Nonparametric Bayesian Statistics: Part III

Tamara Broderick
ITT Career Development Assistant Professor
Electrical Engineering & Computer Science
MIT

Recall: Part I

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$$\rho = (\rho_1, \rho_2, \ldots) \sim \mathrm{GEM}(\alpha)$$

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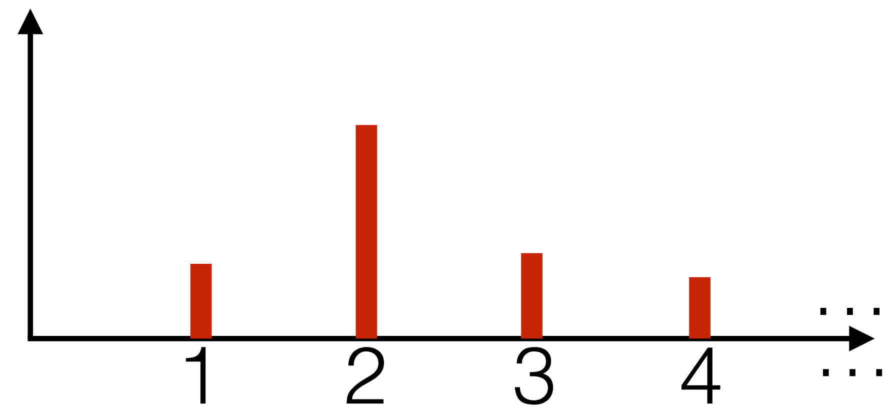

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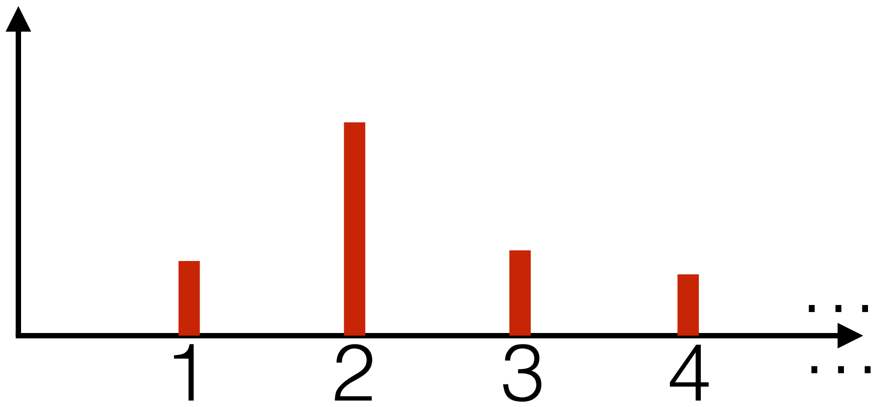
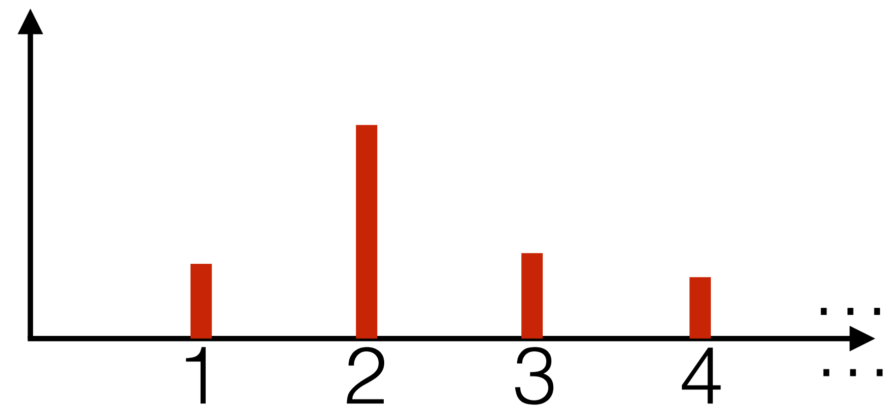


$$z_n \stackrel{iid}{\sim} \text{Categorical}(\rho)$$

Recall: Part I and Part II



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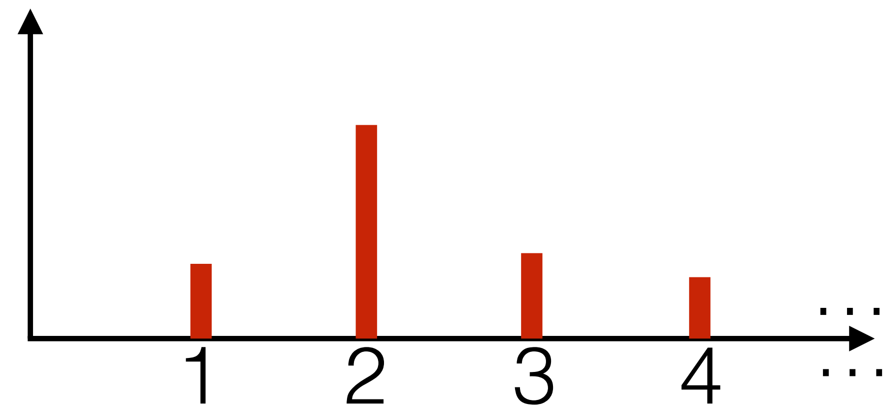

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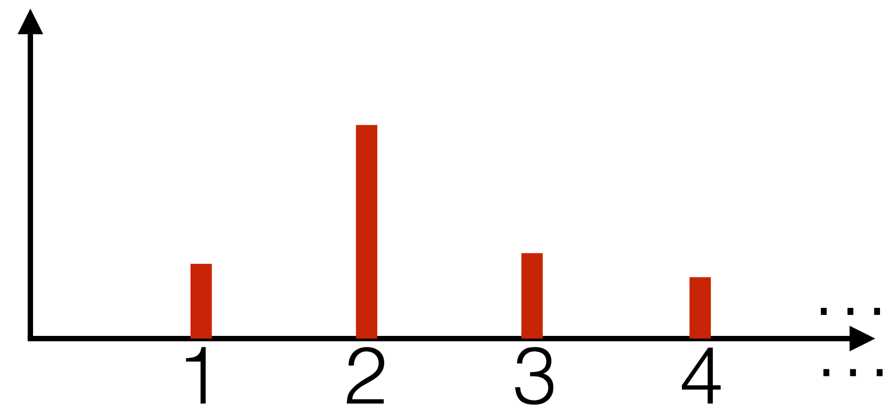

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- Part of Dirichlet Process mixture model

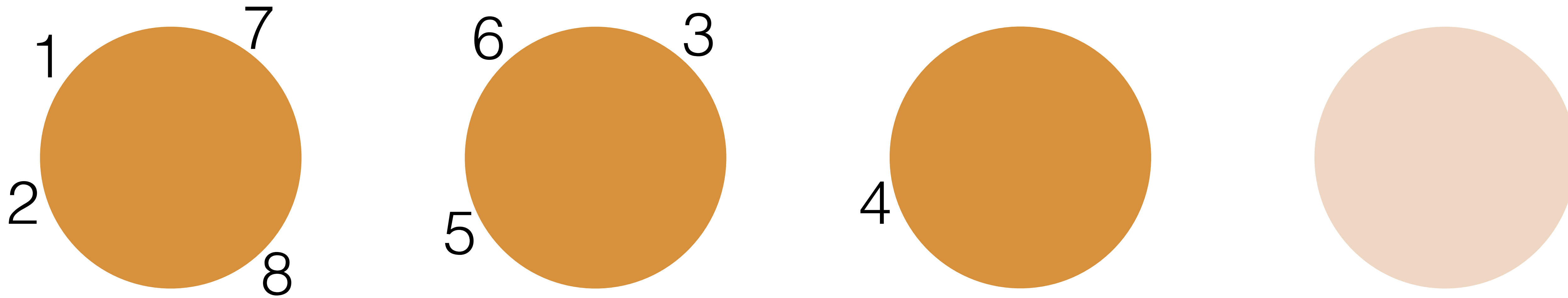
- Finite representation for inference?

Recall: Part II

Recall: Part II

- Chinese restaurant process

Recall: Part II



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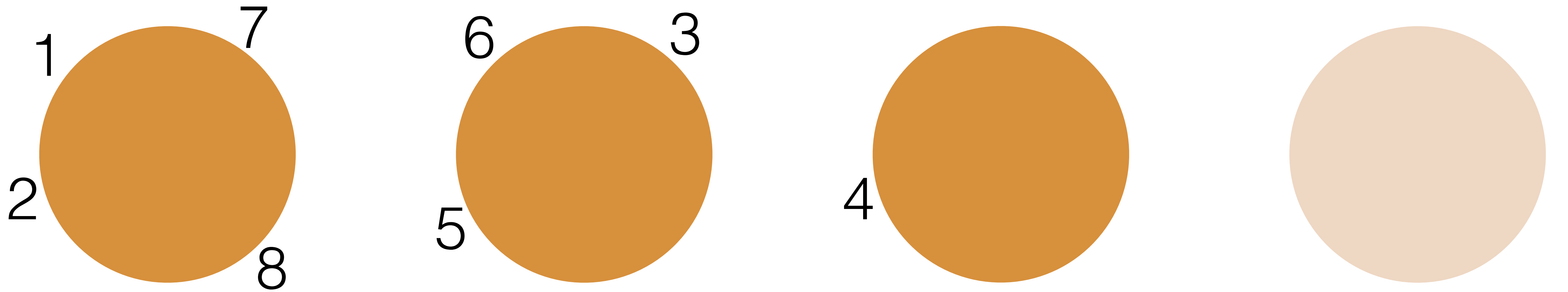
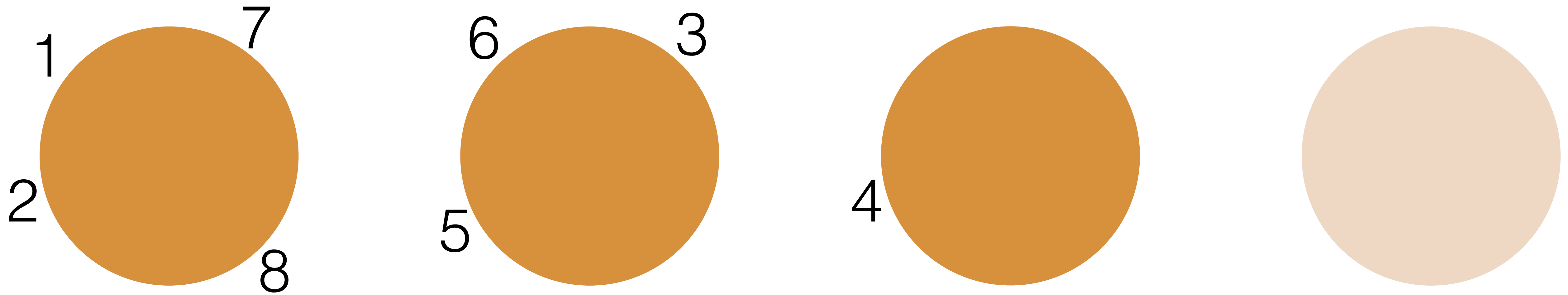
Recall: Part II



- Chinese restaurant process
- Each customer walks into the restaurant

Recall: Part II



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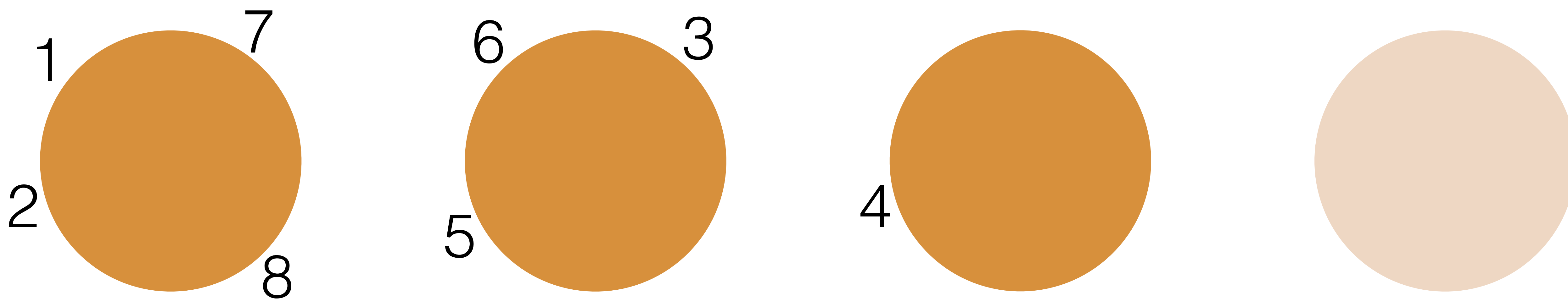

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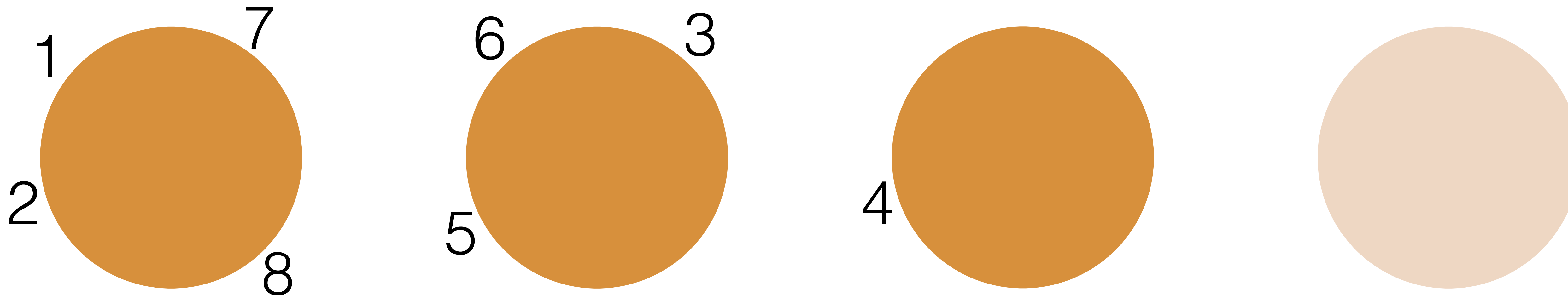

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- "Partition" $\Pi_8 = \{\{1,2,7,8\},\{3,5,6\},\{4\}\}$

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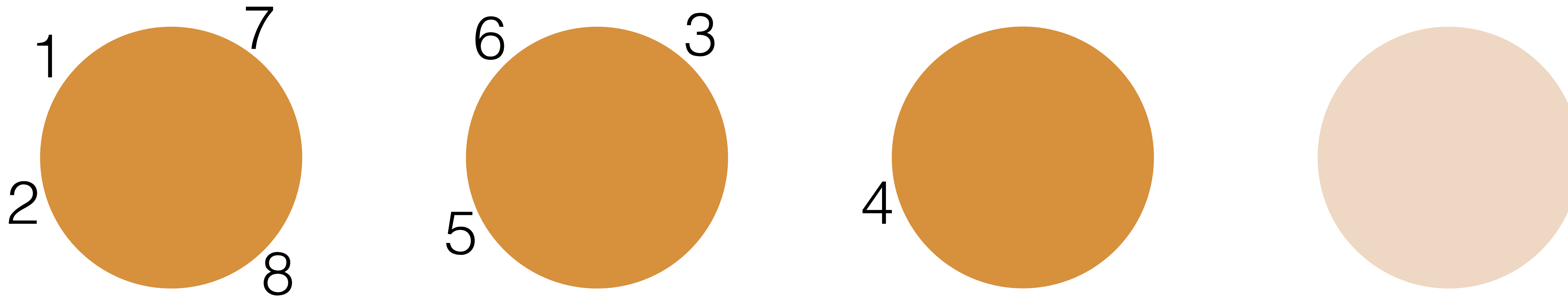

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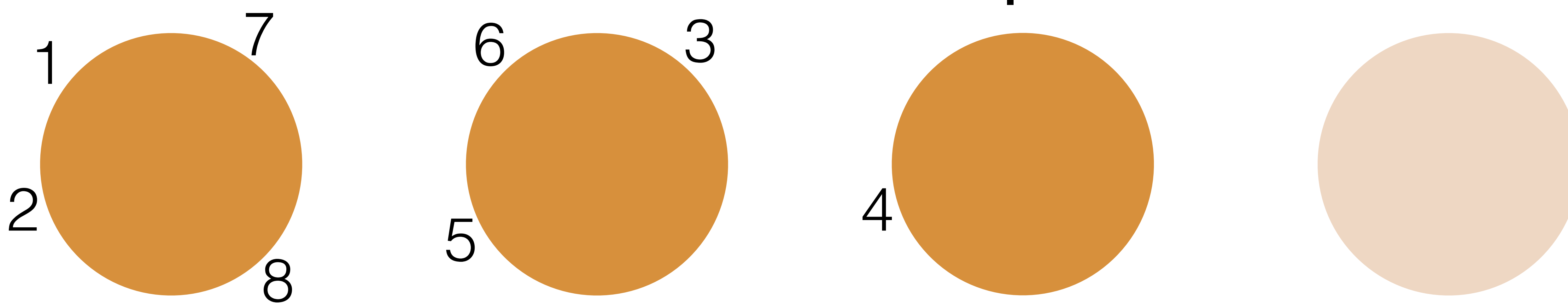
- "Partition" $\Pi_8 = \{\{1,2,7,8\},\{3,5,6\},\{4\}\}$
- Partition from a GEM(α) with categorical draws = same distribution as partition from a CRP(α)

Chinese restaurant process

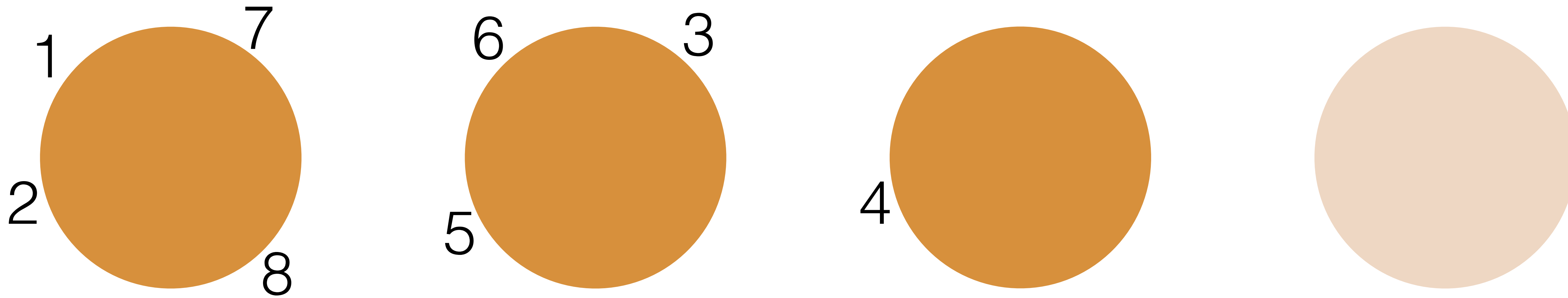


- Probability of this seating:

$$\frac{\alpha}{\alpha} \cdot \frac{1}{\alpha+1} \cdot \frac{\alpha}{\alpha+2} \cdot \frac{\alpha}{\alpha+3} \cdot \frac{1}{\alpha+4} \cdot \frac{2}{\alpha+5} \cdot \frac{2}{\alpha+6} \cdot \frac{3}{\alpha+7}$$

- Probability of *N* customers (K_N tables, n_k at table *k*):

Chinese restaurant process



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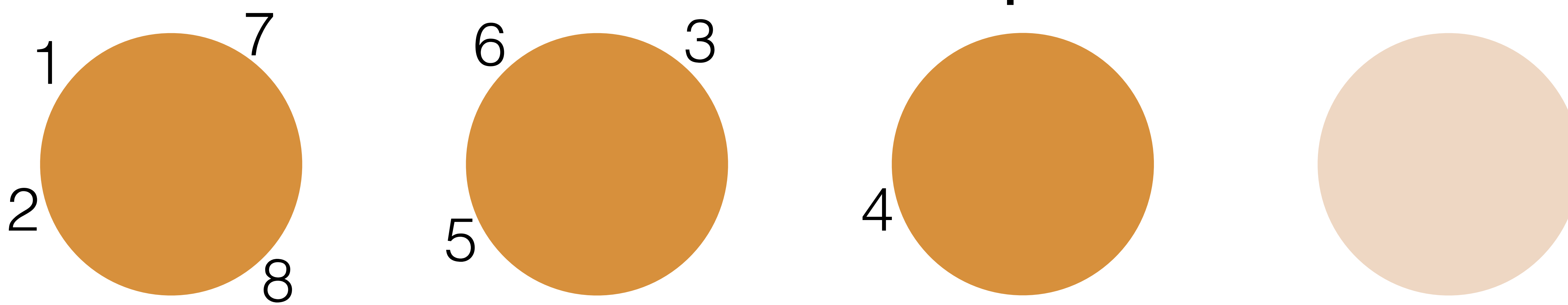

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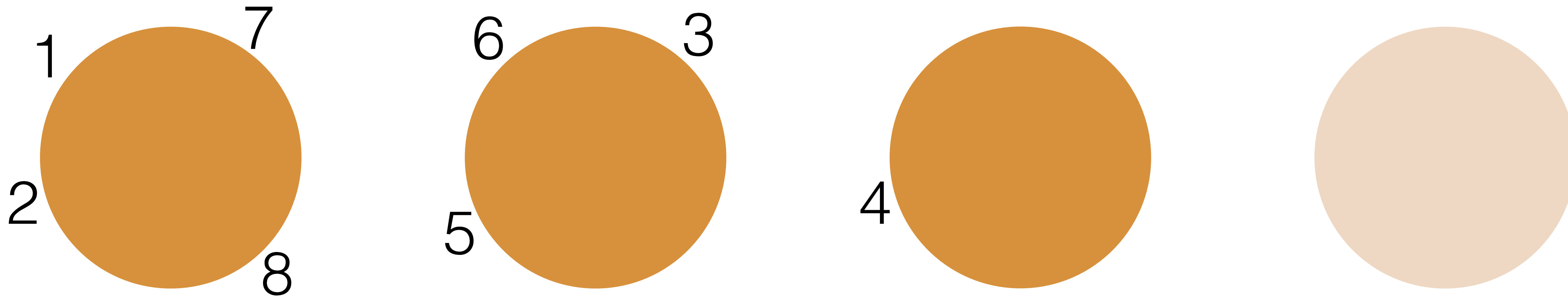

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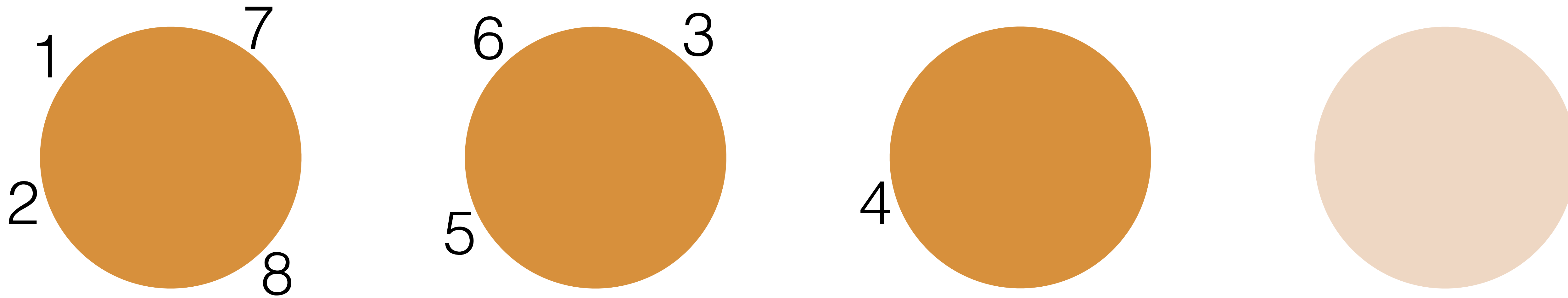

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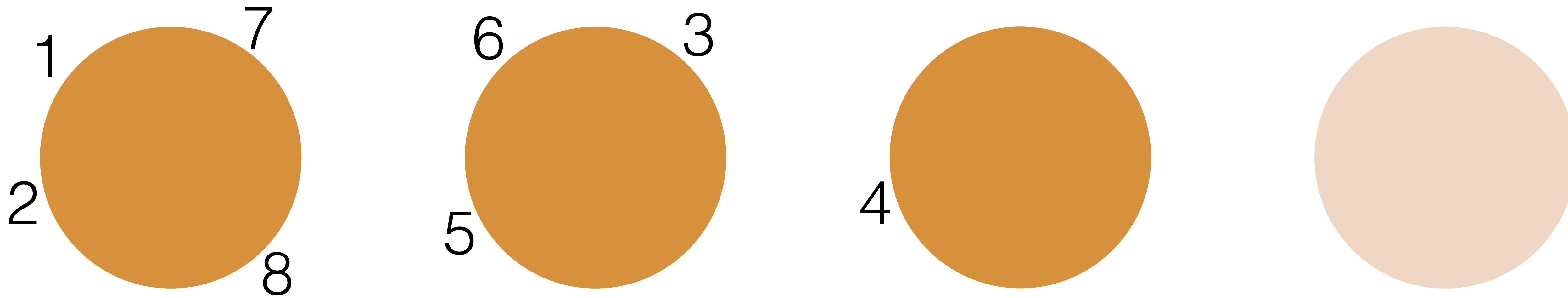

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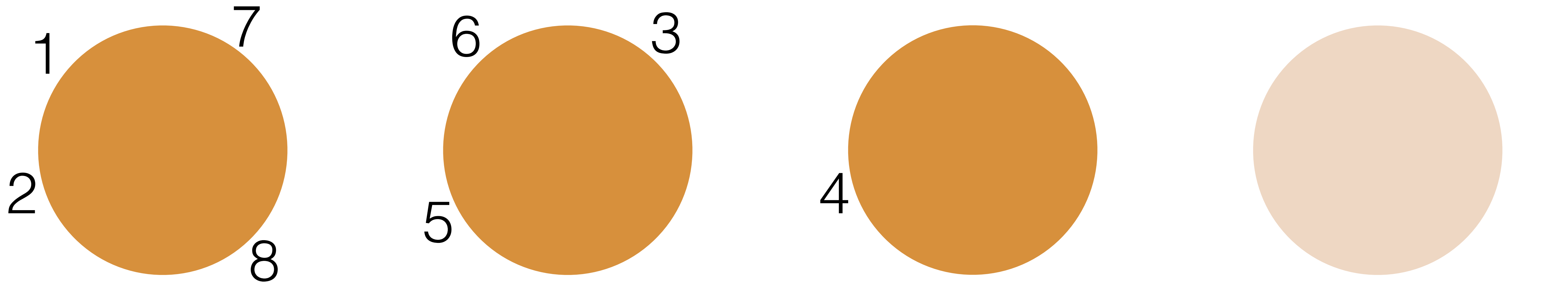

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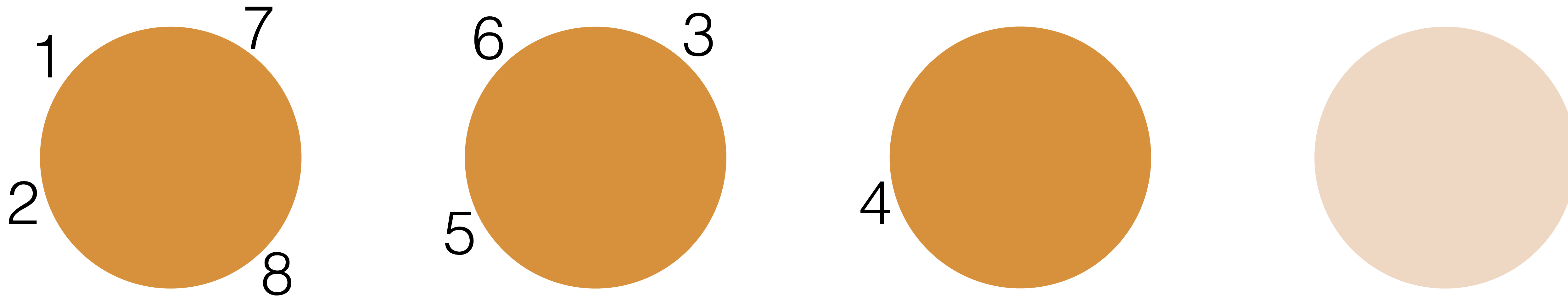

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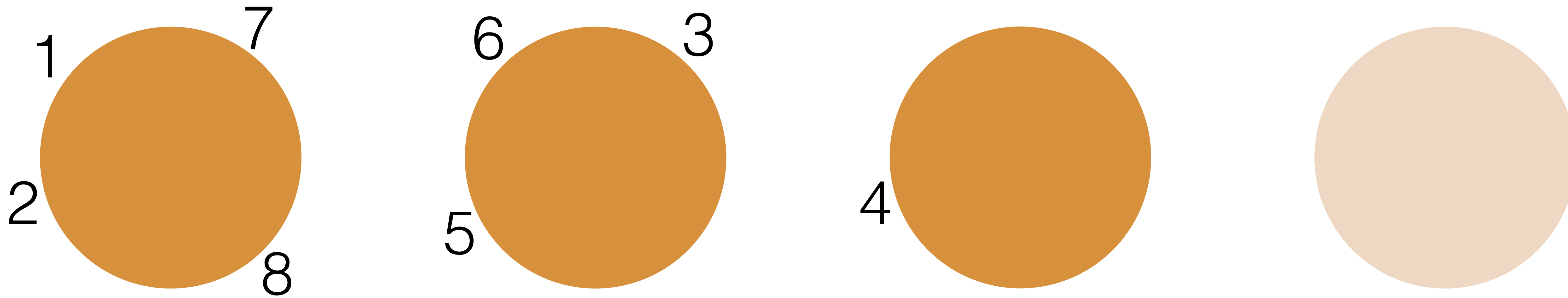

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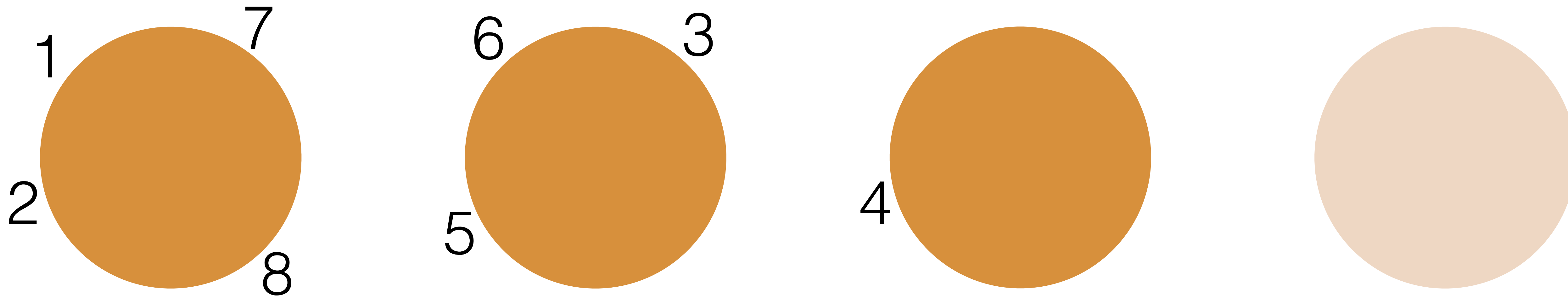

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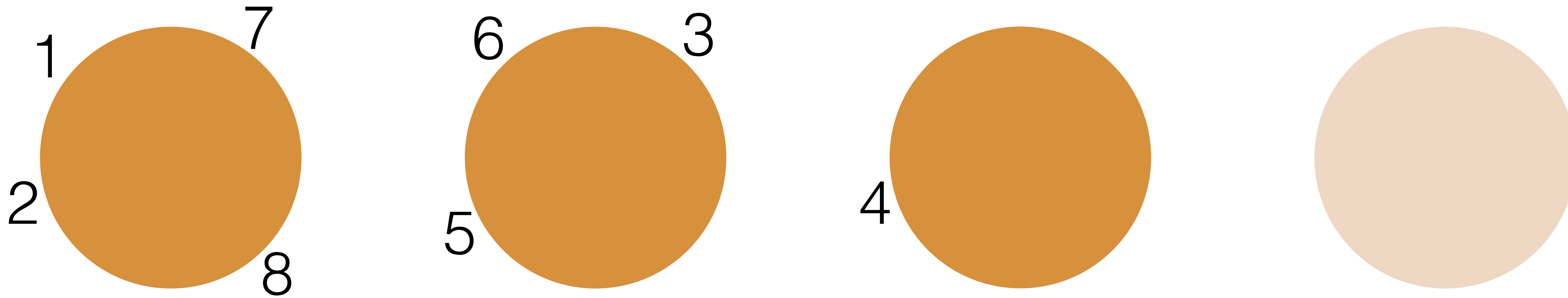

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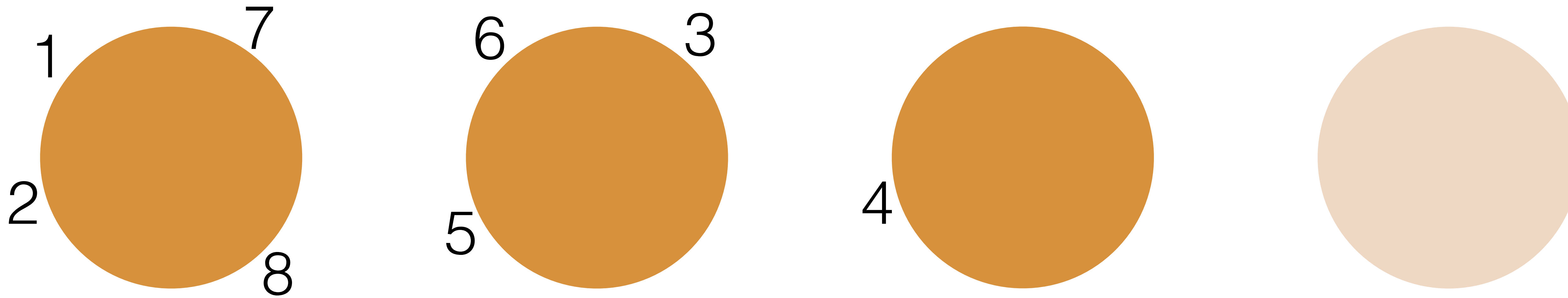

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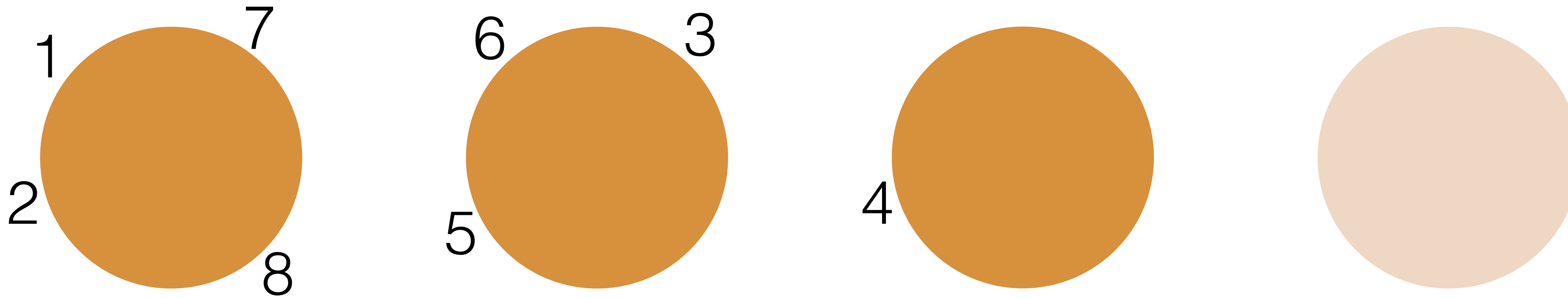
Chinese restaurant process



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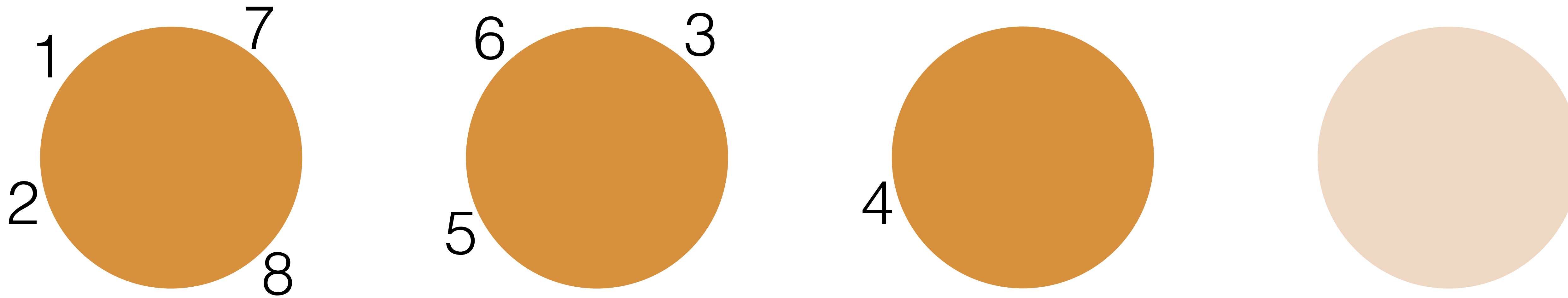
Chinese restaurant process



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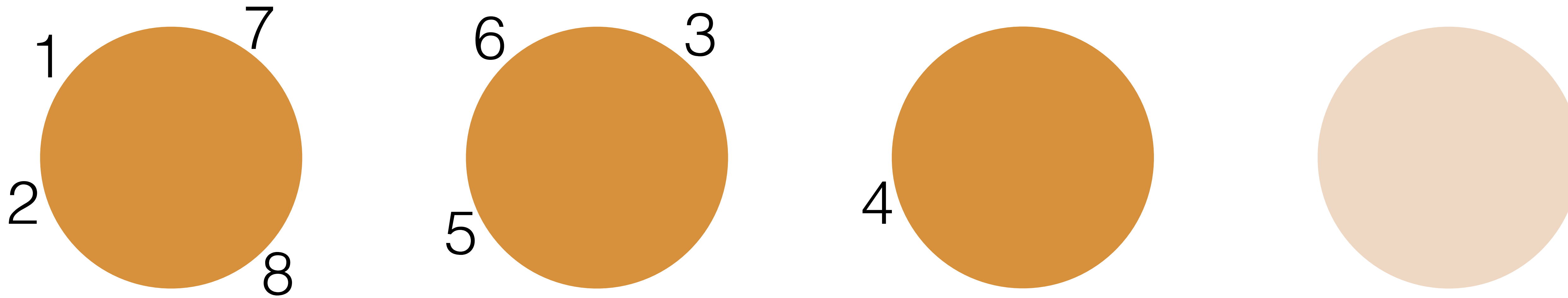
Chinese restaurant process



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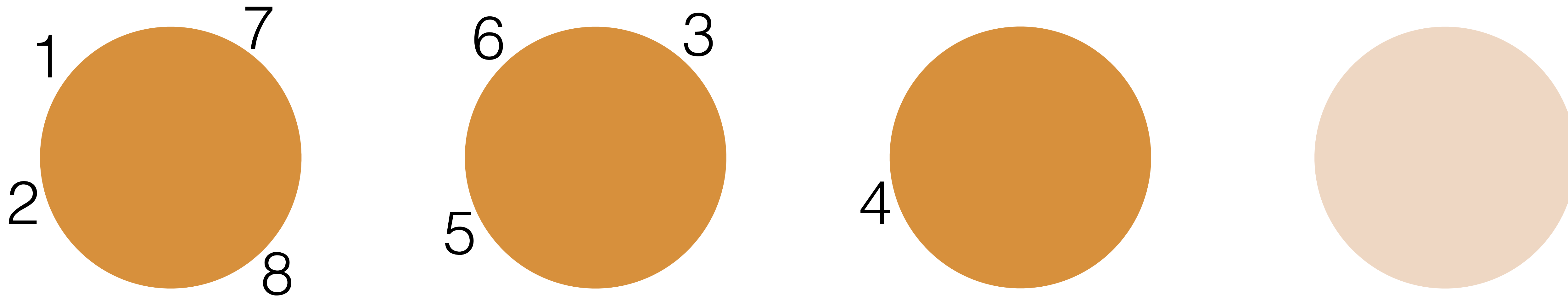
    - e.g. $\Pi_{8,\text{-}5} = \{\{1,2,7,8\},\{3,6\},\{4\}\}$

Chinese restaurant process



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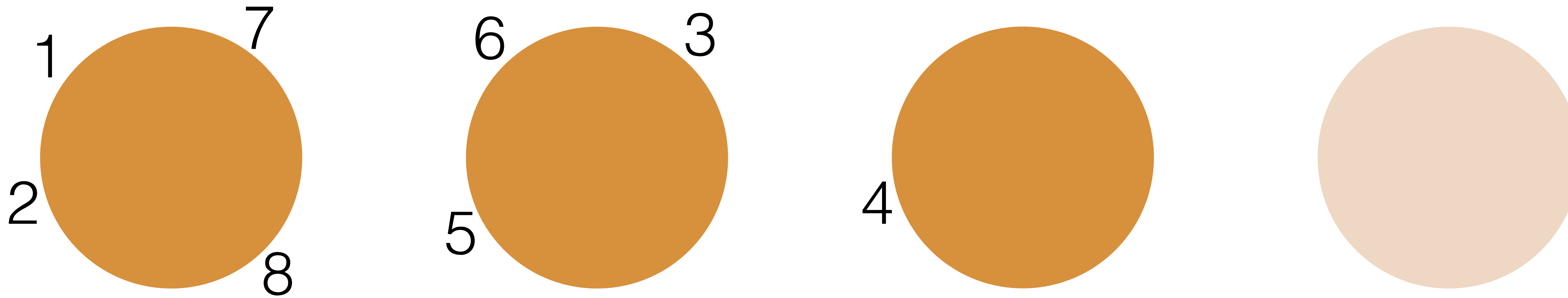
Chinese restaurant process



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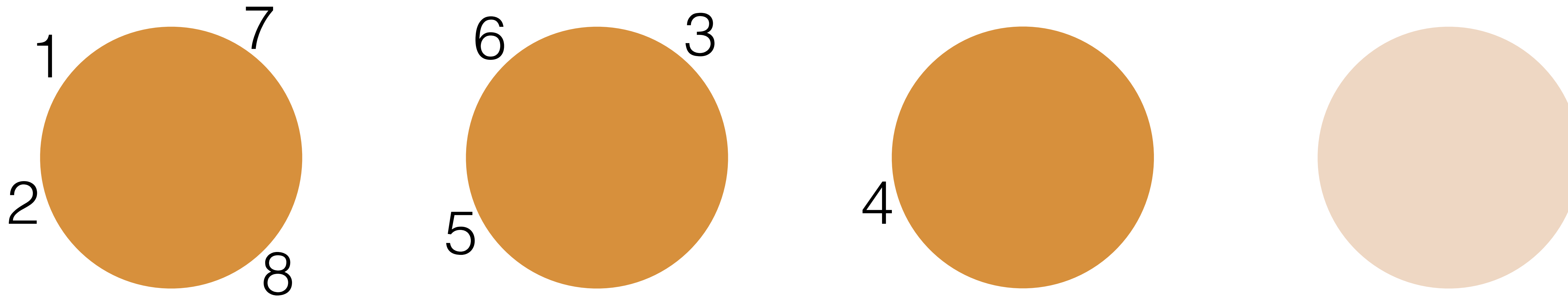

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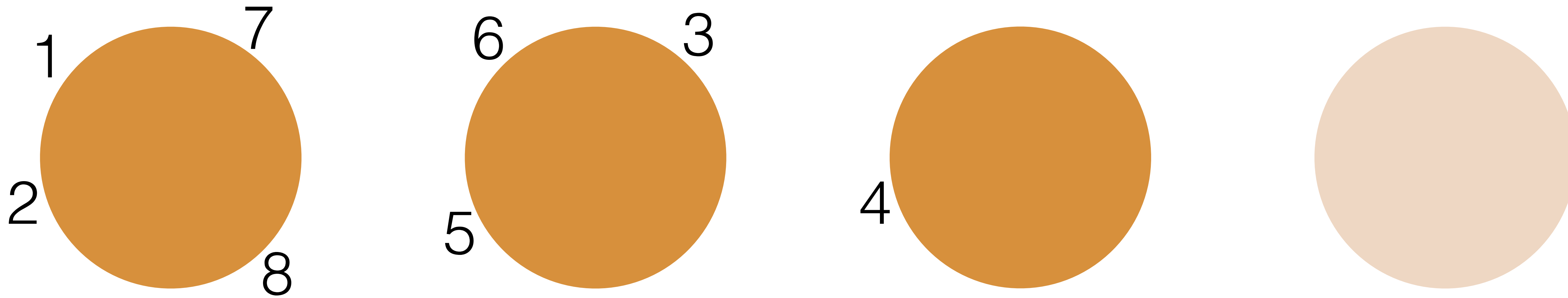
Chinese restaurant process



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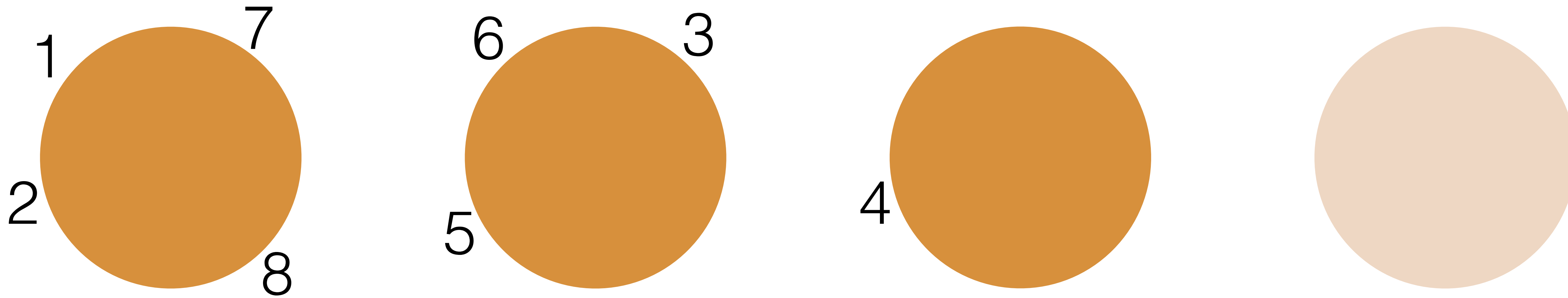

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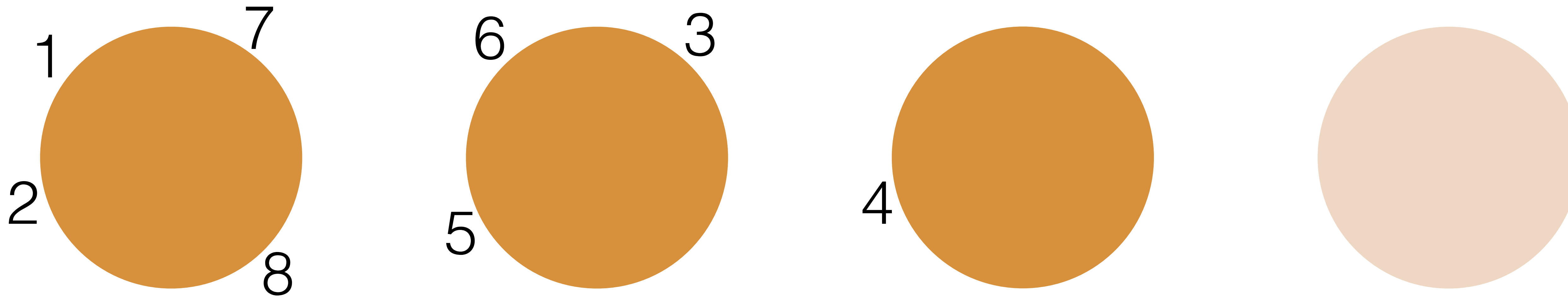

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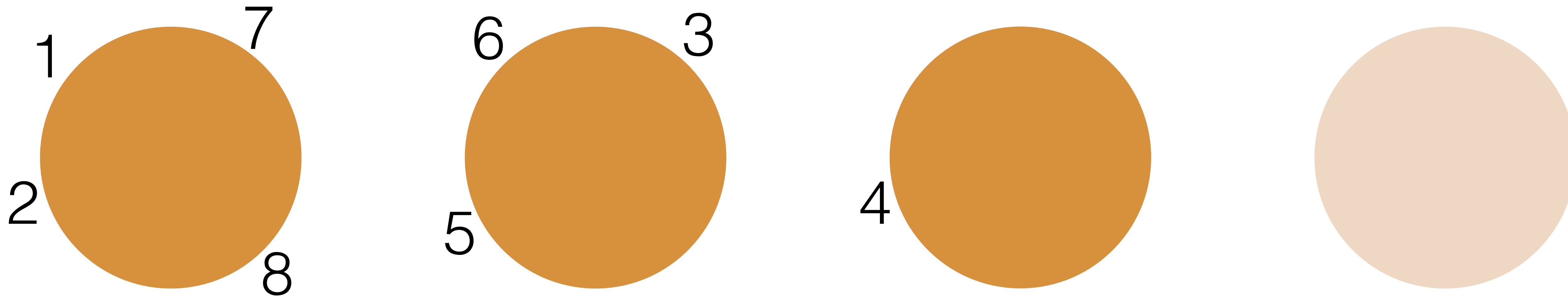
- Gibbs sampling review:

Chinese restaurant process



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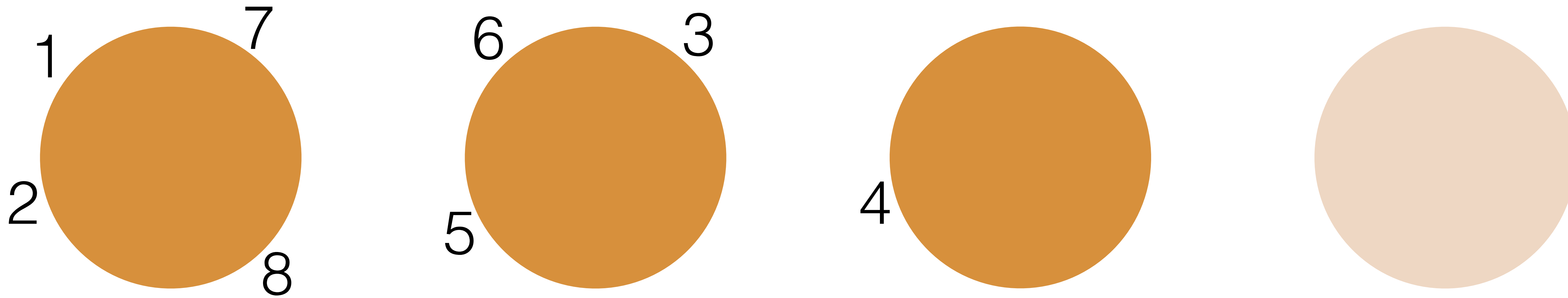
- Gibbs sampling review: target distribution $p(v_1, v_2, v_3)$

Chinese restaurant process



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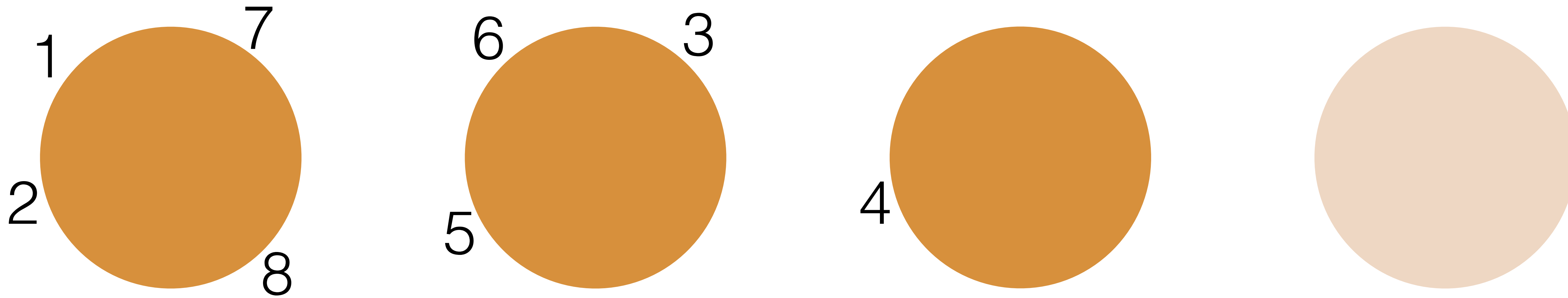
- So: $$p(\Pi_N | \Pi_{N,-n}) = \begin{cases} \frac{\#C}{\alpha + N - 1} & \text{if } n \text{ joins cluster } C \\ \frac{\alpha}{\alpha + N - 1} & \text{if } n \text{ starts a new cluster} \end{cases}$$

- Gibbs sampling review: target distribution $p(v_1, v_2, v_3)$
  - Start: $v_1^{(0)}, v_2^{(0)}, v_3^{(0)}$

Chinese restaurant process



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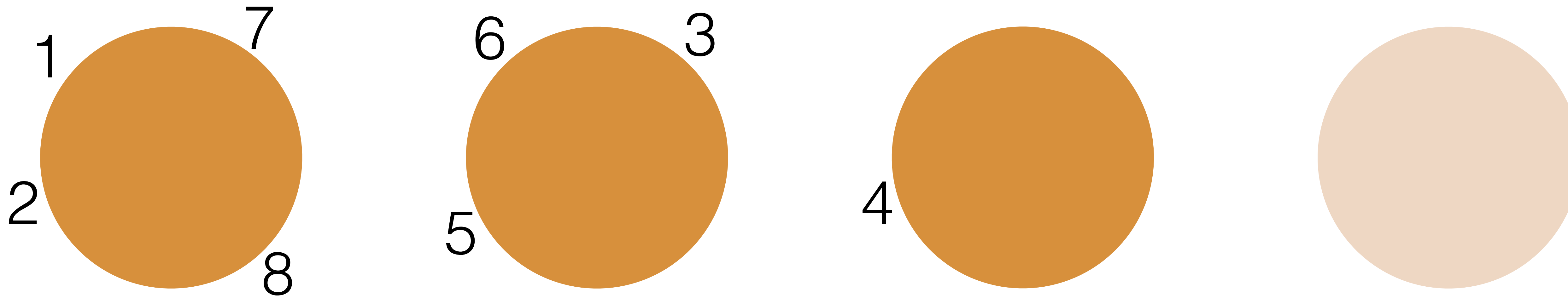

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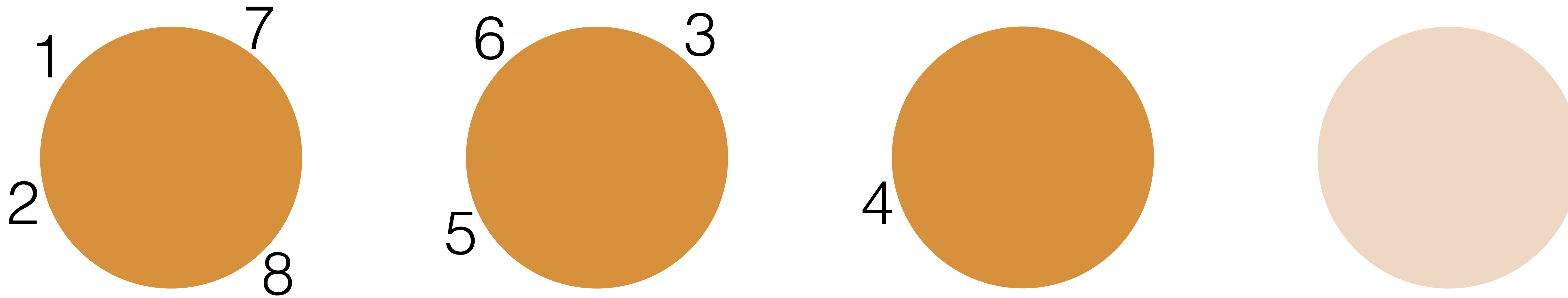

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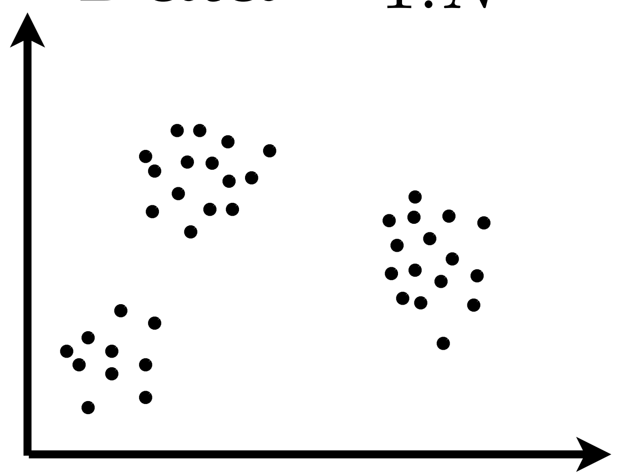
CRP mixture model: inference

CRP mixture model: inference

- Data $x_{1:N}$



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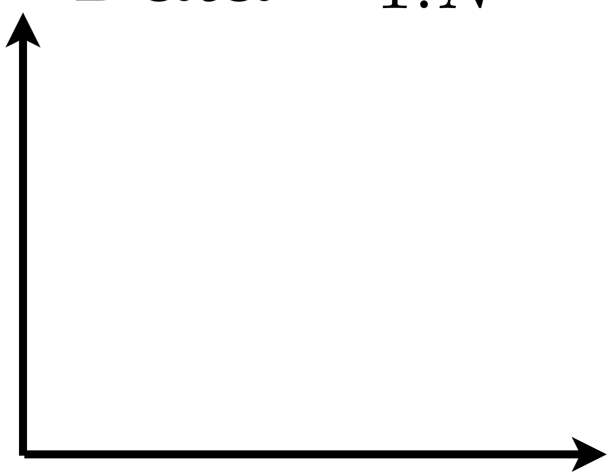

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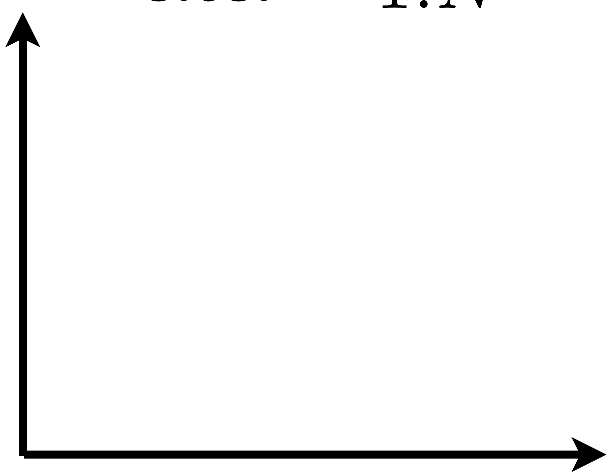

- Generative model

CRP mixture model: inference

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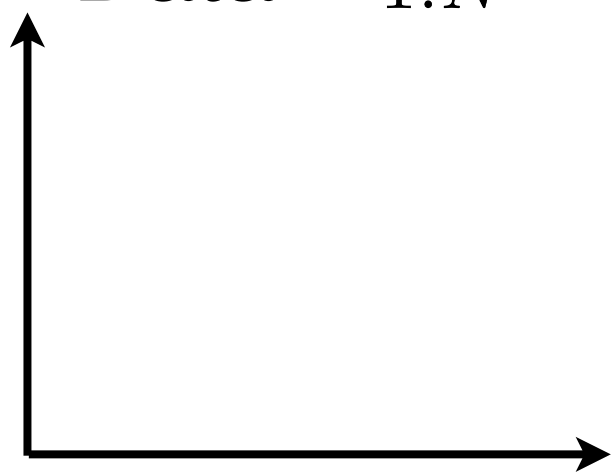

- Generative model
  $\Pi_N \sim \mathrm{CRP}(N, \alpha)$

CRP mixture model: inference

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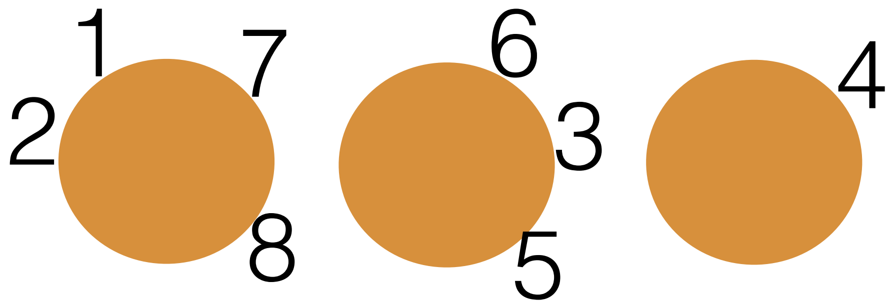

CRP mixture model: inference

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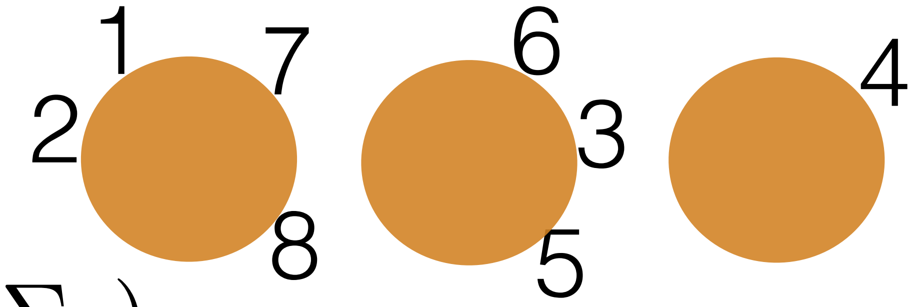

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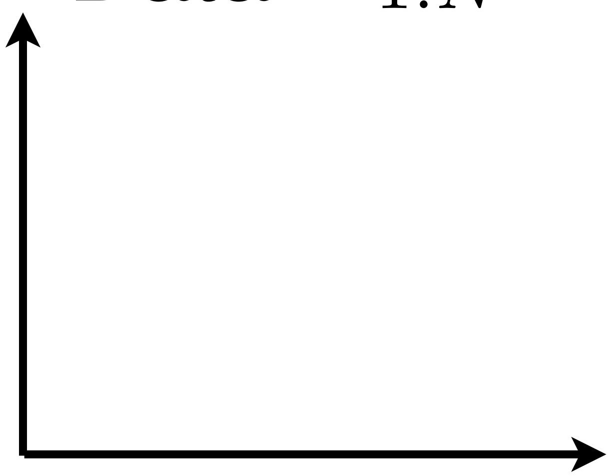

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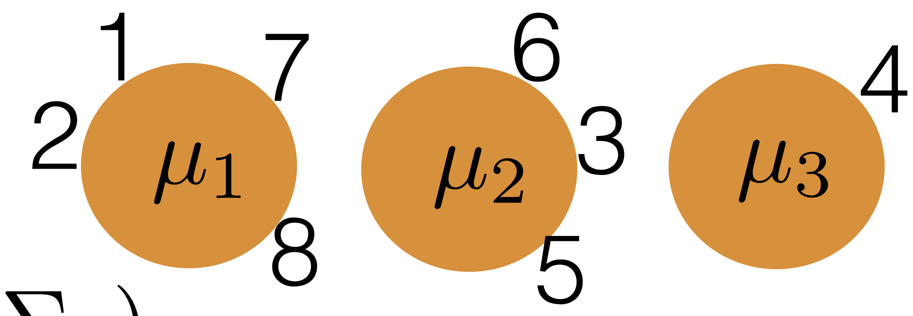

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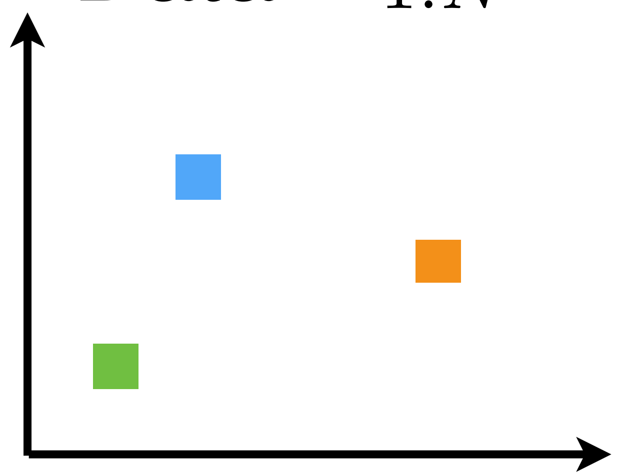

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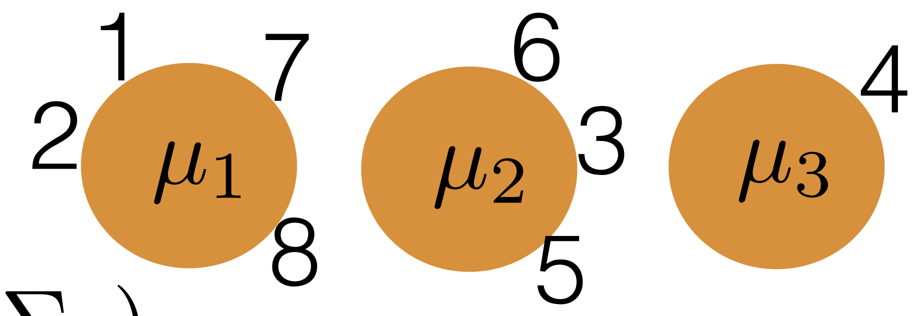

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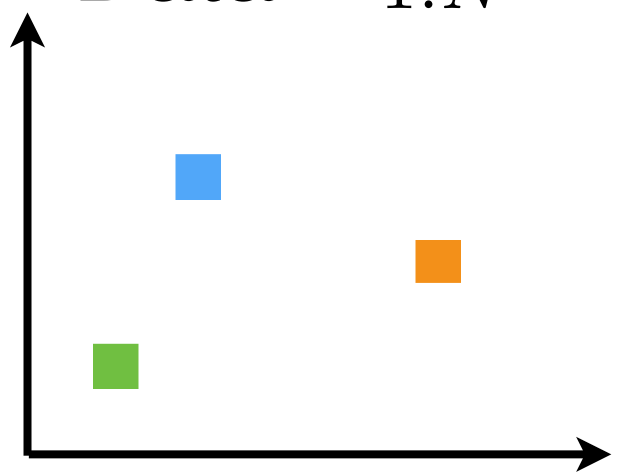

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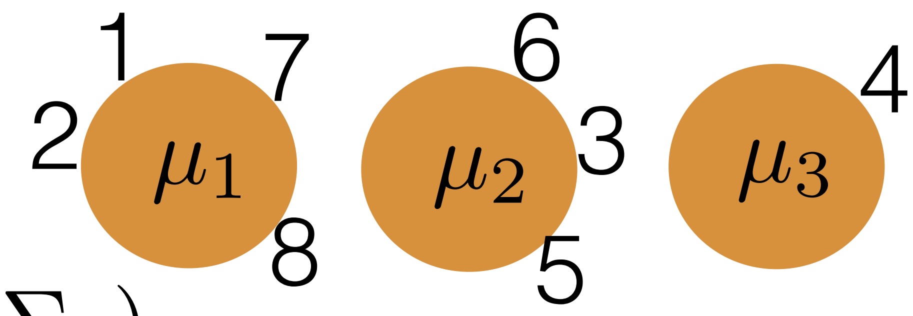

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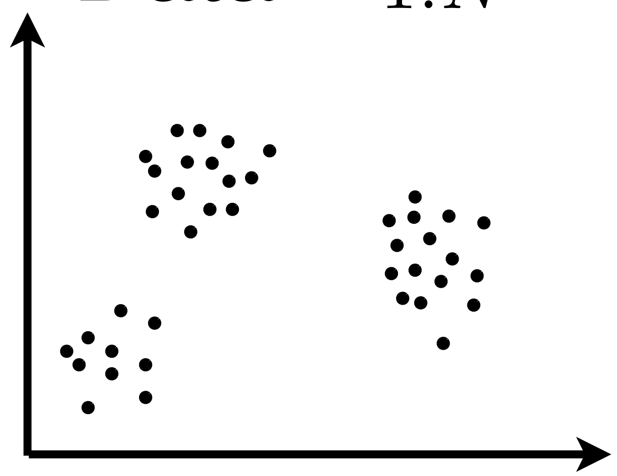

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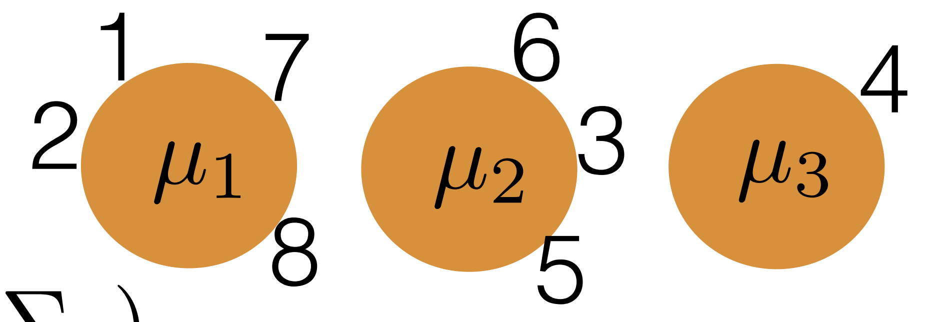

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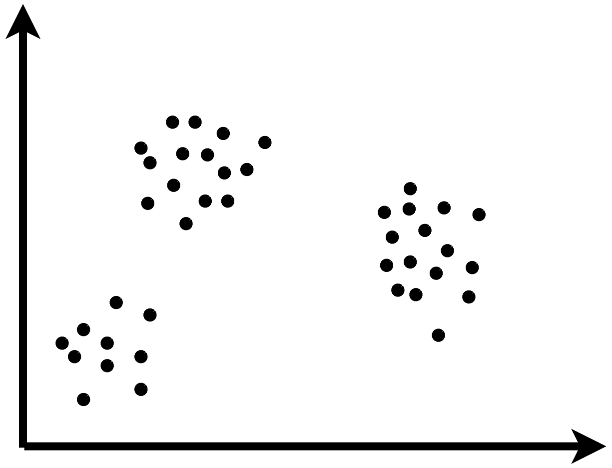
- Data $x_{1:N}$



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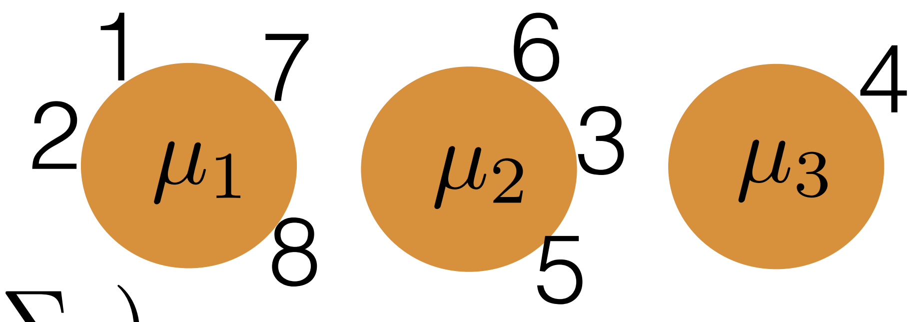
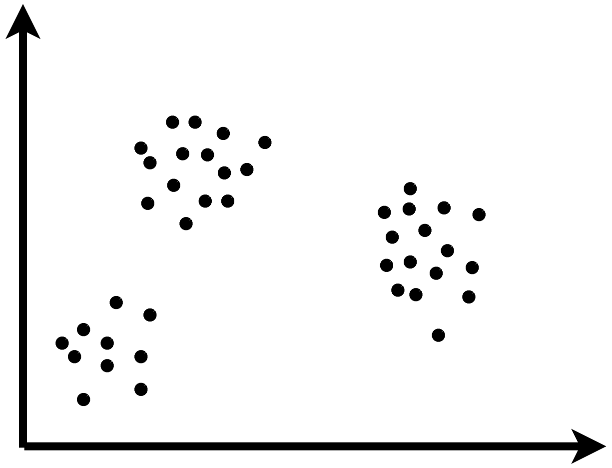


- Want: posterior

CRP mixture model: inference

- Data $x_{1:N}$



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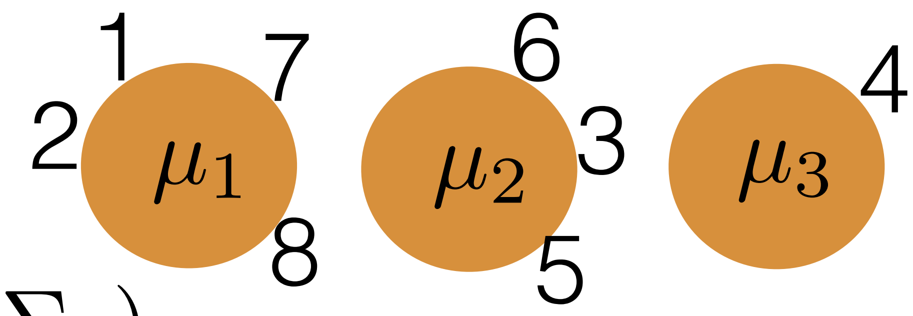
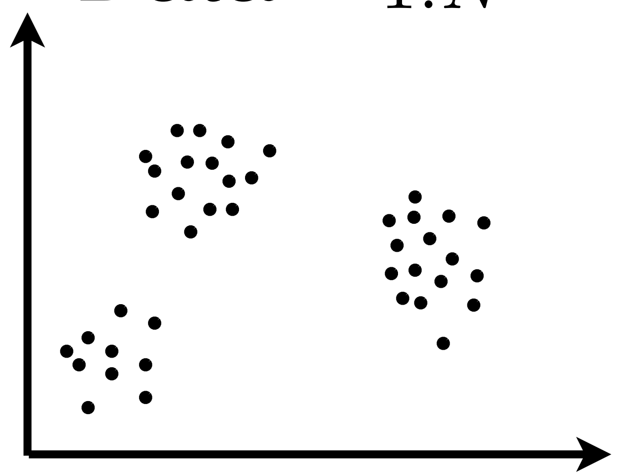


- Want: posterior $p(\Pi_N | x_{1:N})$

CRP mixture model: inference

- Data $x_{1:N}$



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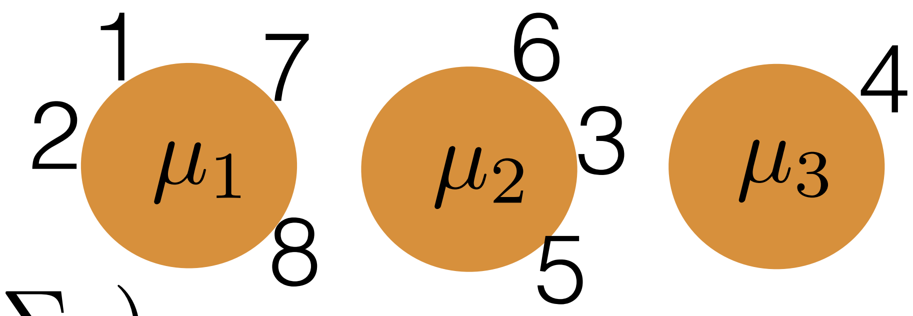
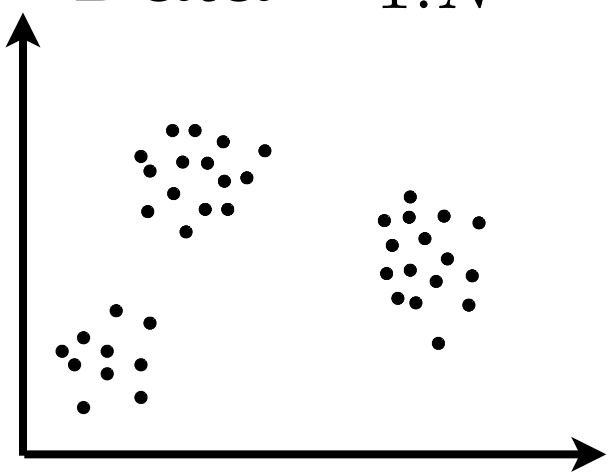


- Want: posterior $p(\Pi_N | x_{1:N})$

- Gibbs sampler:

CRP mixture model: inference

- Data $x_{1:N}$



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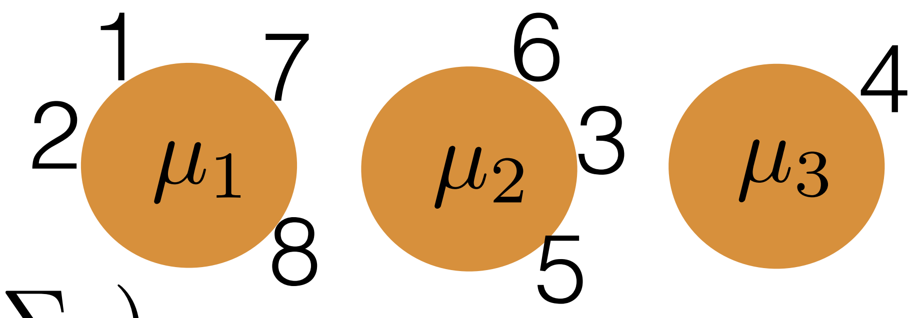

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$$p(\Pi_N | \Pi_{N,\text{-}n}, x)$$

CRP mixture model: inference

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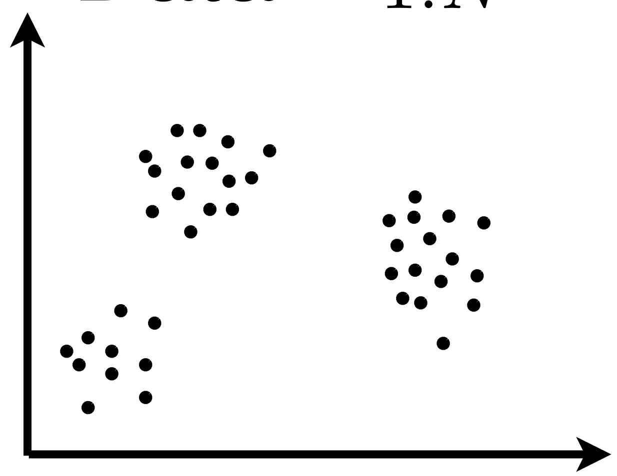

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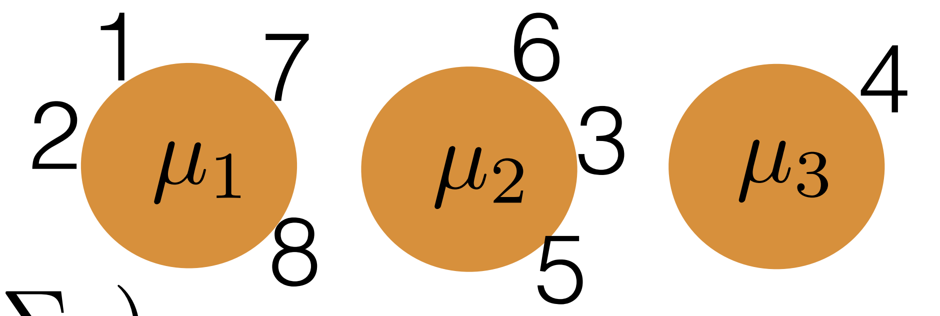

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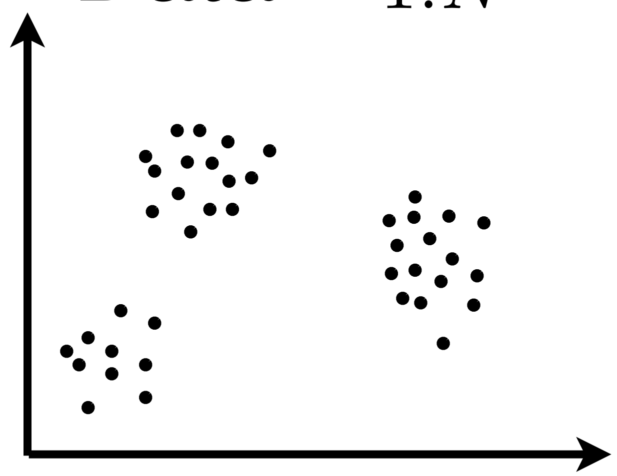
CRP mixture model: inference

- Data $x_{1:N}$



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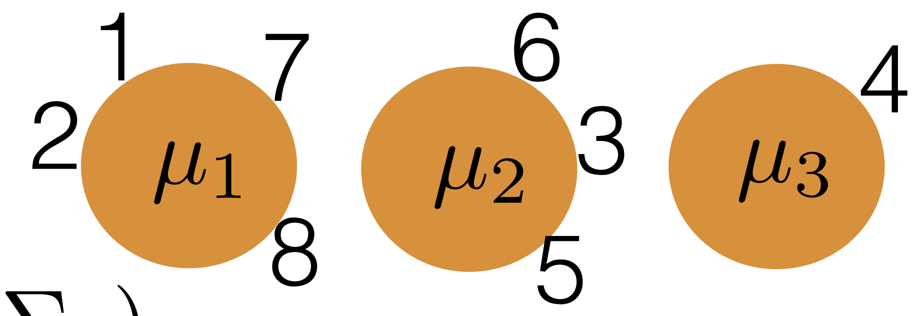

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CRP mixture model: inference

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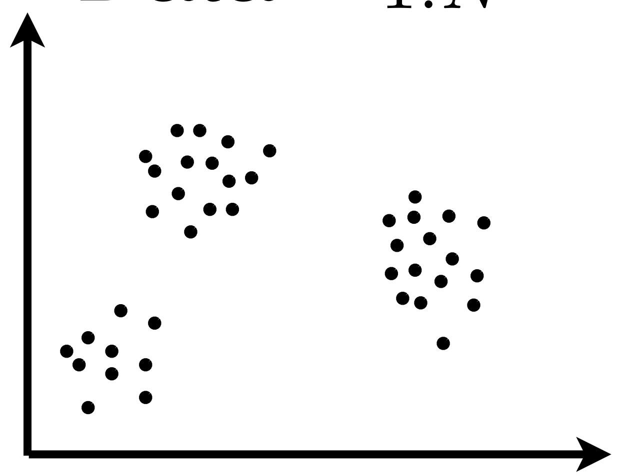

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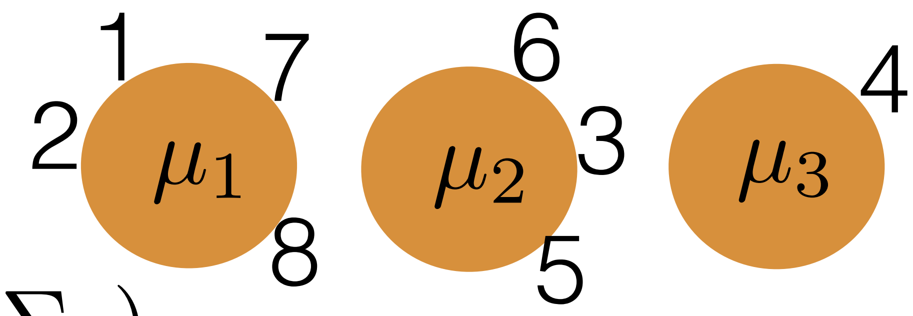

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CRP mixture model: inference

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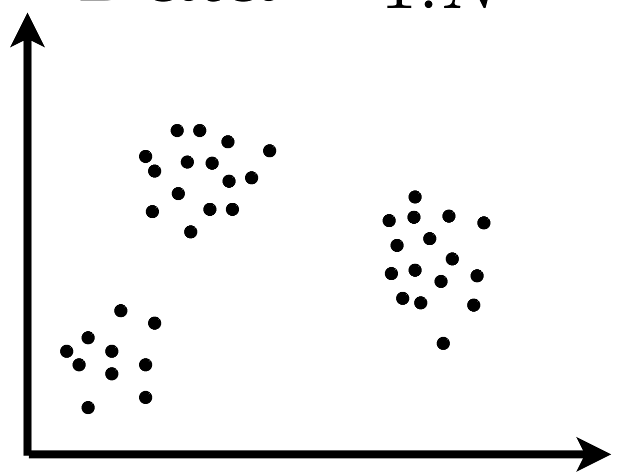

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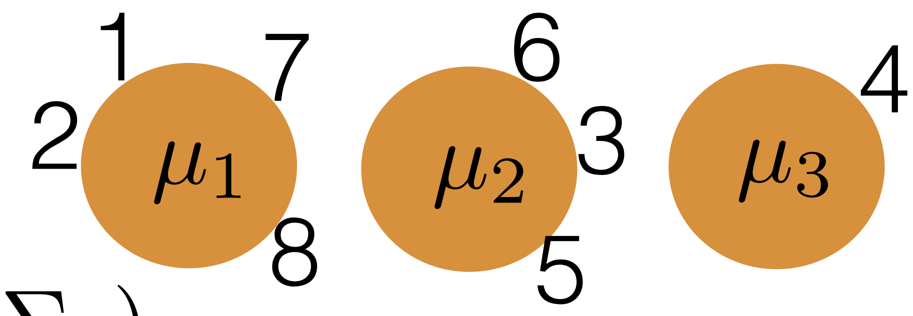

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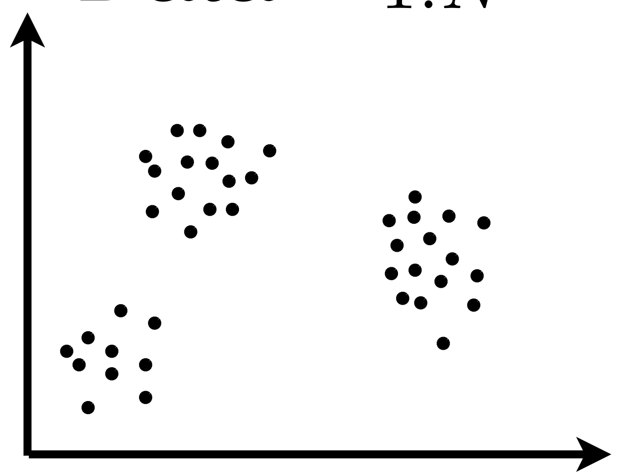

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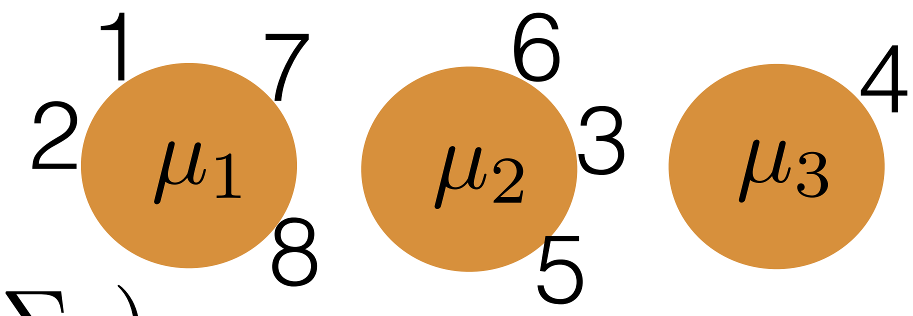

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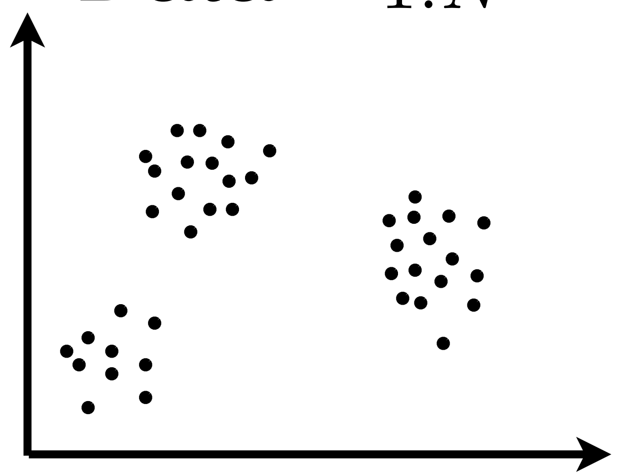

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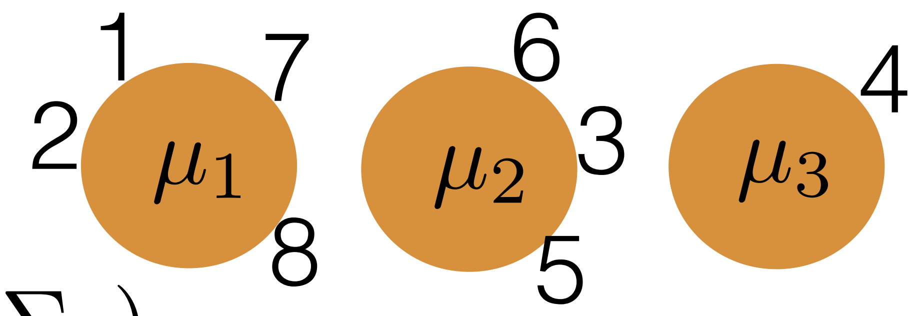

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CRP mixture model: inference

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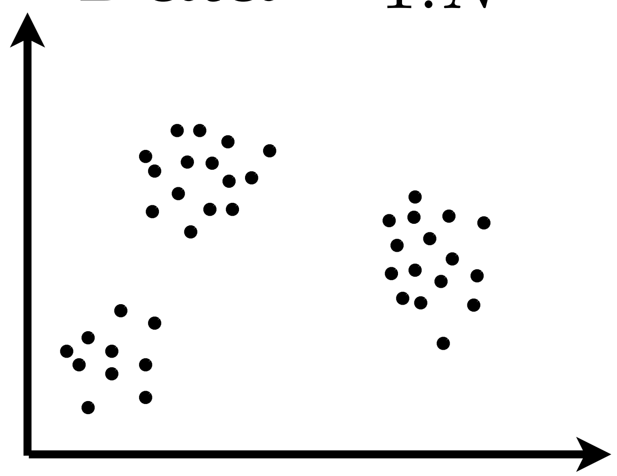

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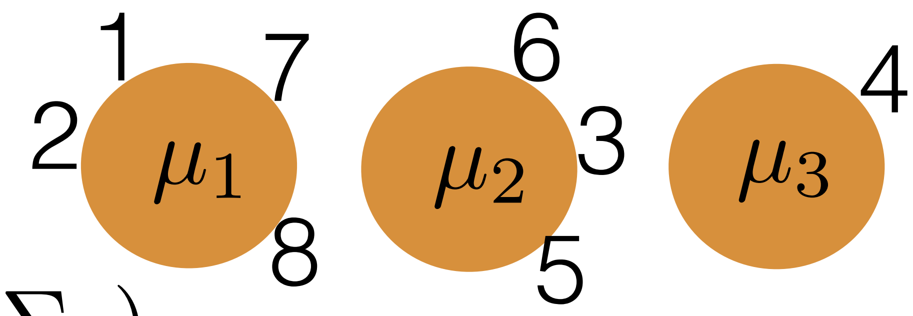

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- Gibbs sampler:

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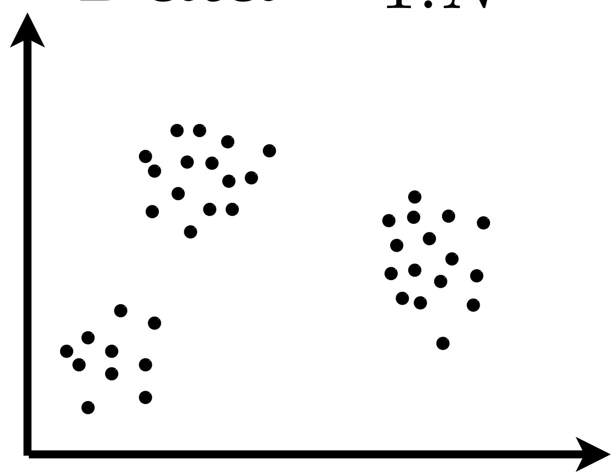
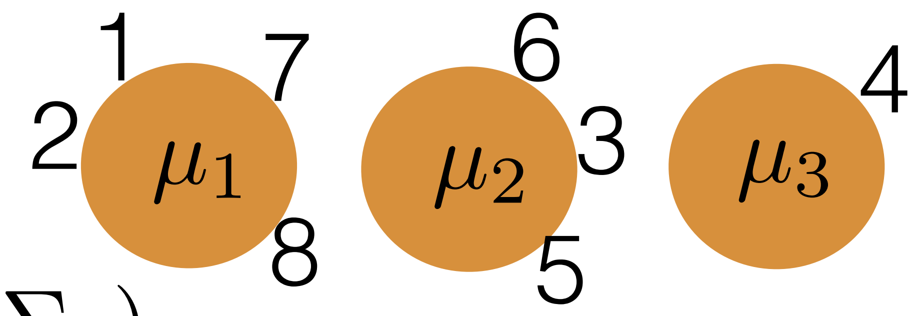
CRP mixture model: inference

- Data $x_{1:N}$



- Generative model



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CRP mixture model: inference

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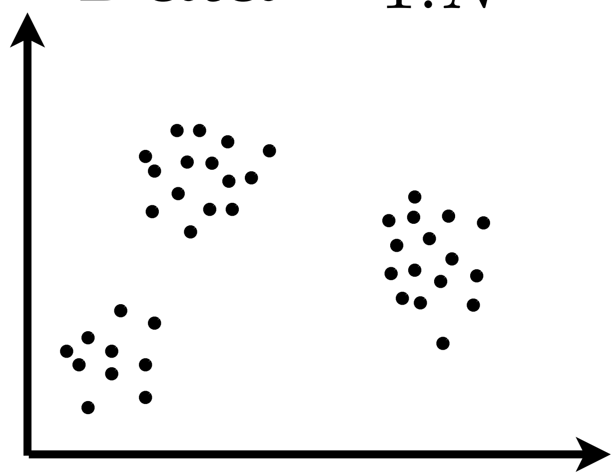

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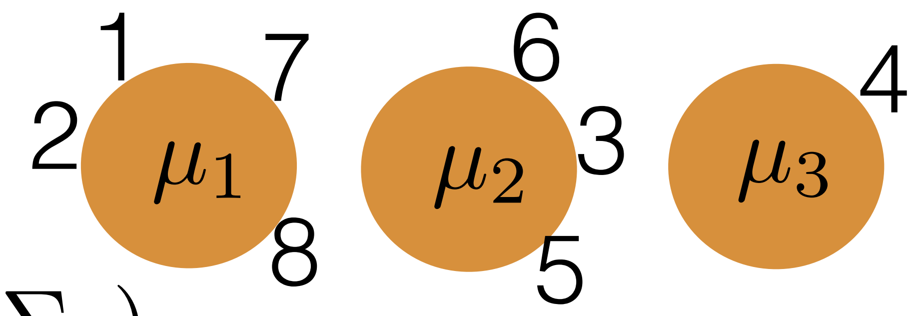

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[MacEachern 1994; Neal 1992; Neal 2000]

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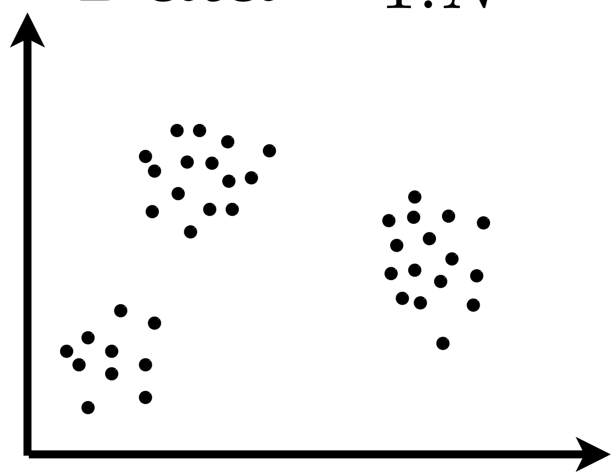

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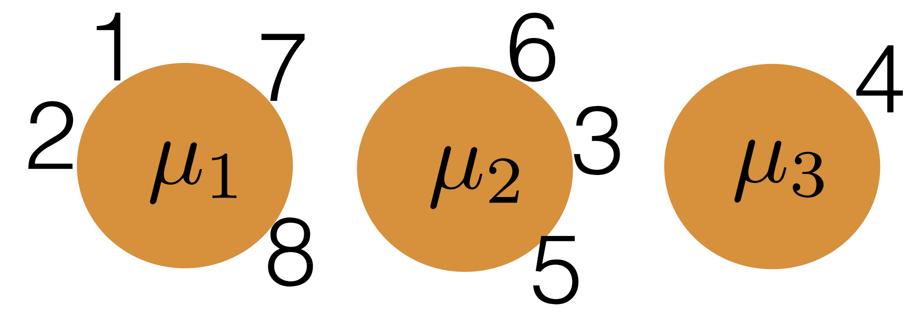

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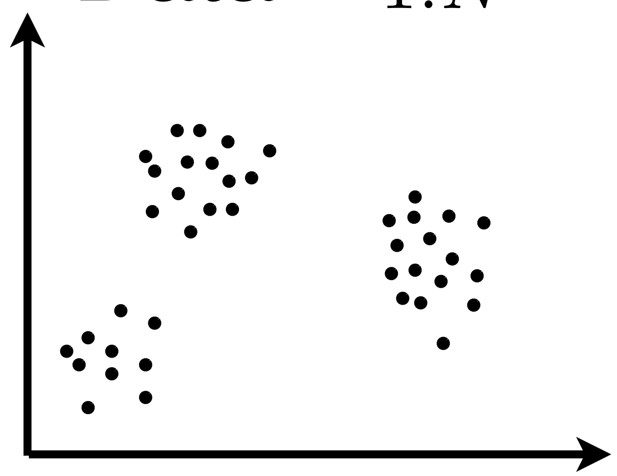

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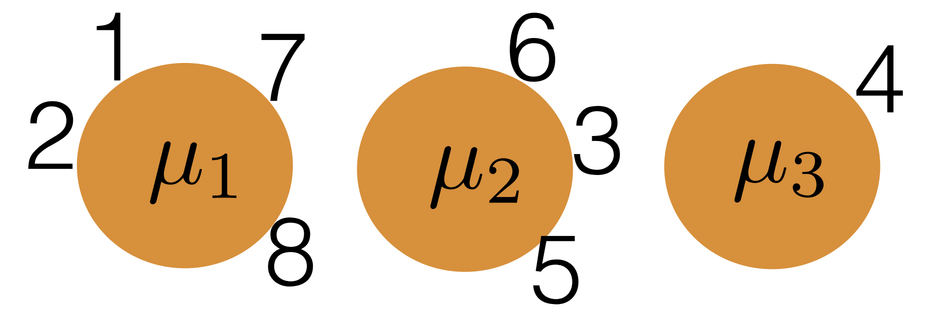

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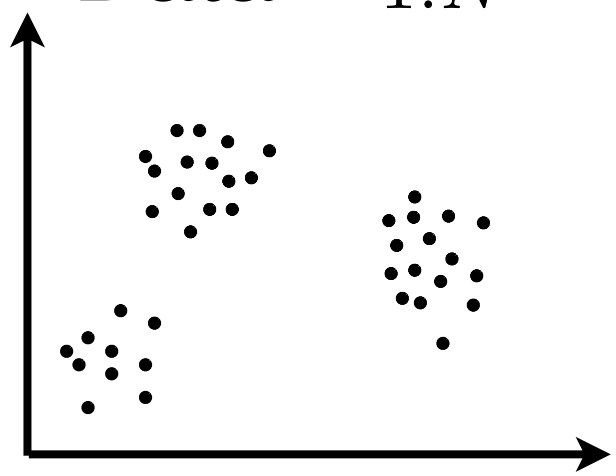

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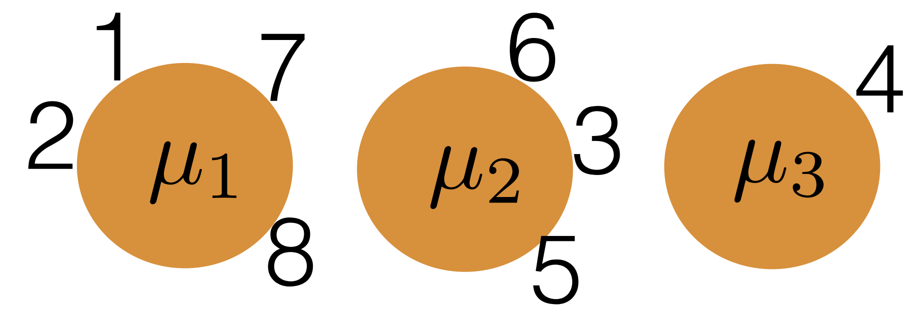

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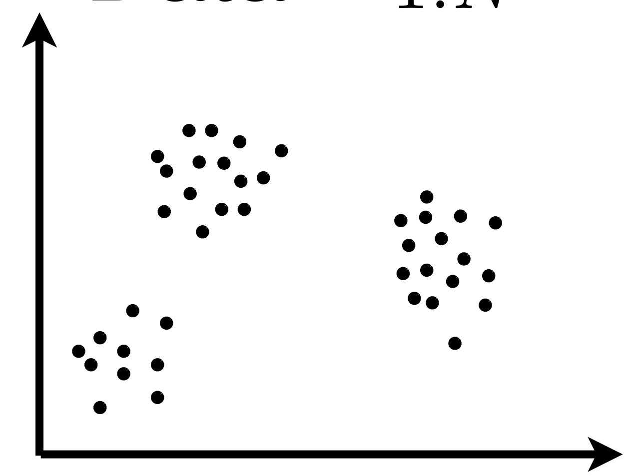

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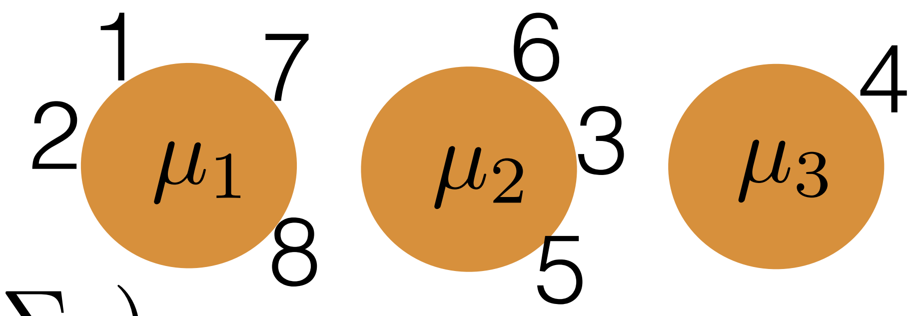

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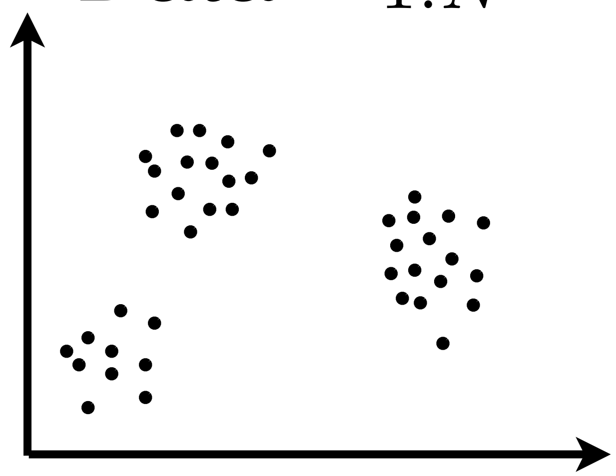

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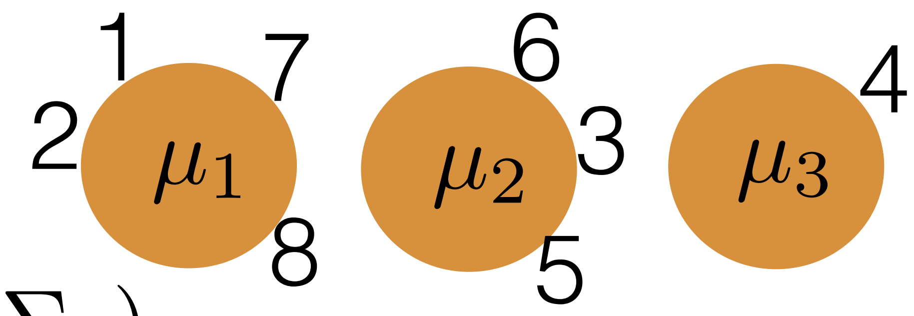

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[demo]

[MacEachern 1994; Neal 1992; Neal 2000]

CRP mixture model exercises

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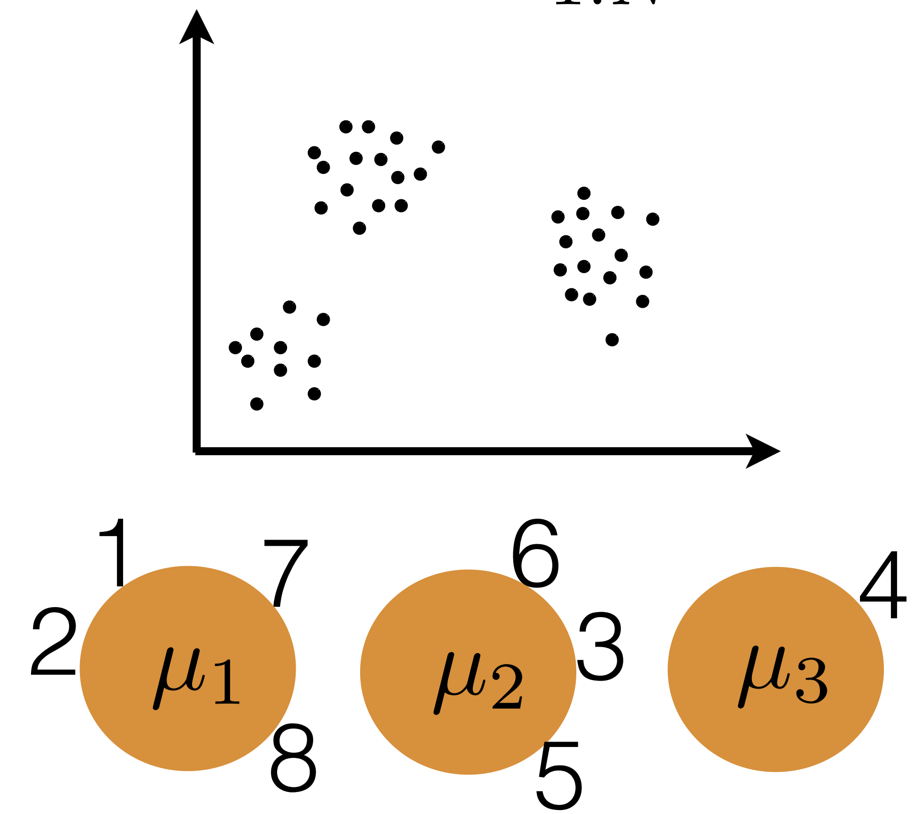

CRP mixture model exercises

- Code a CRP mixture model simulator

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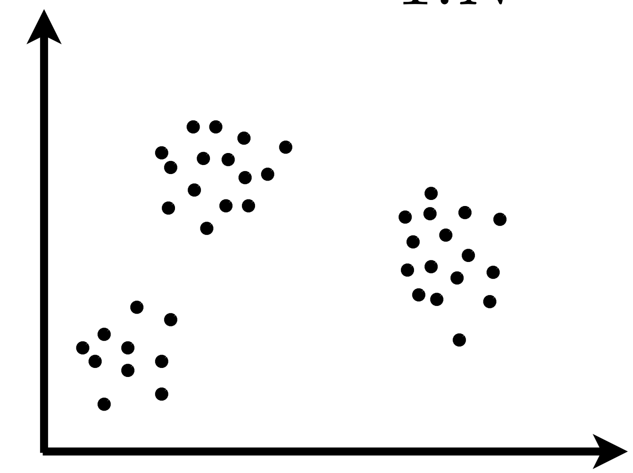

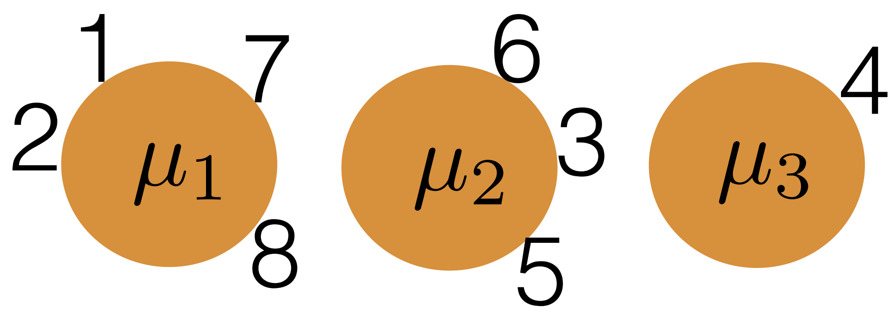

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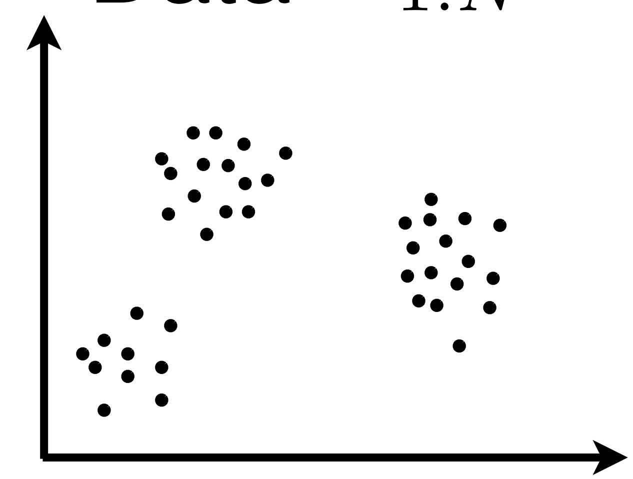

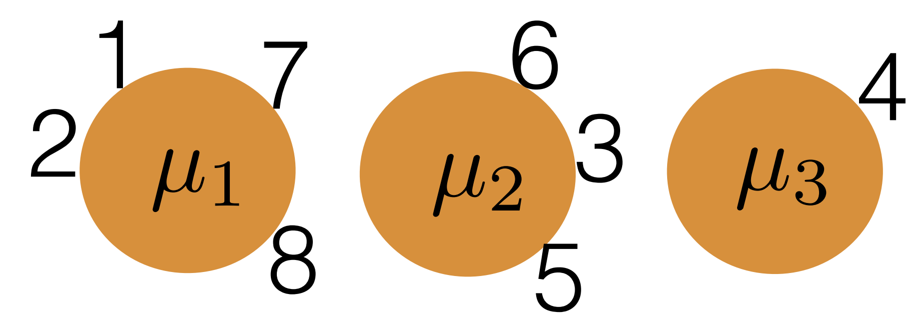

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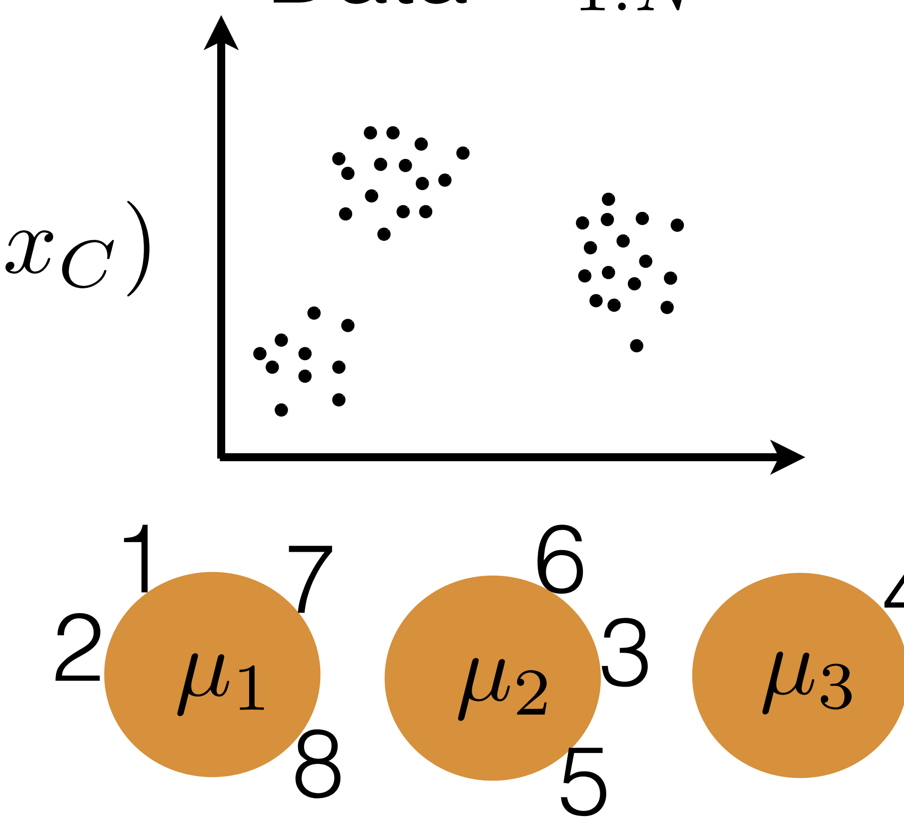

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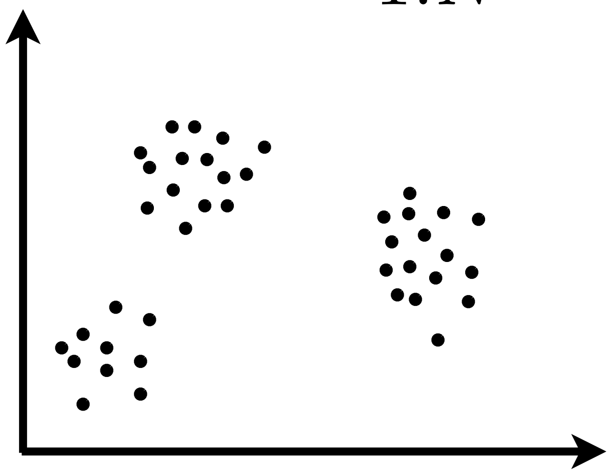

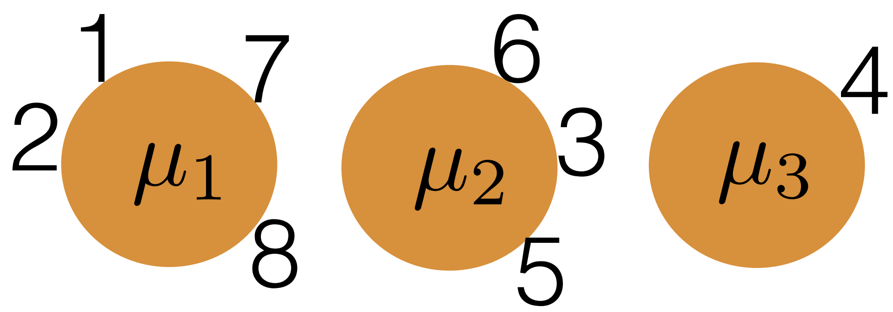

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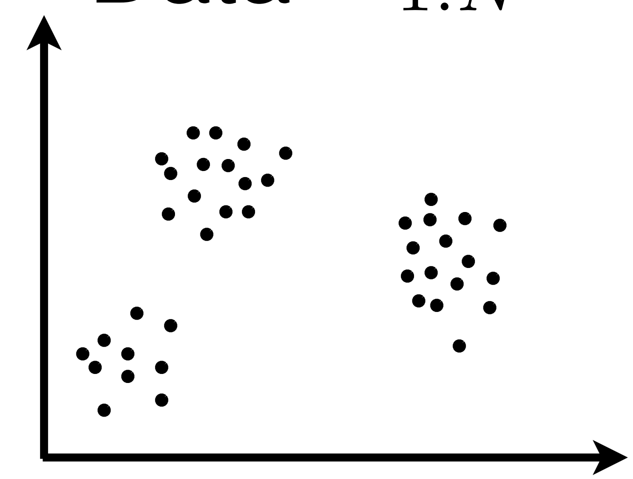

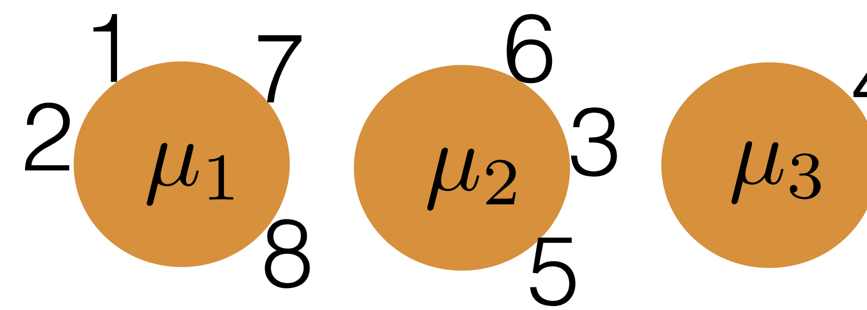

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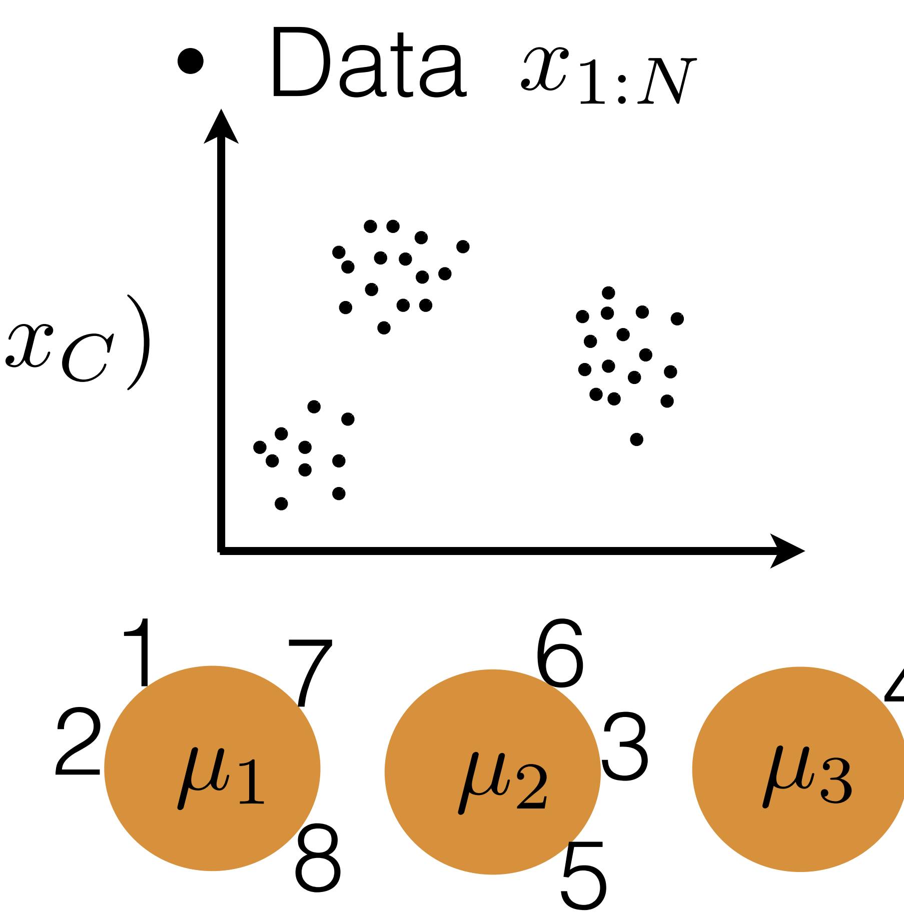

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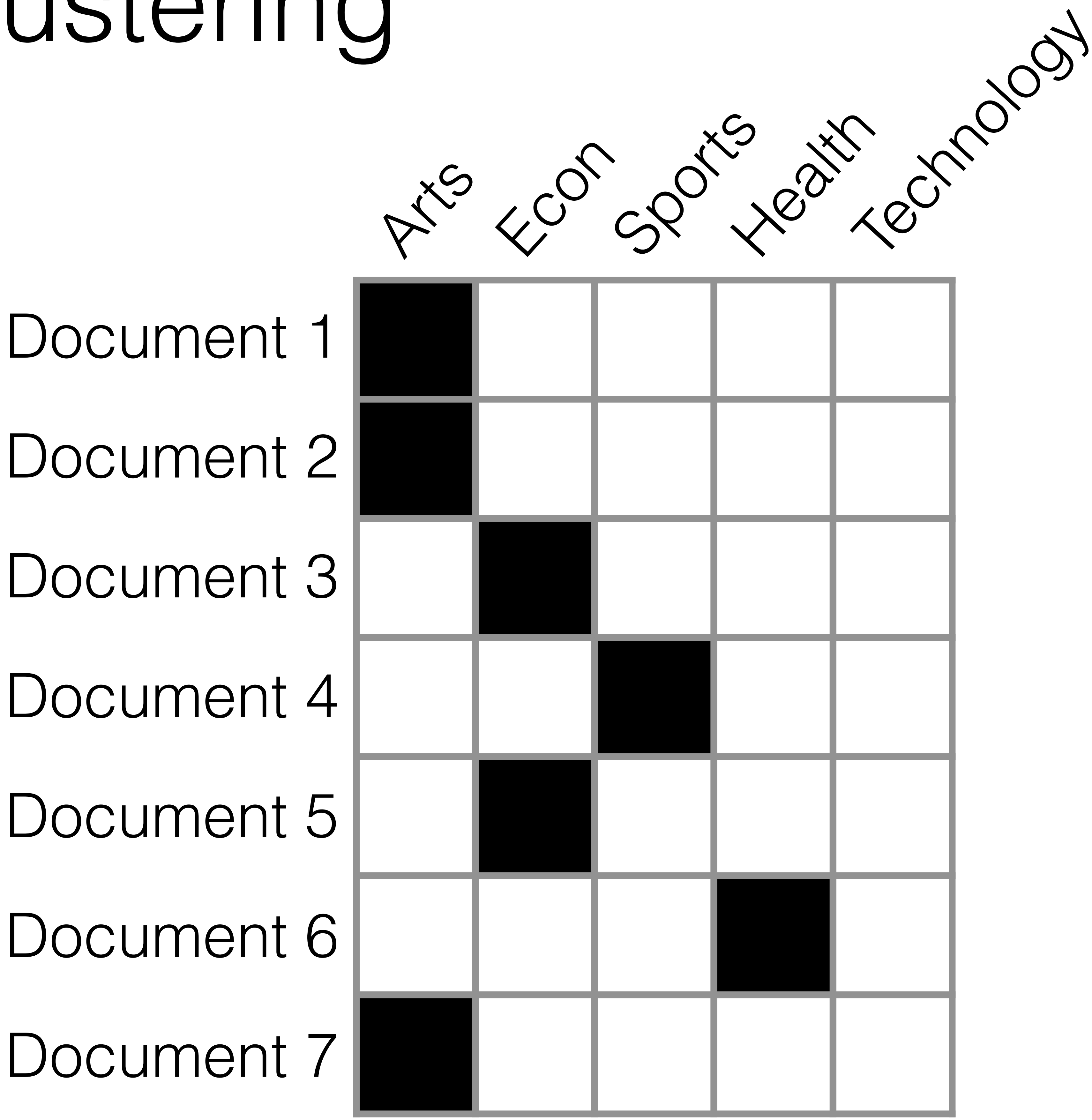
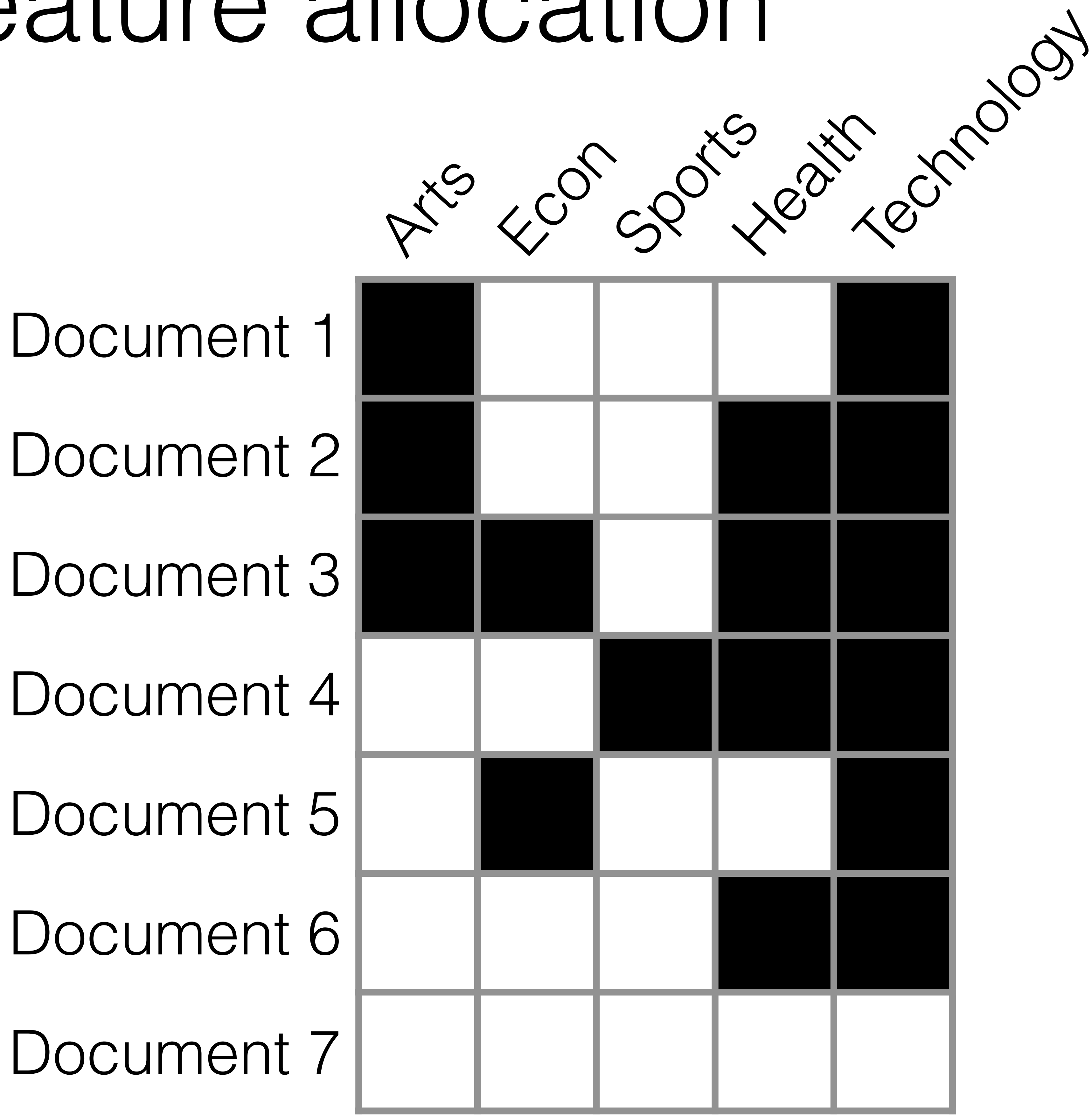
- Read Broderick, Jordan, Pitman 2013 "Cluster and feature modeling […]" for more background/extensions

[Tiia Monto, https://commons.wikimedia.org/wiki/File:Trail_and_mountain.jpg]

Clustering


Arts
Econ
Sports
Health
Technology
Document 1
Document 2
Document 3
Document 4
Document 5
Document 6
Document 7

Feature allocation



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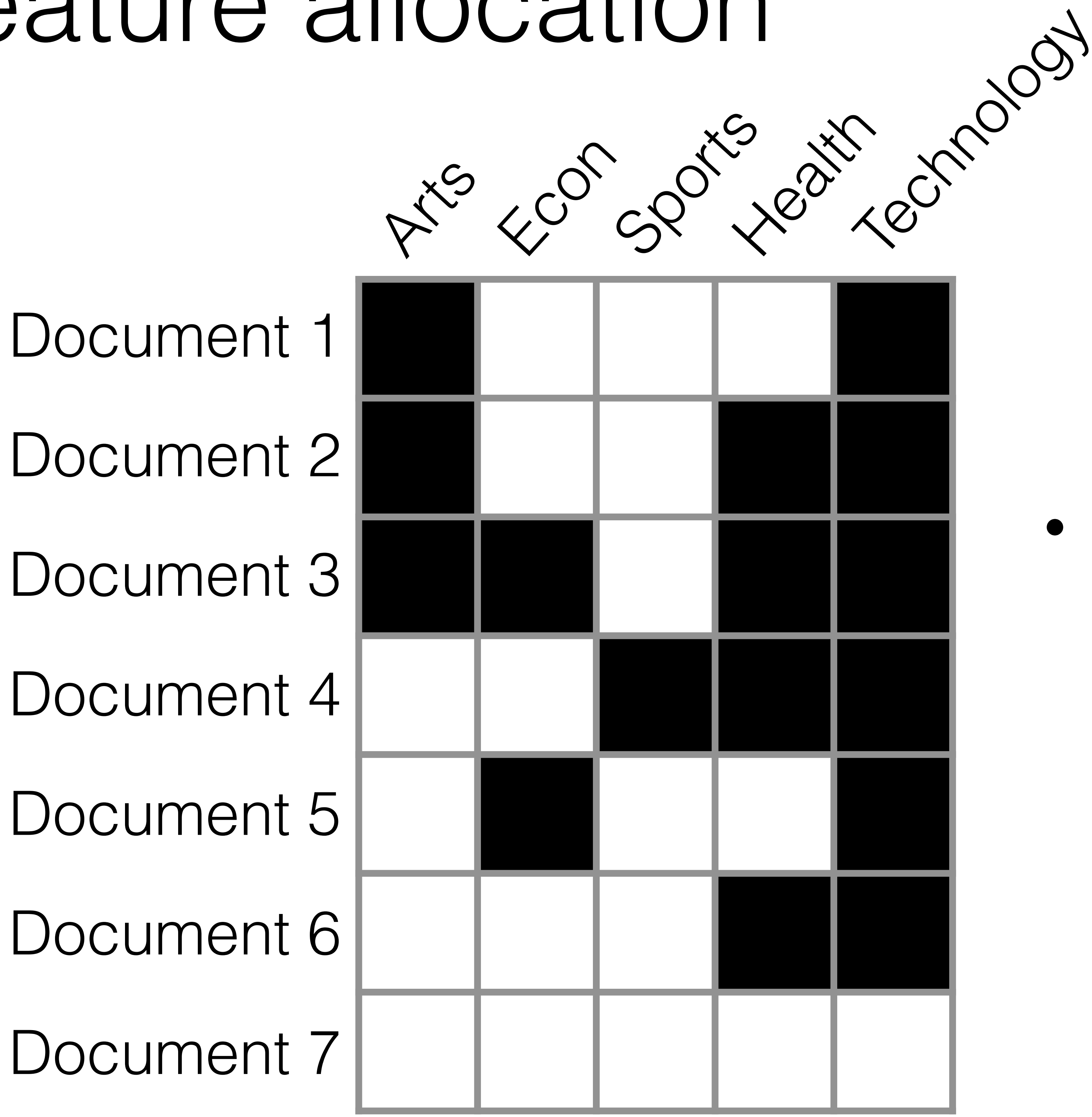

- Indian buffet process

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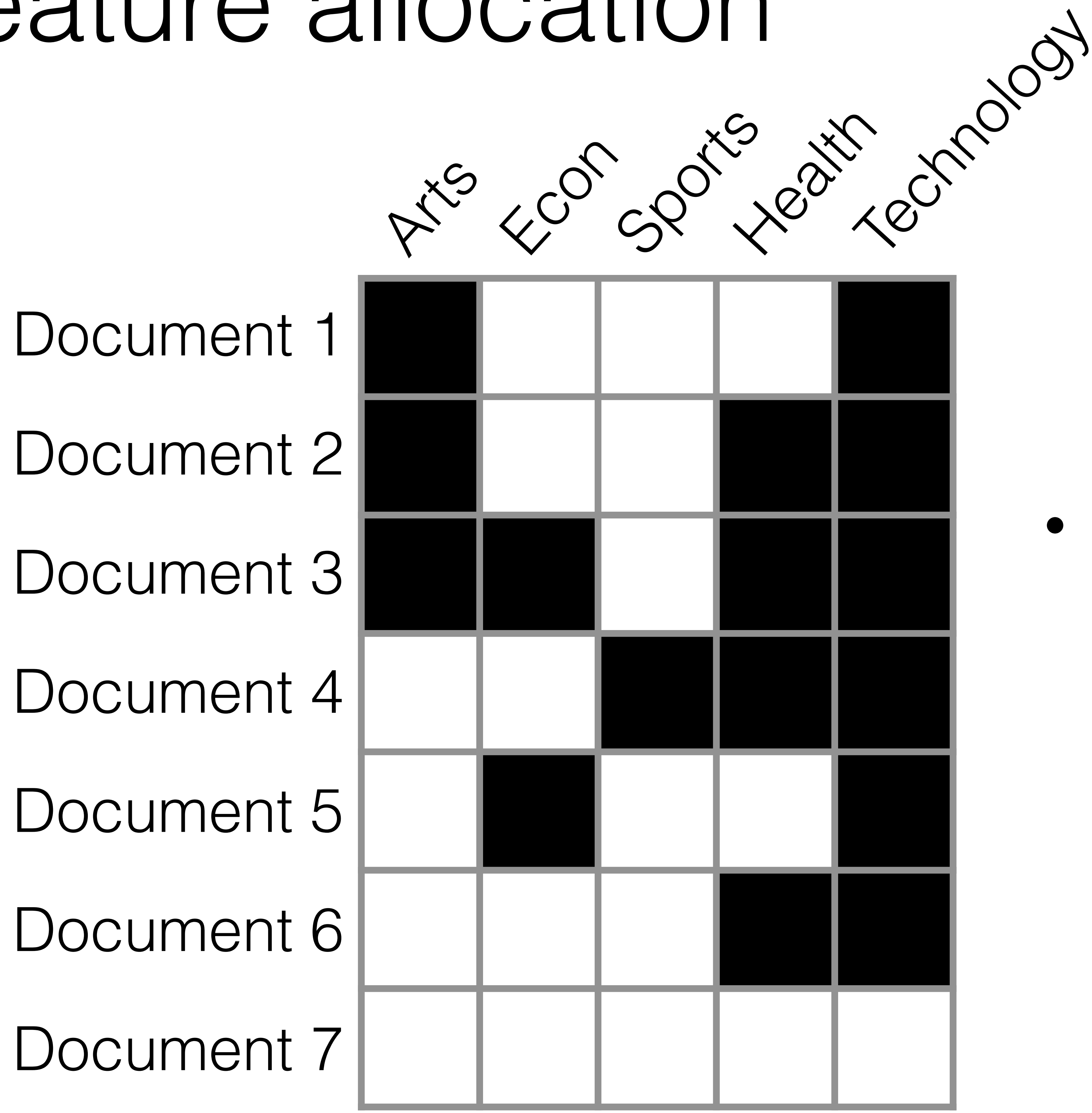

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[Griffiths, Ghahramani 2005

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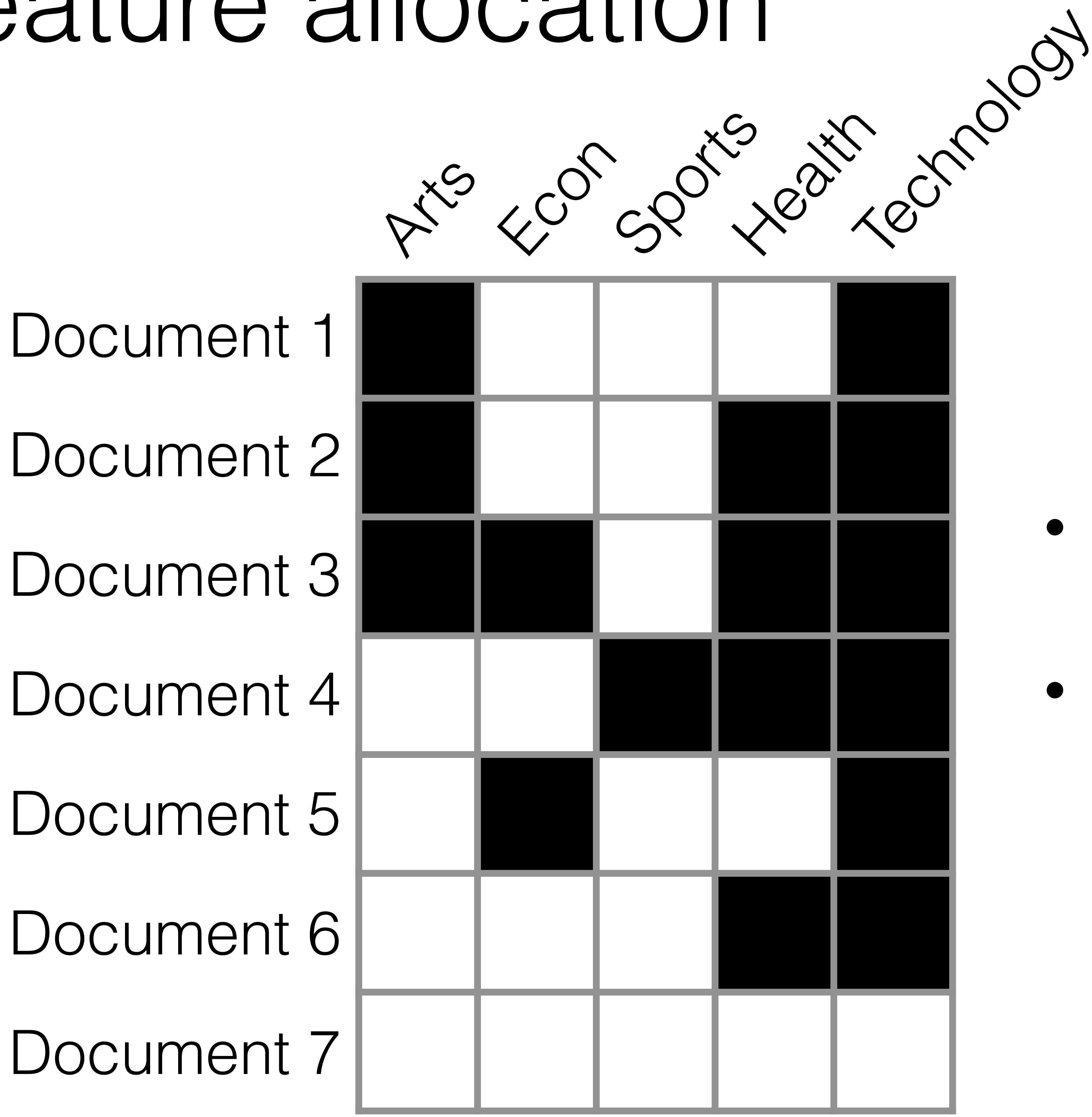

- Indian buffet process
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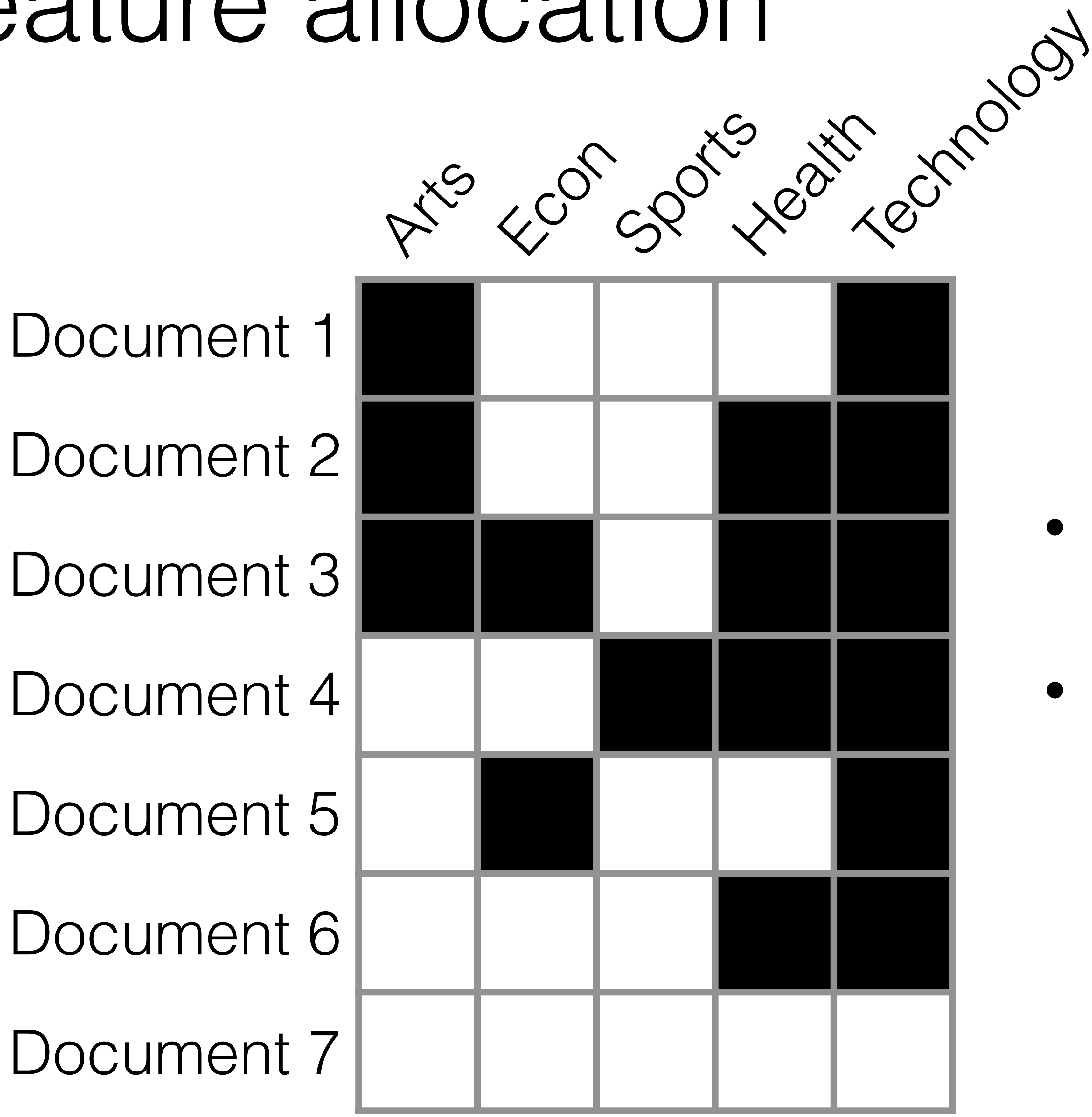

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[Griffiths, Ghahramani 2005, Hjort 1990

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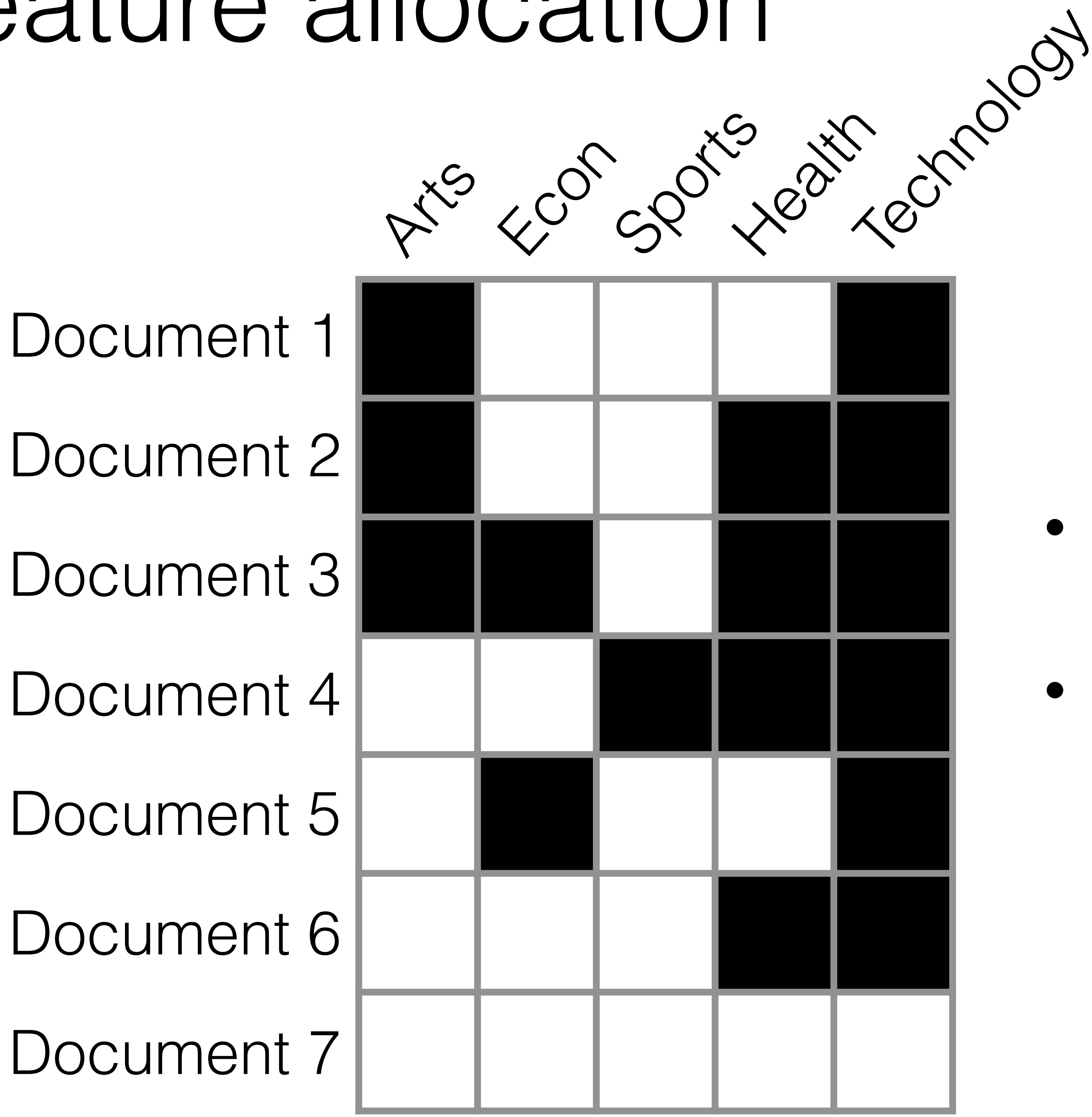

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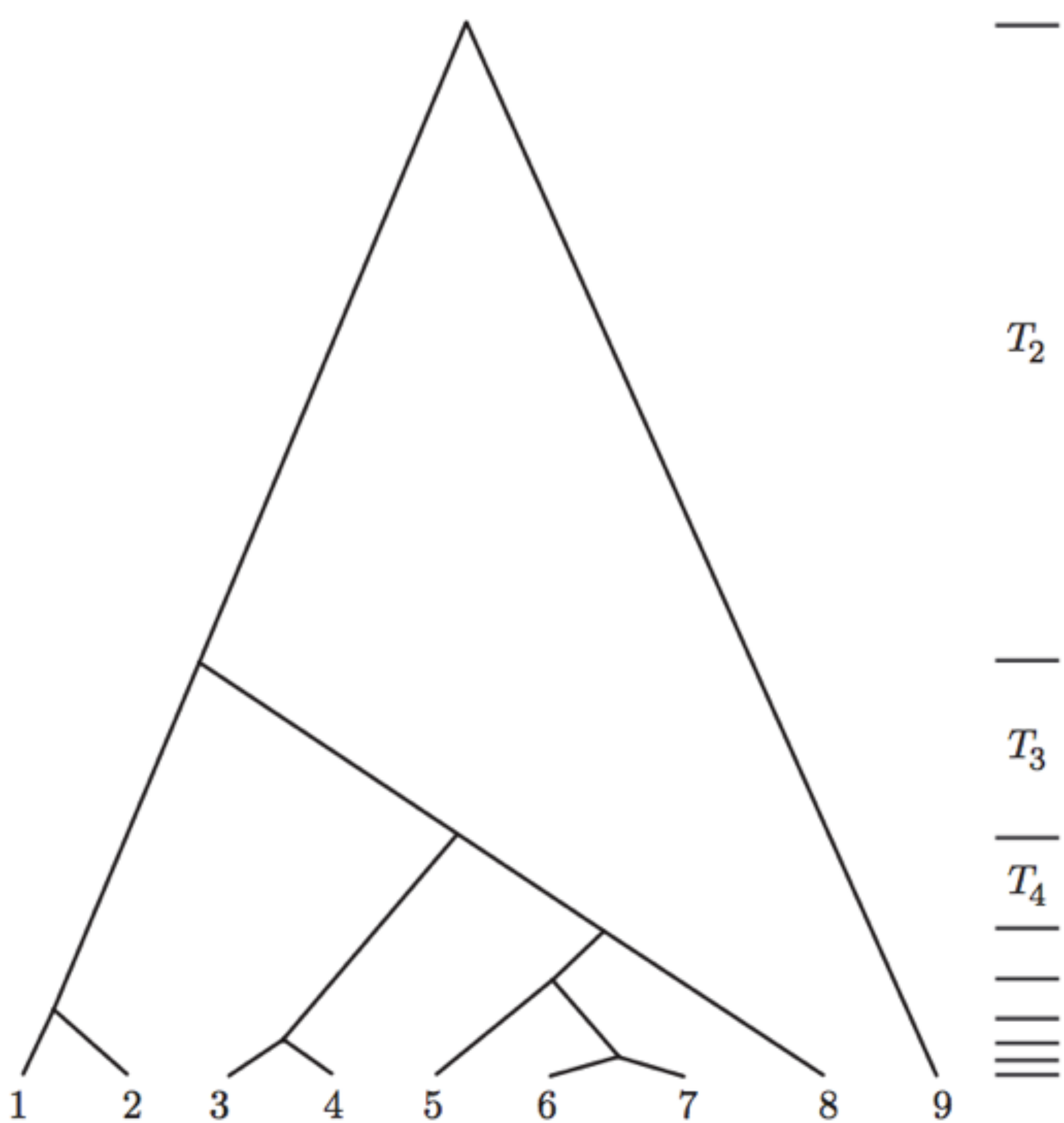
[Griffiths, Ghahramani 2005, Hjort 1990, Kim 1999, Thibaux, Jordan 2007, Broderick, Jordan, Pitman 2013]

Genealogy, trees, beyond trees



[Wakeley 2008]

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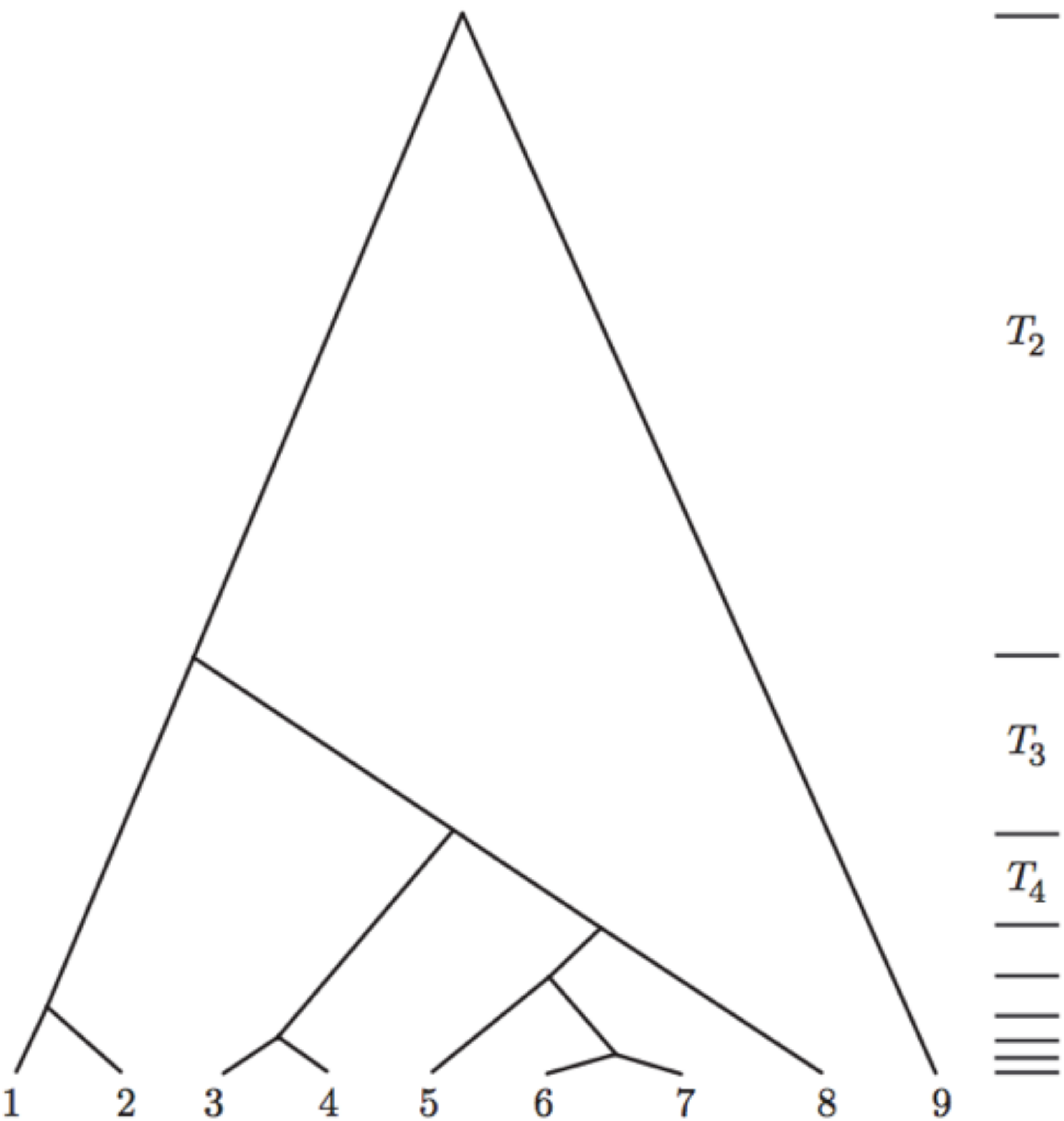

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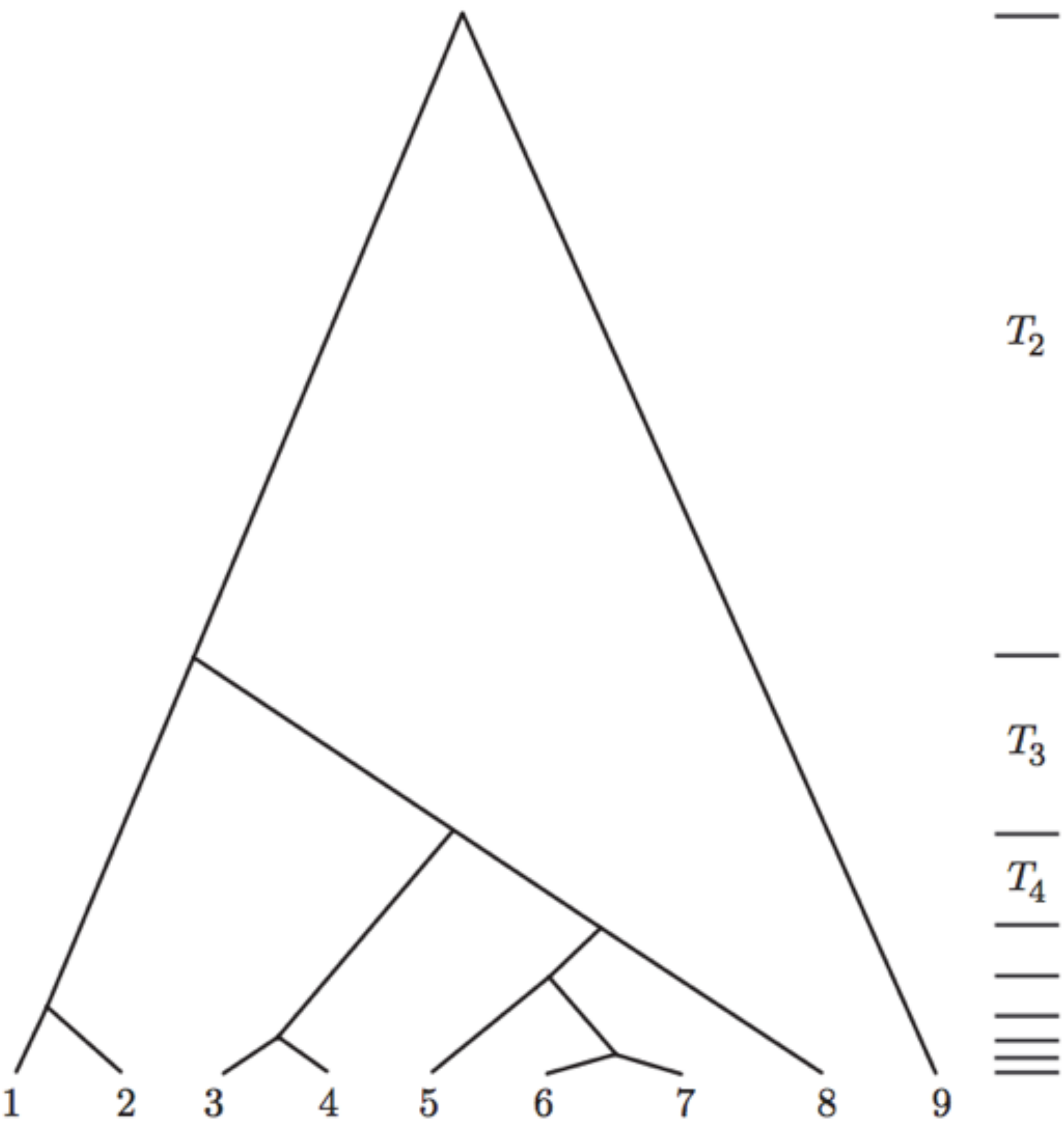

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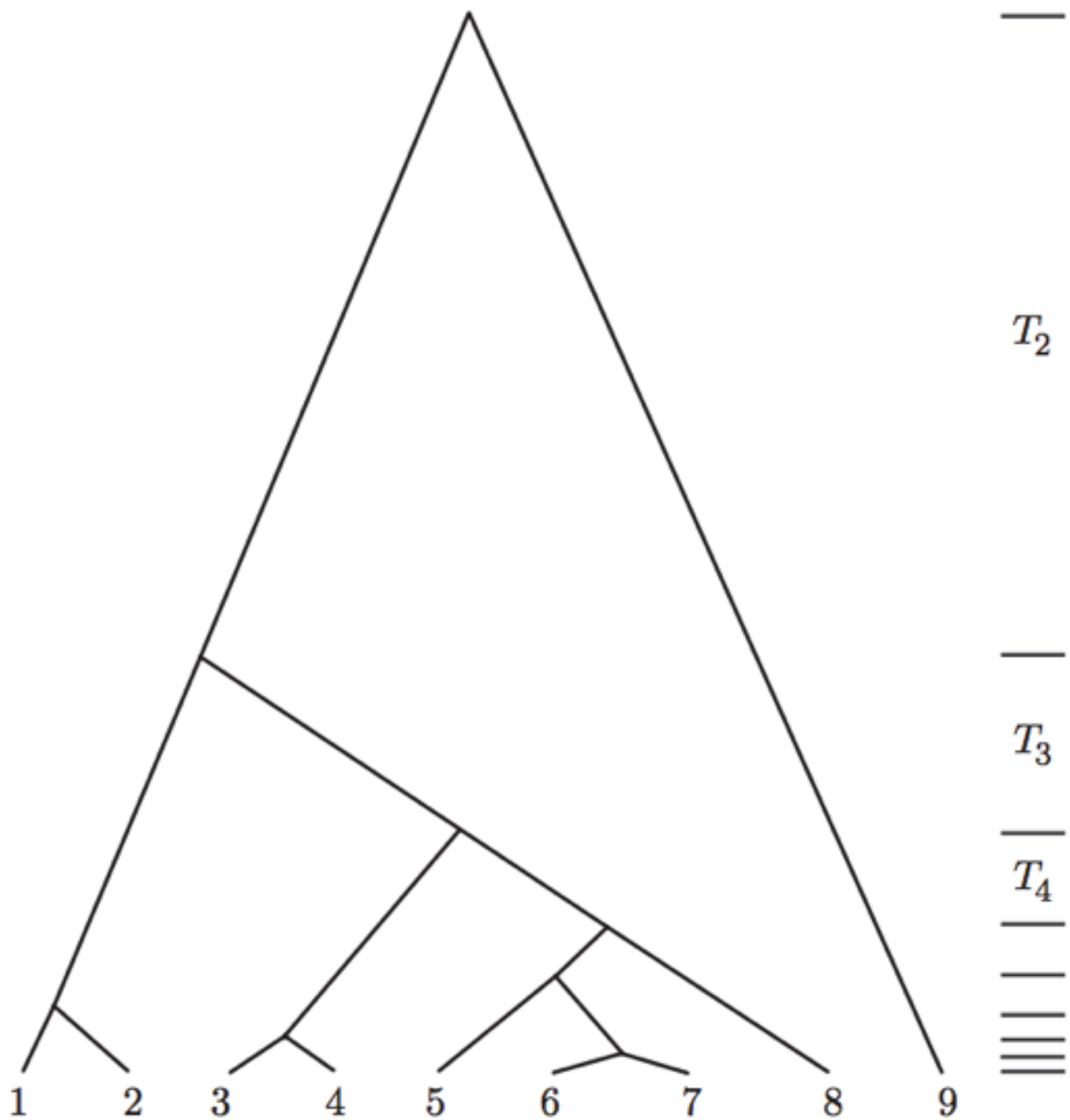

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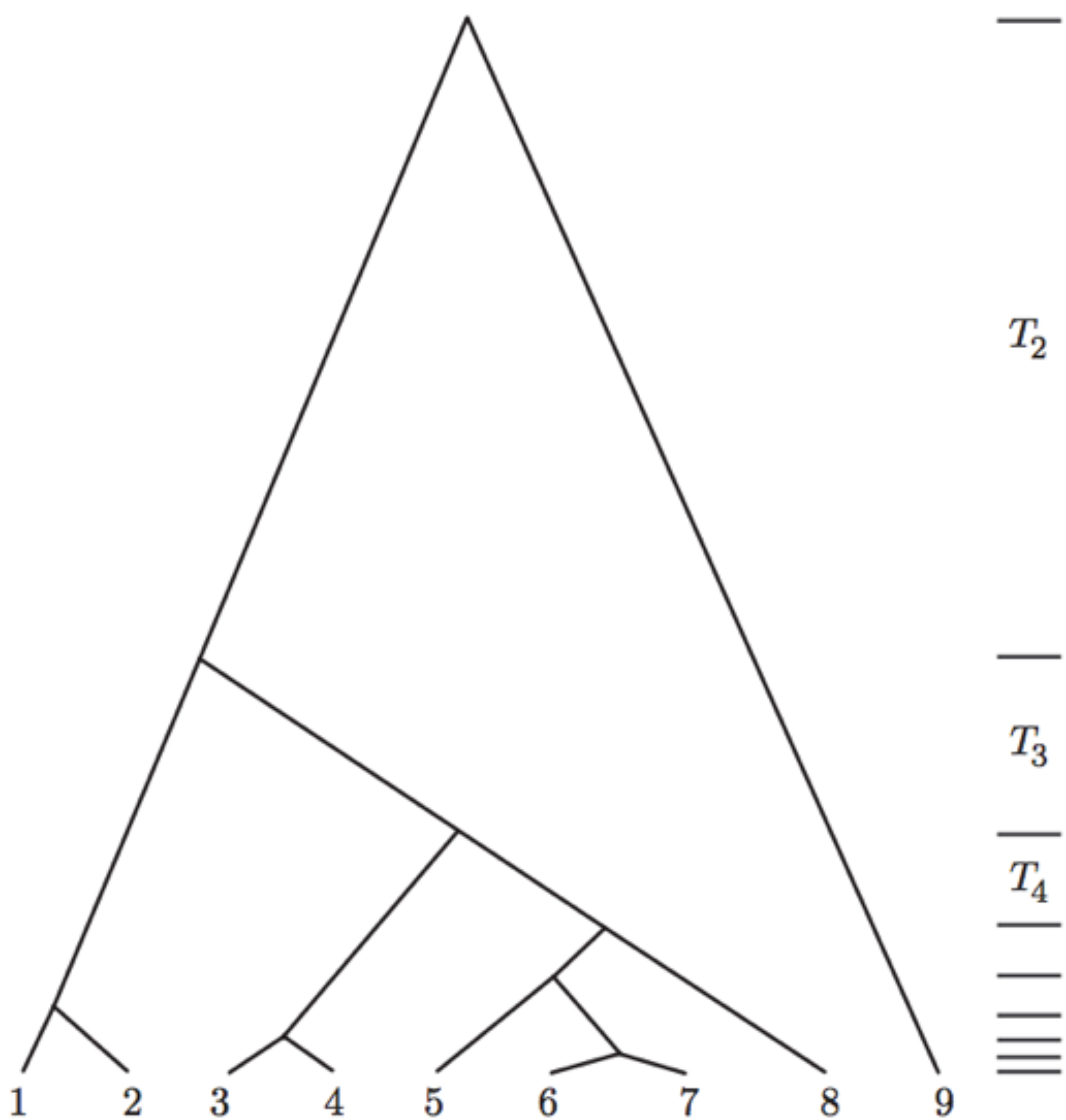

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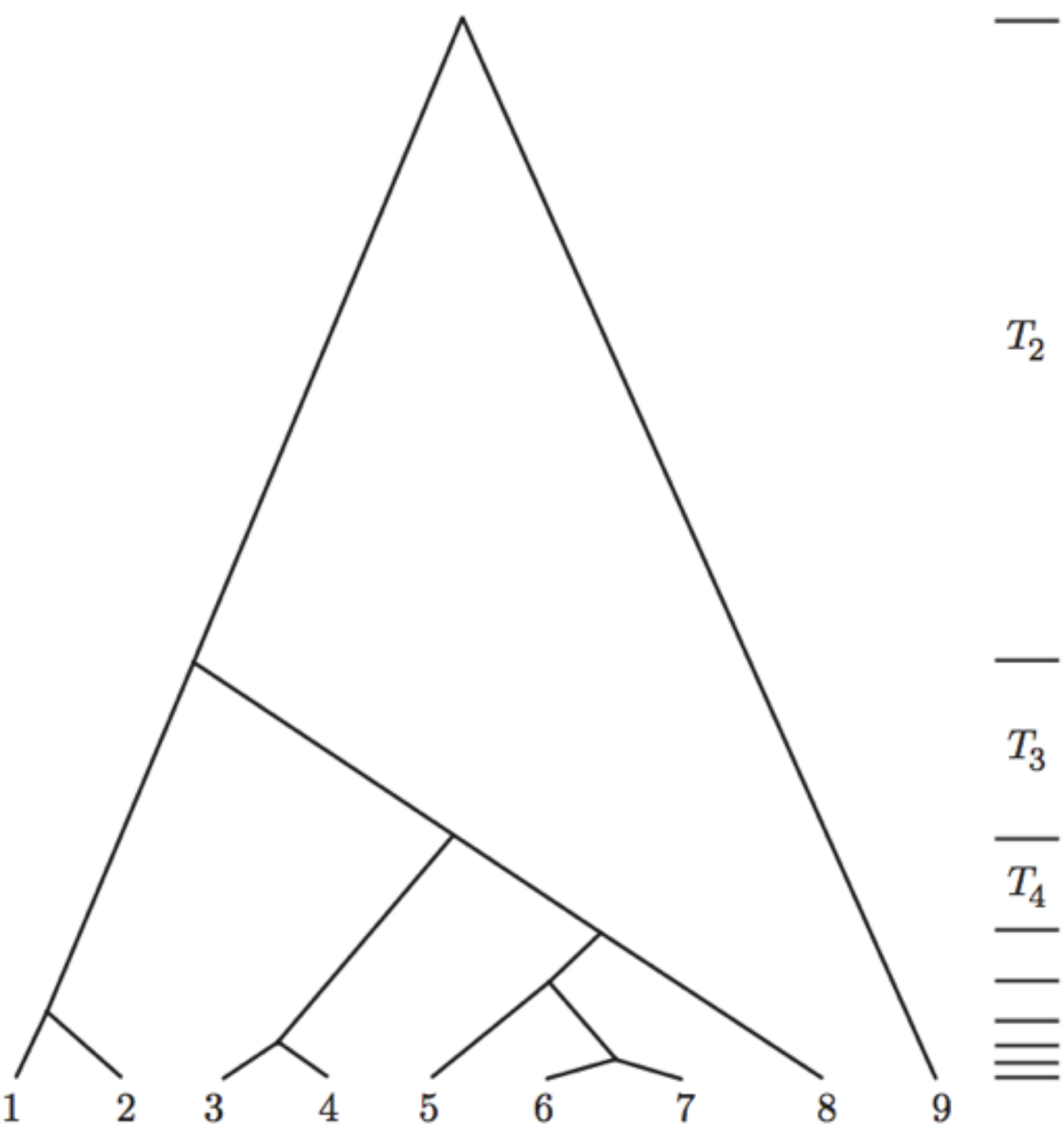

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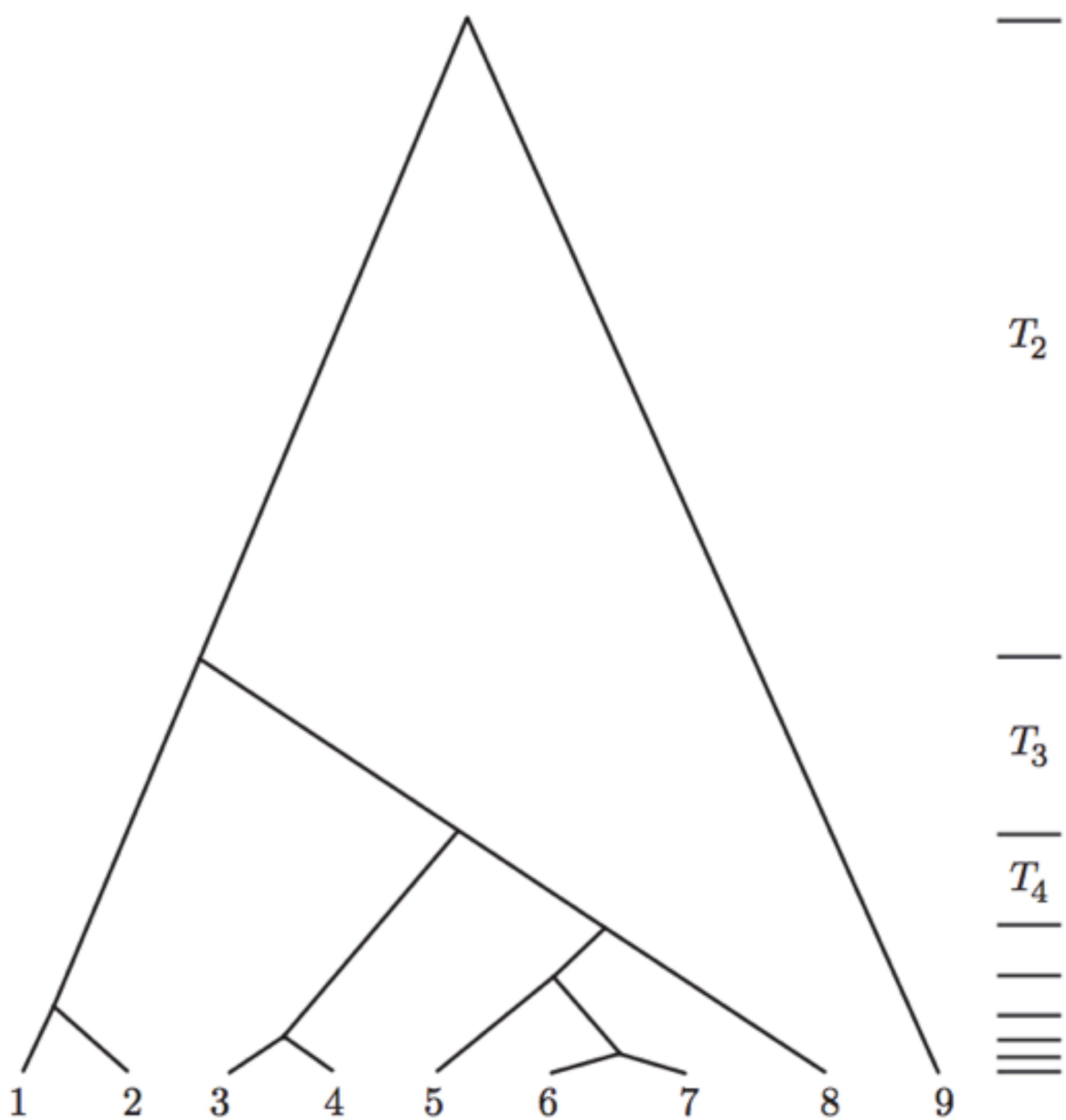

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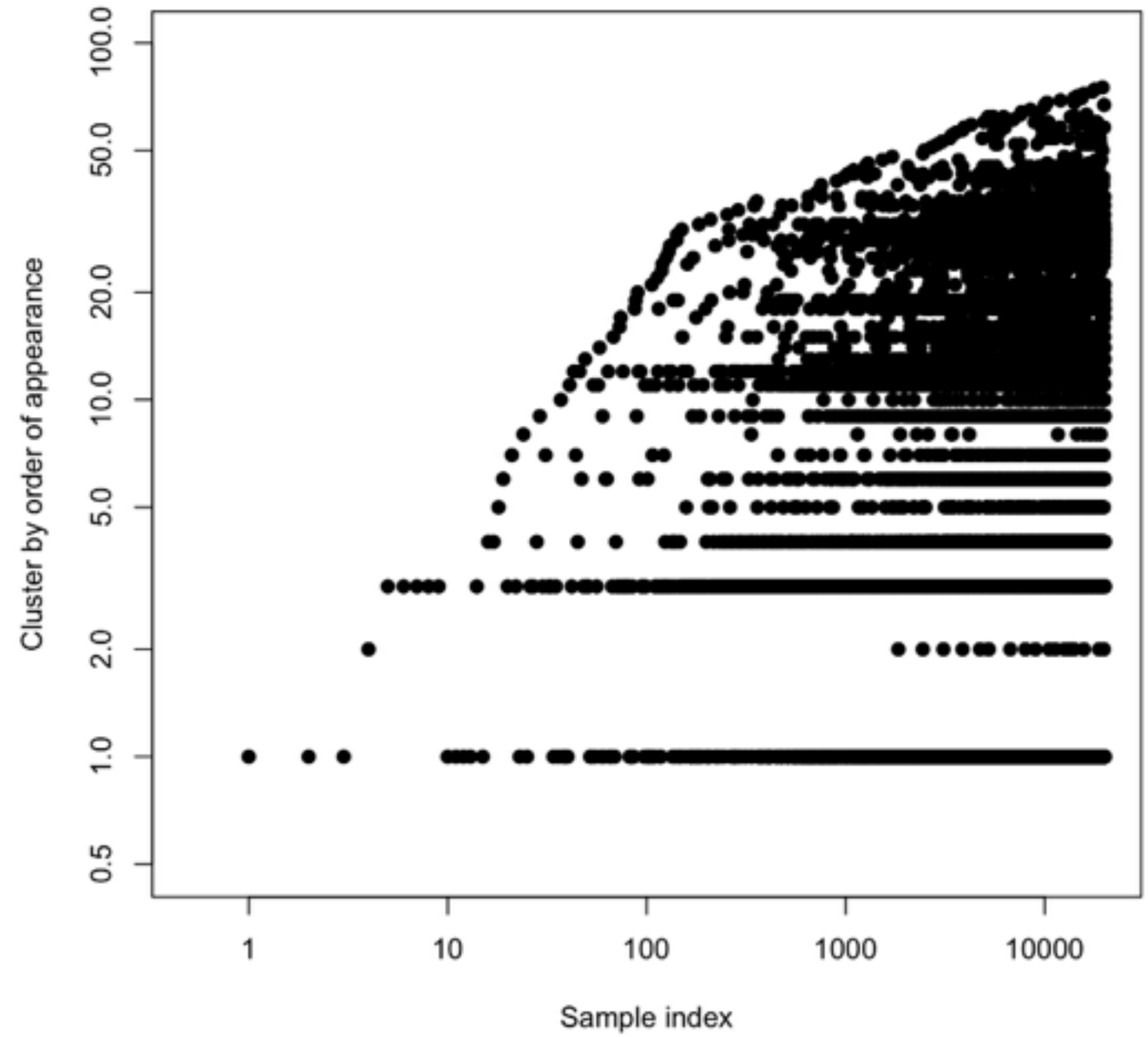
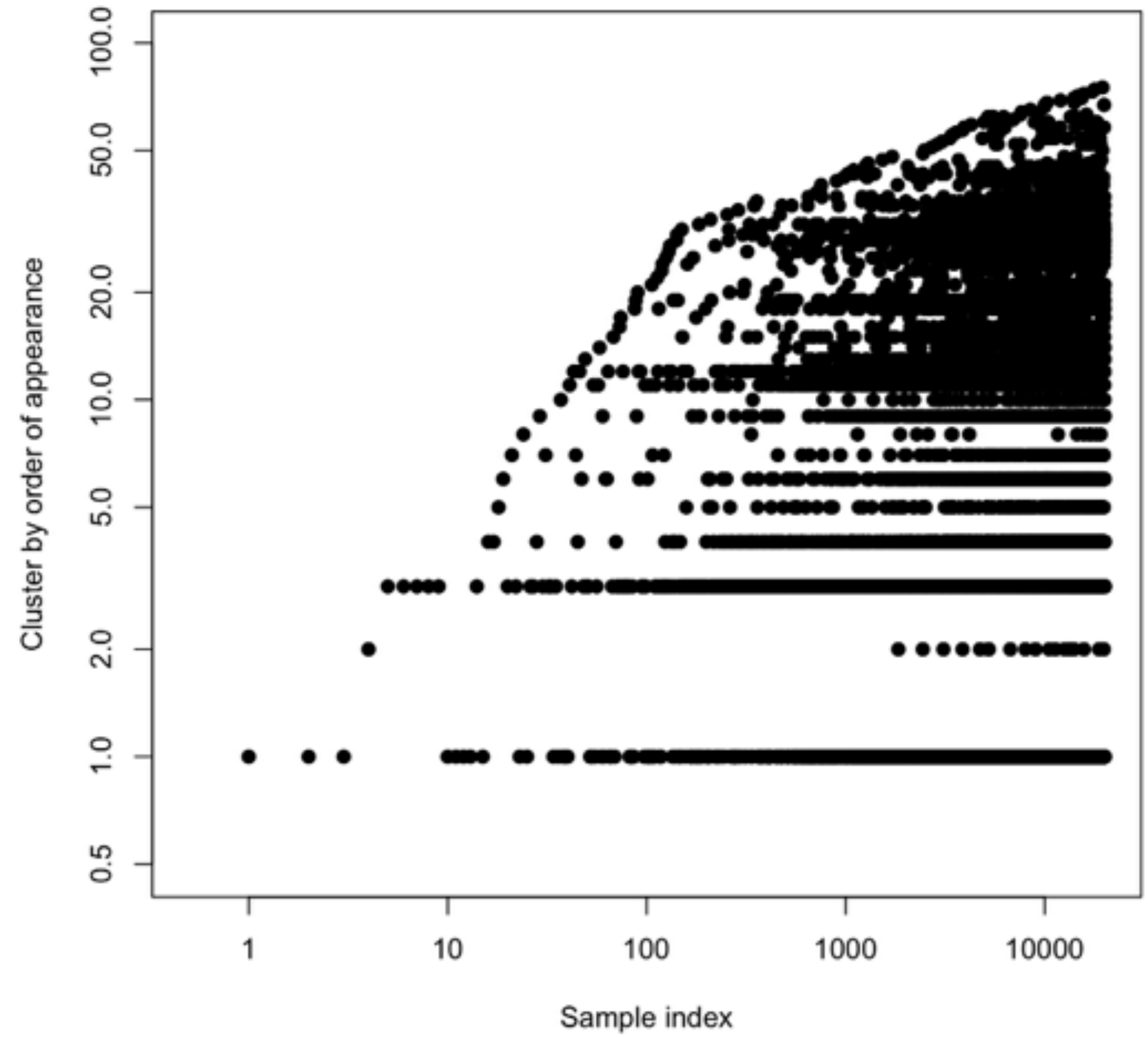
Power laws



Cluster by order of appearance
100.0
50.0
20.0
10.0
5.0
2.0
1.0
0.5
1
10
100
1000
10000
Sample index

Power laws

- K_N := # clusters occupied by N data points



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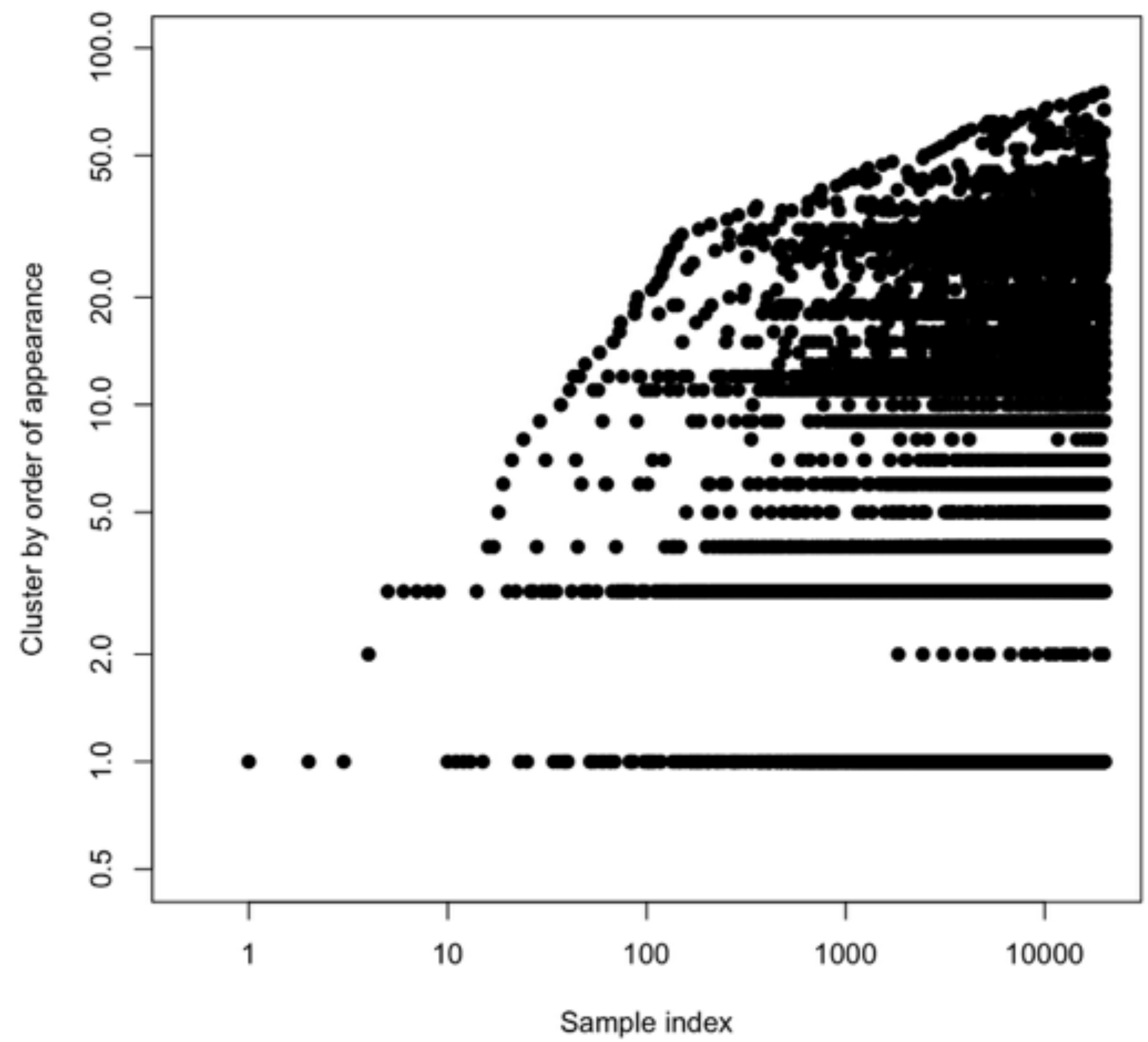

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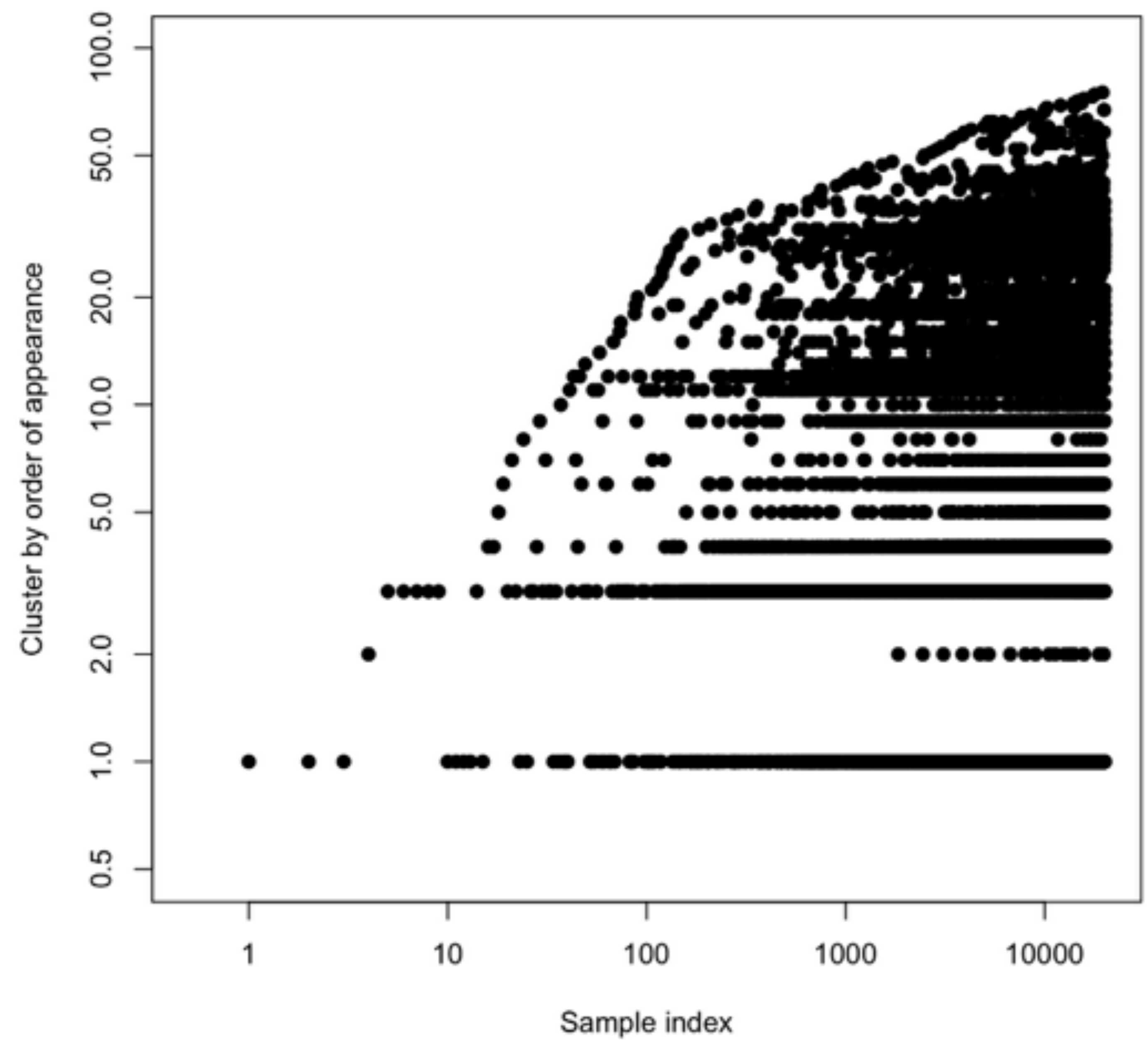

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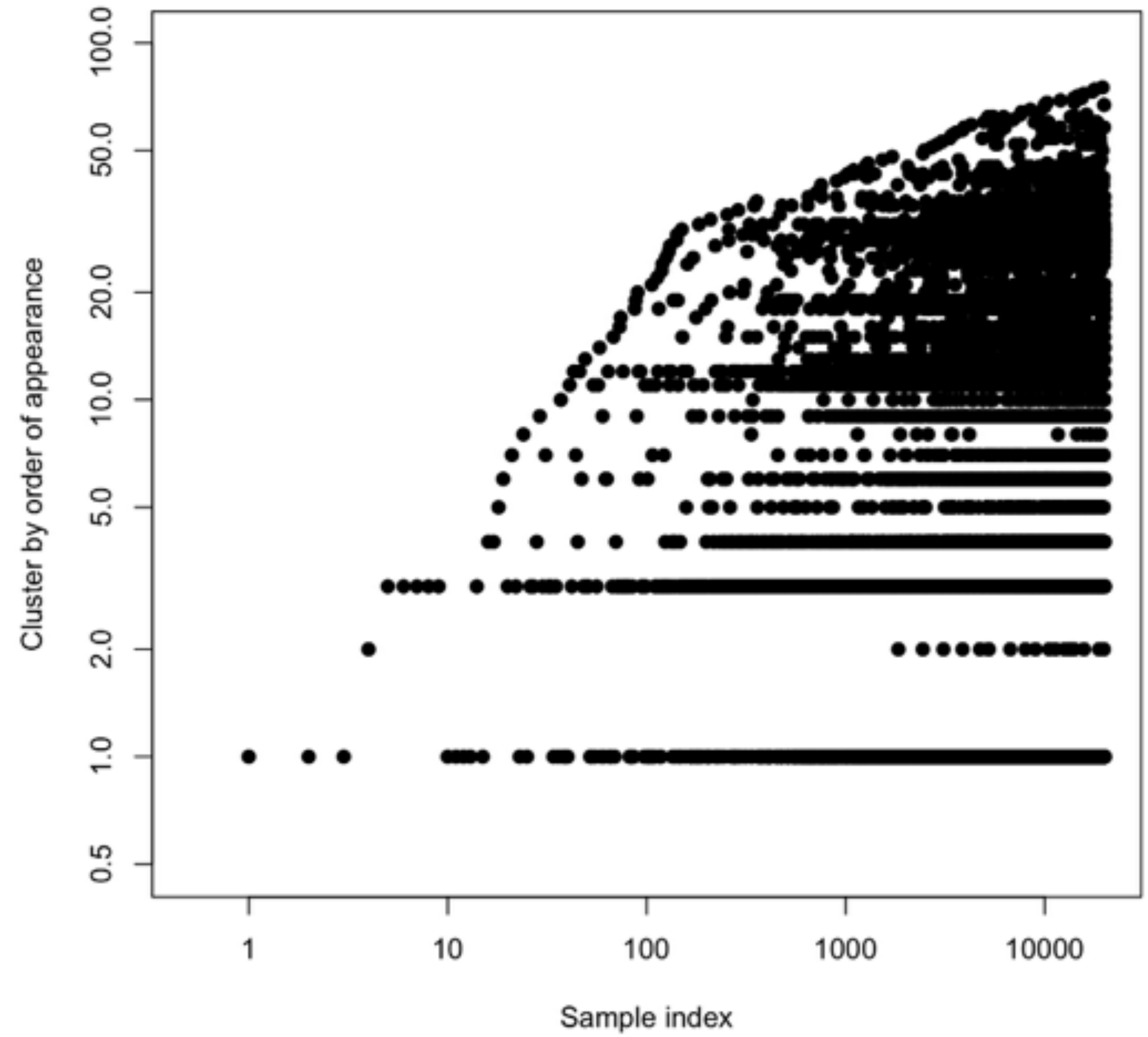

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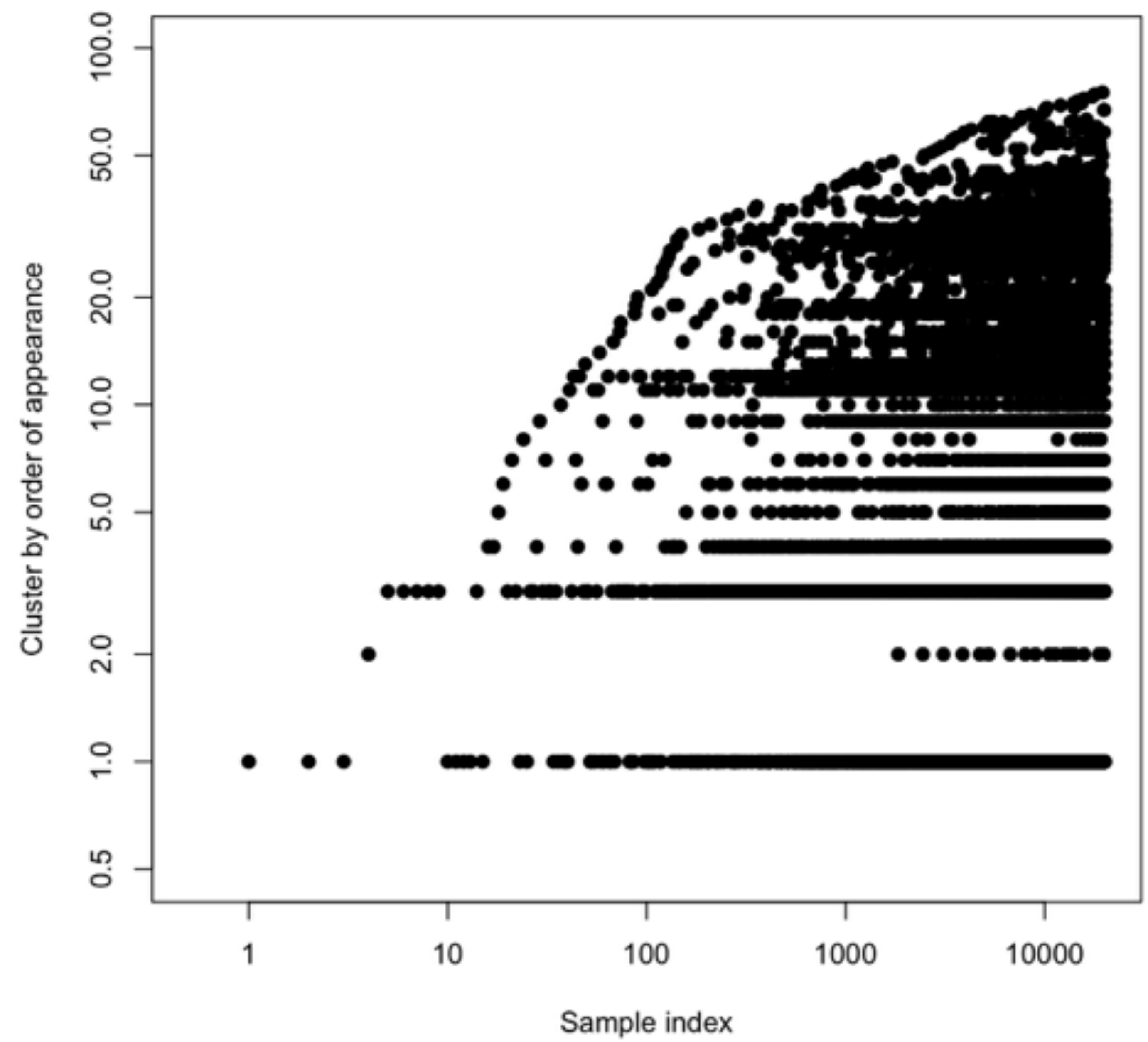

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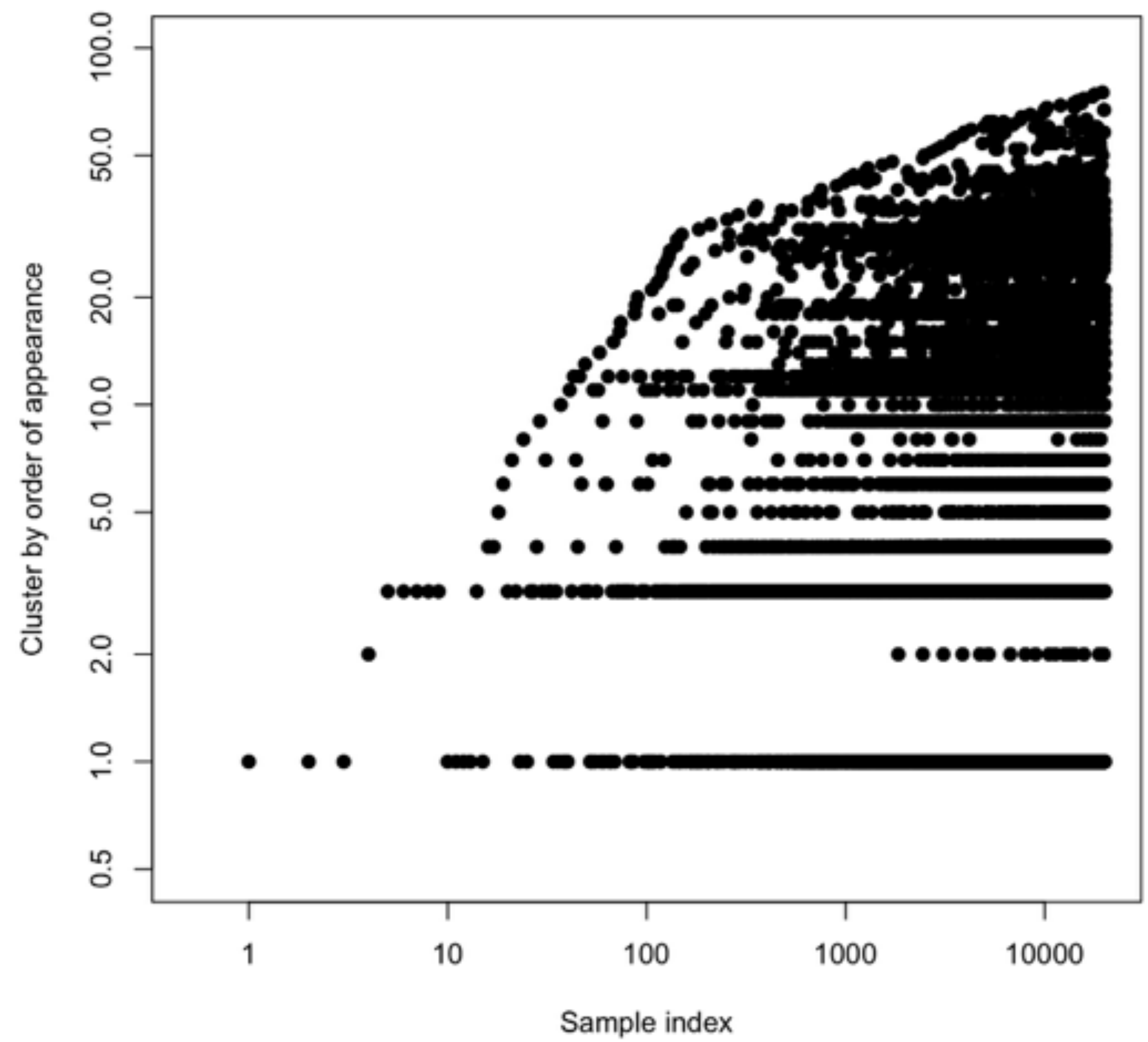

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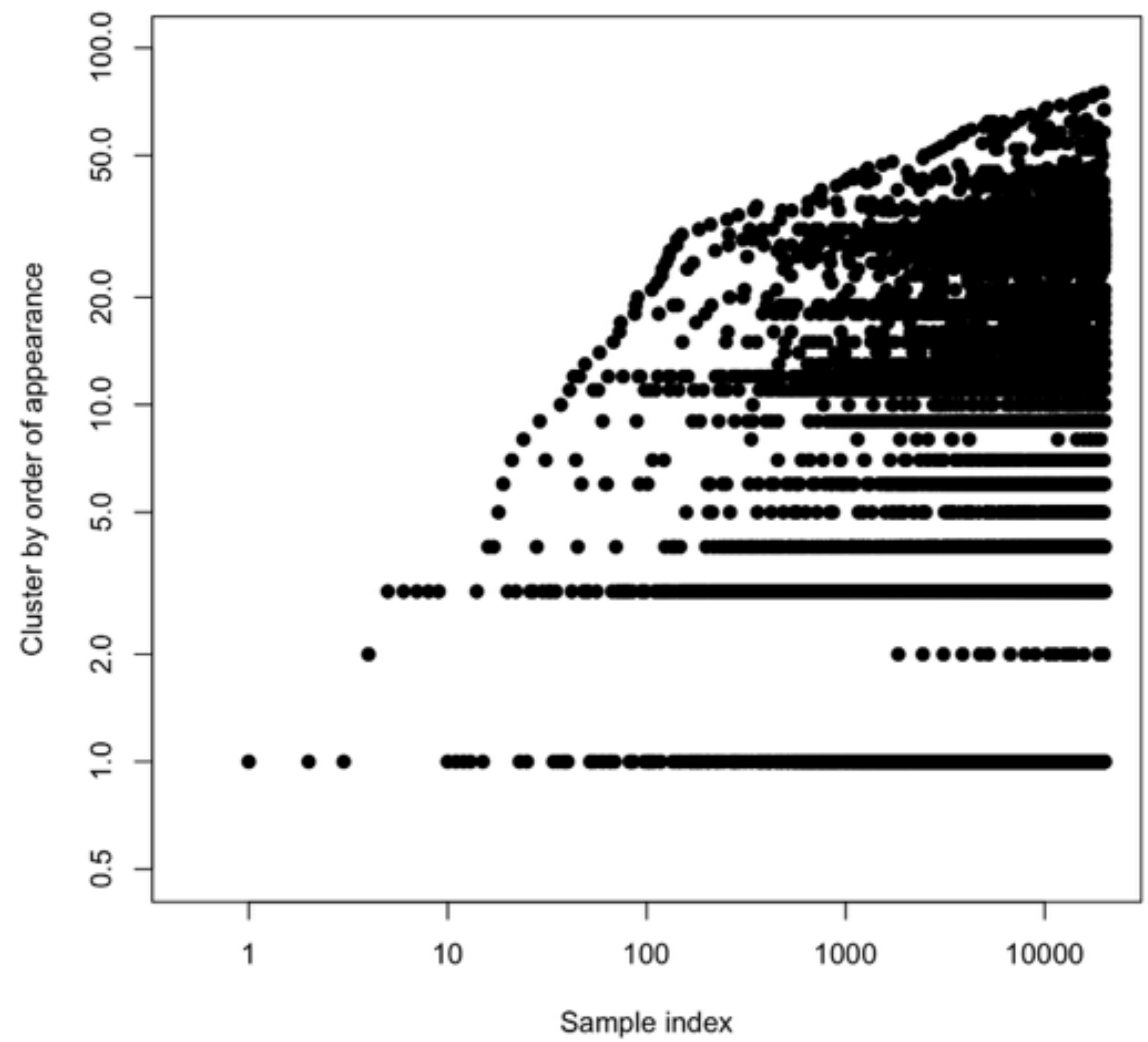

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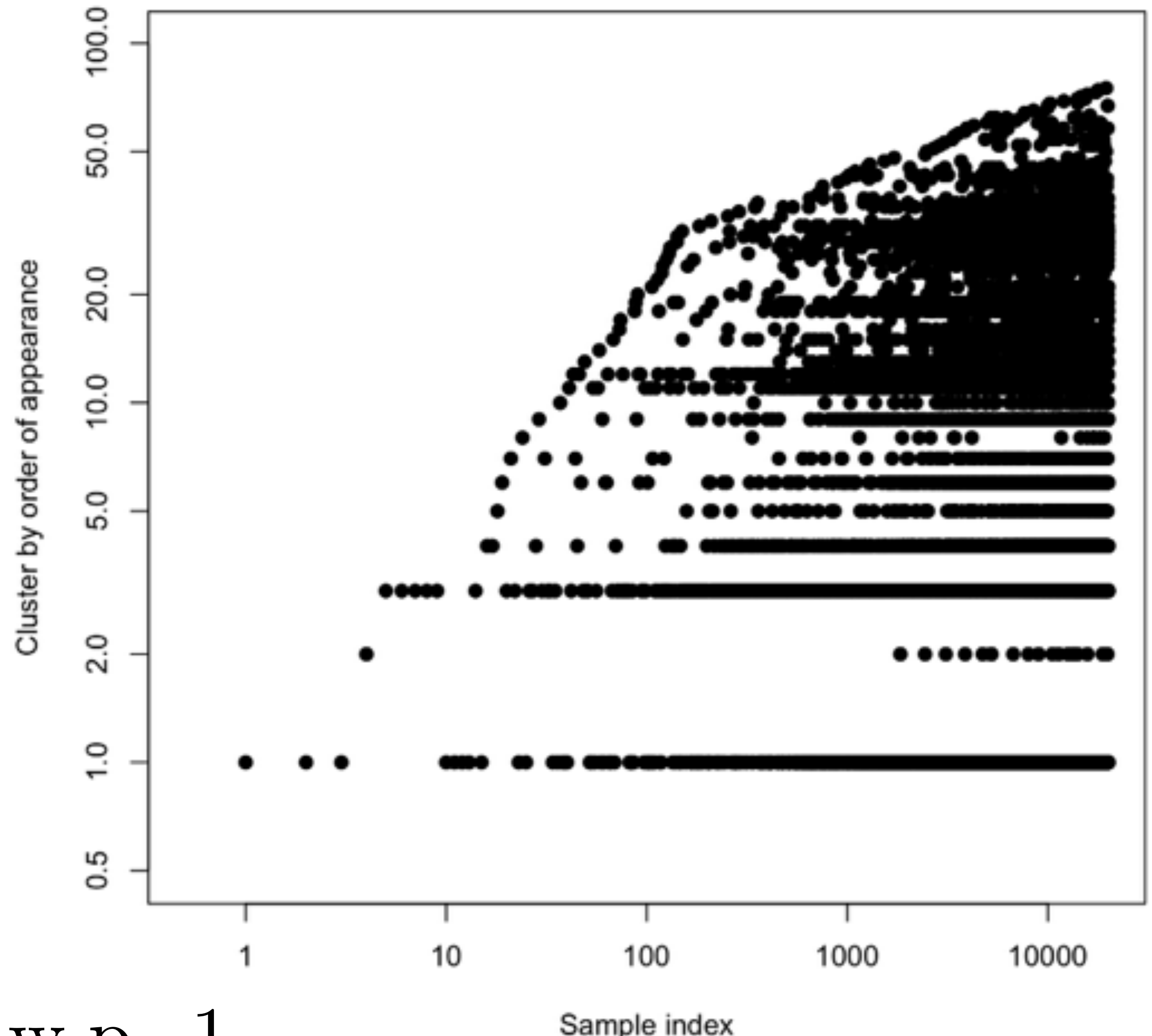

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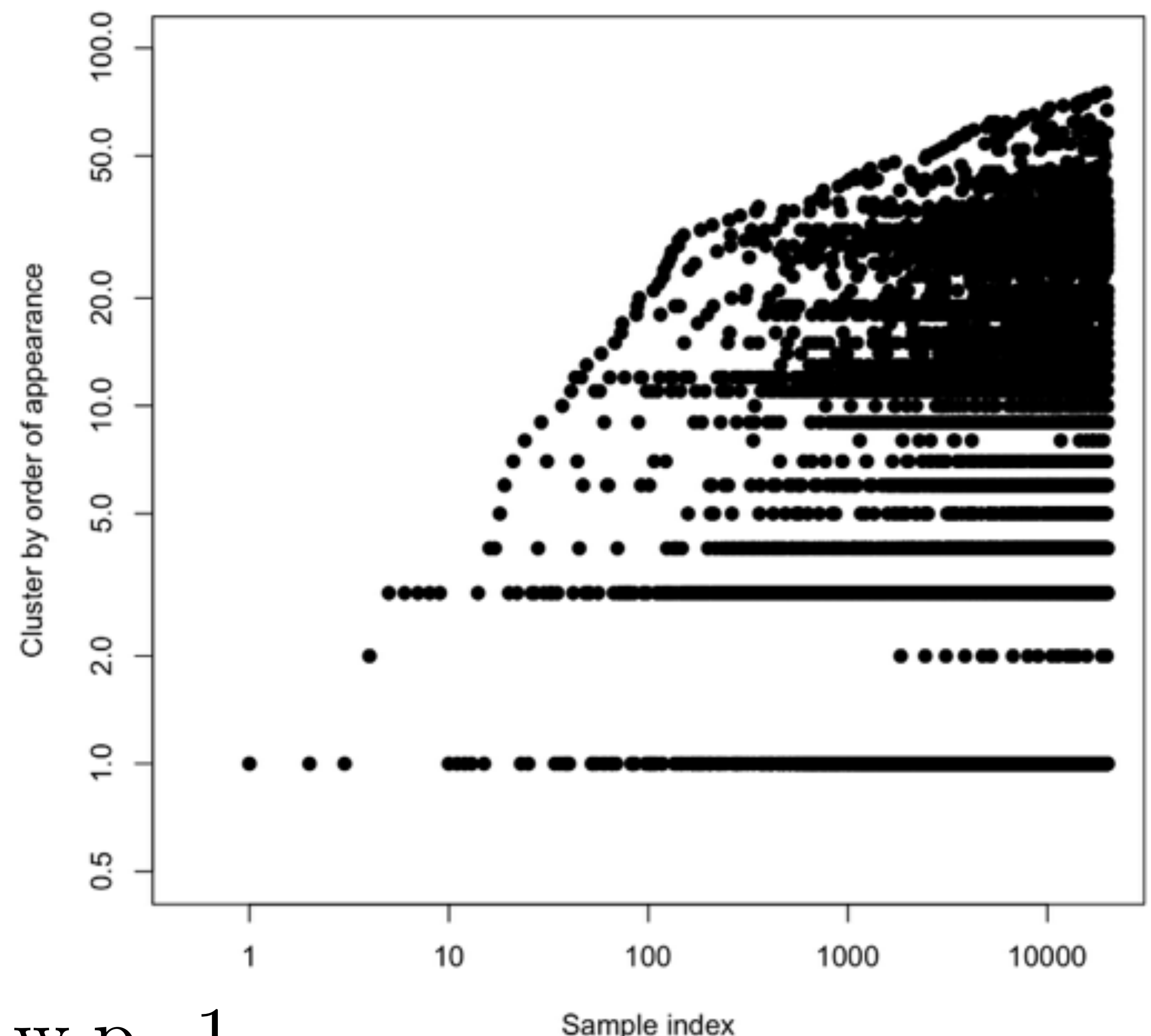

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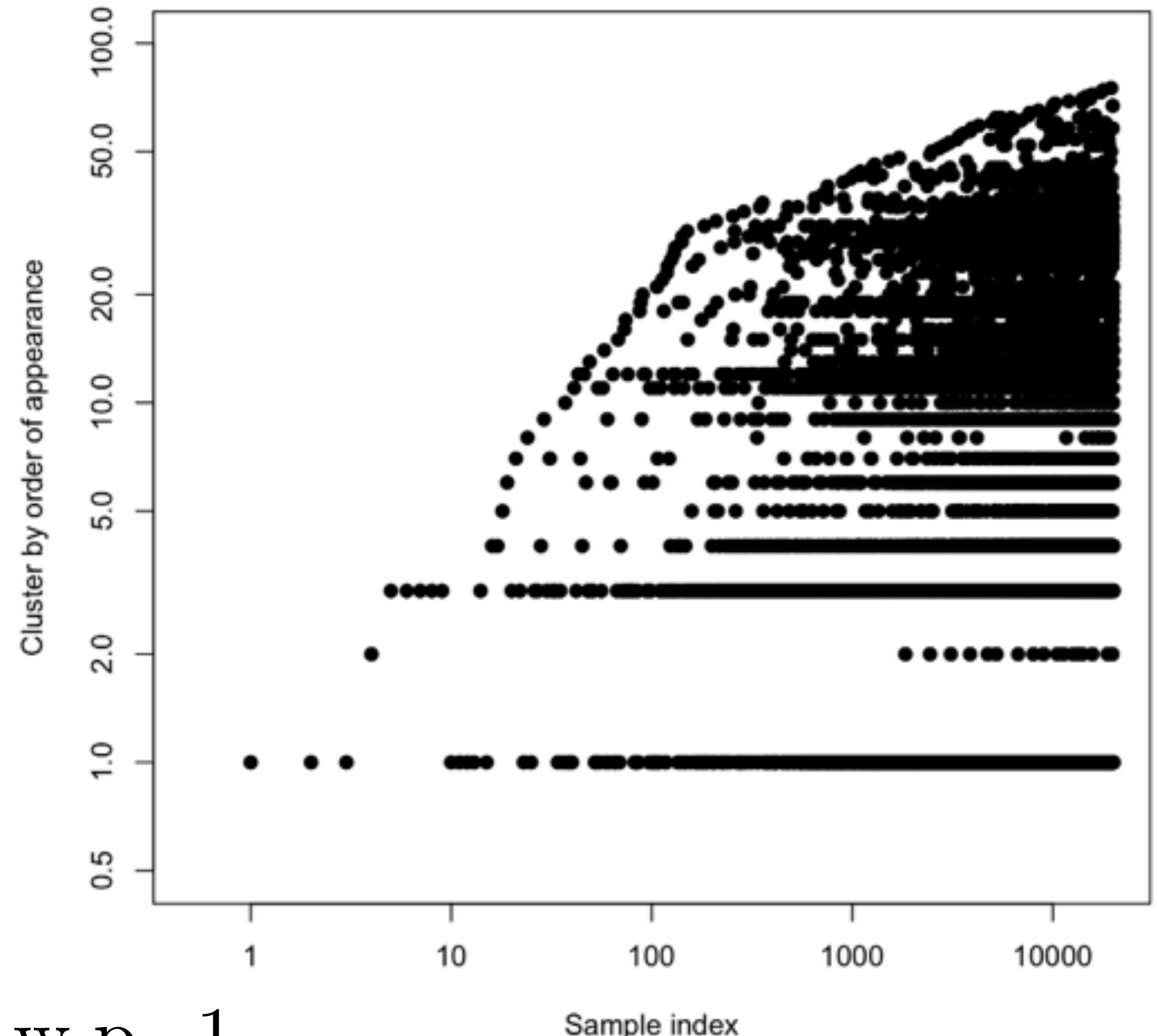

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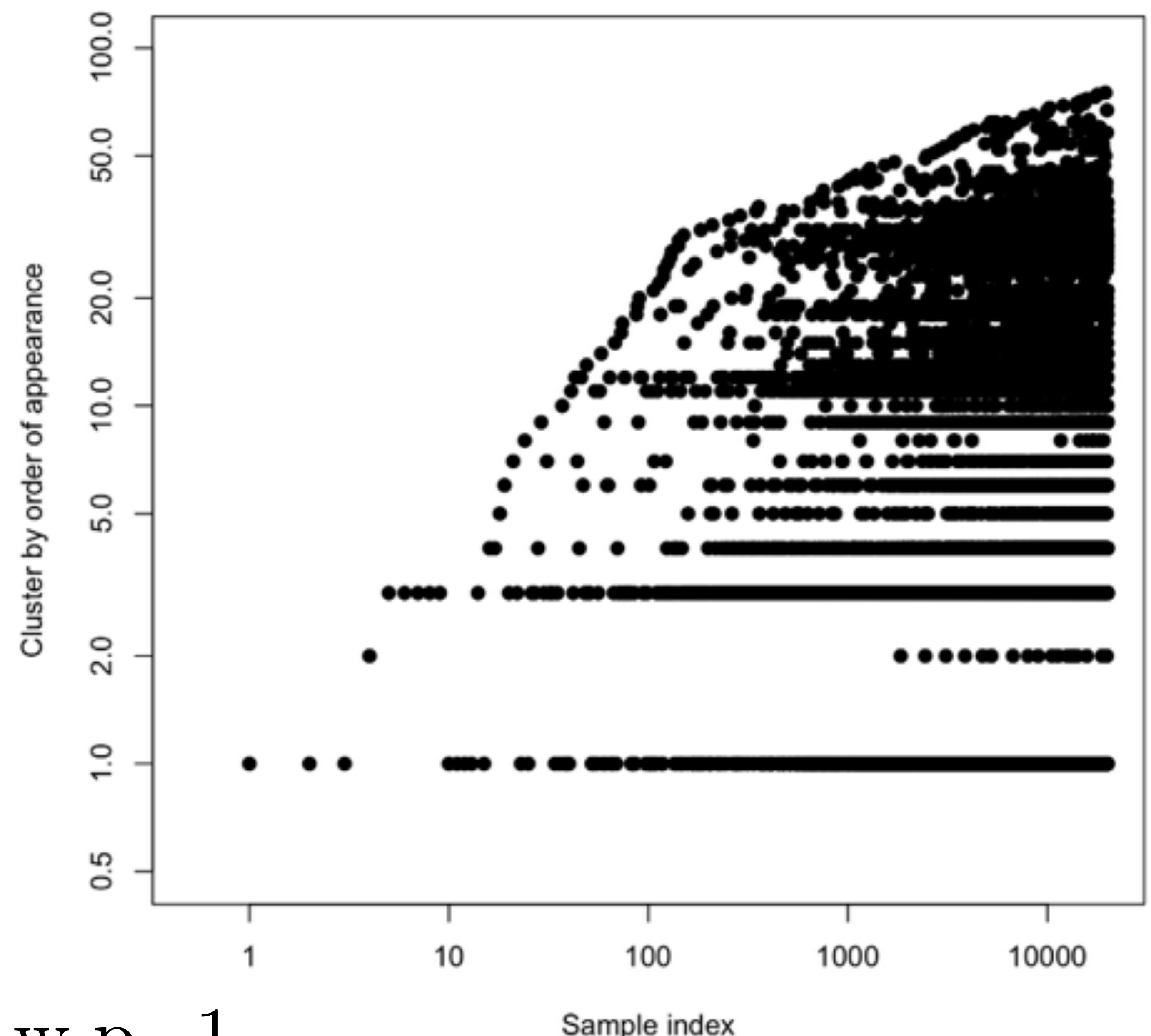

[Gnedin, et al 2007, Pitman, Yor 1997, Goldwater et al 2005, Teh 2006, Broderick et al 2012]

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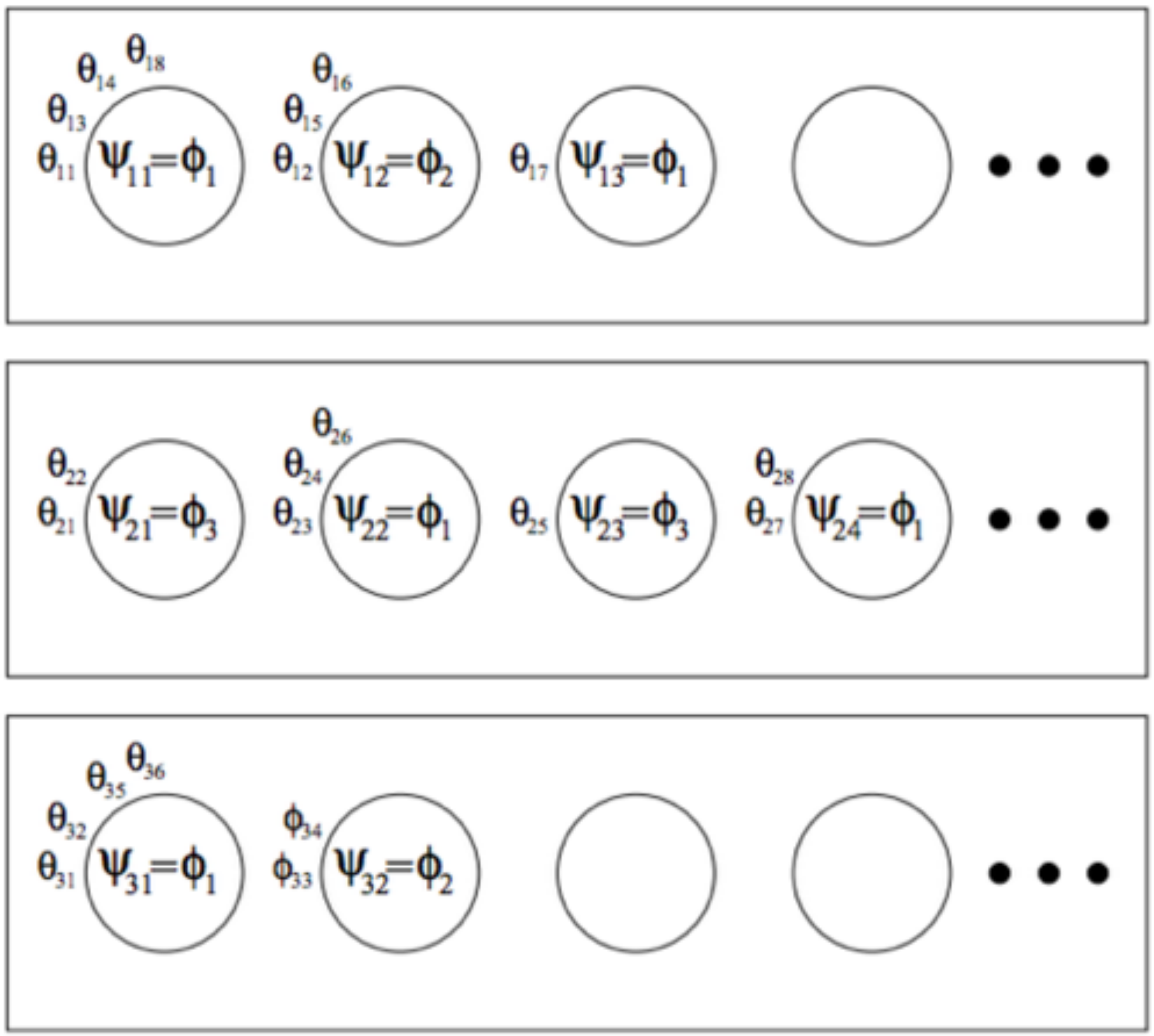

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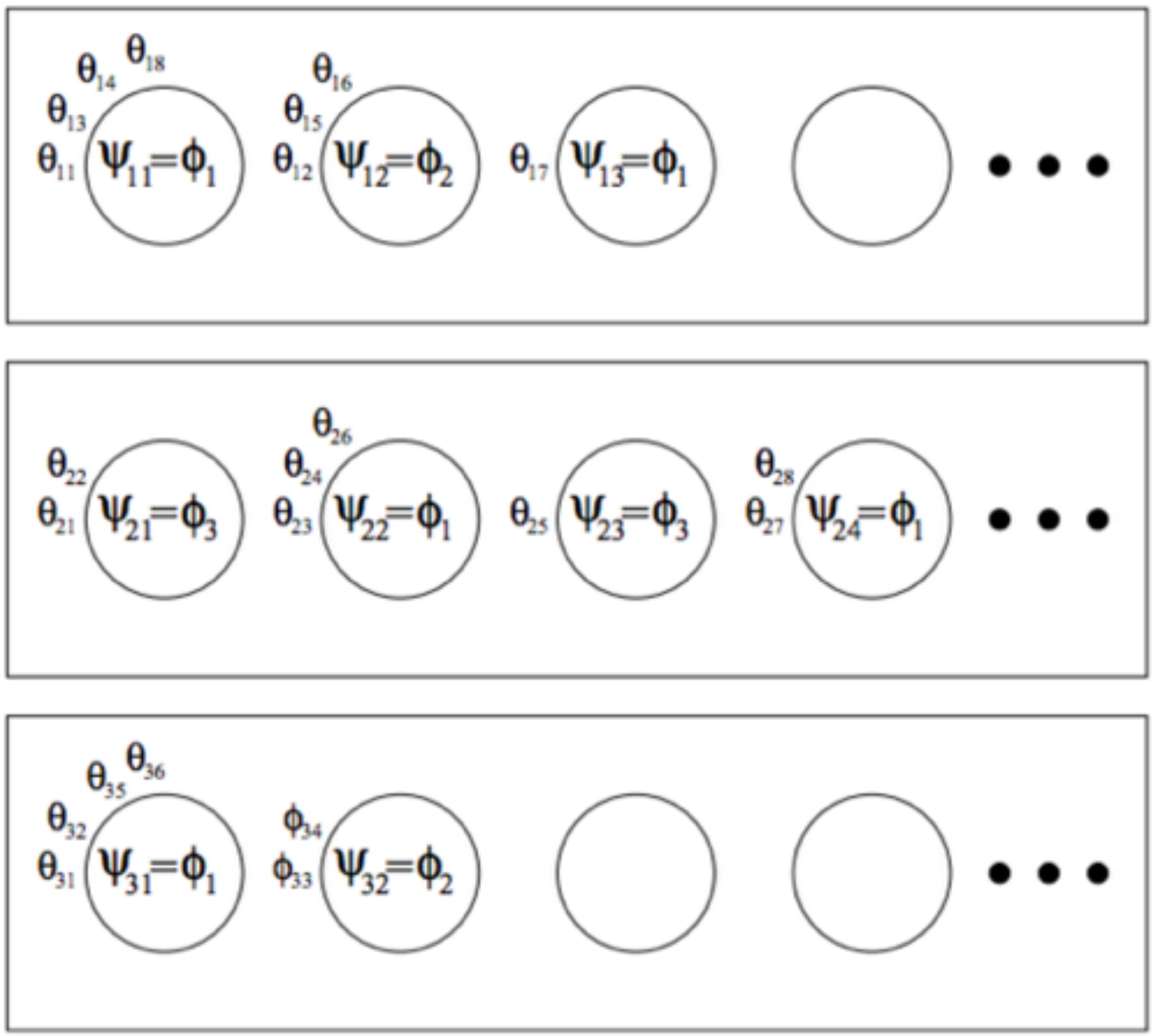

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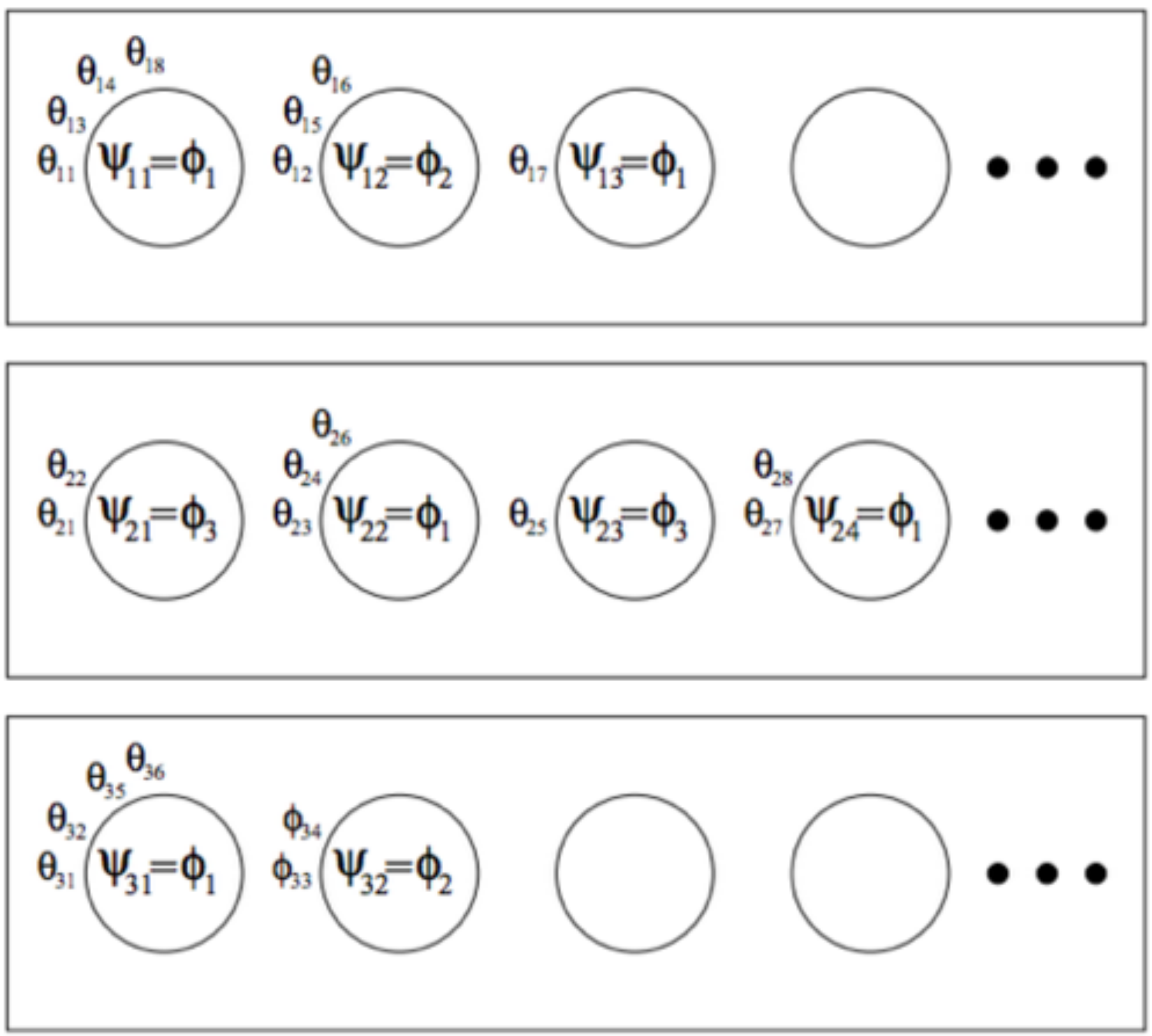

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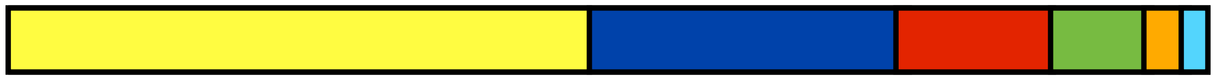

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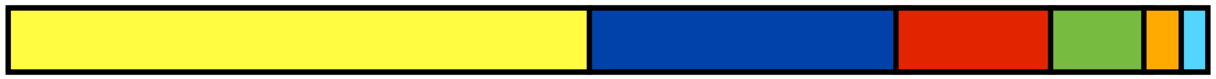

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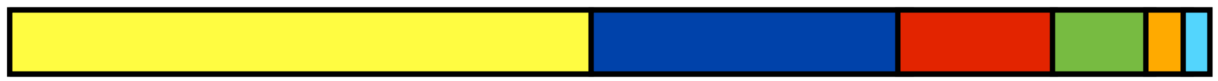

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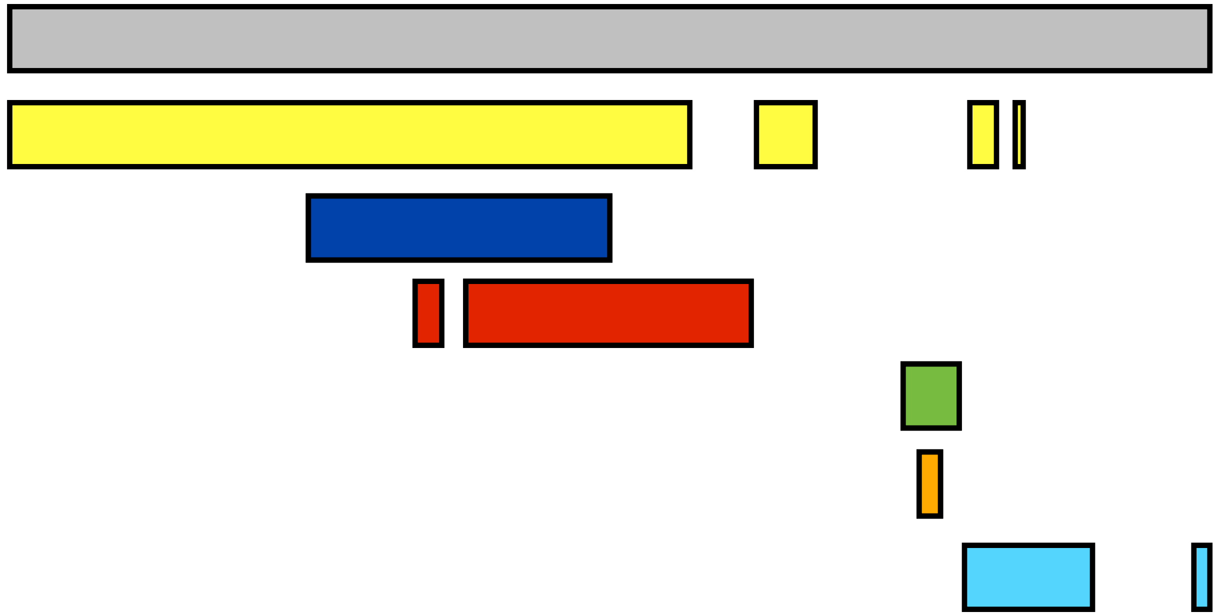

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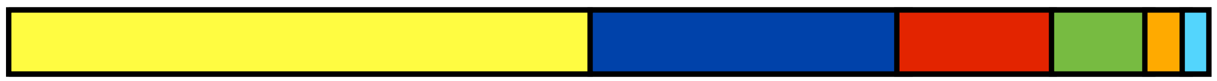

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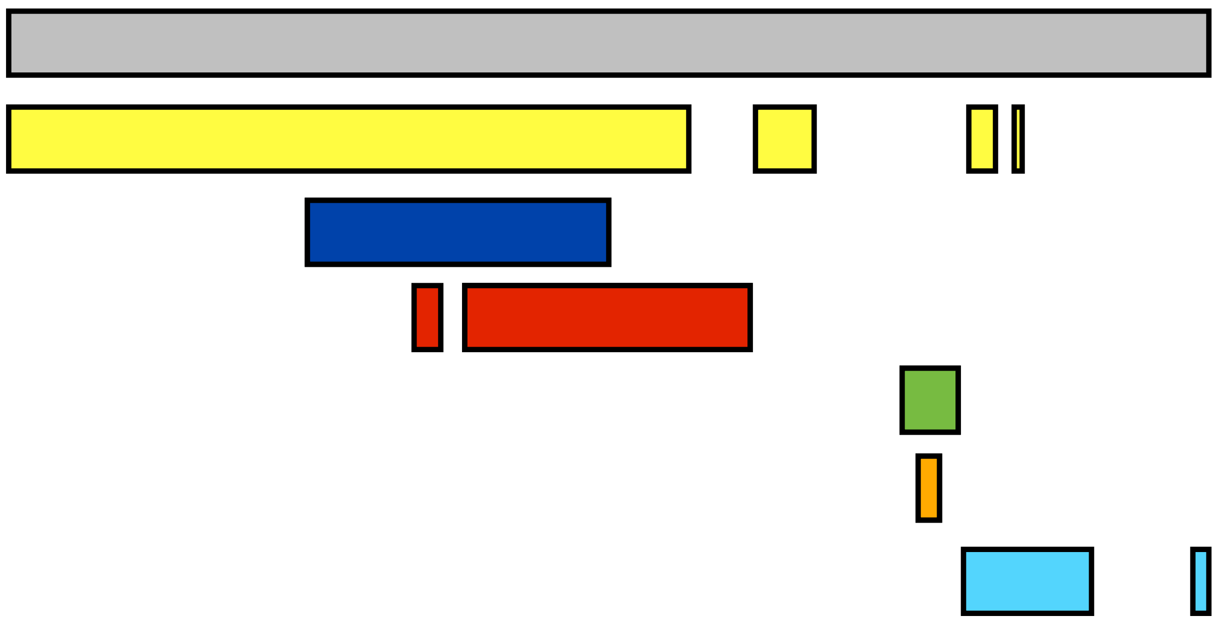

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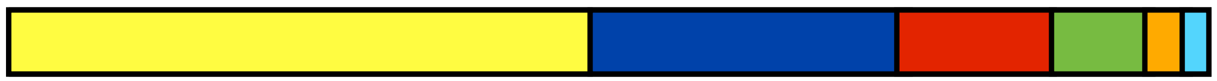

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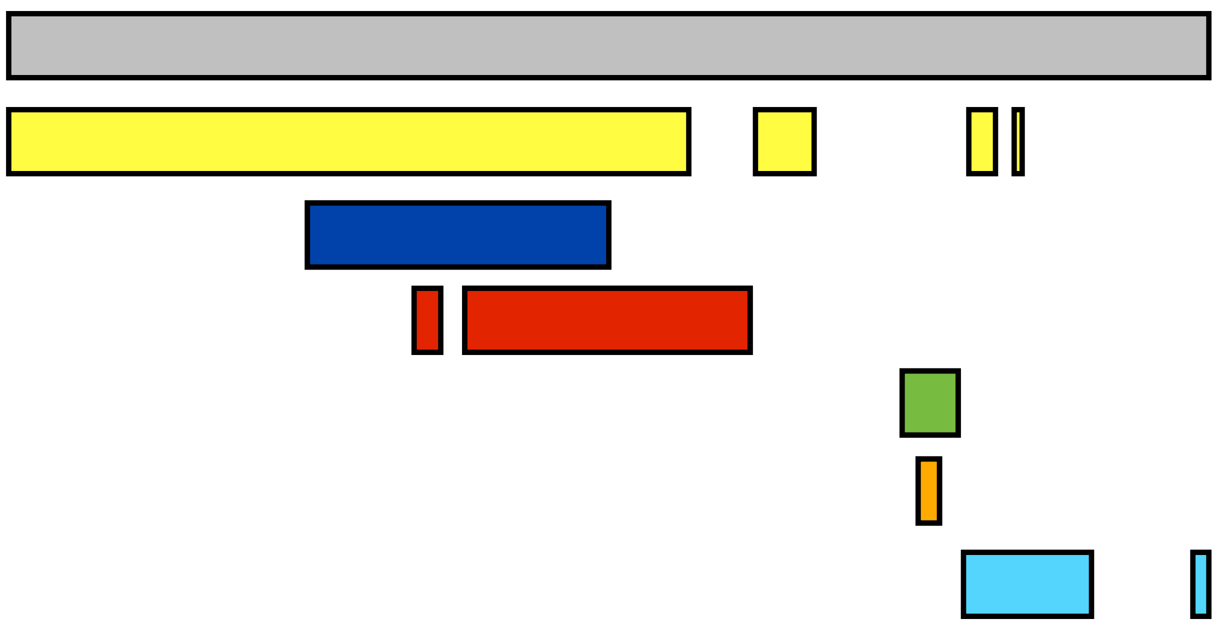

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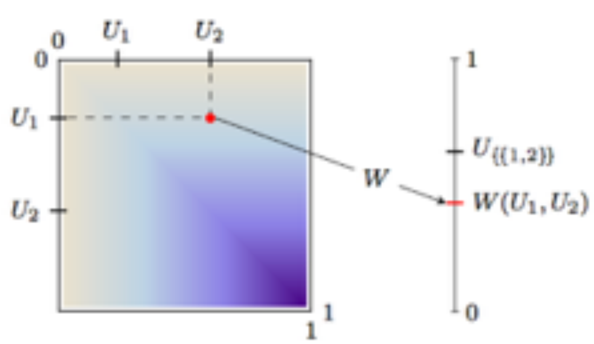

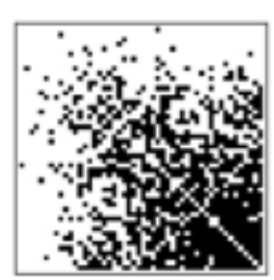

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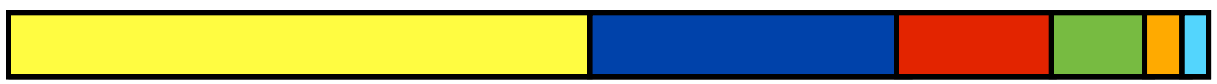

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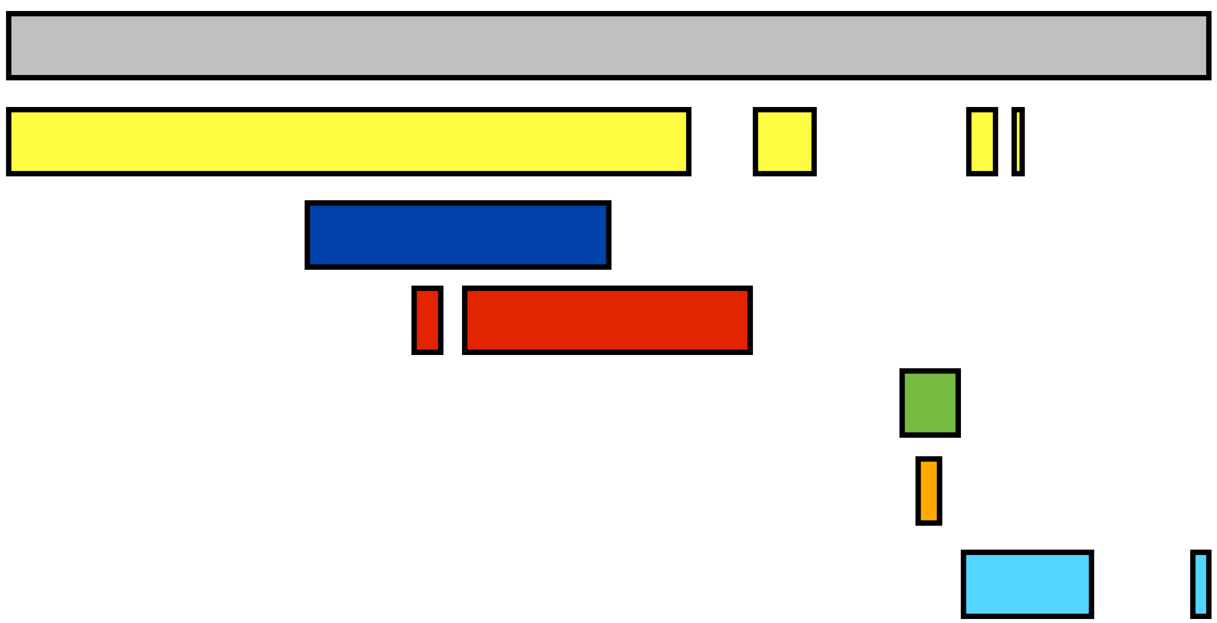

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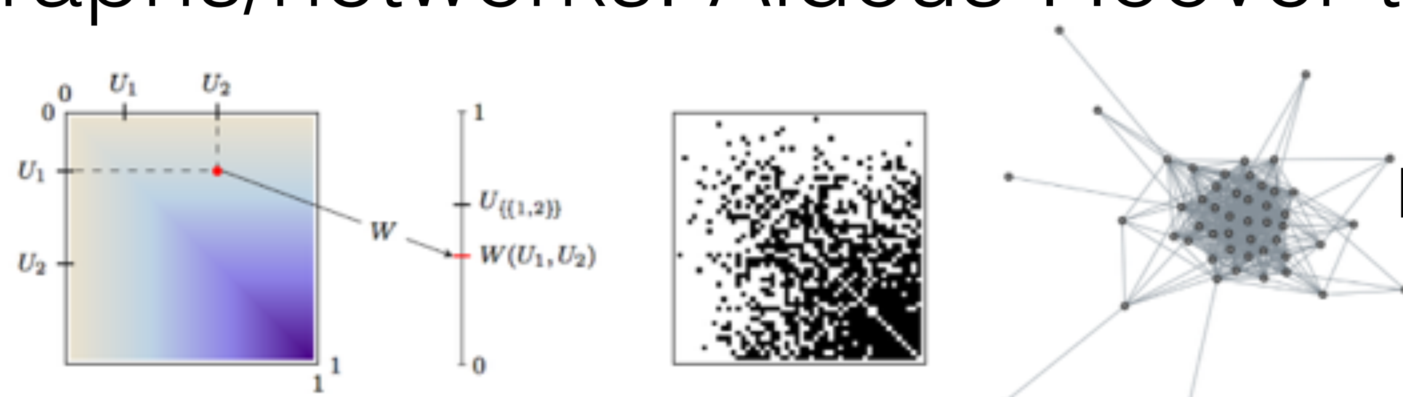

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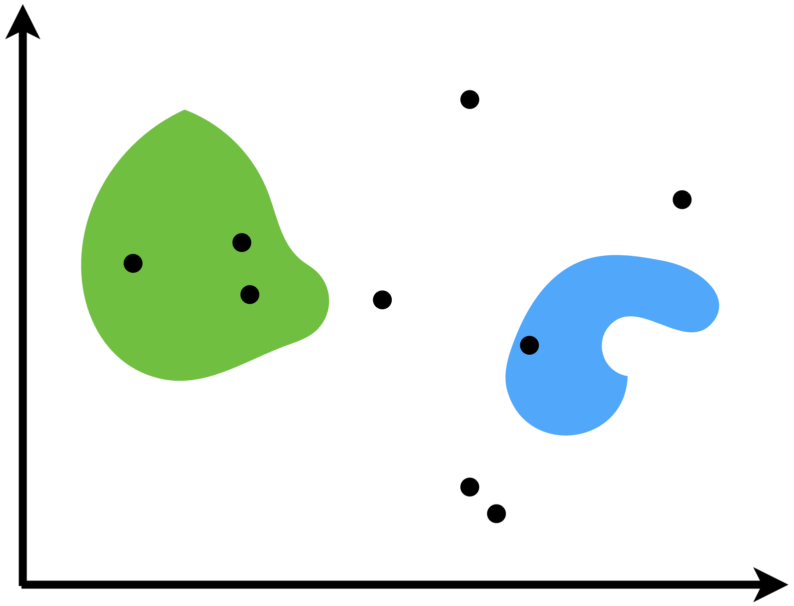

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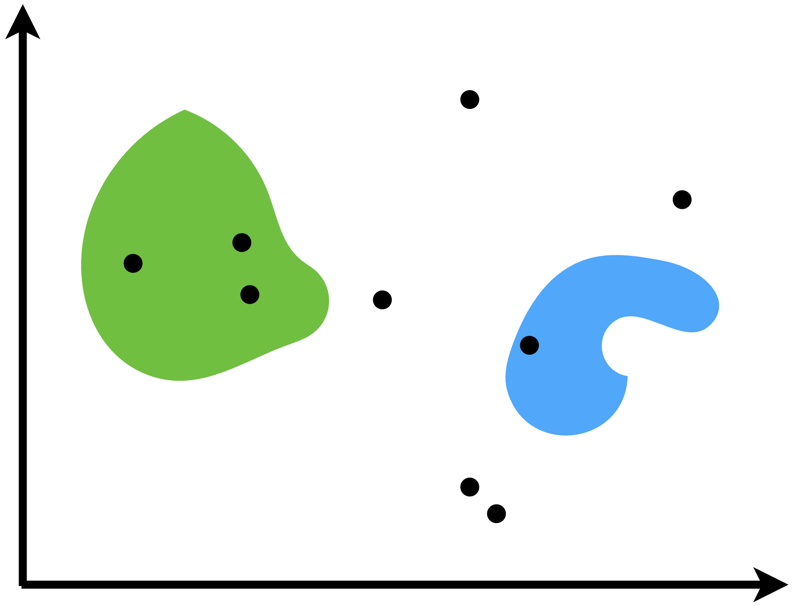

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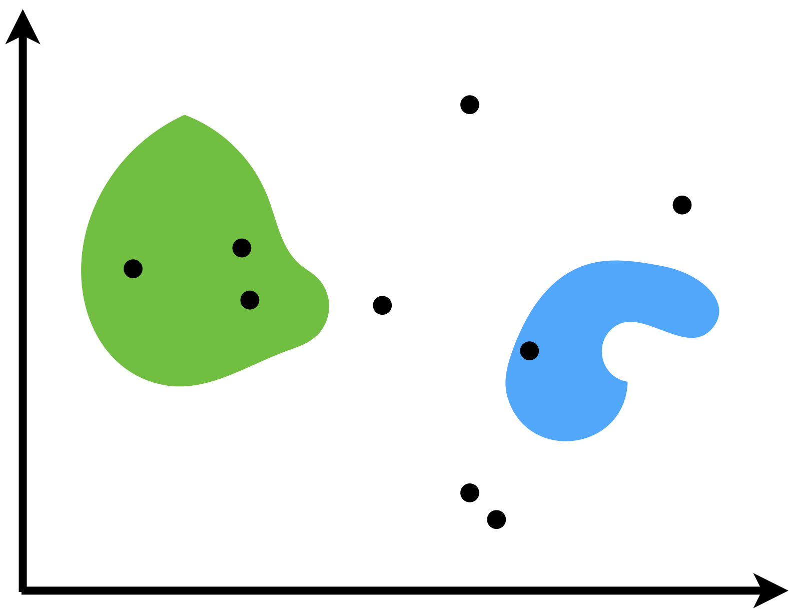

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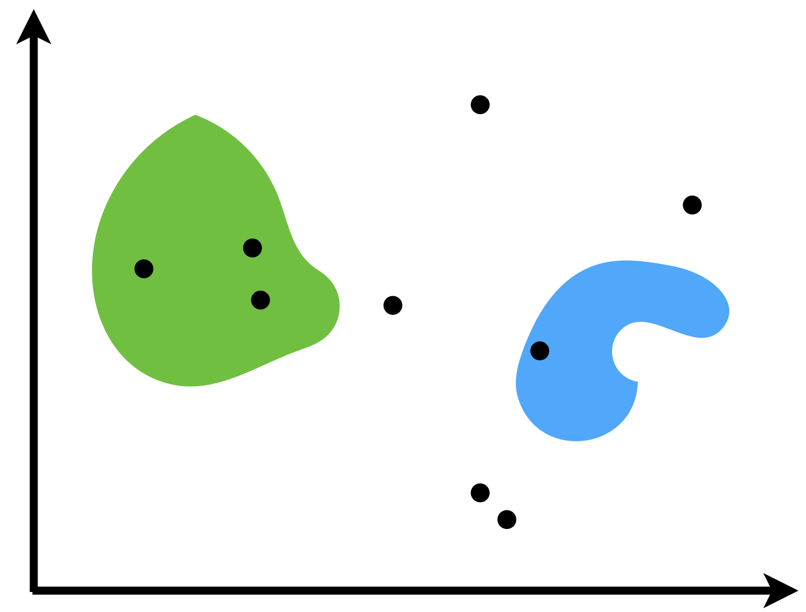

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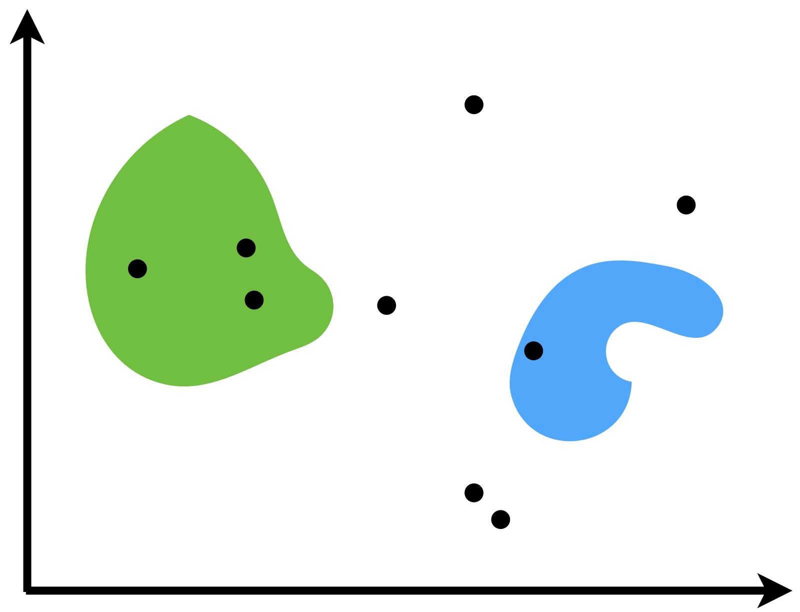

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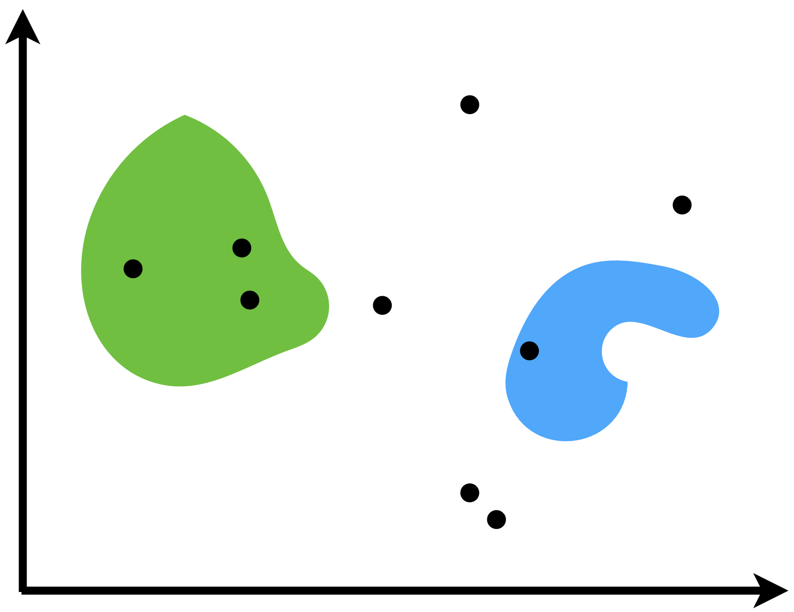

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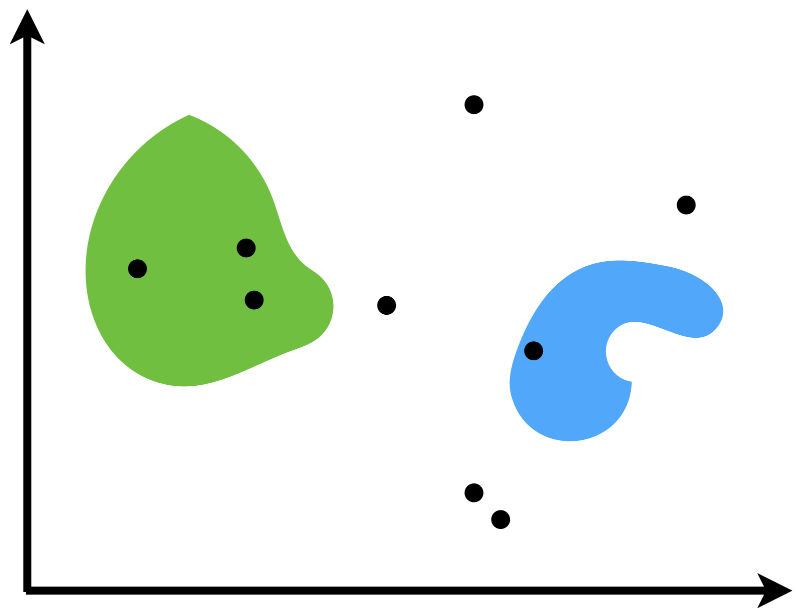

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[Kingman 1992, Orbanz 2009, Broderick et al 2014, James 2014]

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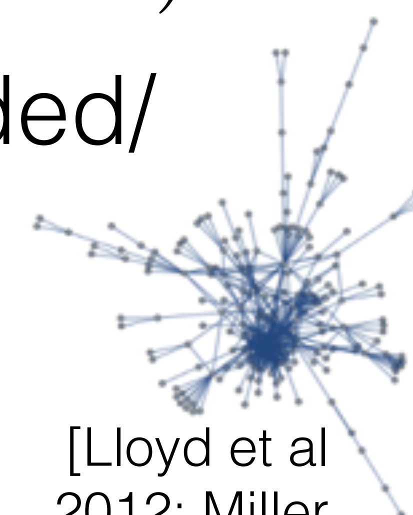


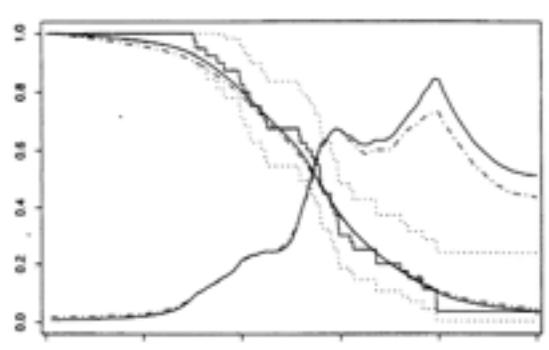
[Lloyd et al 2012; Miller et al, 2010]

[wikipedia.org]

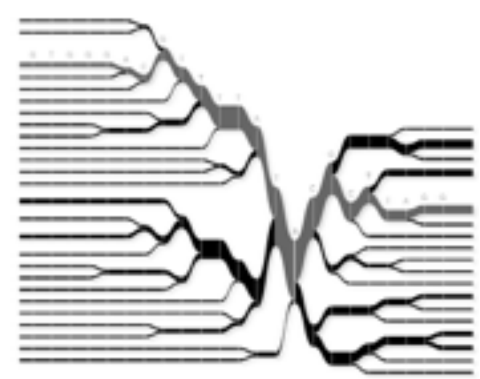
[Ed Bowlby, NOAA]

[Fox, et al 2014]

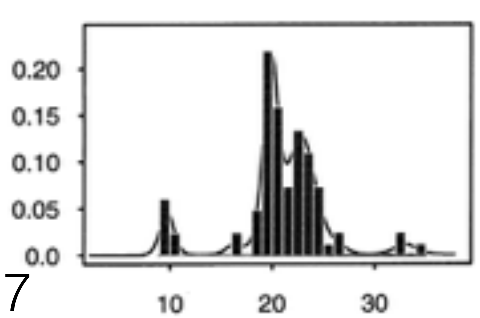
[Sudderth, Jordan 2009]



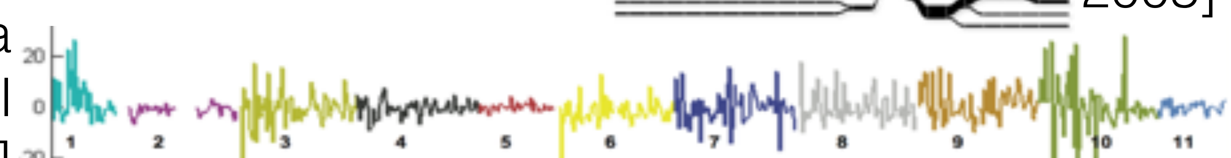
[Arjas, Gasbarra 1994]



[Ewens, 1972; Hartl, Clark 2003]



[Escobar, West 1995; Ghosal, et al 1999]



[Saria et al 2010]

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