Nonparametric Bayesian Statistics: Part II

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Electrical Engineering & Computer Science
MIT

Nonparametric Bayes: Part II

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  - Understand what it means to have a growing number of parameters

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  - A finite representation of an infinite process

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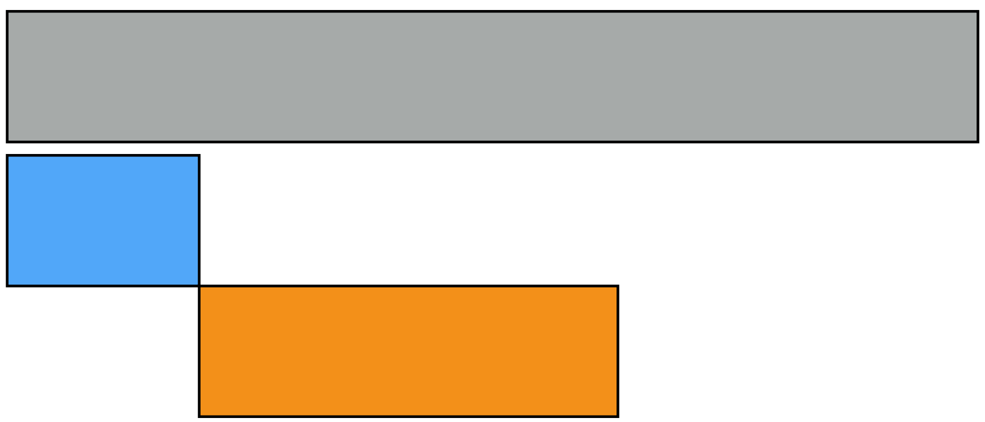

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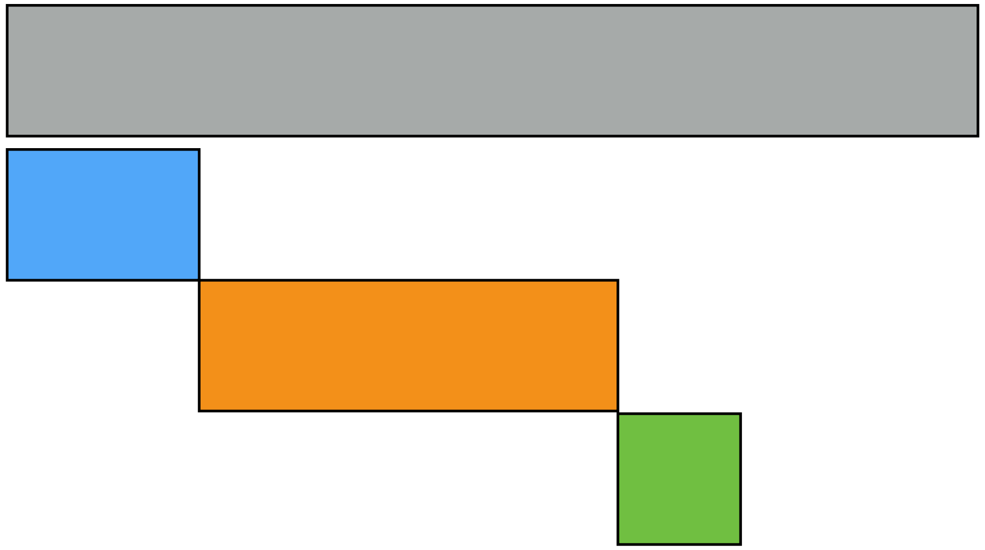

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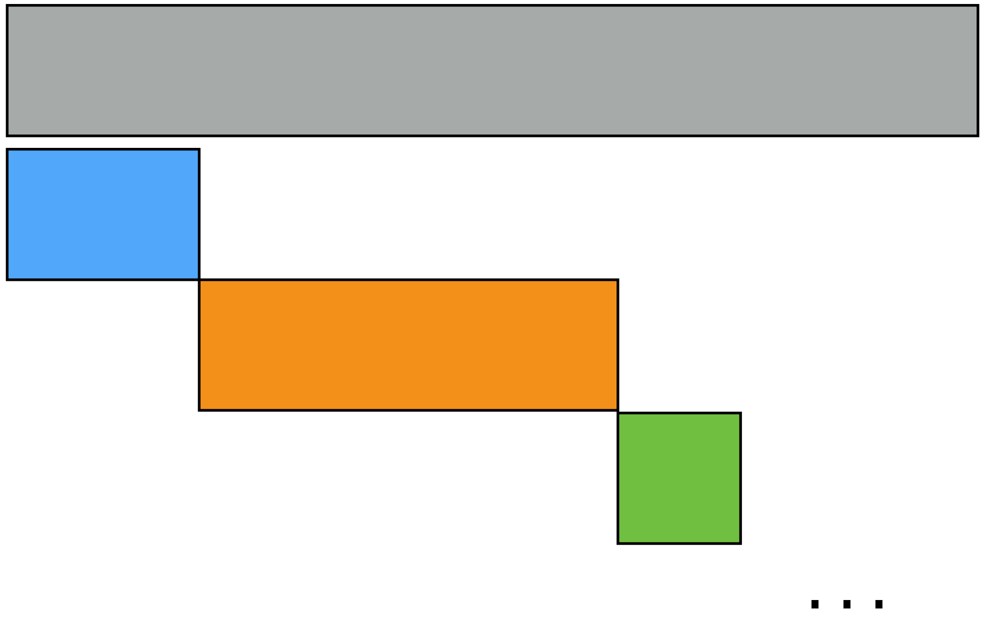

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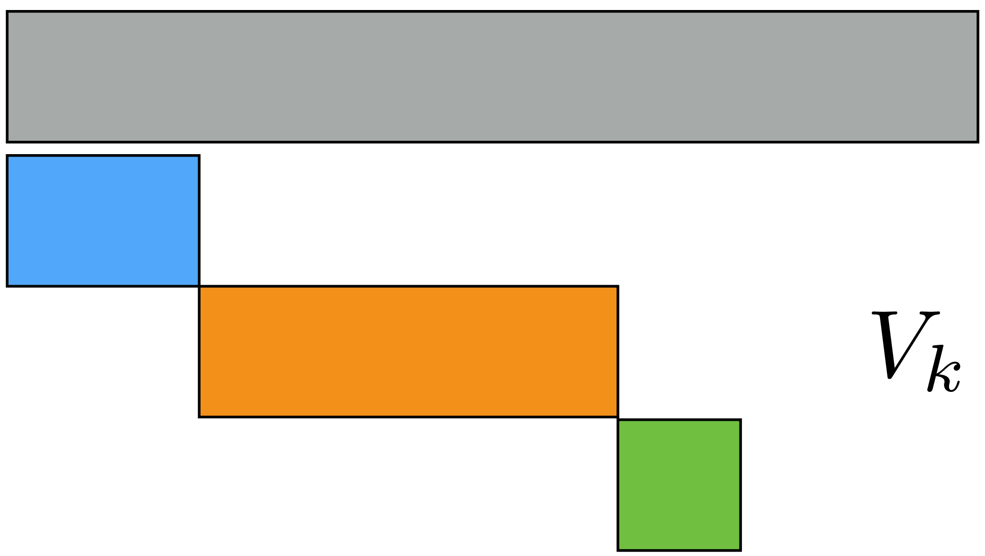

$$V_k \overset{iid}{\sim} \mathrm{Beta}(1, \alpha)$$

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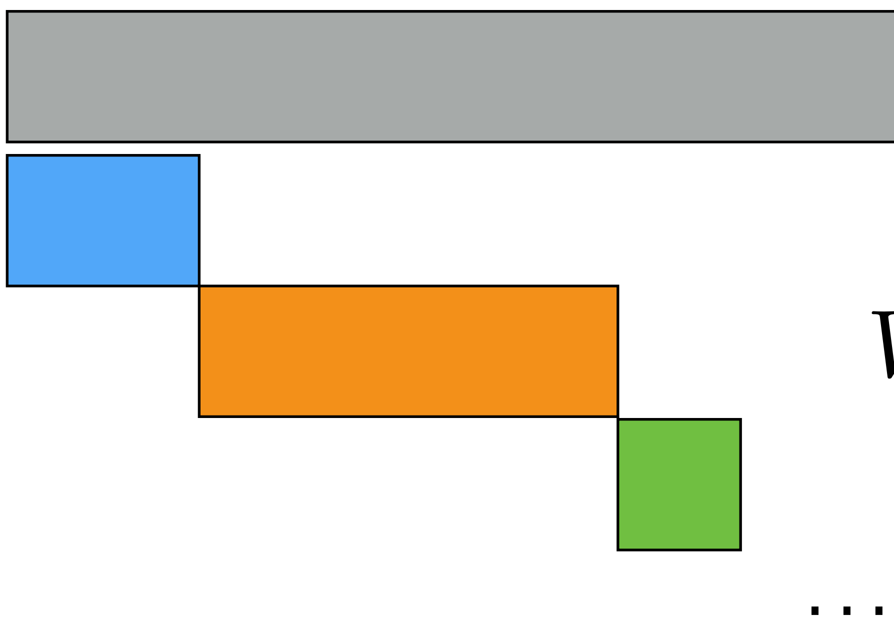

$$V_k \overset{iid}{\sim} \mathrm{Beta}(1, \alpha) \qquad \rho_k = \left[\prod_{j=1}^{k-1} (1 - V_j)\right] V_k$$

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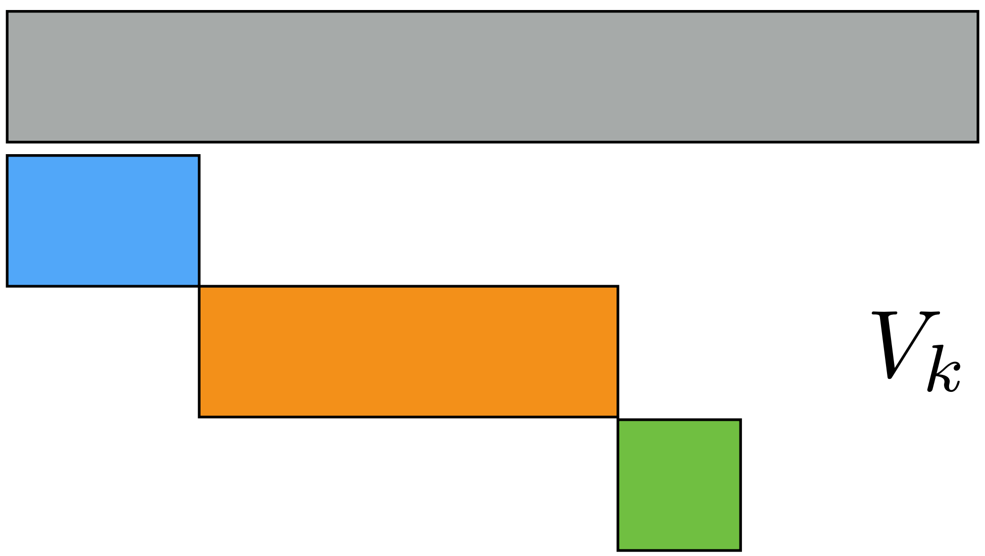

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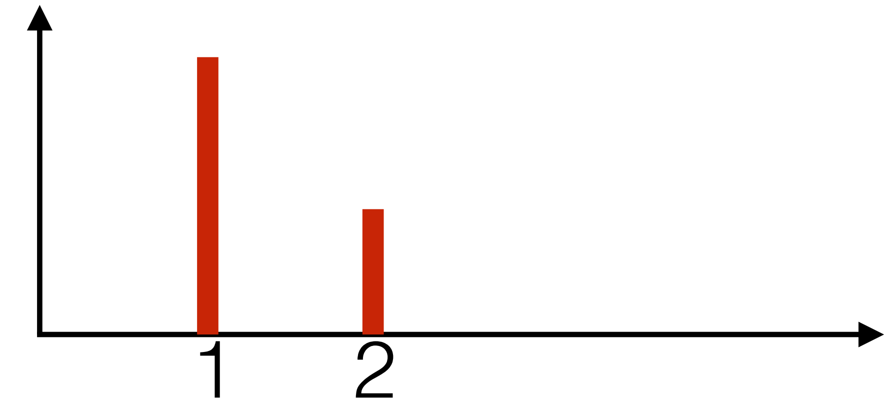

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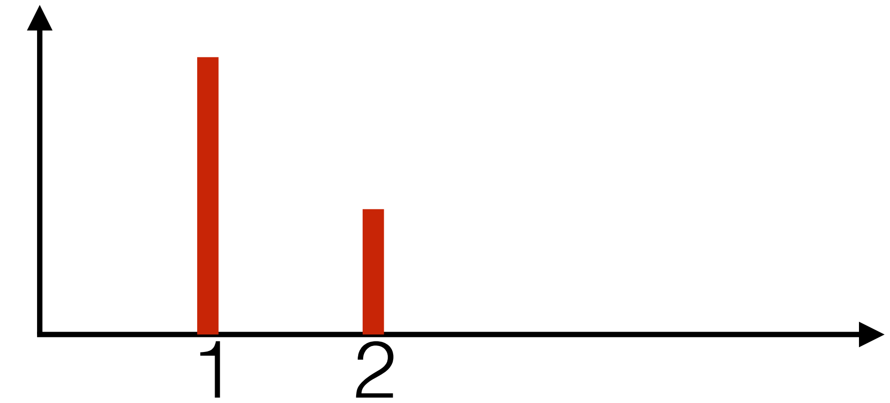

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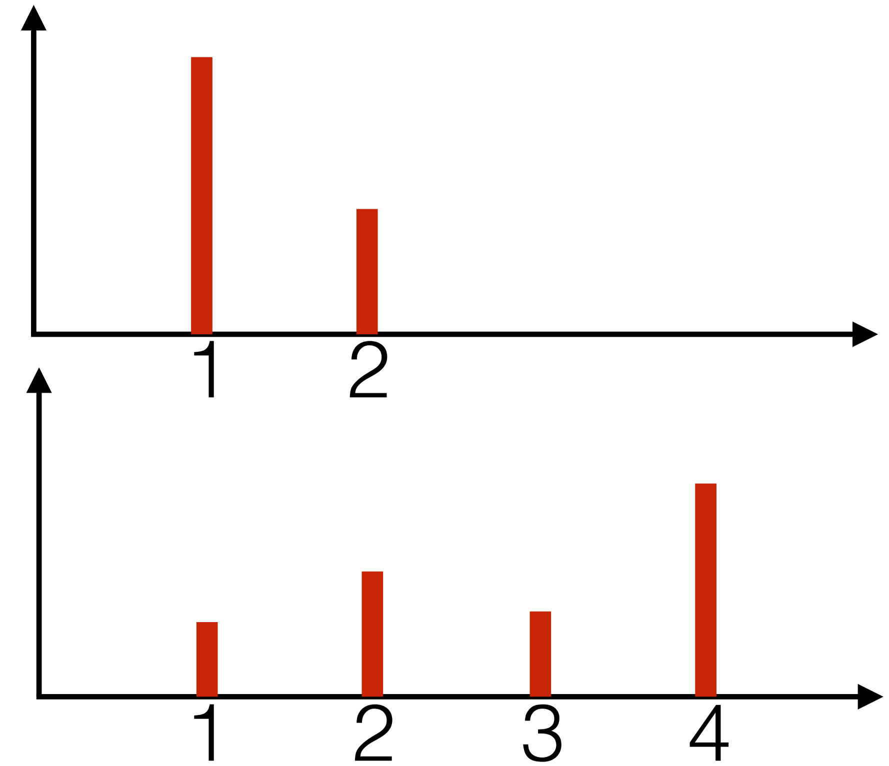

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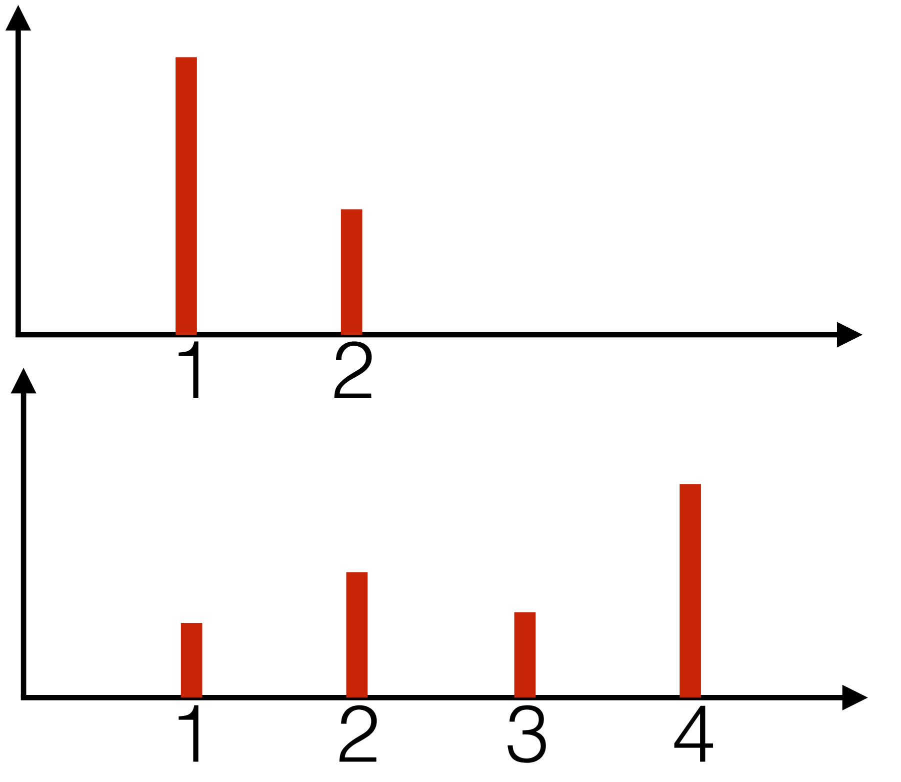

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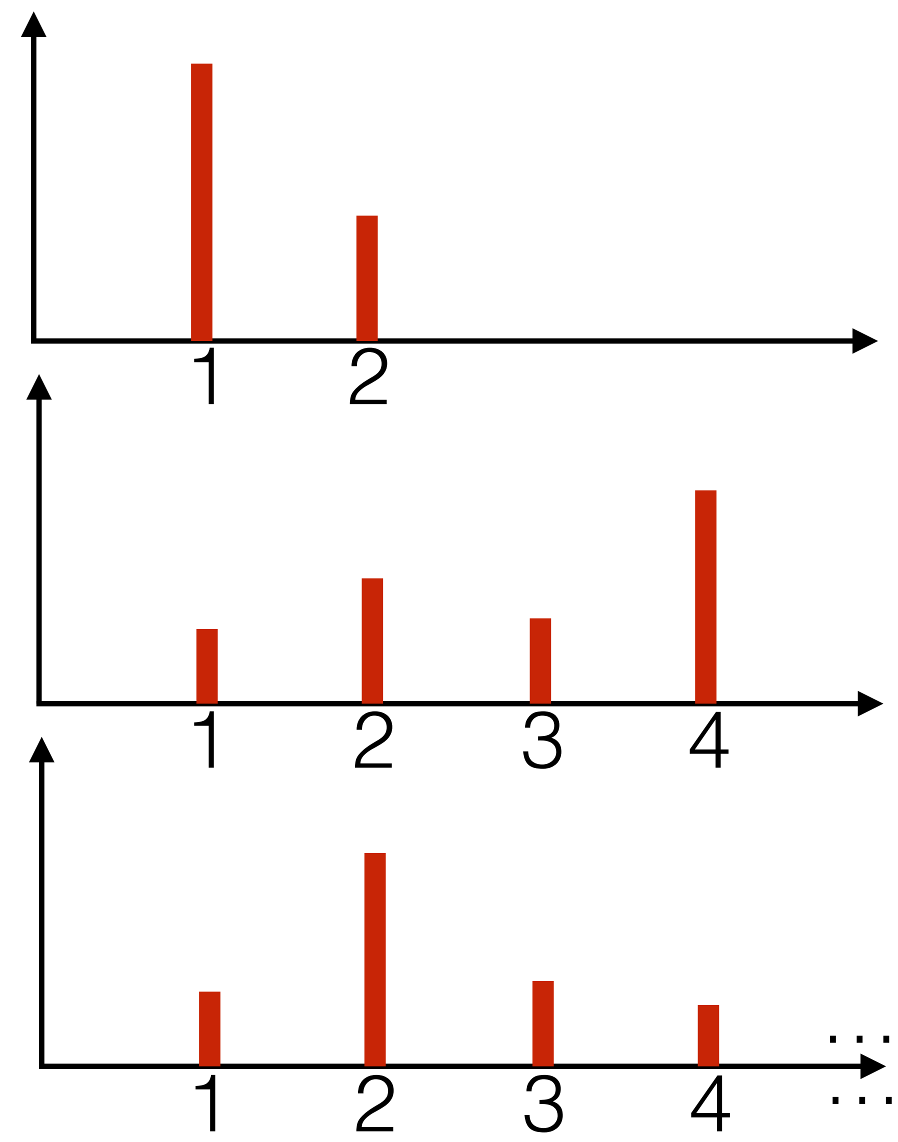

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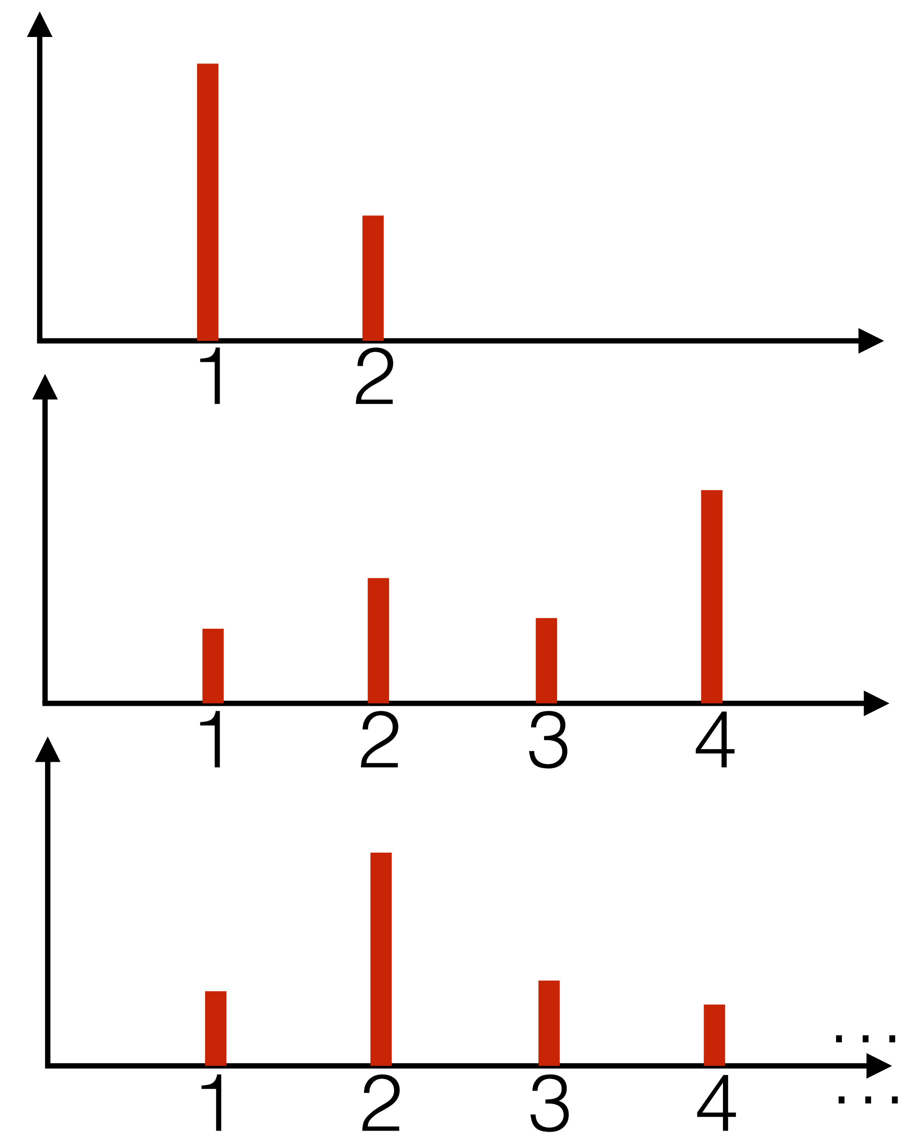

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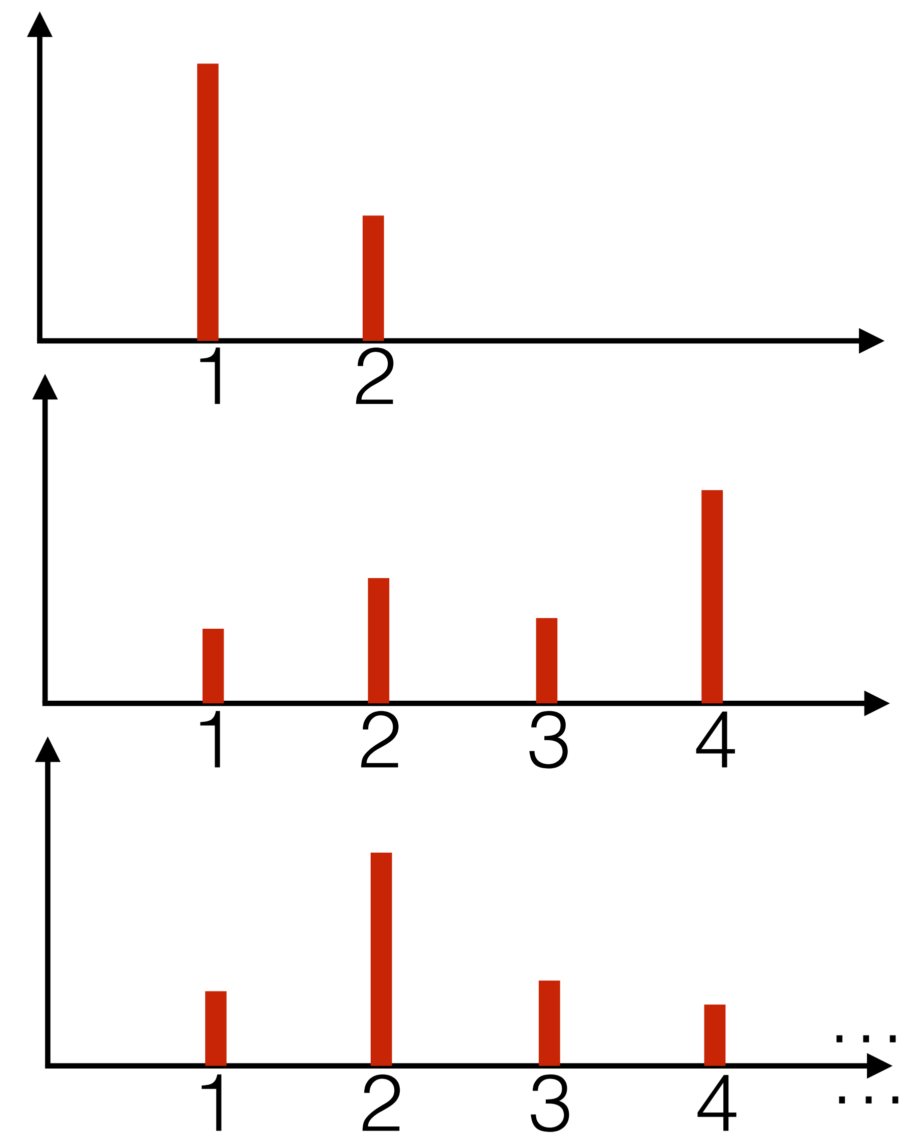

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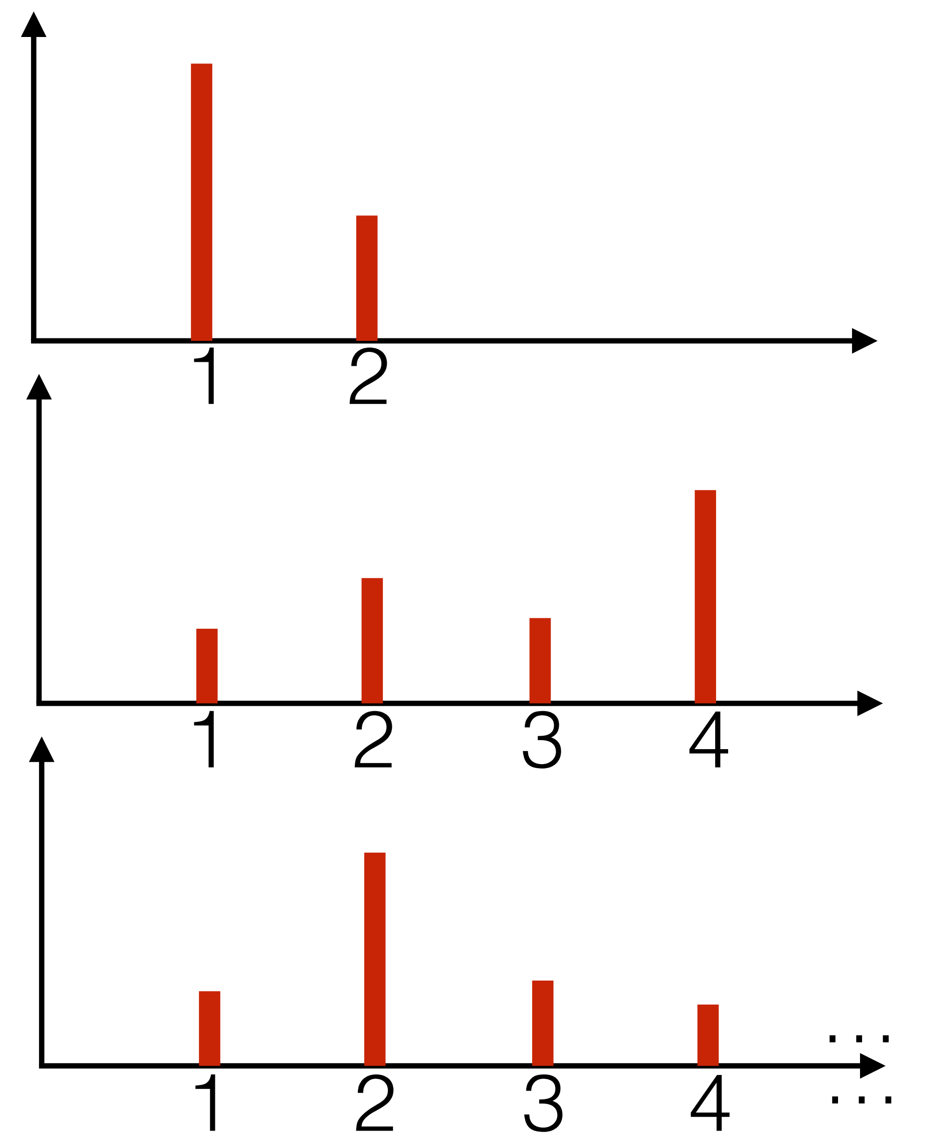

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$$G = \sum_{k=1}^{\infty} \rho_k \delta_{\phi_k}$$

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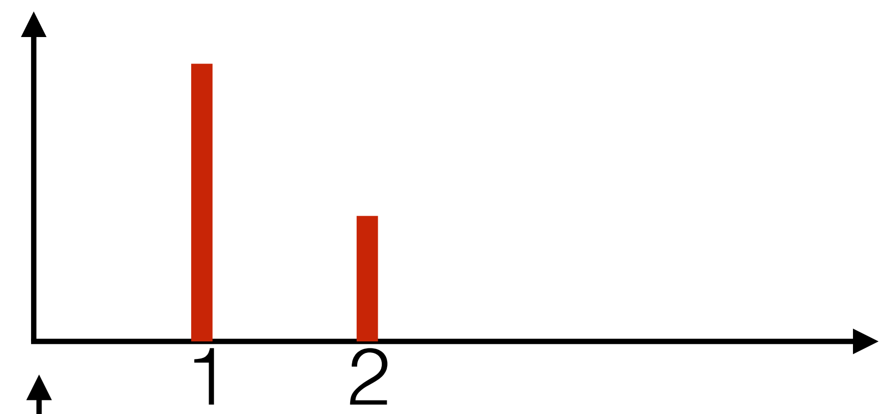

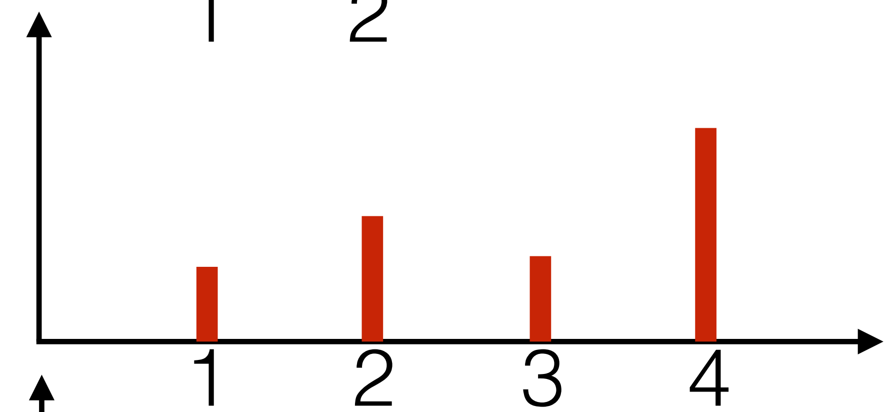

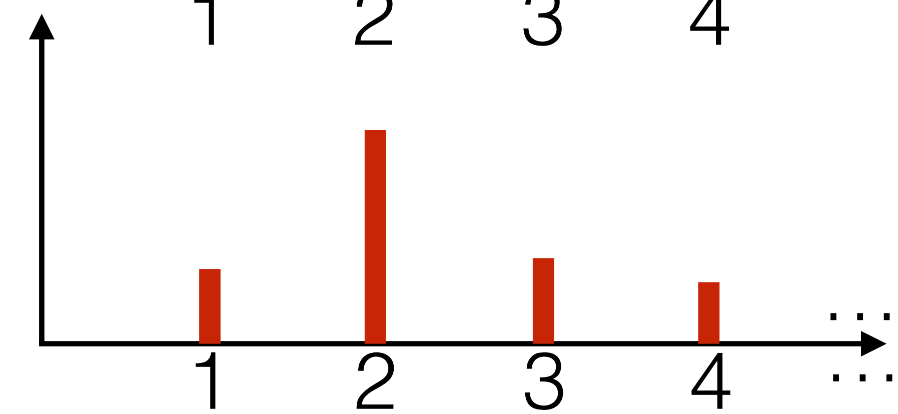

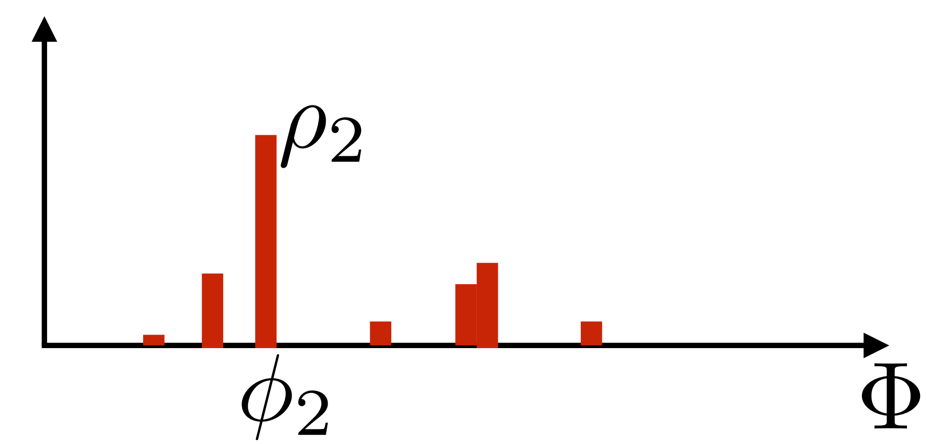

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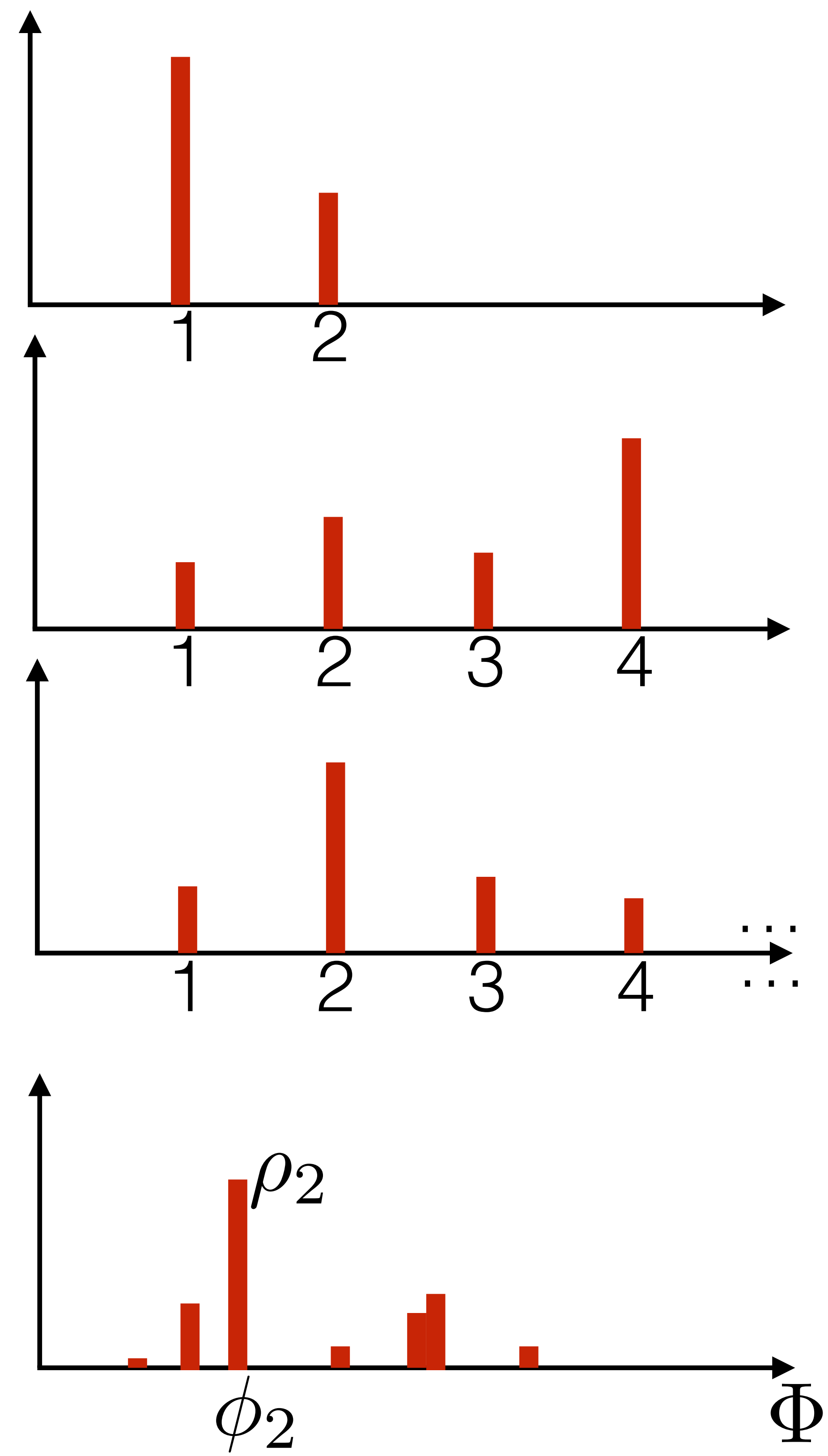

[Ferguson 1973]

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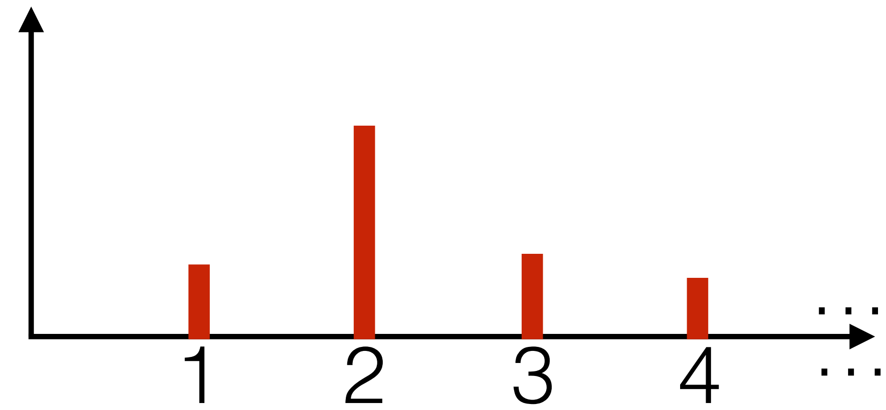

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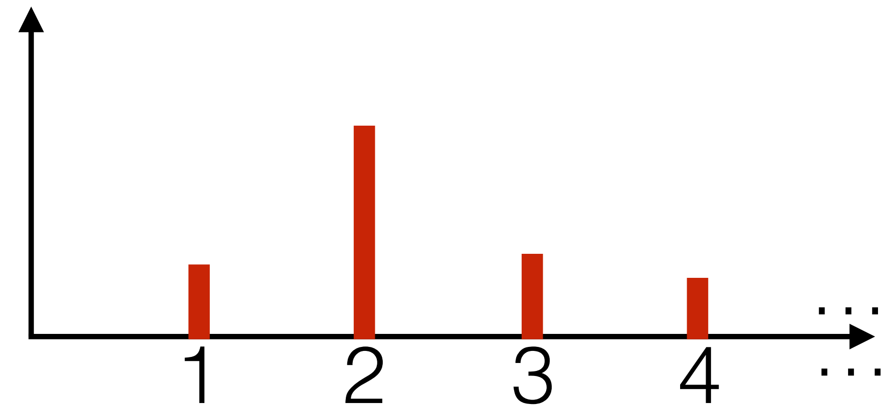

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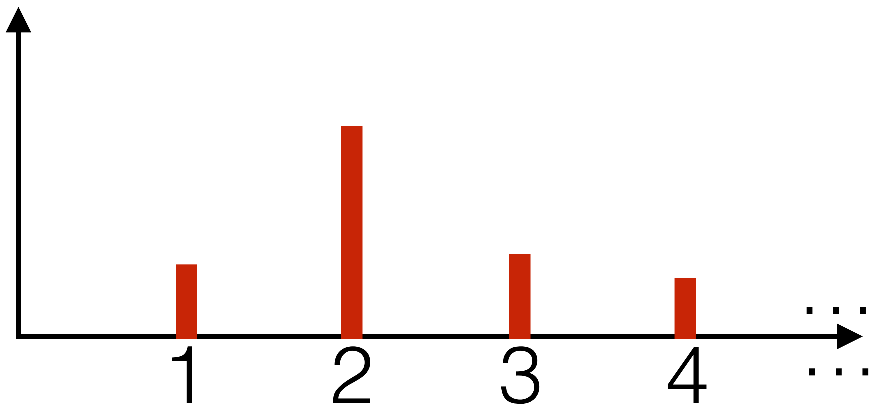

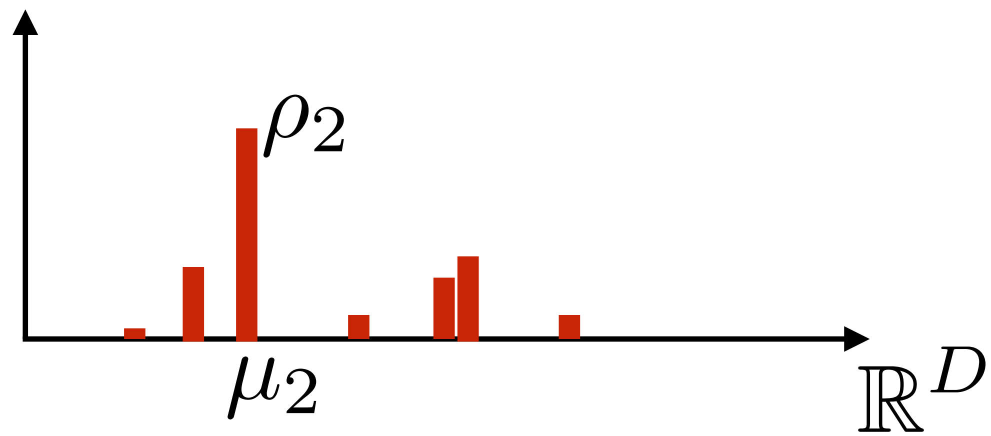

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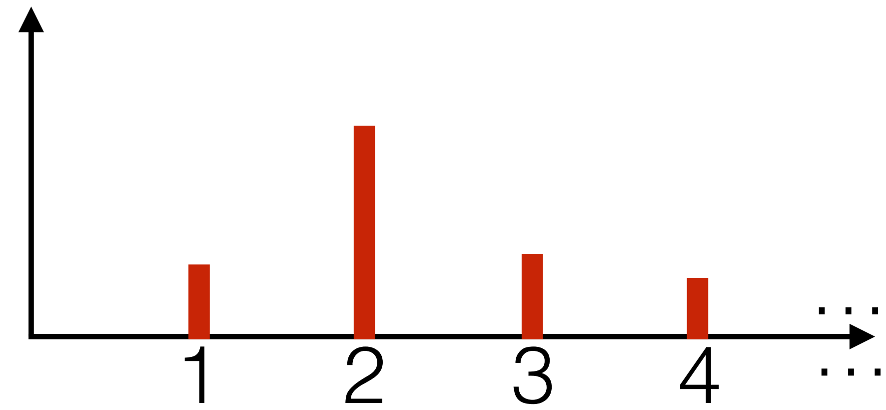

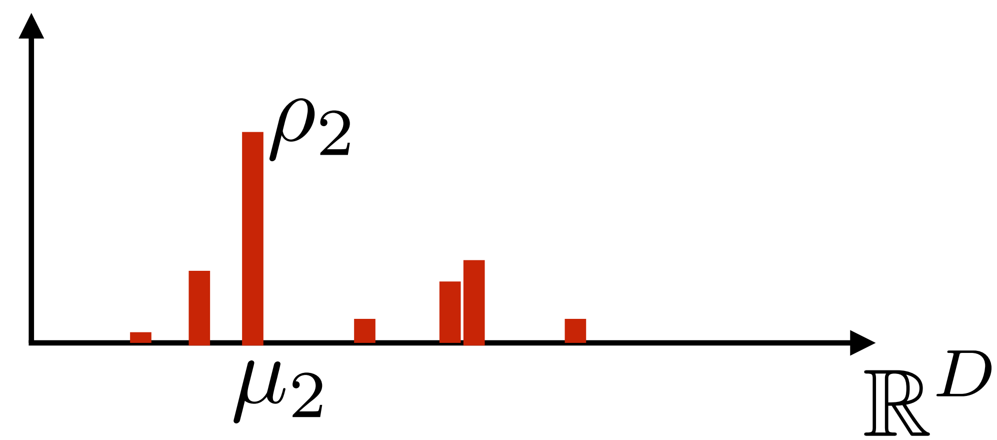

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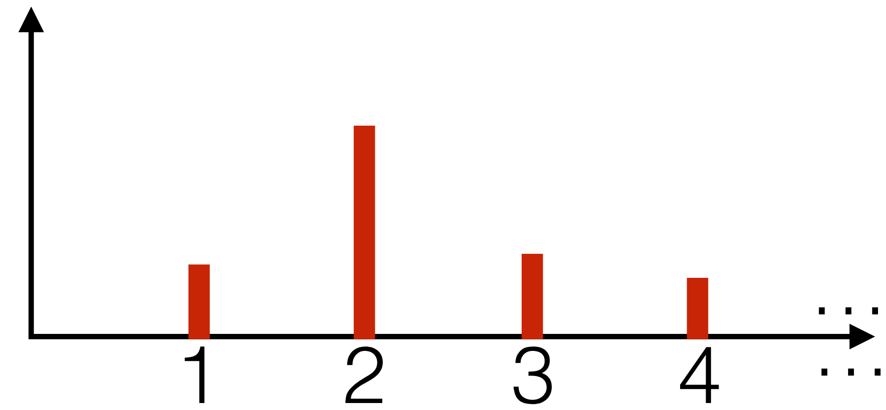

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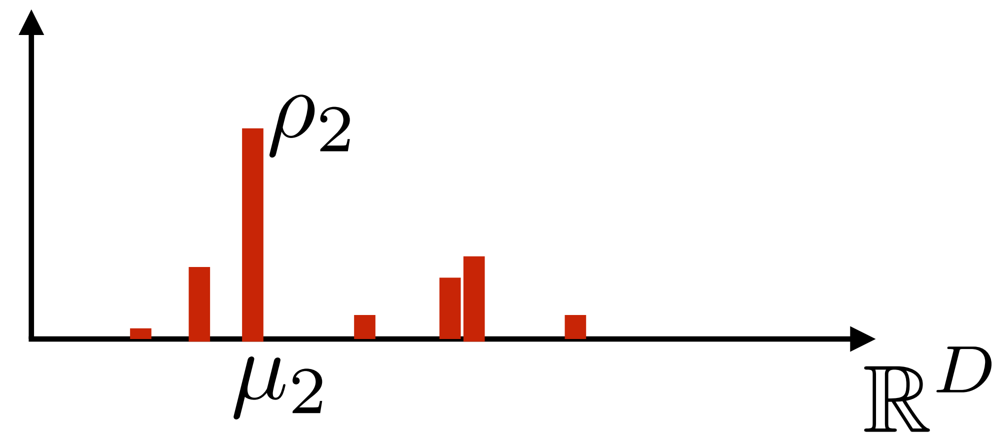

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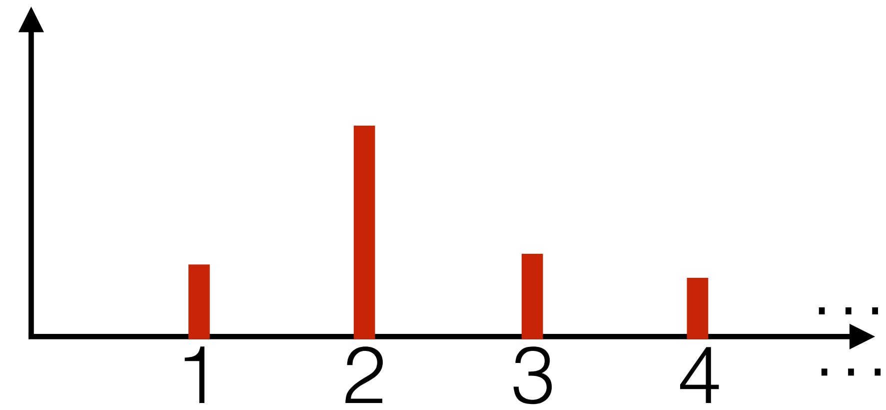

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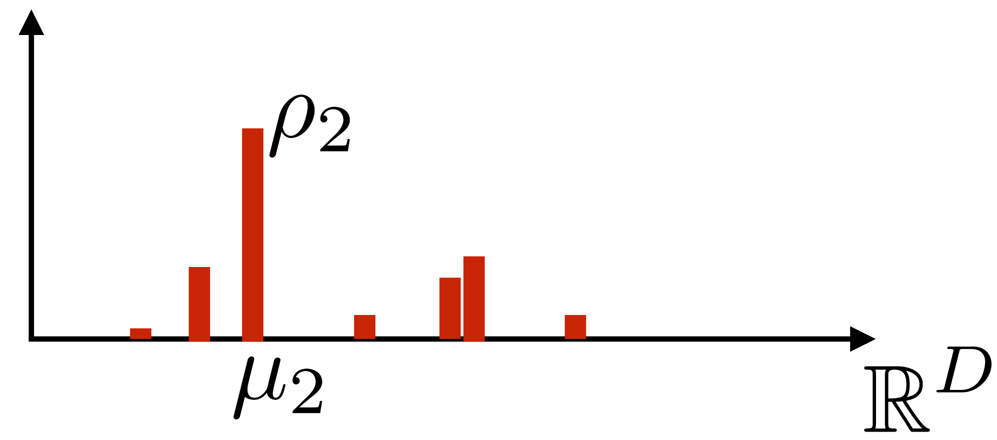

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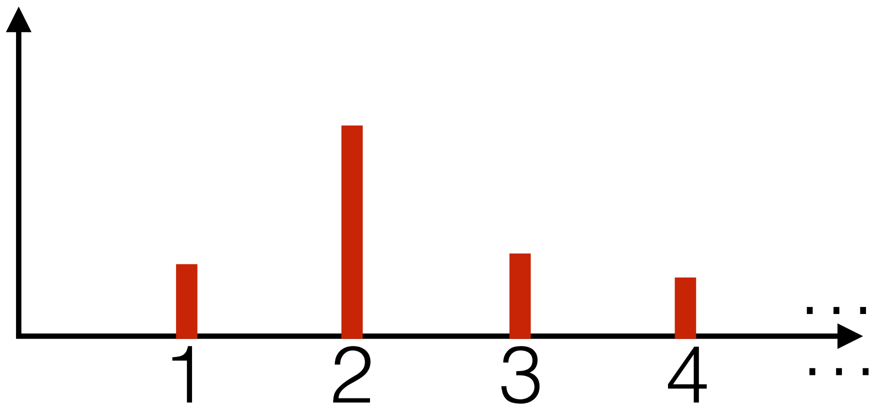

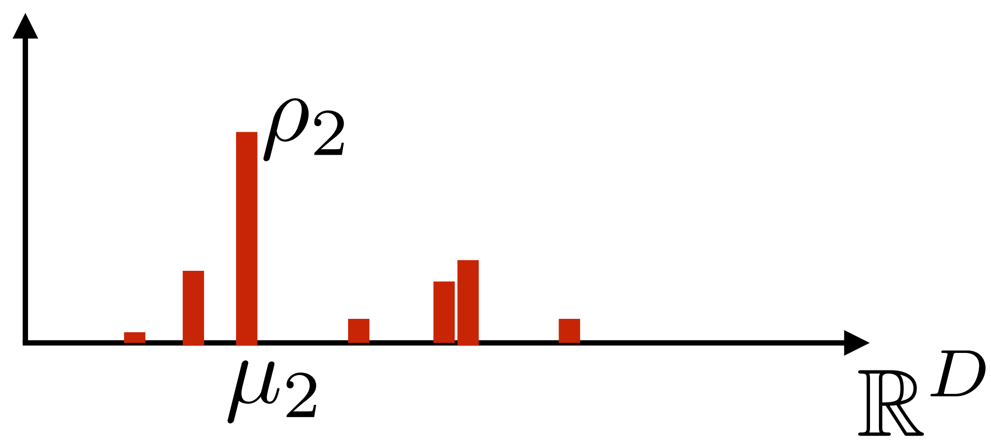

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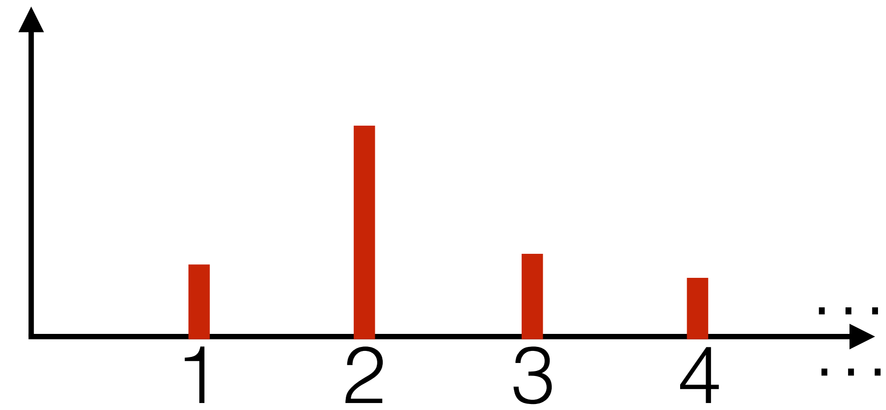

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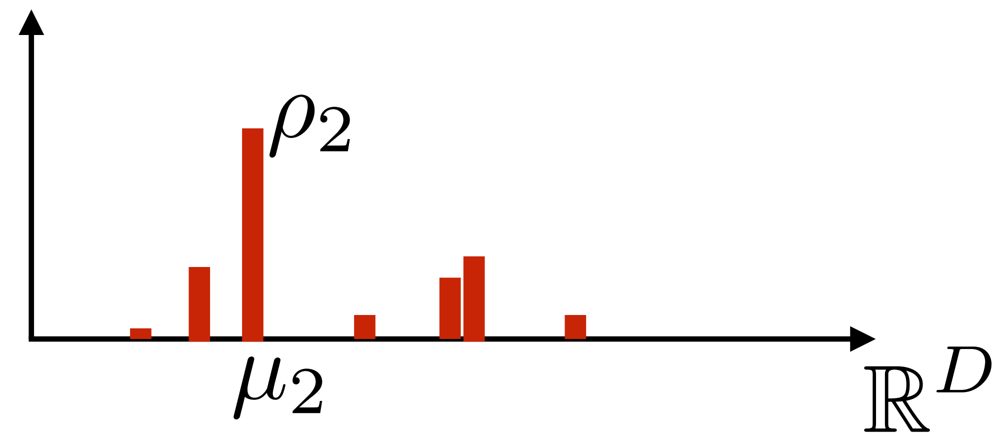

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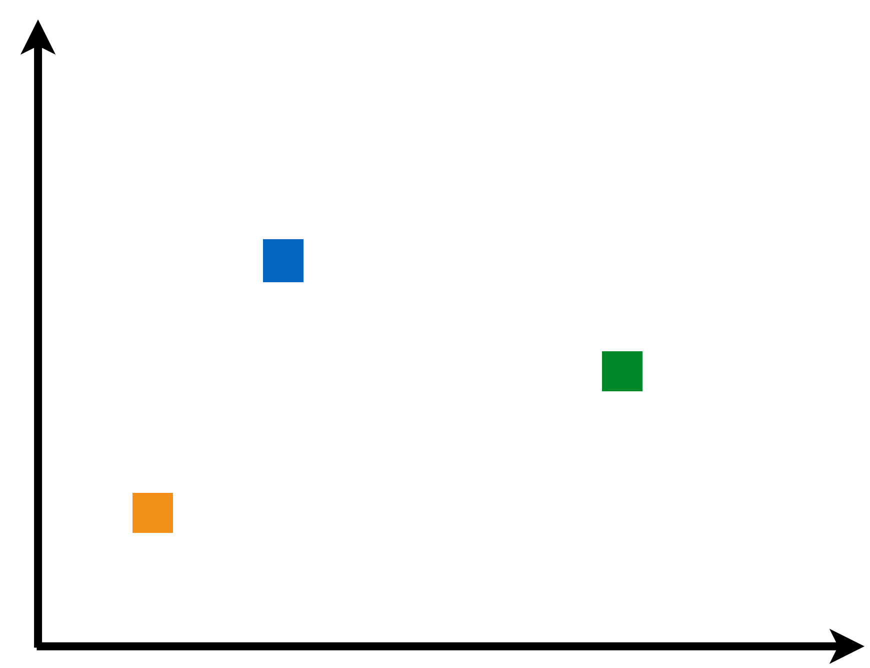
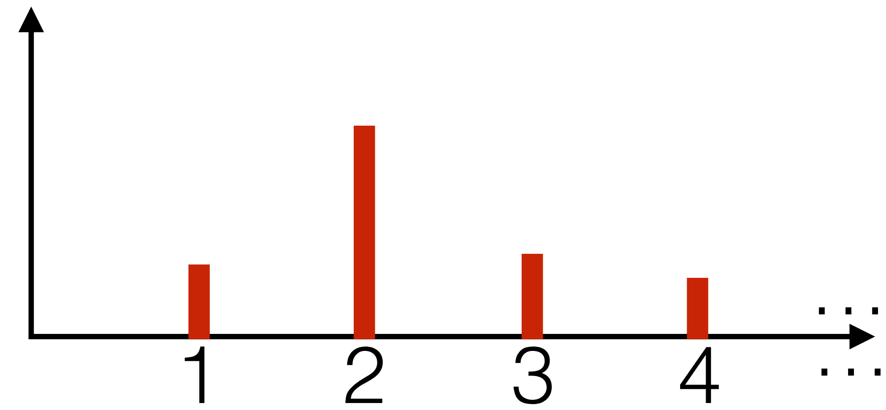


Dirichlet process mixture model



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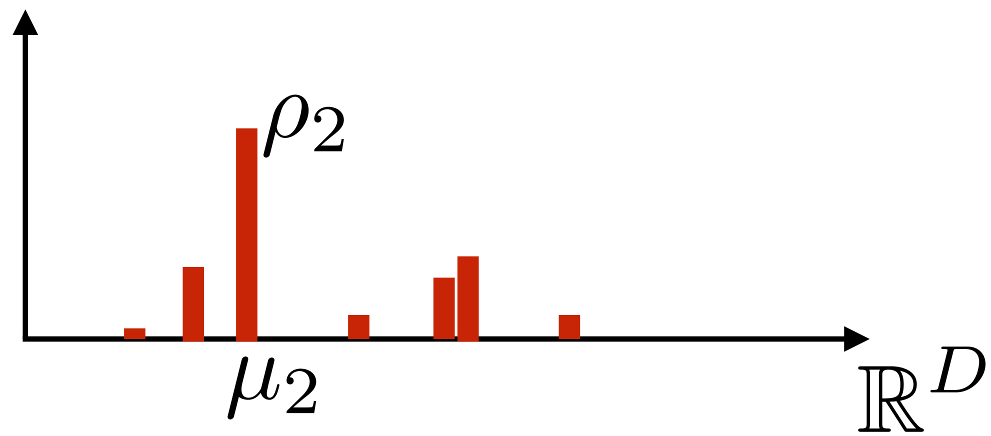

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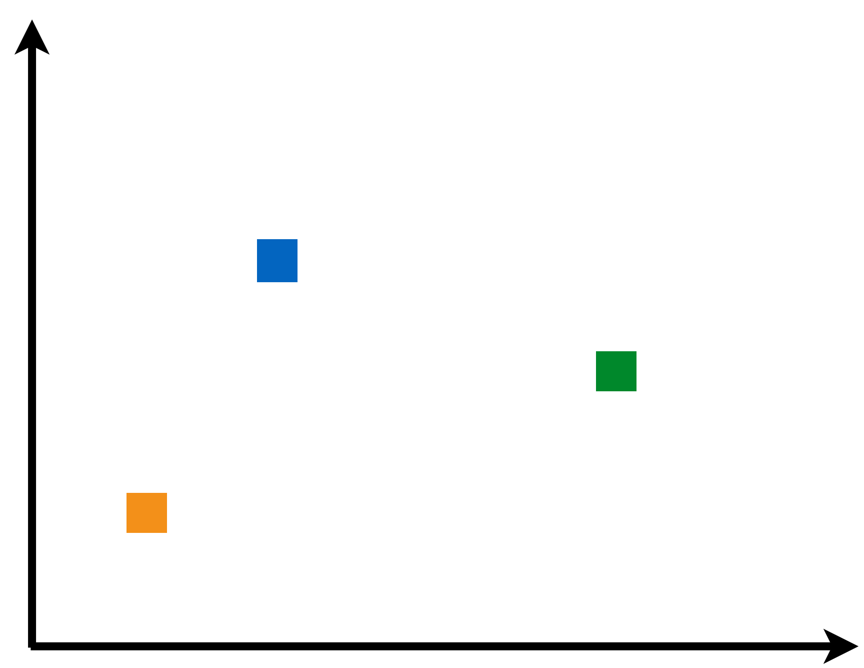

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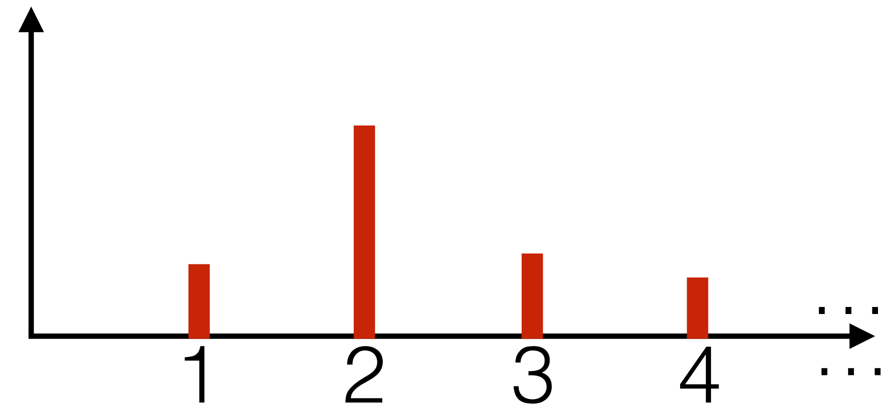

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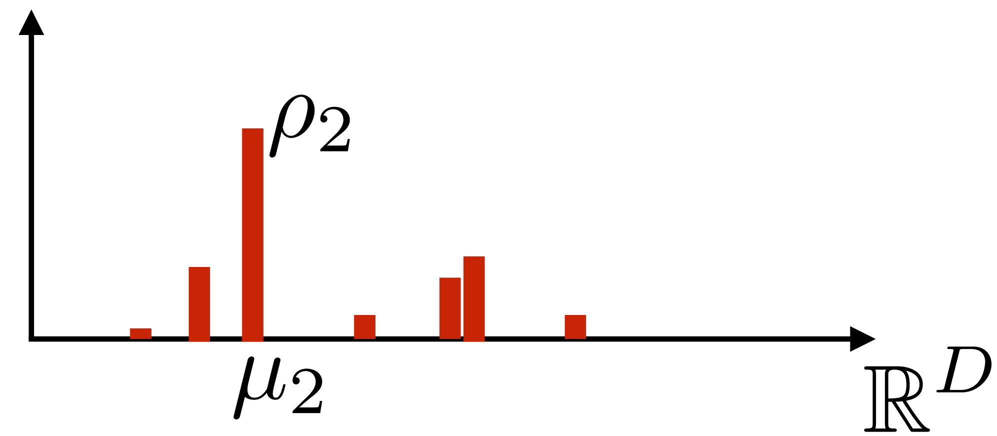
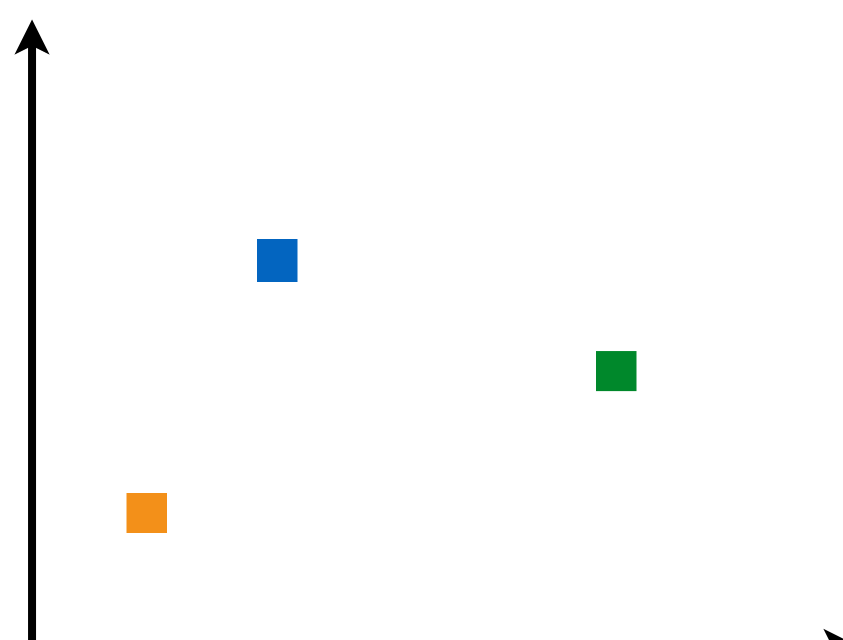


$$z_n \overset{iid}{\sim} \mathrm{Categorical}(\rho)$$

$$\mu_n^* = \mu_{z_n}$$

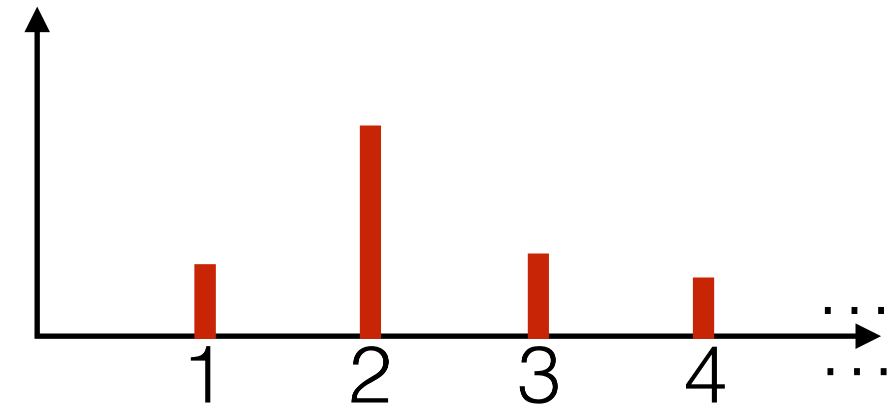
- i.e. $\mu_n^* \overset{iid}{\sim} G$



$$x_n \overset{indep}{\sim} \mathcal{N}(\mu_n^*, \Sigma)$$

Dirichlet process mixture model

- Gaussian mixture model



$$\rho = (\rho_1, \rho_2, \ldots) \sim \mathrm{GEM}(\alpha)$$

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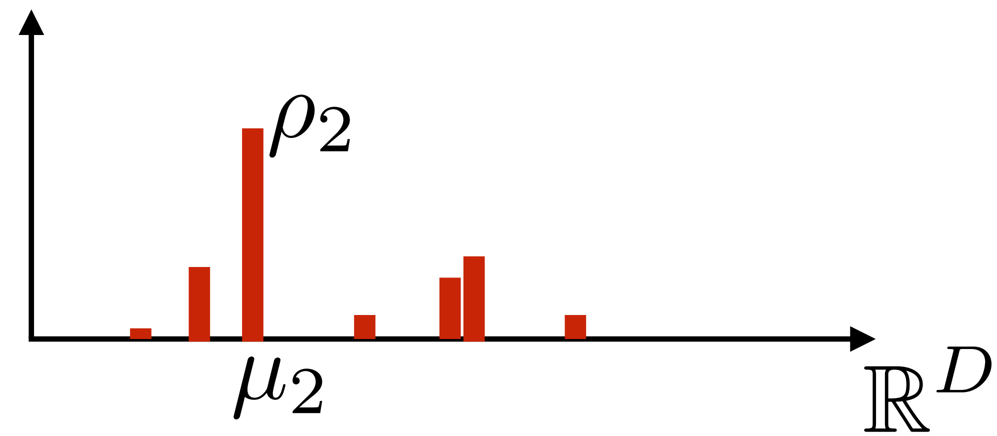

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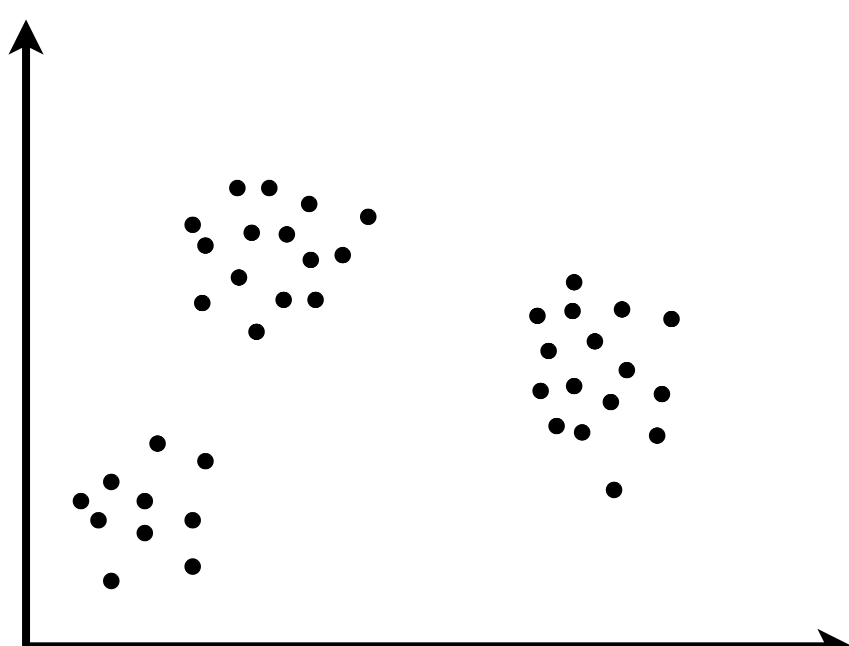

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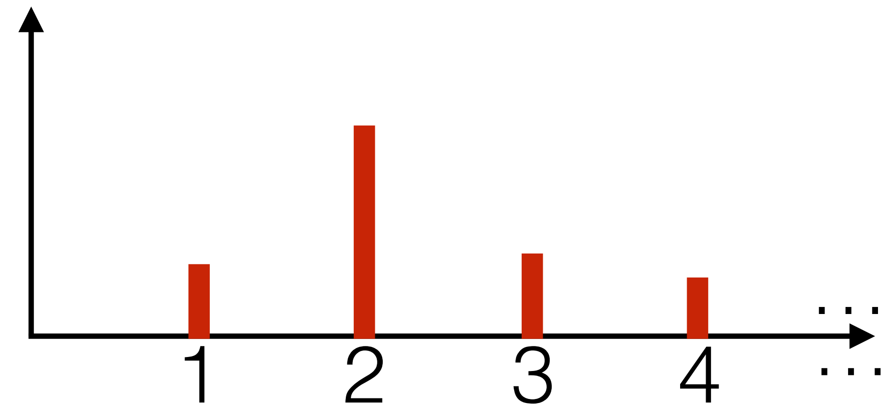

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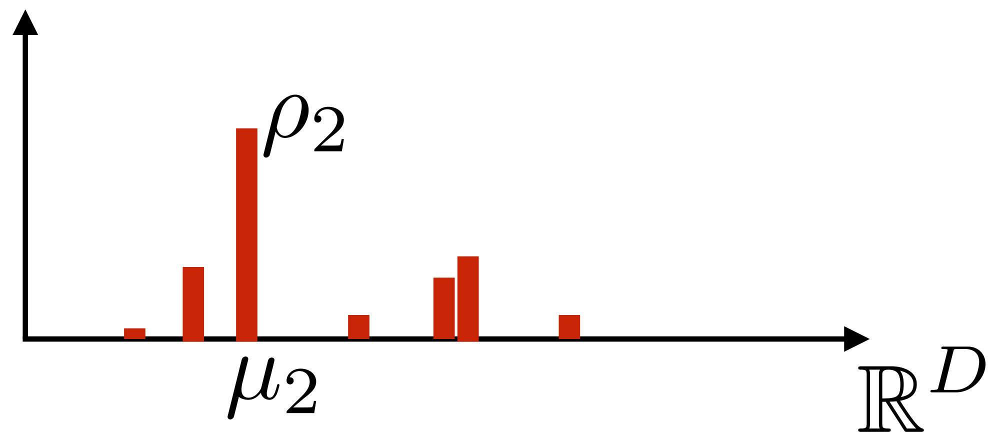

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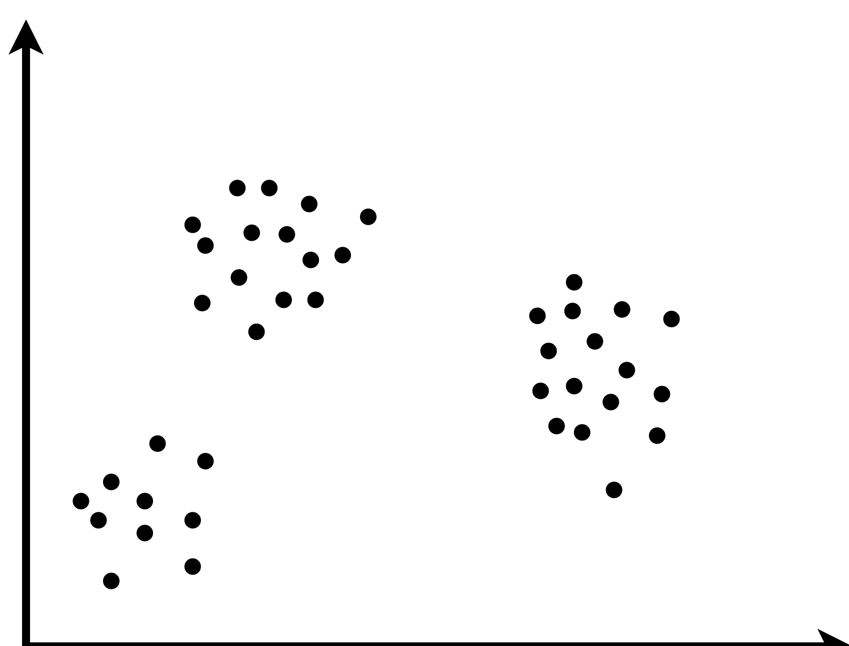

$$x_n \overset{indep}{\sim} \mathcal{N}(\mu_n^*, \Sigma)$$

[demo]

Dirichlet process mixture model

- More generally

$$\rho = (\rho_1, \rho_2, \ldots) \sim \mathrm{GEM}(\alpha)$$

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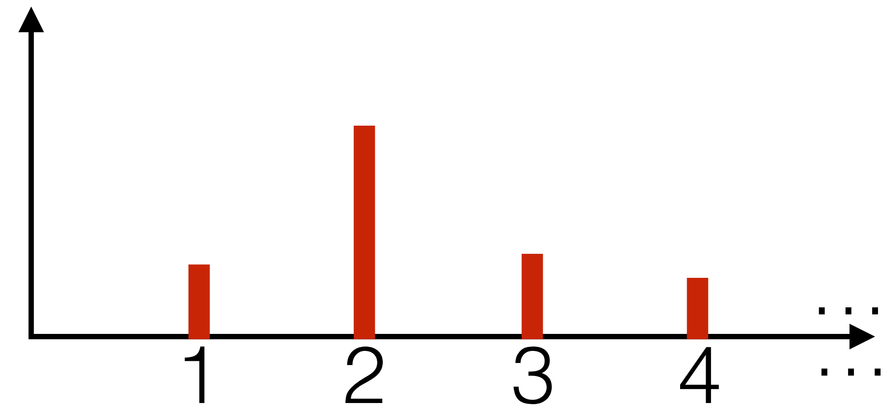

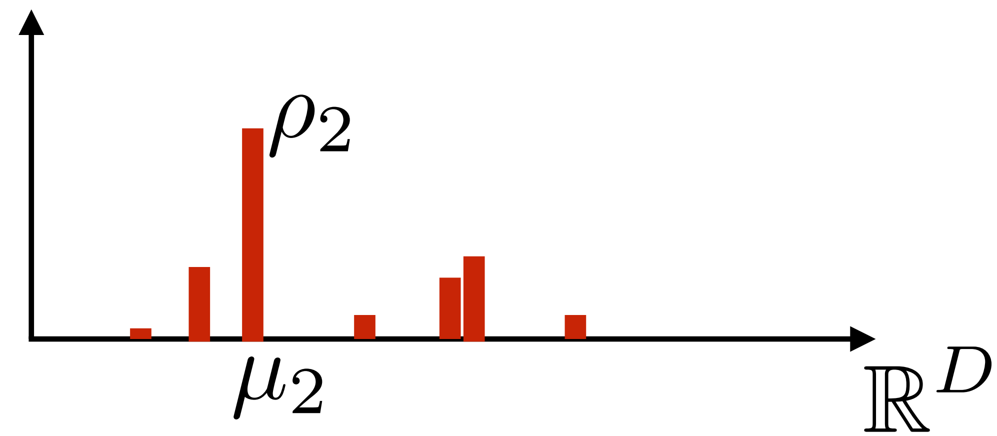

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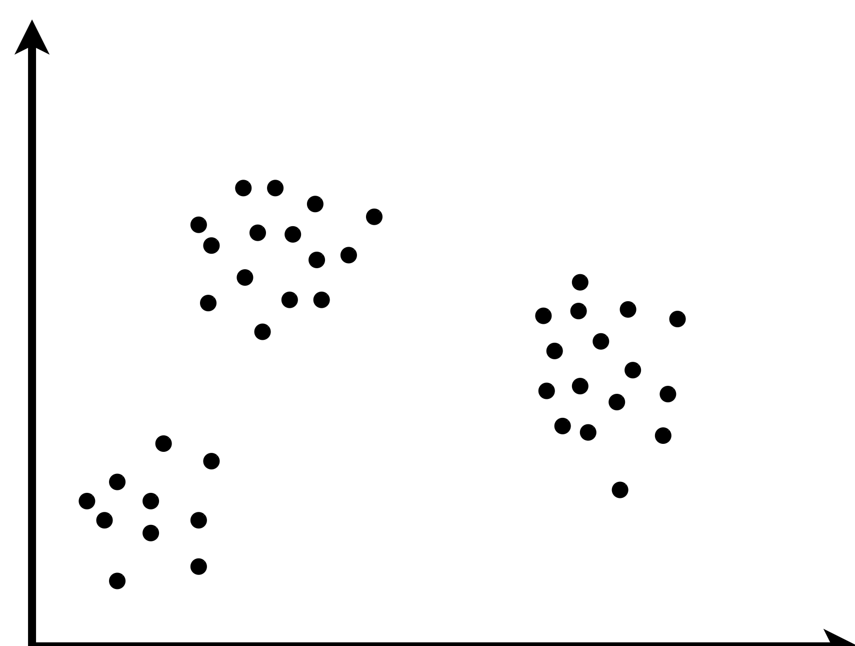

Dirichlet process mixture model

- More generally

$$\rho = (\rho_1, \rho_2, \ldots) \sim \mathrm{GEM}(\alpha)$$

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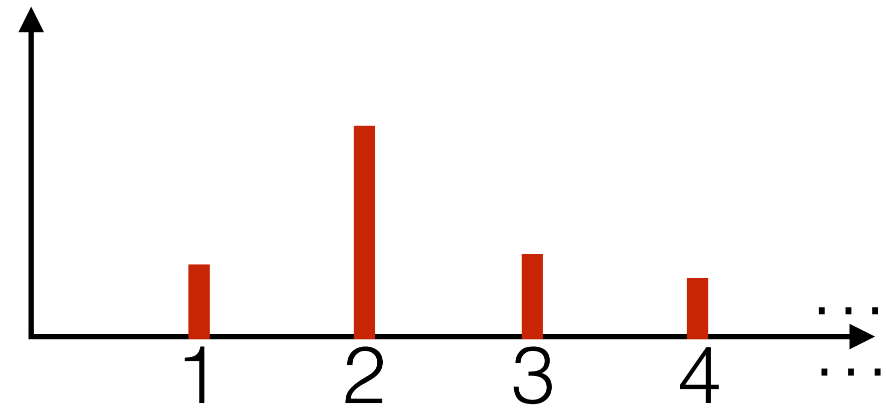

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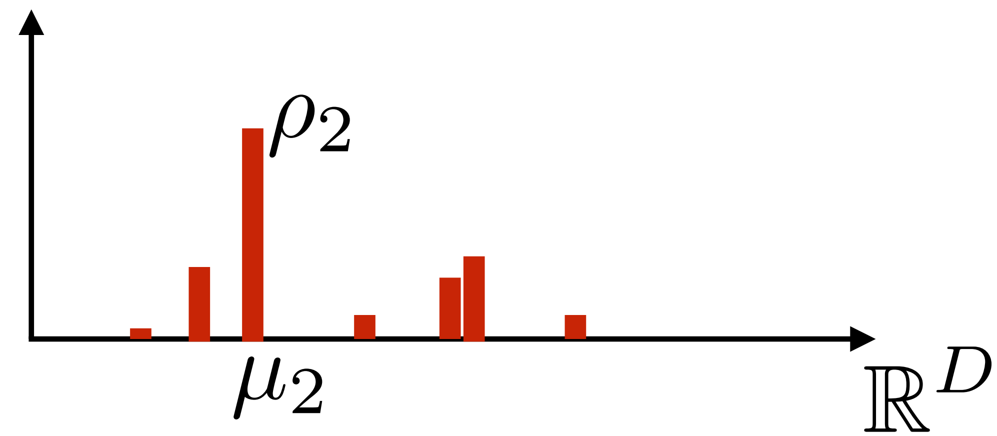

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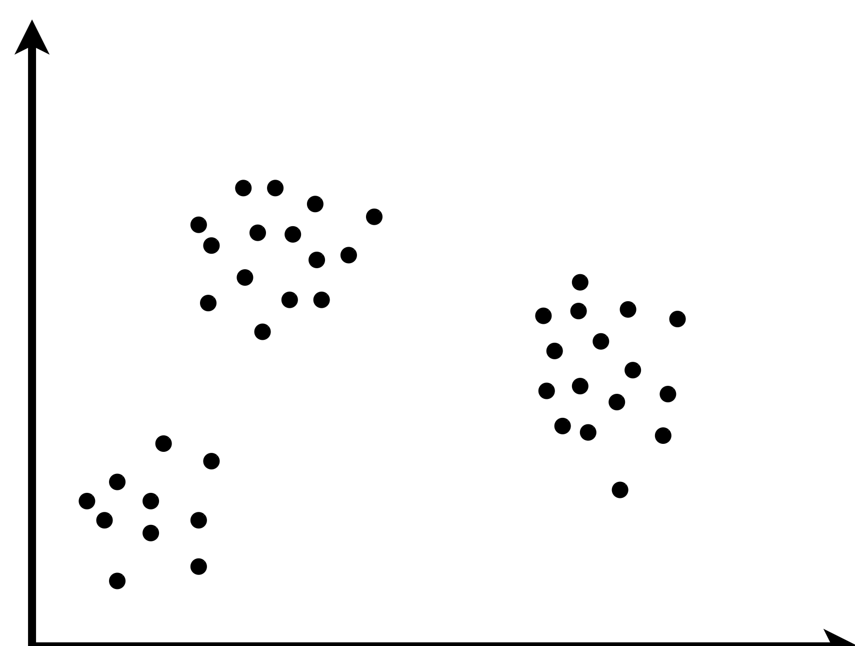

Dirichlet process mixture model

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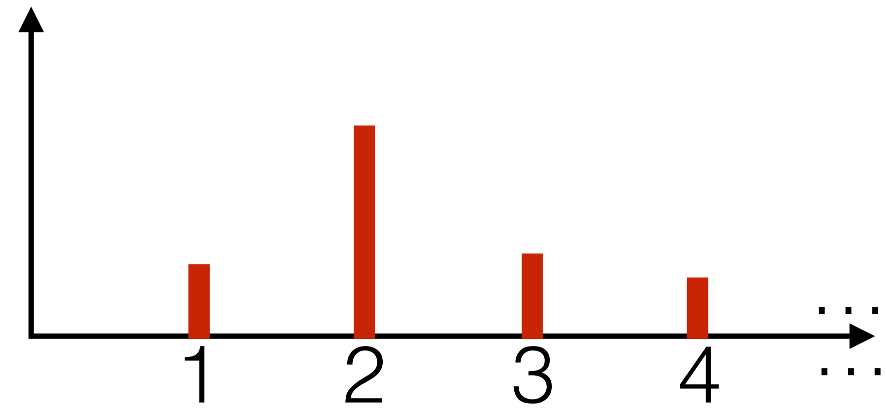

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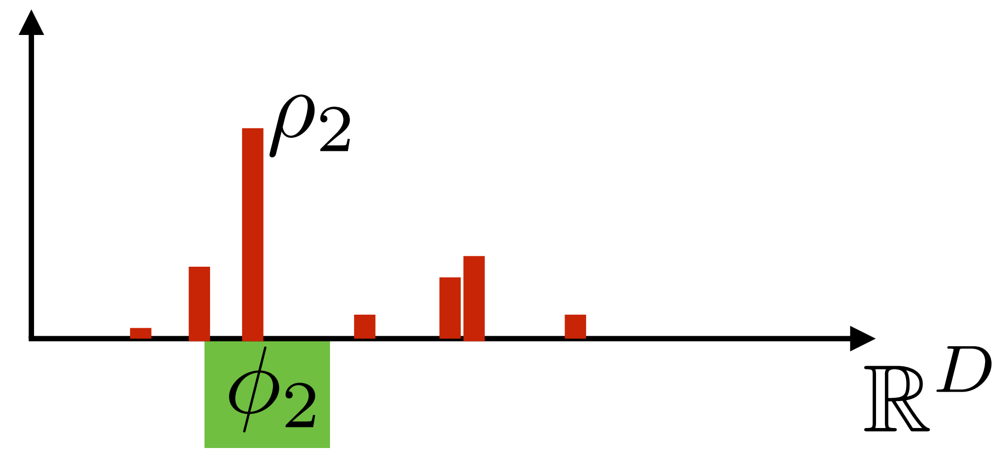

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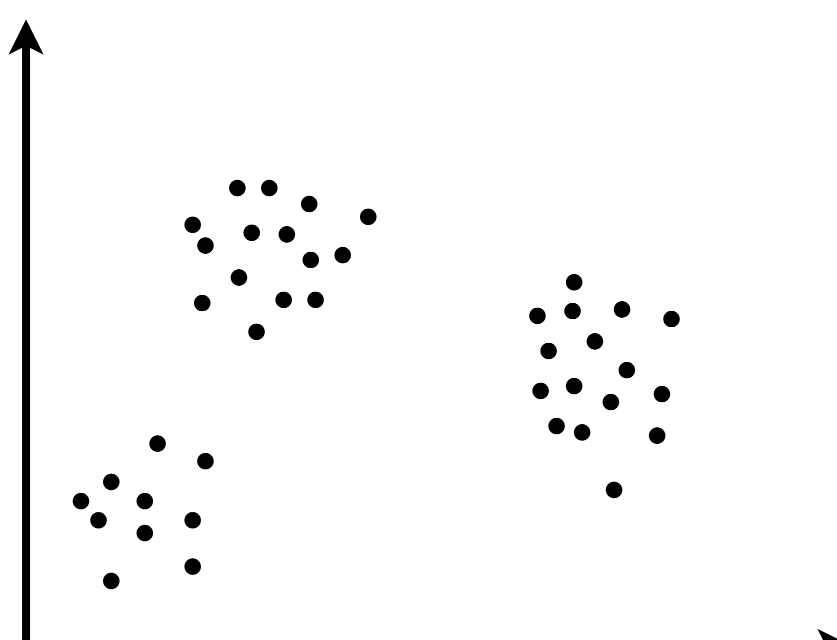

Dirichlet process mixture model

- More generally

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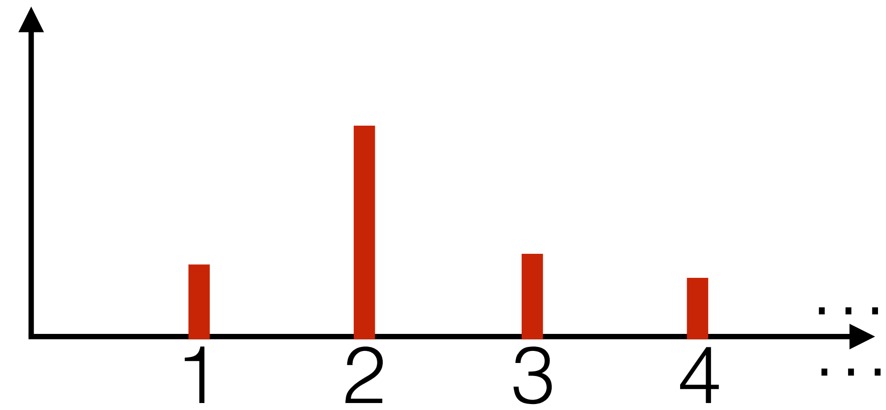

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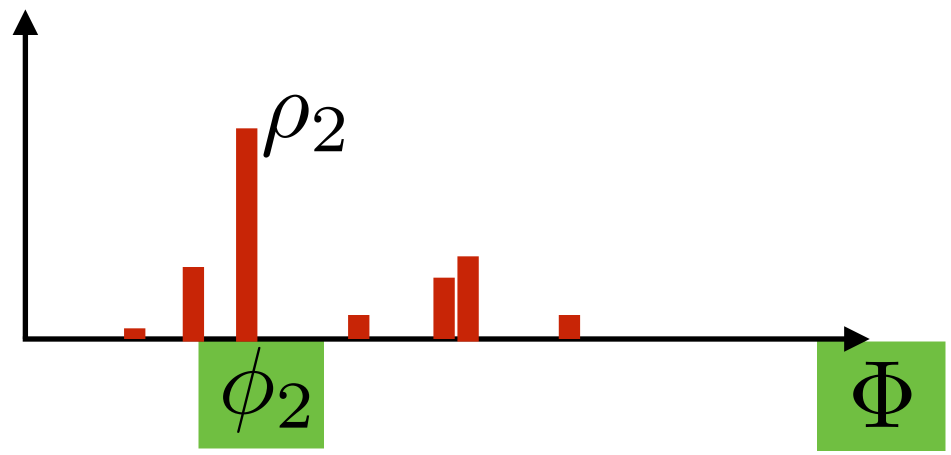

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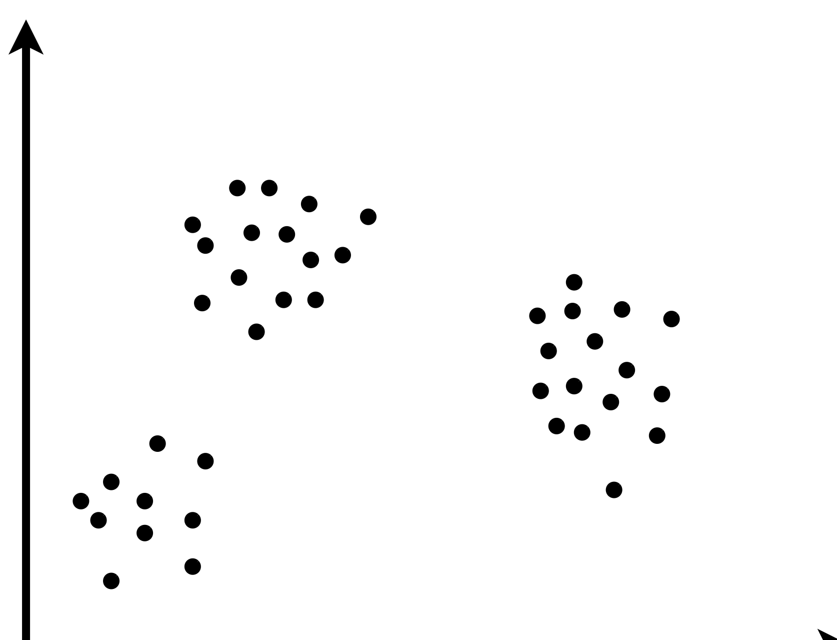

Dirichlet process mixture model

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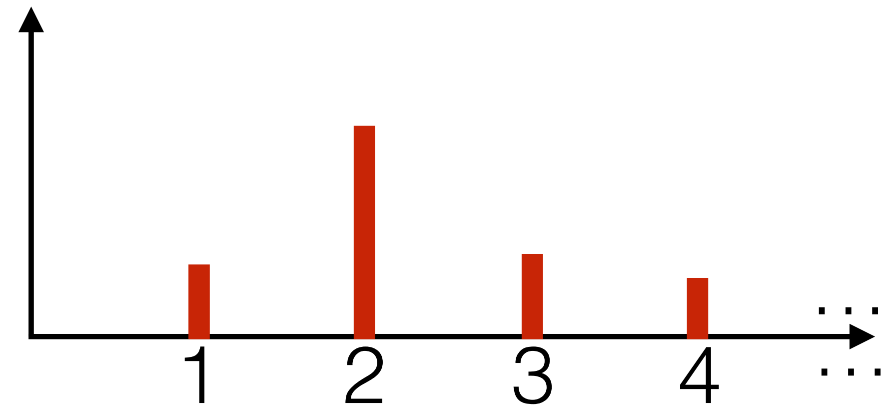

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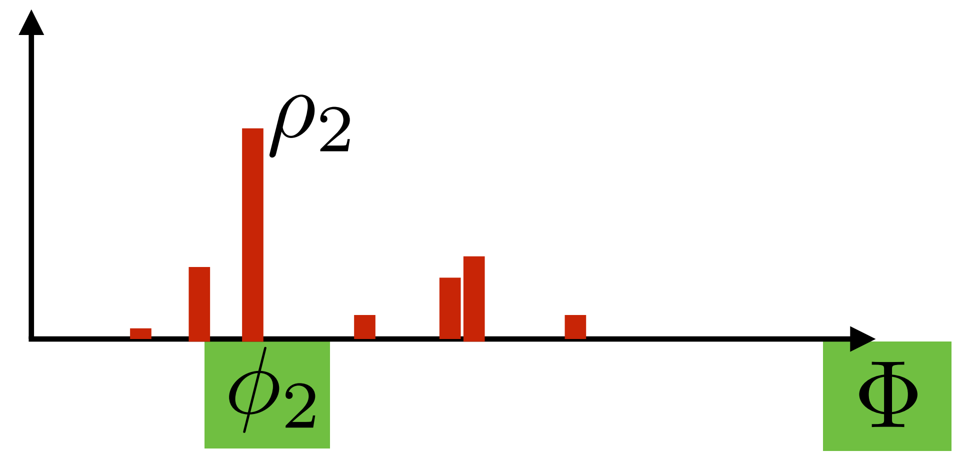

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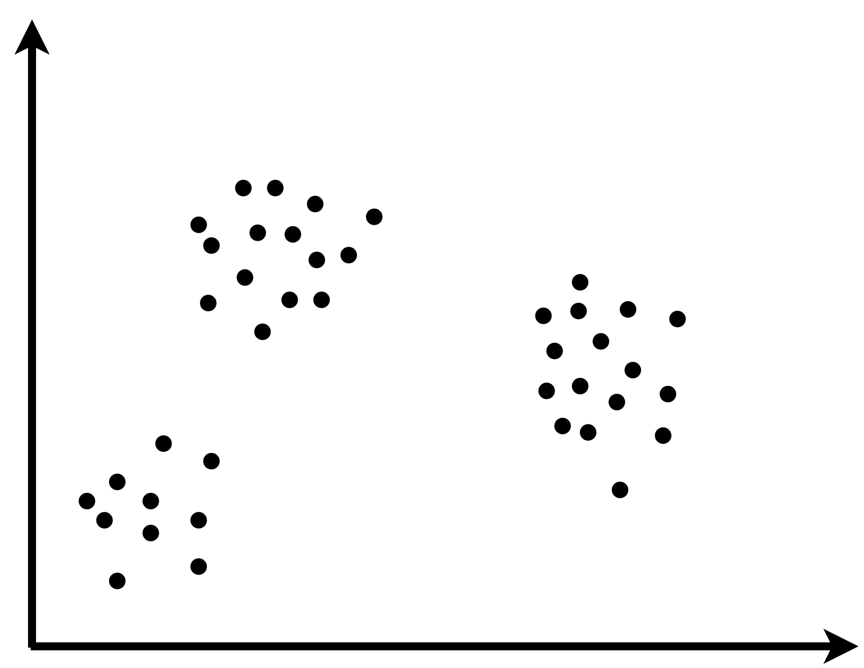

Dirichlet process mixture model

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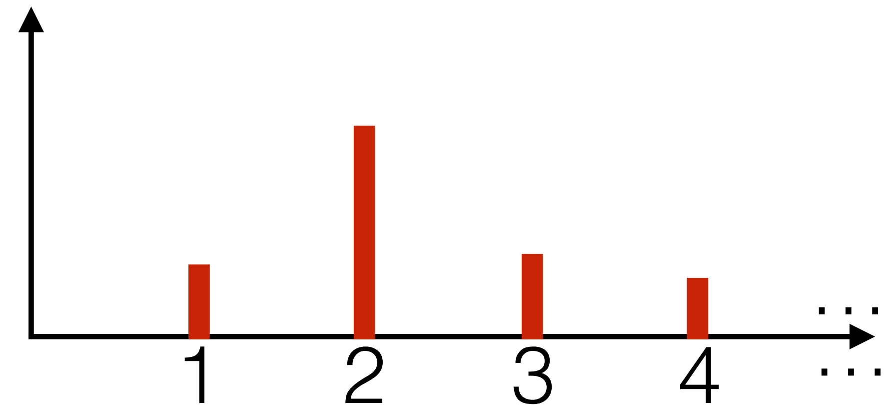

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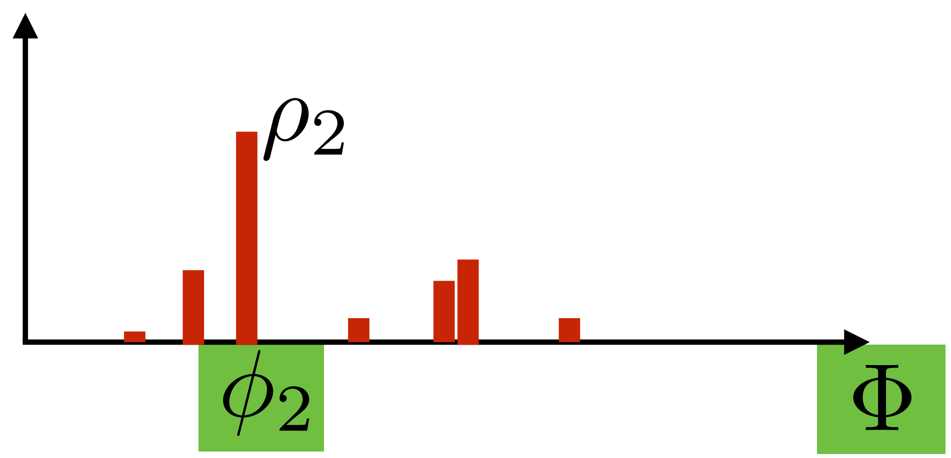

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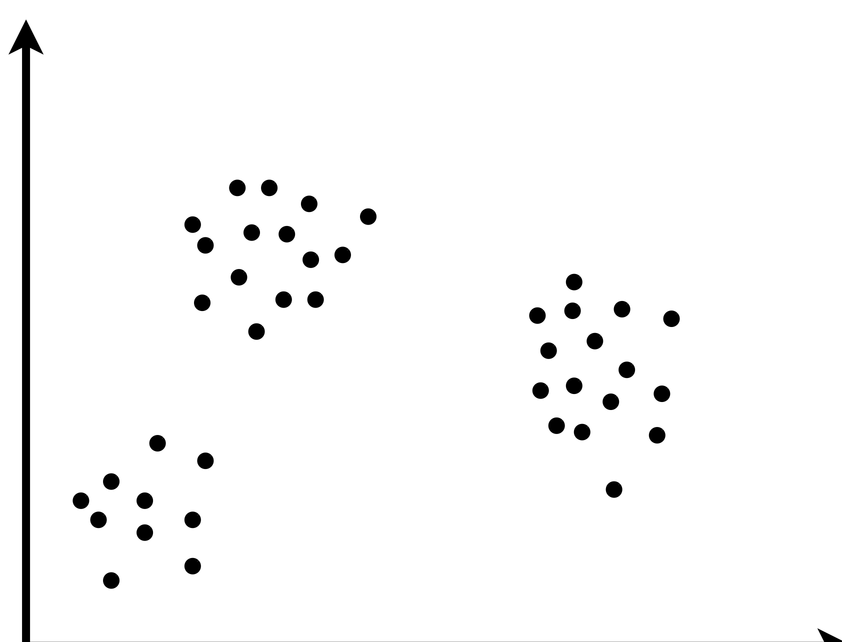

Dirichlet process mixture model

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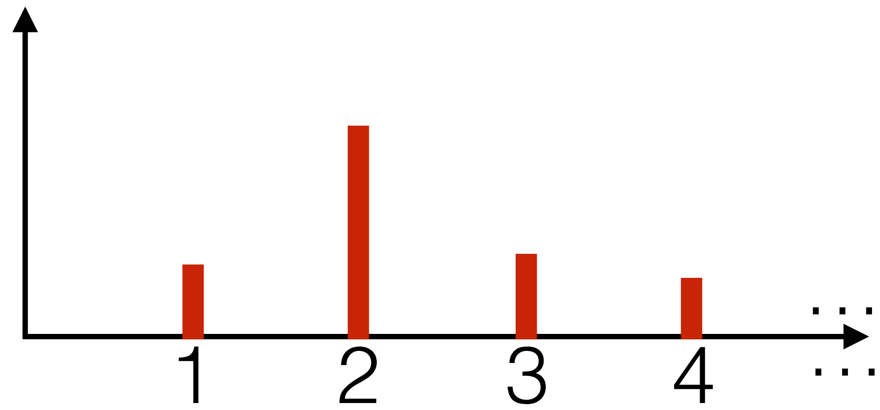

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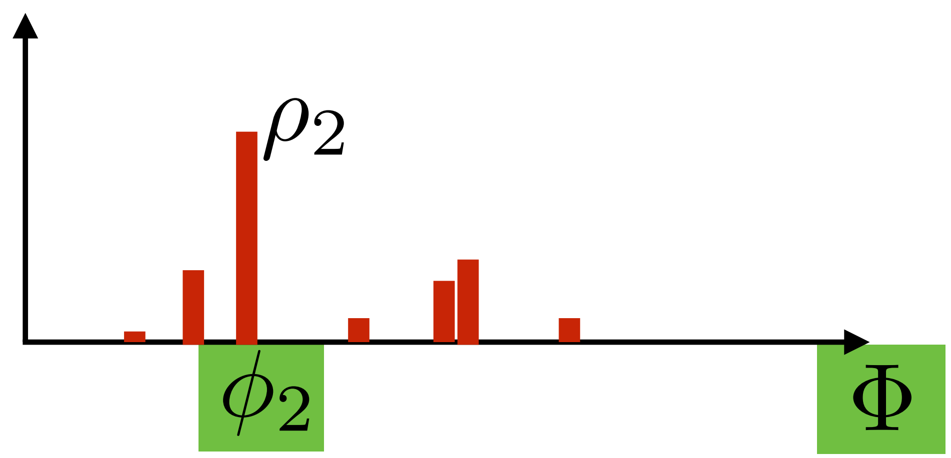

$$z_n \overset{iid}{\sim} \mathrm{Categorical}(\rho)$$

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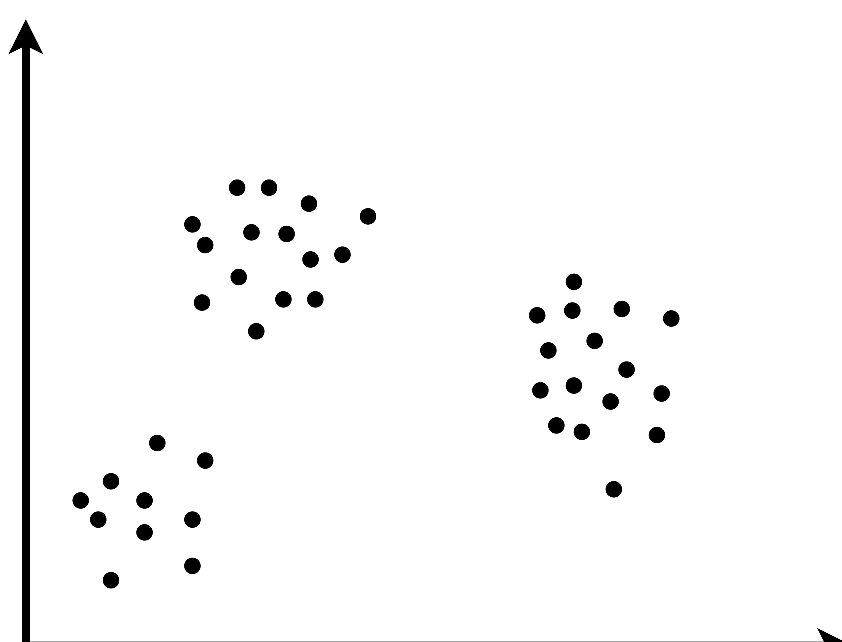

Dirichlet process mixture model

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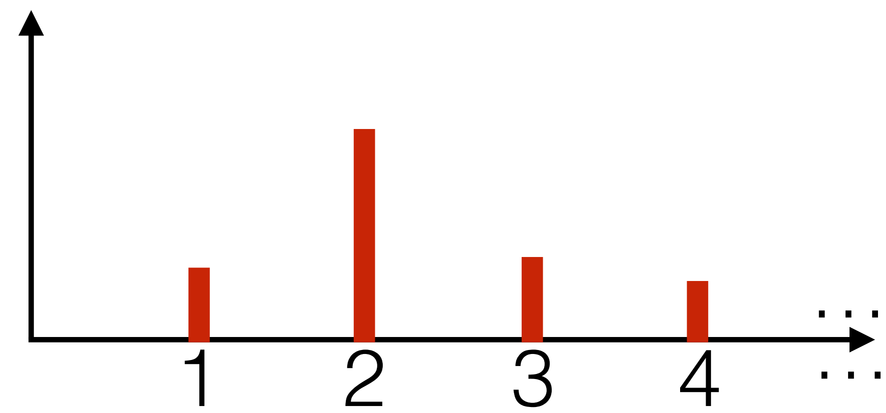

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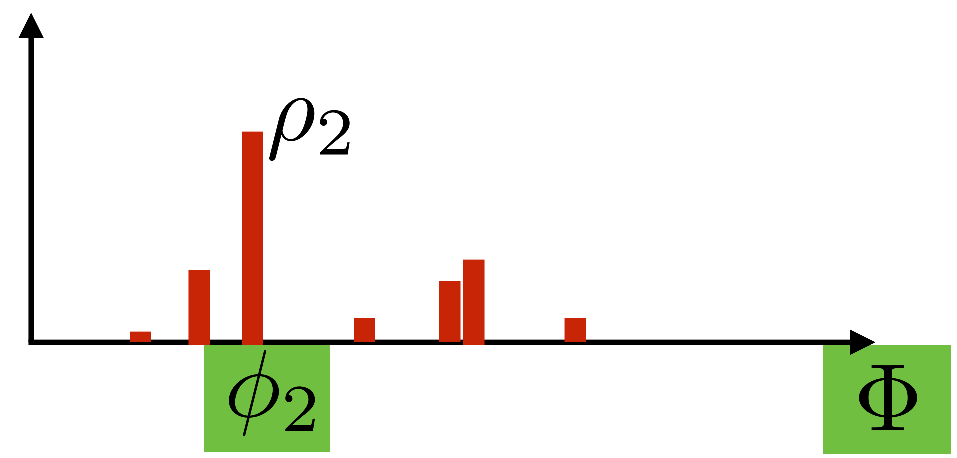

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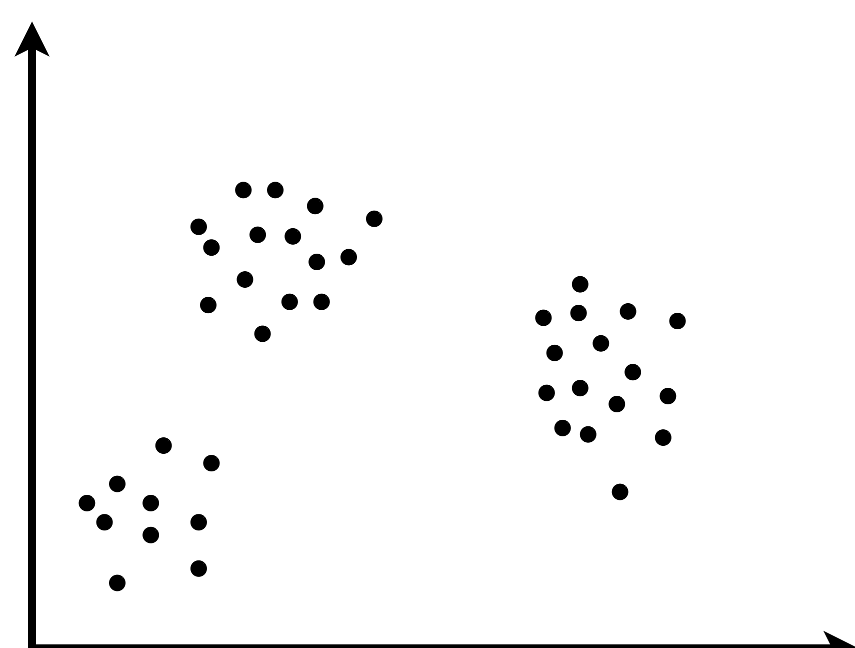

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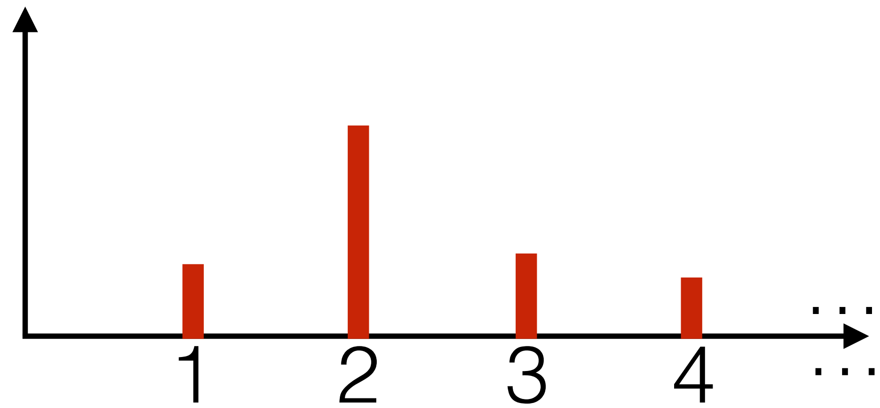

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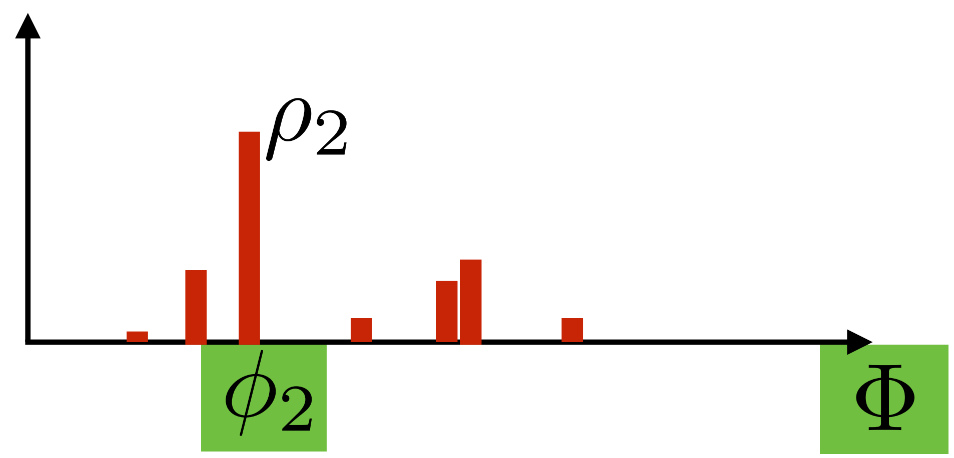

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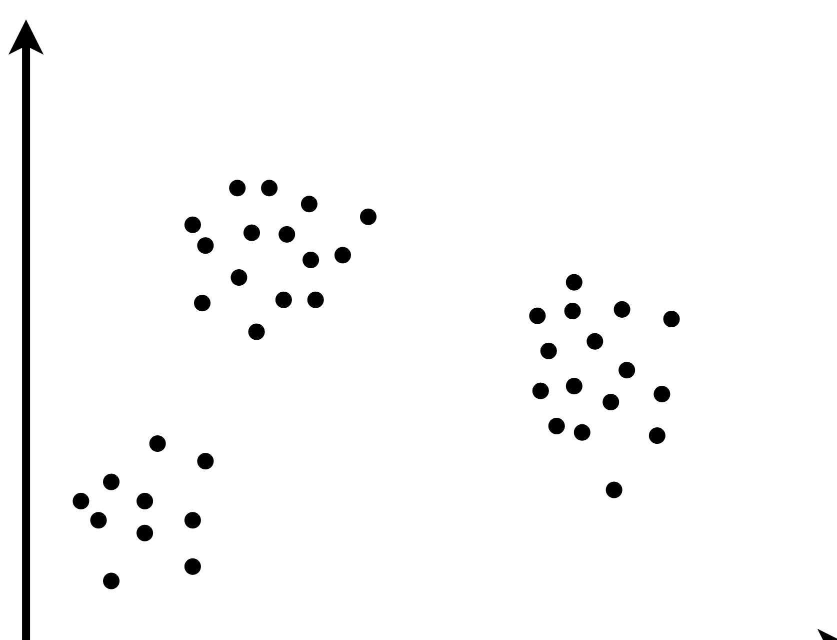

Dirichlet process mixture model

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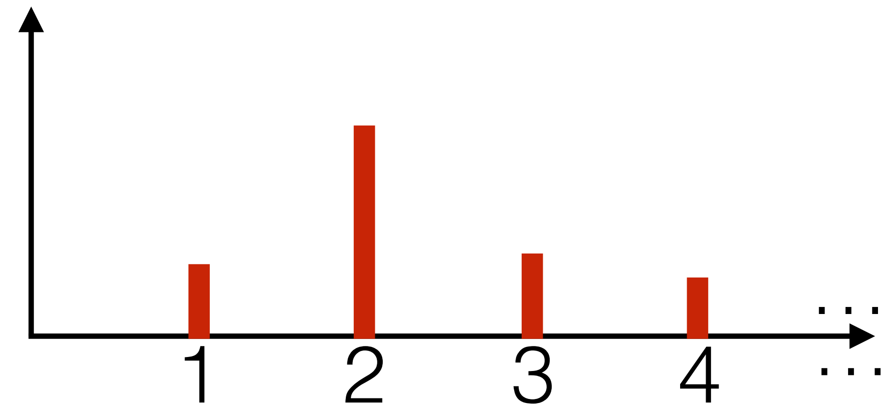

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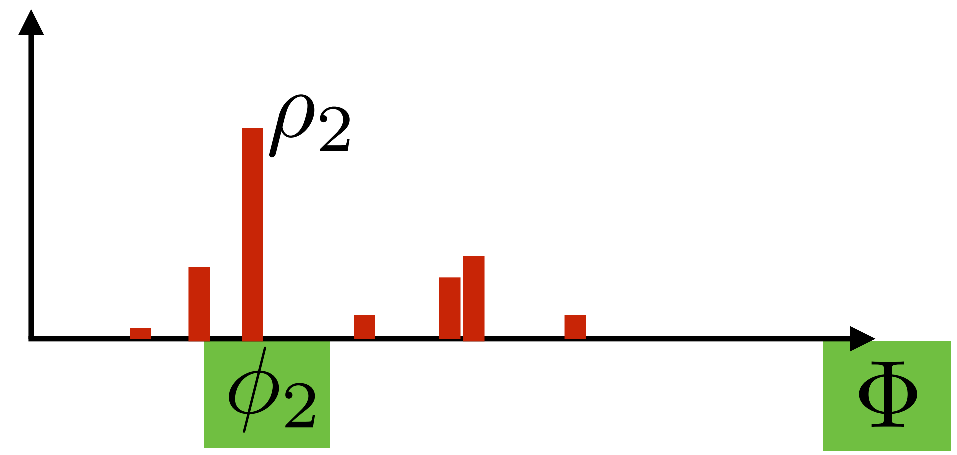

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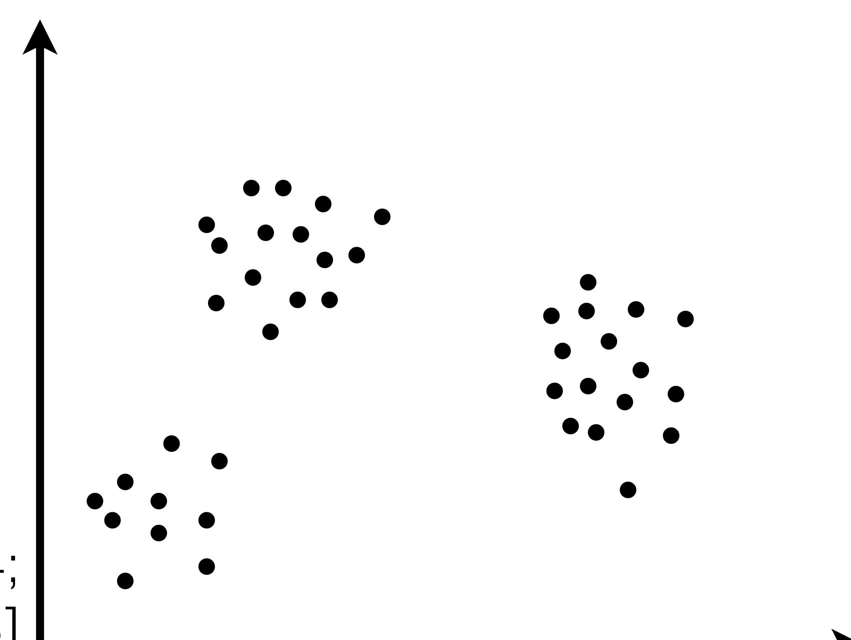
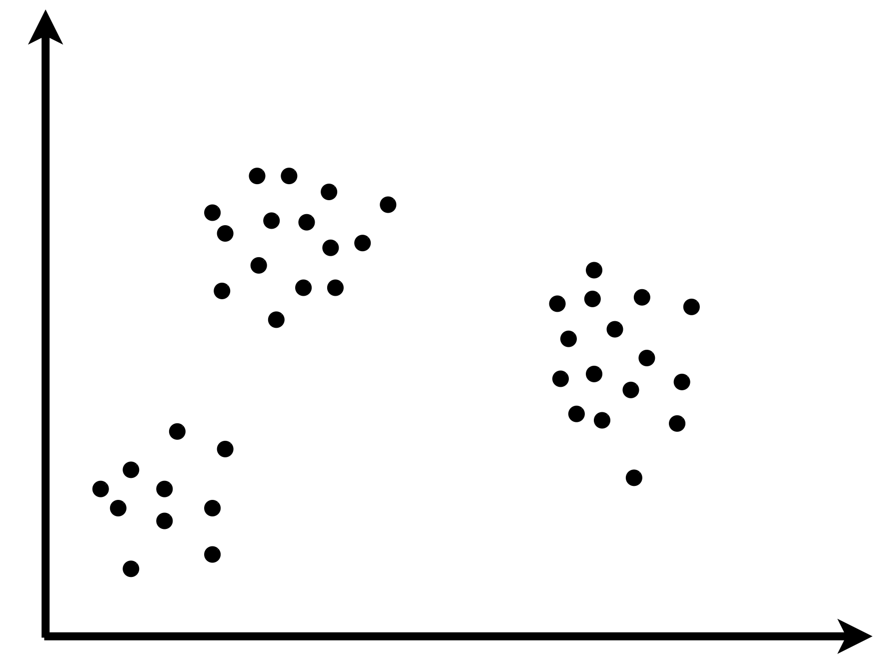


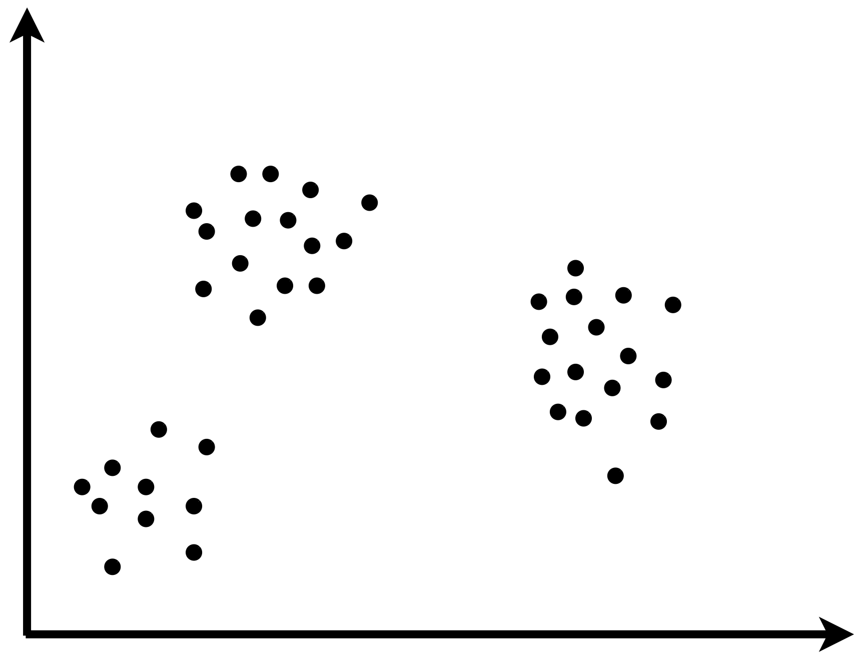
[Antoniak 1974; Ferguson 1983; West, Müller, Escobar 1994; Escobar, West 1995; MacEachern, Müller 1998]

DP or not DP, that is the question



DP or not DP, that is the question

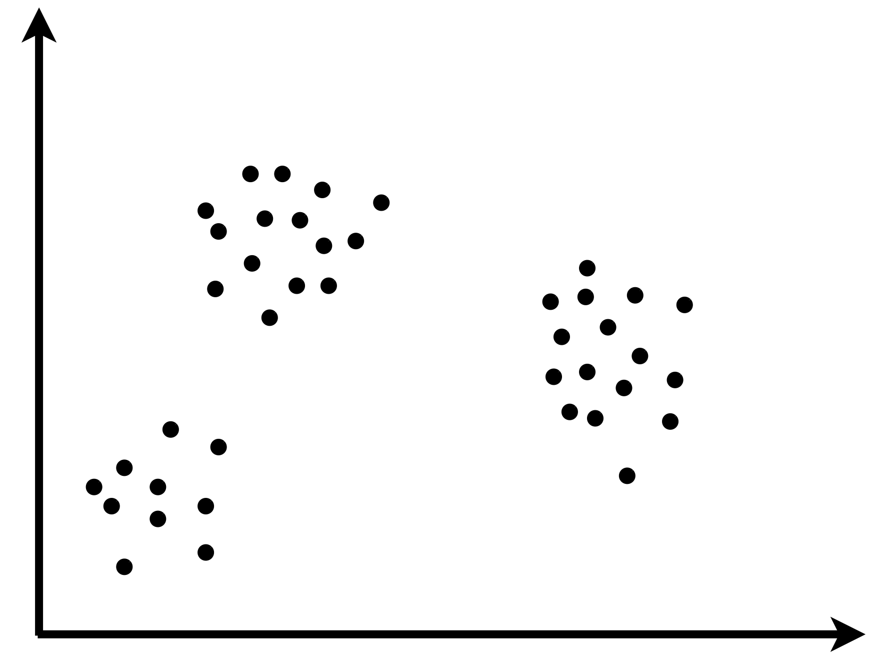
- GEM:



DP or not DP, that is the question

- GEM: 

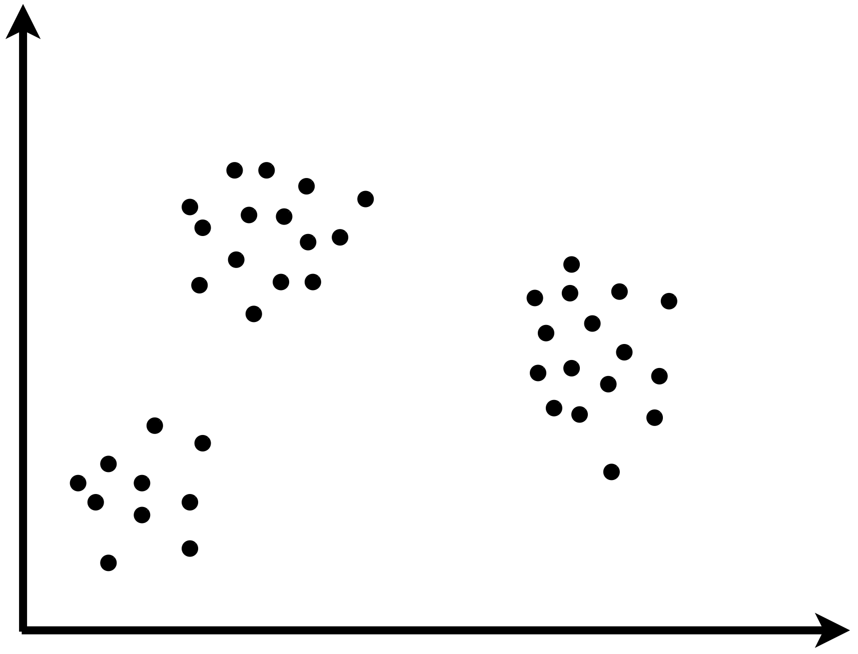
- Compare to:



DP or not DP, that is the question

- GEM: 

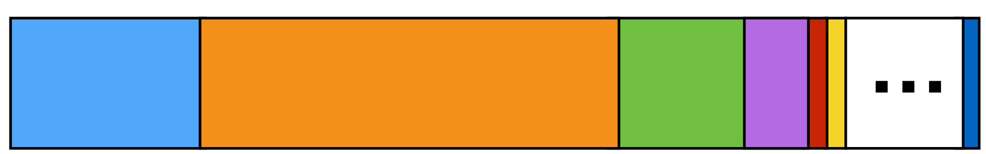
- Compare to:
  - Finite (small *K*) mixture model

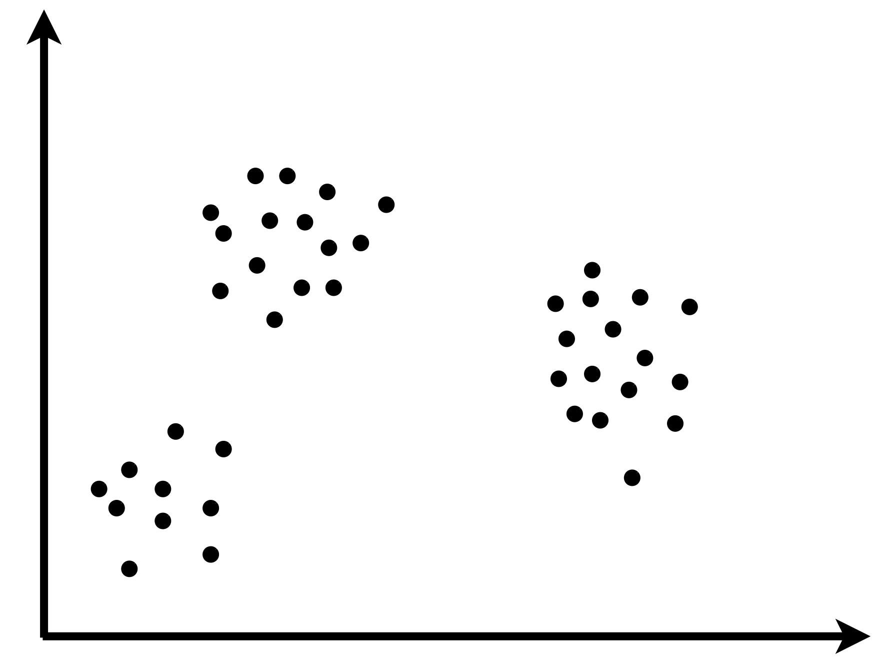


DP or not DP, that is the question

- GEM: 
- Compare to:
    - Finite (small *K*) mixture model

    - Finite (large *K*) mixture model





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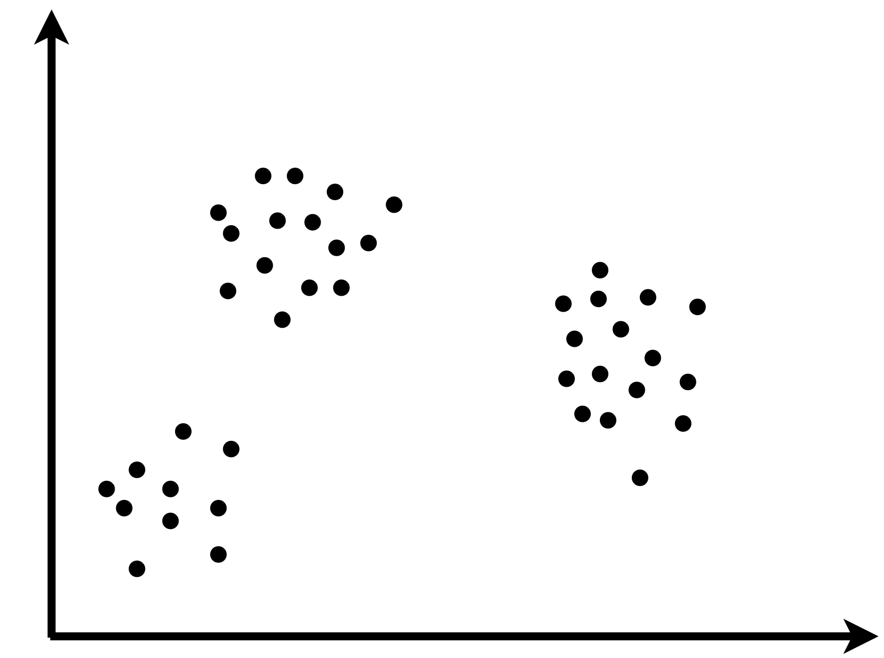
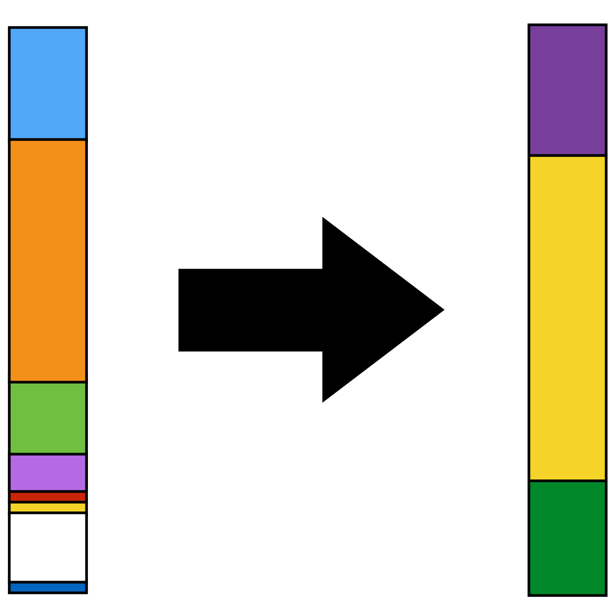
- GEM: 



- Compare to:
  - Finite (small *K*) mixture model

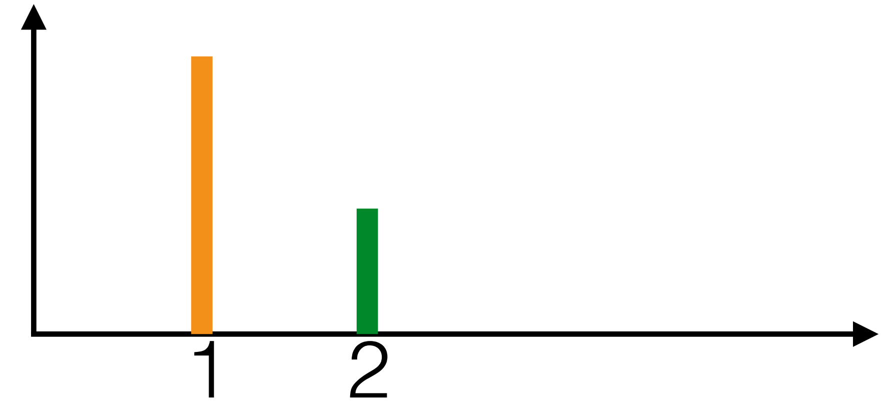
  - Finite (large *K*) mixture model

  - Time series



Marginal cluster assignments

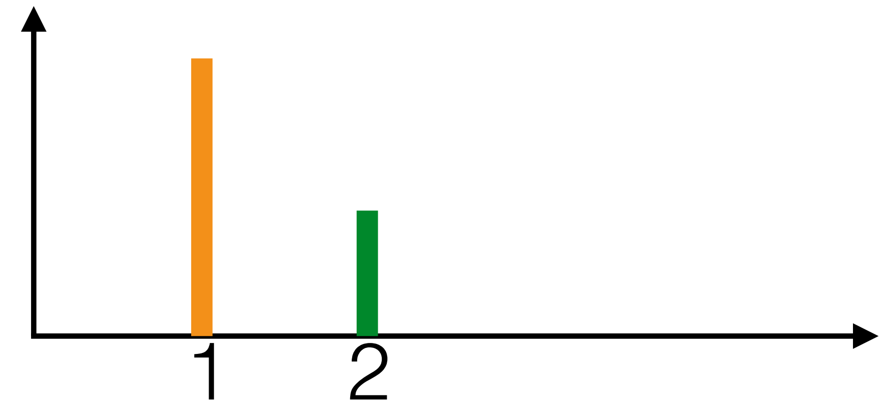
$$\rho_1 \sim \mathrm{Beta}(a_1, a_2), z_n \overset{iid}{\sim} \mathrm{Cat}(\rho_1, \rho_2)$$



Marginal cluster assignments

- Integrate out the frequencies

$\rho_1 \sim \mathrm{Beta}(a_1, a_2), z_n \overset{iid}{\sim} \mathrm{Cat}(\rho_1, \rho_2)$

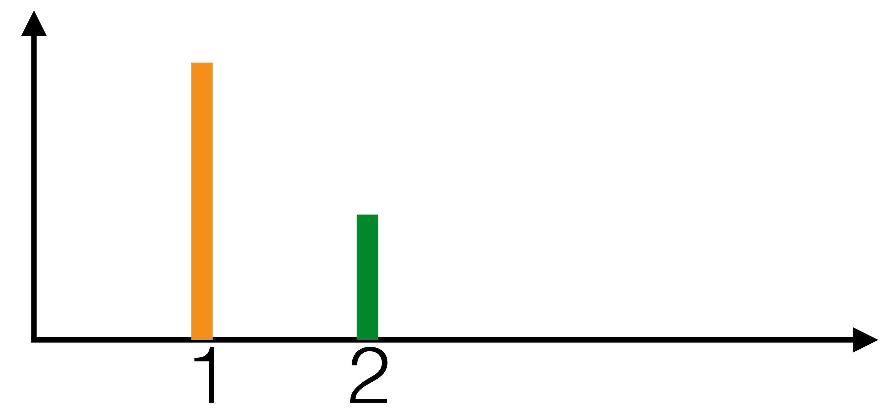


Marginal cluster assignments

- Integrate out the frequencies

$$\rho_1 \sim \mathrm{Beta}(a_1, a_2), z_n \overset{iid}{\sim} \mathrm{Cat}(\rho_1, \rho_2)$$

$$p(z_n = 1 | z_1, \ldots, z_{n-1})$$

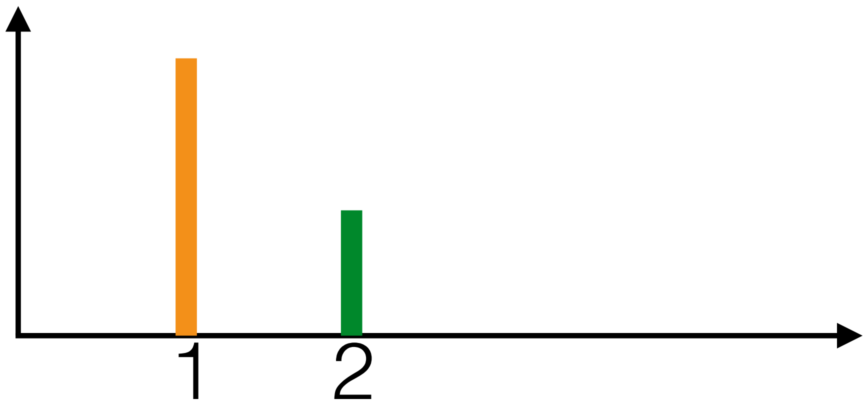


Marginal cluster assignments

- Integrate out the frequencies

$$\rho_1 \sim \mathrm{Beta}(a_1, a_2), z_n \overset{iid}{\sim} \mathrm{Cat}(\rho_1, \rho_2)$$

$$p(z_n = 1 | z_1, \ldots, z_{n-1})$$
$$= \int p(z_n = 1, \rho_1 | z_1, \ldots, z_{n-1}) d\rho_1$$

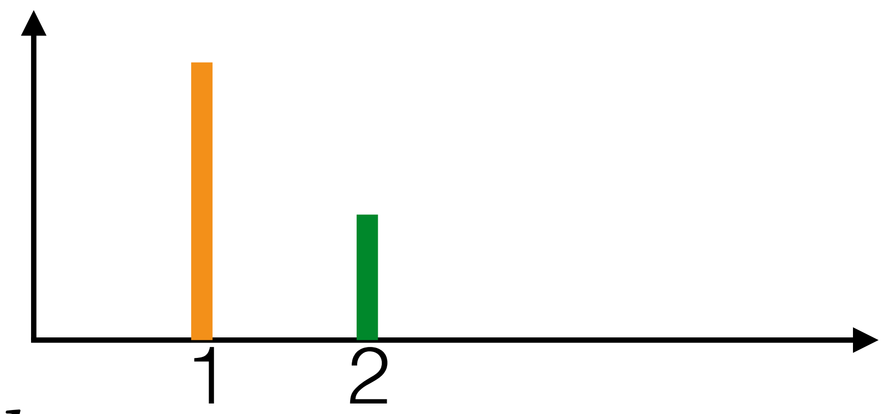


Marginal cluster assignments

- Integrate out the frequencies

$$\rho_1 \sim \mathrm{Beta}(a_1, a_2), z_n \overset{iid}{\sim} \mathrm{Cat}(\rho_1, \rho_2)$$

$$p(z_n = 1 | z_1, \ldots, z_{n-1})$$
$$= \int p(z_n = 1 | \rho_1) p(\rho_1 | z_1, \ldots, z_{n-1}) d\rho_1$$

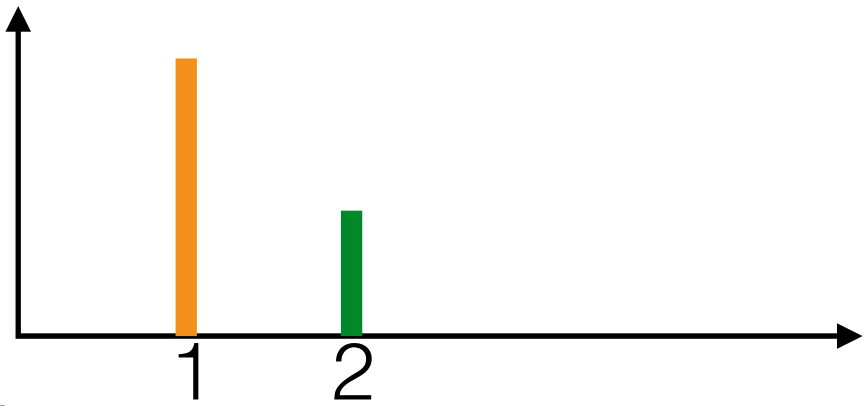


Marginal cluster assignments

- Integrate out the frequencies

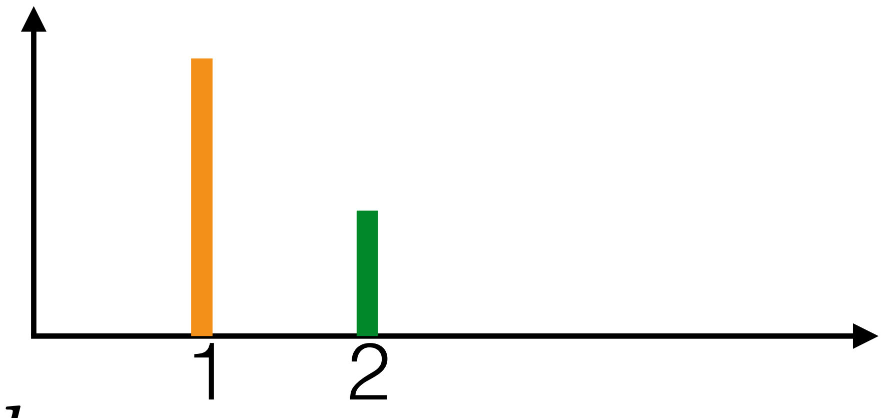
$$\rho_1 \sim \text{Beta}(a_1, a_2), z_n \stackrel{iid}{\sim} \text{Cat}(\rho_1, \rho_2)$$

$$p(z_n = 1 | z_1, \ldots, z_{n-1})$$
$$= \int p(z_n = 1 | \rho_1) p(\rho_1 | z_1, \ldots, z_{n-1}) d\rho_1$$
$$= \int \rho_1 \text{Beta}(\rho_1 | a_{1,n}, a_{2,n}) d\rho_1$$



Marginal cluster assignments

- Integrate out the frequencies



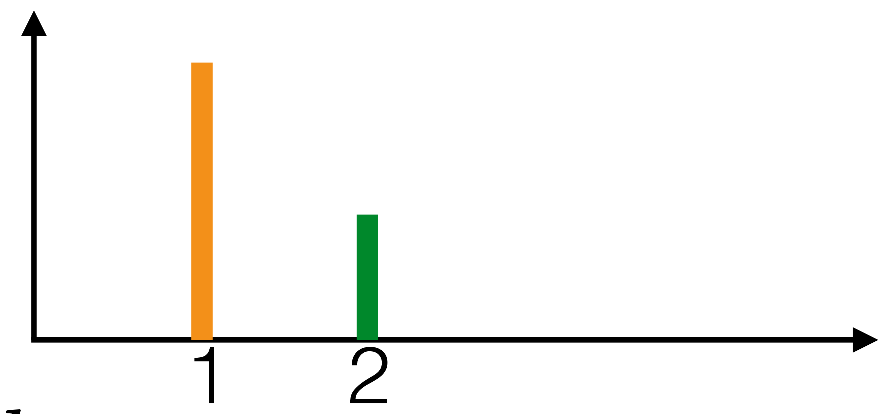
$$\rho_1 \sim \mathrm{Beta}(a_1, a_2), z_n \stackrel{iid}{\sim} \mathrm{Cat}(\rho_1, \rho_2)$$

$$p(z_n = 1 | z_1, \ldots, z_{n-1})$$
$$= \int p(z_n = 1 | \rho_1) p(\rho_1 | z_1, \ldots, z_{n-1}) d\rho_1$$
$$= \int \rho_1 \mathrm{Beta}(\rho_1 | a_{1,n}, a_{2,n}) d\rho_1$$

$$a_{1,n} := a_1 + \sum_{m=1}^{n-1} \mathbf{1}\{z_m = 1\}, a_{2,n} = a_2 + \sum_{m=1}^{n-1} \mathbf{1}\{z_m = 2\}$$

Marginal cluster assignments

- Integrate out the frequencies



$$\rho_1 \sim \mathrm{Beta}(a_1, a_2), z_n \overset{iid}{\sim} \mathrm{Cat}(\rho_1, \rho_2)$$

$$p(z_n = 1 | z_1, \ldots, z_{n-1})$$

$$= \int p(z_n = 1 | \rho_1) p(\rho_1 | z_1, \ldots, z_{n-1}) d\rho_1$$

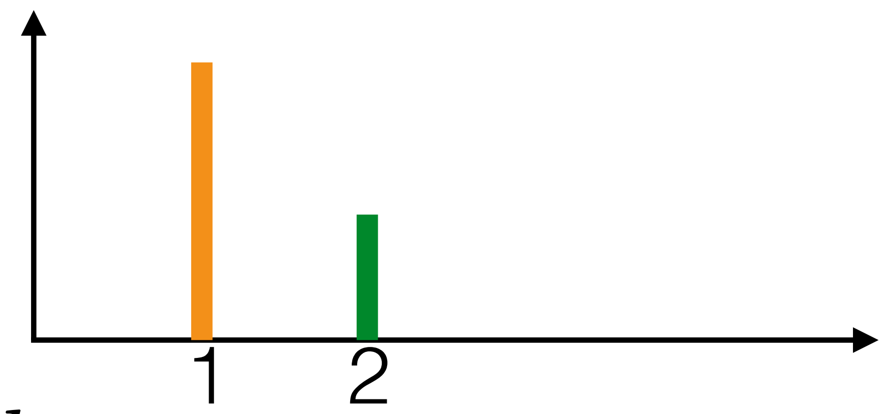
$$= \int \rho_1 \mathrm{Beta}(\rho_1 | a_{1,n}, a_{2,n}) d\rho_1$$

$$a_{1,n} := a_1 + \sum_{m=1}^{n-1} \mathbf{1}\{z_m = 1\}, a_{2,n} = a_2 + \sum_{m=1}^{n-1} \mathbf{1}\{z_m = 2\}$$

$$= \int \rho_1 \frac{\Gamma(a_{1,n} + a_{2,n})}{\Gamma(a_{1,n})\Gamma(a_{2,n})} \rho_1^{a_{1,n}-1} (1 - \rho_1)^{a_{2,n}-1} d\rho_1$$

Marginal cluster assignments

- Integrate out the frequencies



$$\rho_1 \sim \text{Beta}(a_1, a_2),\, z_n \overset{iid}{\sim} \text{Cat}(\rho_1, \rho_2)$$

$$p(z_n = 1 | z_1, \ldots, z_{n-1})$$

$$= \int p(z_n = 1 | \rho_1) p(\rho_1 | z_1, \ldots, z_{n-1}) d\rho_1$$

$$= \int \rho_1 \text{Beta}(\rho_1 | a_{1,n}, a_{2,n}) d\rho_1$$

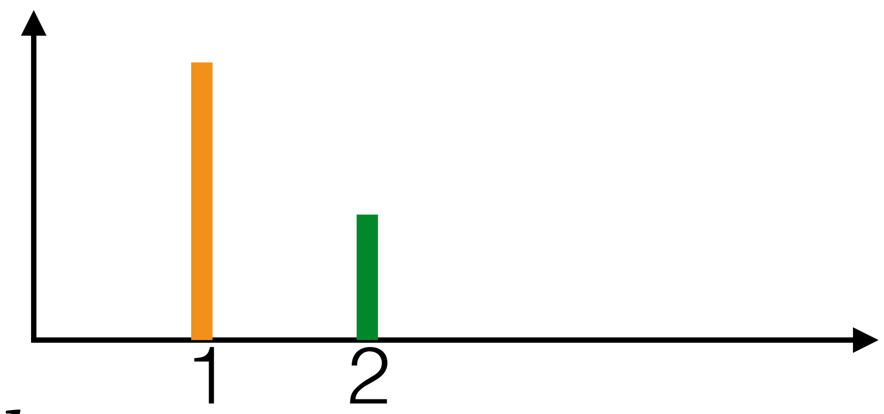
$$a_{1,n} := a_1 + \sum_{m=1}^{n-1} \mathbf{1}\{z_m = 1\},\, a_{2,n} = a_2 + \sum_{m=1}^{n-1} \mathbf{1}\{z_m = 2\}$$

$$= \int \rho_1 \frac{\Gamma(a_{1,n} + a_{2,n})}{\Gamma(a_{1,n})\Gamma(a_{2,n})} \rho_1^{a_{1,n}-1} (1 - \rho_1)^{a_{2,n}-1} d\rho_1$$

$$= \frac{\Gamma(a_{1,n} + a_{2,n})}{\Gamma(a_{1,n})\Gamma(a_{2,n})} \frac{\Gamma(a_{1,n} + 1)\Gamma(a_{2,n})}{\Gamma(a_{1,n} + a_{2,n} + 1)}$$

Marginal cluster assignments

- Integrate out the frequencies



$$\rho_1 \sim \mathrm{Beta}(a_1, a_2), z_n \stackrel{iid}{\sim} \mathrm{Cat}(\rho_1, \rho_2)$$

$$p(z_n = 1 | z_1, \ldots, z_{n-1})$$

$$= \int p(z_n = 1 | \rho_1) p(\rho_1 | z_1, \ldots, z_{n-1}) d\rho_1$$

$$= \int \rho_1 \mathrm{Beta}(\rho_1 | a_{1,n}, a_{2,n}) d\rho_1$$

$$a_{1,n} := a_1 + \sum_{m=1}^{n-1} \mathbf{1}\{z_m = 1\}, a_{2,n} = a_2 + \sum_{m=1}^{n-1} \mathbf{1}\{z_m = 2\}$$

$$= \int \rho_1 \frac{\Gamma(a_{1,n} + a_{2,n})}{\Gamma(a_{1,n})\Gamma(a_{2,n})} \rho_1^{a_{1,n}-1} (1 - \rho_1)^{a_{2,n}-1} d\rho_1$$

$$= \frac{\Gamma(a_{1,n} + a_{2,n})}{\Gamma(a_{1,n})\Gamma(a_{2,n})} \frac{\Gamma(a_{1,n} + 1)\Gamma(a_{2,n})}{\Gamma(a_{1,n} + a_{2,n} + 1)}$$

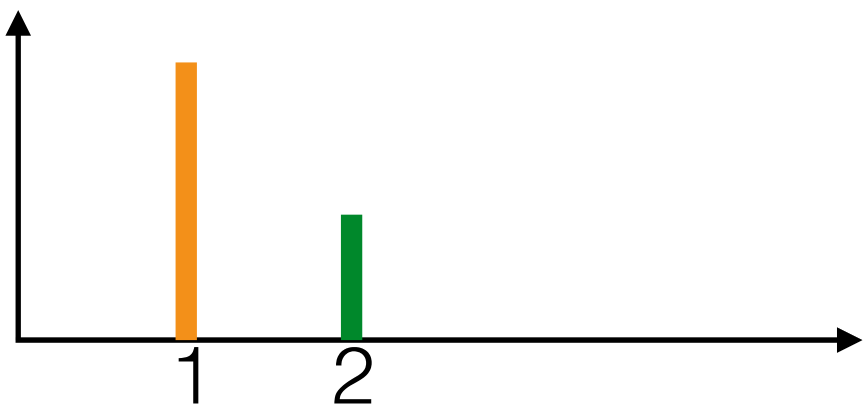
$$= \frac{a_{1,n}}{a_{1,n} + a_{2,n}}$$

Marginal cluster assignments

- Integrate out the frequencies

$$\rho_1 \sim \mathrm{Beta}(a_1, a_2), z_n \stackrel{iid}{\sim} \mathrm{Cat}(\rho_1, \rho_2)$$

$$p(z_n = 1 | z_1, \ldots, z_{n-1}) = \frac{a_{1,n}}{a_{1,n} + a_{2,n}}$$



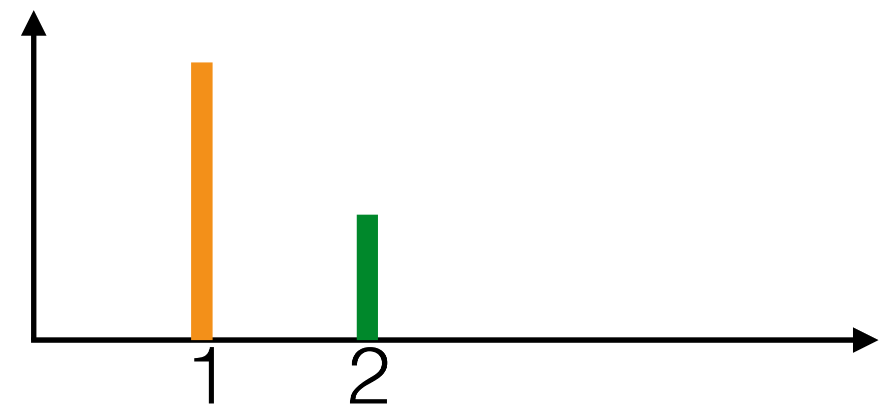
$$a_{1,n} := a_1 + \sum_{m=1}^{n-1} \mathbf{1}\{z_m = 1\}, a_{2,n} = a_2 + \sum_{m=1}^{n-1} \mathbf{1}\{z_m = 2\}$$

Marginal cluster assignments

- Integrate out the frequencies

$$\rho_1 \sim \text{Beta}(a_1, a_2), z_n \overset{iid}{\sim} \text{Cat}(\rho_1, \rho_2)$$

$$p(z_n = 1 | z_1, \ldots, z_{n-1}) = \frac{a_{1,n}}{a_{1,n} + a_{2,n}}$$

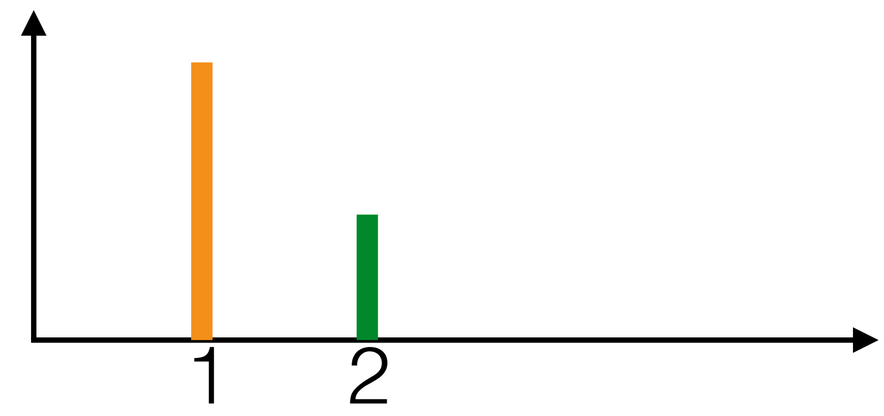


$$a_{1,n} := a_1 + \sum_{m=1}^{n-1} \mathbf{1}\{z_m = 1\}, a_{2,n} = a_2 + \sum_{m=1}^{n-1} \mathbf{1}\{z_m = 2\}$$

- Pólya urn

Marginal cluster assignments

- Integrate out the frequencies

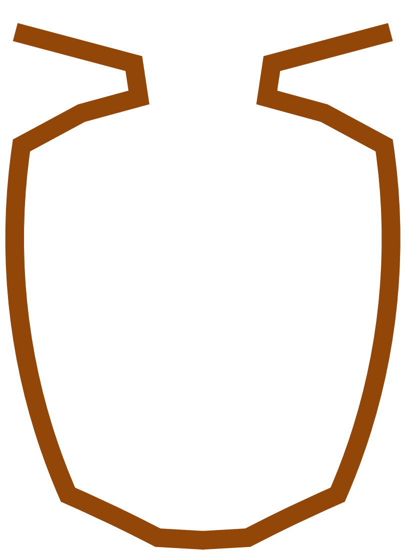


$$\rho_1 \sim \mathrm{Beta}(a_1, a_2),\, z_n \overset{iid}{\sim} \mathrm{Cat}(\rho_1, \rho_2)$$

$$p(z_n = 1 | z_1, \ldots, z_{n-1}) = \frac{a_{1,n}}{a_{1,n} + a_{2,n}}$$

$$a_{1,n} := a_1 + \sum_{m=1}^{n-1} \mathbf{1}\{z_m = 1\},\, a_{2,n} = a_2 + \sum_{m=1}^{n-1} \mathbf{1}\{z_m = 2\}$$

- Pólya urn

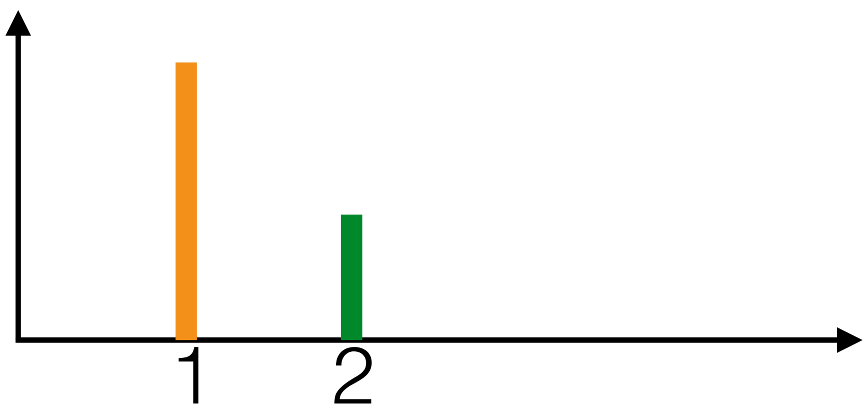


Marginal cluster assignments

- Integrate out the frequencies

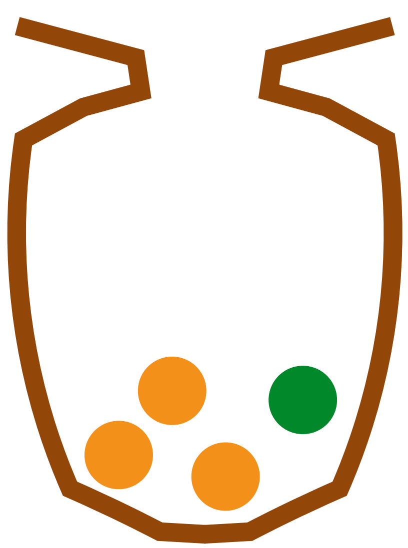
$$\rho_1 \sim \mathrm{Beta}(a_1, a_2), z_n \overset{iid}{\sim} \mathrm{Cat}(\rho_1, \rho_2)$$

$$p(z_n = 1 | z_1, \ldots, z_{n-1}) = \frac{a_{1,n}}{a_{1,n} + a_{2,n}}$$



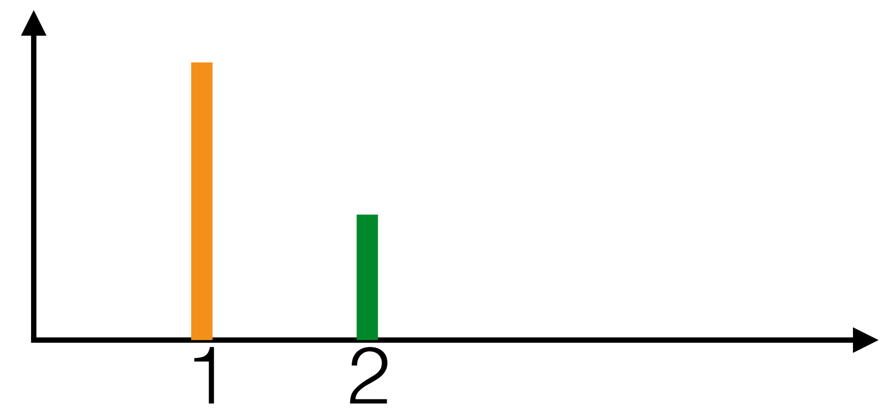
$$a_{1,n} := a_1 + \sum_{m=1}^{n-1} \mathbf{1}\{z_m = 1\}, a_{2,n} = a_2 + \sum_{m=1}^{n-1} \mathbf{1}\{z_m = 2\}$$

- Pólya urn



Marginal cluster assignments

- Integrate out the frequencies



$$\rho_1 \sim \mathrm{Beta}(a_1, a_2), z_n \stackrel{iid}{\sim} \mathrm{Cat}(\rho_1, \rho_2)$$

$$p(z_n = 1 | z_1, \ldots, z_{n-1}) = \frac{a_{1,n}}{a_{1,n} + a_{2,n}}$$

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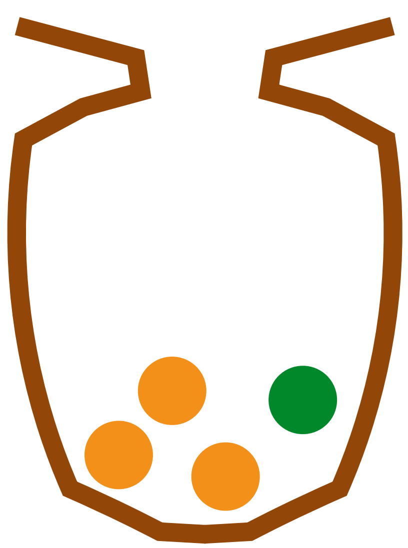
- Pólya urn
  - Choose any ball with equal probability



Marginal cluster assignments

- Integrate out the frequencies

$$\rho_1 \sim \mathrm{Beta}(a_1, a_2), z_n \stackrel{iid}{\sim} \mathrm{Cat}(\rho_1, \rho_2)$$

$$p(z_n = 1 | z_1, \ldots, z_{n-1}) = \frac{a_{1,n}}{a_{1,n} + a_{2,n}}$$

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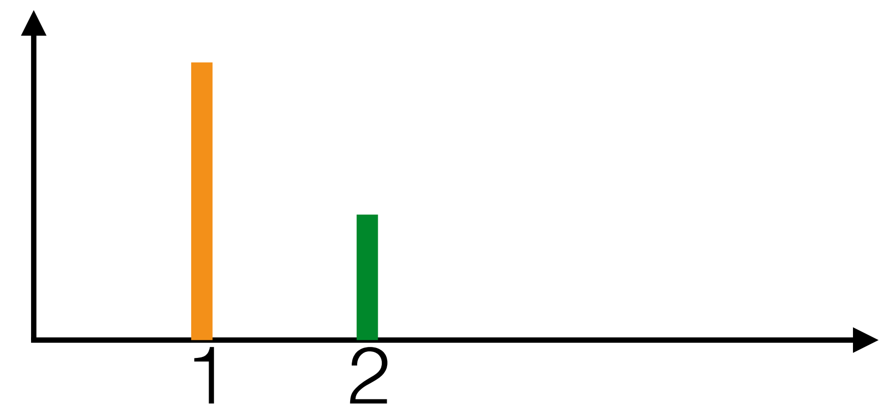
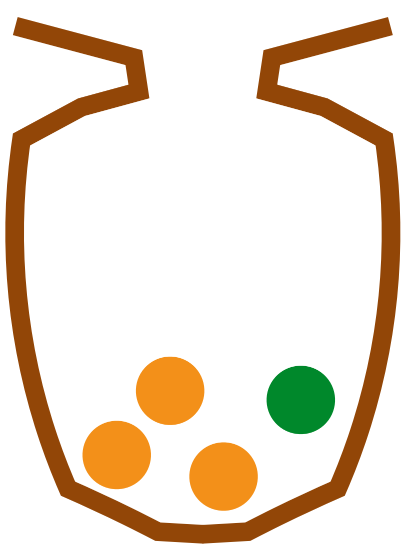


- Pólya urn
  - Choose any ball with equal probability
  - Replace and add ball of same color

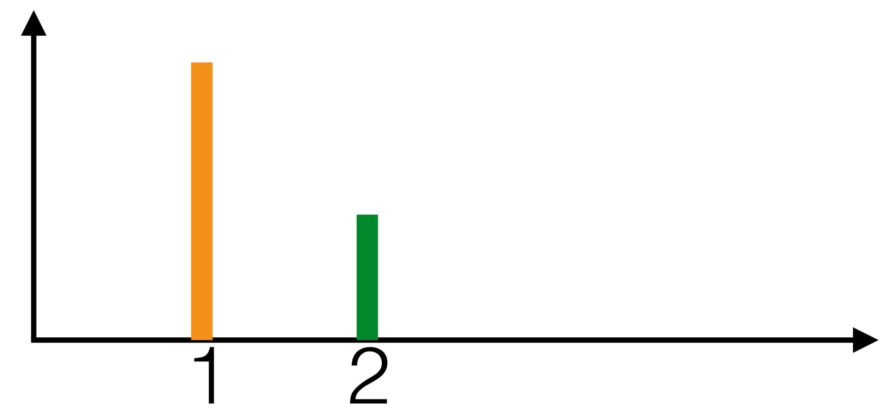


Marginal cluster assignments

- Integrate out the frequencies

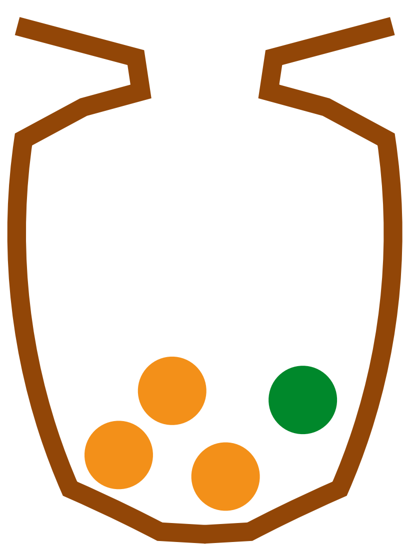
$$\rho_1 \sim \mathrm{Beta}(a_1, a_2),\, z_n \overset{iid}{\sim} \mathrm{Cat}(\rho_1, \rho_2)$$

$$p(z_n = 1 | z_1, \ldots, z_{n-1}) = \frac{a_{1,n}}{a_{1,n} + a_{2,n}}$$



$$a_{1,n} := a_1 + \sum_{m=1}^{n-1} \mathbf{1}\{z_m = 1\}, a_{2,n} = a_2 + \sum_{m=1}^{n-1} \mathbf{1}\{z_m = 2\}$$

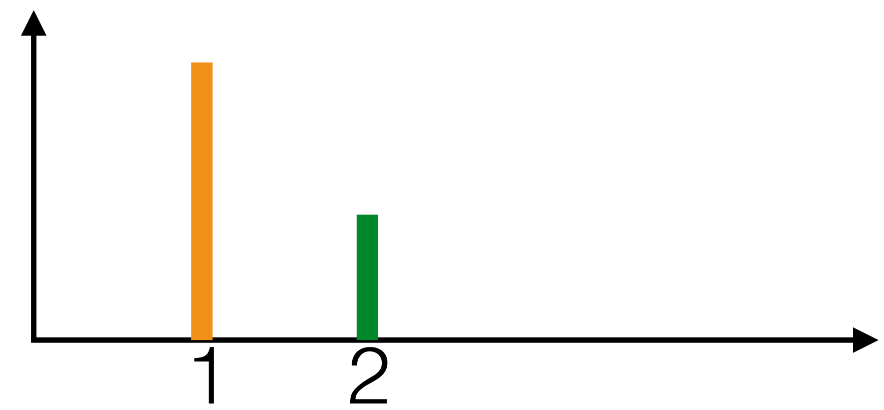
- Pólya urn
    - Choose any ball with equal probability
    - Replace and add ball of same color



$$\lim_{n \to \infty} \frac{\#\ \text{orange}}{\#\ \text{total}}$$

Marginal cluster assignments

- Integrate out the frequencies

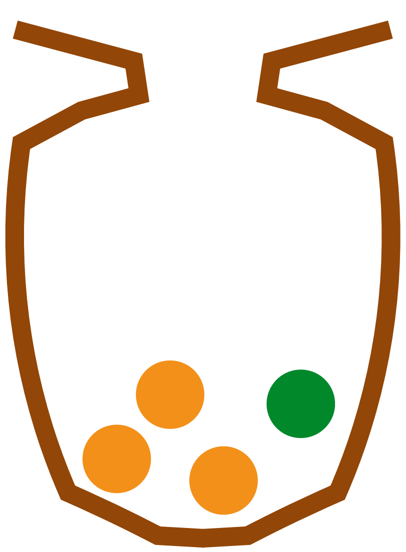


$$\rho_1 \sim \mathrm{Beta}(a_1, a_2),\, z_n \overset{iid}{\sim} \mathrm{Cat}(\rho_1, \rho_2)$$

$$p(z_n = 1 | z_1, \ldots, z_{n-1}) = \frac{a_{1,n}}{a_{1,n} + a_{2,n}}$$

$$a_{1,n} := a_1 + \sum_{m=1}^{n-1} \mathbf{1}\{z_m = 1\},\, a_{2,n} = a_2 + \sum_{m=1}^{n-1} \mathbf{1}\{z_m = 2\}$$

- Pólya urn
  - Choose any ball with equal probability
  - Replace and add ball of same color



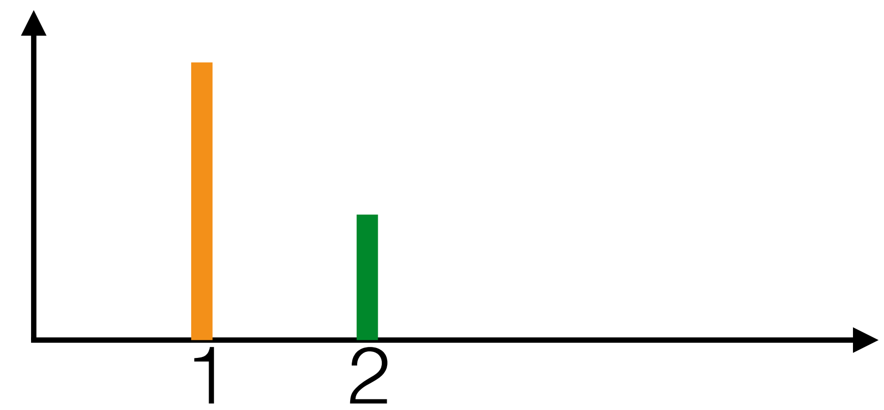
$$\lim_{n \to \infty} \frac{\#\ \text{orange}}{\#\ \text{total}} = \rho_{\text{orange}}$$

Marginal cluster assignments

- Integrate out the frequencies

$$\rho_1 \sim \text{Beta}(a_1, a_2),\, z_n \overset{iid}{\sim} \text{Cat}(\rho_1, \rho_2)$$

$$p(z_n = 1 | z_1, \ldots, z_{n-1}) = \frac{a_{1,n}}{a_{1,n} + a_{2,n}}$$



$$a_{1,n} := a_1 + \sum_{m=1}^{n-1} \mathbf{1}\{z_m = 1\},\, a_{2,n} = a_2 + \sum_{m=1}^{n-1} \mathbf{1}\{z_m = 2\}$$

- Pólya urn
  - Choose any ball with equal probability
  - Replace and add ball of same color

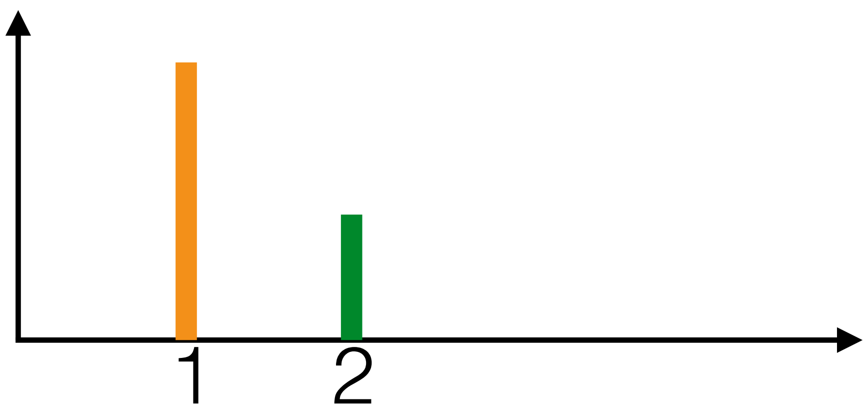
$$\lim_{n \to \infty} \frac{\#\ \text{orange}}{\#\ \text{total}} = \rho_{\text{orange}} \overset{d}{=} \text{Beta}(a_{\text{orange}}, a_{\text{green}})$$

Marginal cluster assignments

- Integrate out the frequencies

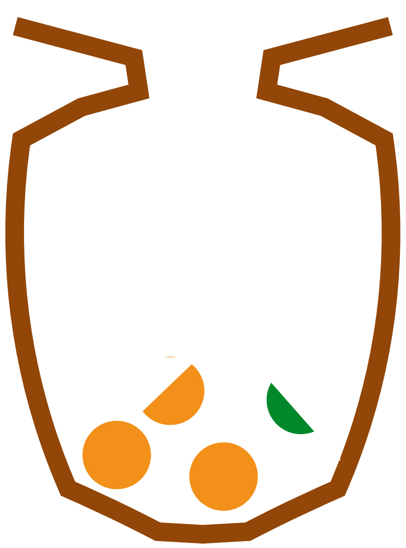
$$\rho_1 \sim \mathrm{Beta}(a_1, a_2),\, z_n \overset{iid}{\sim} \mathrm{Cat}(\rho_1, \rho_2)$$

$$p(z_n = 1 | z_1, \ldots, z_{n-1}) = \frac{a_{1,n}}{a_{1,n} + a_{2,n}}$$



$$a_{1,n} := a_1 + \sum_{m=1}^{n-1} \mathbf{1}\{z_m = 1\},\, a_{2,n} = a_2 + \sum_{m=1}^{n-1} \mathbf{1}\{z_m = 2\}$$

- Pólya urn
  - Choose any ball with equal probability
  - Replace and add ball of same color



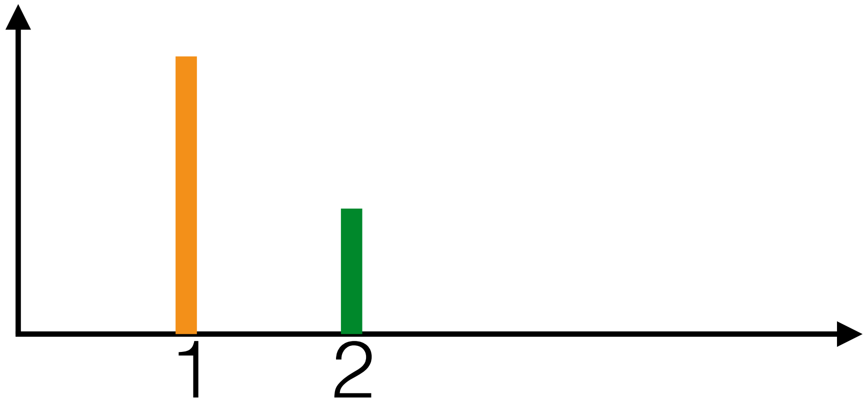
$$\lim_{n \to \infty} \frac{\#\ \mathrm{orange}}{\#\ \mathrm{total}} = \rho_{\mathrm{orange}} \overset{d}{=} \mathrm{Beta}(a_{\mathrm{orange}}, a_{\mathrm{green}})$$

Marginal cluster assignments

- Integrate out the frequencies

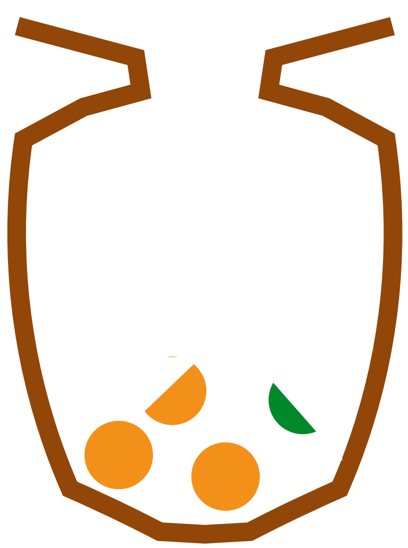
$$\rho_1 \sim \mathrm{Beta}(a_1, a_2),\, z_n \stackrel{iid}{\sim} \mathrm{Cat}(\rho_1, \rho_2)$$

$$p(z_n = 1 | z_1, \ldots, z_{n-1}) = \frac{a_{1,n}}{a_{1,n} + a_{2,n}}$$



$$a_{1,n} := a_1 + \sum_{m=1}^{n-1} \mathbf{1}\{z_m = 1\},\, a_{2,n} = a_2 + \sum_{m=1}^{n-1} \mathbf{1}\{z_m = 2\}$$

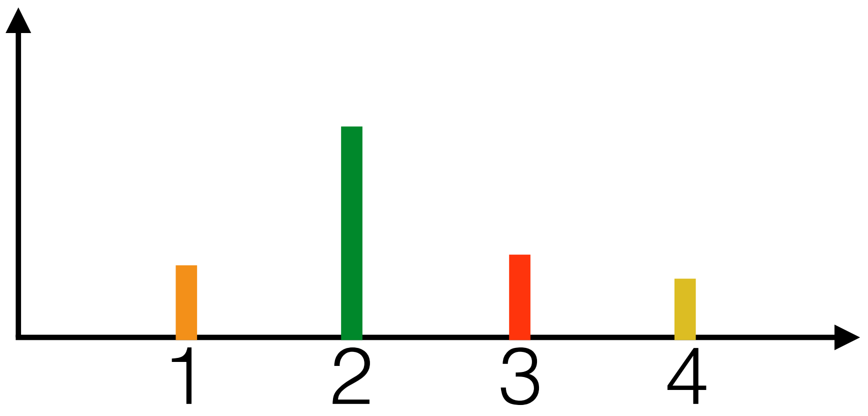
- Pólya urn
  - Choose any ball with prob proportional to its mass
  - Replace and add ball of same color



$$\lim_{n \to \infty} \frac{\#\ \text{orange}}{\#\ \text{total}} = \rho_{\text{orange}} \stackrel{d}{=} \mathrm{Beta}(a_{\text{orange}}, a_{\text{green}})$$

Marginal cluster assignments

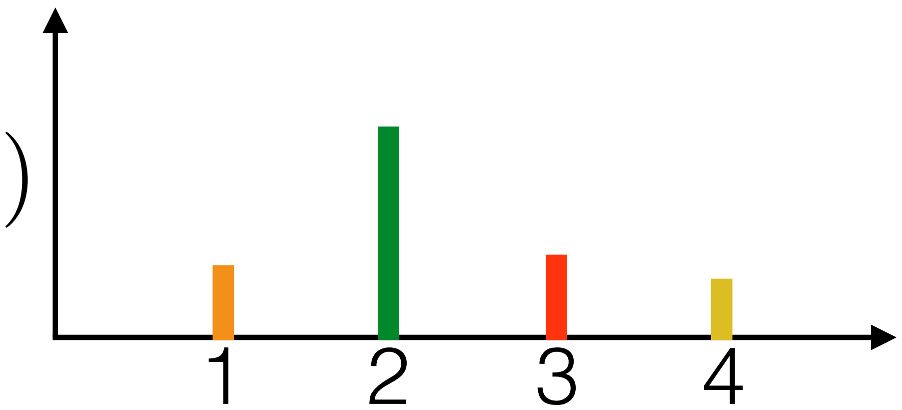
- Integrate out the frequencies



Marginal cluster assignments

- Integrate out the frequencies

$$\rho_{1:K} \sim \mathrm{Dirichlet}(a_{1:K}),\ z_n \overset{iid}{\sim} \mathrm{Cat}(\rho_{1:K})$$

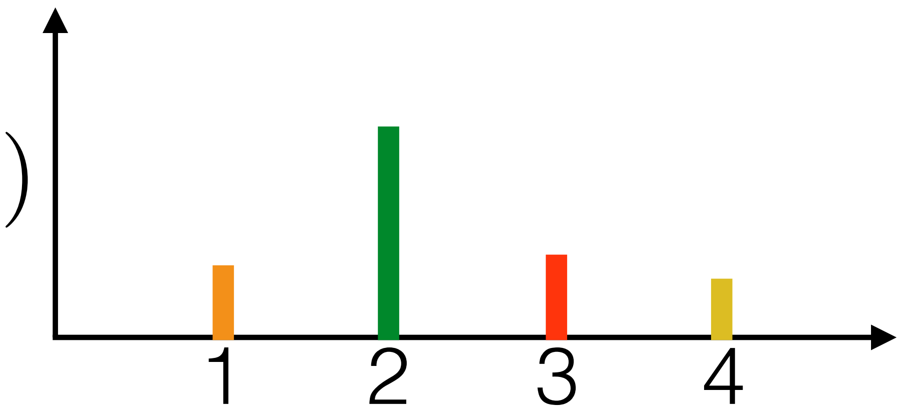


Marginal cluster assignments

- Integrate out the frequencies

$$\rho_{1:K} \sim \mathrm{Dirichlet}(a_{1:K}), z_n \stackrel{iid}{\sim} \mathrm{Cat}(\rho_{1:K})$$

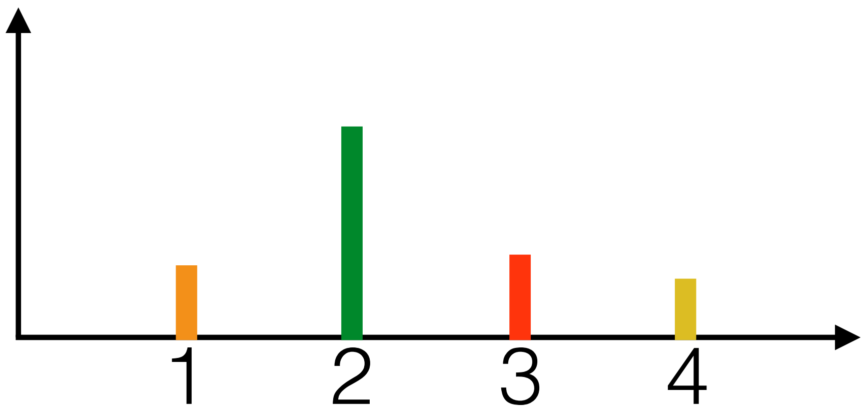
$$p(z_n = k | z_1, \ldots, z_{n-1}) = \frac{a_{k,n}}{\sum_{j=1}^{K} a_{j,n}}$$



Marginal cluster assignments

- Integrate out the frequencies

$$\rho_{1:K} \sim \text{Dirichlet}(a_{1:K}),\, z_n \stackrel{iid}{\sim} \text{Cat}(\rho_{1:K})$$



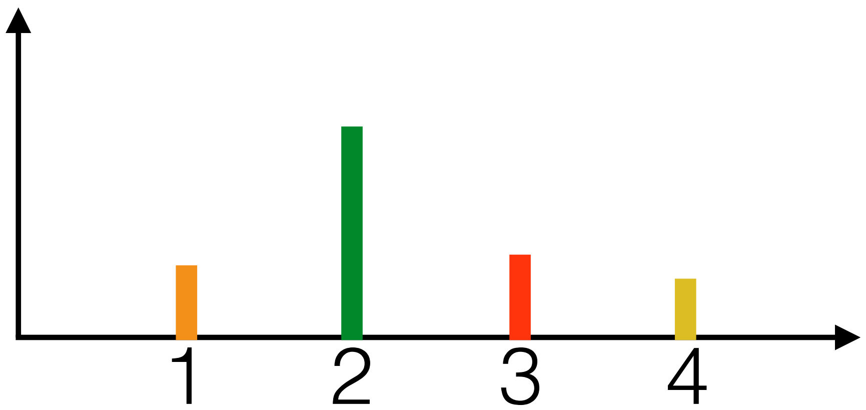
$$p(z_n = k | z_1, \ldots, z_{n-1}) = \frac{a_{k,n}}{\sum_{j=1}^{K} a_{j,n}}$$

$$a_{k,n} := a_k + \sum_{m=1}^{n-1} \mathbf{1}\{z_m = k\}$$

Marginal cluster assignments

- Integrate out the frequencies

$$\rho_{1:K} \sim \mathrm{Dirichlet}(a_{1:K}), z_n \overset{iid}{\sim} \mathrm{Cat}(\rho_{1:K})$$



$$p(z_n = k | z_1, \ldots, z_{n-1}) = \frac{a_{k,n}}{\sum_{j=1}^{K} a_{j,n}}$$

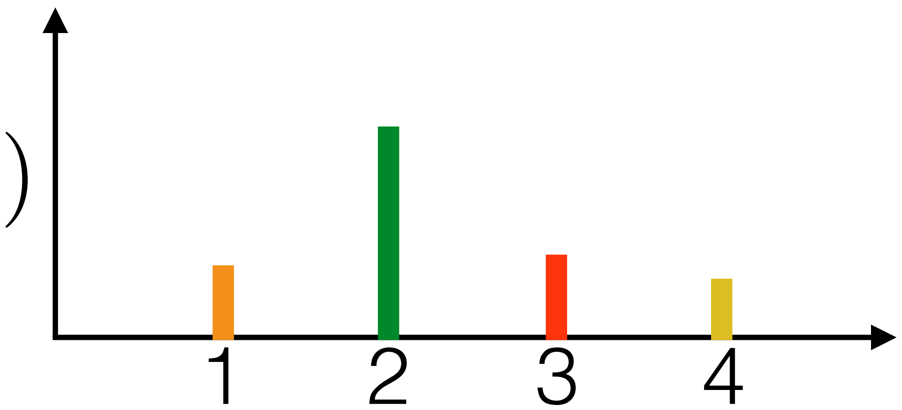
$$a_{k,n} := a_k + \sum_{m=1}^{n-1} \mathbf{1}\{z_m = k\}$$

- multivariate Pólya urn

Marginal cluster assignments

- Integrate out the frequencies

$$\rho_{1:K} \sim \text{Dirichlet}(a_{1:K}), z_n \overset{iid}{\sim} \text{Cat}(\rho_{1:K})$$



$$p(z_n = k | z_1, \ldots, z_{n-1}) = \frac{a_{k,n}}{\sum_{j=1}^{K} a_{j,n}}$$

$$a_{k,n} := a_k + \sum_{m=1}^{n-1} \mathbf{1}\{z_m = k\}$$

- multivariate Pólya urn

Marginal cluster assignments

- Integrate out the frequencies

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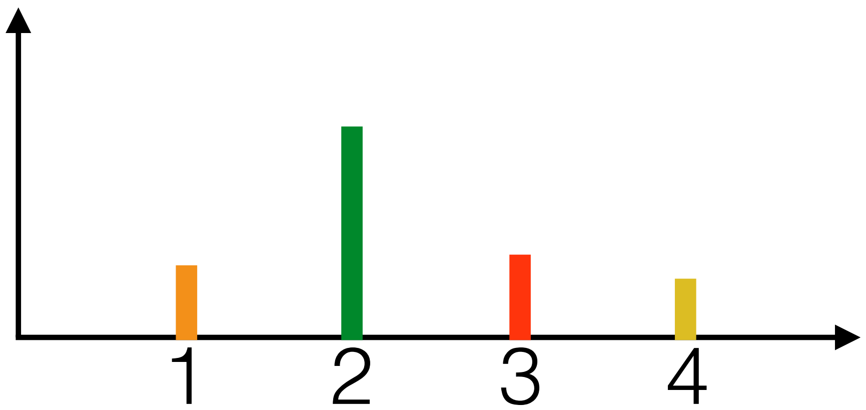

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  - Choose any ball with prob proportional to its mass

Marginal cluster assignments

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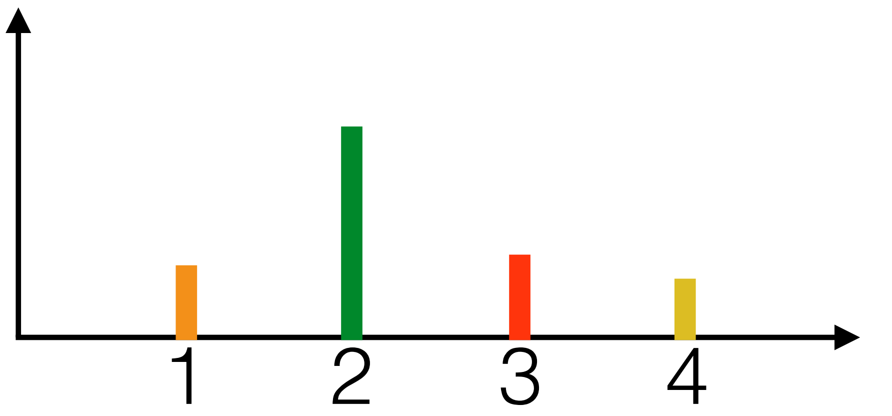

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Marginal cluster assignments

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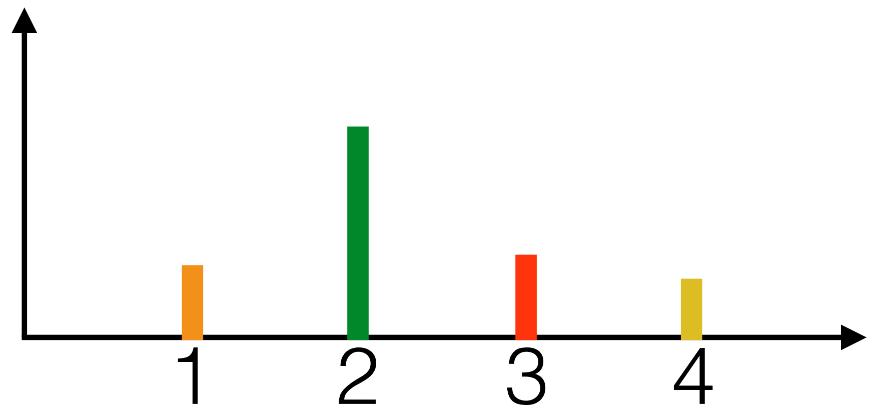

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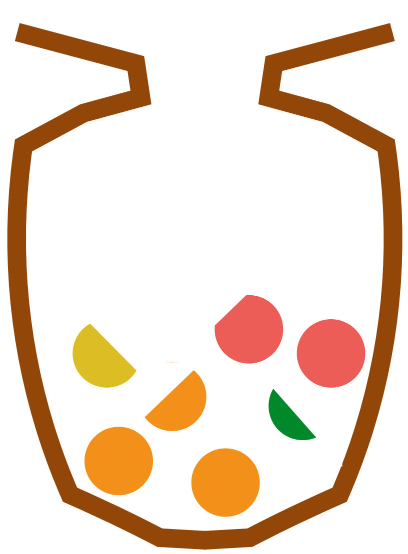
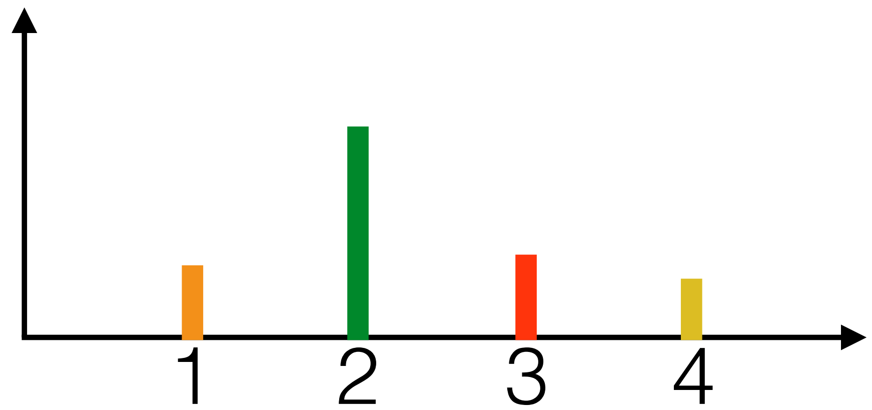


$$\lim_{n \to \infty} \frac{(\#\ \text{orange},\ \#\ \text{green},\ \#\ \text{red},\ \#\ \text{yellow})}{\#\ \text{total}}$$

Marginal cluster assignments

- Integrate out the frequencies



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- multivariate Pólya urn
  - Choose any ball with prob proportional to its mass
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$$\lim_{n \to \infty} \frac{(\#\text{ orange}, \#\text{ green}, \#\text{ red}, \#\text{ yellow})}{\#\text{ total}}$$

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Marginal cluster assignments

- Integrate out the frequencies

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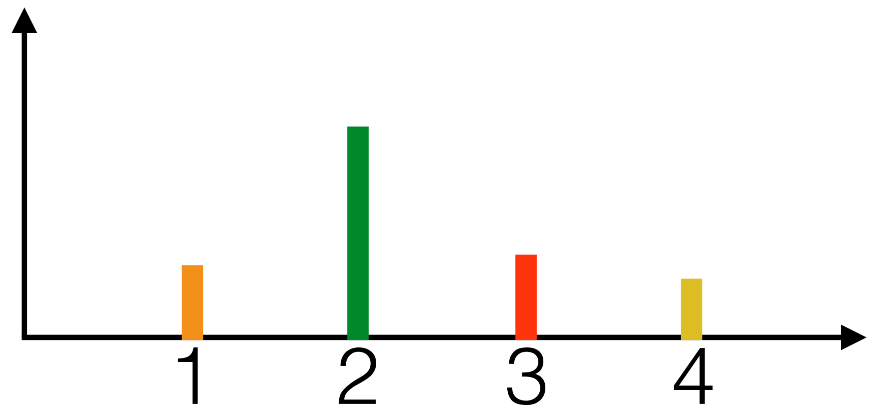

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$$\to (\rho_{\text{orange}}, \rho_{\text{green}}, \rho_{\text{red}}, \rho_{\text{yellow}})$$

$$\overset{d}{=} \text{Dirichlet}(a_{\text{orange}}, a_{\text{green}}, a_{\text{red}}, a_{\text{yellow}})$$

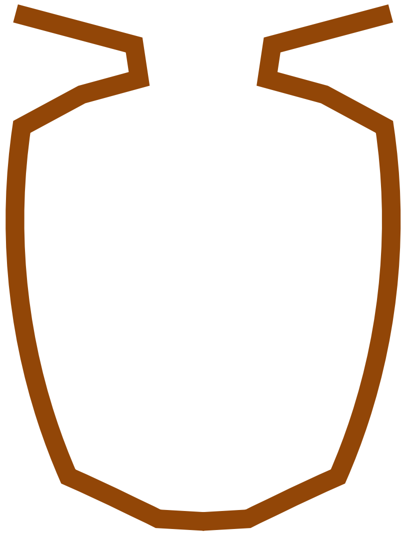
Marginal cluster assignments

- Hoppe urn / Blackwell-MacQueen urn

[Blackwell, MacQueen 1973; Hoppe 1984]

Marginal cluster assignments

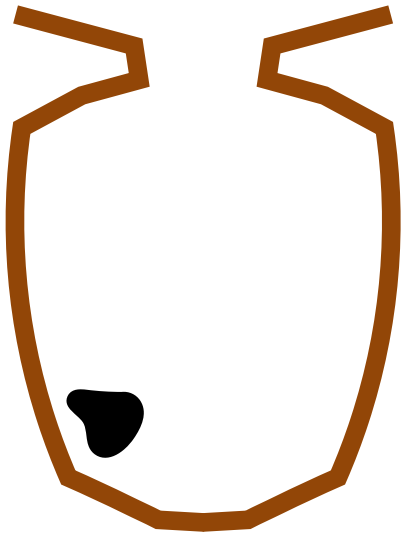
- Hoppe urn / Blackwell-MacQueen urn



[Blackwell, MacQueen 1973; Hoppe 1984]

Marginal cluster assignments

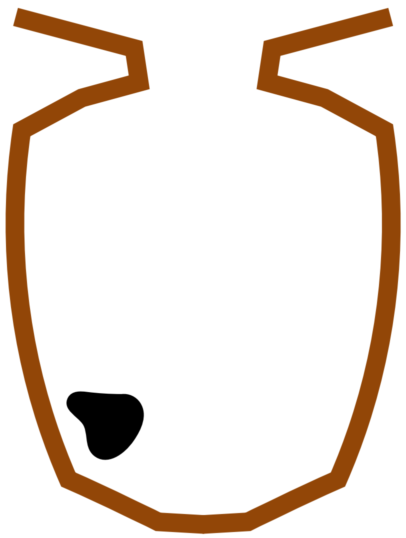
- Hoppe urn / Blackwell-MacQueen urn



[Blackwell, MacQueen 1973; Hoppe 1984]

Marginal cluster assignments

- Hoppe urn / Blackwell-MacQueen urn

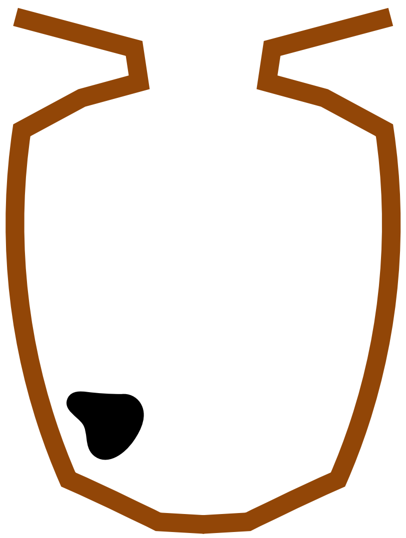


- Choose ball with prob proportional to its mass

[Blackwell, MacQueen 1973; Hoppe 1984]

Marginal cluster assignments

- Hoppe urn / Blackwell-MacQueen urn



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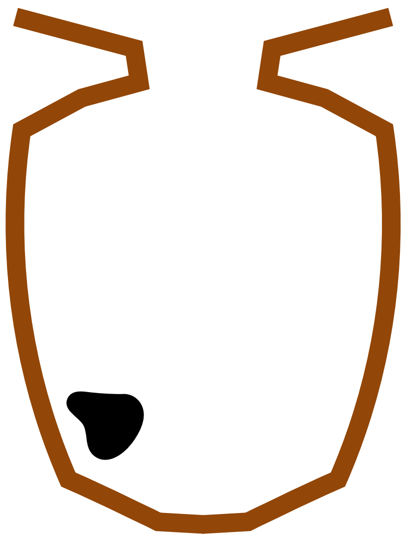
[Blackwell, MacQueen 1973; Hoppe 1984]

Marginal cluster assignments

- Hoppe urn / Blackwell-MacQueen urn



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Marginal cluster assignments

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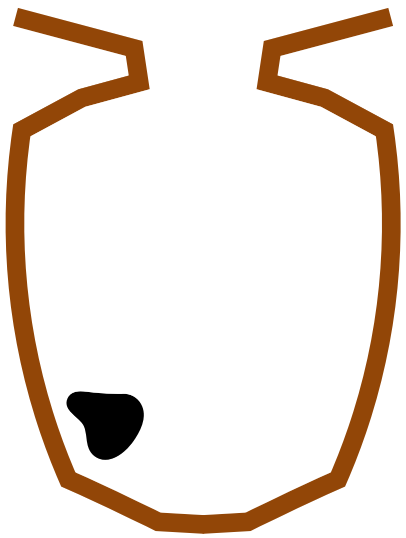

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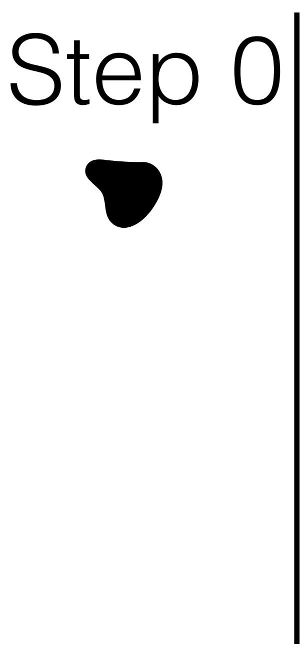
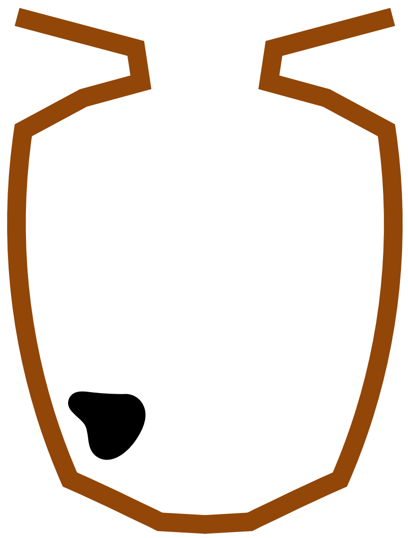


[Blackwell, MacQueen 1973; Hoppe 1984]

Marginal cluster assignments

- Hoppe urn / Blackwell-MacQueen urn



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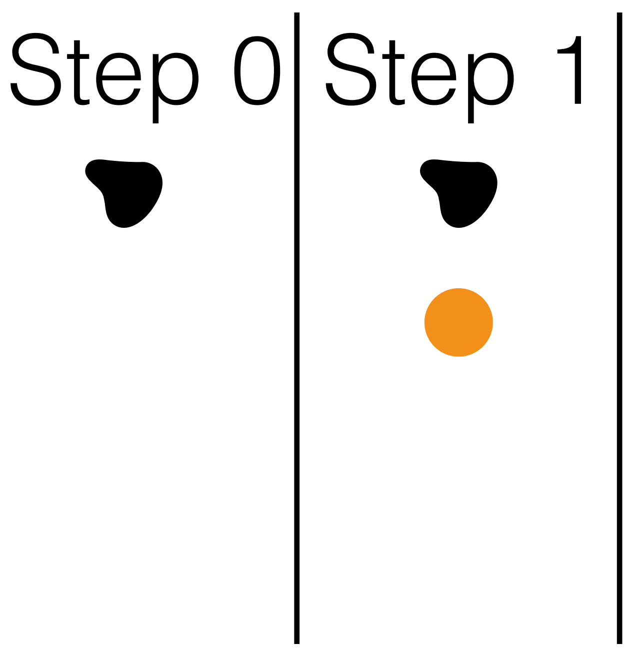
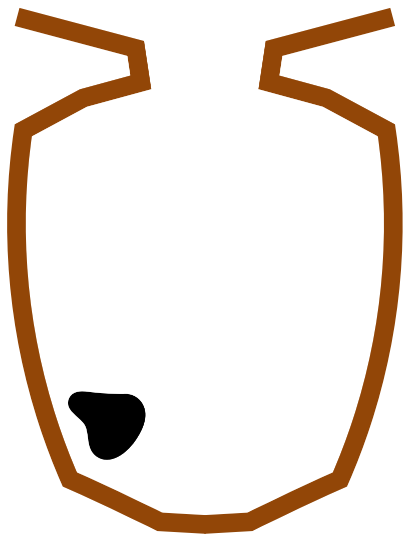


[Blackwell, MacQueen 1973; Hoppe 1984]

Marginal cluster assignments

- Hoppe urn / Blackwell-MacQueen urn



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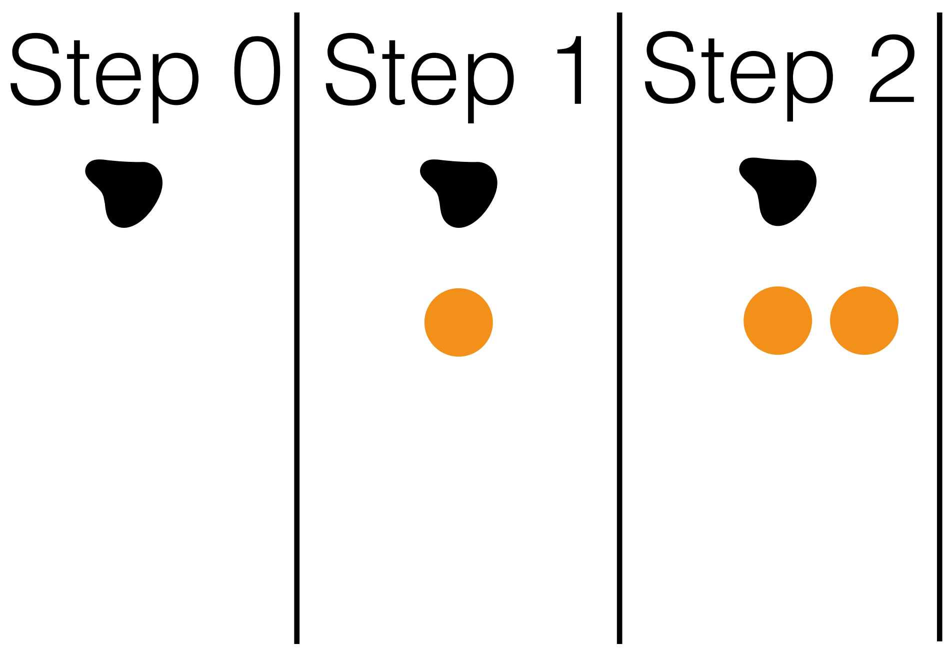

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Marginal cluster assignments

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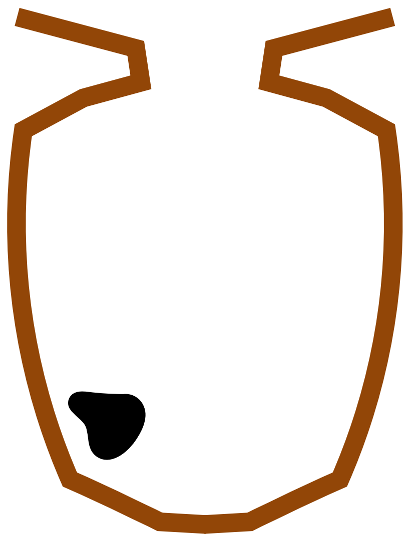

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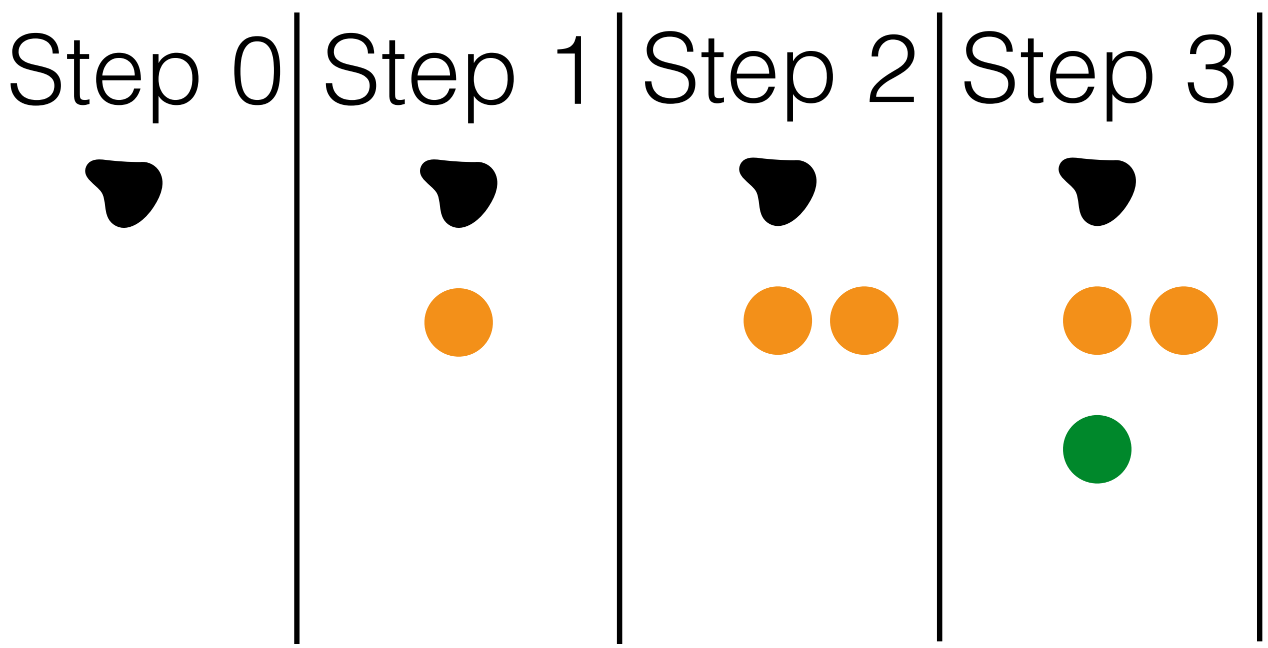

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Marginal cluster assignments

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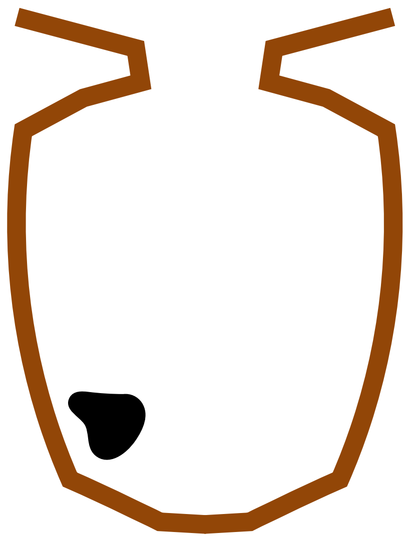

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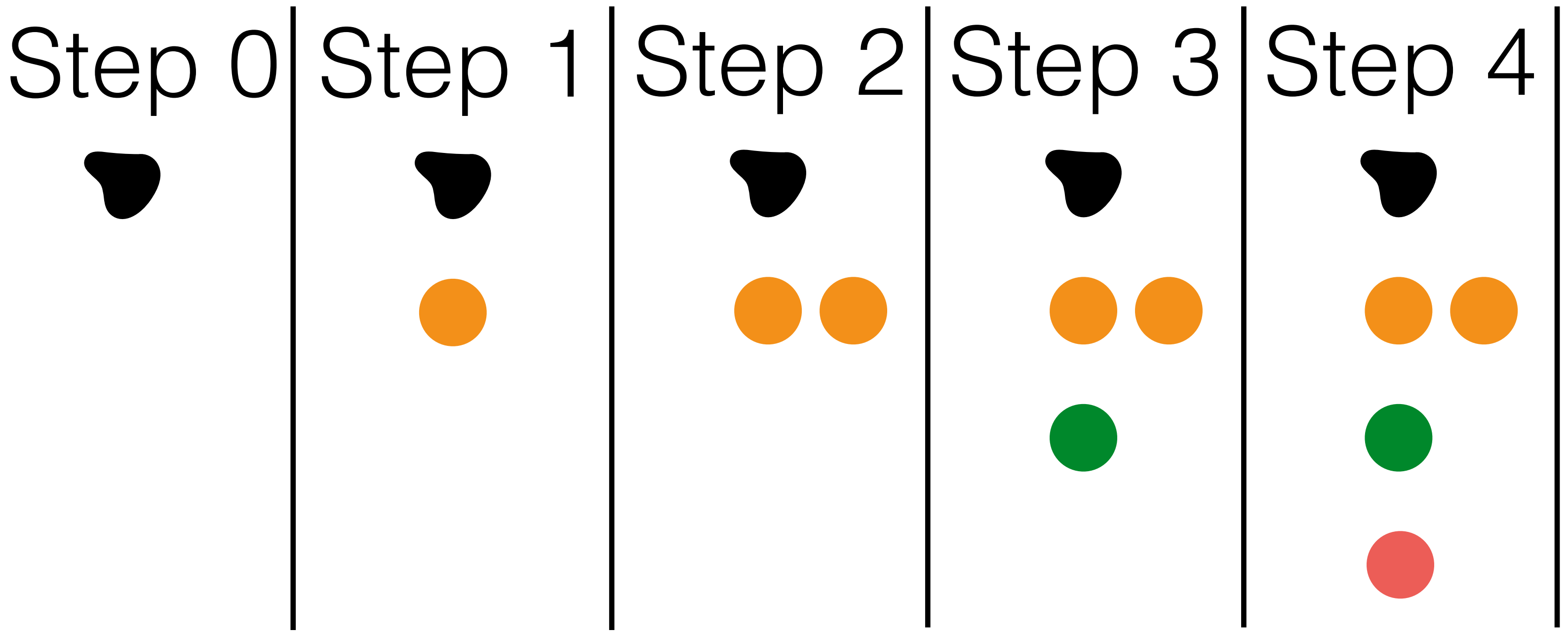

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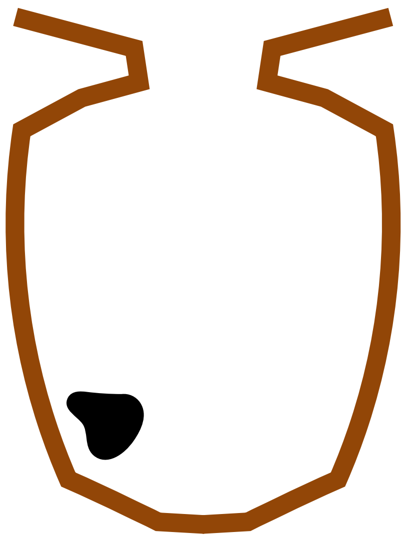
Marginal cluster assignments

- Hoppe urn / Blackwell-MacQueen urn



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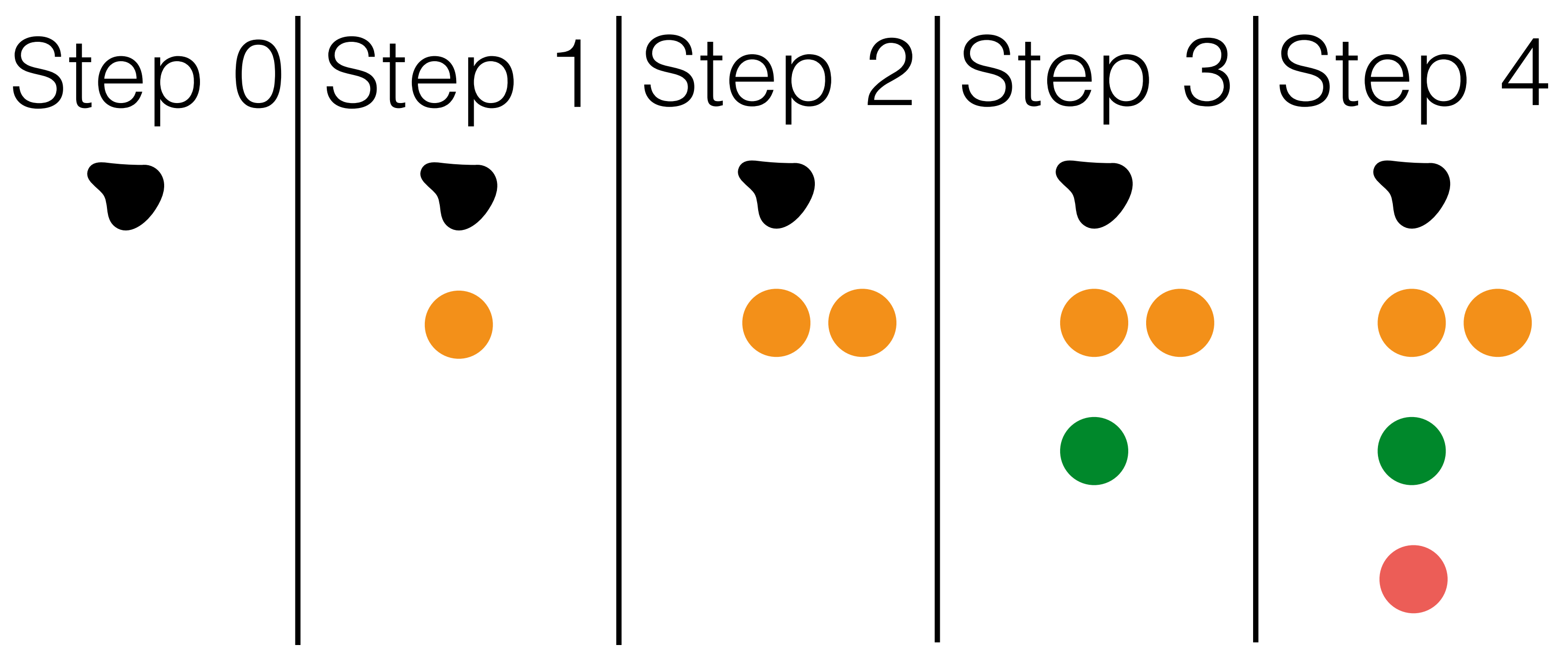
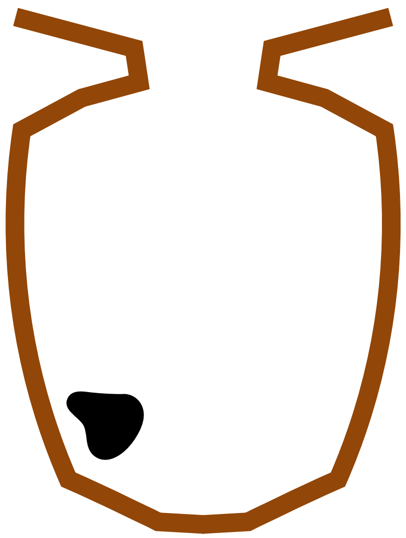


$$(\#\text{orange}, \#\text{other}) = \text{PolyaUrn}(1, \alpha)$$

Marginal cluster assignments

- Hoppe urn / Blackwell-MacQueen urn



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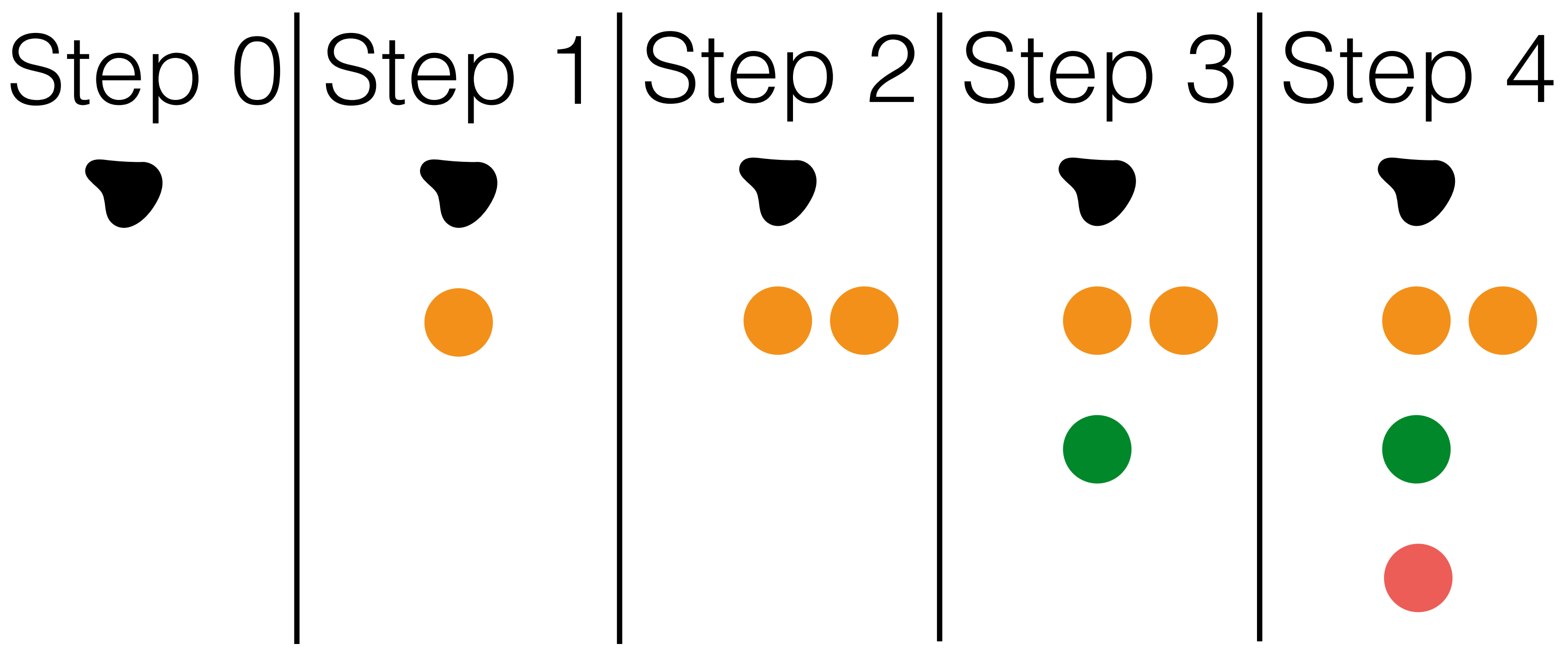
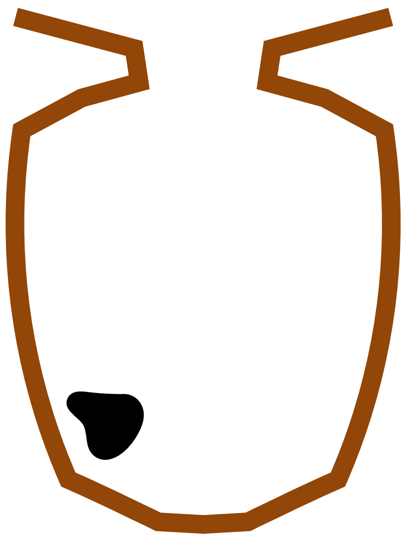


$$(\#\text{orange}, \#\text{other}) = \text{PolyaUrn}(1, \alpha)$$

- not orange: $(\#\text{green}, \#\text{other}) = \text{PolyaUrn}(1, \alpha)$

Marginal cluster assignments

- Hoppe urn / Blackwell-MacQueen urn



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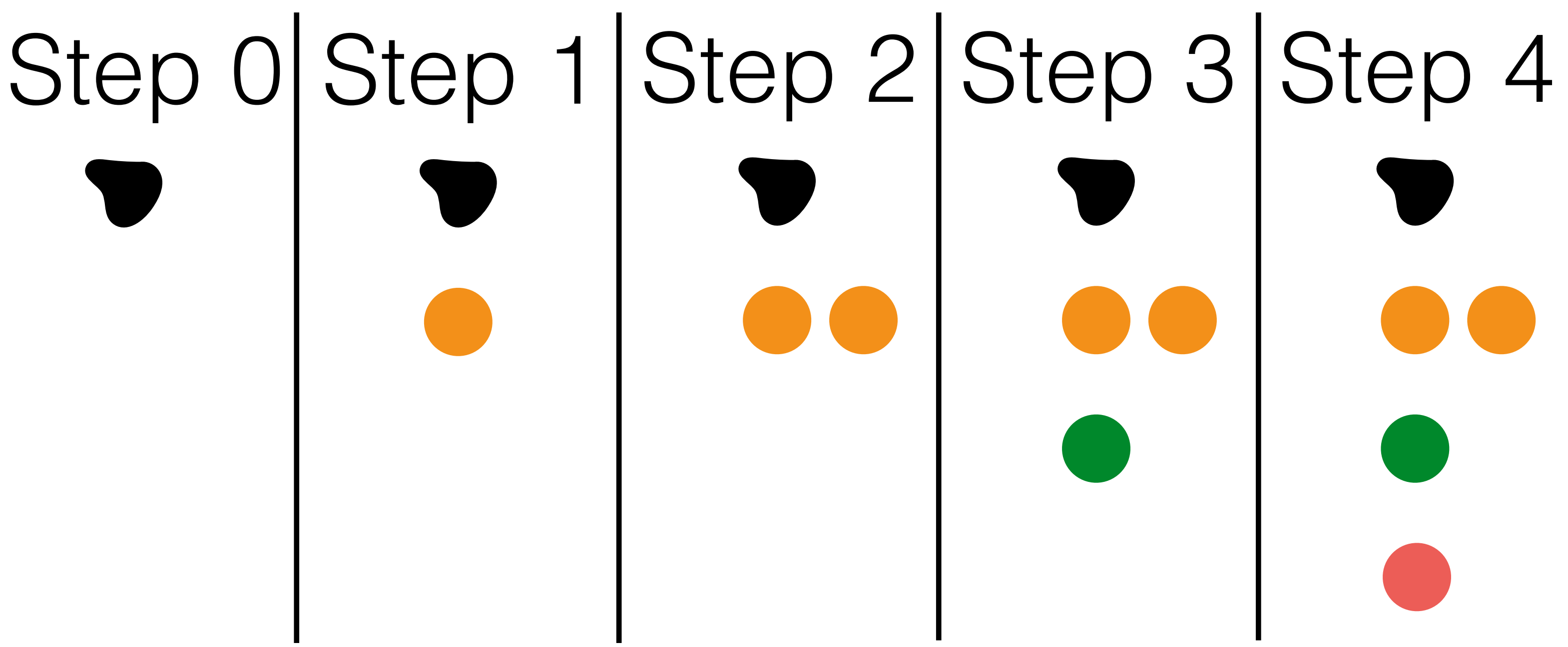

$$(\#\text{orange}, \#\text{other}) = \text{PolyaUrn}(1, \alpha)$$

- not orange: $(\#\text{green}, \#\text{other}) = \text{PolyaUrn}(1, \alpha)$
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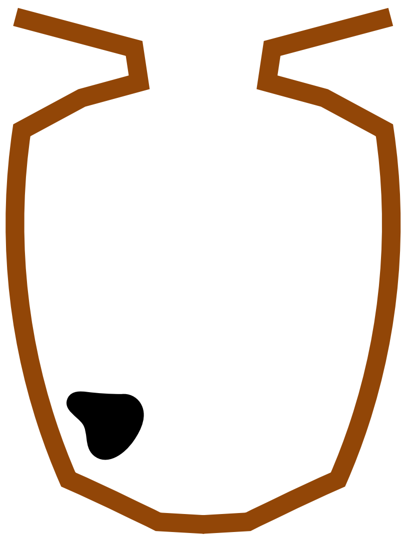
Marginal cluster assignments

- Hoppe urn / Blackwell-MacQueen urn



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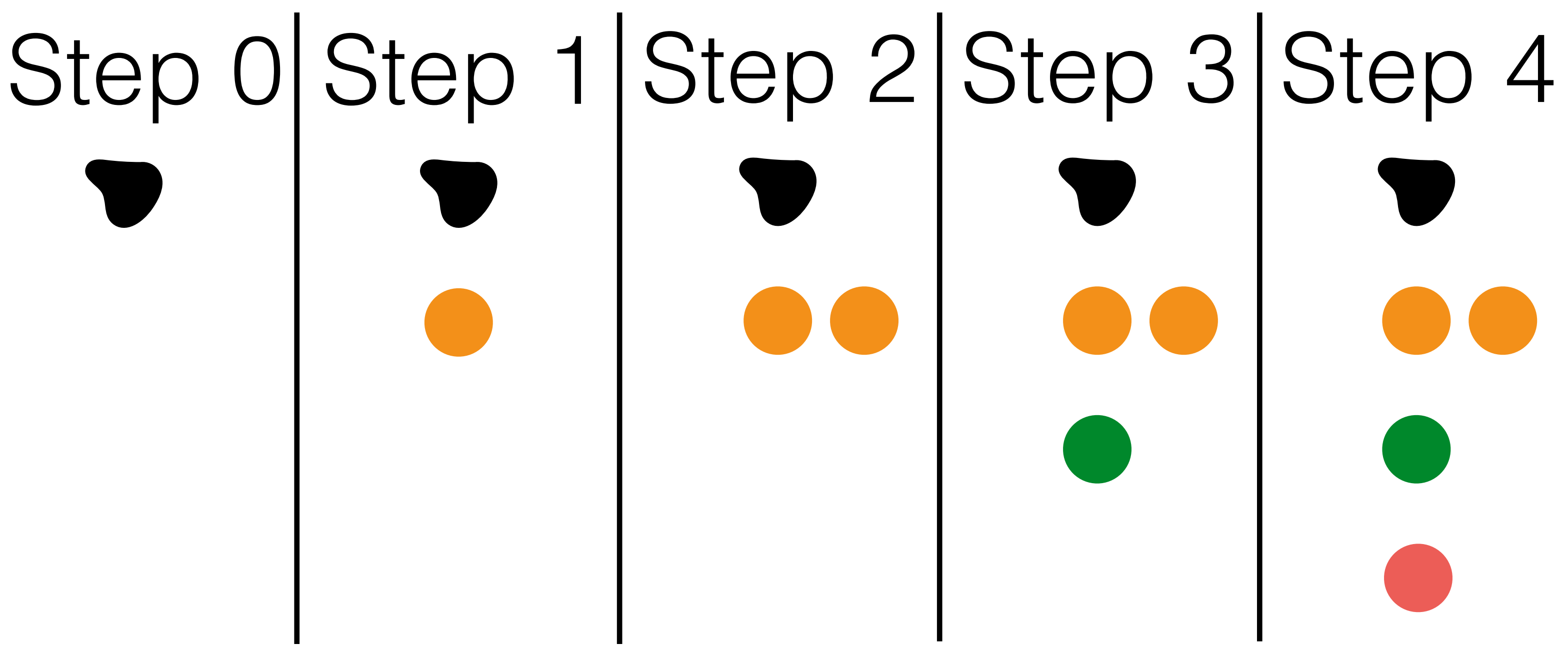

$$V_k \overset{iid}{\sim} \mathrm{Beta}(1, \alpha)$$

$$(\#\text{orange}, \#\text{other}) = \mathrm{PolyaUrn}(1, \alpha)$$

- not orange: $(\#\text{green}, \#\text{other}) = \mathrm{PolyaUrn}(1, \alpha)$
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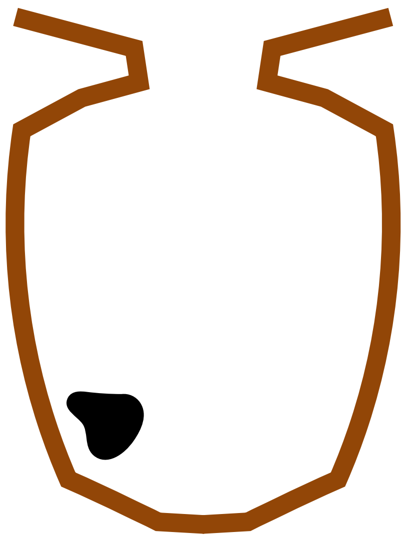
Marginal cluster assignments

- Hoppe urn / Blackwell-MacQueen urn



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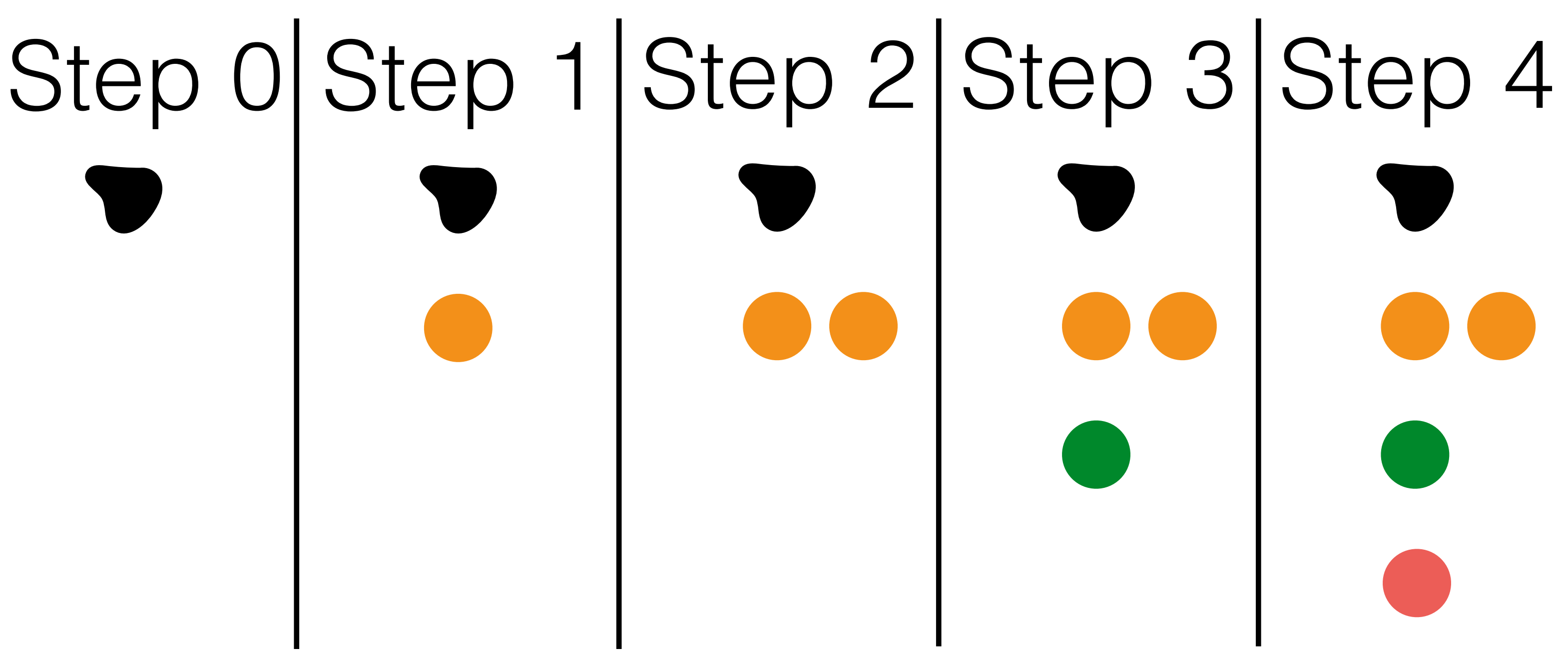

$$V_k \overset{iid}{\sim} \mathrm{Beta}(1, \alpha)$$

$$\rho_1 = V_1$$

$$(\#\text{orange}, \#\text{other}) = \mathrm{PolyaUrn}(1, \alpha)$$

- not orange: $(\#\text{green}, \#\text{other}) = \mathrm{PolyaUrn}(1, \alpha)$
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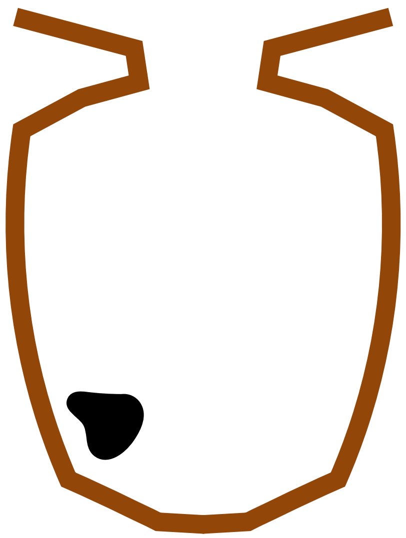
Marginal cluster assignments

- Hoppe urn / Blackwell-MacQueen urn



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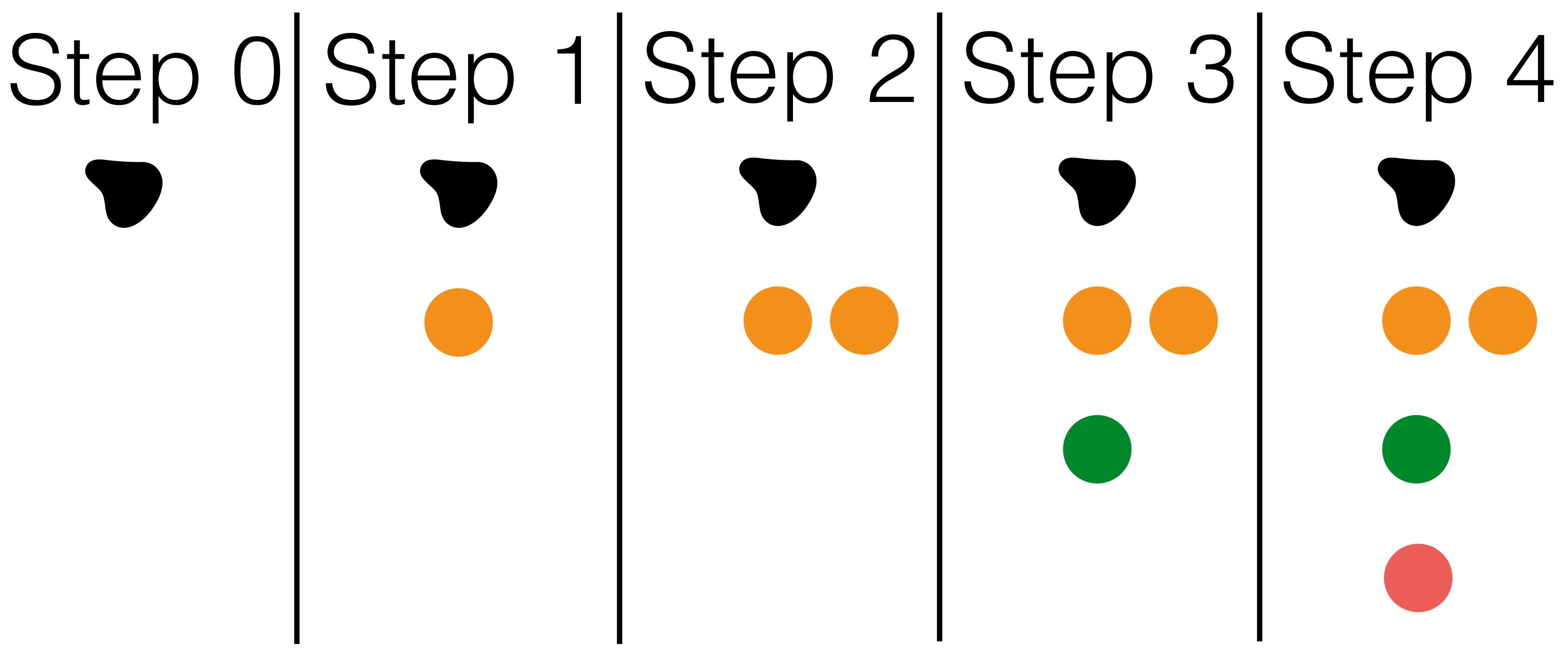

$$V_k \overset{iid}{\sim} \mathrm{Beta}(1, \alpha)$$

$$\rho_1 = V_1$$

$$\rho_2 = (1 - V_1) V_2$$

$$(\#\mathrm{orange}, \#\mathrm{other}) = \mathrm{PolyaUrn}(1, \alpha)$$

- not orange: $(\#\mathrm{green}, \#\mathrm{other}) = \mathrm{PolyaUrn}(1, \alpha)$
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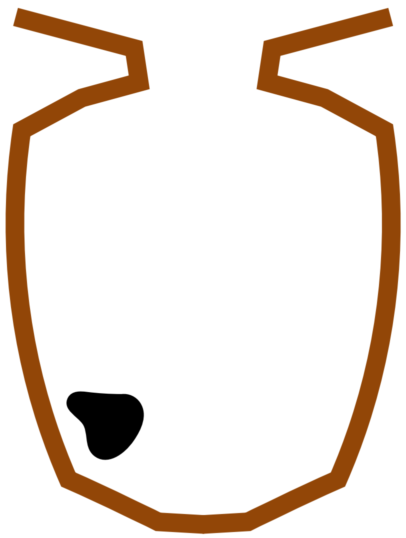
Marginal cluster assignments

- Hoppe urn / Blackwell-MacQueen urn



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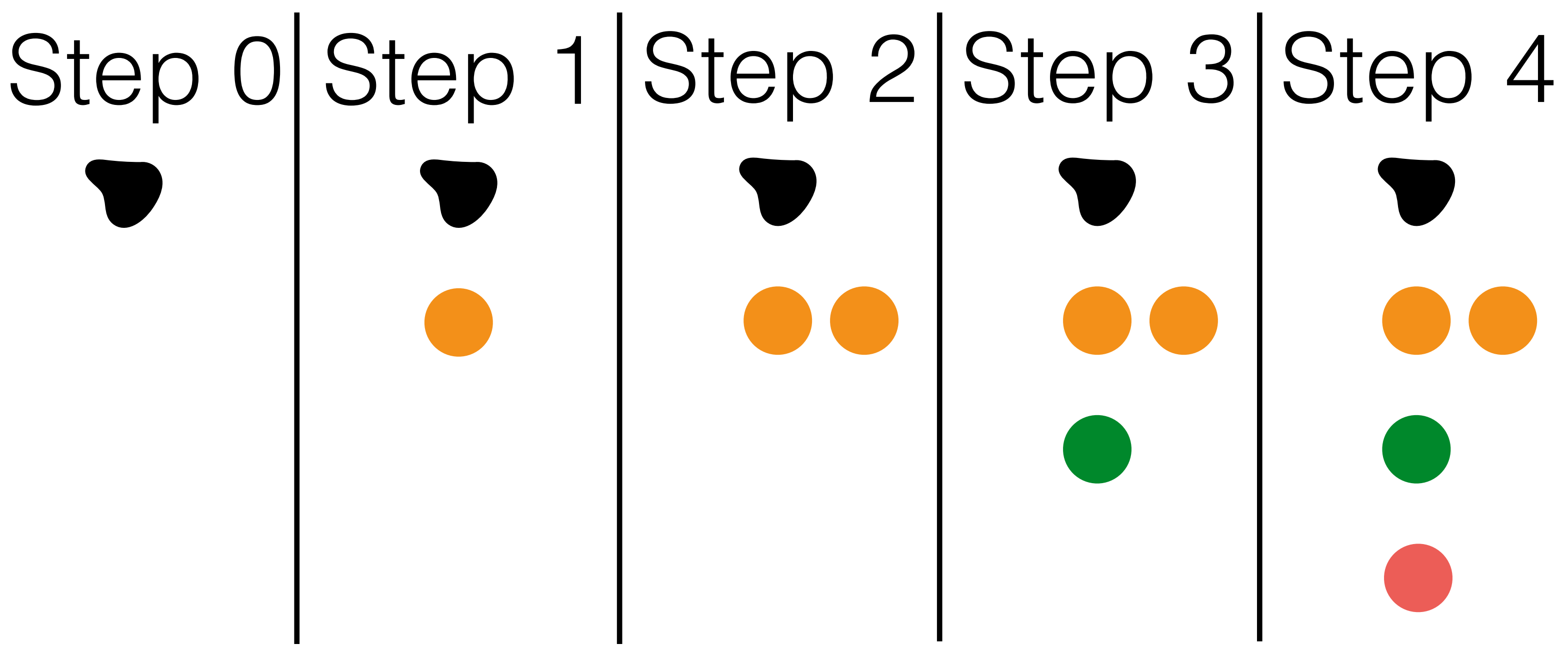

$$V_k \overset{iid}{\sim} \mathrm{Beta}(1, \alpha)$$

$$\rho_1 = V_1$$

$$\rho_2 = (1 - V_1)V_2$$

$$\rho_3 = [\prod_{k=1}^{2}(1 - V_k)]V_3$$

$$(\#\text{orange}, \#\text{other}) = \mathrm{PolyaUrn}(1, \alpha)$$

- not orange: $(\#\text{green}, \#\text{other}) = \mathrm{PolyaUrn}(1, \alpha)$
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Chinese restaurant process

Chinese restaurant process

- Same thing we just did

Chinese restaurant process

- Same thing we just did
- Each customer walks into the restaurant

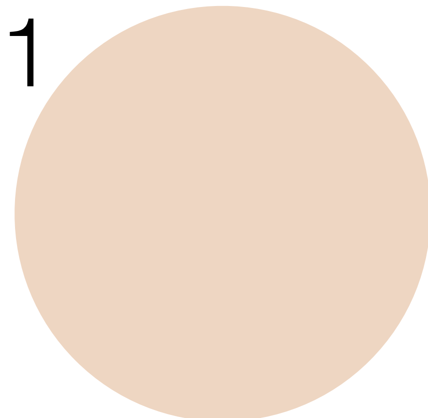
Chinese restaurant process

- Same thing we just did
- Each customer walks into the restaurant
  - Sits at existing table with prob proportional to # people there

Chinese restaurant process

- Same thing we just did
- Each customer walks into the restaurant
  - Sits at existing table with prob proportional to # people there
  - Forms new table with prob proportional to α

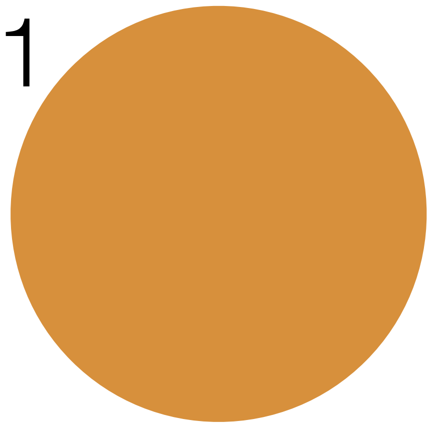
Chinese restaurant process



- Same thing we just did
- Each customer walks into the restaurant
    - Sits at existing table with prob proportional to # people there
    - Forms new table with prob proportional to α

Chinese restaurant process

1



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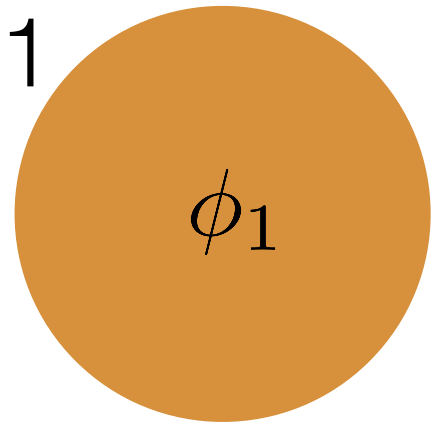
Chinese restaurant process



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Chinese restaurant process

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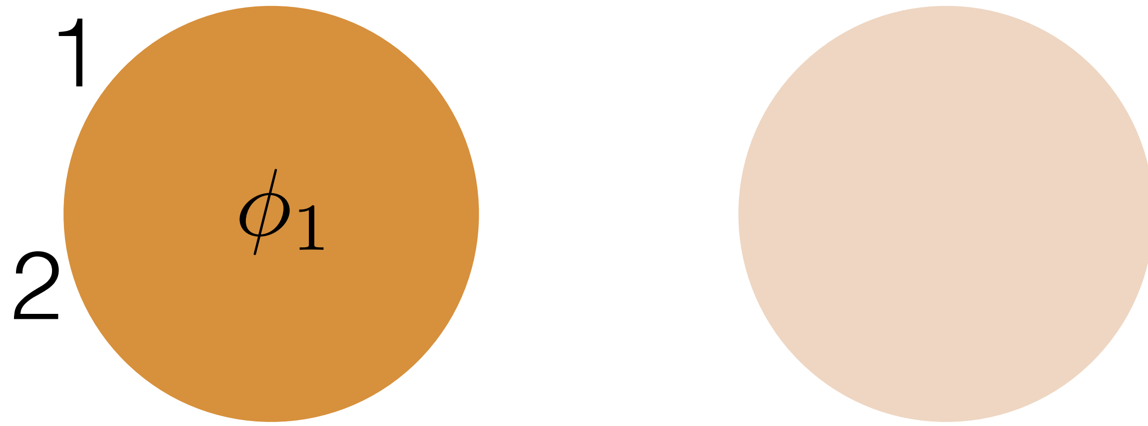
Chinese restaurant process



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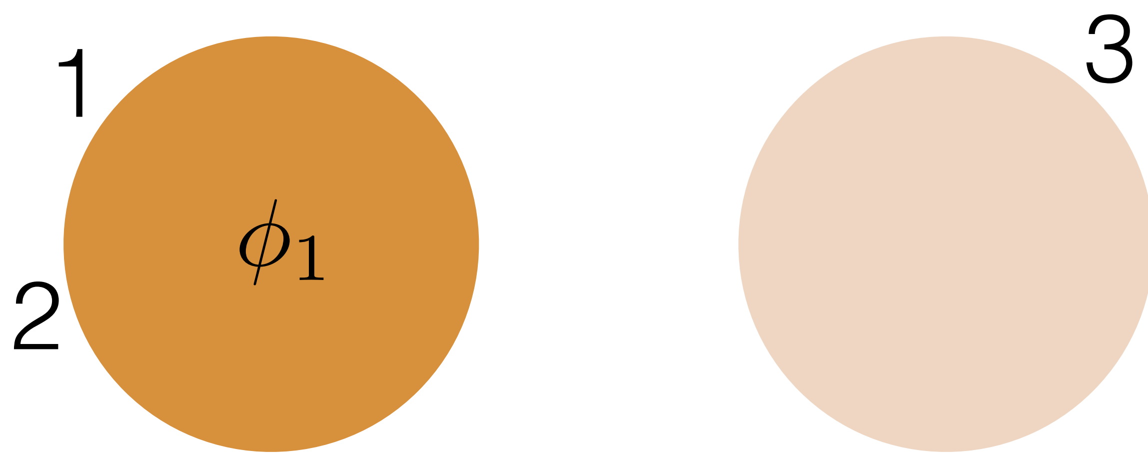
Chinese restaurant process



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Chinese restaurant process

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Chinese restaurant process

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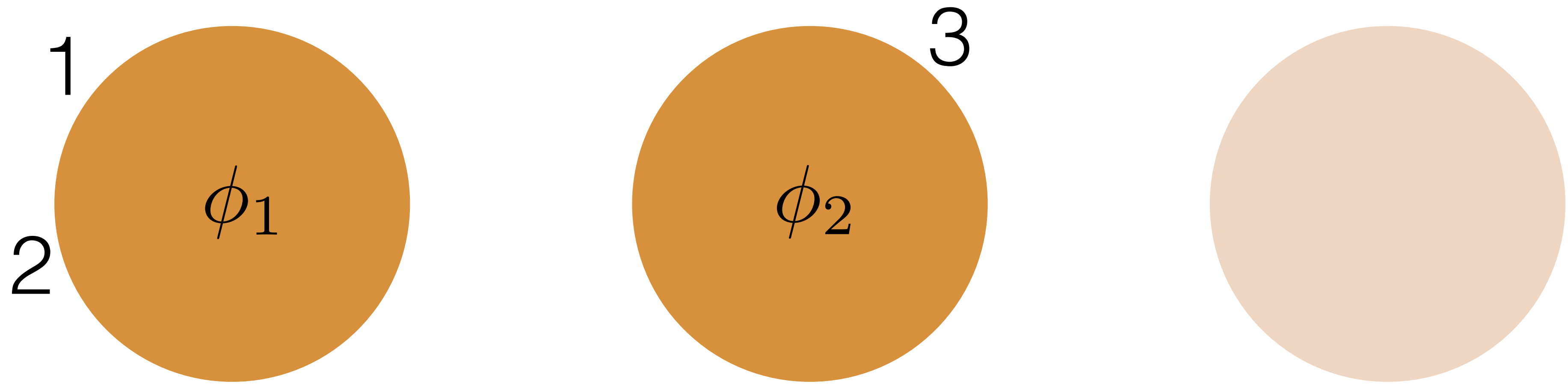
Chinese restaurant process



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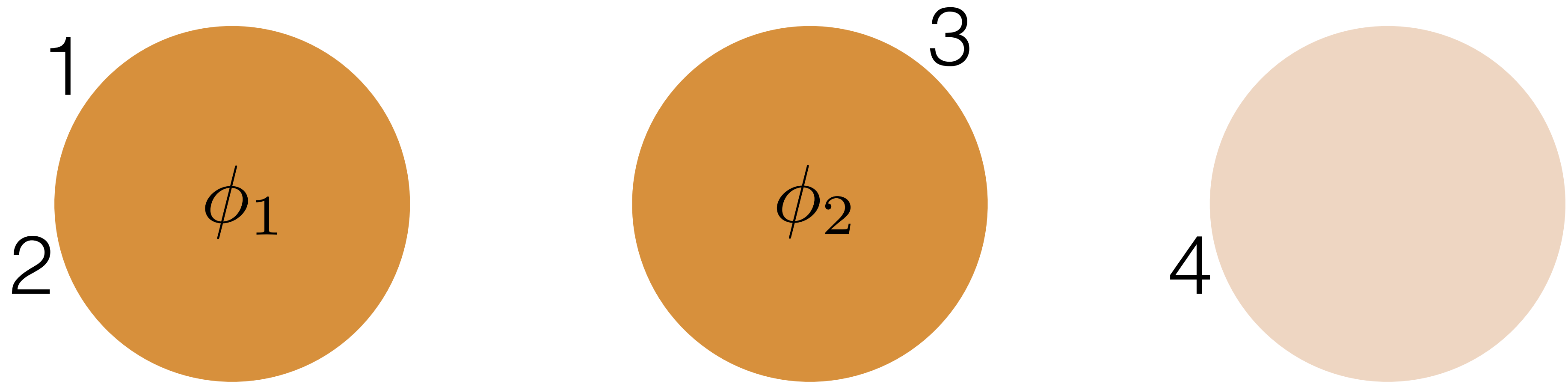

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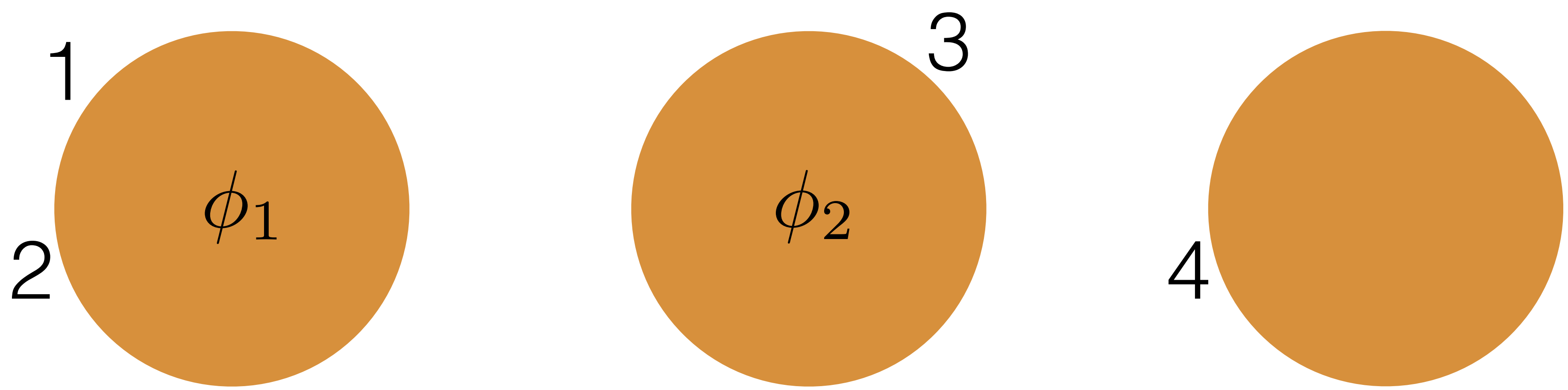
    - Forms new table with prob proportional to α

Chinese restaurant process



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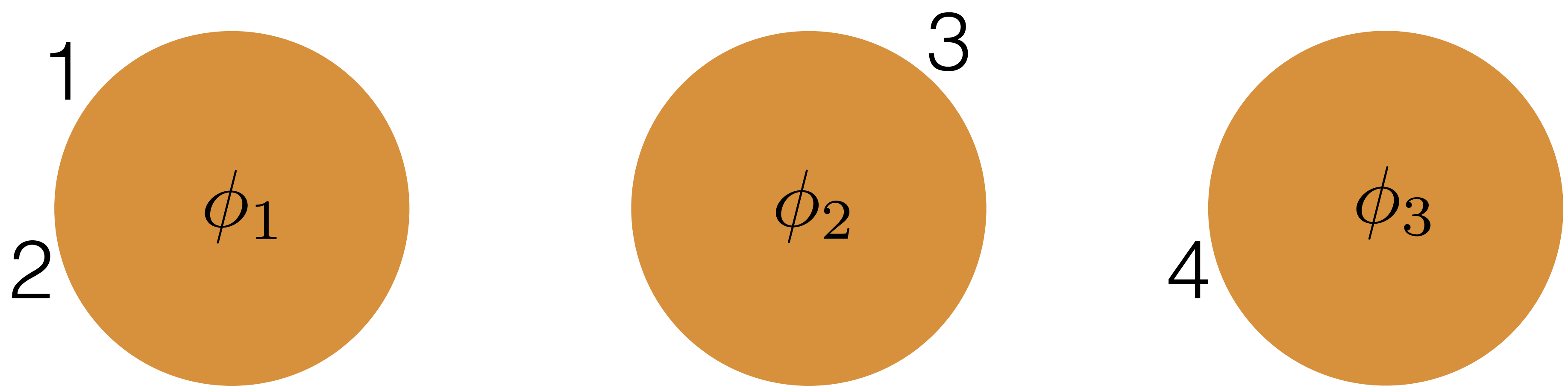
Chinese restaurant process



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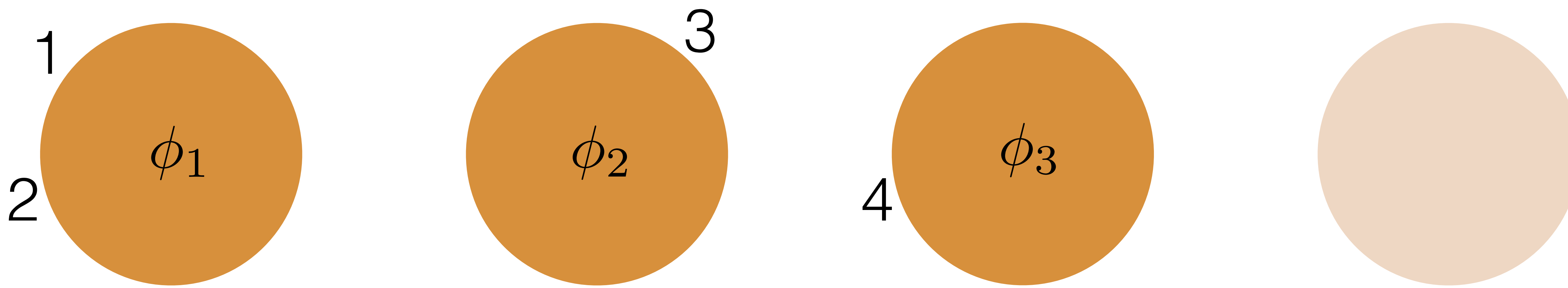

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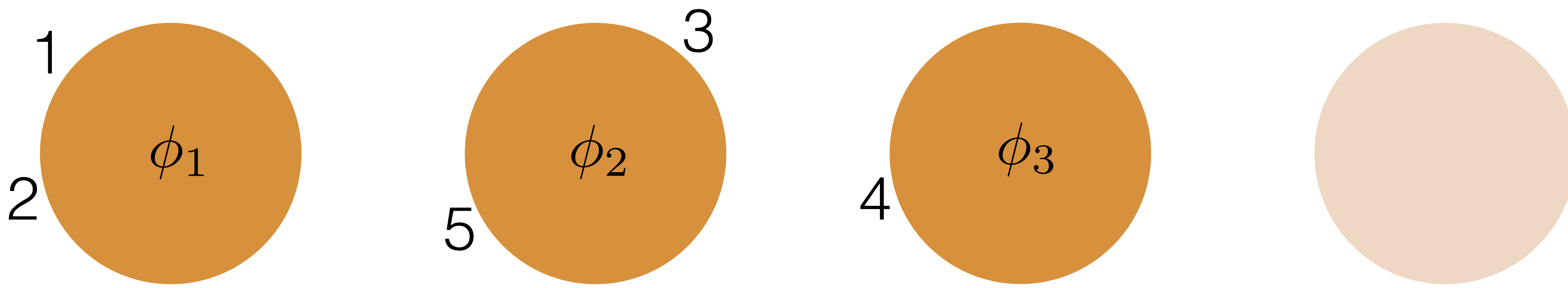

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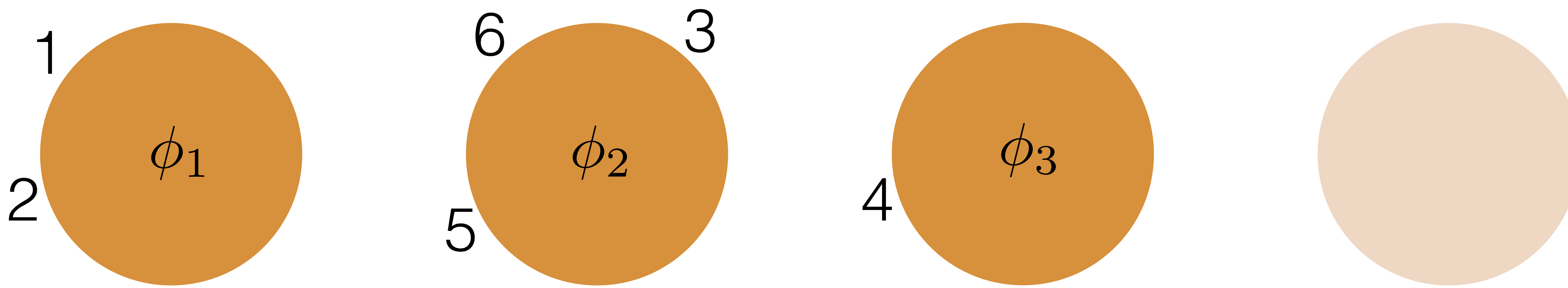

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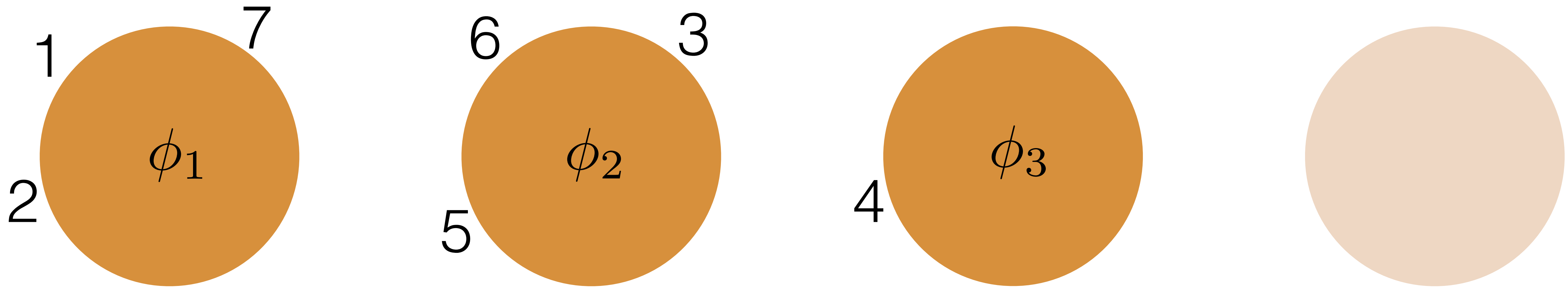

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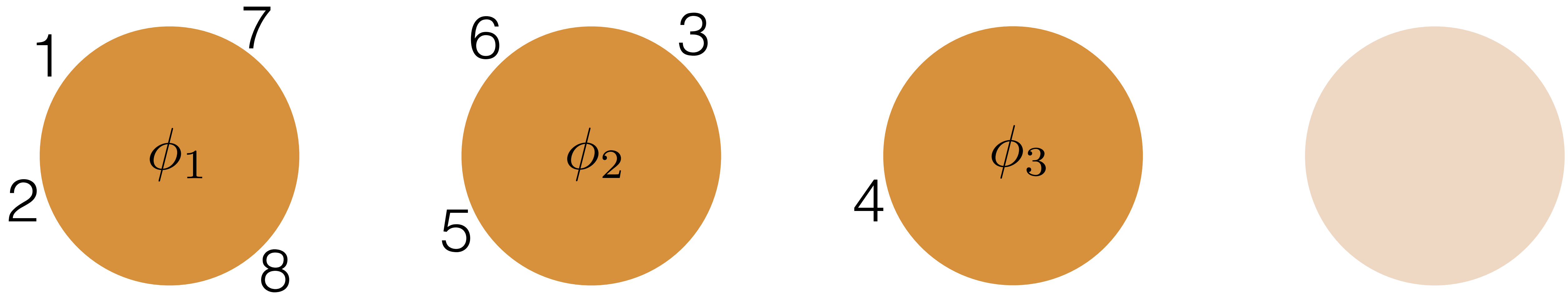
Chinese restaurant process



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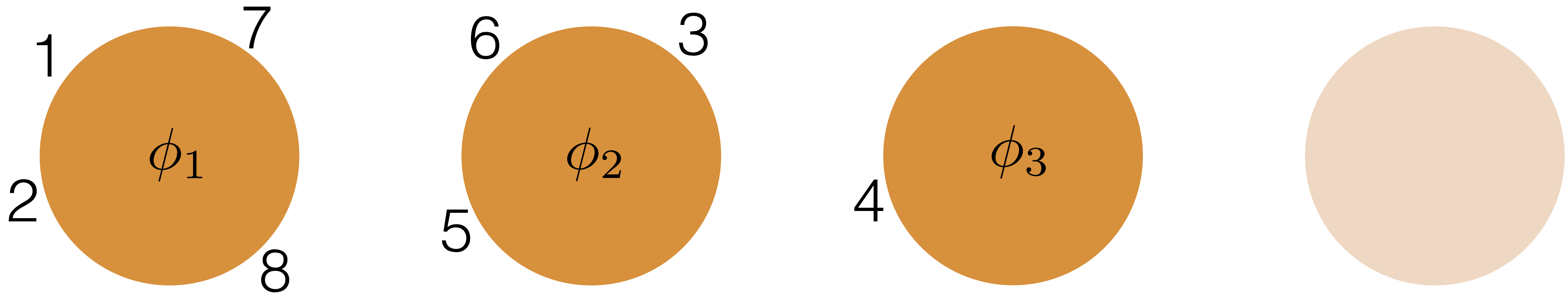
Chinese restaurant process



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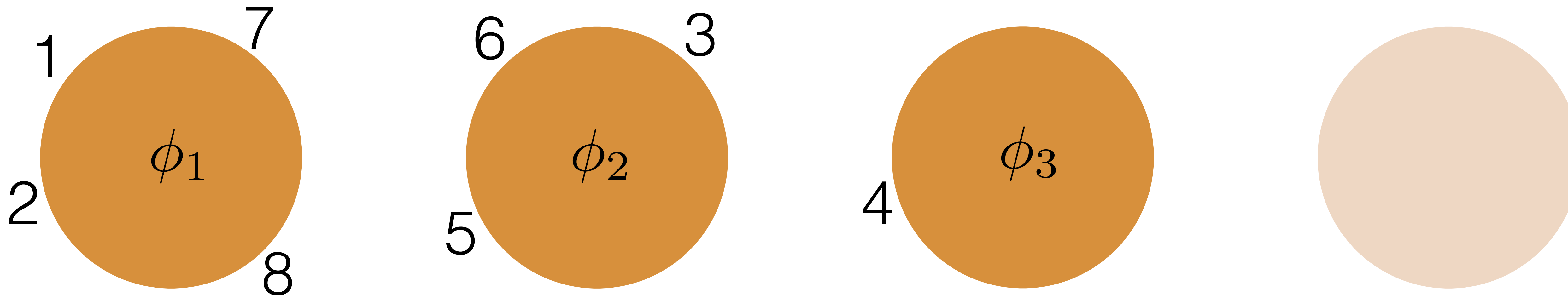
[Aldous 1983]

Chinese restaurant process



- Same thing we just did
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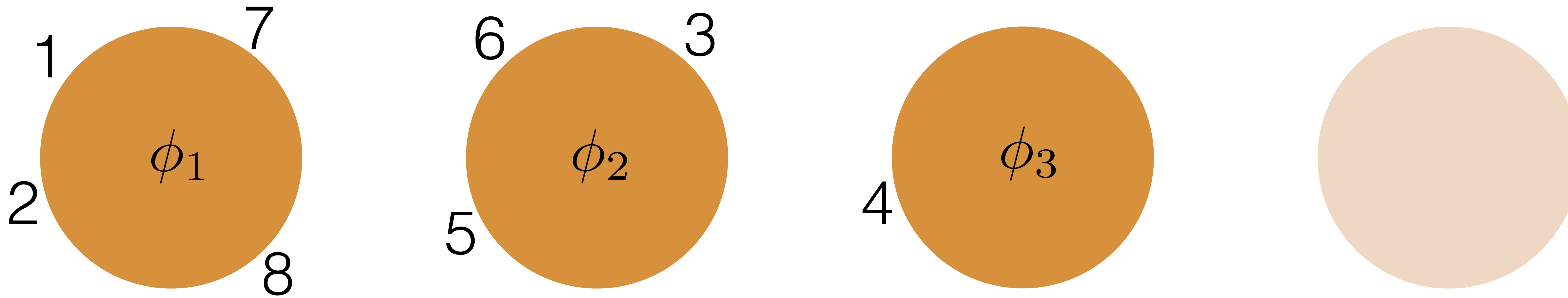
- Marginal for the Categorical likelihood with GEM prior

Chinese restaurant process



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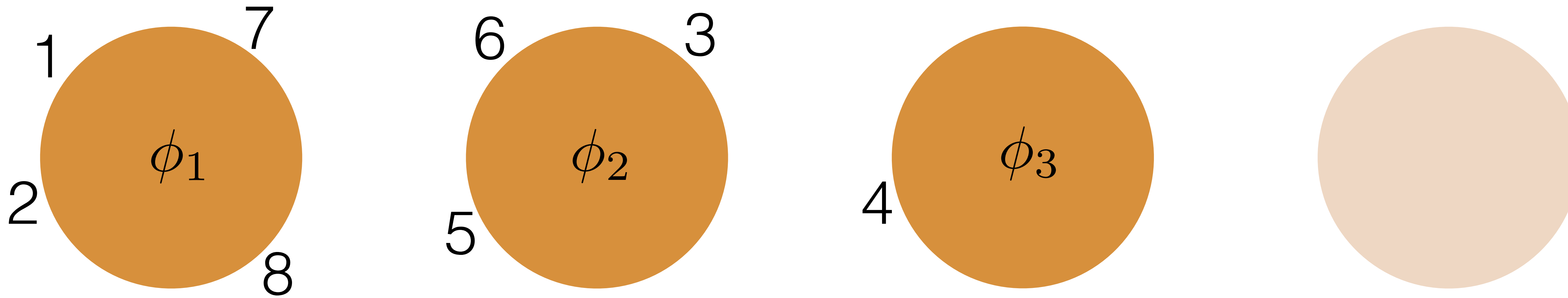
- Marginal for the Categorical likelihood with GEM prior

$$z_1 = z_2 = z_7 = z_8 = 1, z_3 = z_5 = z_6 = 2, z_4 = 3$$

Chinese restaurant process



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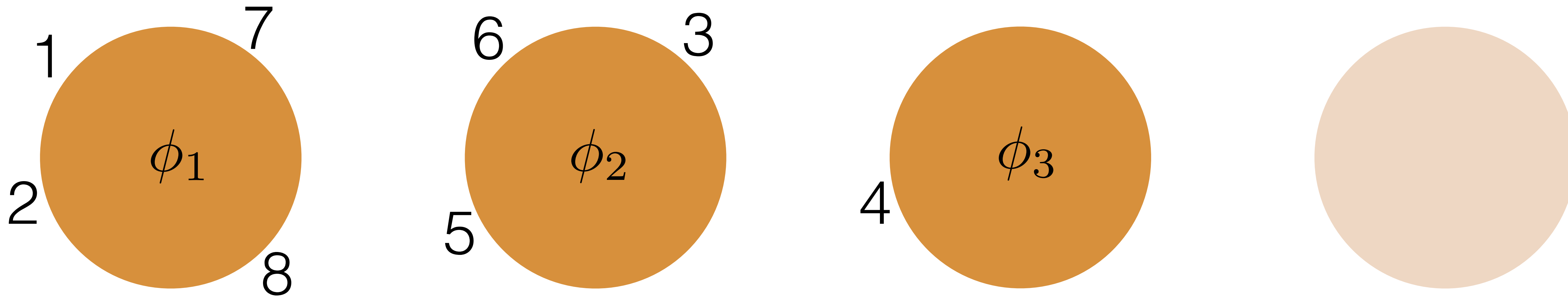
$$z_1 = z_2 = z_7 = z_8 = 1, z_3 = z_5 = z_6 = 2, z_4 = 3$$
$$\Rightarrow \Pi_8 = \{\{1, 2, 7, 8\}, \{3, 5, 6\}, \{4\}\}$$

Chinese restaurant process



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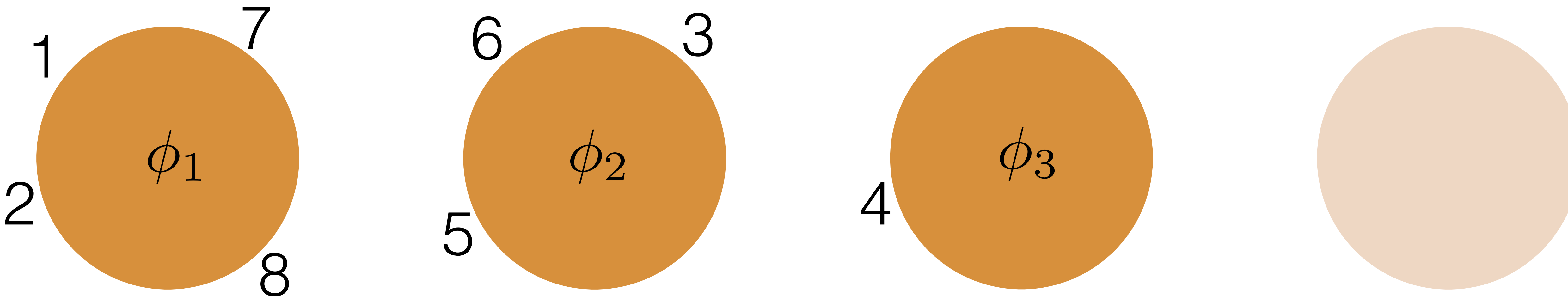
$$\Rightarrow \Pi_8 = \{\{1, 2, 7, 8\}, \{3, 5, 6\}, \{4\}\}$$

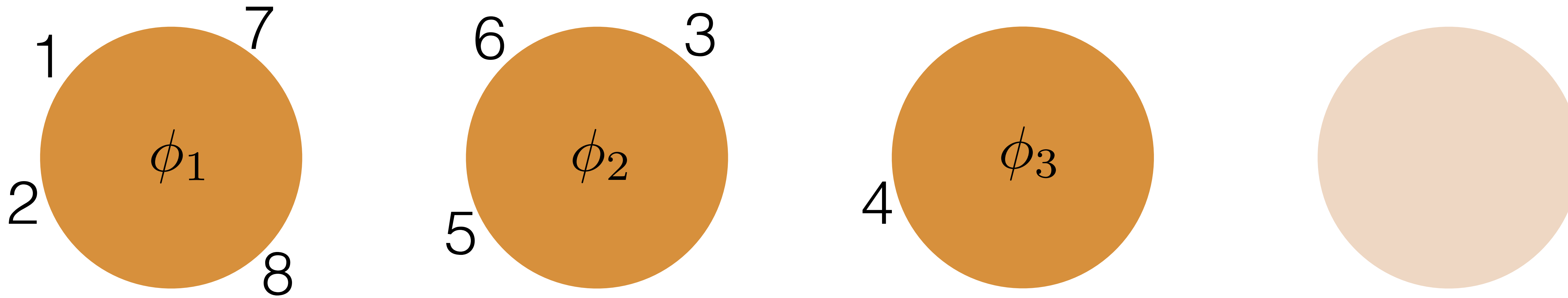
- *Partition of [8]*: set of mutually exclusive & exhaustive sets of $[8] = \{1, \ldots, 8\}$

Chinese restaurant process



- Probability of this seating:

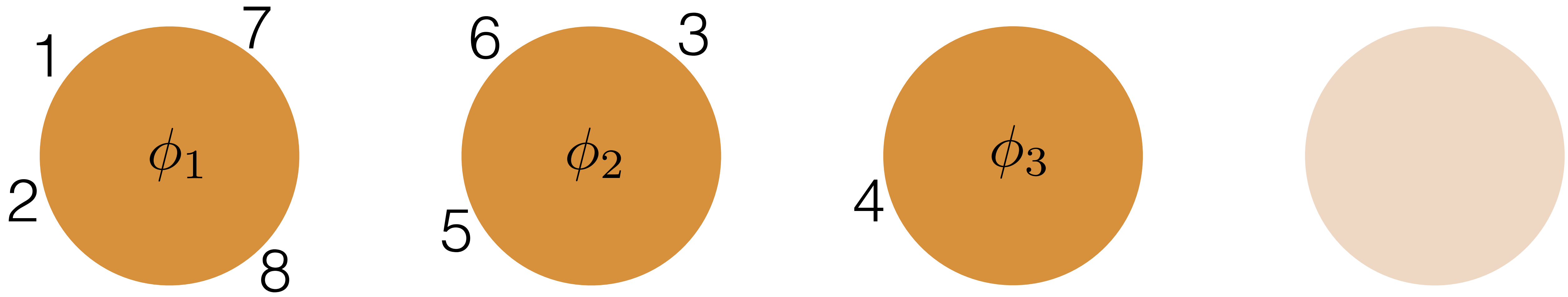
Chinese restaurant process



- Probability of this seating:

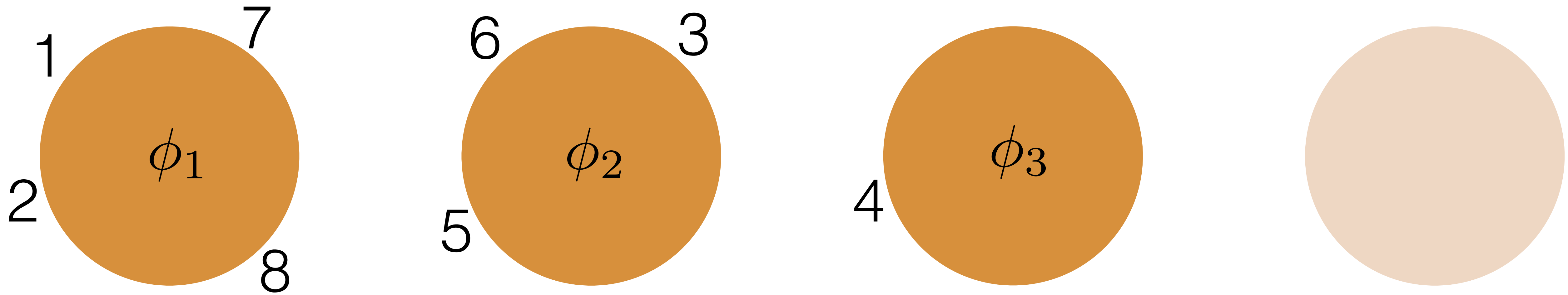
$$\frac{\alpha}{\alpha}$$

Chinese restaurant process



- Probability of this seating:
  $$\frac{\alpha}{\alpha} \cdot \frac{1}{\alpha + 1}$$

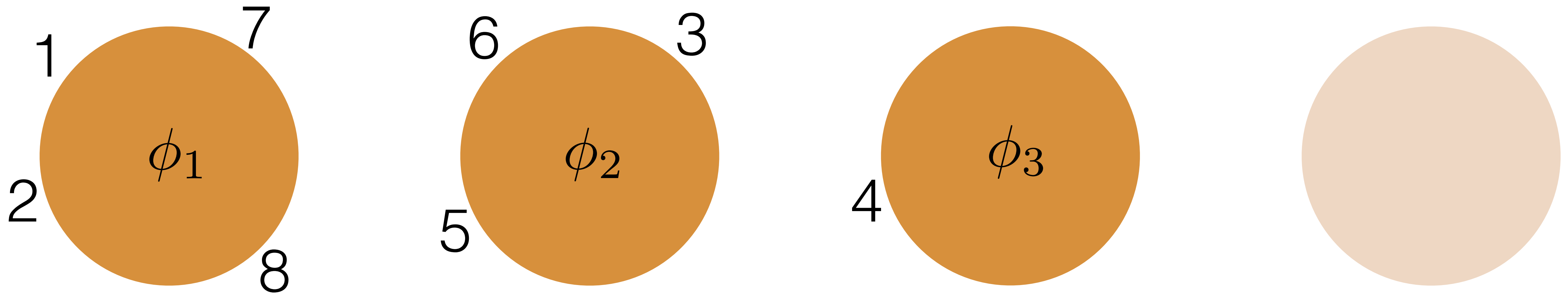
Chinese restaurant process



- Probability of this seating:

$$\frac{\alpha}{\alpha} \cdot \frac{1}{\alpha + 1} \cdot \frac{\alpha}{\alpha + 2}$$

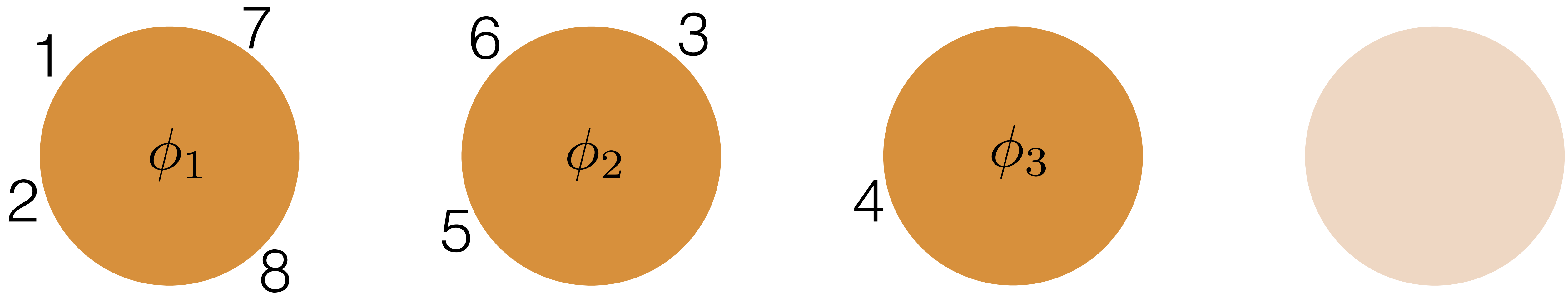
Chinese restaurant process



- Probability of this seating:

$$\frac{\alpha}{\alpha} \cdot \frac{1}{\alpha+1} \cdot \frac{\alpha}{\alpha+2} \cdot \frac{\alpha}{\alpha+3}$$

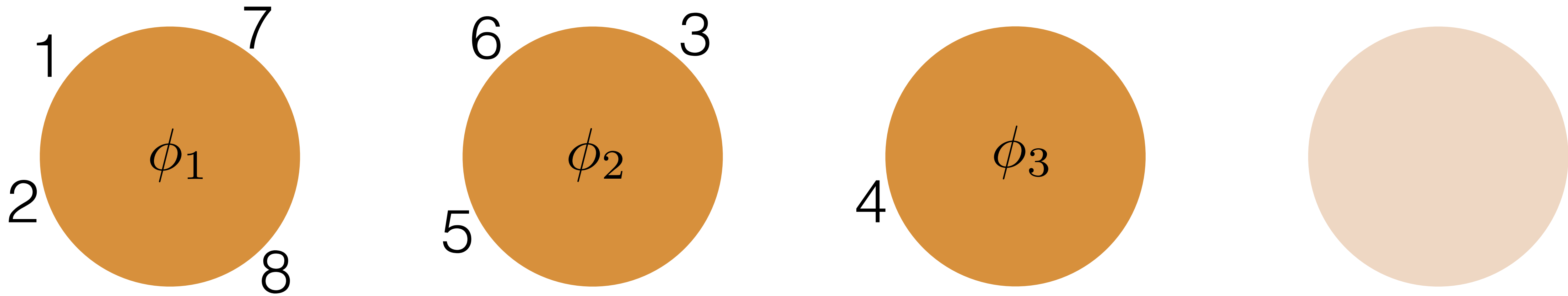
Chinese restaurant process



- Probability of this seating:

$$\frac{\alpha}{\alpha} \cdot \frac{1}{\alpha+1} \cdot \frac{\alpha}{\alpha+2} \cdot \frac{\alpha}{\alpha+3} \cdot \frac{1}{\alpha+4}$$

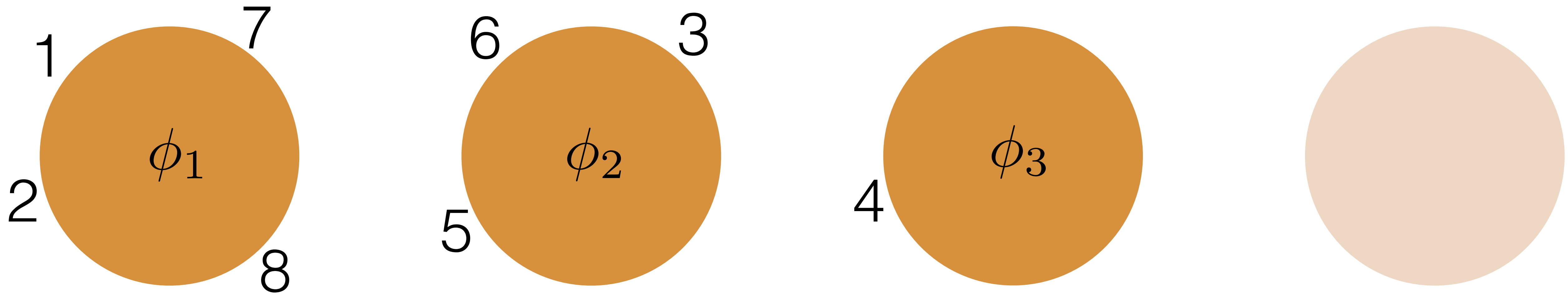
Chinese restaurant process



- Probability of this seating:

$$\frac{\alpha}{\alpha} \cdot \frac{1}{\alpha+1} \cdot \frac{\alpha}{\alpha+2} \cdot \frac{\alpha}{\alpha+3} \cdot \frac{1}{\alpha+4} \cdot \frac{2}{\alpha+5}$$

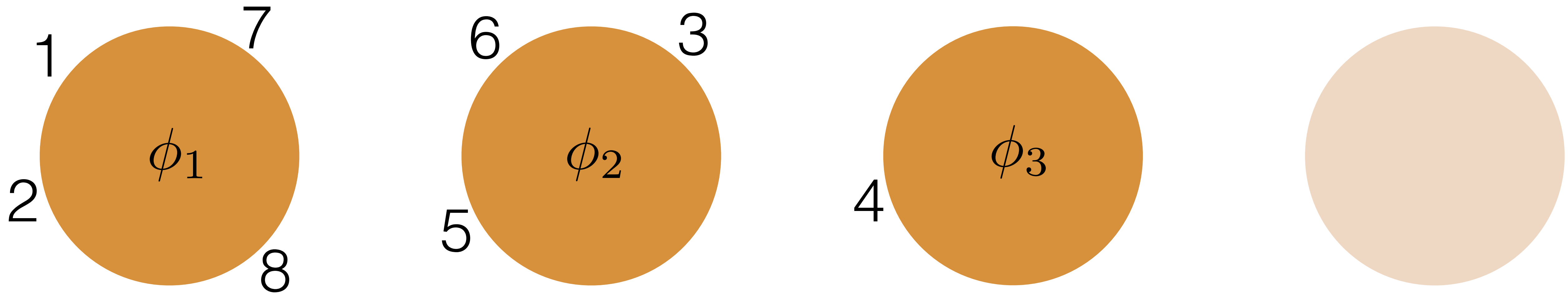
Chinese restaurant process



- Probability of this seating:

$$\frac{\alpha}{\alpha} \cdot \frac{1}{\alpha+1} \cdot \frac{\alpha}{\alpha+2} \cdot \frac{\alpha}{\alpha+3} \cdot \frac{1}{\alpha+4} \cdot \frac{2}{\alpha+5} \cdot \frac{2}{\alpha+6}$$

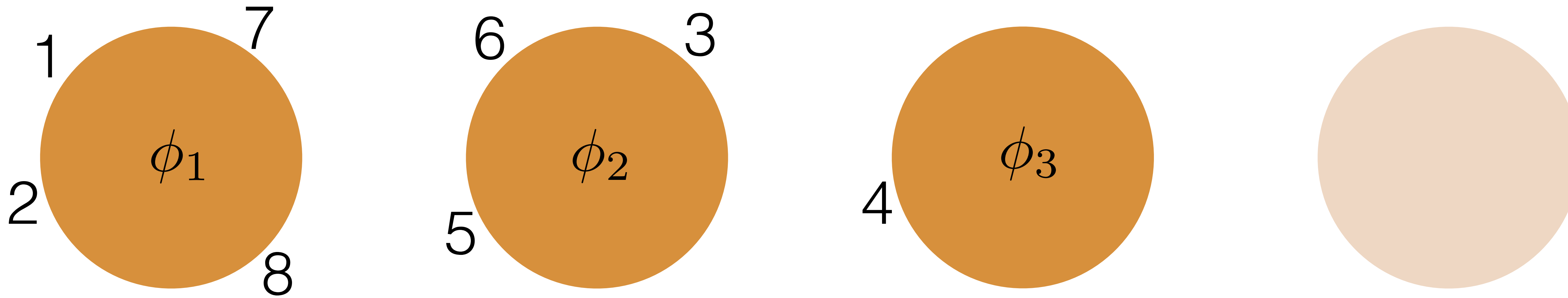
Chinese restaurant process



- Probability of this seating:

$$\frac{\alpha}{\alpha} \cdot \frac{1}{\alpha+1} \cdot \frac{\alpha}{\alpha+2} \cdot \frac{\alpha}{\alpha+3} \cdot \frac{1}{\alpha+4} \cdot \frac{2}{\alpha+5} \cdot \frac{2}{\alpha+6} \cdot \frac{3}{\alpha+7}$$

Exercises



- If I run the Chinese restaurant process (CRP) for N customers, what is the probability of the customers sitting at K_N tables where the kth table to form has n_k ($n_k > 0$) people at it?
- What is the expected number of clusters generated by a CRP(α) after N data points?
- What is the distribution of the number of clusters under a CRP(α) after N data points? (consider simulations and/or theory)

References (Part II), Page 1

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