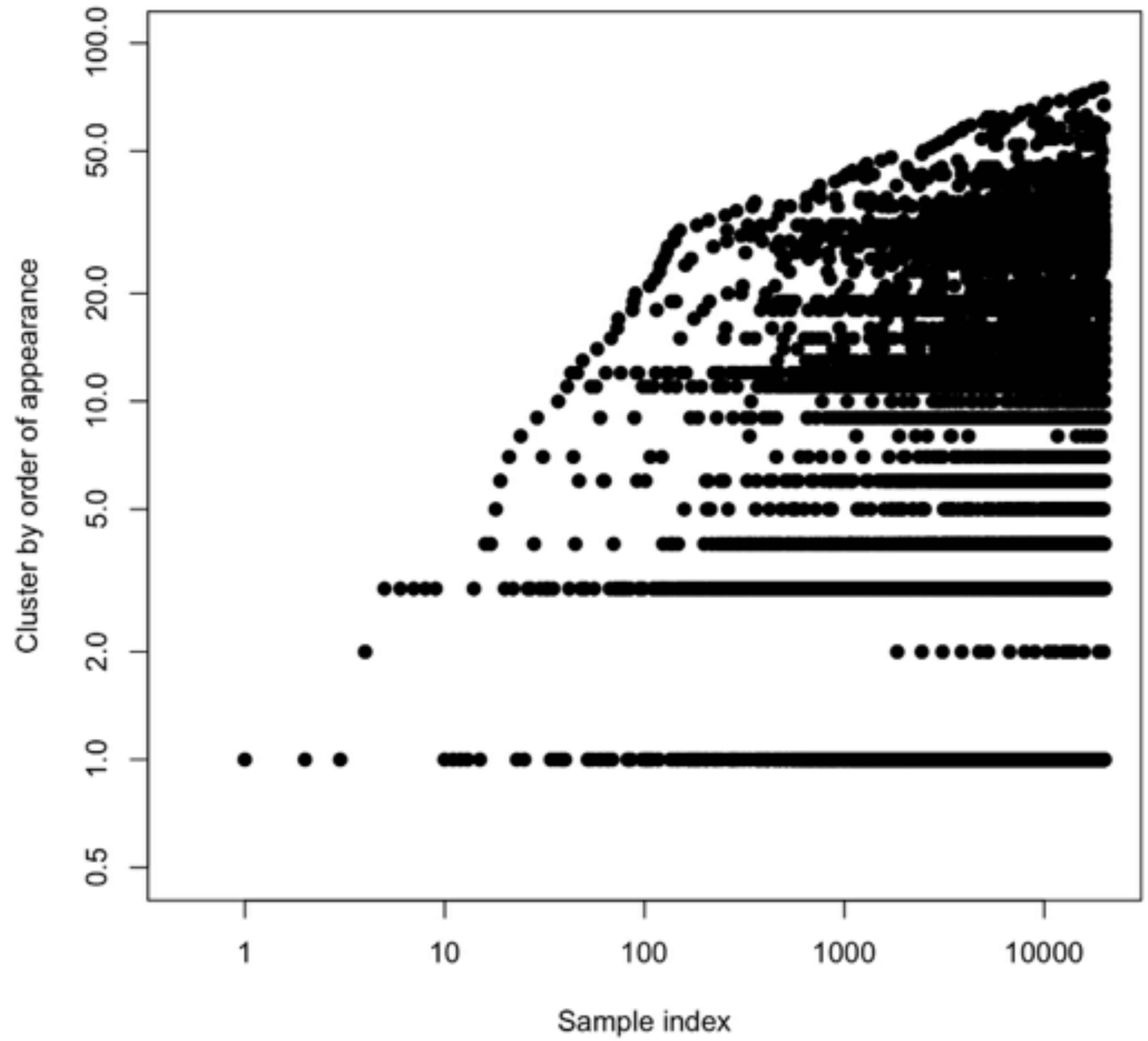

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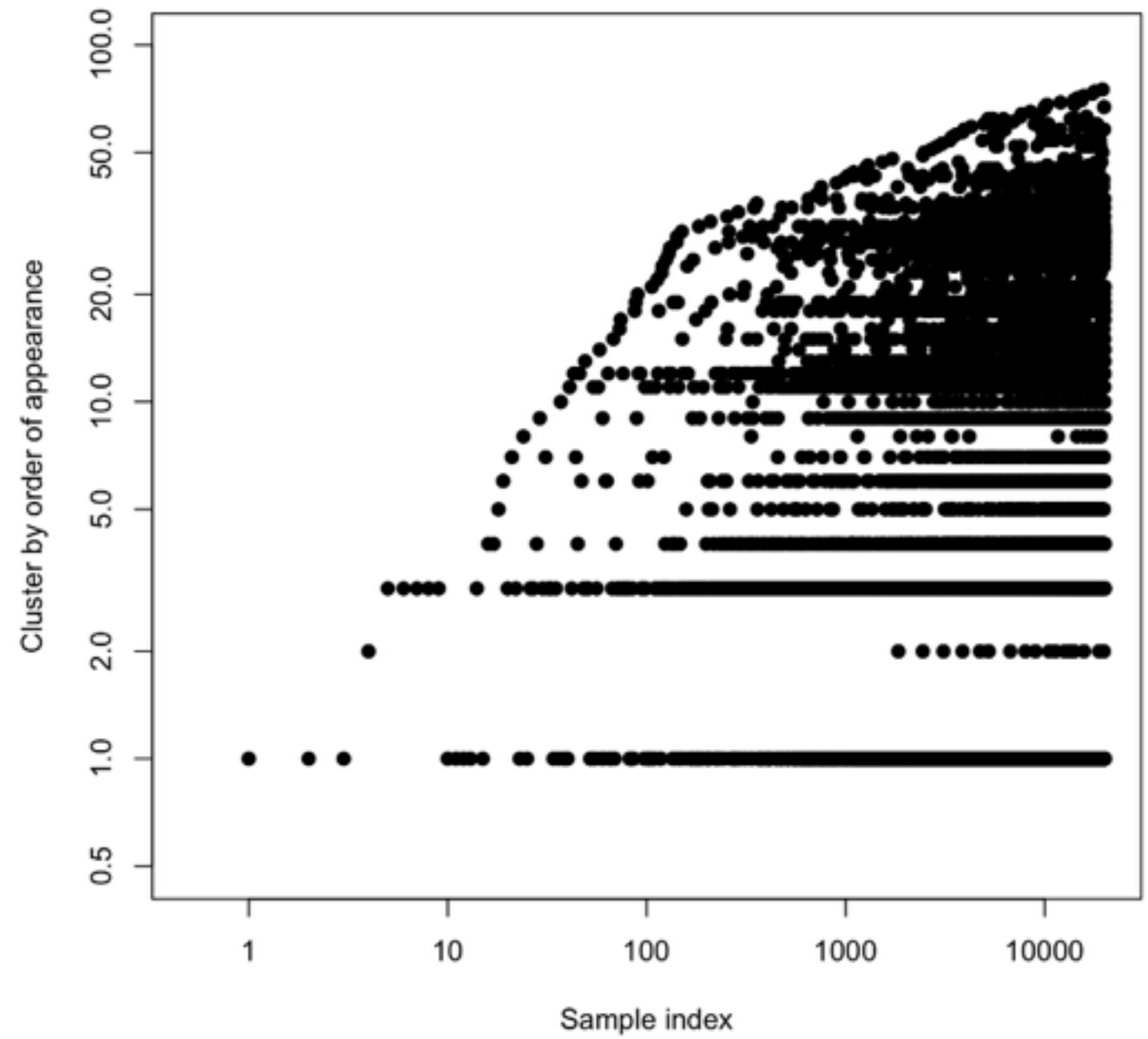

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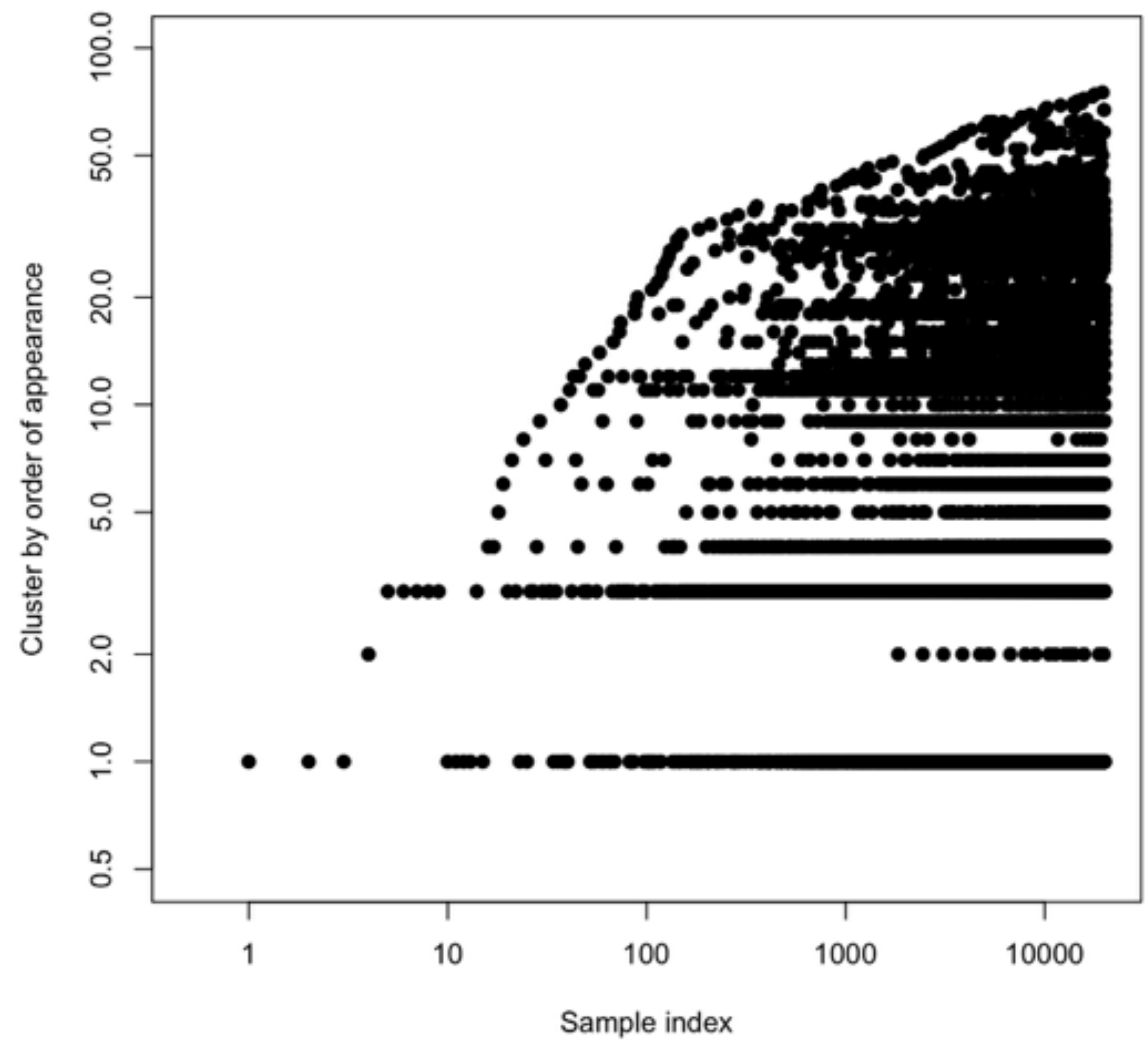

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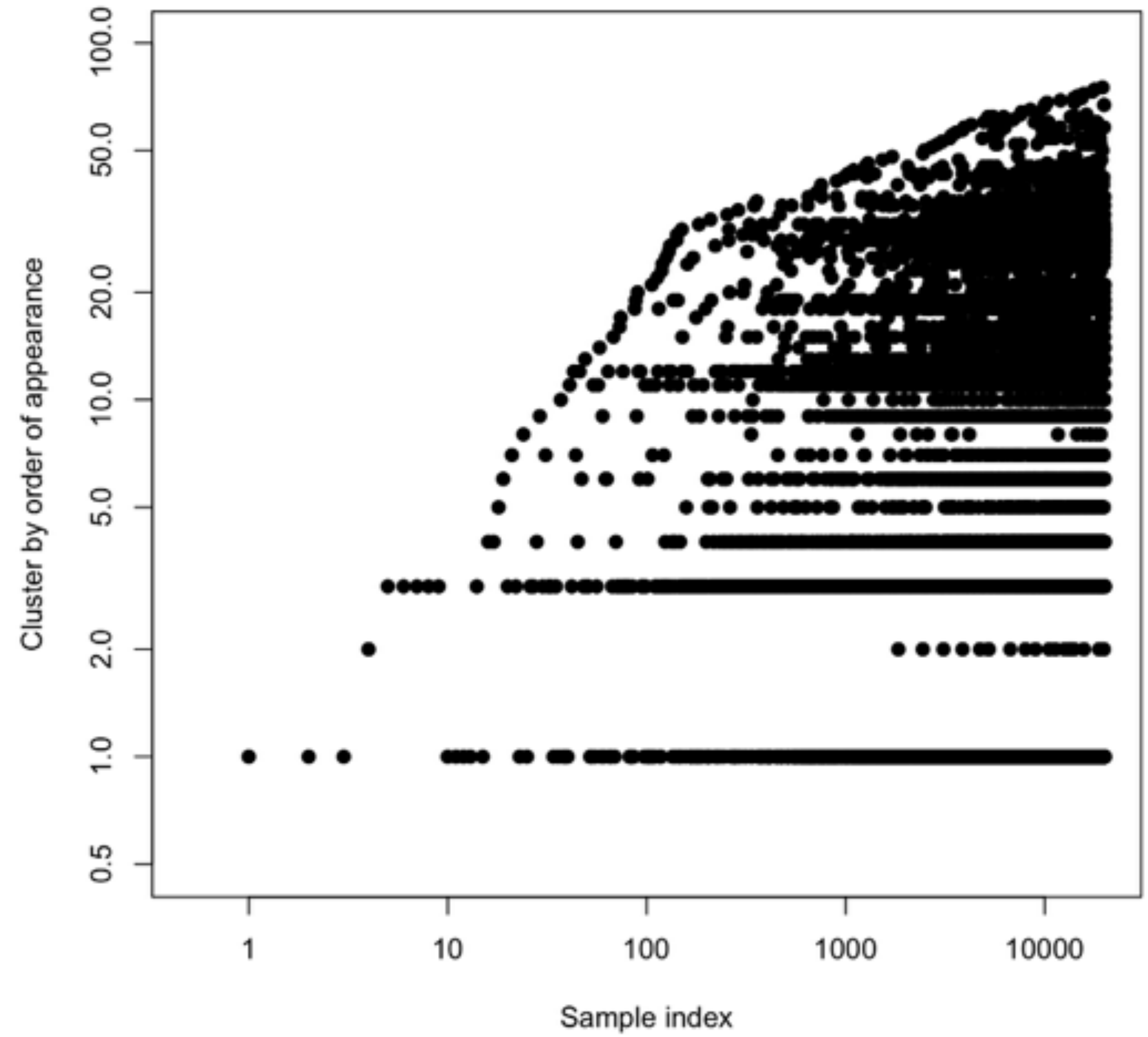

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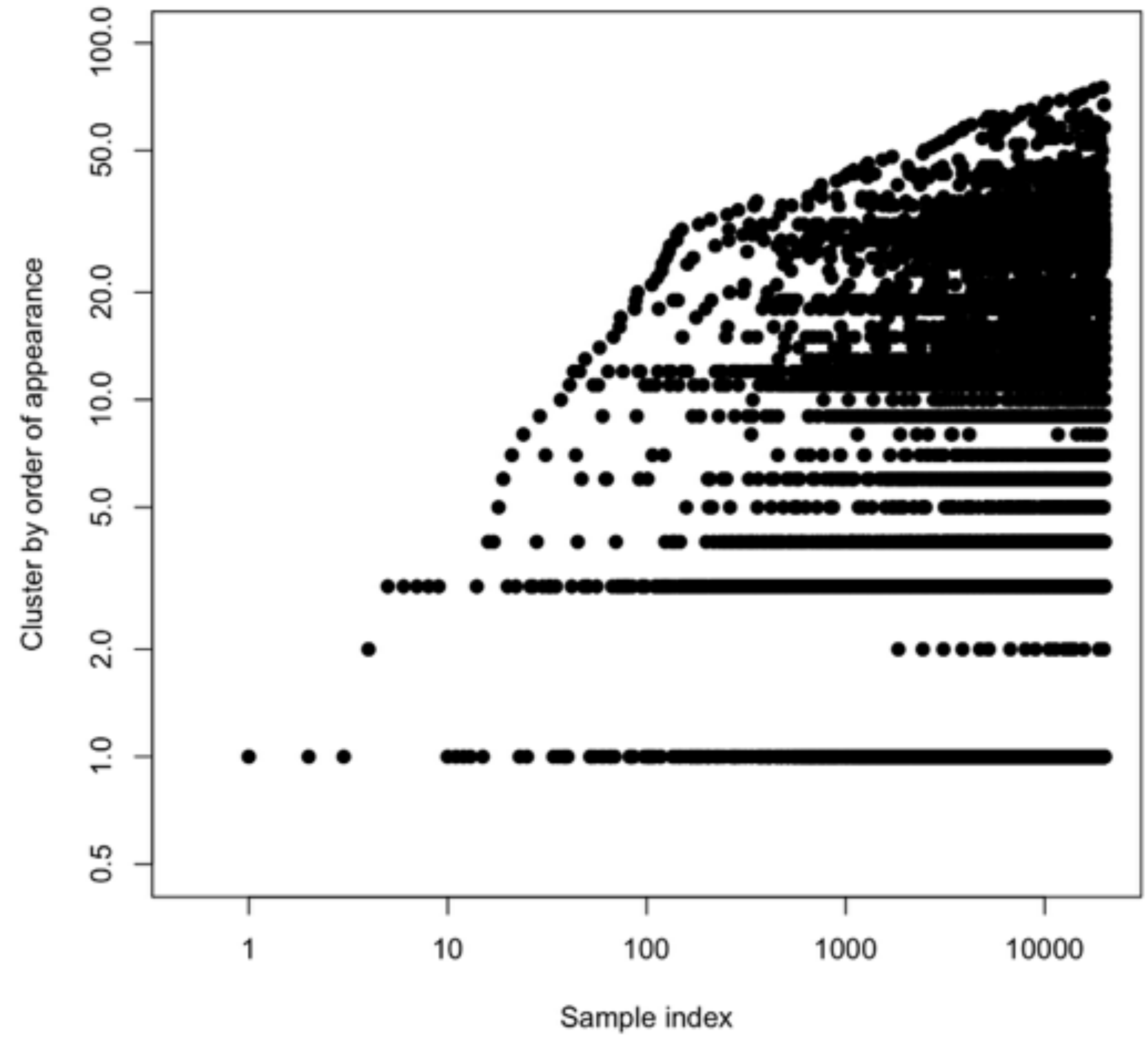

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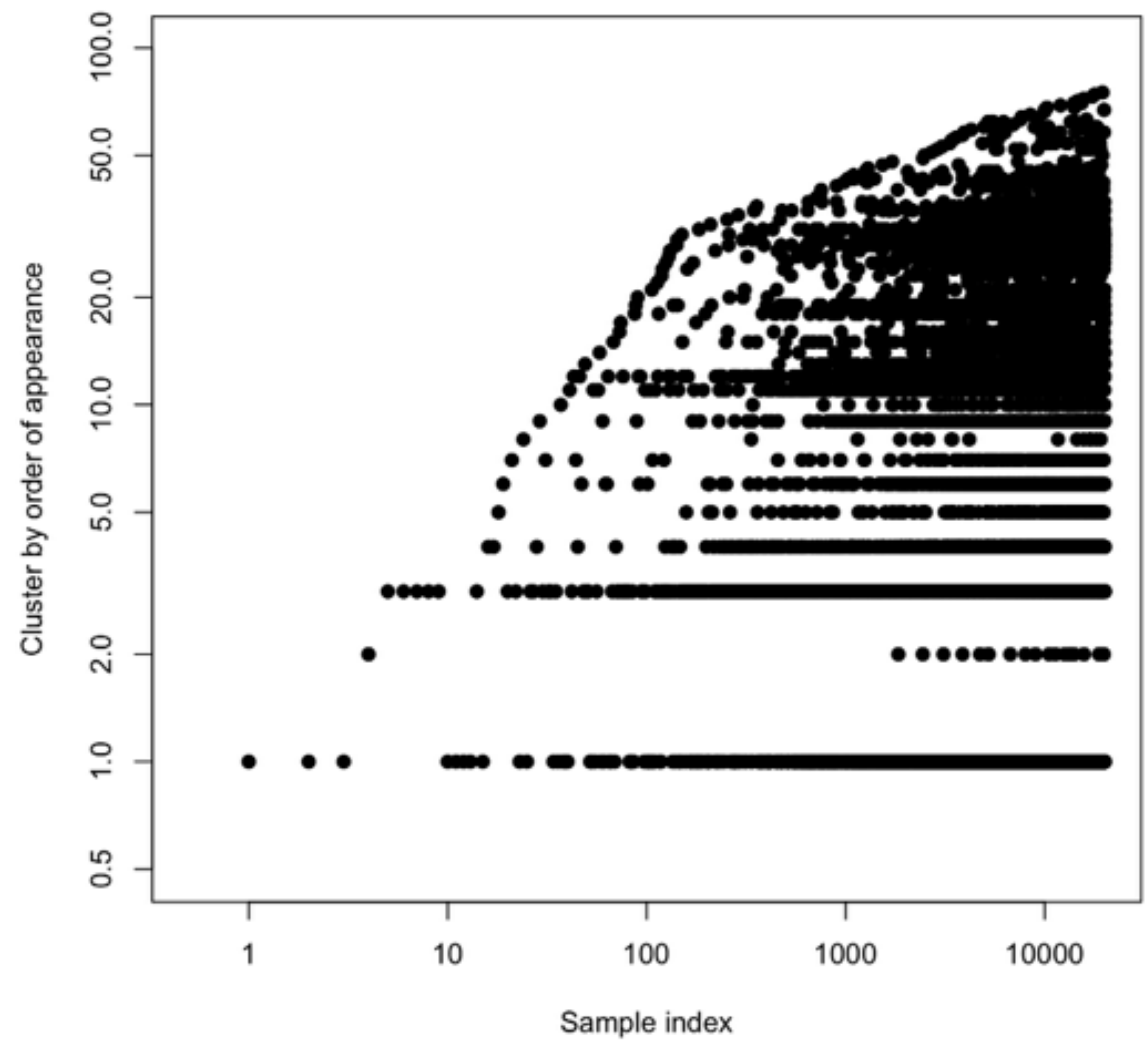

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Hierarchies

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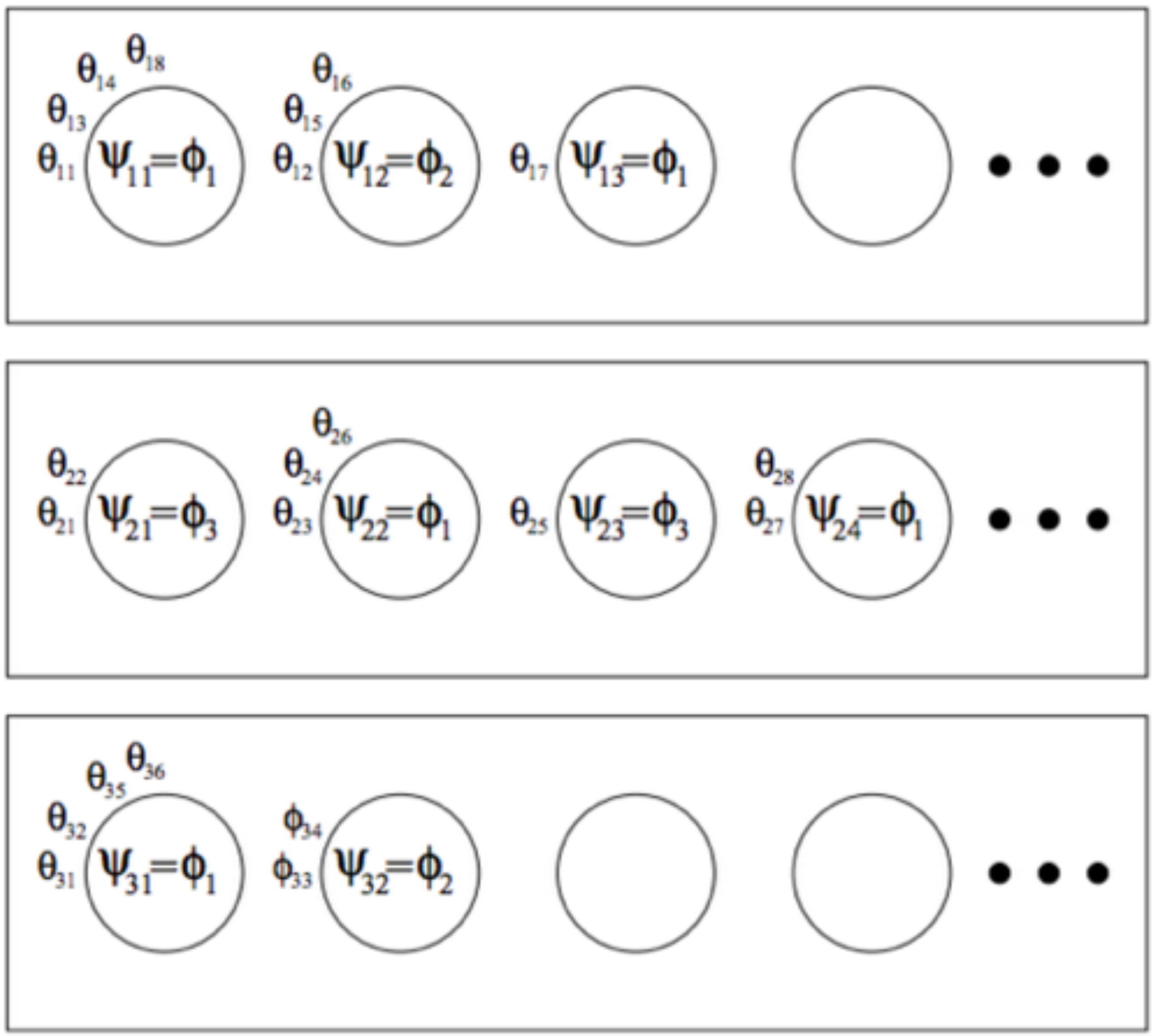

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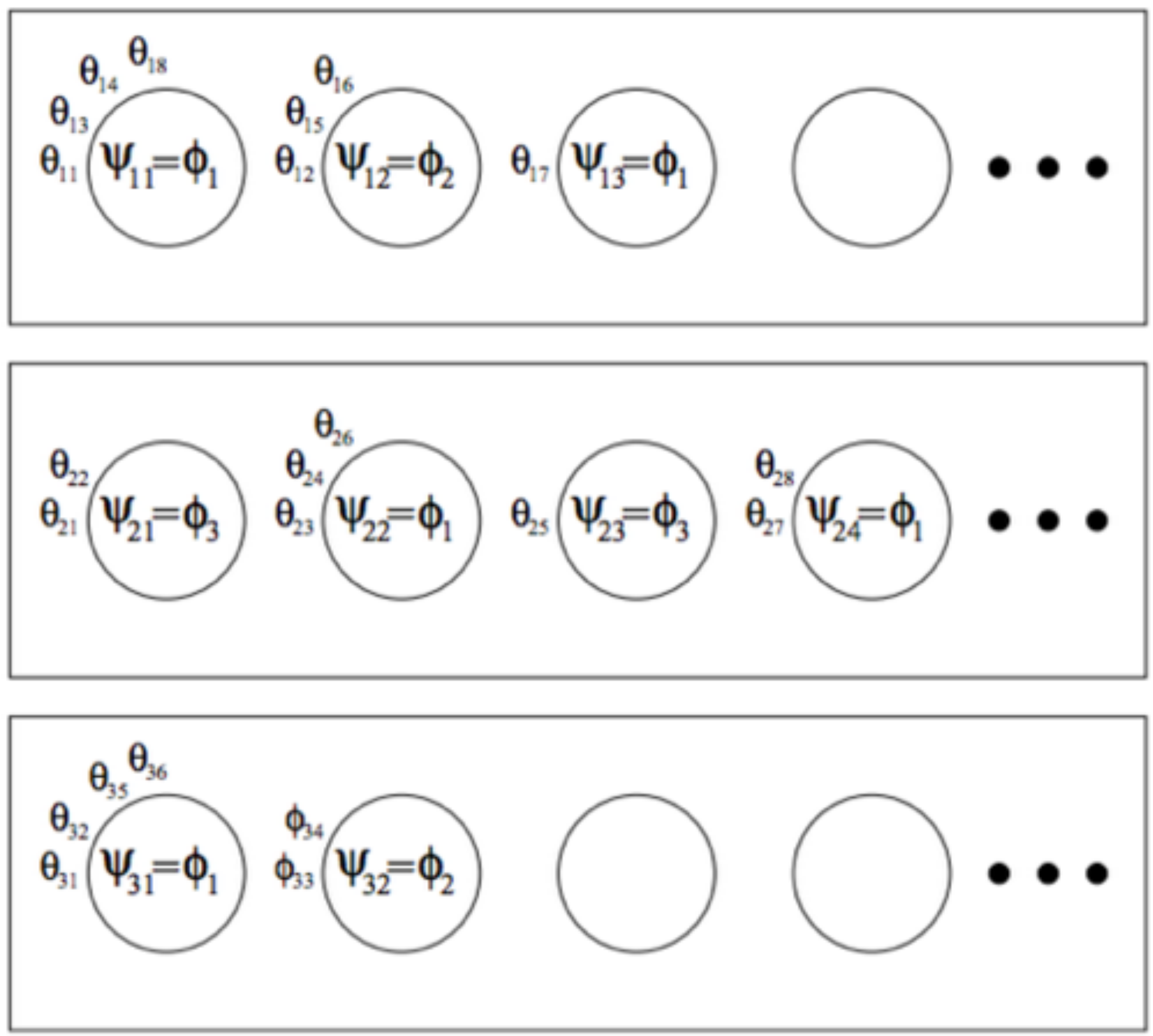

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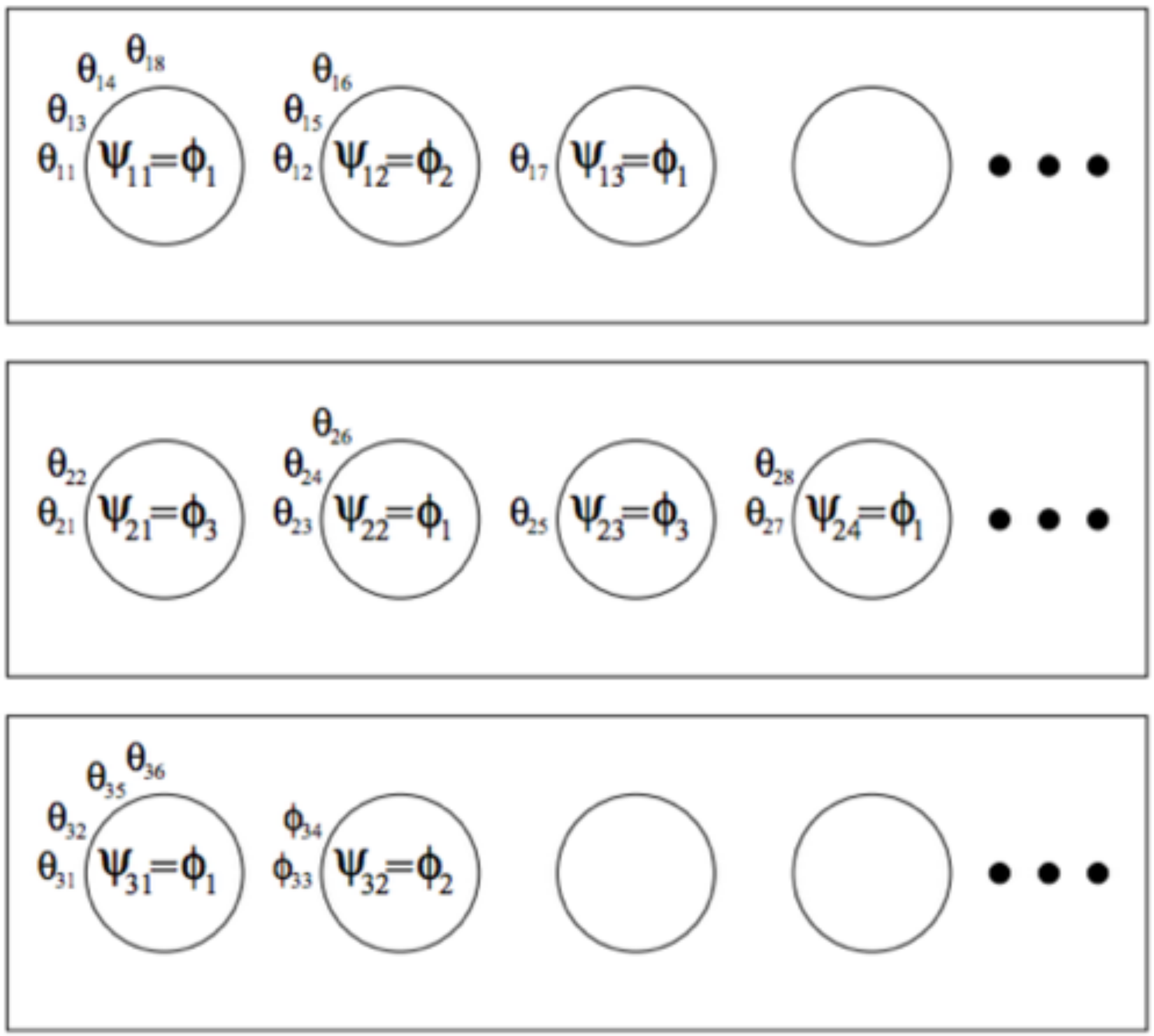

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De Finetti mixing measures

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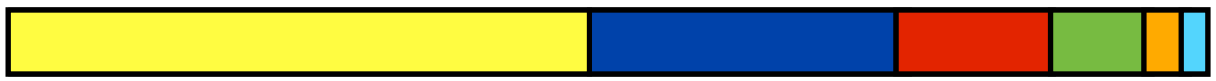

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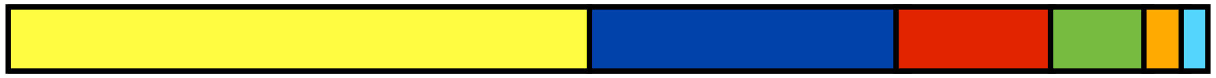

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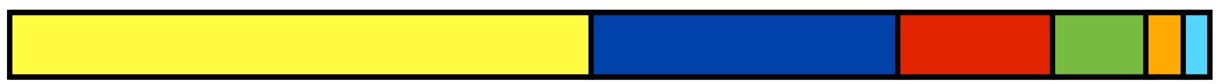

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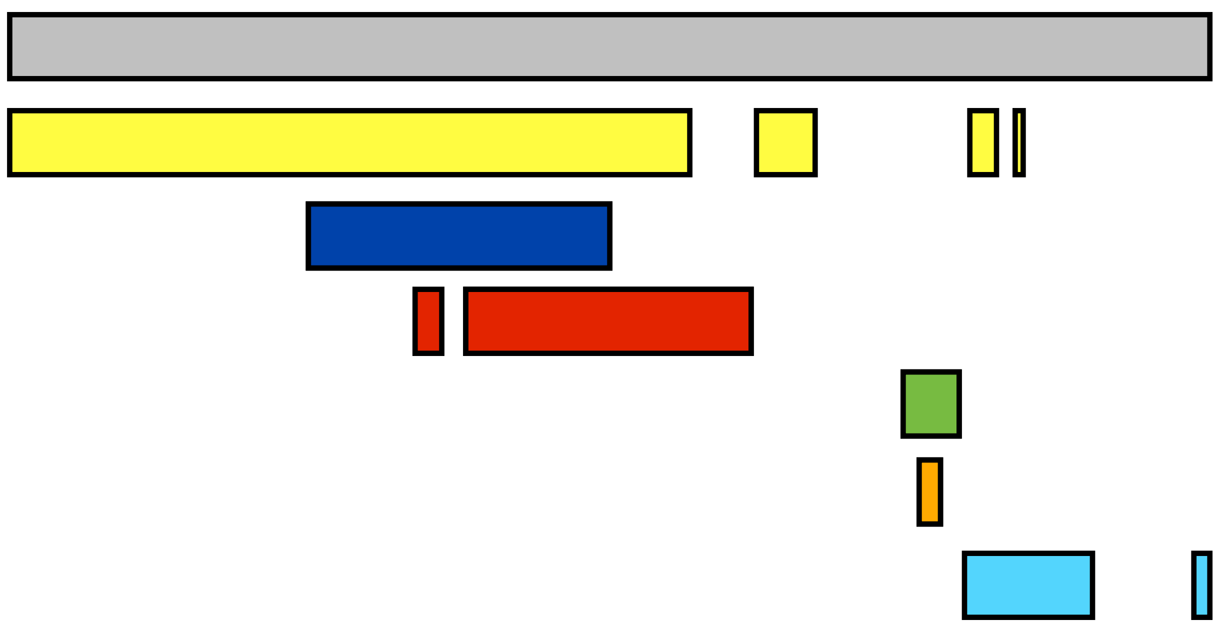

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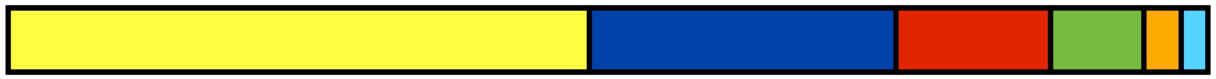

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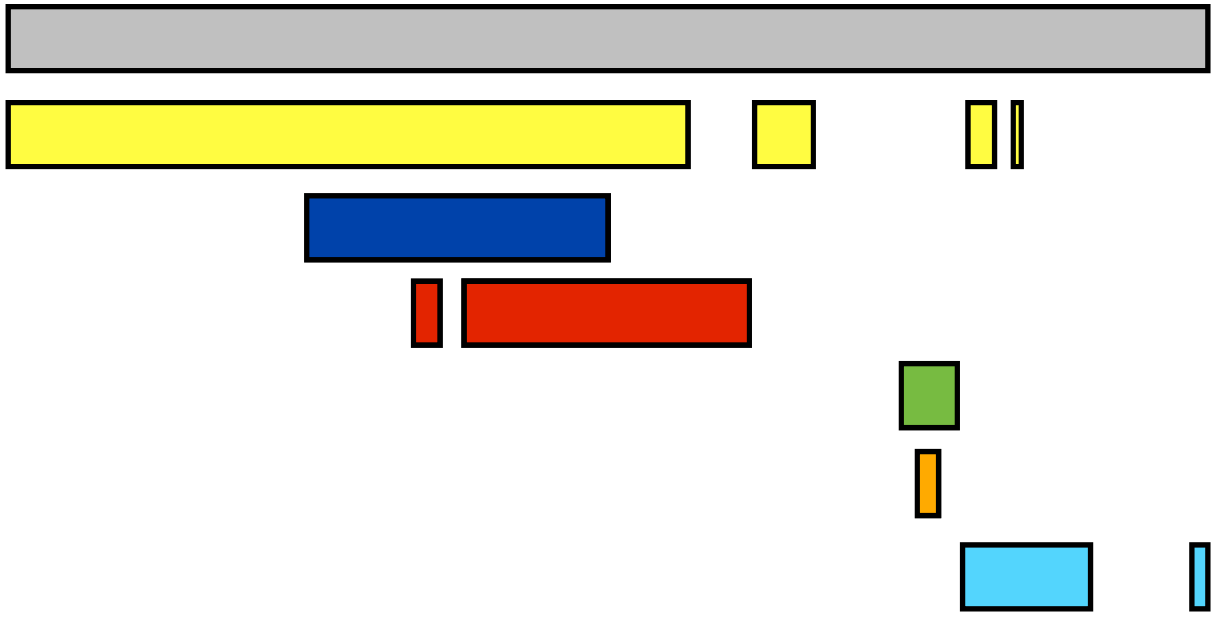

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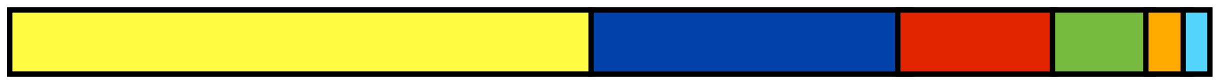

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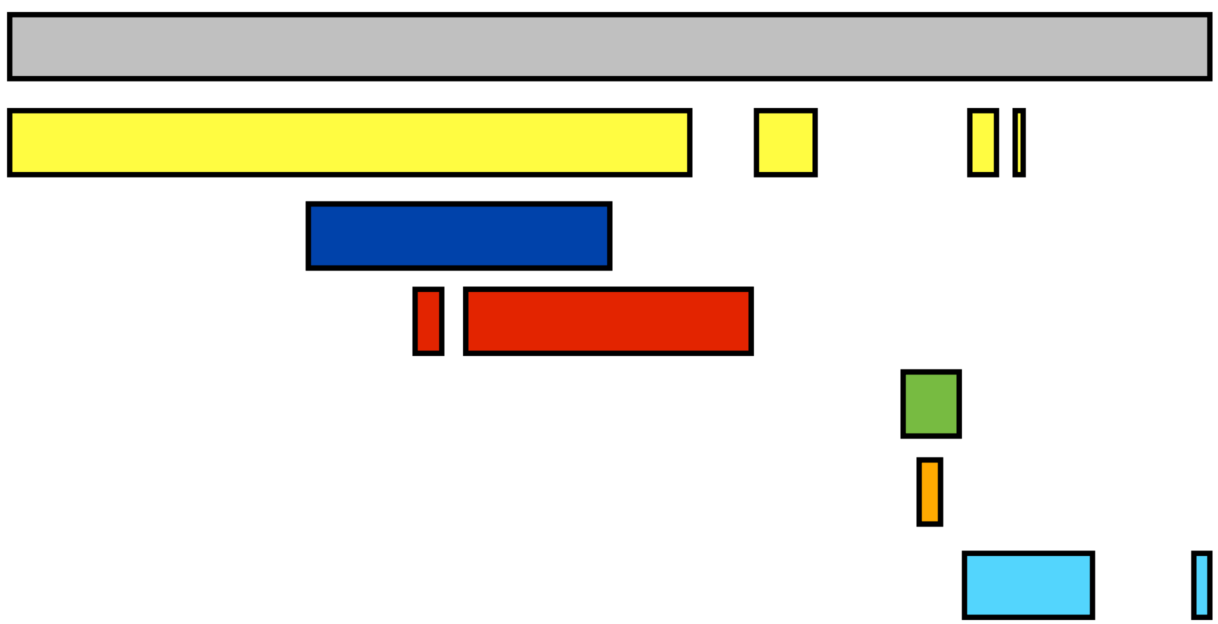

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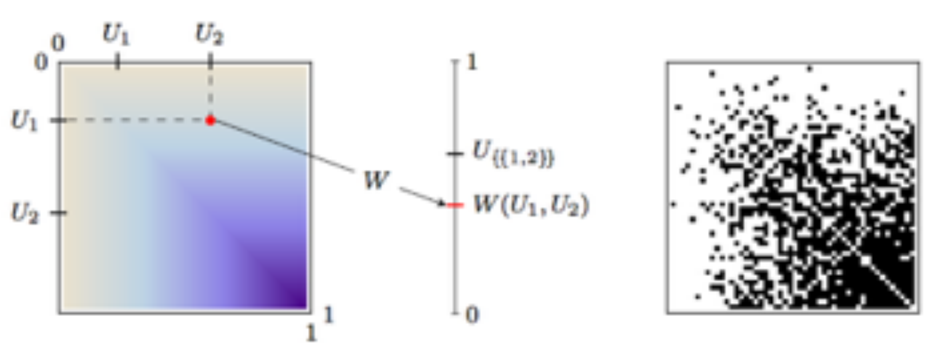

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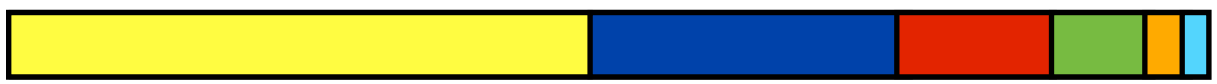

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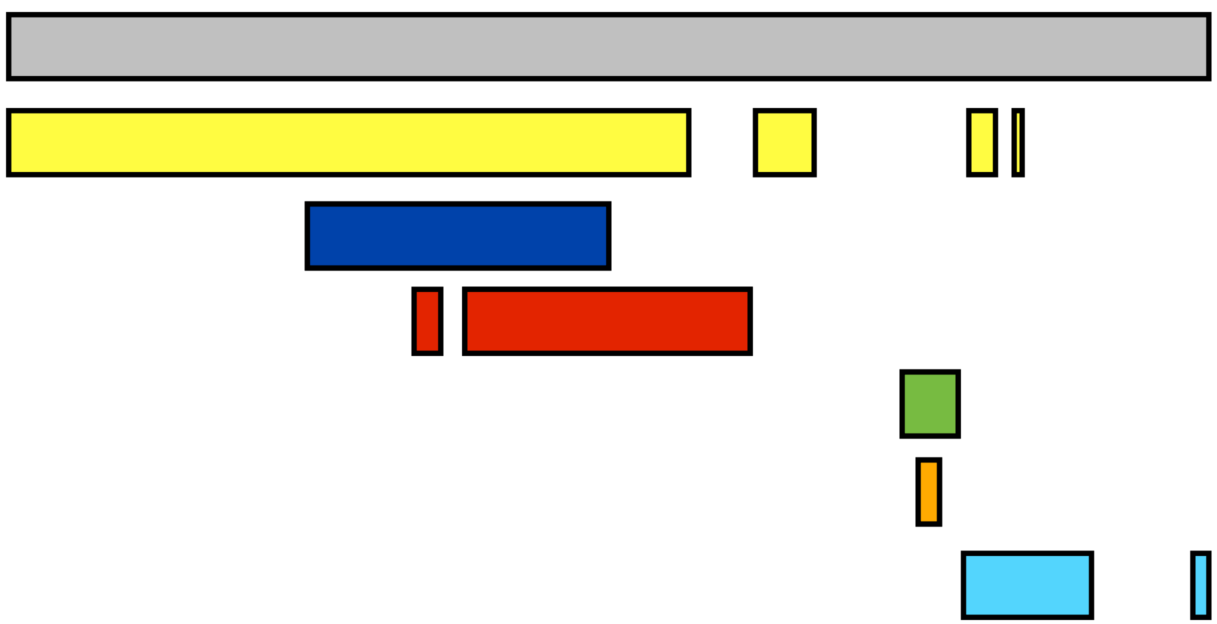

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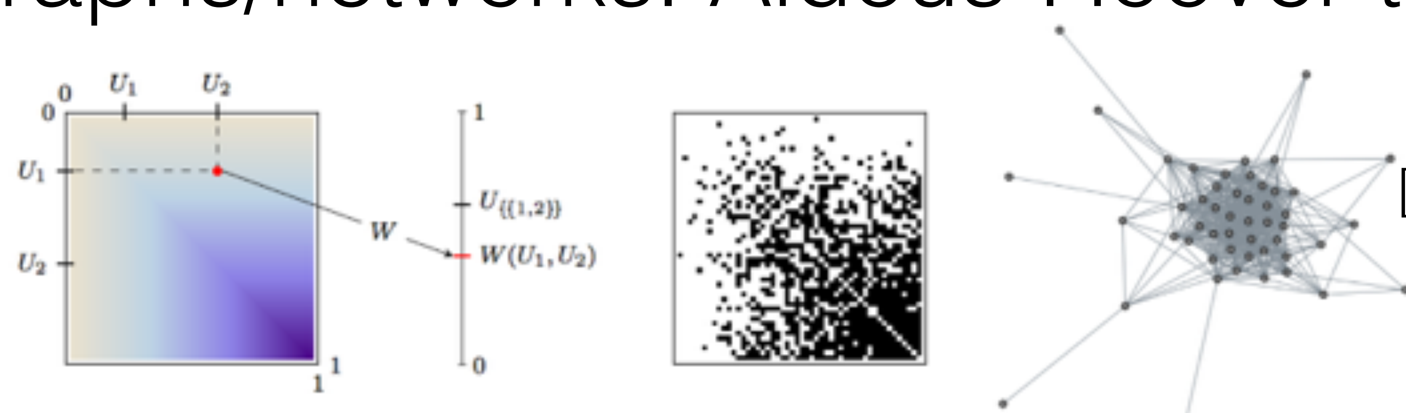

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Conjugacy & Poisson point processes

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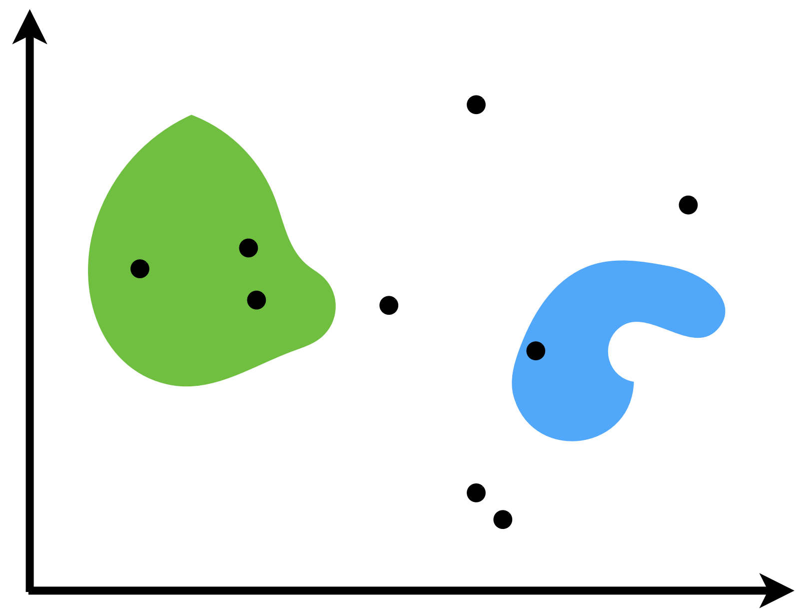

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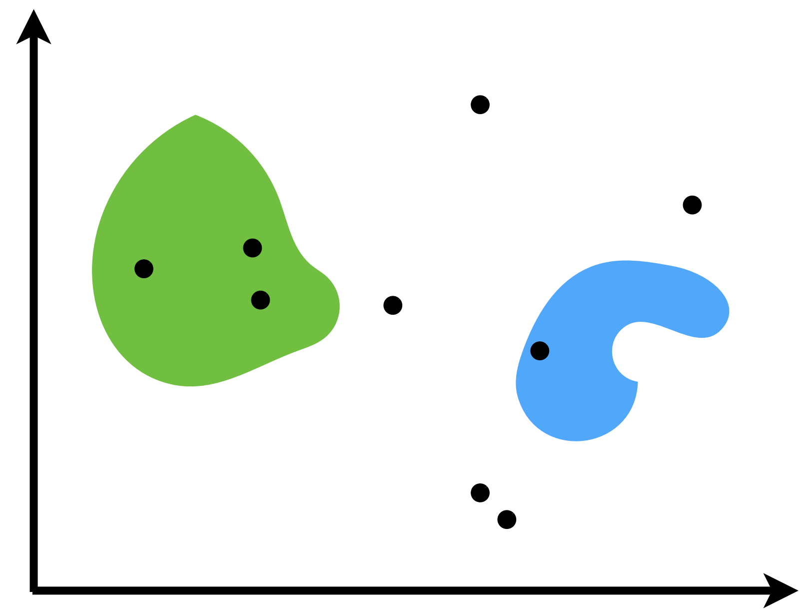

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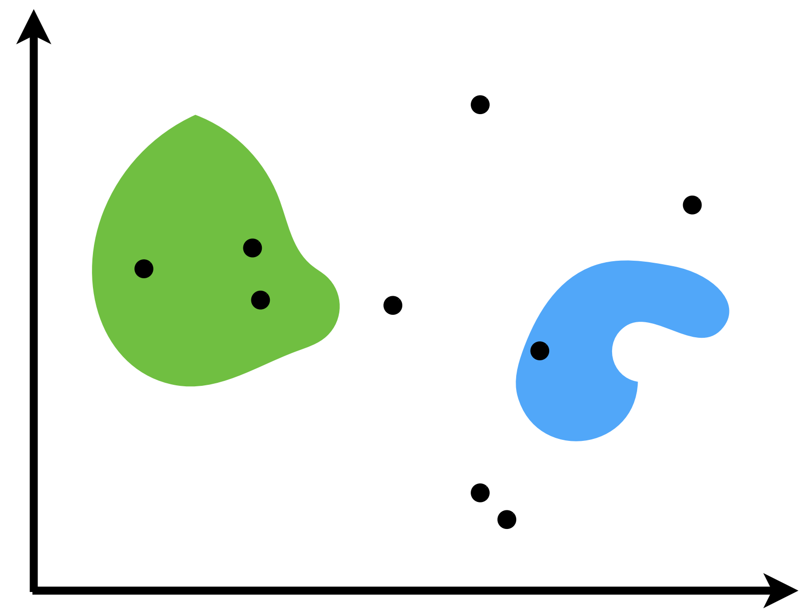

- Posteriors, conjugacy, and exponential families for completely random measures

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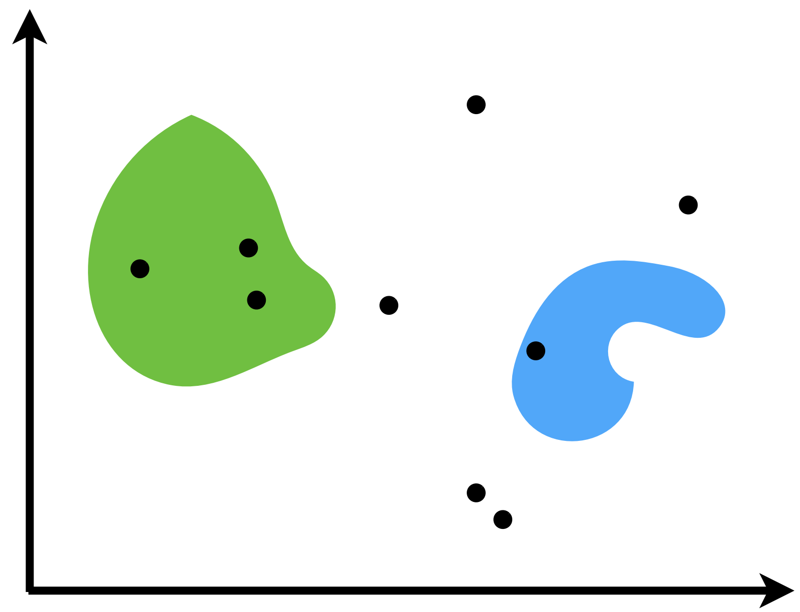

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Nonparametric Bayes

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- Bayesian statistics that is not parametric

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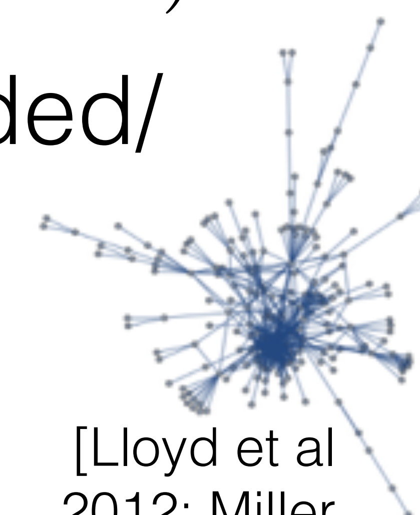
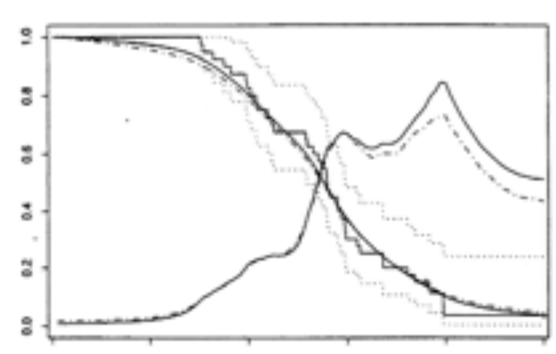


[Lloyd et al 2012; Miller et al, 2010]

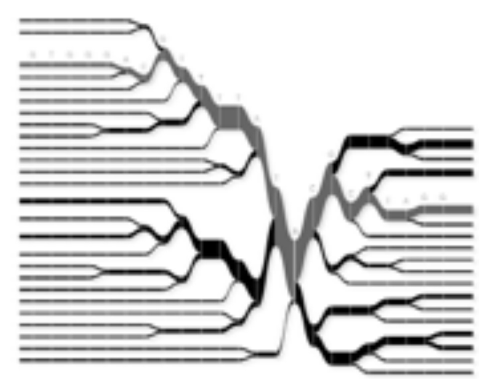
[wikipedia.org]

[Ed Bowlby, NOAA]

[Fox, et al 2014]

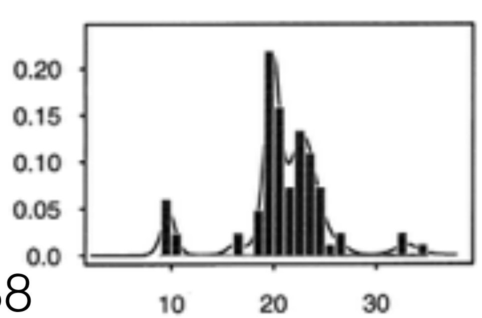


[Arjas, Gasbarra 1994]



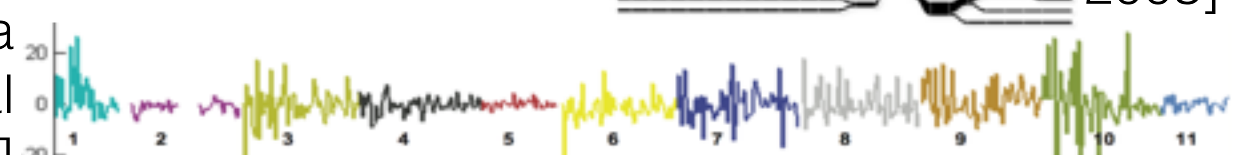
[Ewens, 1972; Hartl, Clark 2003]

[Sudderth, Jordan 2009]



[Escobar, West 1995; Ghosal, et al 1999]

[Saria et al 2010]



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