Nonparametric Bayesian Statistics: Part II

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MIT

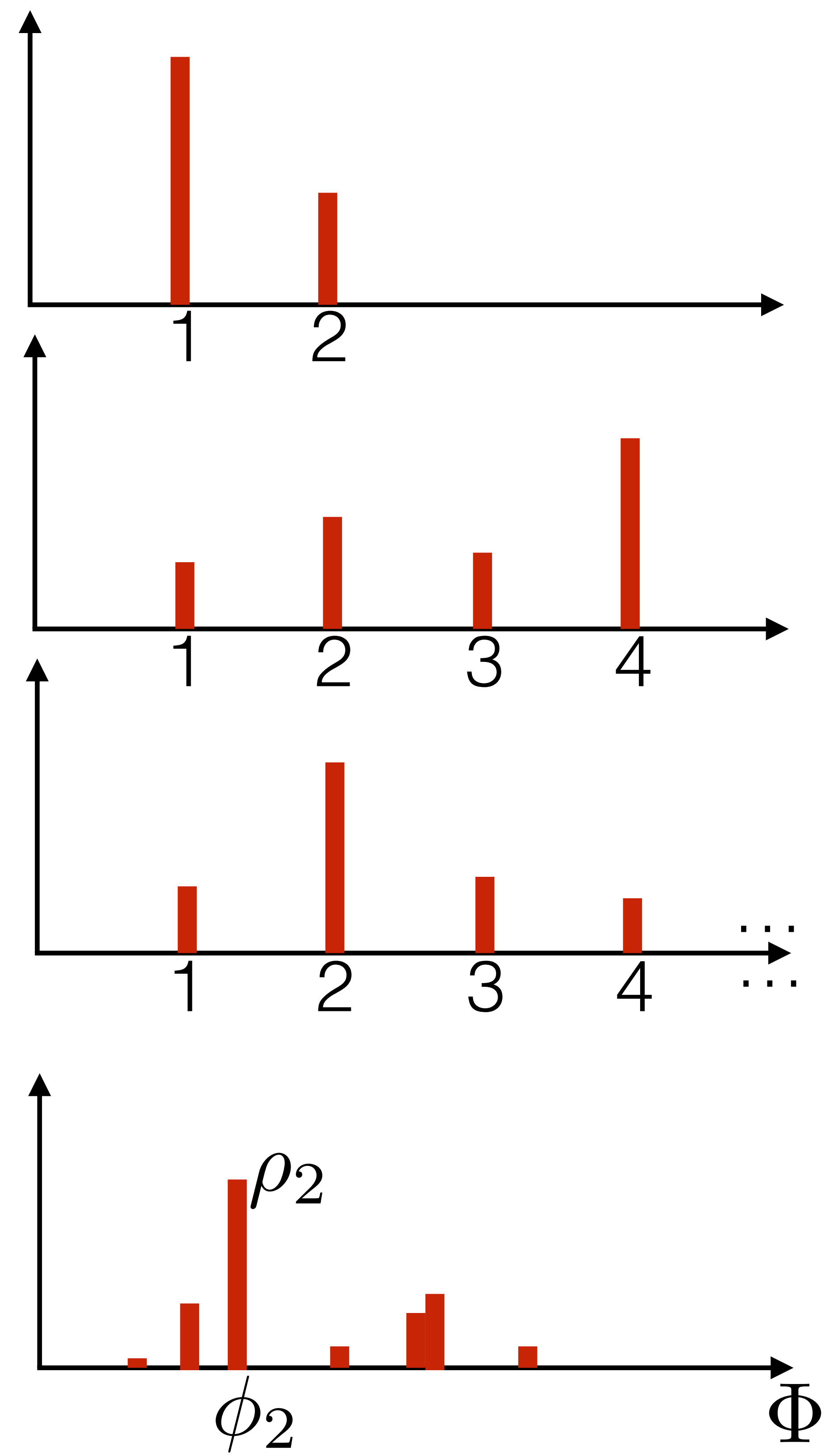
Outline

- Dirichlet process
  - Background for intuition
  - Generative model
  - What does a growing/infinite number of parameters really mean (in Nonparametric Bayes)?
- Chinese restaurant process
- Inference
- Venture further into the wild world of Nonparametric Bayesian statistics

[slides, code: www.tamarabroderick.com/tutorials.html]

Distributions

- Beta → random distribution over $1, 2$
- Dirichlet → random distribution over $1, 2, \ldots, K$
- GEM / Dirichlet process stick-breaking → random distribution over $1, 2, \ldots$
- **Dirichlet process →** random distribution over Φ:
  $\rho = (\rho_1, \rho_2, \ldots) \sim \mathrm{GEM}(\alpha)$
  $\phi_k \overset{iid}{\sim} G_0$
  $G = \sum_{k=1}^{\infty} \rho_k \delta_{\phi_k}$



[Ferguson 1973]

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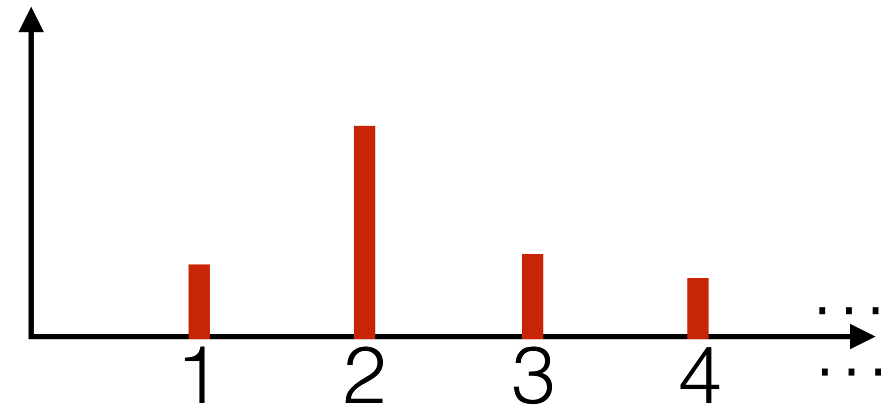

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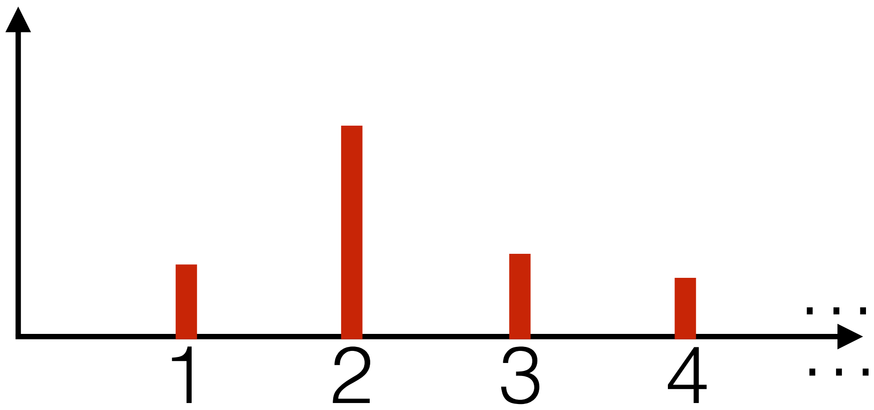

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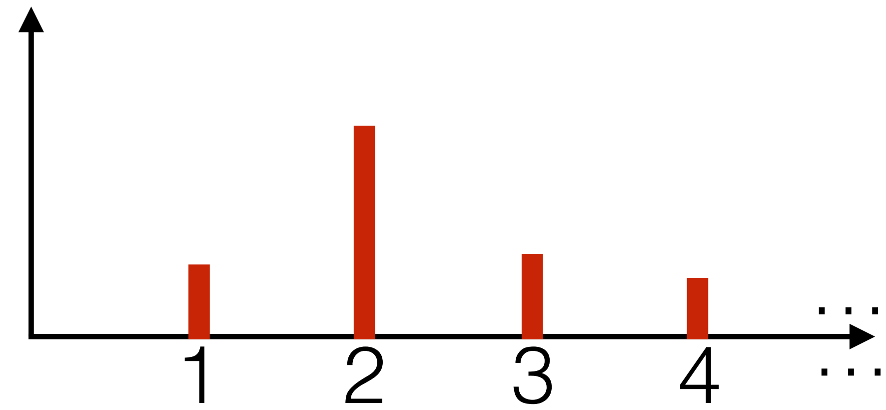

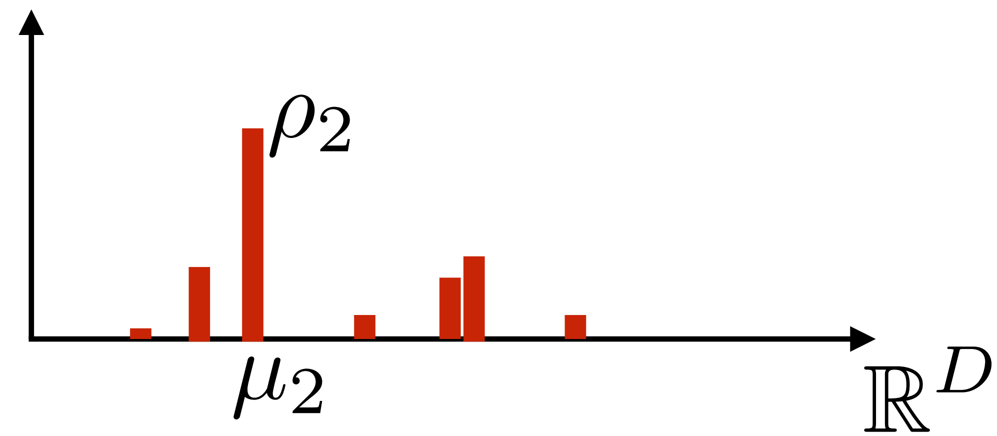

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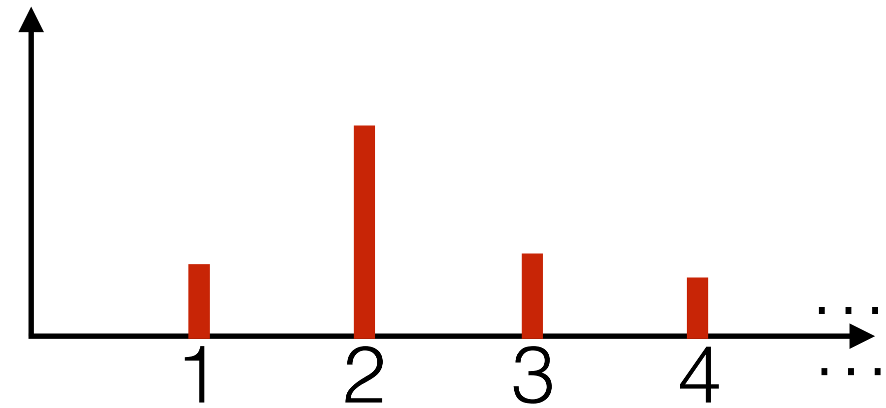

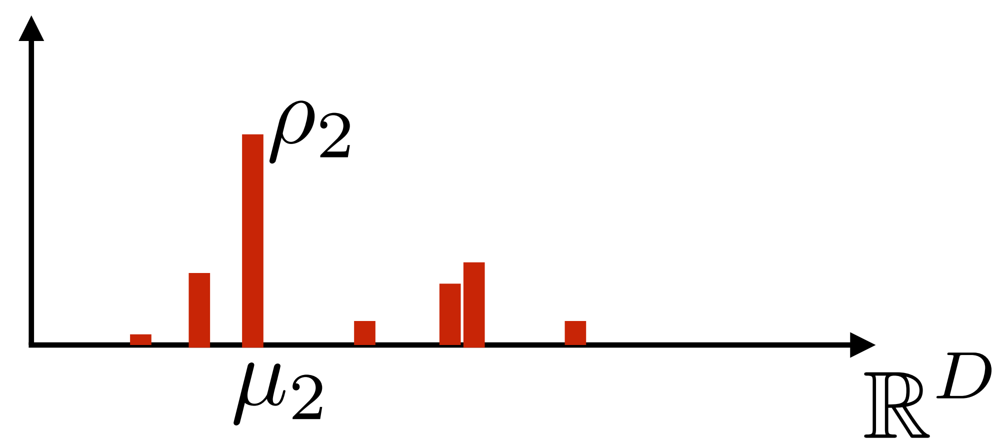

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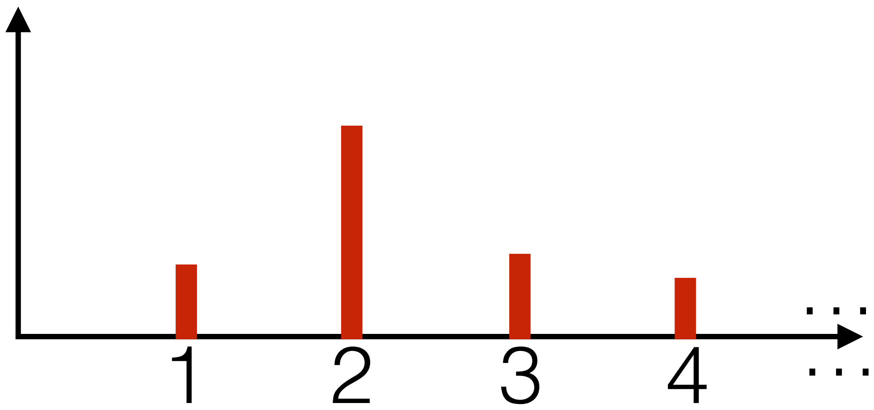

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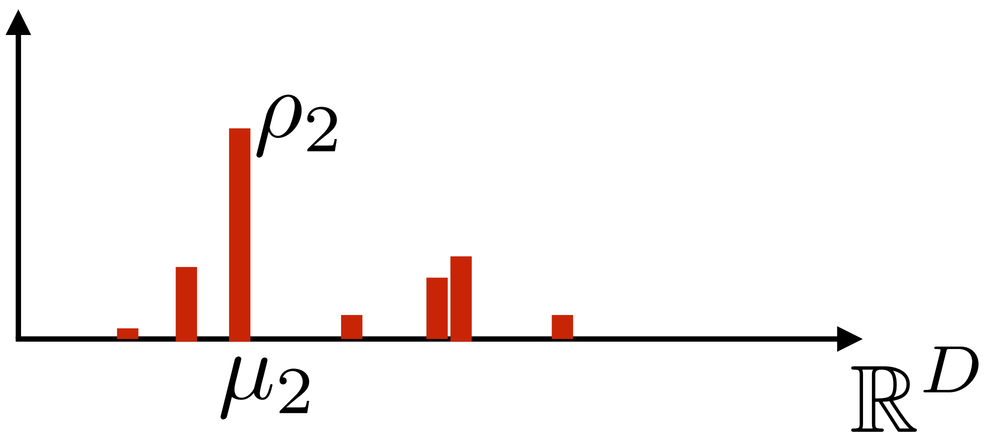

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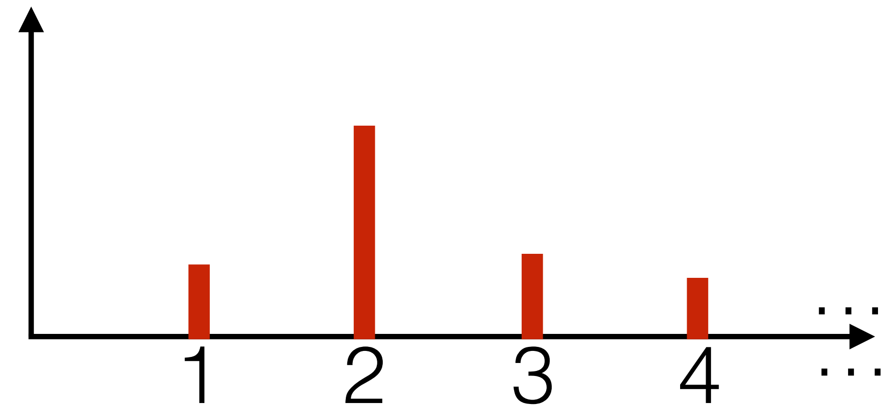

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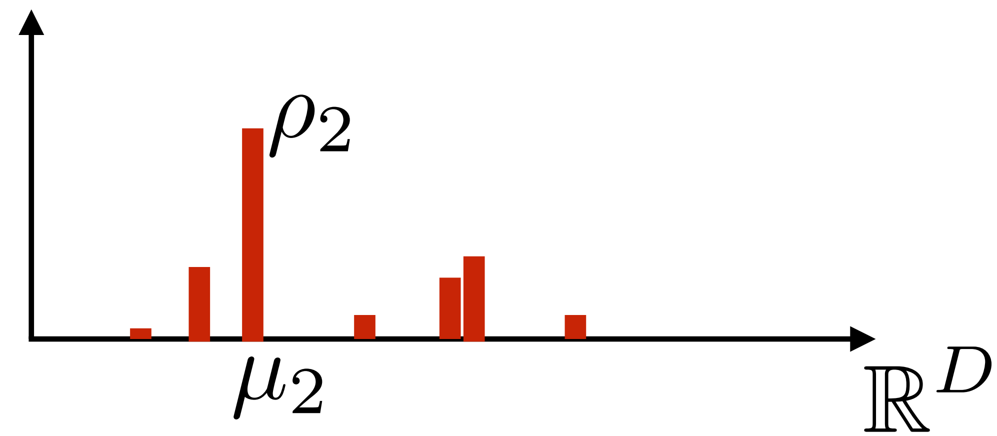

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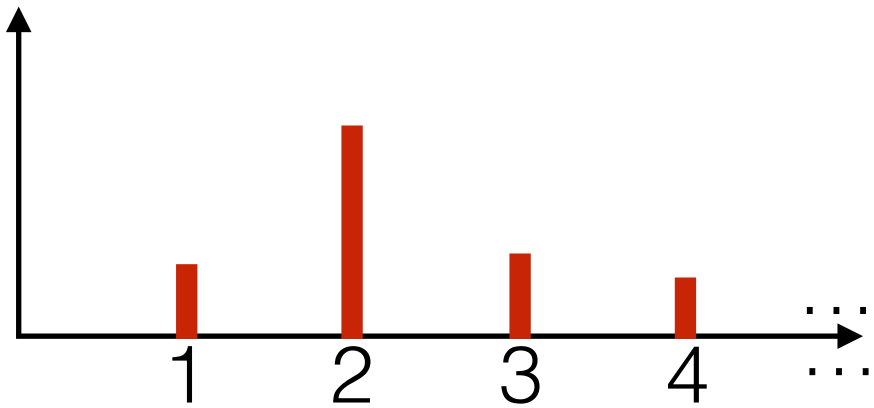

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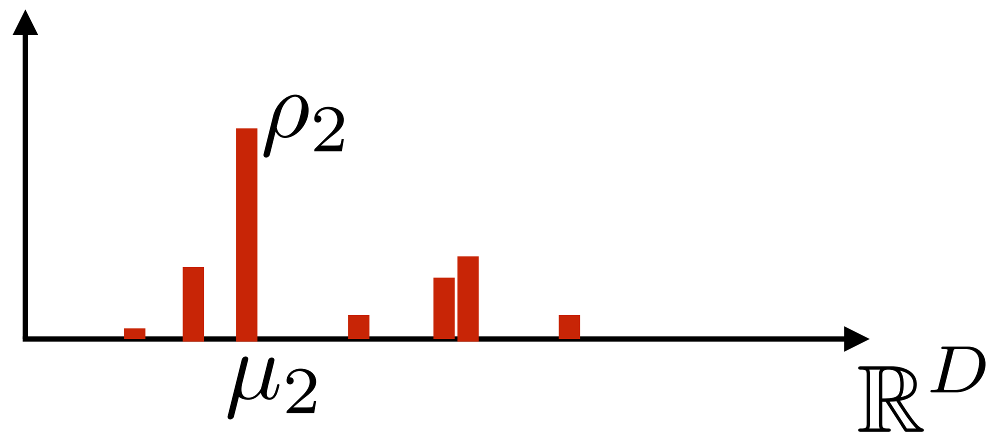

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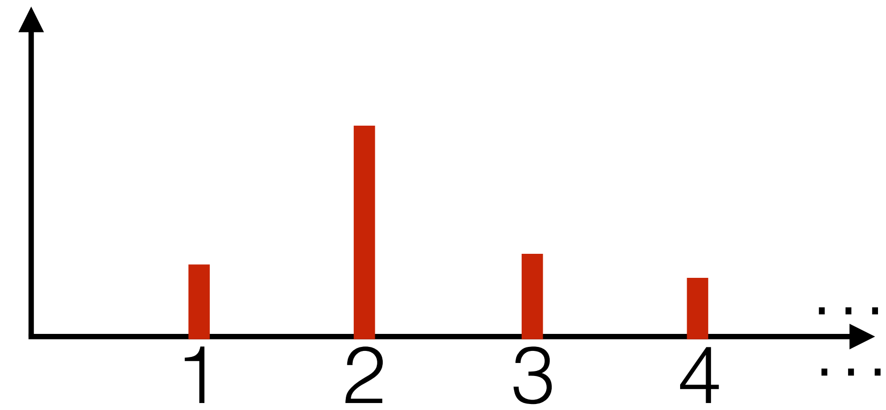

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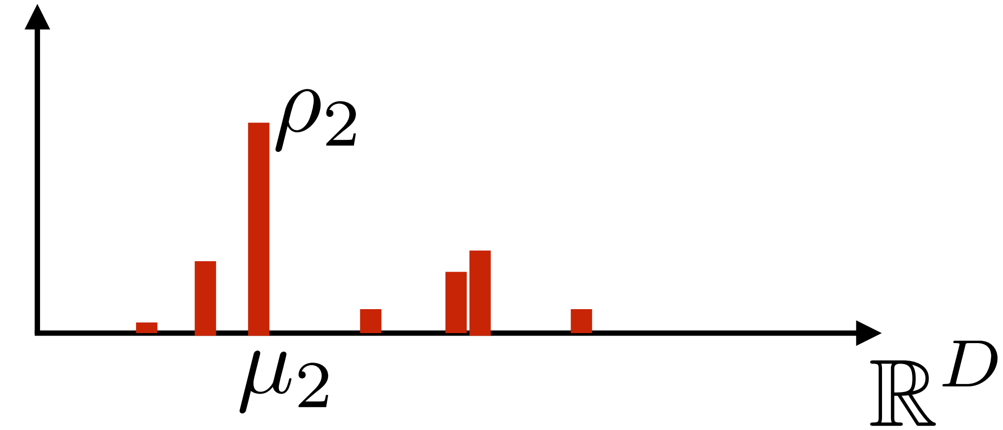

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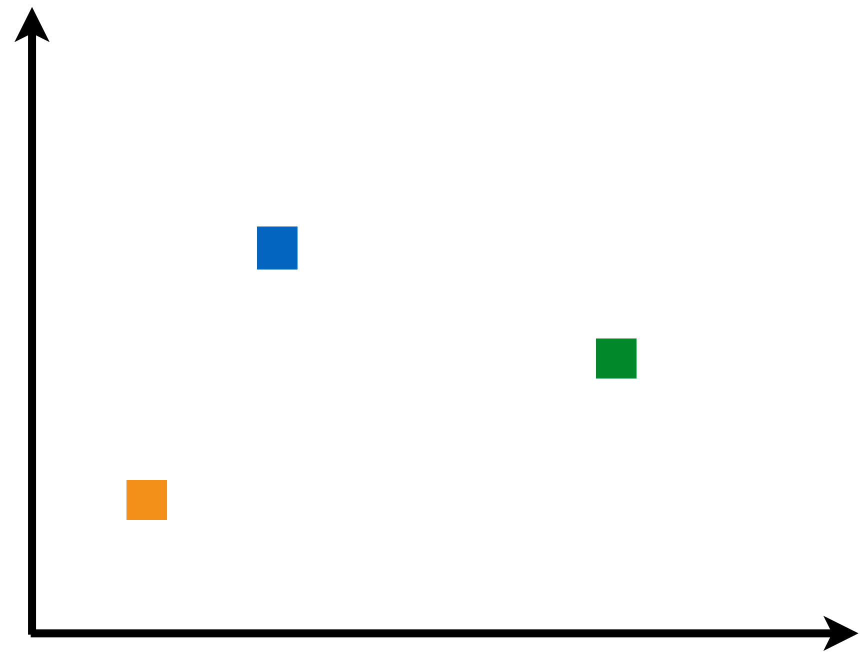

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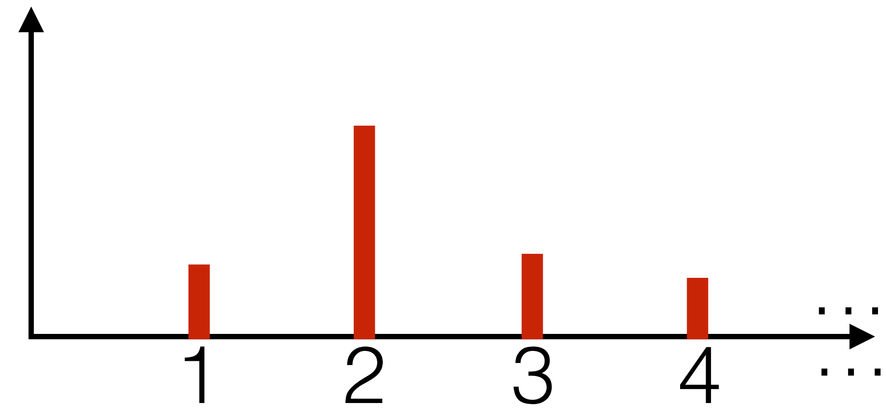

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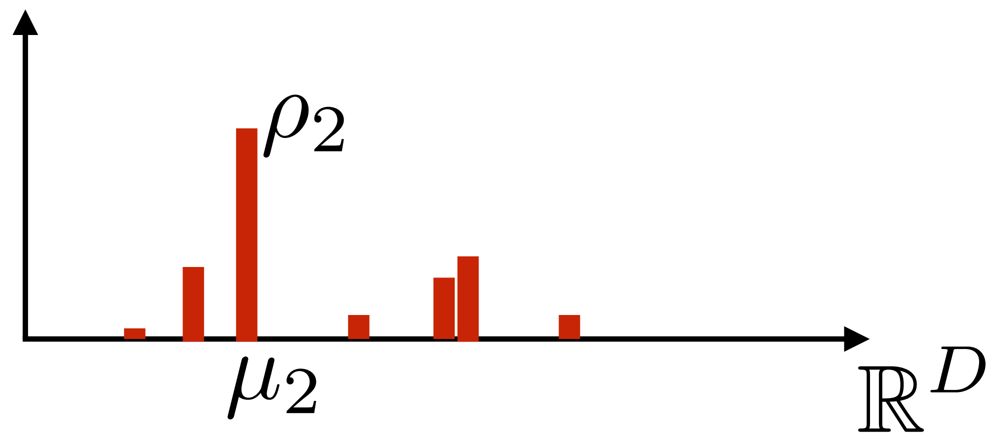

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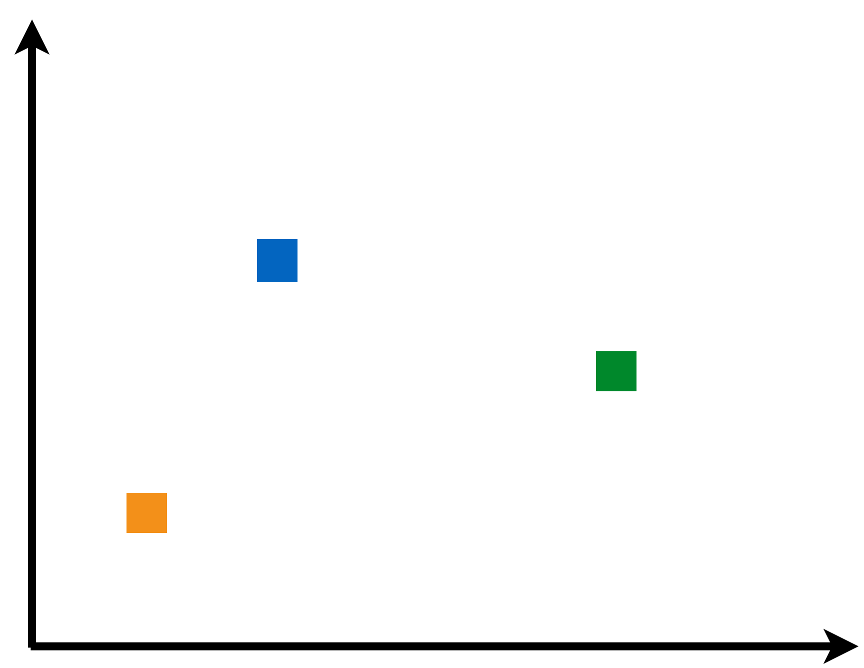

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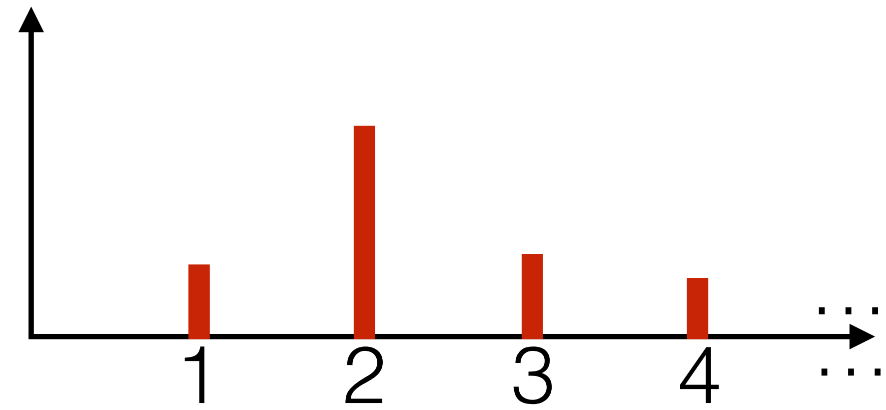

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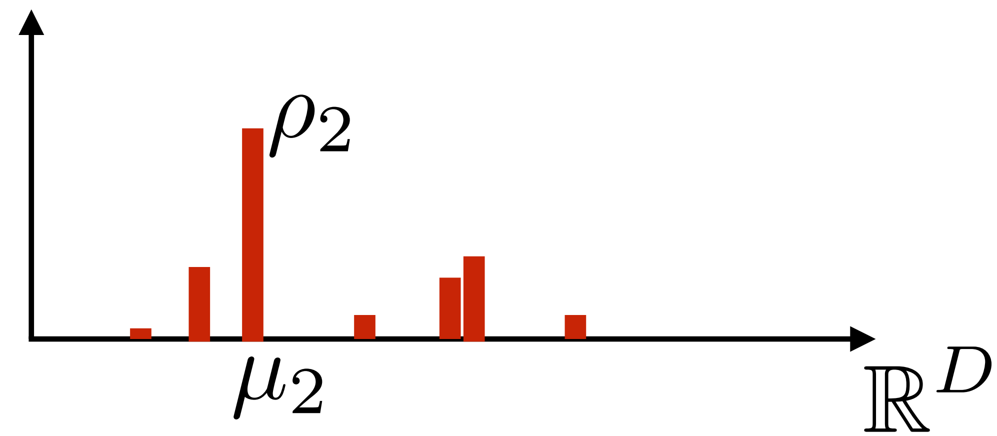

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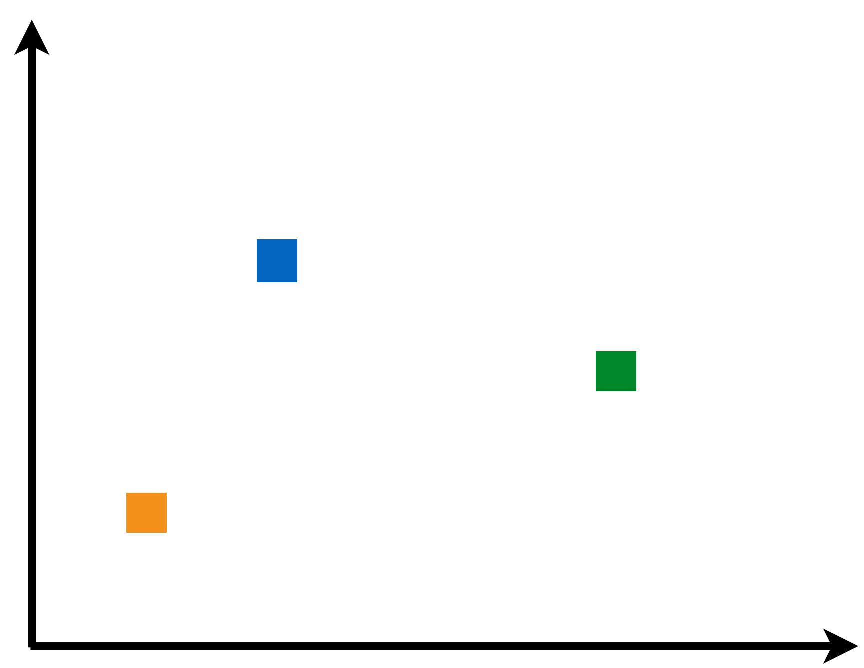

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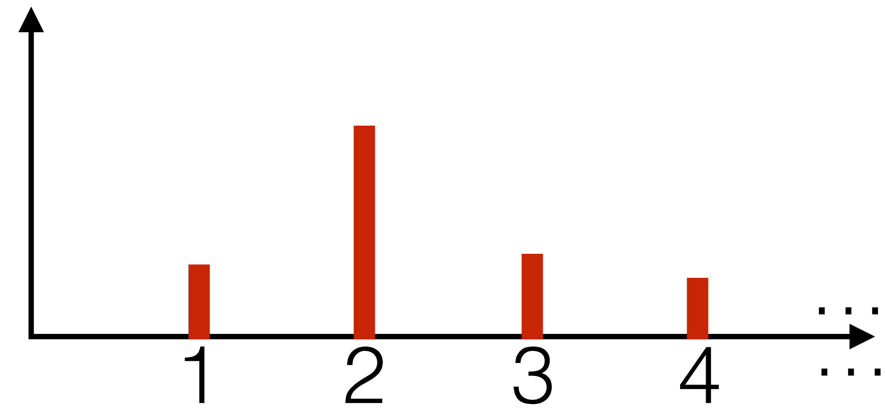

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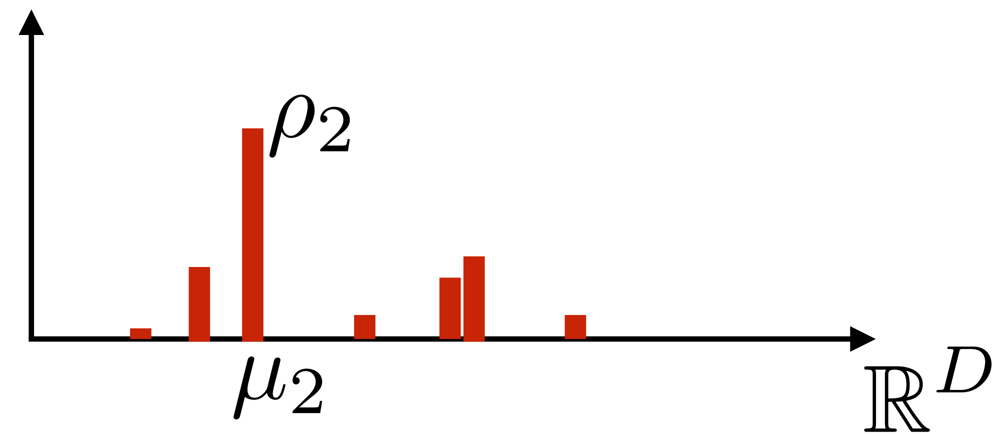

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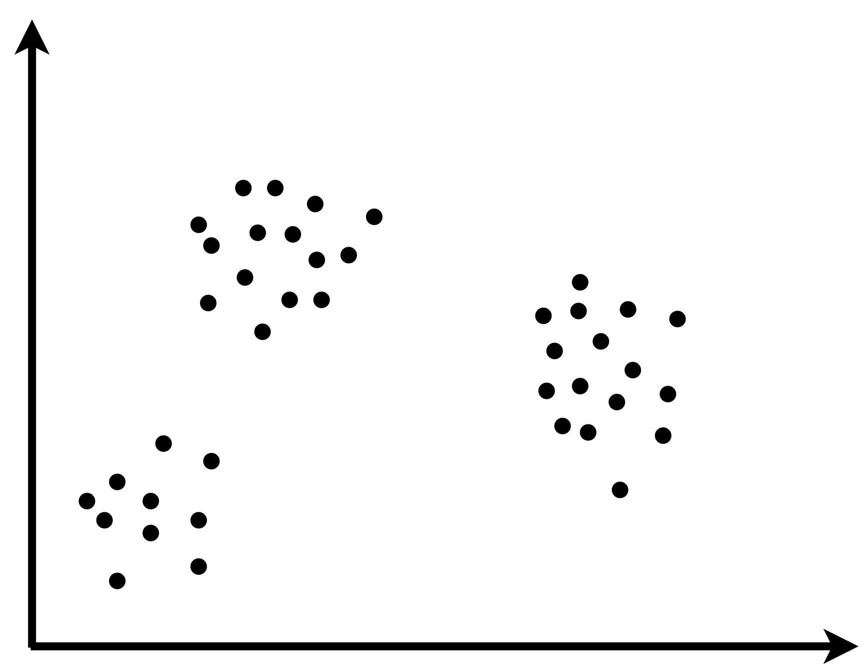

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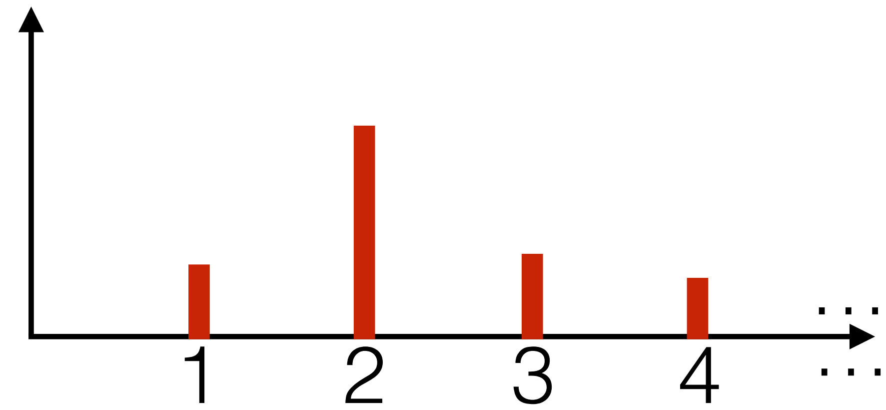

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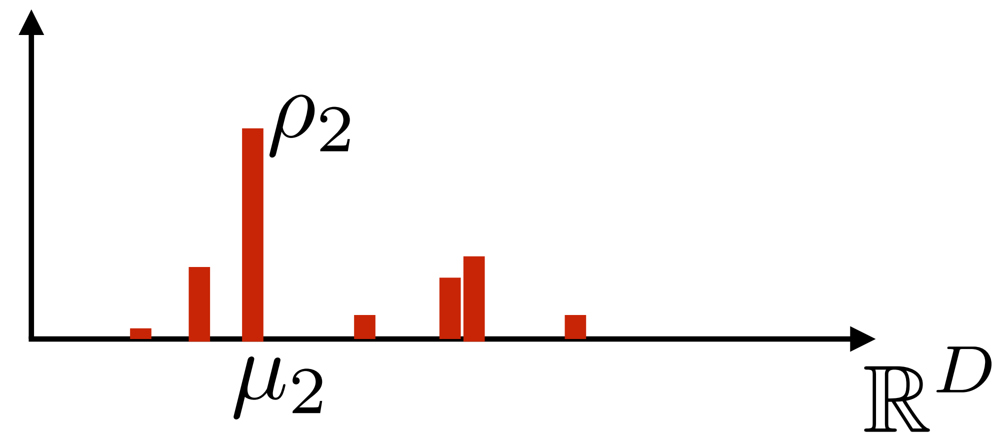

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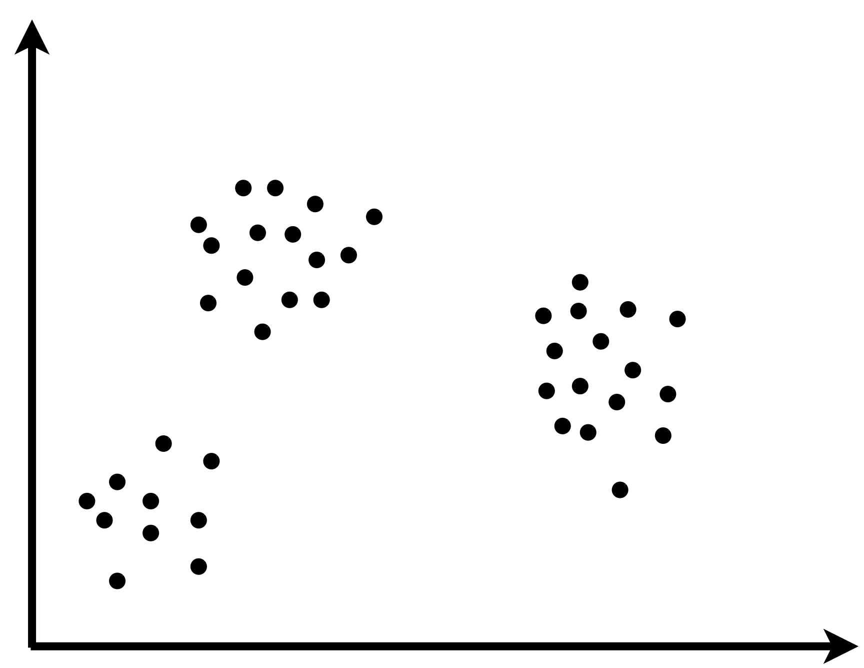

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[demo]

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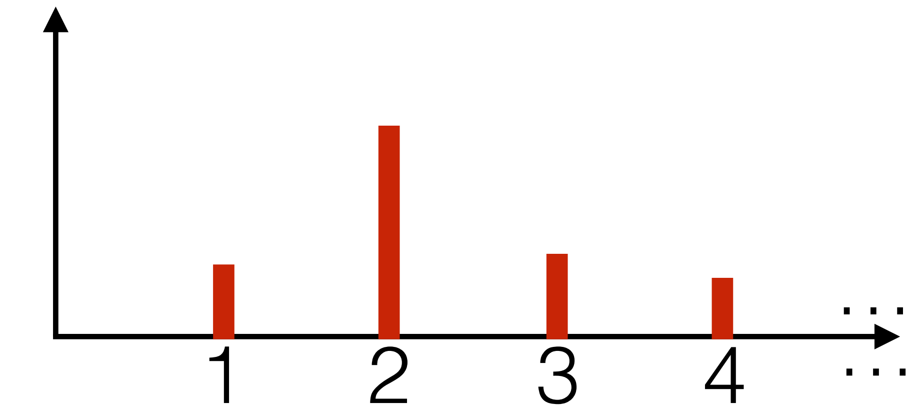

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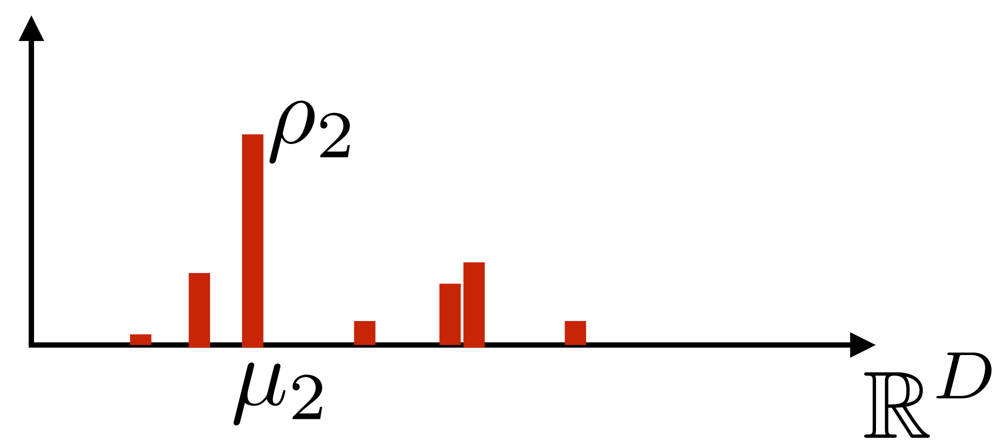

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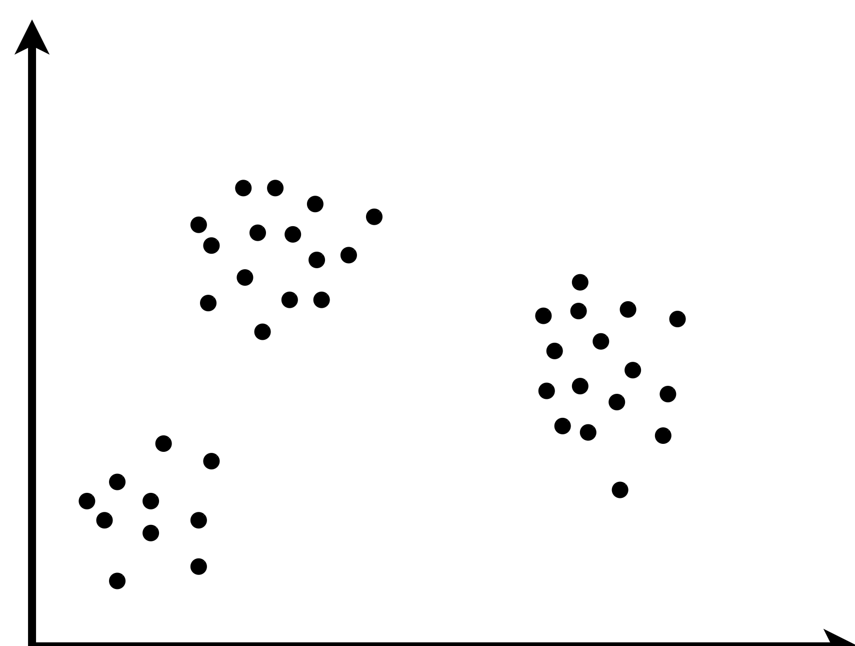

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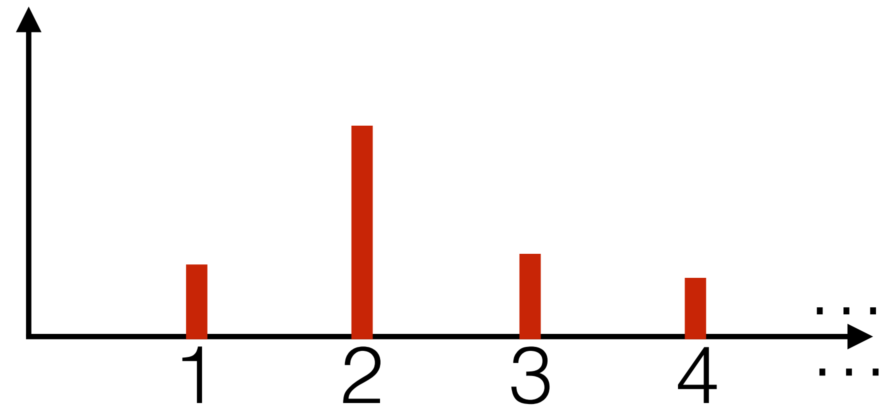

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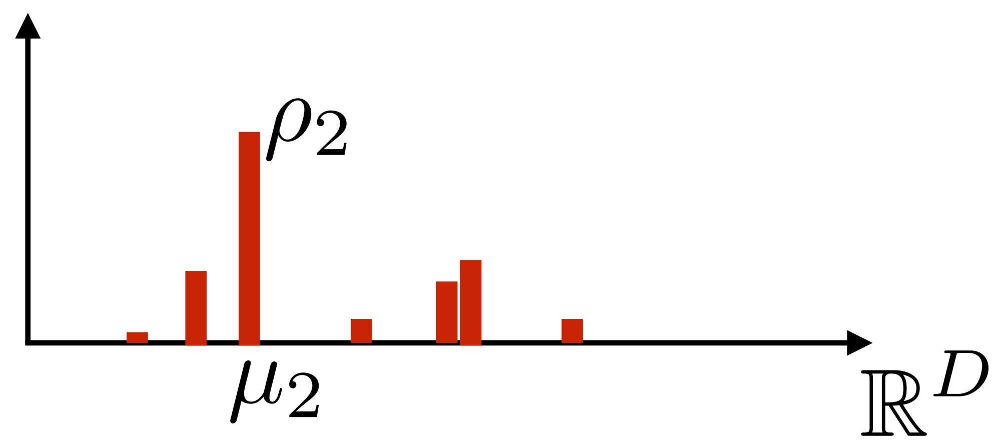

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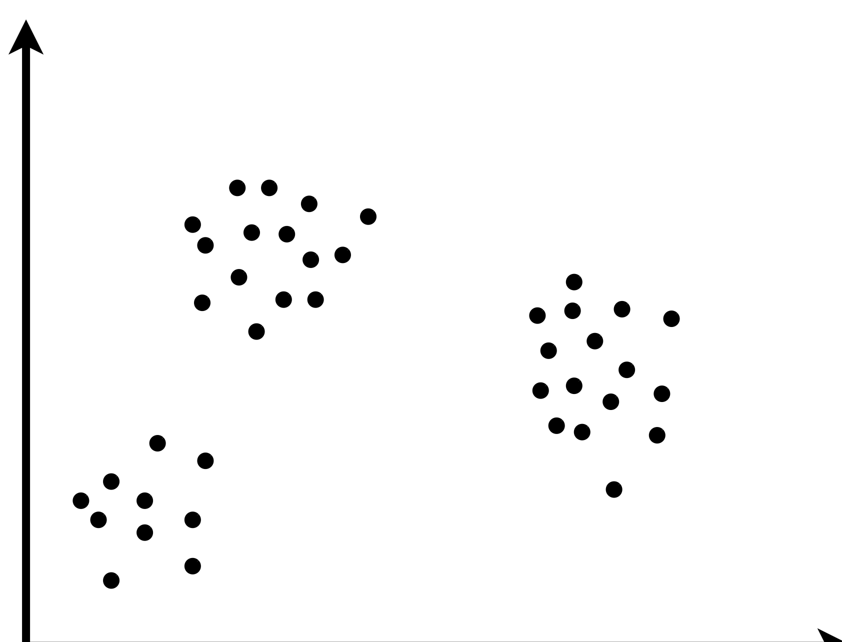

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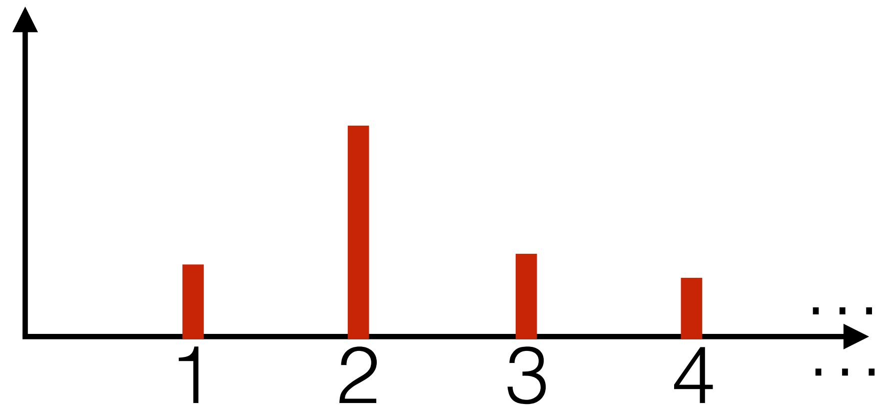

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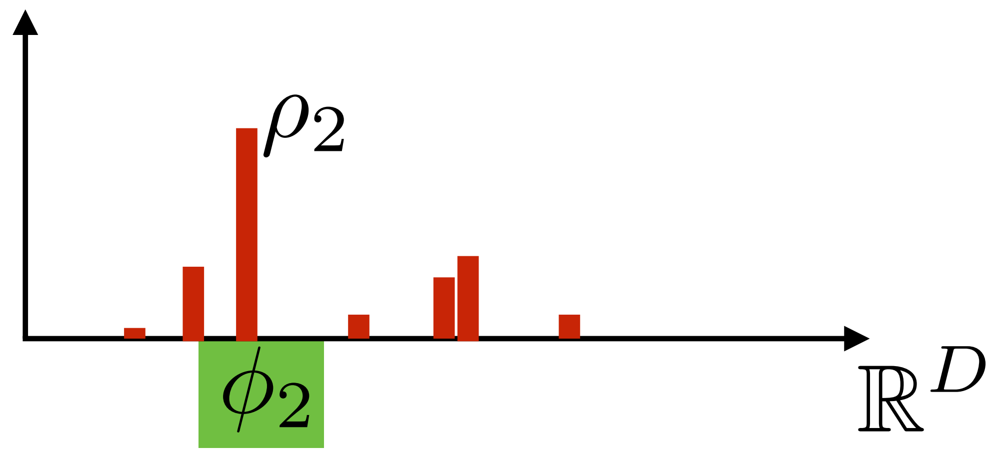

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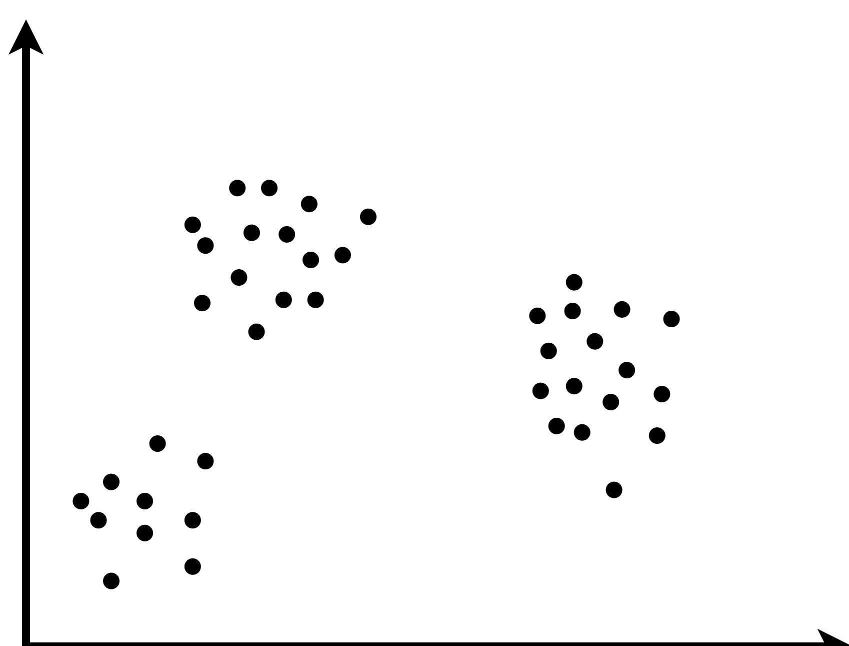

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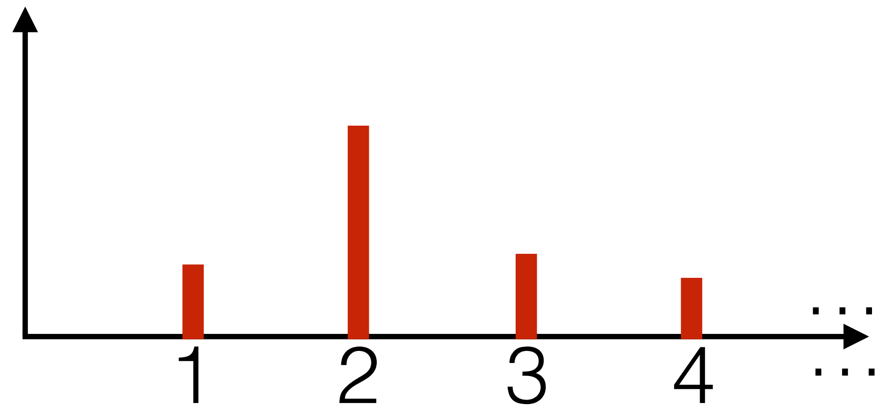

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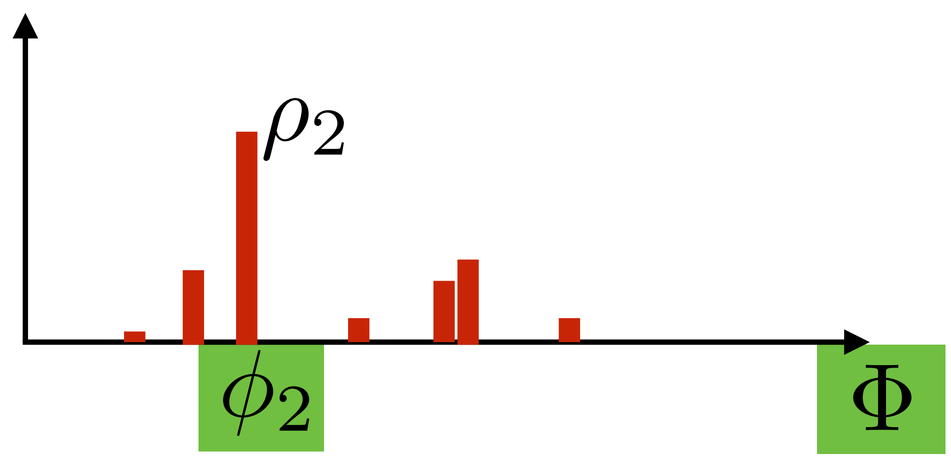

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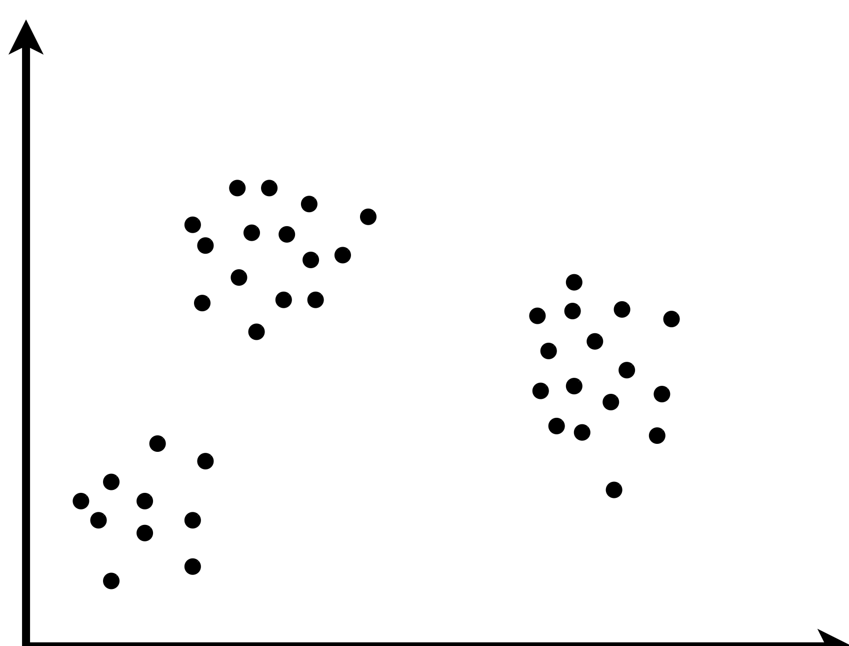

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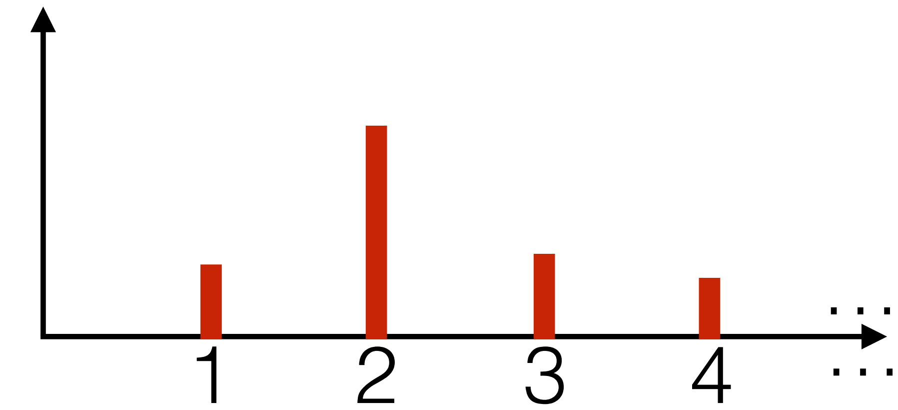

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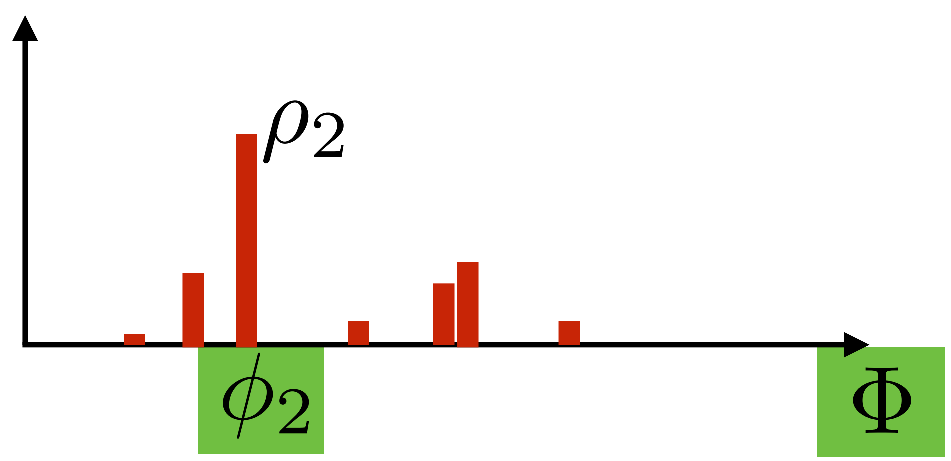

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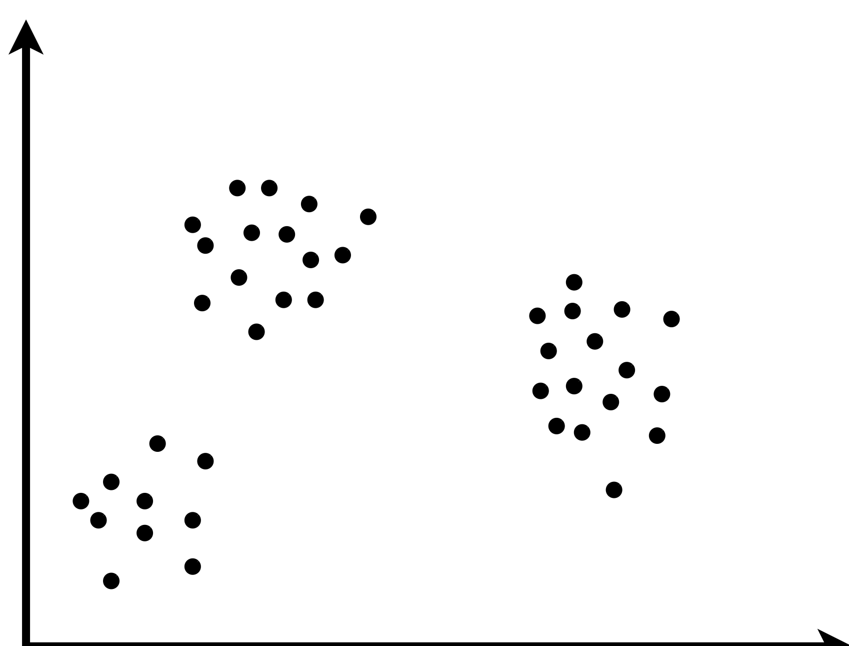

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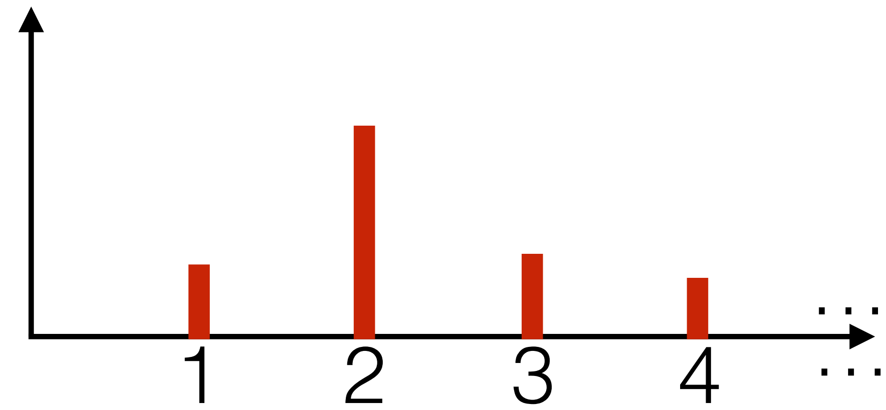

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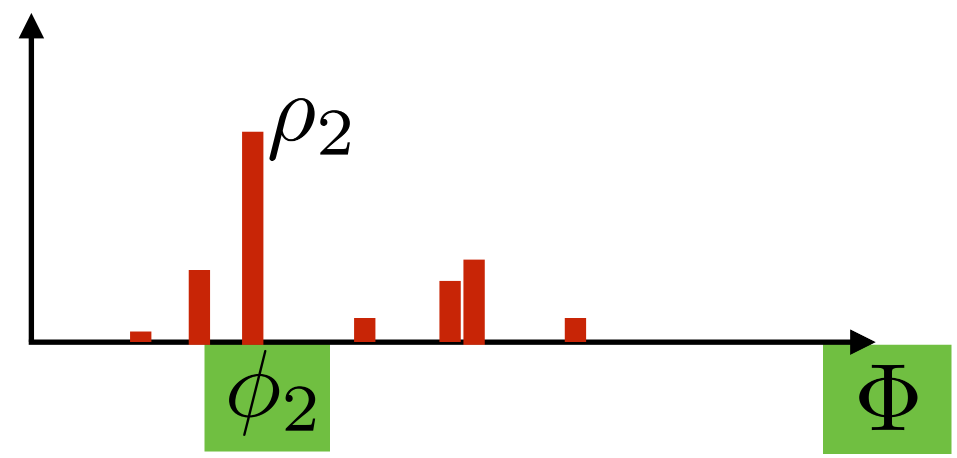

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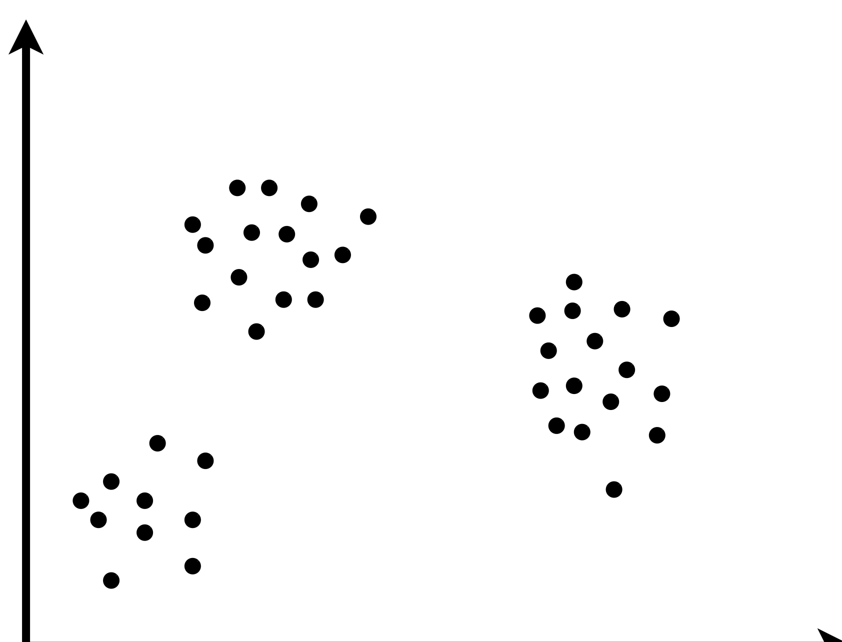

Dirichlet process mixture model

- More generally

$$\rho = (\rho_1, \rho_2, \ldots) \sim \text{GEM}(\alpha)$$

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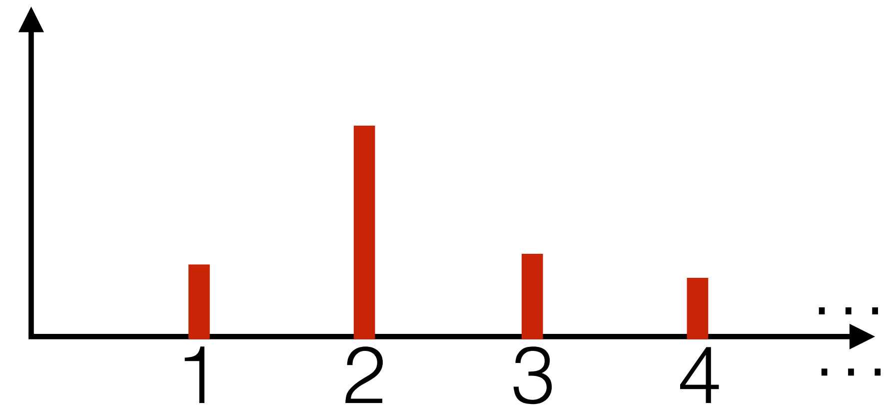

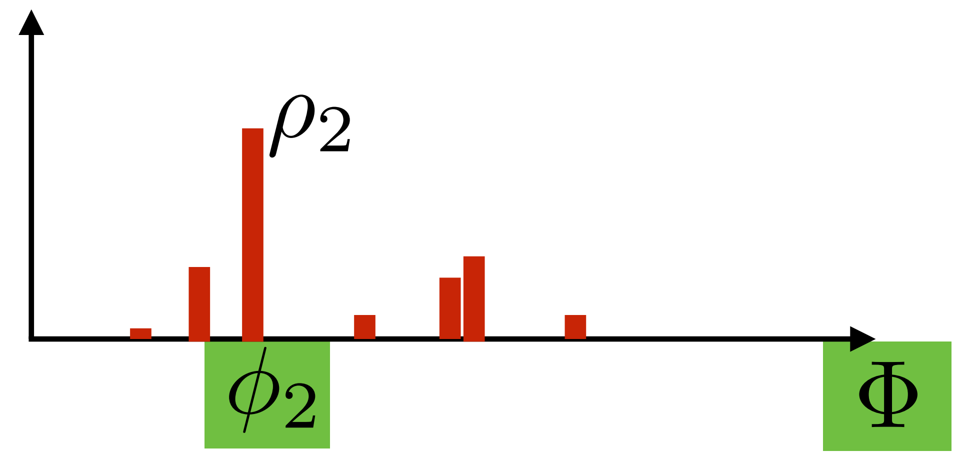

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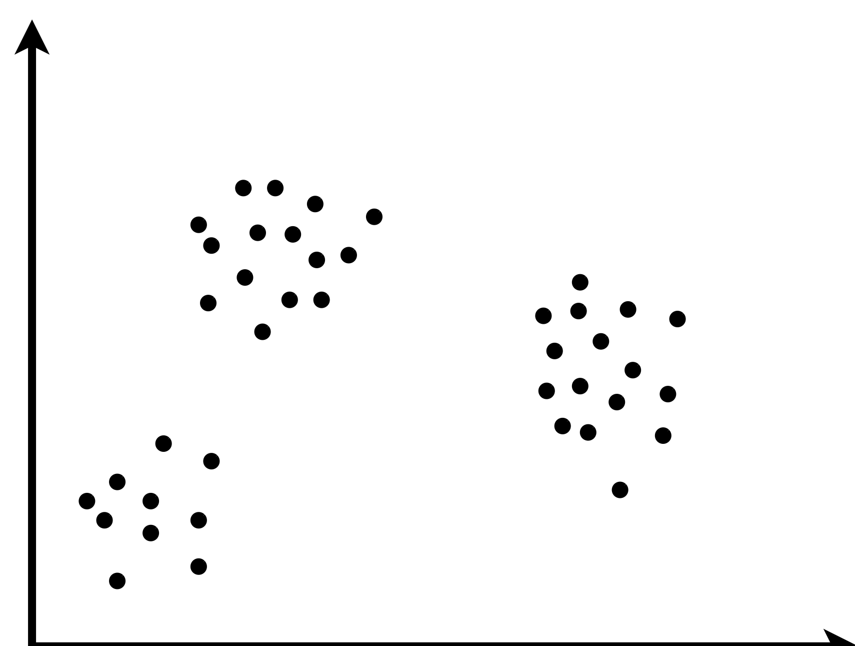

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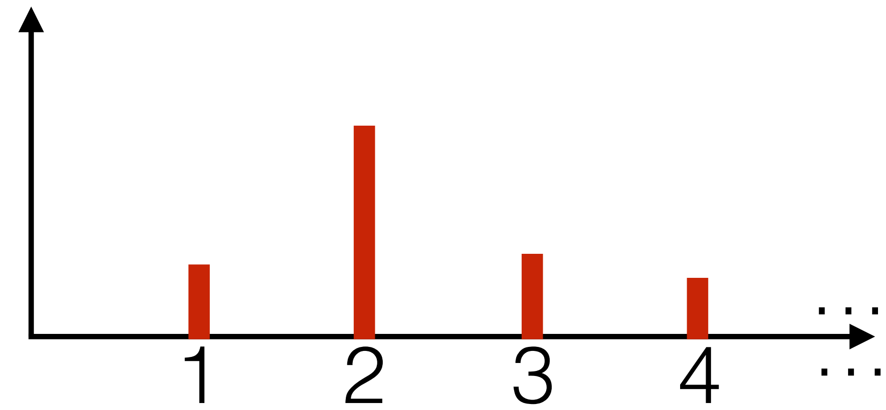

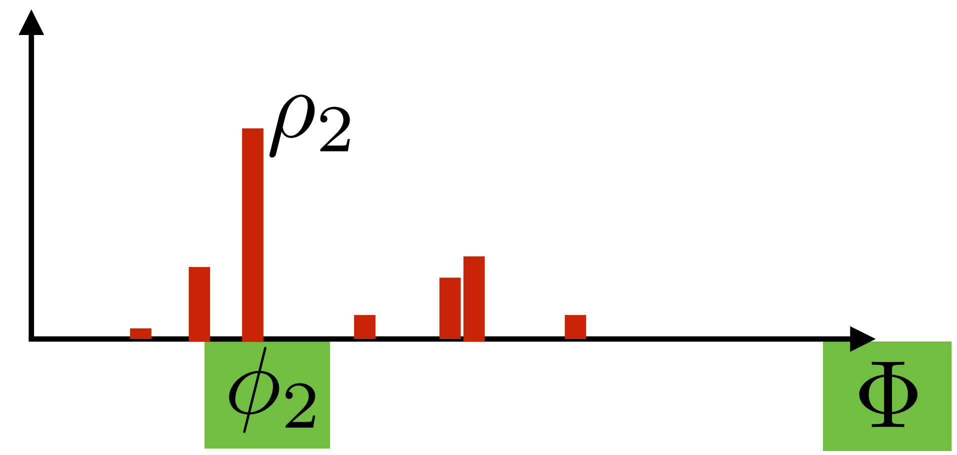

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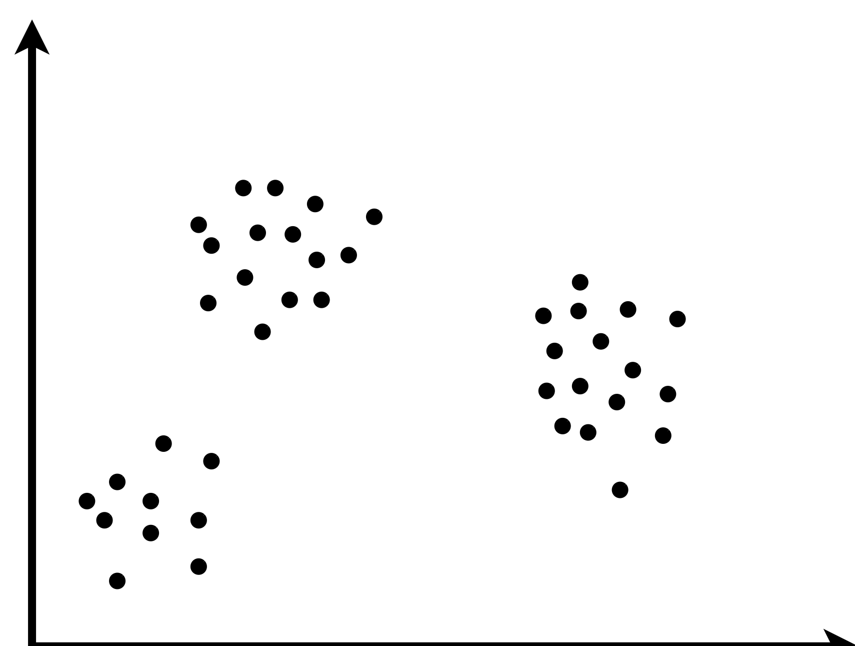

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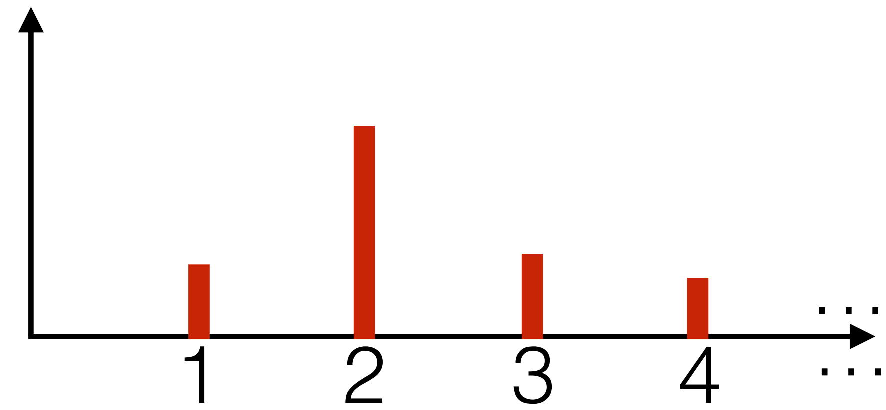

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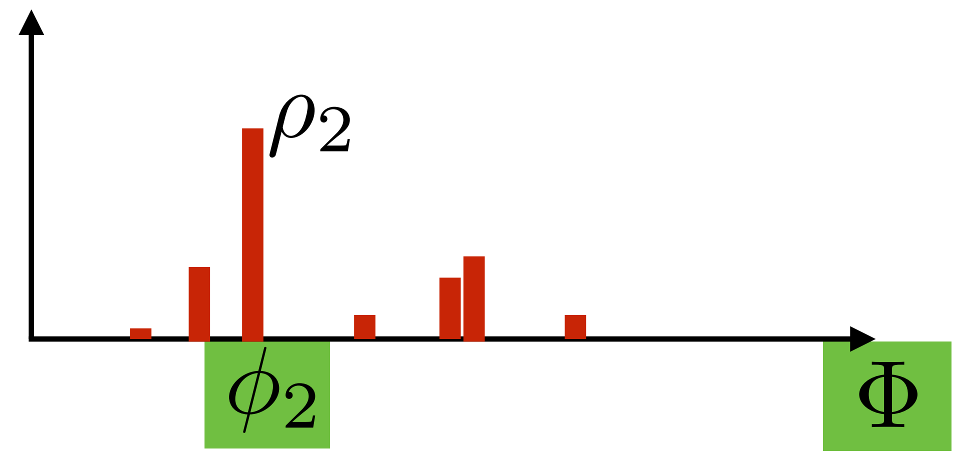

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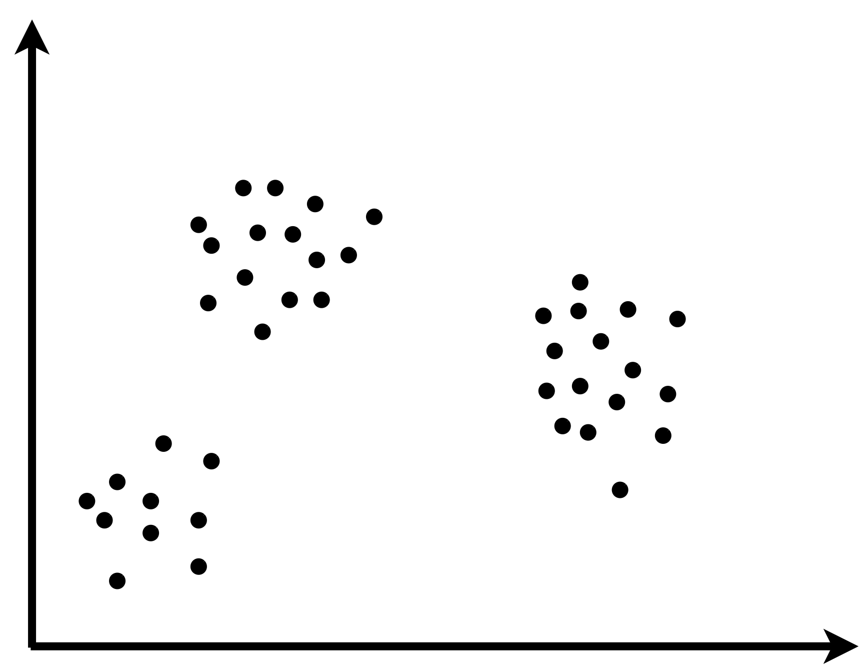

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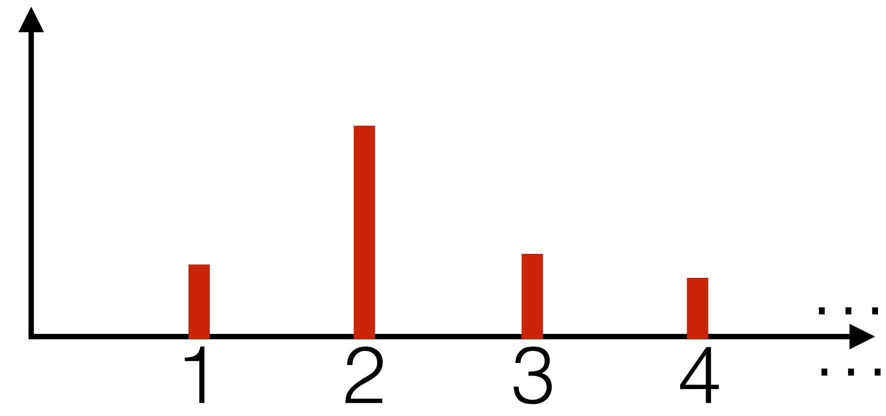

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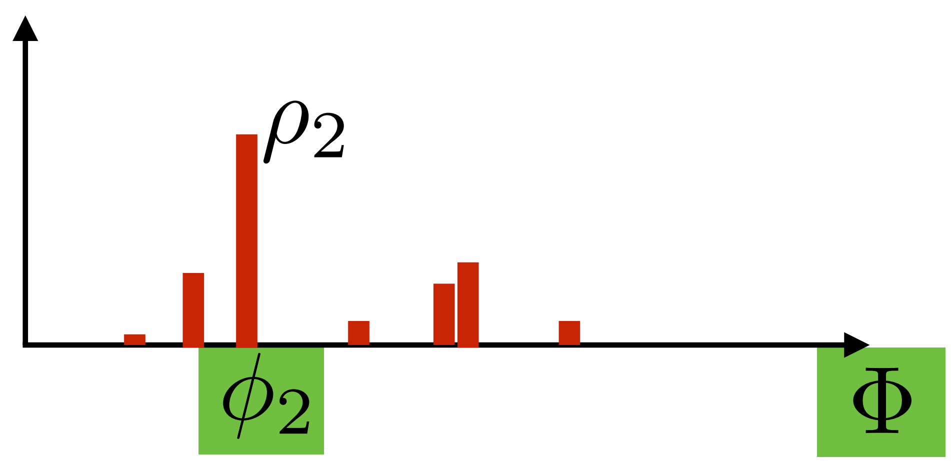

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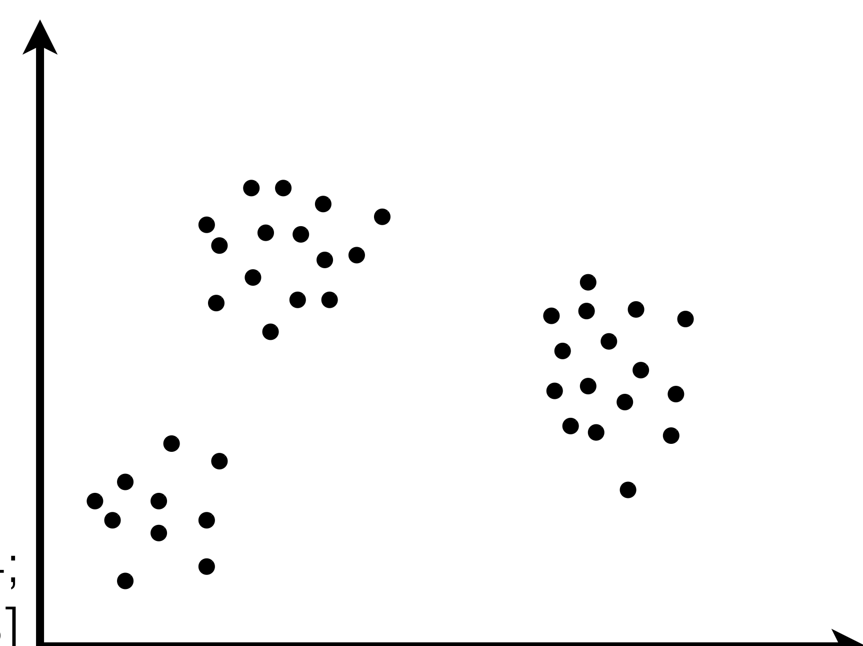
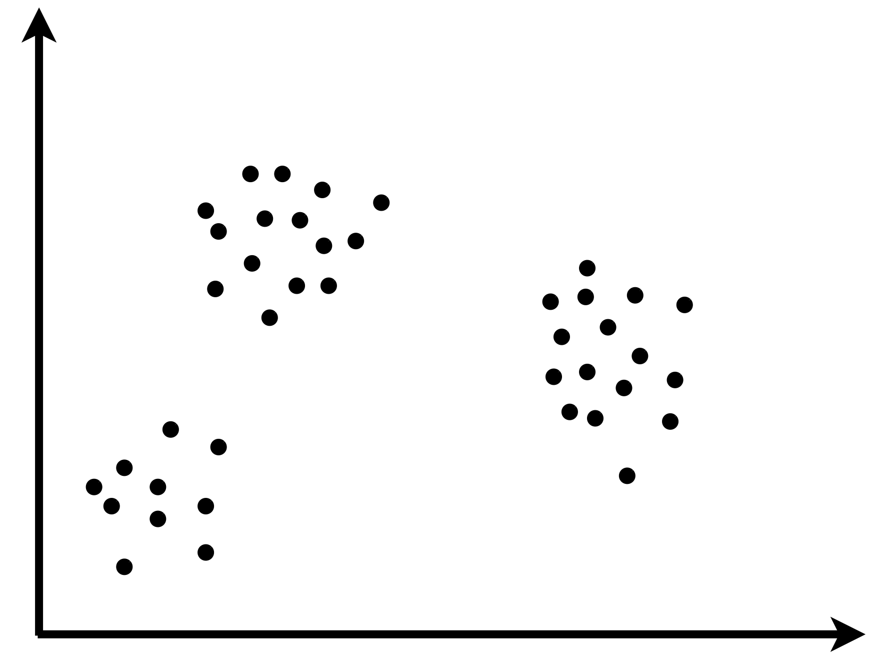


$$x_n \overset{indep}{\sim} F(\theta_n)$$

[Antoniak 1974; Ferguson 1983; West, Müller, Escobar 1994; Escobar, West 1995; MacEachern, Müller 1998]

DP or not DP, that is the question



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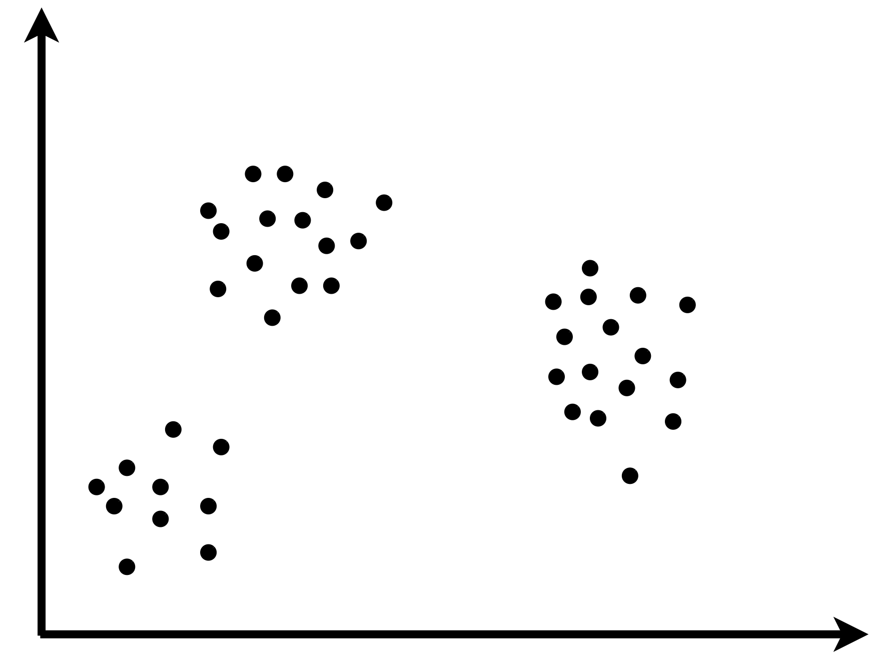
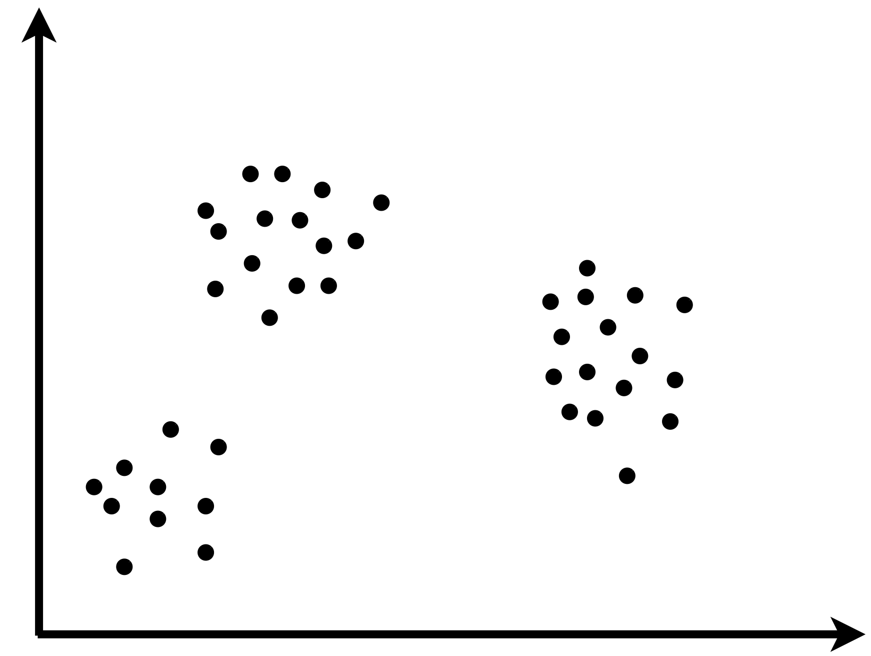
- GEM: 



DP or not DP, that is the question

- GEM: 

- Compare to:



DP or not DP, that is the question

- GEM: 
- Compare to:
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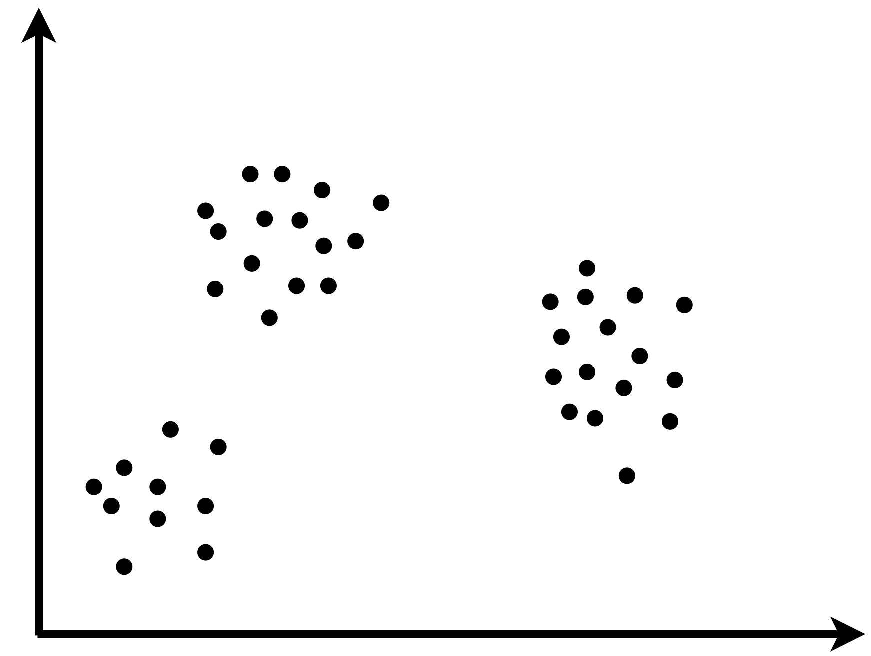

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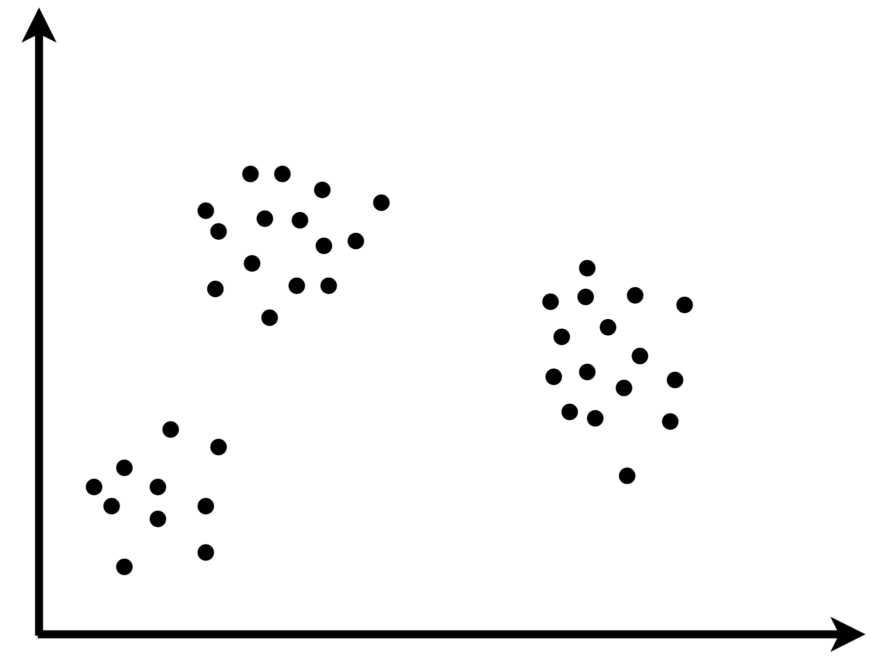

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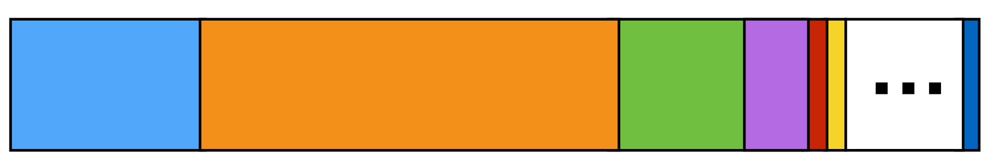
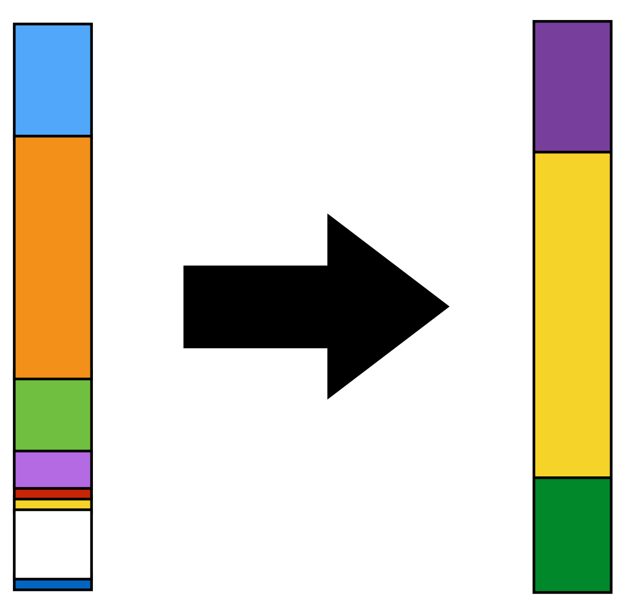
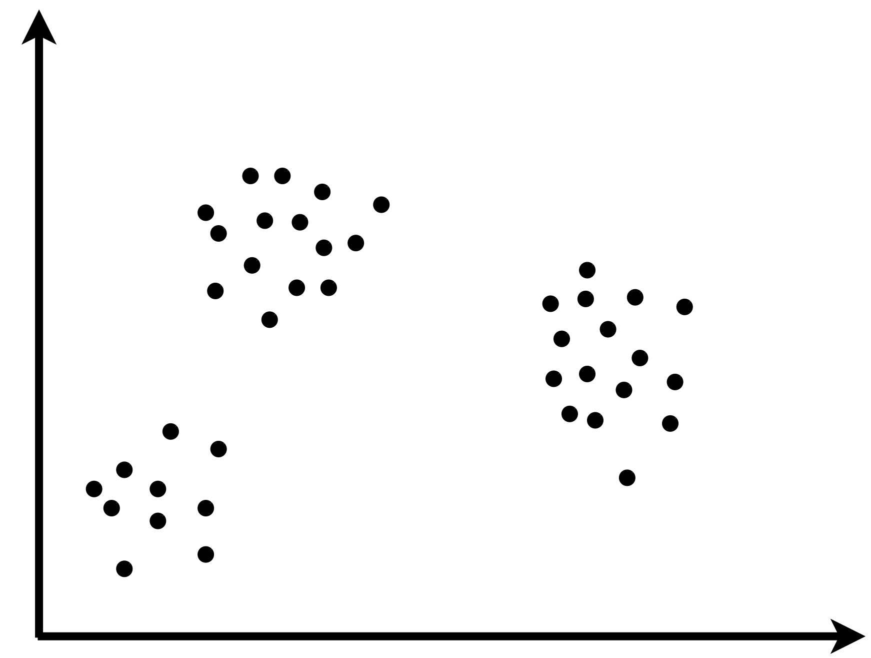


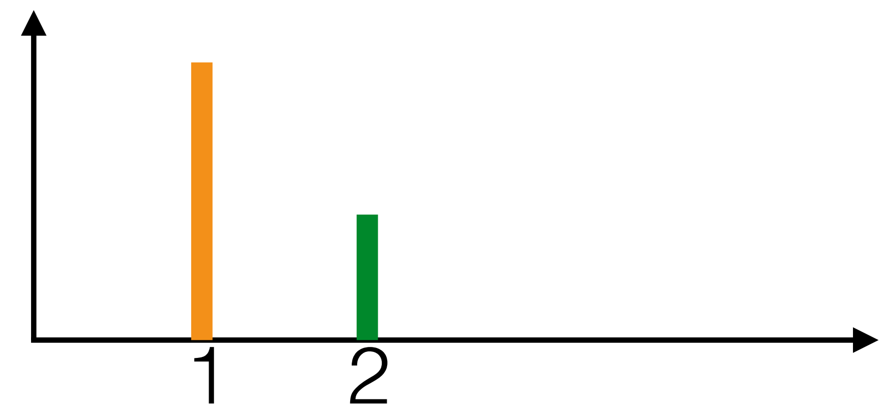
  - Time series





Marginal cluster assignments

$$\rho_1 \sim \mathrm{Beta}(a_1, a_2), z_n \stackrel{iid}{\sim} \mathrm{Cat}(\rho_1, \rho_2)$$



Marginal cluster assignments

- Integrate out the frequencies

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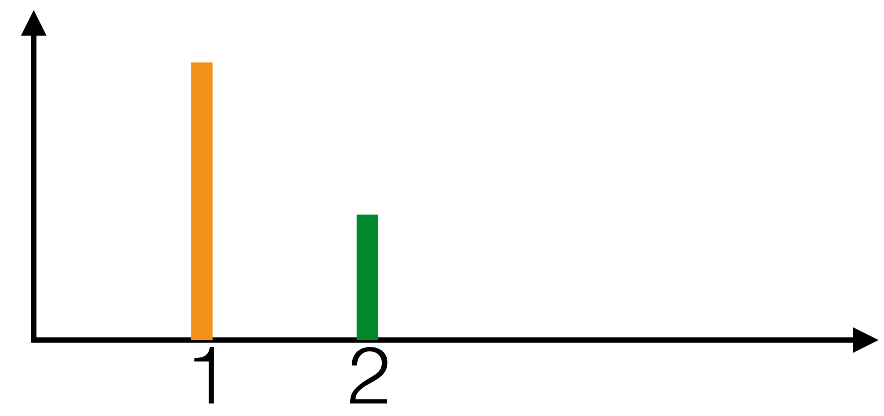

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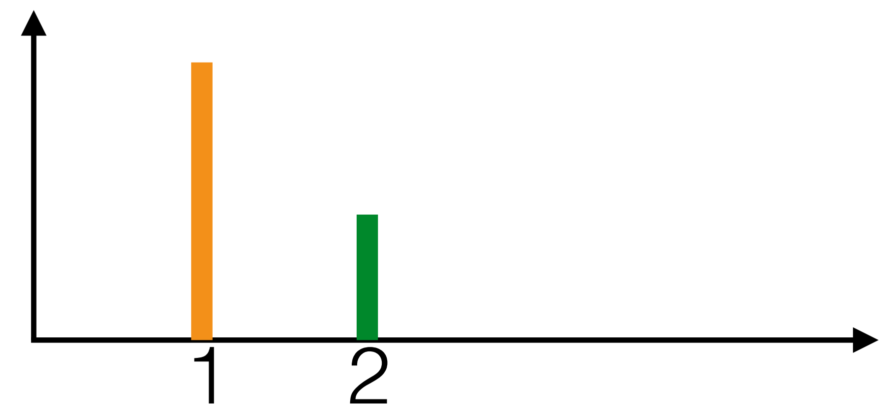

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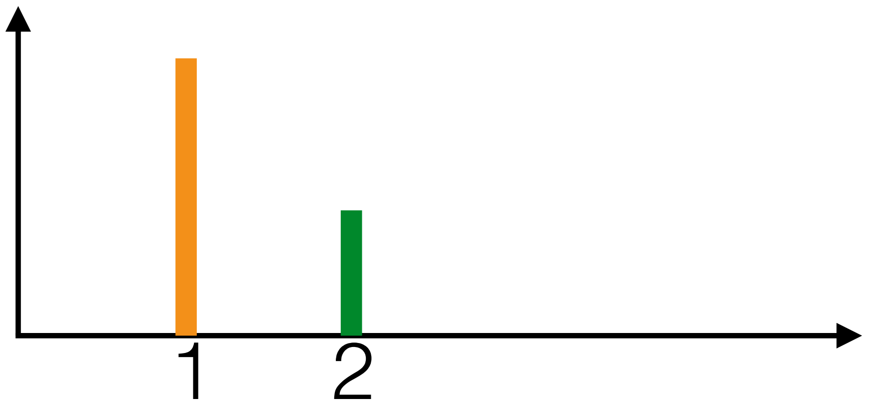

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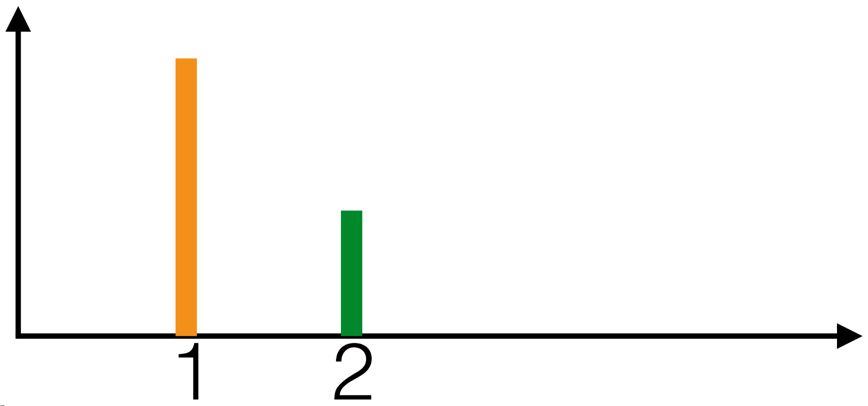

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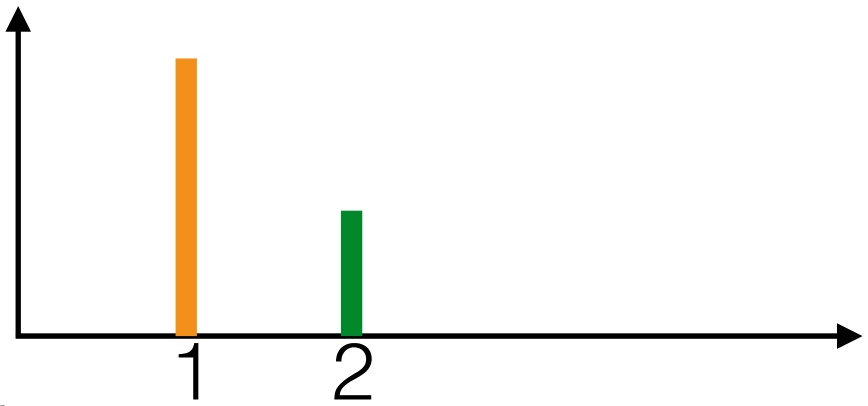

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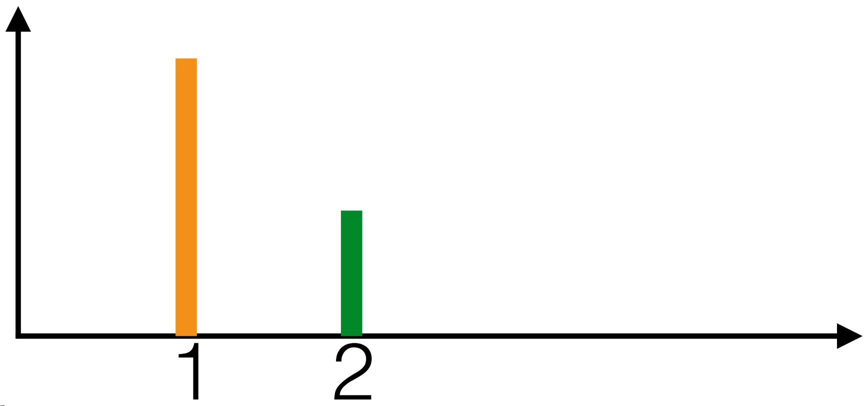

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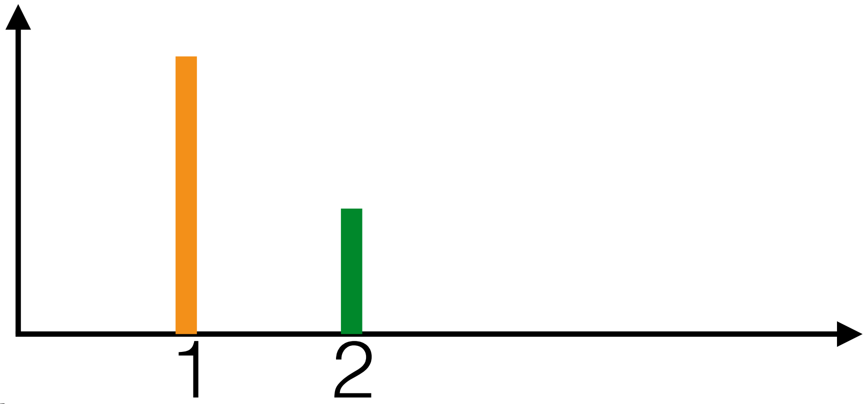

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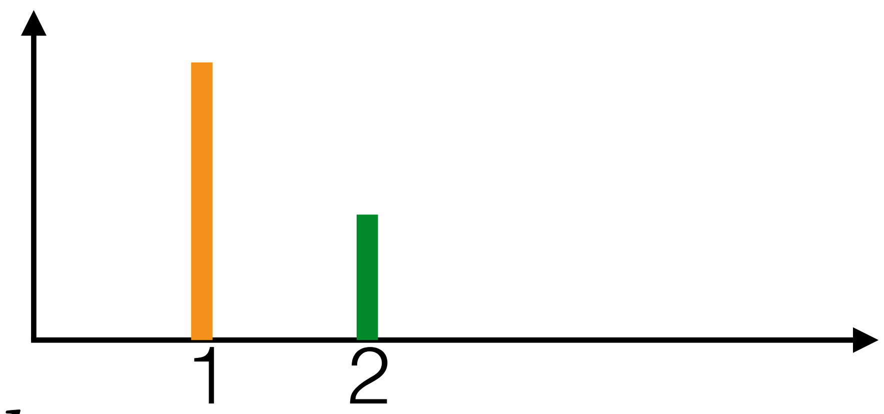

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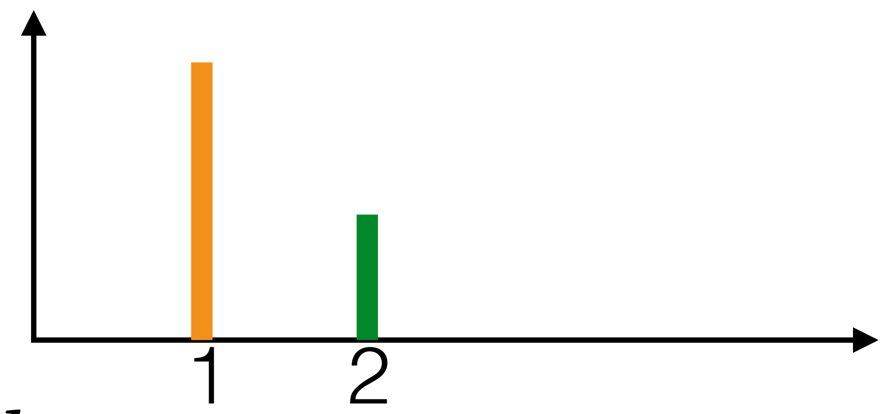

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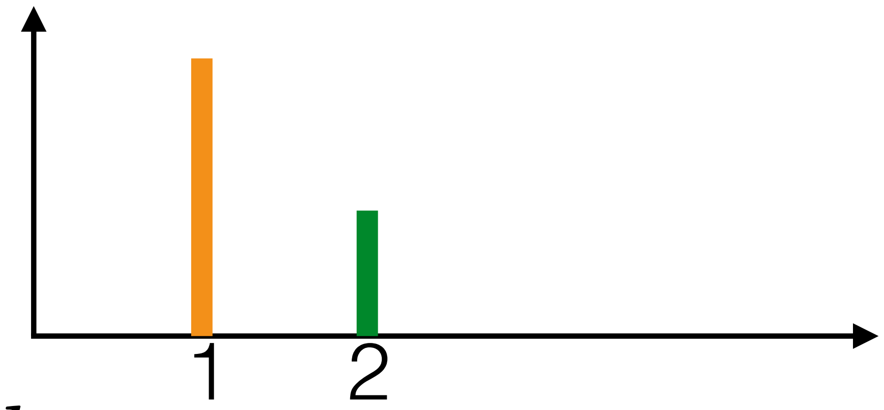

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Marginal cluster assignments

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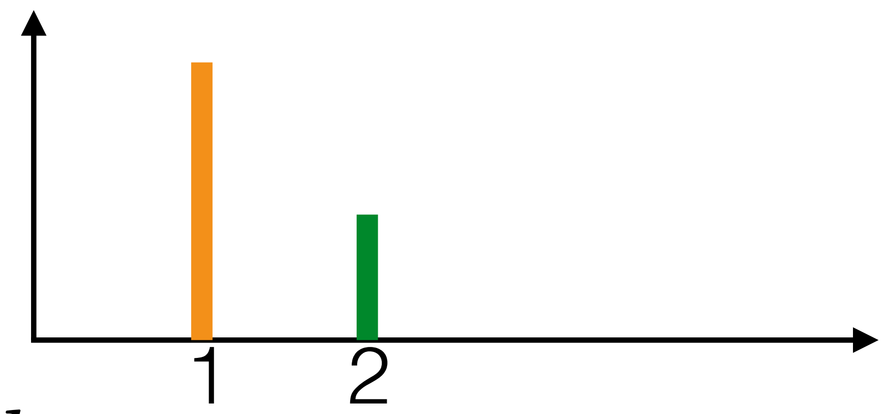

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Marginal cluster assignments

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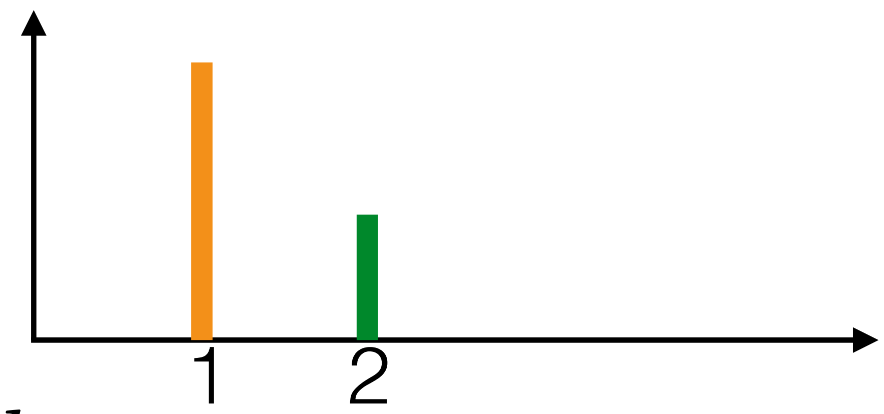

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Marginal cluster assignments

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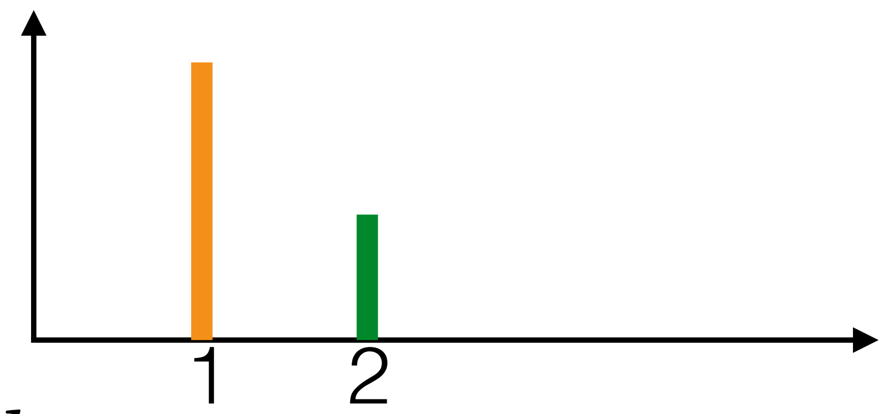

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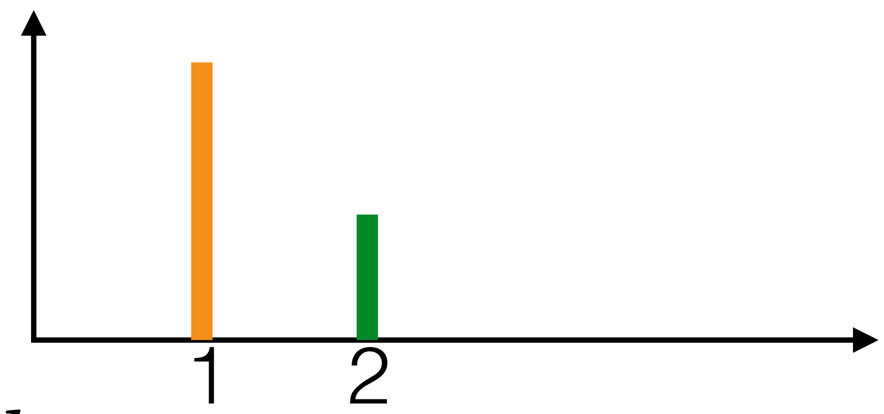
Recall
$\Gamma(x+1) = x\Gamma(x)$

Marginal cluster assignments

- Integrate out the frequencies



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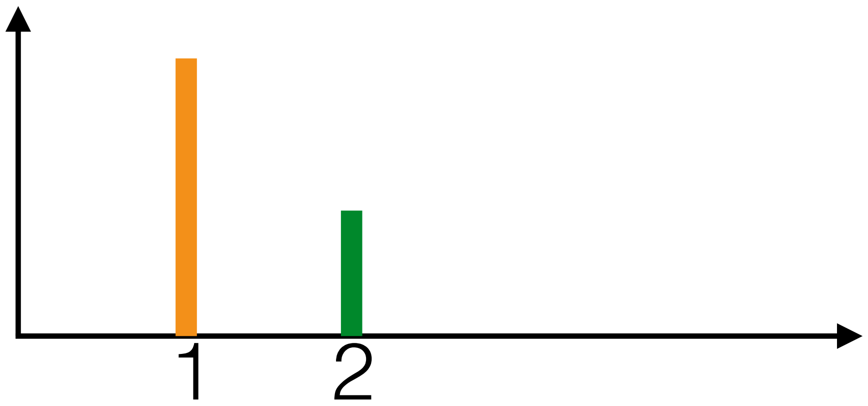
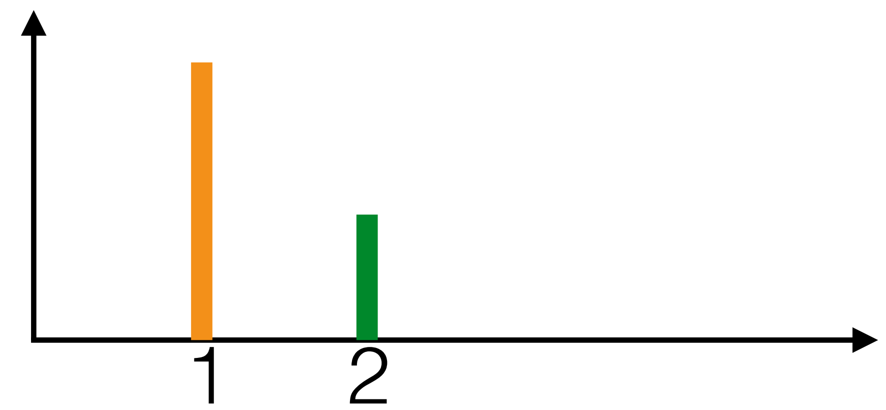


$$a_{1,n} := a_1 + \sum_{m=1}^{n-1} \mathbf{1}\{z_m = 1\}, a_{2,n} = a_2 + \sum_{m=1}^{n-1} \mathbf{1}\{z_m = 2\}$$

Marginal cluster assignments

- Integrate out the frequencies

$$\rho_1 \sim \text{Beta}(a_1, a_2), z_n \overset{iid}{\sim} \text{Cat}(\rho_1, \rho_2)$$



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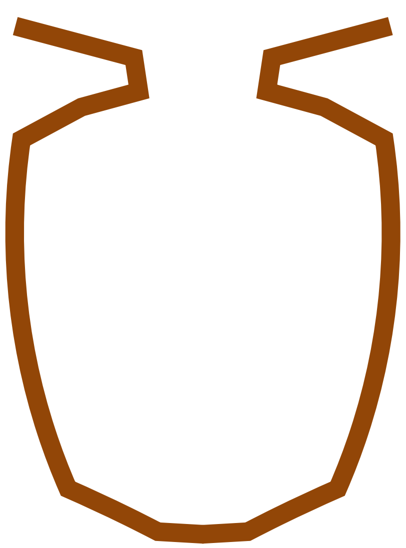

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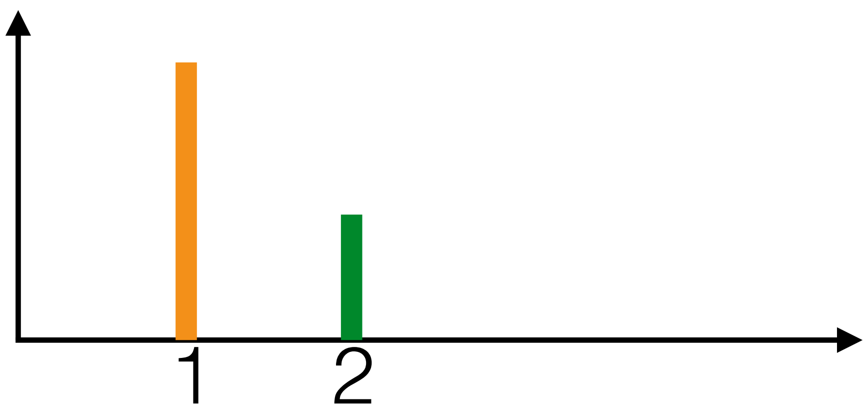

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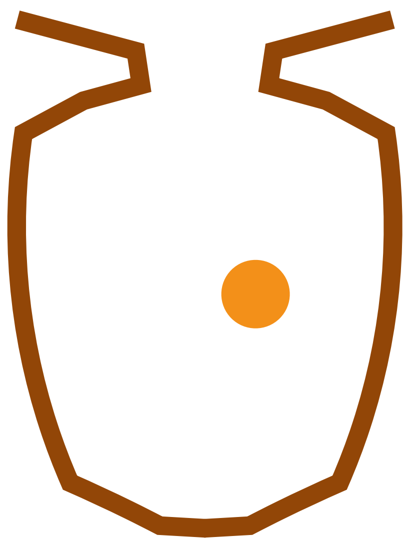

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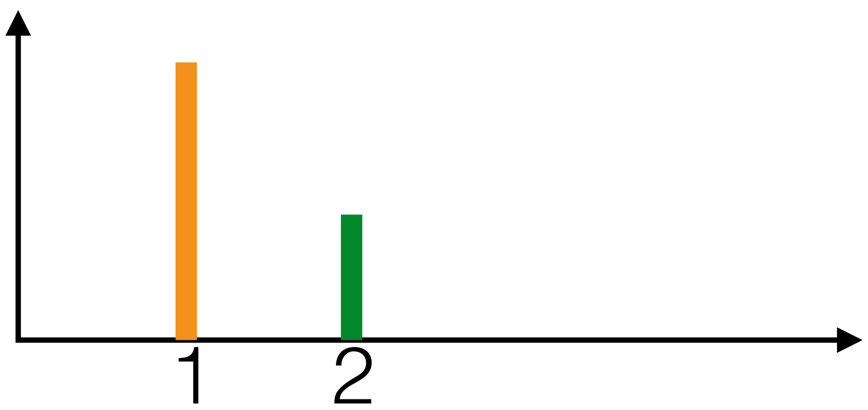

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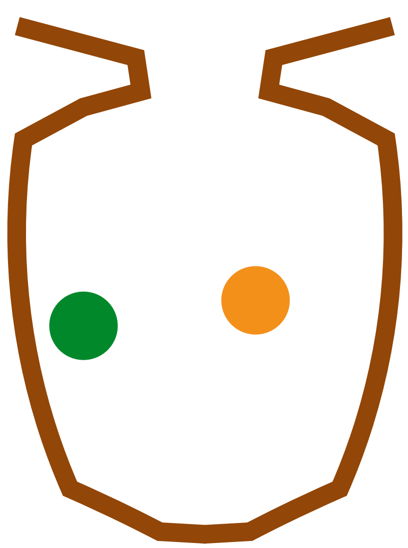

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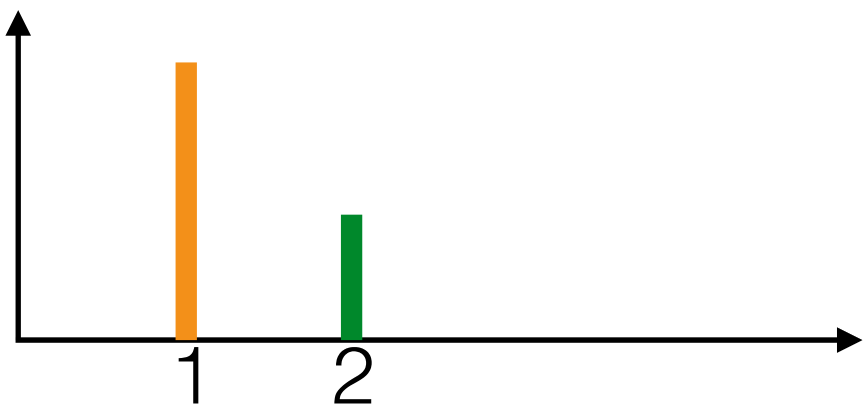

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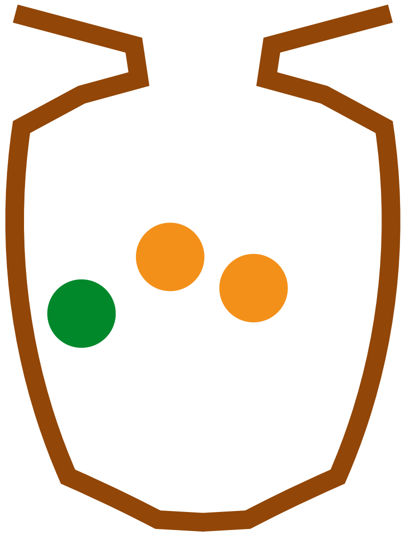

Marginal cluster assignments

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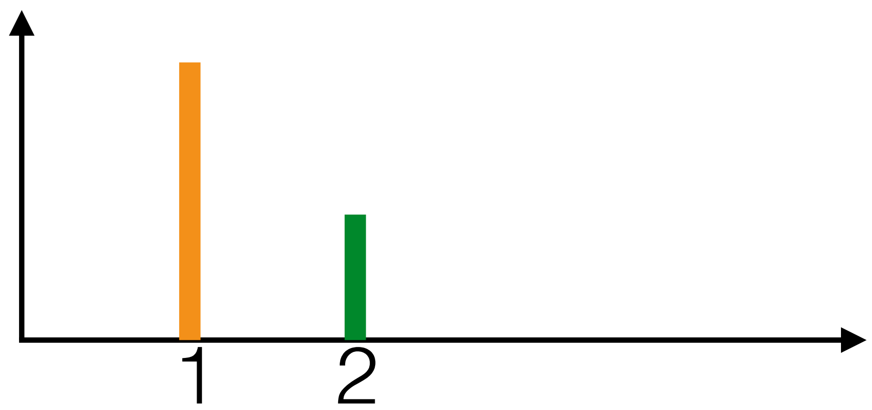

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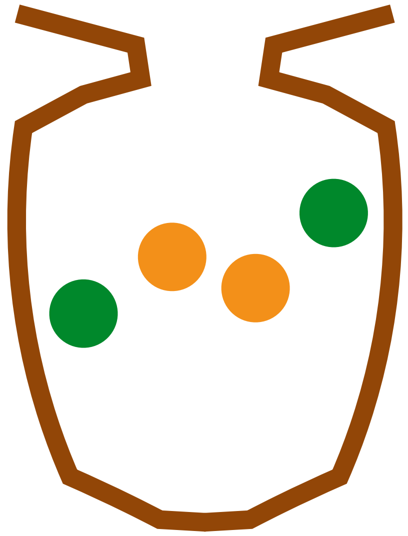

Marginal cluster assignments

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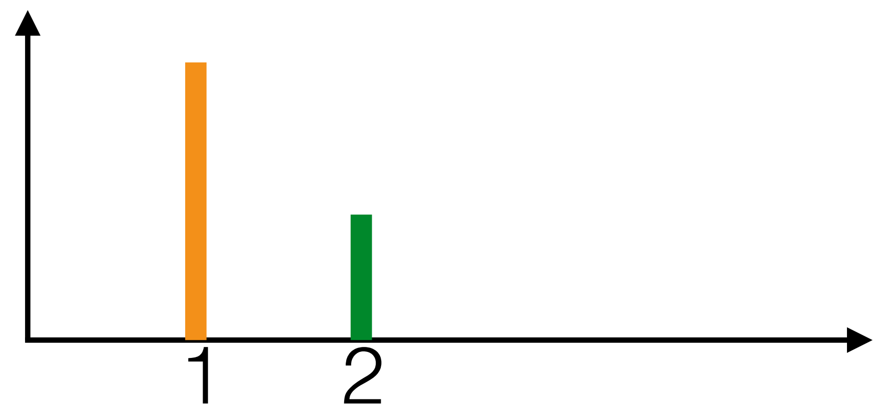

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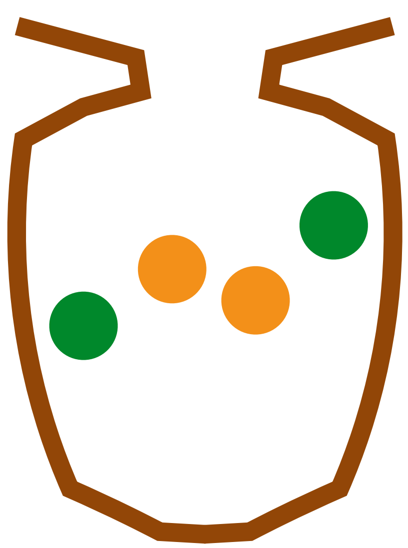

$$\lim_{n \to \infty} \frac{\#\ \text{orange}}{\#\ \text{total}}$$

Marginal cluster assignments

- Integrate out the frequencies

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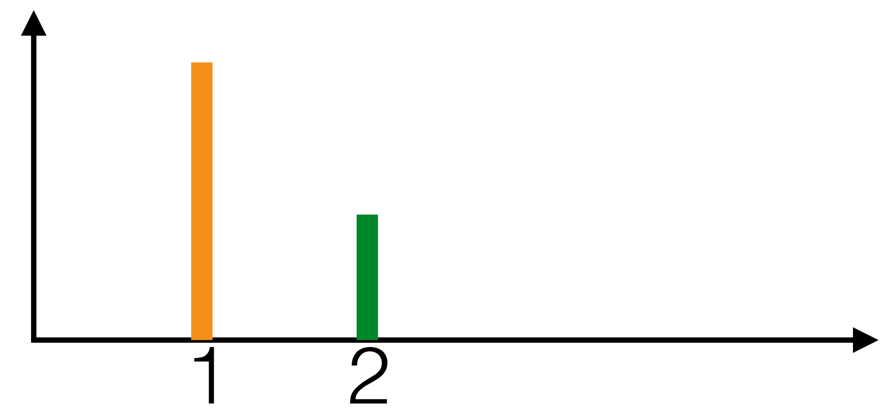

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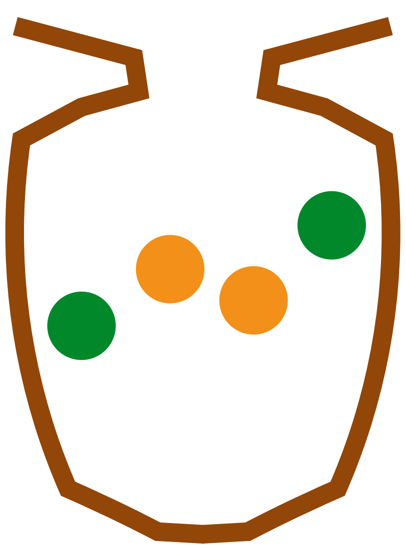

$$\lim_{n \to \infty} \frac{\#\ \text{orange}}{\#\ \text{total}} = \rho_{\text{orange}}$$

Marginal cluster assignments

- Integrate out the frequencies

$$\rho_1 \sim \text{Beta}(a_1, a_2), z_n \overset{iid}{\sim} \text{Cat}(\rho_1, \rho_2)$$

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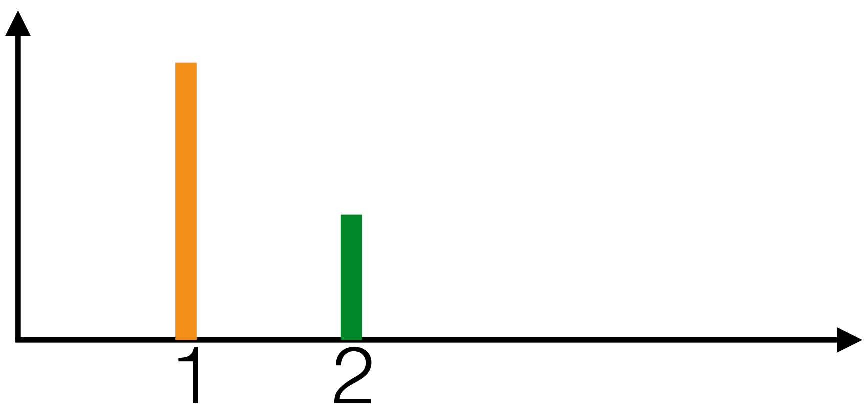

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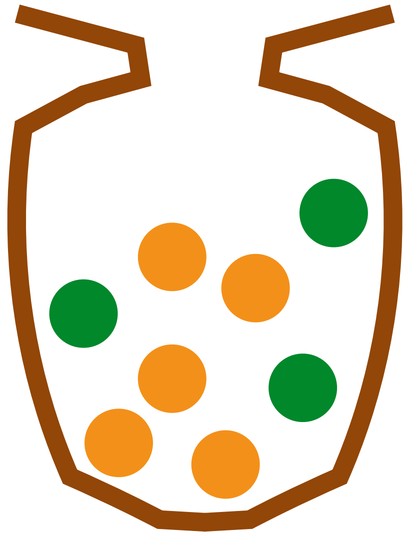

Marginal cluster assignments

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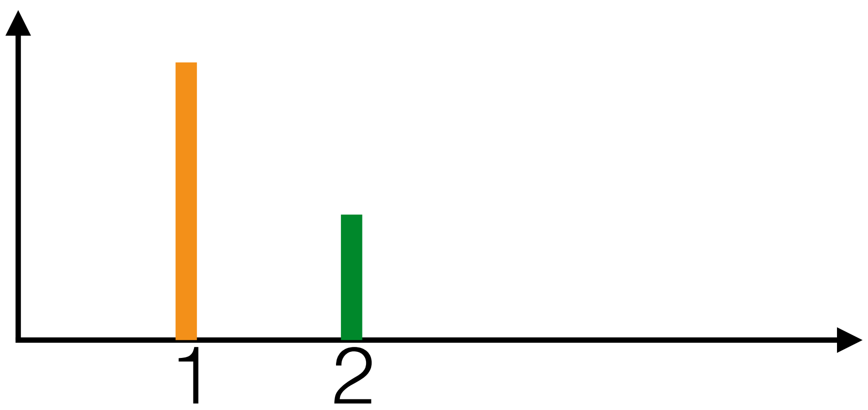

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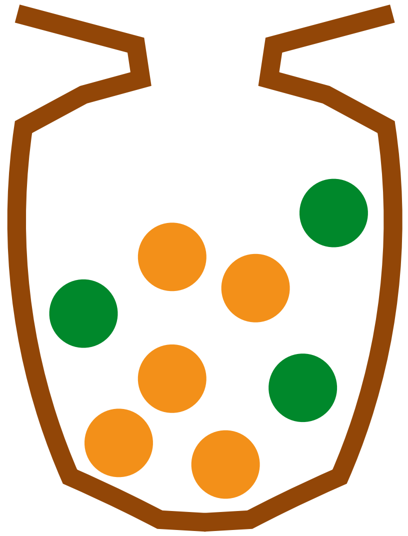

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Marginal cluster assignments

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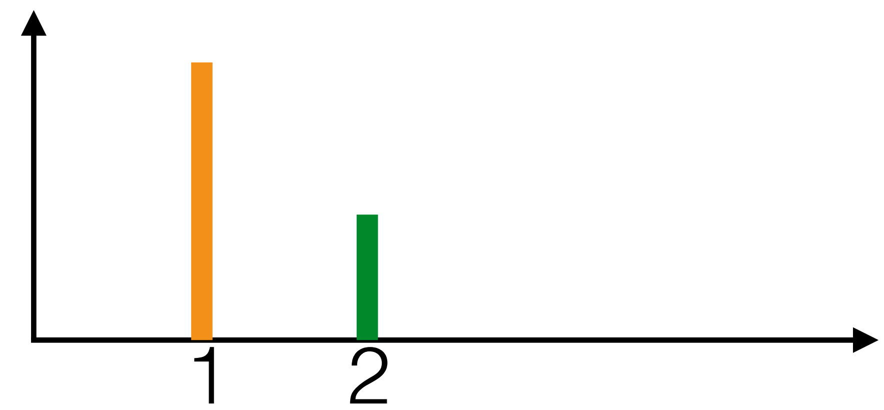

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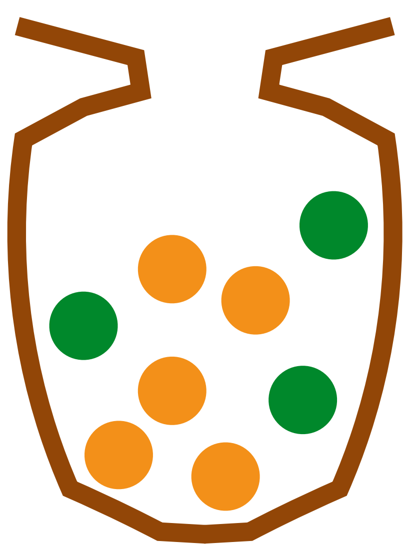

$$\lim_{n \to \infty} \frac{\# \text{ orange}}{\# \text{ total}} = \rho_{\text{orange}}$$

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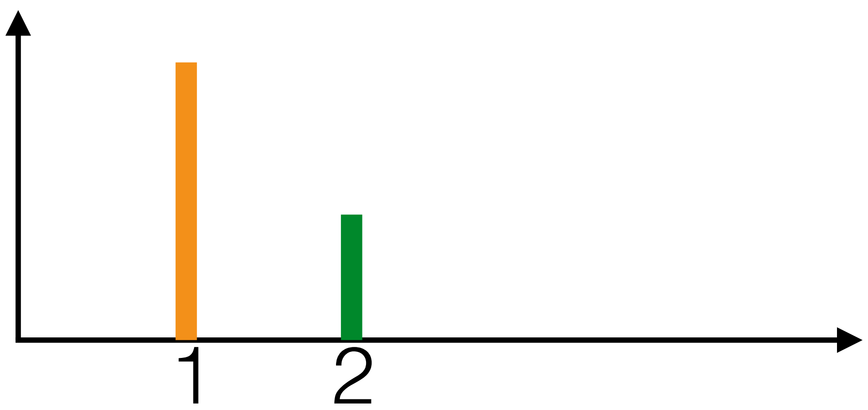

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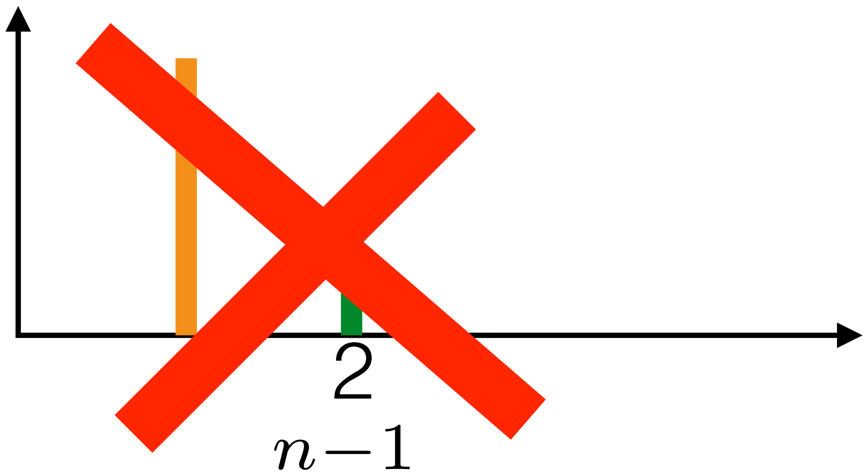

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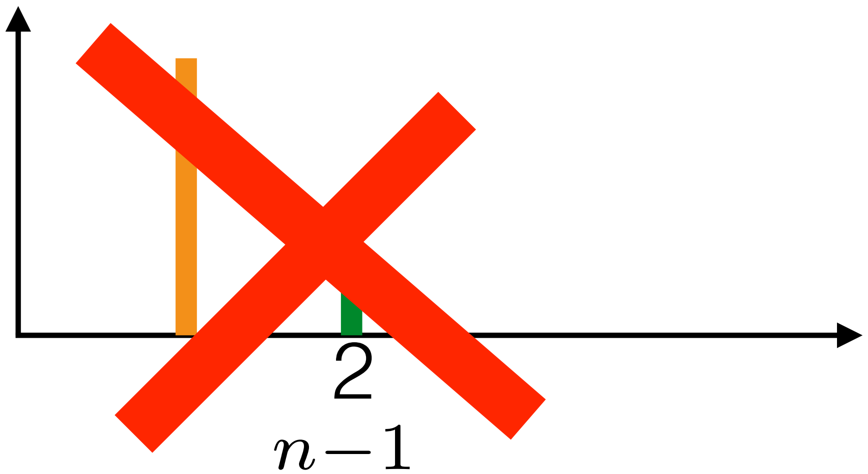

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- Pólya urn

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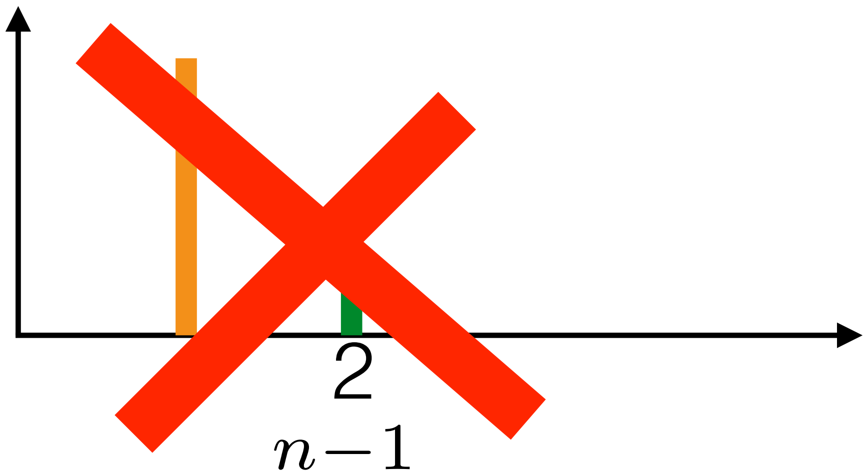

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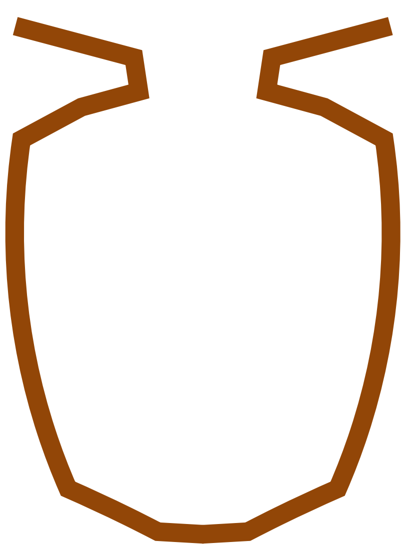

Marginal cluster assignments

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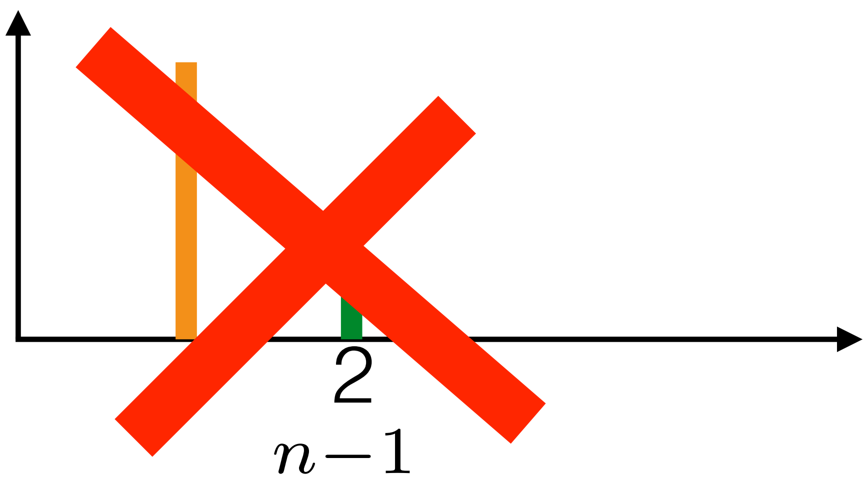

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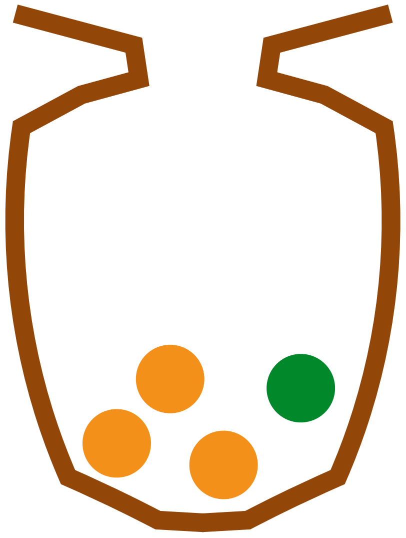

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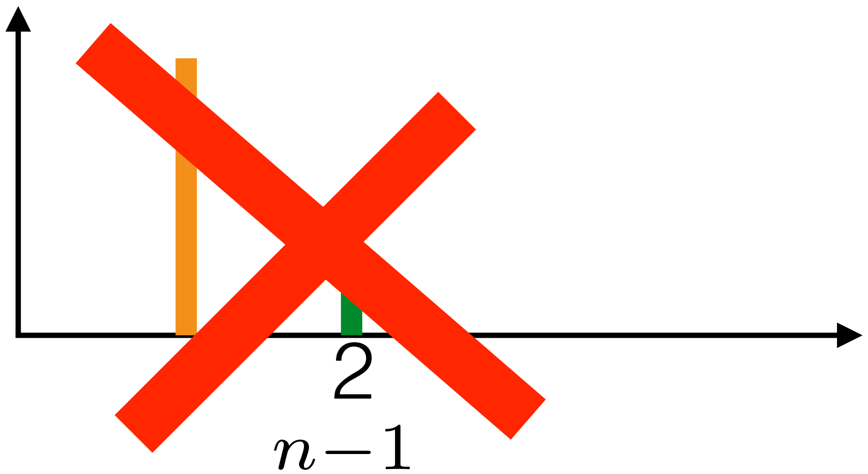

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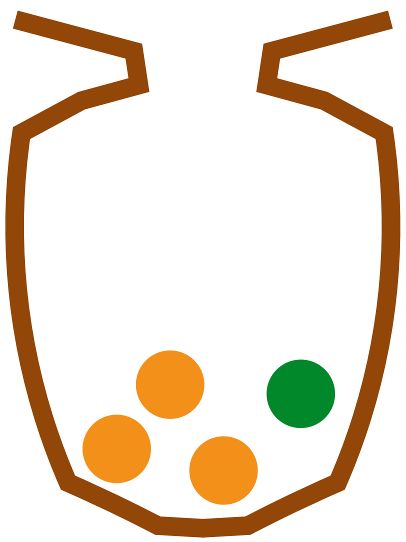
- Pólya urn
  - Choose any ball with equal probability



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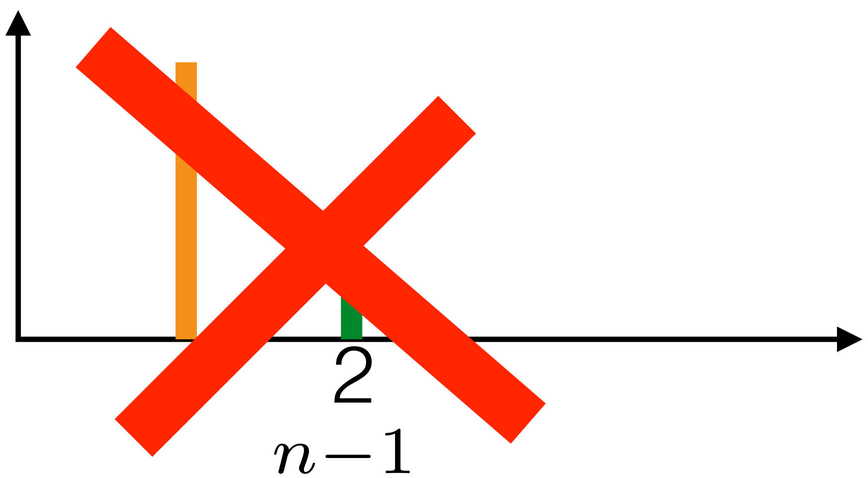

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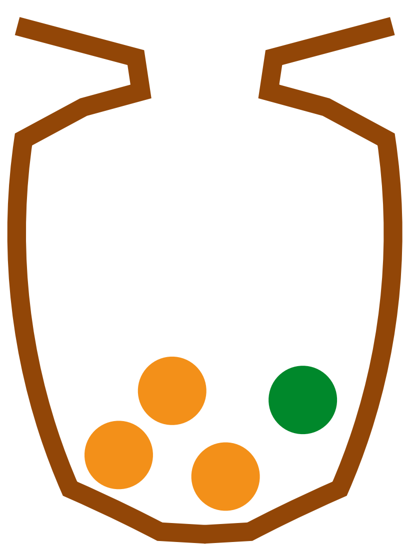
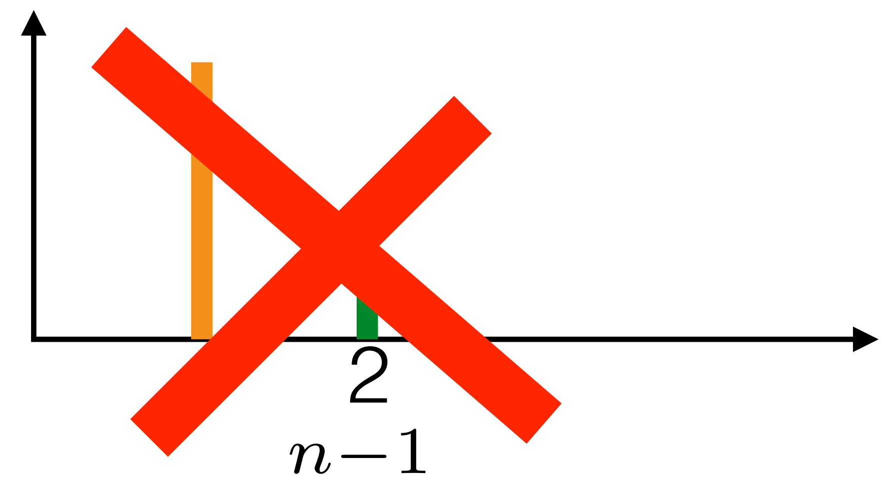
- Pólya urn
  - Choose any ball with equal probability
  - Replace and add ball of same color



Marginal cluster assignments

- Integrate out the frequencies



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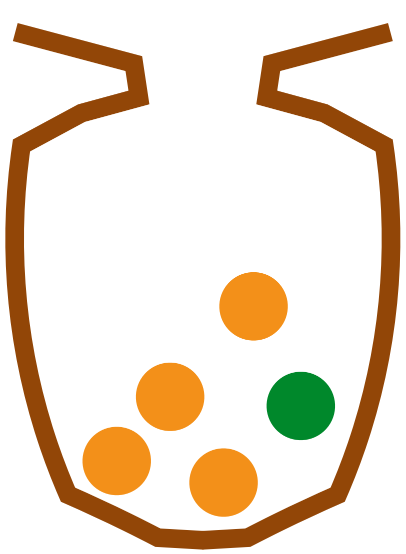

Marginal cluster assignments

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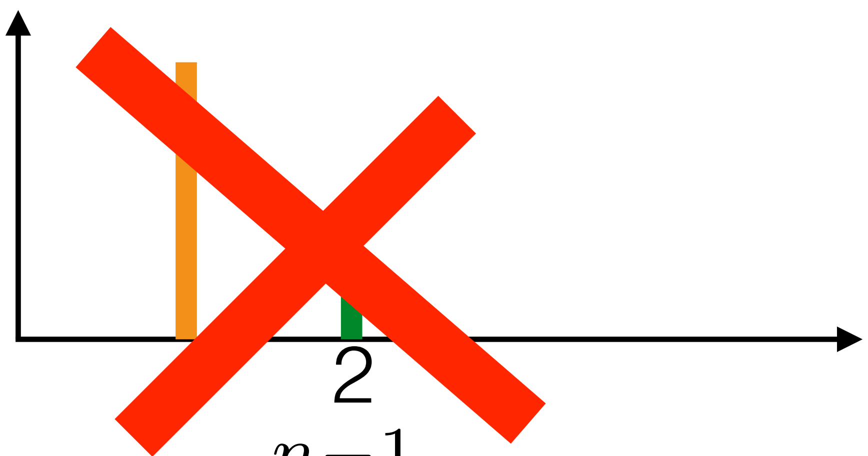

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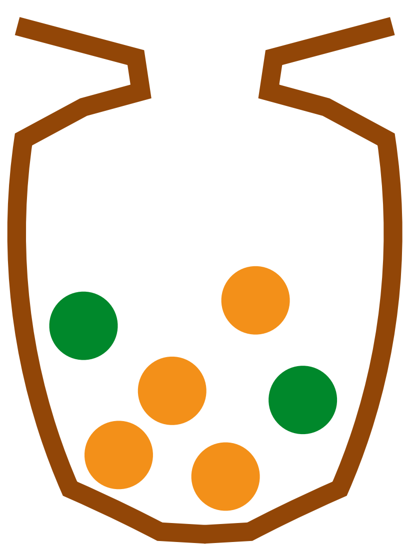

Marginal cluster assignments

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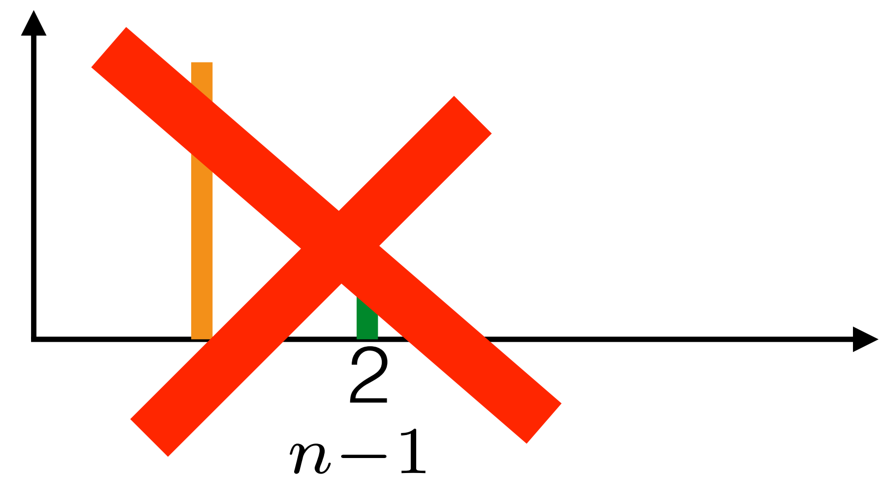

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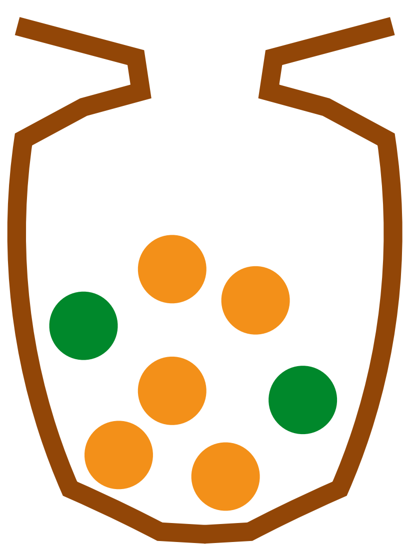

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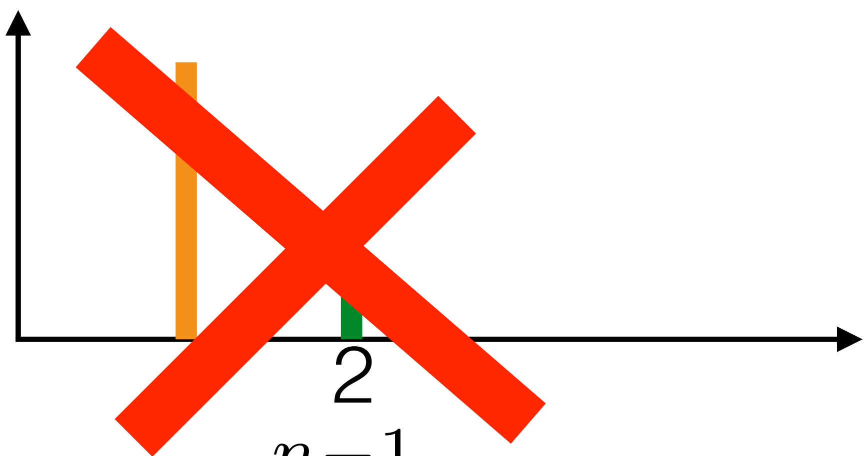
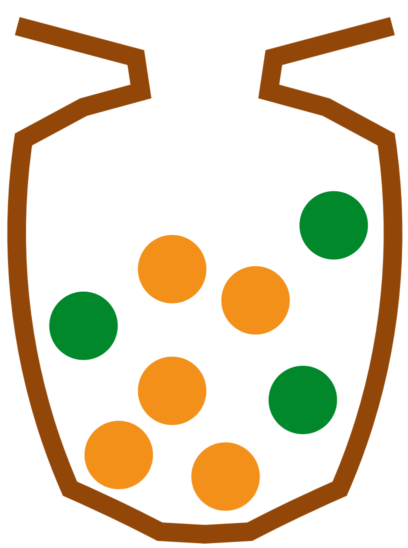


$$p(z_n = 1 | z_1, \ldots, z_{n-1}) = \frac{a_{1,n}}{a_{1,n} + a_{2,n}}$$

$$a_{1,n} := a_1 + \sum_{m=1}^{n-1} \mathbf{1}\{z_m = 1\},\ a_{2,n} = a_2 + \sum_{m=1}^{n-1} \mathbf{1}\{z_m = 2\}$$

- Pólya urn
  - Choose any ball with equal probability
  - Replace and add ball of same color



Marginal cluster assignments

- Integrate out the frequencies

$$\rho_1 \sim \mathrm{Beta}(a_1, a_2),\ z_n \overset{iid}{\sim} \mathrm{Cat}(\rho_1, \rho_2)$$

$$p(z_n = 1 | z_1, \ldots, z_{n-1}) = \frac{a_{1,n}}{a_{1,n} + a_{2,n}}$$

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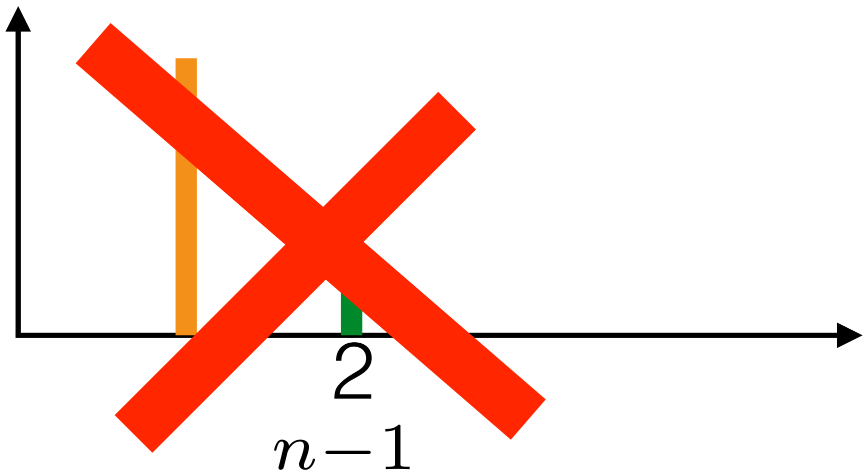

- Pólya urn
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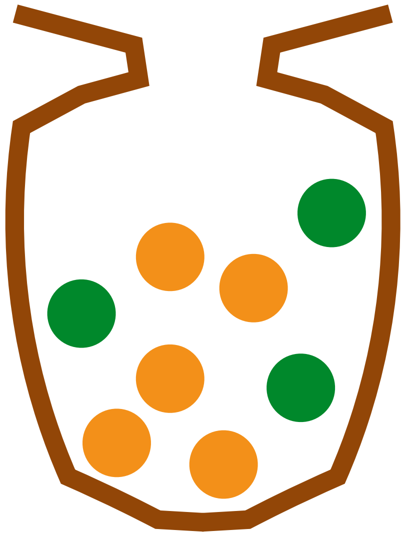

$$\lim_{n \to \infty} \frac{\#\ \text{orange}}{\#\ \text{total}}$$

Marginal cluster assignments

- Integrate out the frequencies

$$\rho_1 \sim \text{Beta}(a_1, a_2),\ z_n \overset{iid}{\sim} \text{Cat}(\rho_1, \rho_2)$$

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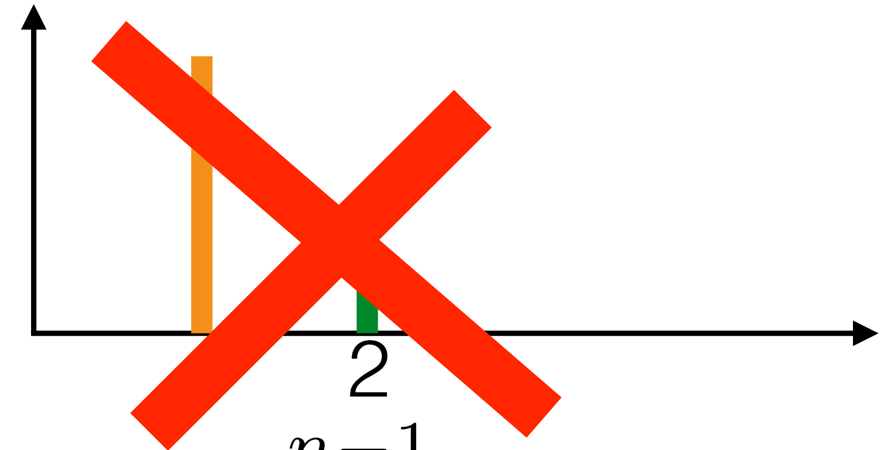

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    - Choose any ball with equal probability
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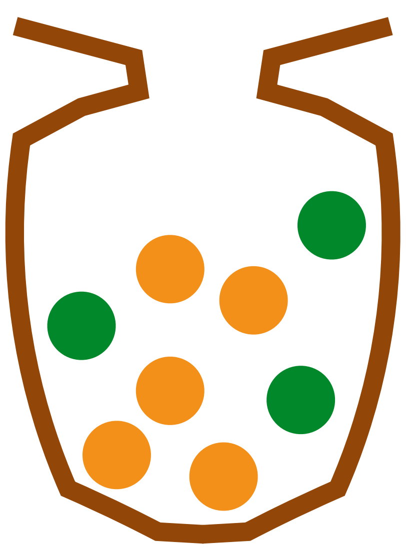

$$\lim_{n \to \infty} \frac{\#\ \text{orange}}{\#\ \text{total}} = \rho_{\text{orange}}$$

Marginal cluster assignments

- Integrate out the frequencies

$$\rho_1 \sim \text{Beta}(a_1, a_2),\, z_n \overset{iid}{\sim} \text{Cat}(\rho_1, \rho_2)$$

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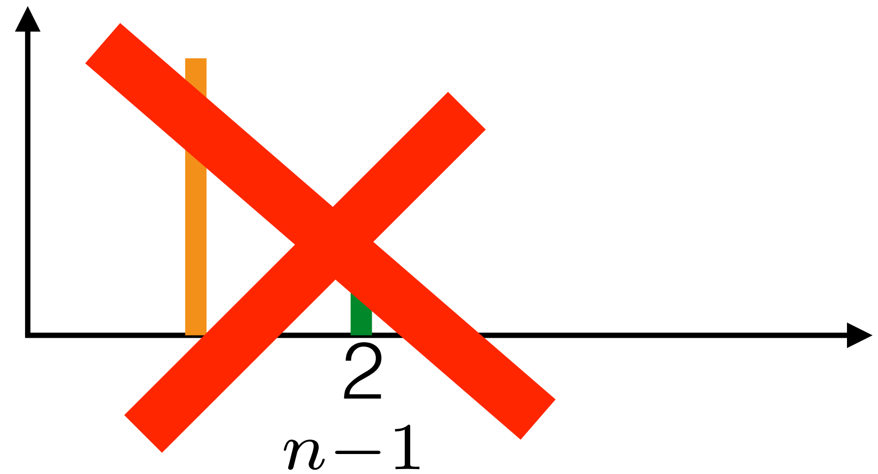

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- Pólya urn
  - Choose any ball with equal probability
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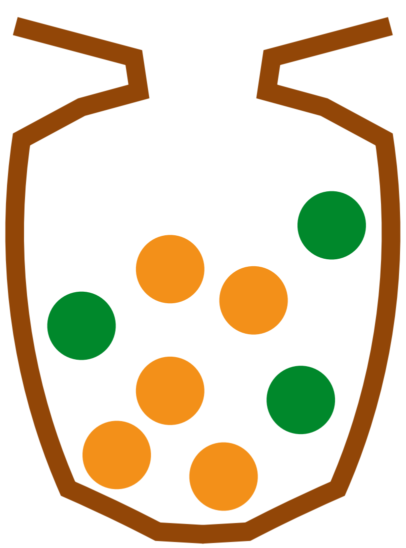

$$\lim_{n \to \infty} \frac{\#\ \text{orange}}{\#\ \text{total}} = \rho_{\text{orange}} \overset{d}{=} \text{Beta}(a_{\text{orange}}, a_{\text{green}})$$

Marginal cluster assignments

- Integrate out the frequencies

$$\rho_1 \sim \text{Beta}(a_1, a_2),\ z_n \overset{iid}{\sim} \text{Cat}(\rho_1, \rho_2)$$

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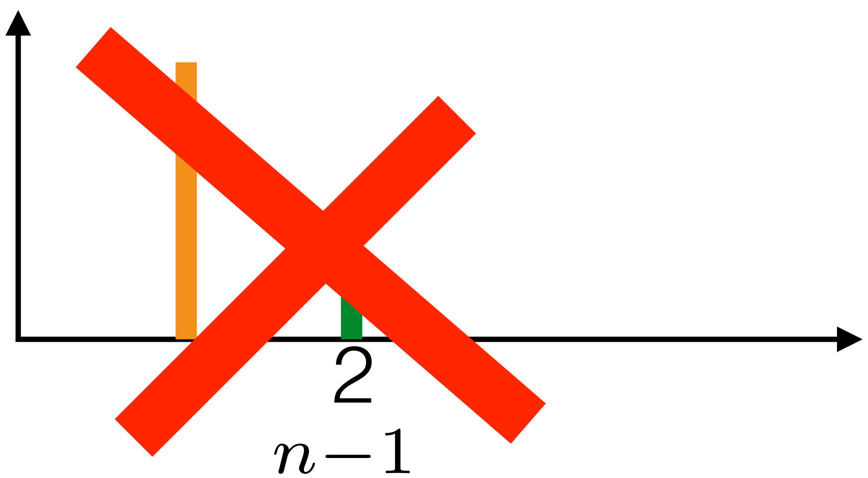

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  - Choose any ball with equal probability
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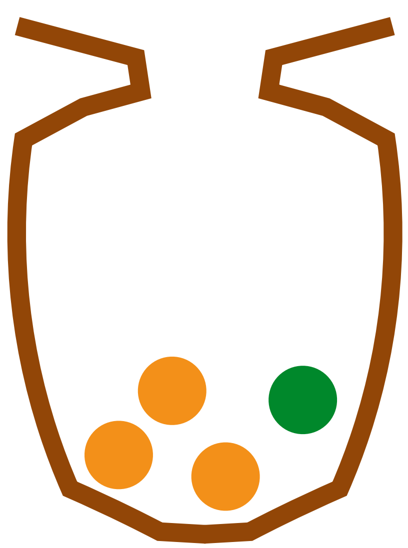

$$\lim_{n \to \infty} \frac{\#\ \text{orange}}{\#\ \text{total}} = \rho_{\text{orange}} \overset{d}{=} \text{Beta}(a_{\text{orange}}, a_{\text{green}})$$

Marginal cluster assignments

- Integrate out the frequencies

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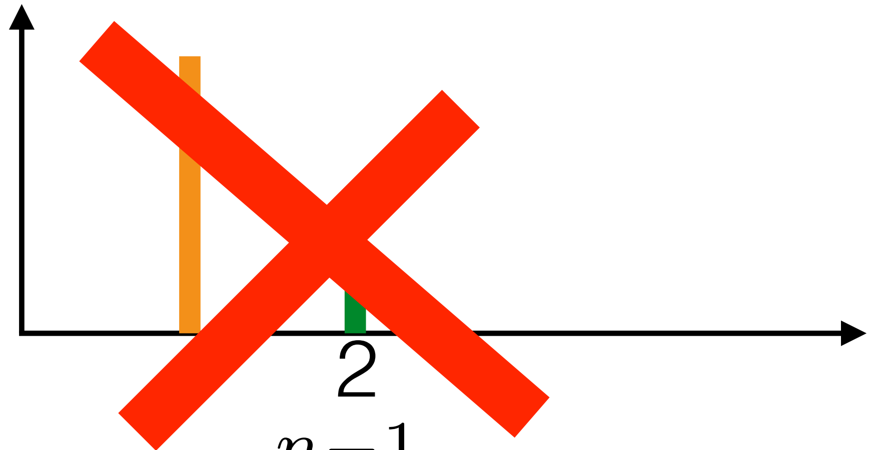

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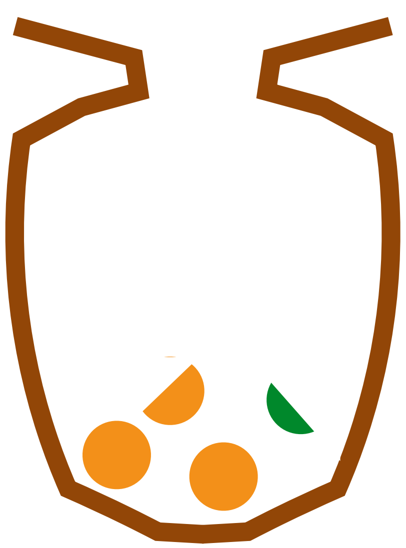

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Marginal cluster assignments

- Integrate out the frequencies

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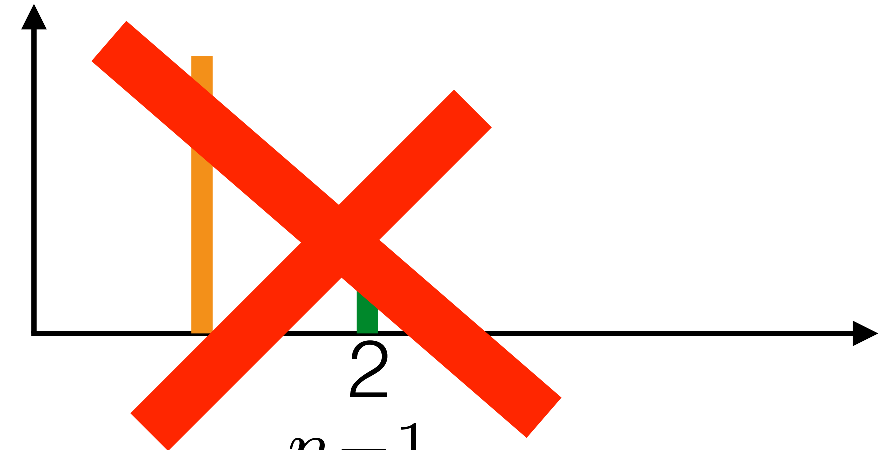

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- Pólya urn
  - Choose any ball with prob proportional to its mass
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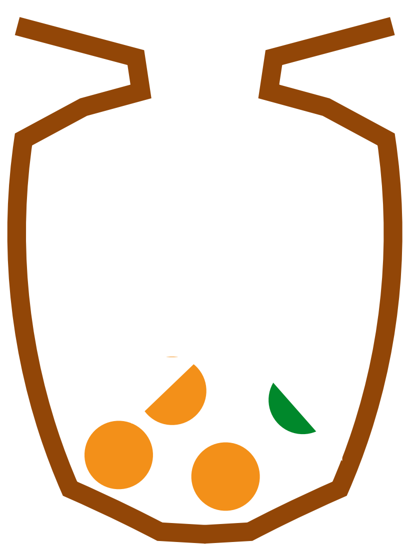

$$\lim_{n \to \infty} \frac{\#\ \text{orange}}{\#\ \text{total}} = \rho_{\text{orange}} \overset{d}{=} \mathrm{Beta}(a_{\text{orange}}, a_{\text{green}})$$

Marginal cluster assignments

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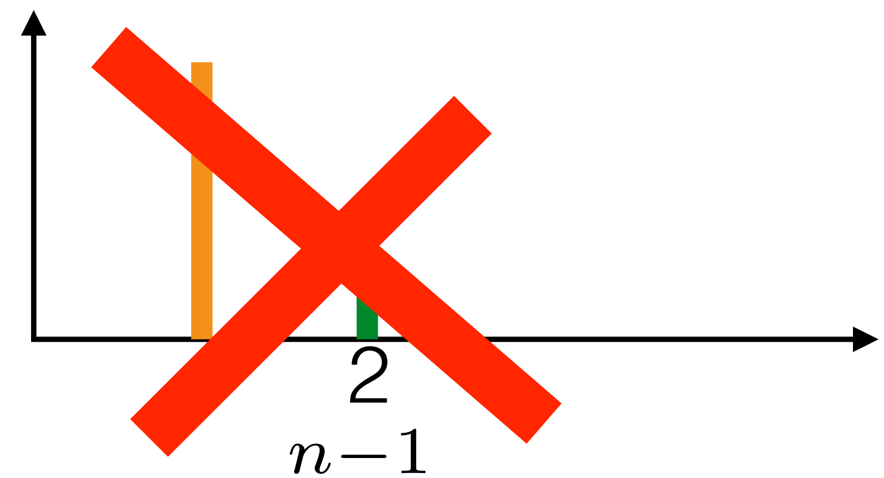

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- Pólya urn
  - Choose any ball with prob proportional to its mass
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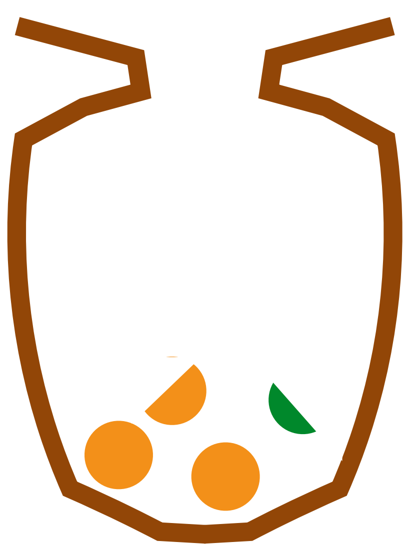
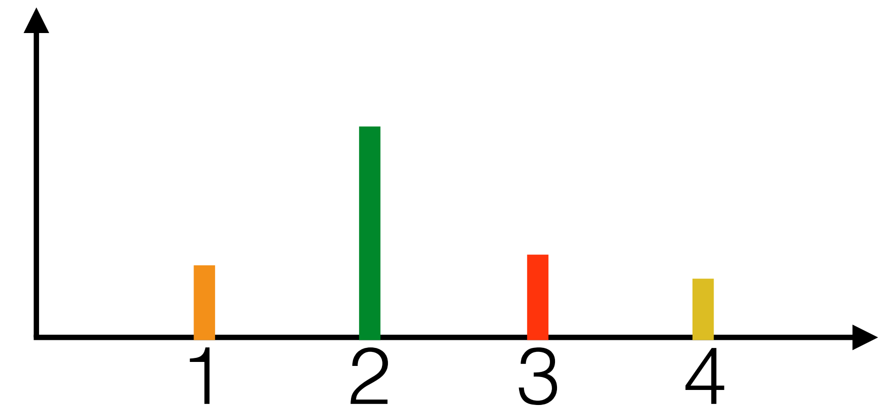


$$\lim_{n \to \infty} \frac{\#\ \text{orange}}{\#\ \text{total}} = \rho_{\text{orange}} \overset{d}{=} \text{Beta}(a_{\text{orange}}, a_{\text{green}})$$

$$\text{PolyaUrn}(a_{\text{orange}}, a_{\text{green}})$$

Marginal cluster assignments

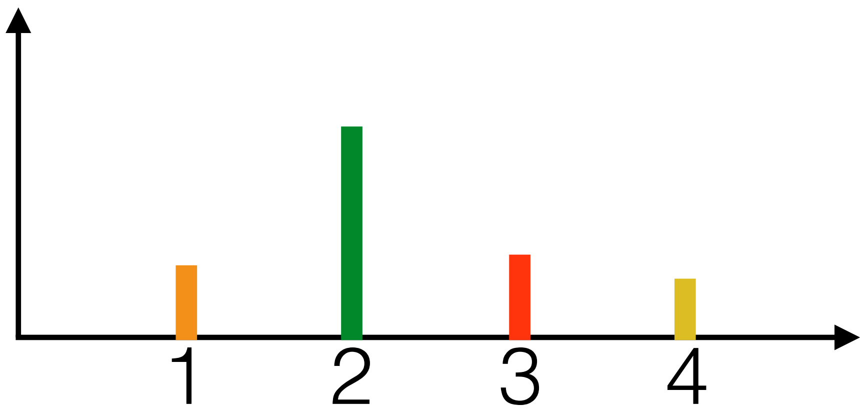
- Integrate out the frequencies



Marginal cluster assignments

- Integrate out the frequencies

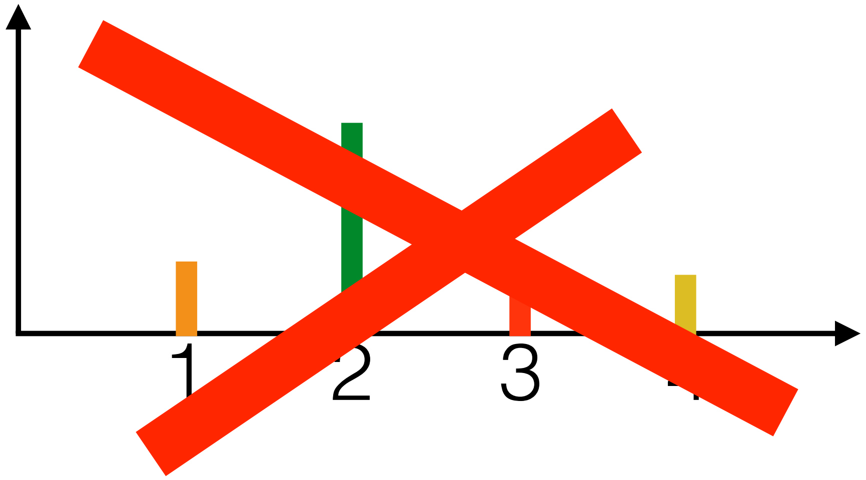
$\rho_{1:K} \sim \text{Dirichlet}(a_{1:K}), z_n \stackrel{iid}{\sim} \text{Cat}(\rho_{1:K})$



Marginal cluster assignments

- Integrate out the frequencies

$\rho_{1:K} \sim \text{Dirichlet}(a_{1:K}), z_n \overset{iid}{\sim} \text{Cat}(\rho_{1:K})$



Marginal cluster assignments

- Integrate out the frequencies

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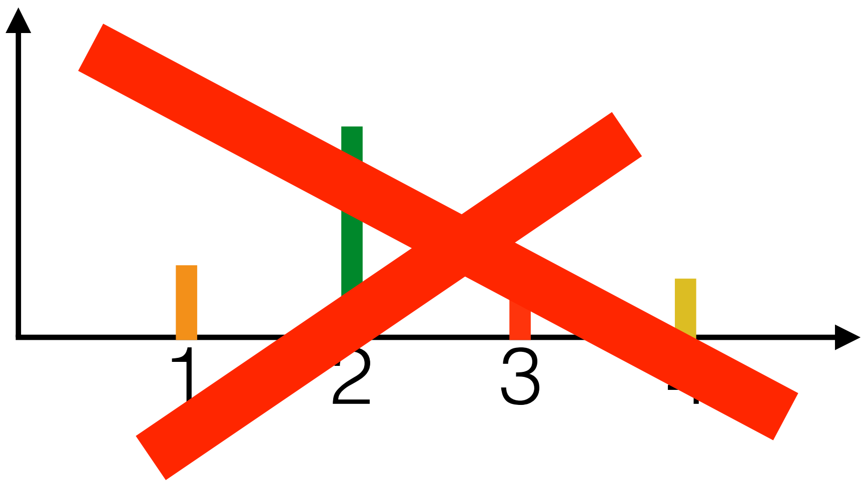
$$p(z_n = k | z_1, \ldots, z_{n-1}) = \frac{a_{k,n}}{\sum_{j=1}^{K} a_{j,n}}$$



Marginal cluster assignments

- Integrate out the frequencies

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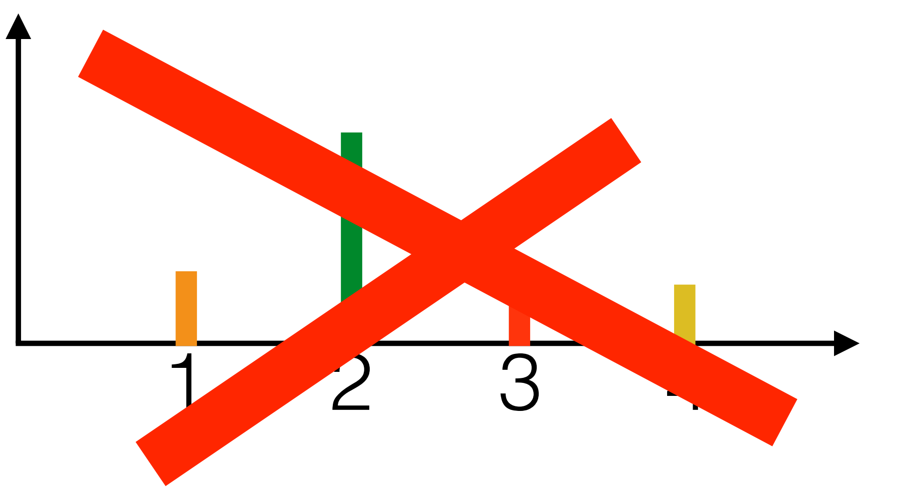

$$p(z_n = k | z_1, \ldots, z_{n-1}) = \frac{a_{k,n}}{\sum_{j=1}^{K} a_{j,n}}$$

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Marginal cluster assignments

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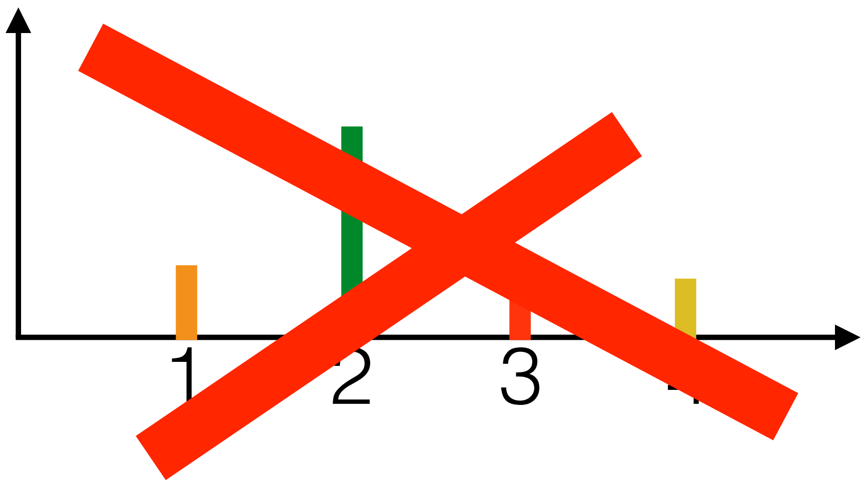
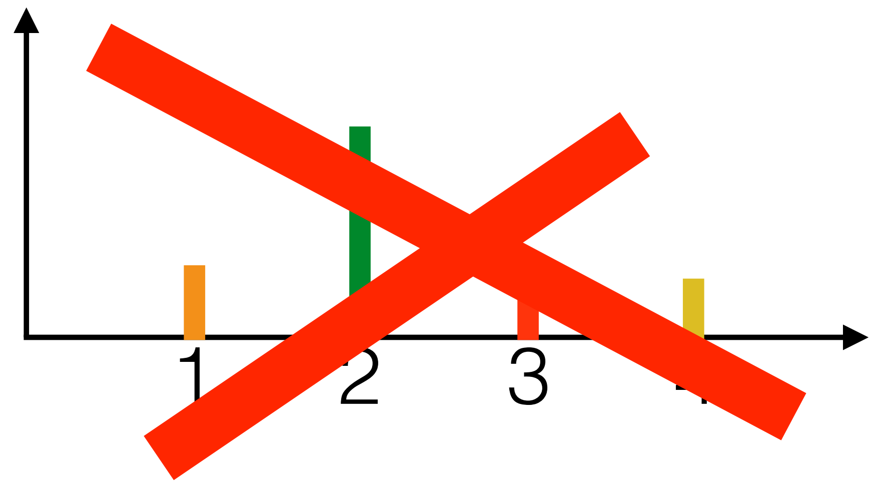


- multivariate Pólya urn

Marginal cluster assignments

- Integrate out the frequencies



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Marginal cluster assignments

- Integrate out the frequencies

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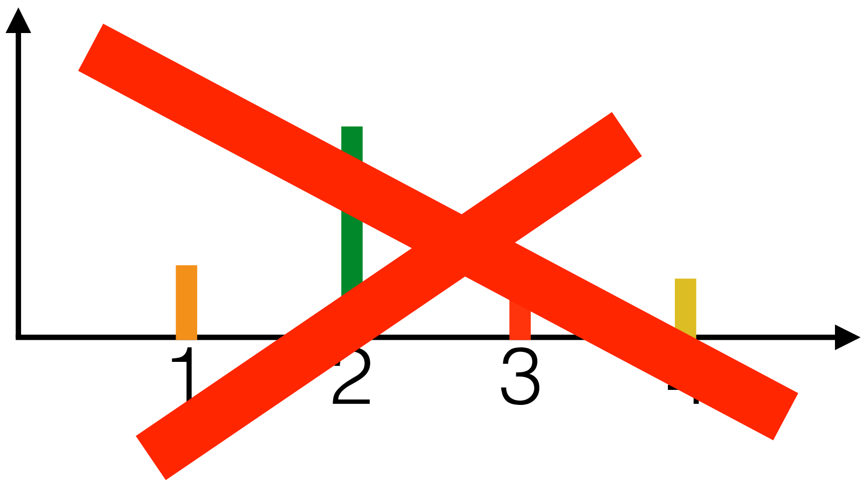

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- multivariate Pólya urn
  - Choose any ball with prob proportional to its mass

Marginal cluster assignments

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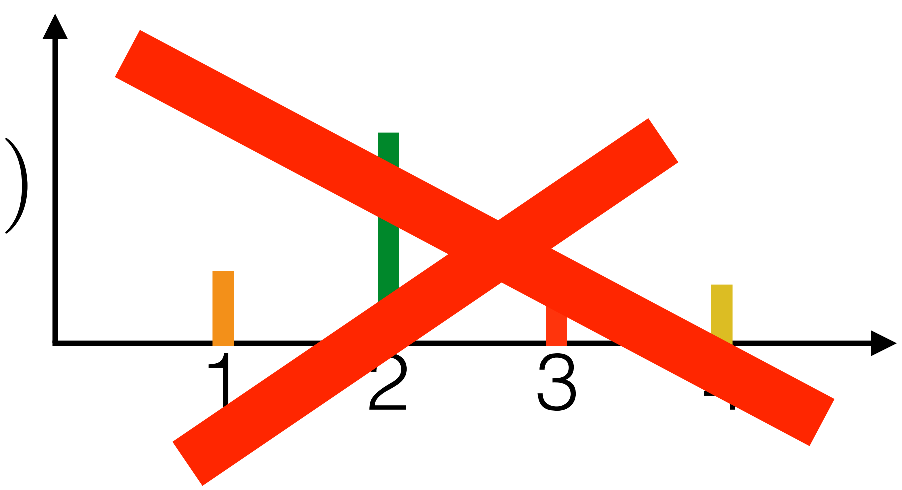
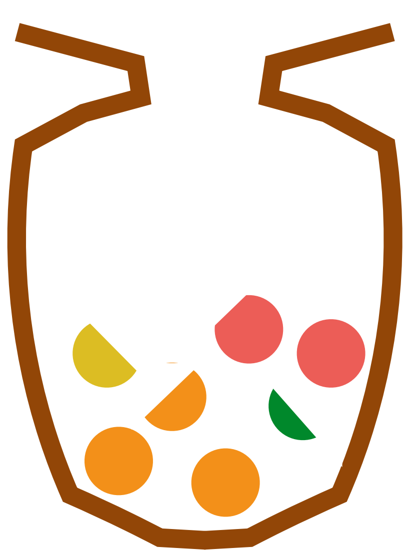
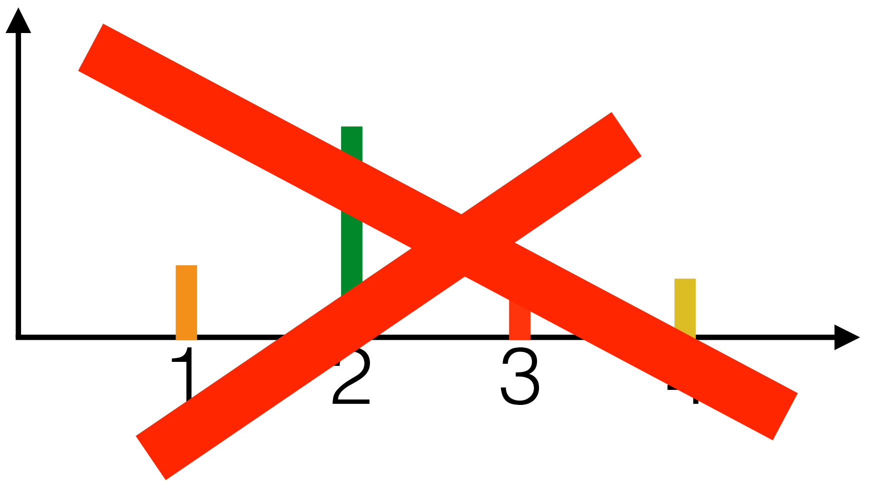


- multivariate Pólya urn
  - Choose any ball with prob proportional to its mass
  - Replace and add ball of same color



Marginal cluster assignments

- Integrate out the frequencies



$$\rho_{1:K} \sim \text{Dirichlet}(a_{1:K}), z_n \overset{iid}{\sim} \text{Cat}(\rho_{1:K})$$

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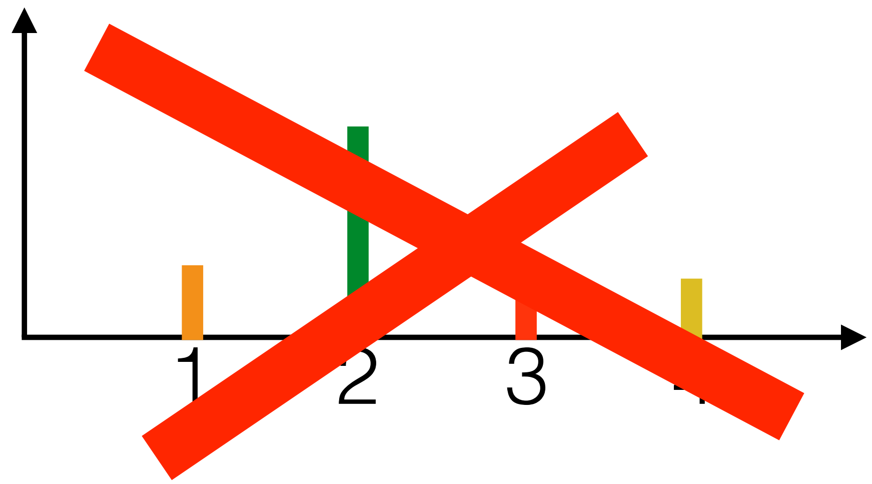
- multivariate Pólya urn
  - Choose any ball with prob proportional to its mass
  - Replace and add ball of same color

$$\lim_{n \to \infty} \frac{(\#\text{ orange}, \#\text{ green}, \#\text{ red}, \#\text{ yellow})}{\#\text{ total}}$$

Marginal cluster assignments

- Integrate out the frequencies



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- multivariate Pólya urn
  - Choose any ball with prob proportional to its mass
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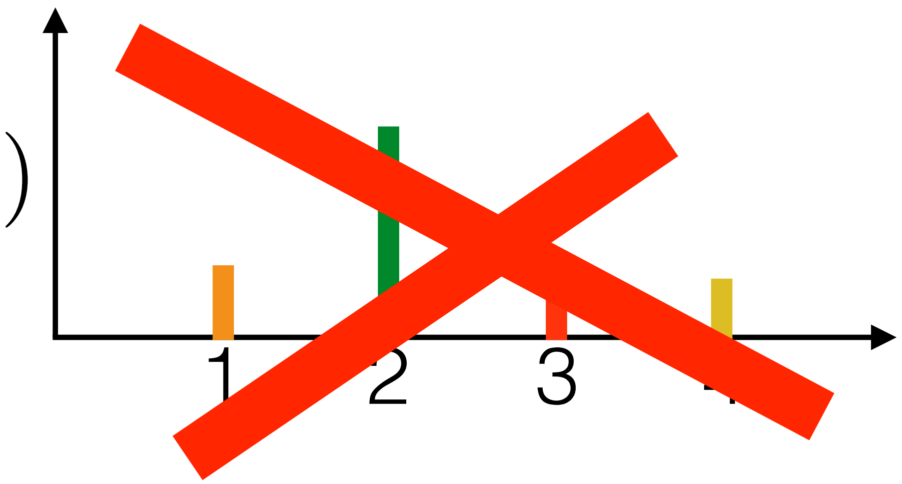
$$\lim_{n \to \infty} \frac{(\#\text{ orange}, \#\text{ green}, \#\text{ red}, \#\text{ yellow})}{\#\text{ total}}$$

$$\to (\rho_{\text{orange}}, \rho_{\text{green}}, \rho_{\text{red}}, \rho_{\text{yellow}})$$

Marginal cluster assignments

- Integrate out the frequencies

$$\rho_{1:K} \sim \text{Dirichlet}(a_{1:K}), z_n \overset{iid}{\sim} \text{Cat}(\rho_{1:K})$$



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- multivariate Pólya urn
  - Choose any ball with prob proportional to its mass
  - Replace and add ball of same color

$$\lim_{n \to \infty} \frac{(\text{\# orange, \# green, \# red, \# yellow})}{\text{\# total}}$$

$$\to (\rho_{\text{orange}}, \rho_{\text{green}}, \rho_{\text{red}}, \rho_{\text{yellow}})$$

$$\overset{d}{=} \text{Dirichlet}(a_{\text{orange}}, a_{\text{green}}, a_{\text{red}}, a_{\text{yellow}})$$

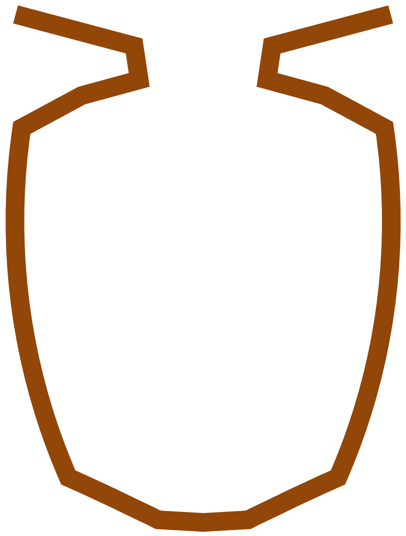
Marginal cluster assignments

- Hoppe urn / Blackwell-MacQueen urn

[Blackwell, MacQueen 1973; Hoppe 1984]

Marginal cluster assignments

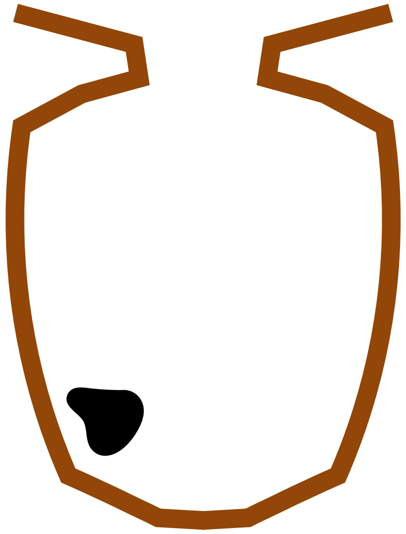
- Hoppe urn / Blackwell-MacQueen urn



[Blackwell, MacQueen 1973; Hoppe 1984]

Marginal cluster assignments

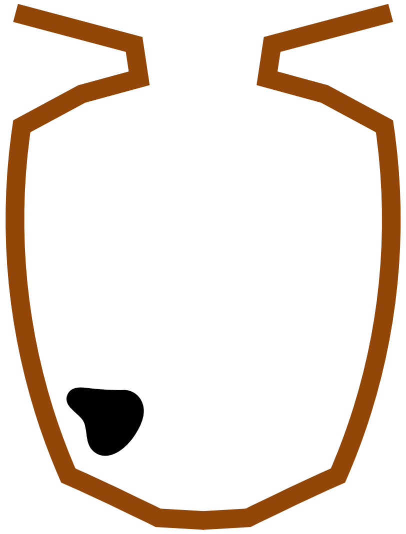
- Hoppe urn / Blackwell-MacQueen urn



[Blackwell, MacQueen 1973; Hoppe 1984]

Marginal cluster assignments

- Hoppe urn / Blackwell-MacQueen urn

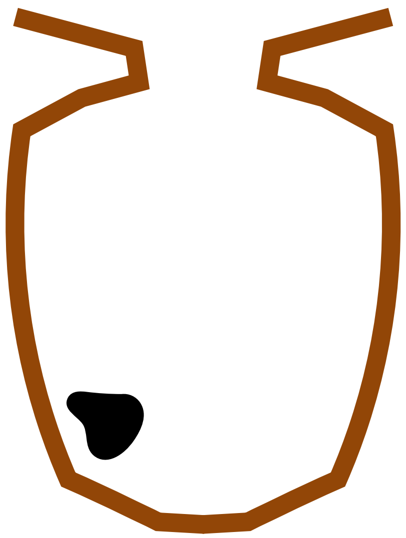


- Choose ball with prob proportional to its mass

[Blackwell, MacQueen 1973; Hoppe 1984]

Marginal cluster assignments

- Hoppe urn / Blackwell-MacQueen urn

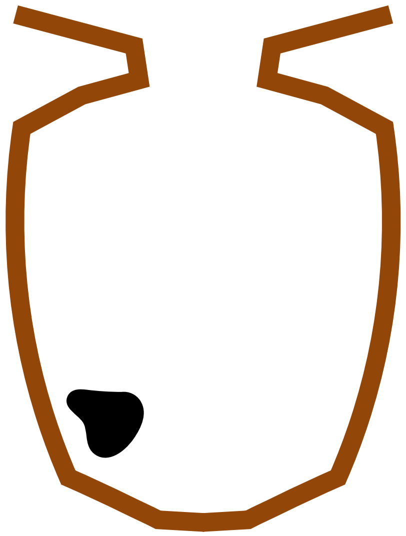


  - Choose ball with prob proportional to its mass
    - If black, replace and add ball of new color

[Blackwell, MacQueen 1973; Hoppe 1984]

Marginal cluster assignments

- Hoppe urn / Blackwell-MacQueen urn



- Choose ball with prob proportional to its mass
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[Blackwell, MacQueen 1973; Hoppe 1984]

Marginal cluster assignments

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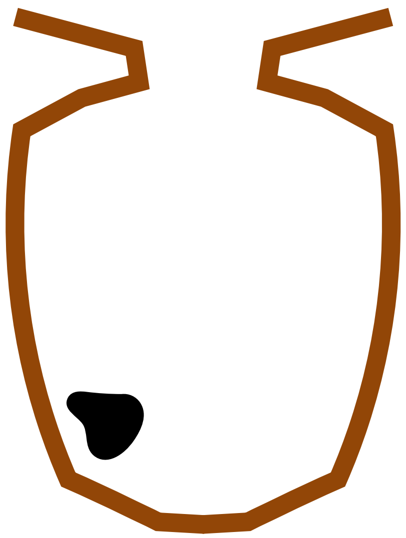

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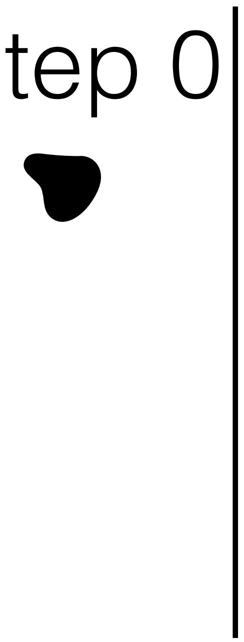
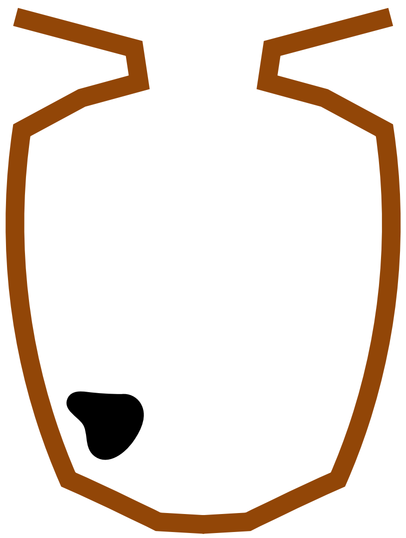
Step 0



[Blackwell, MacQueen 1973; Hoppe 1984]

Marginal cluster assignments

- Hoppe urn / Blackwell-MacQueen urn



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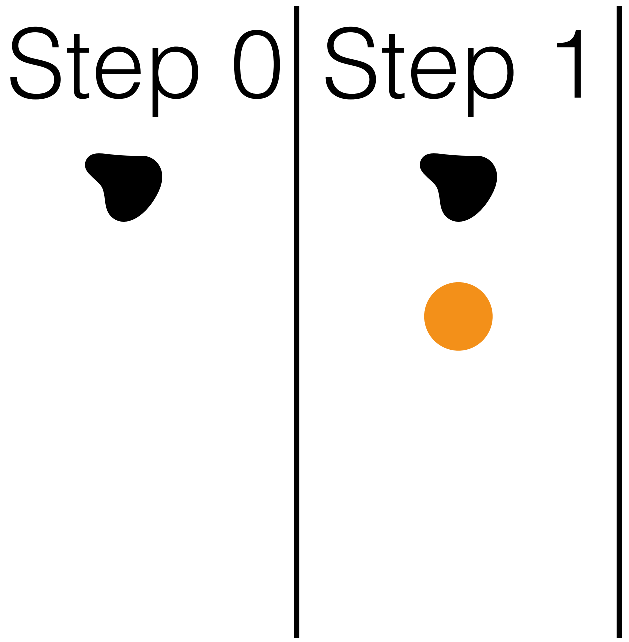
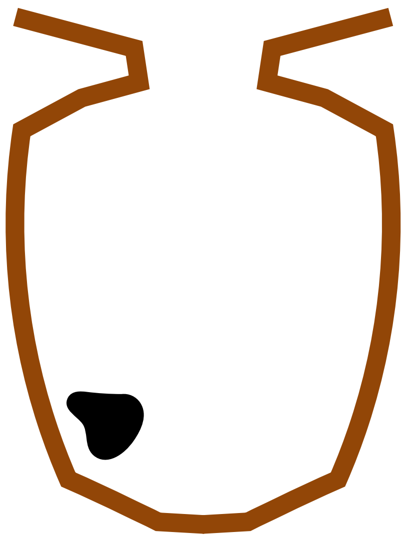


[Blackwell, MacQueen 1973; Hoppe 1984]

Marginal cluster assignments

- Hoppe urn / Blackwell-MacQueen urn



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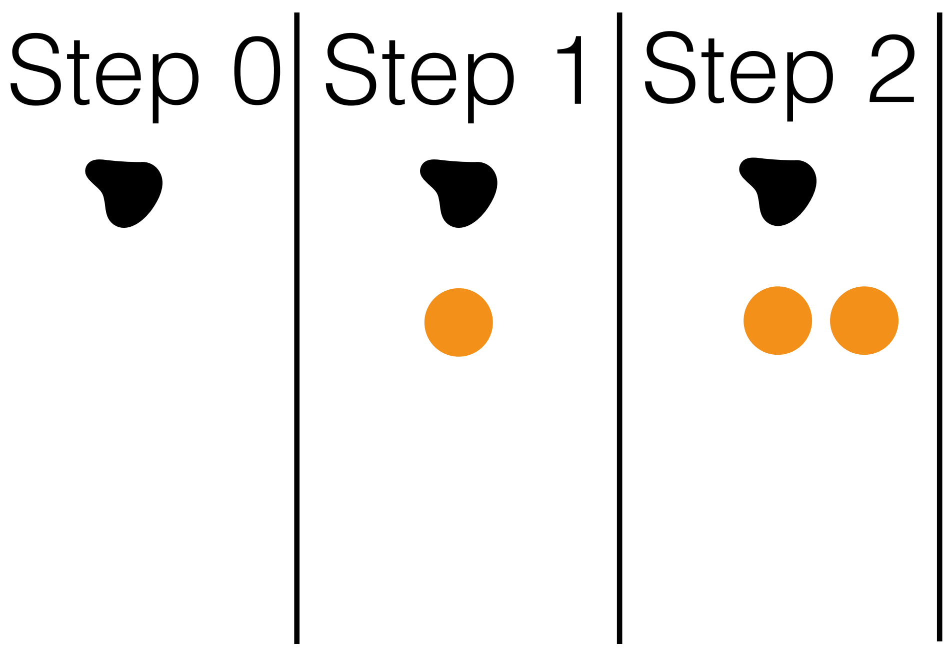

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Marginal cluster assignments

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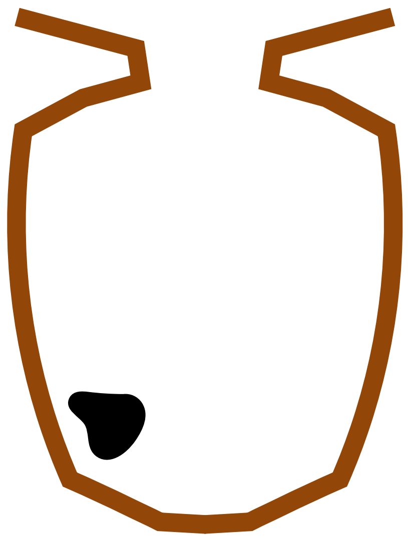

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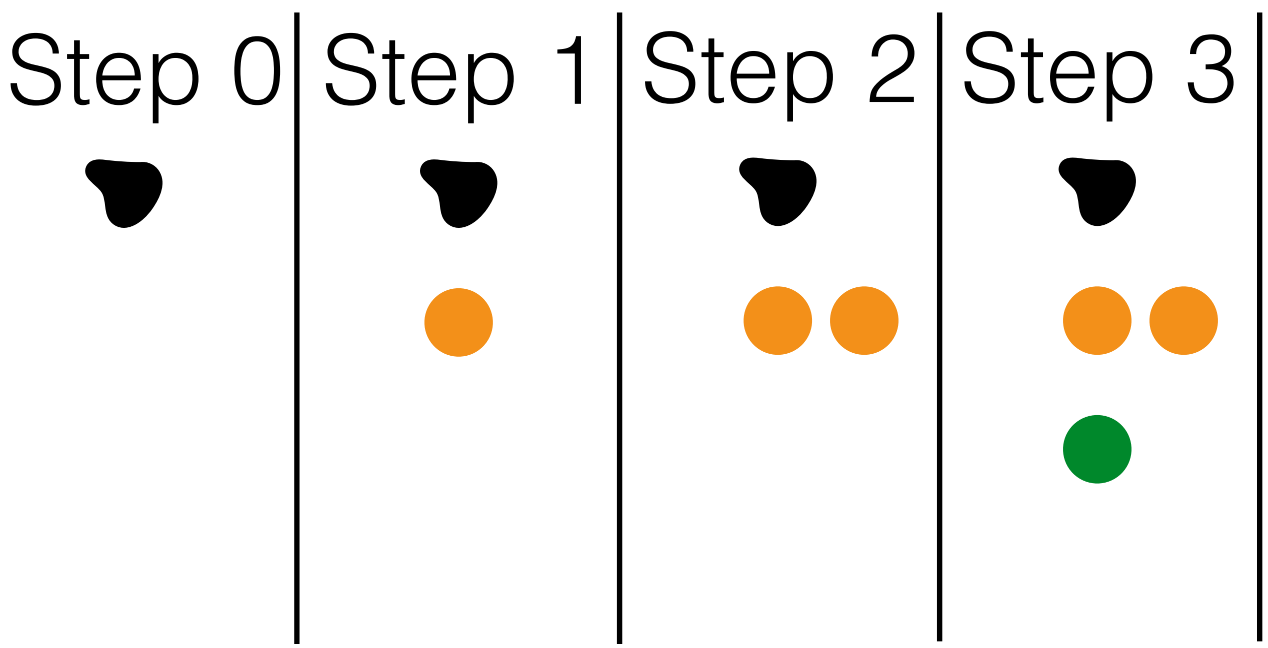

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Marginal cluster assignments

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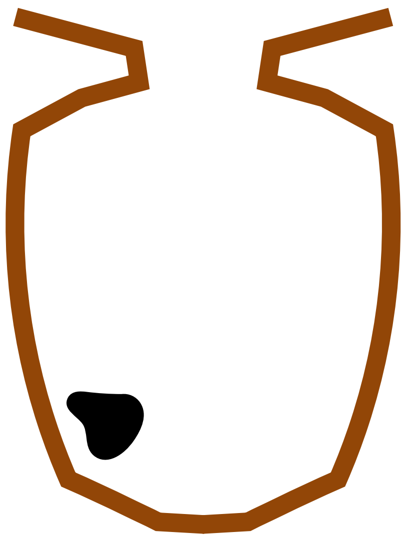

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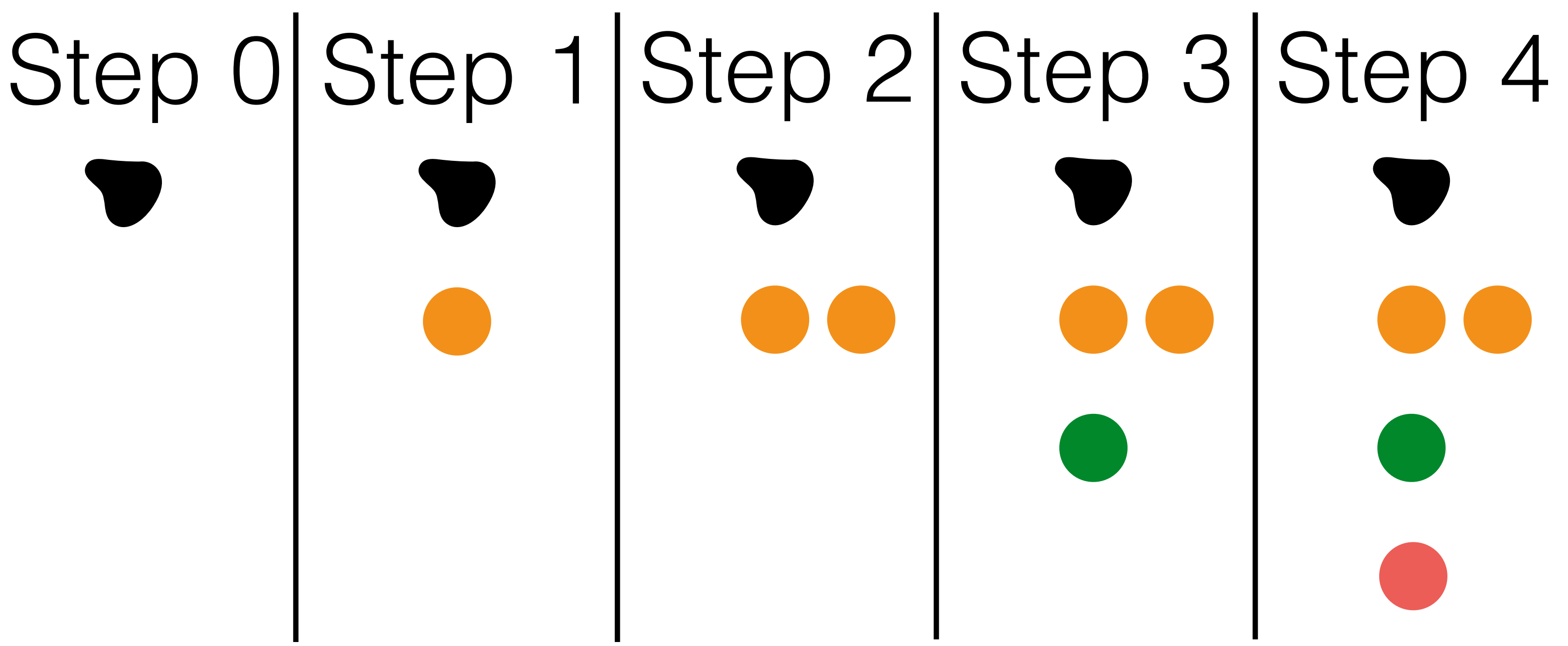

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Marginal cluster assignments

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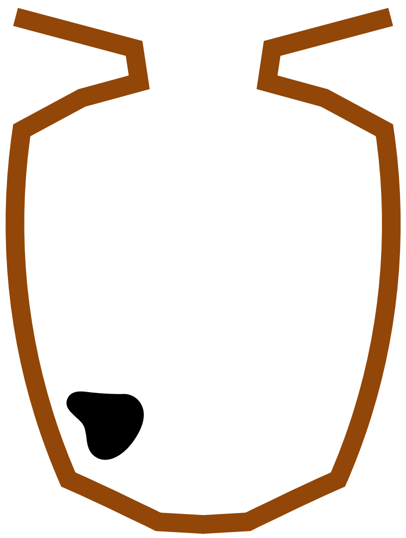

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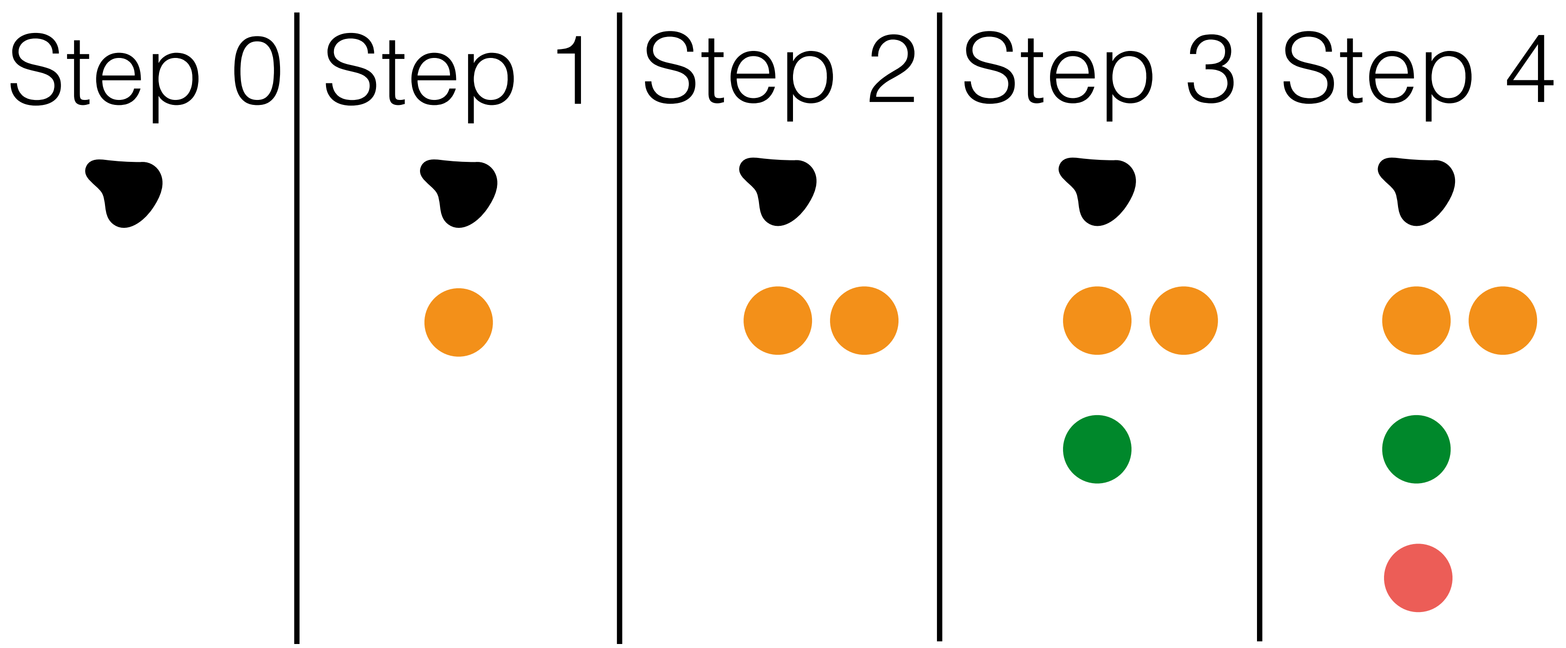
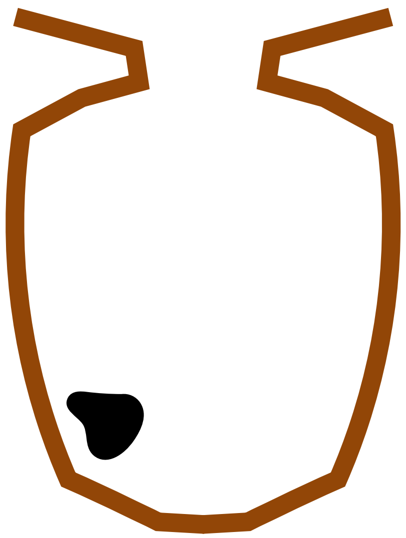


$$(\#\text{orange}, \#\text{other}) = \text{PolyaUrn}(1, \alpha)$$

Marginal cluster assignments

- Hoppe urn / Blackwell-MacQueen urn



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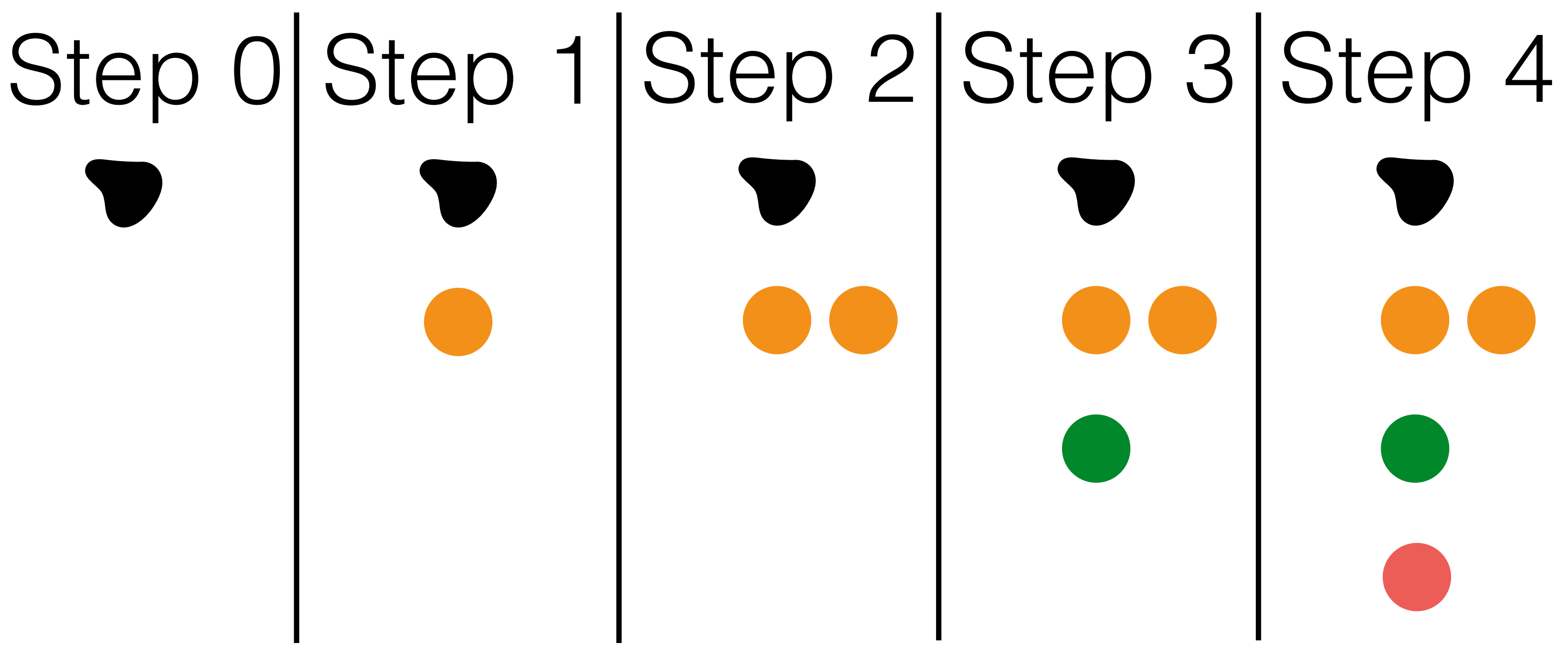

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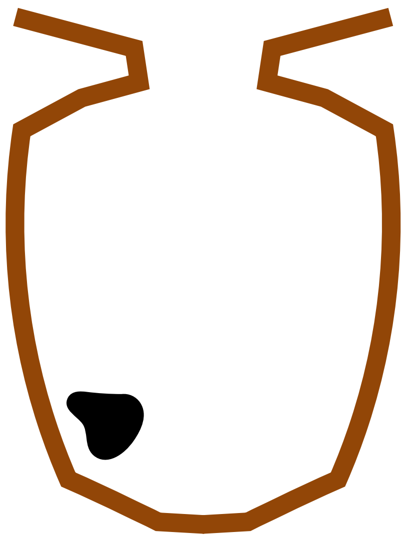
- not orange: $(\#\text{green}, \#\text{other}) = \text{PolyaUrn}(1, \alpha)$

Marginal cluster assignments

- Hoppe urn / Blackwell-MacQueen urn



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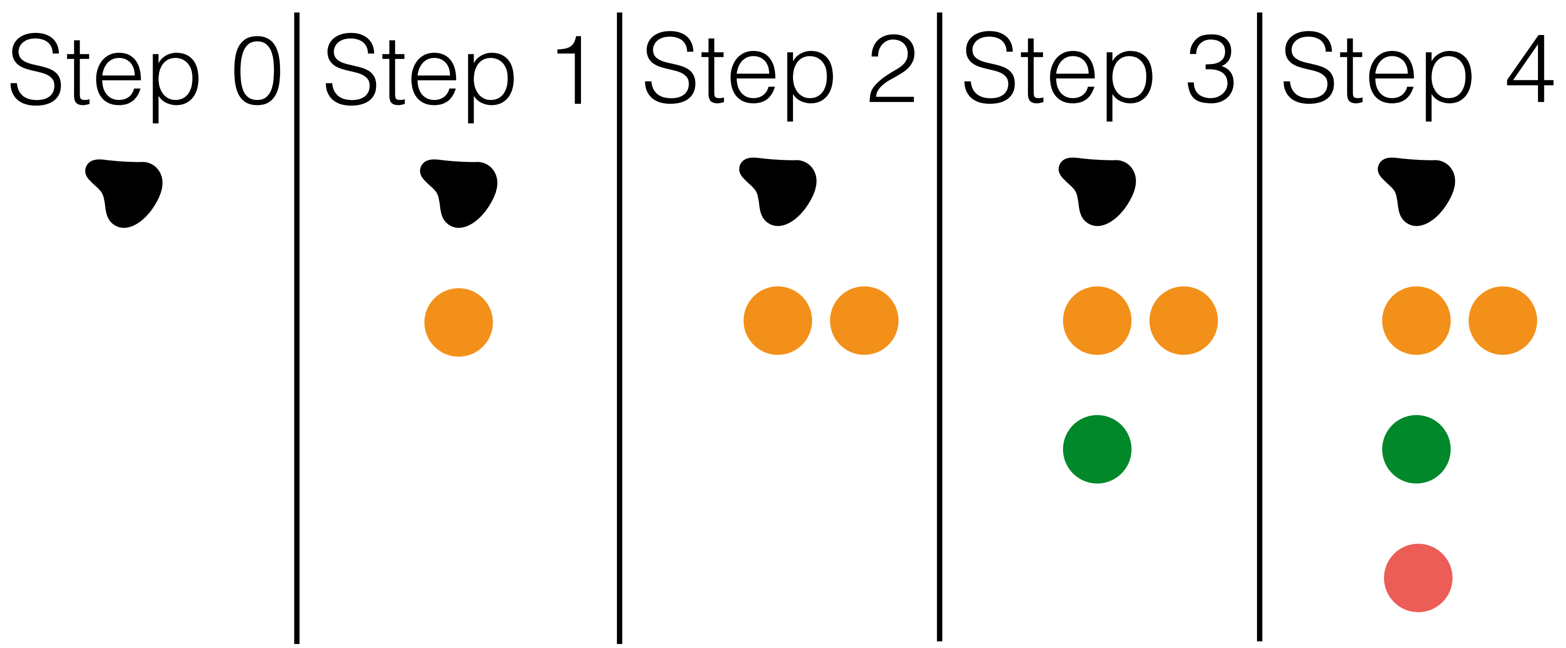

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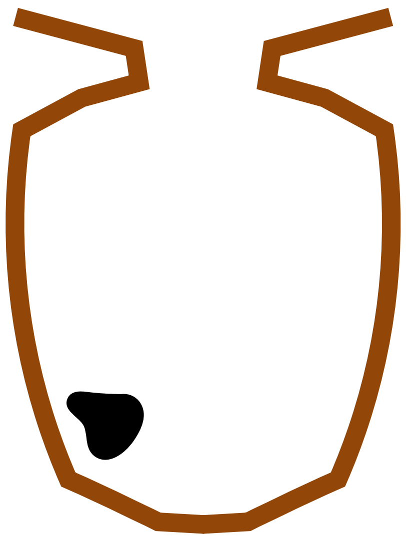
Marginal cluster assignments

- Hoppe urn / Blackwell-MacQueen urn



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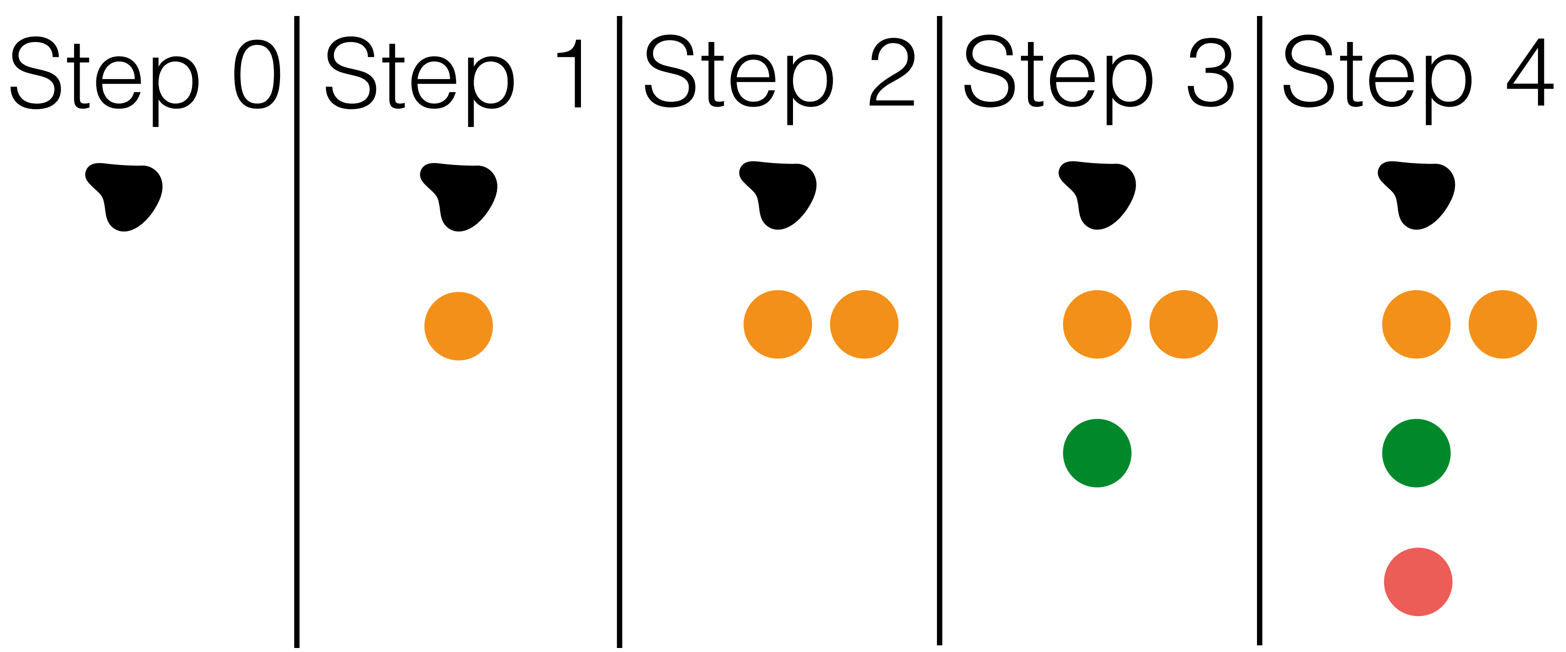

$$V_k \overset{iid}{\sim} \text{Beta}(1, \alpha)$$

$$(\#\text{orange}, \#\text{other}) = \text{PolyaUrn}(1, \alpha)$$

- not orange: $(\#\text{green}, \#\text{other}) = \text{PolyaUrn}(1, \alpha)$
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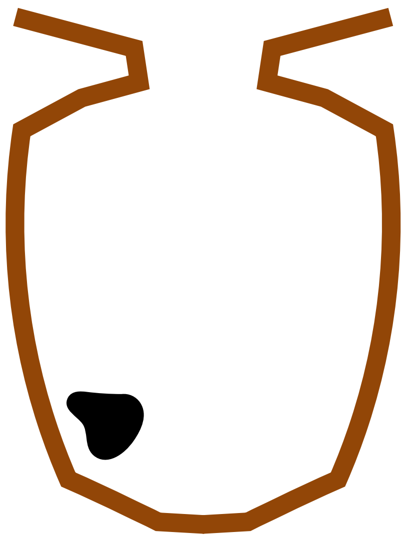
Marginal cluster assignments

- Hoppe urn / Blackwell-MacQueen urn



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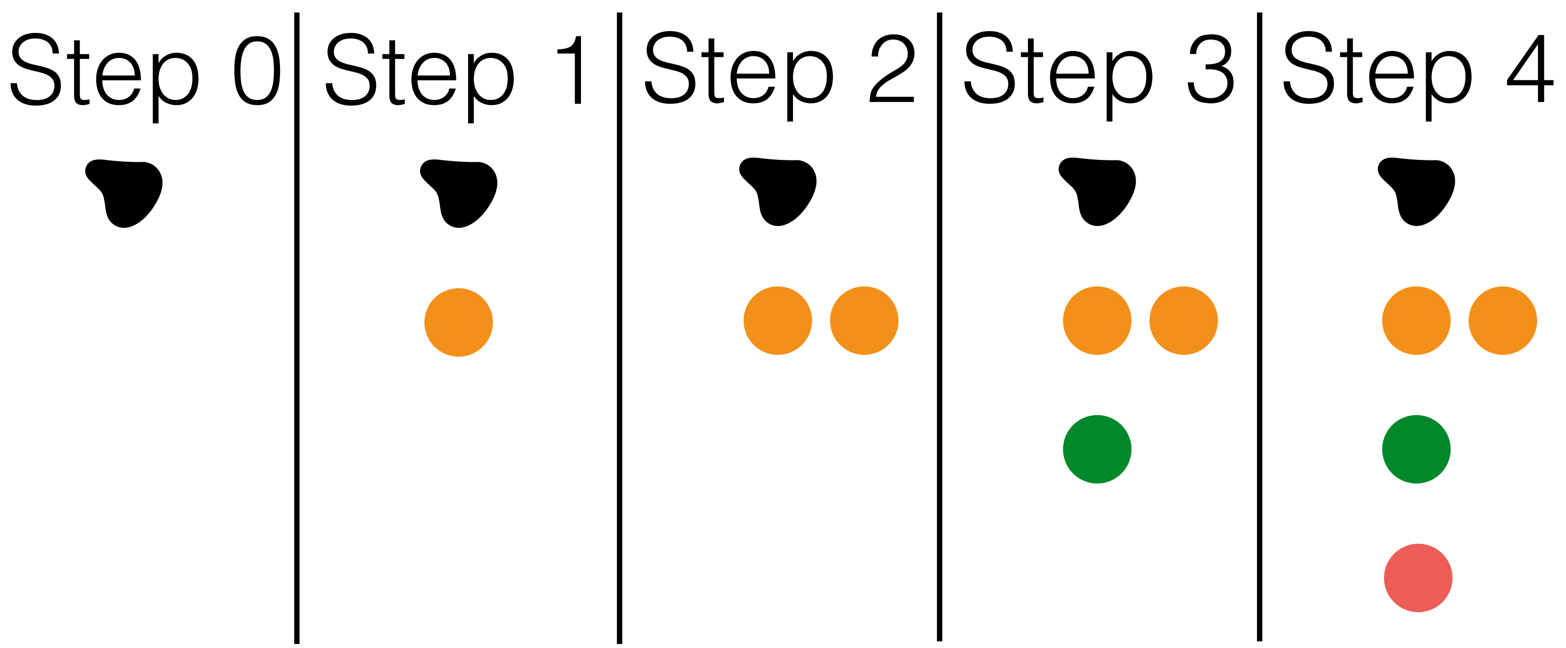

$$V_k \overset{iid}{\sim} \text{Beta}(1, \alpha)$$

$$\rho_1 = V_1$$

$$(\#\text{orange}, \#\text{other}) = \text{PolyaUrn}(1, \alpha)$$

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Marginal cluster assignments

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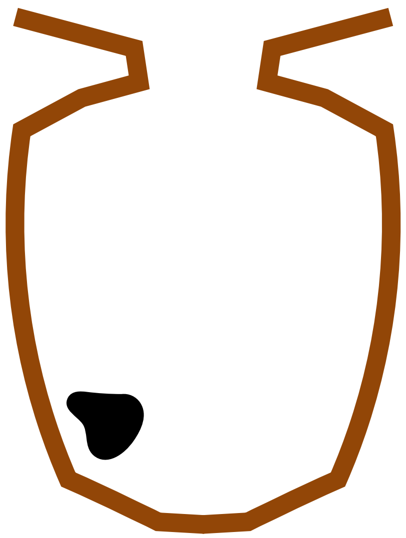

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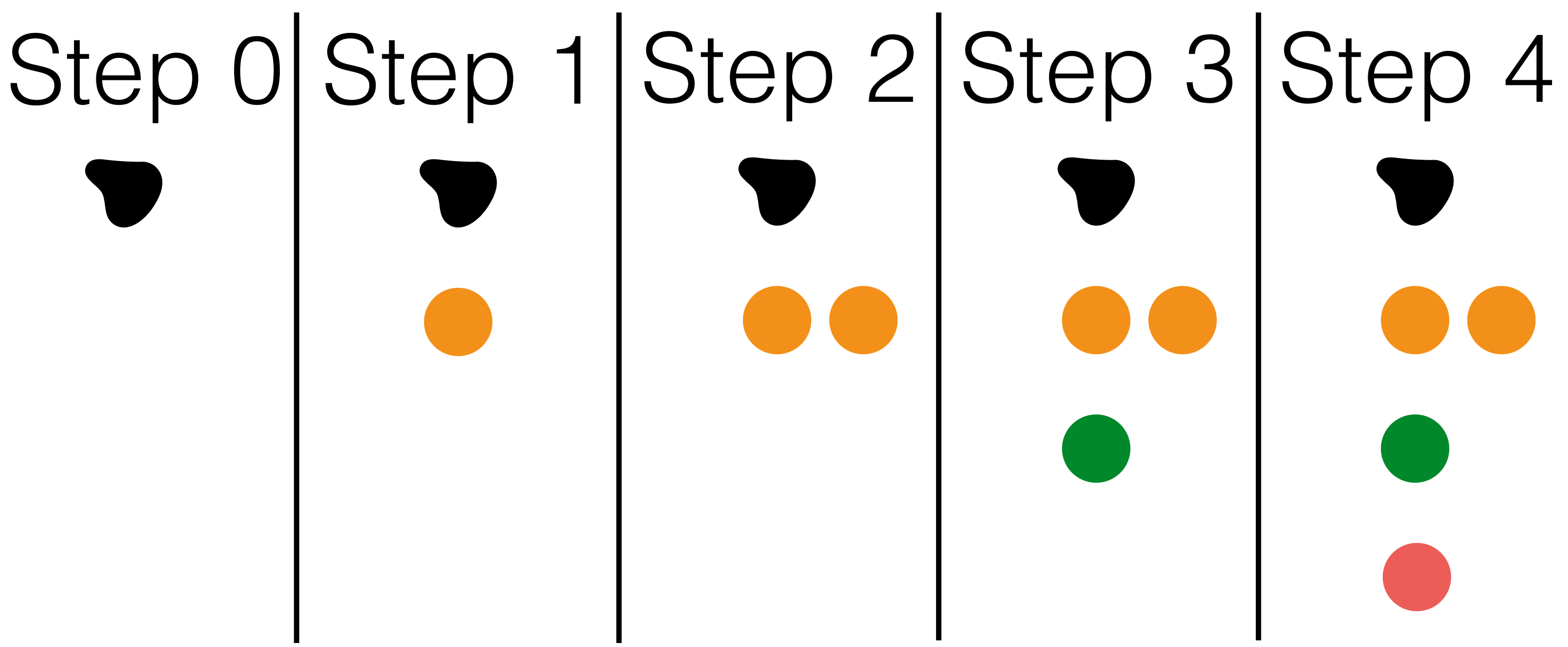

$$V_k \stackrel{iid}{\sim} \text{Beta}(1, \alpha)$$

$$\rho_1 = V_1$$

$$\rho_2 = (1 - V_1)V_2$$

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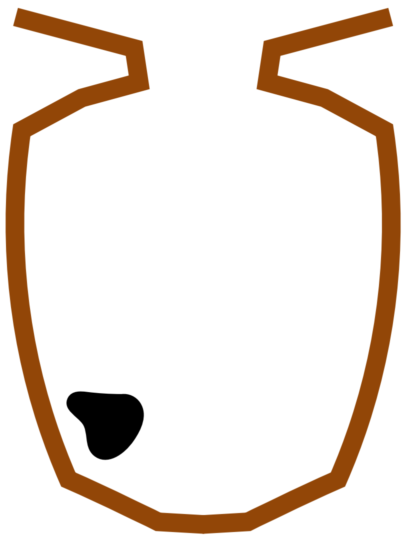
Marginal cluster assignments

- Hoppe urn / Blackwell-MacQueen urn



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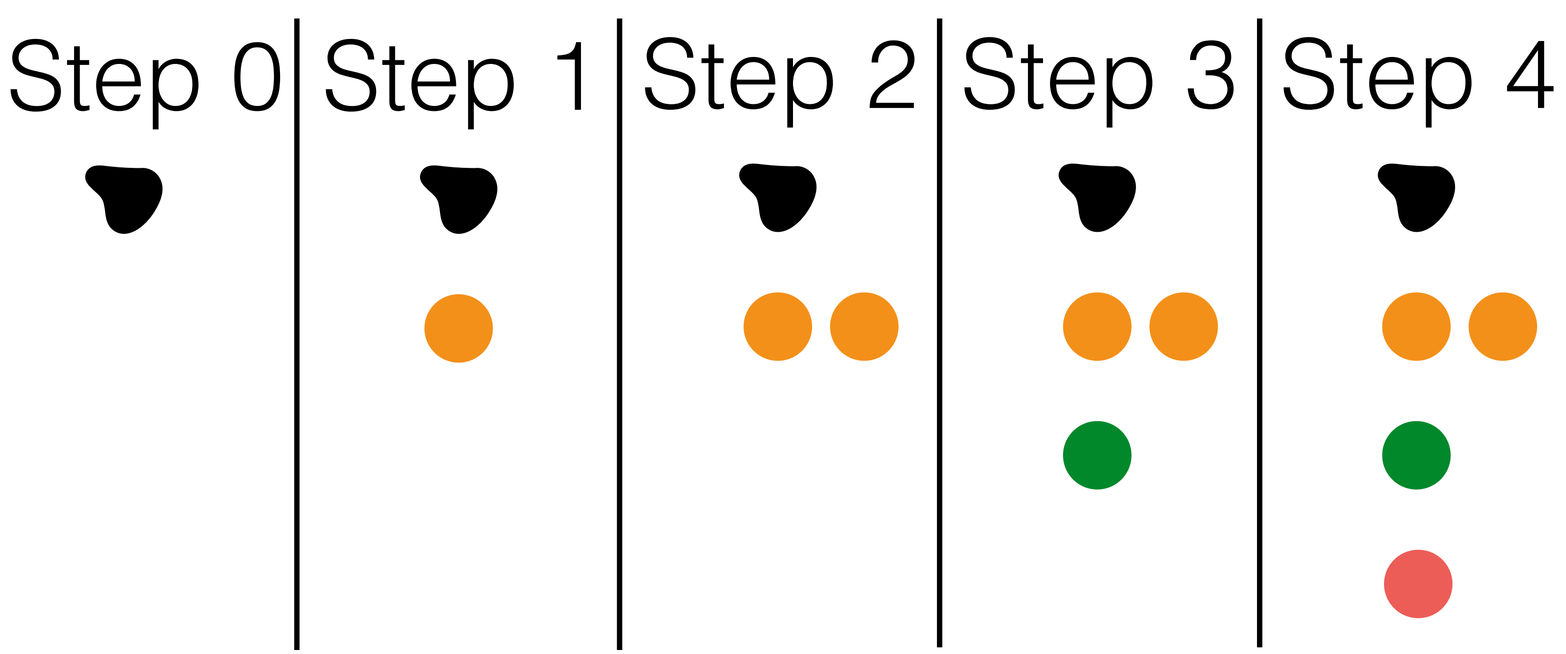

$$V_k \stackrel{iid}{\sim} \text{Beta}(1, \alpha)$$

$$\rho_1 = V_1$$

$$\rho_2 = (1 - V_1) V_2$$

$$\rho_3 = [\prod_{k=1}^{2} (1 - V_k)] V_3$$

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Chinese restaurant process

Chinese restaurant process

- Same thing we just did

Chinese restaurant process

- Same thing we just did
- Each customer walks into the restaurant

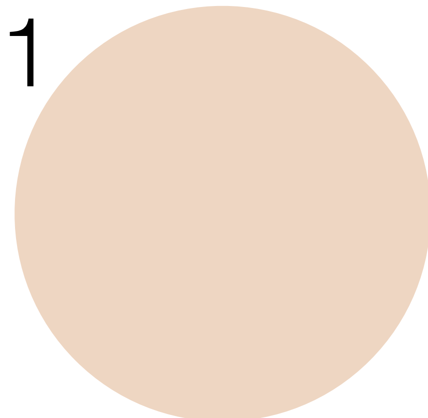
Chinese restaurant process

- Same thing we just did
- Each customer walks into the restaurant
  - Sits at existing table with prob proportional to # people there

Chinese restaurant process

- Same thing we just did
- Each customer walks into the restaurant
  - Sits at existing table with prob proportional to # people there
  - Forms new table with prob proportional to α

Chinese restaurant process



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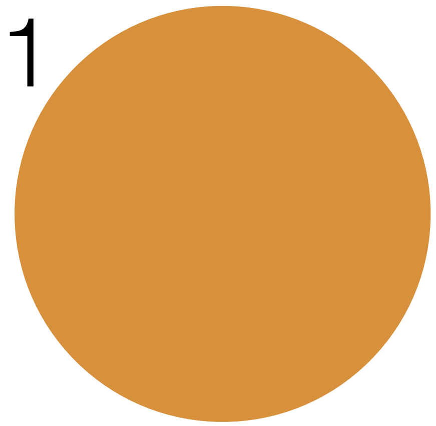
Chinese restaurant process



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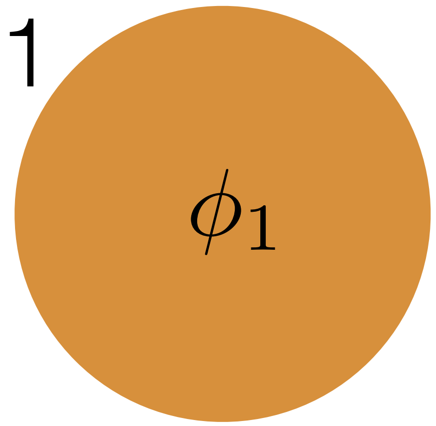
Chinese restaurant process



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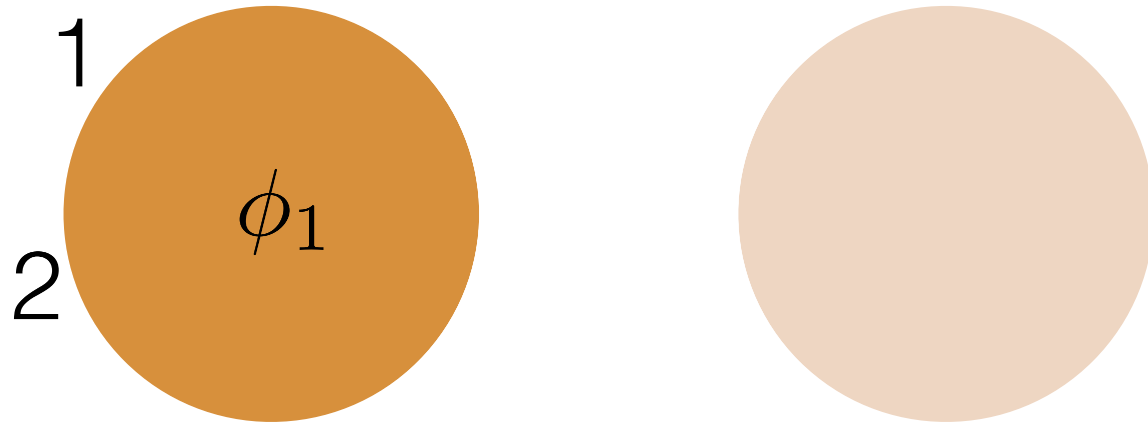

- Same thing we just did
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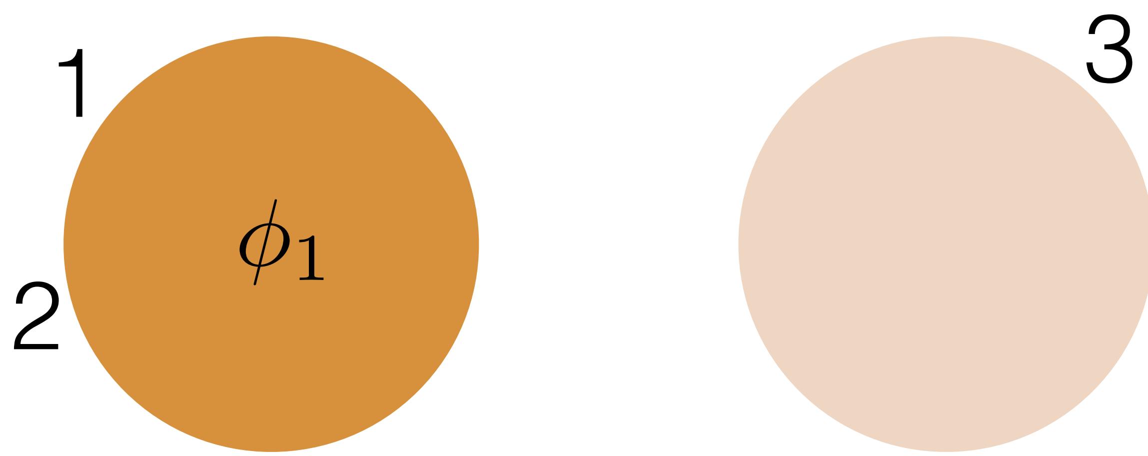
Chinese restaurant process



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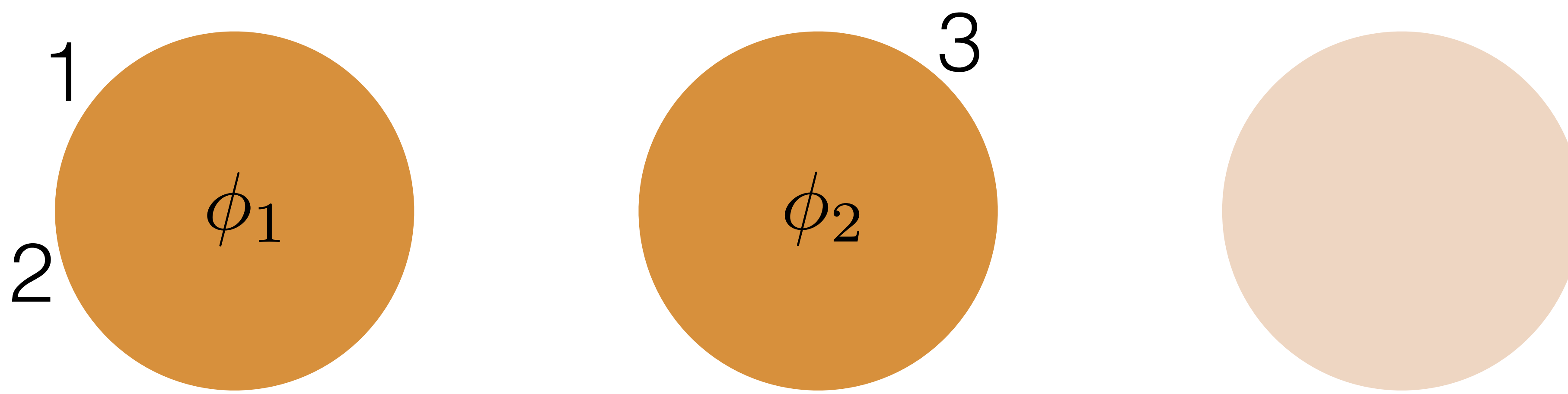
Chinese restaurant process



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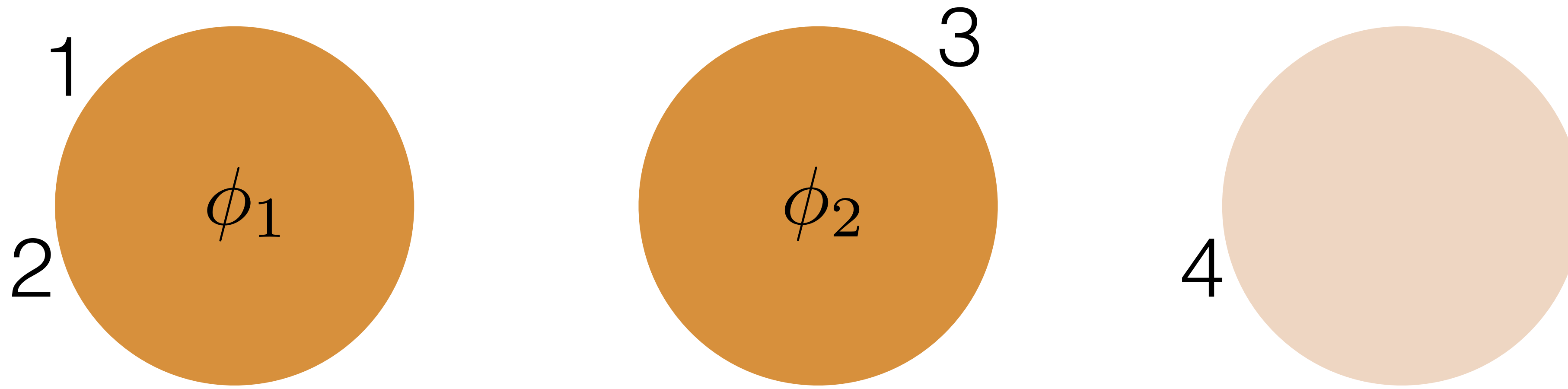
Chinese restaurant process



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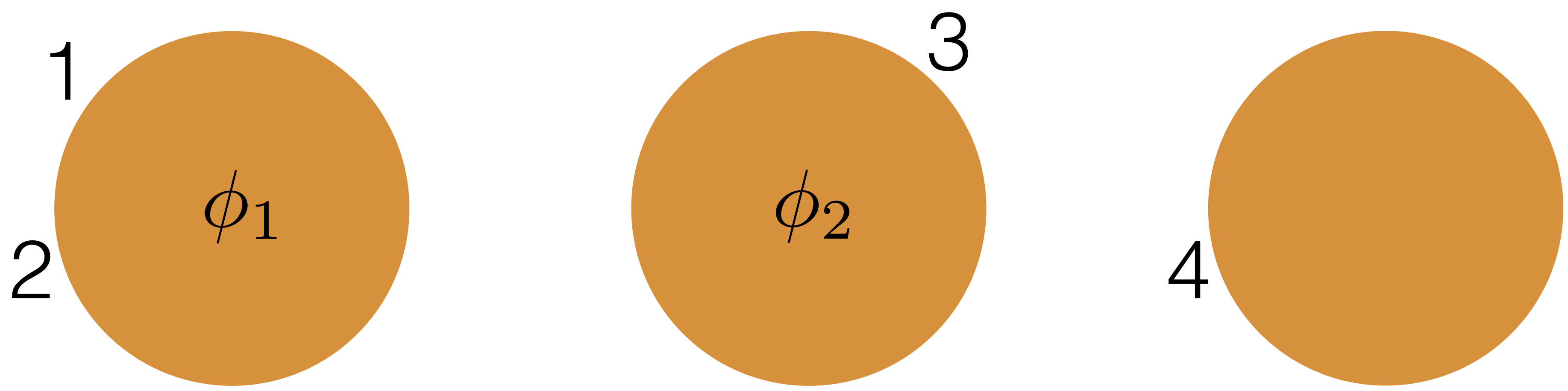
Chinese restaurant process



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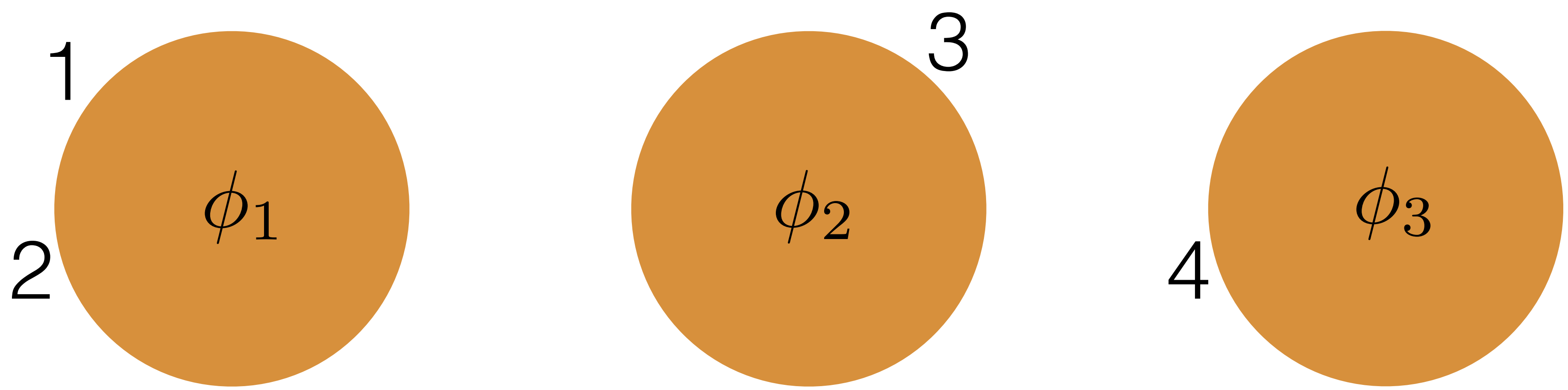
Chinese restaurant process



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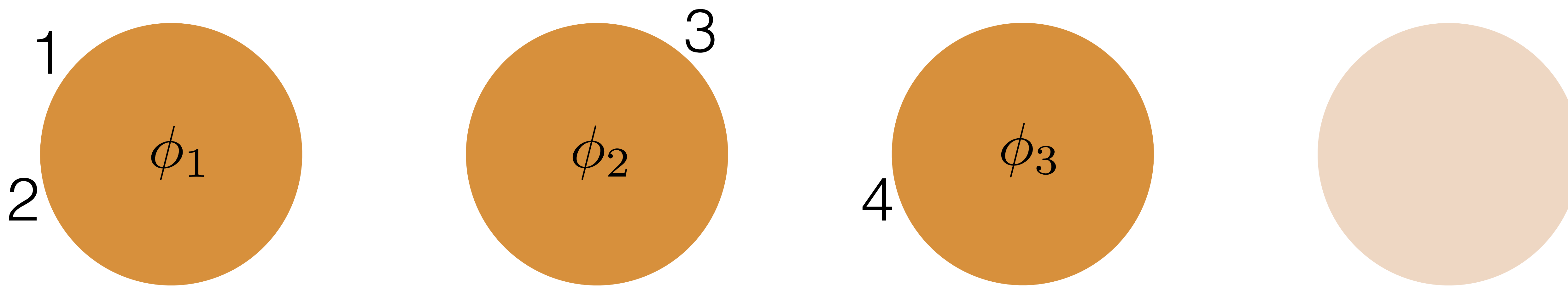
Chinese restaurant process



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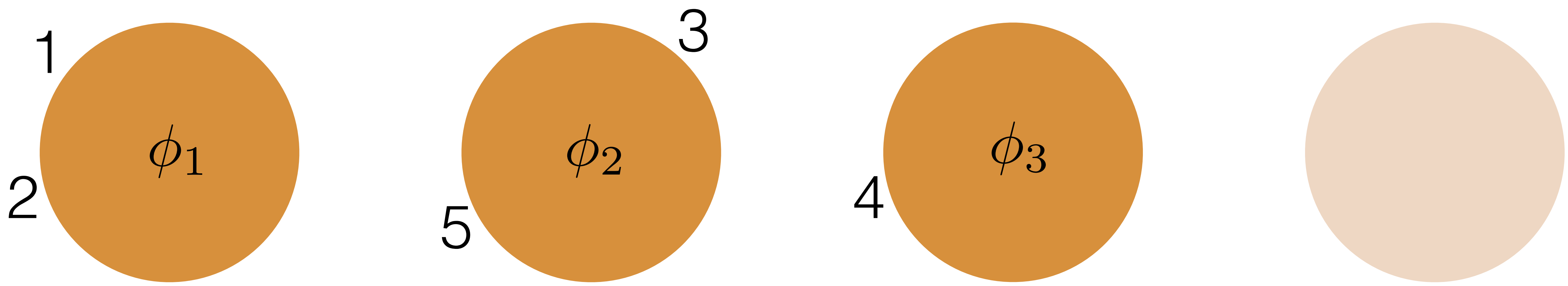
Chinese restaurant process



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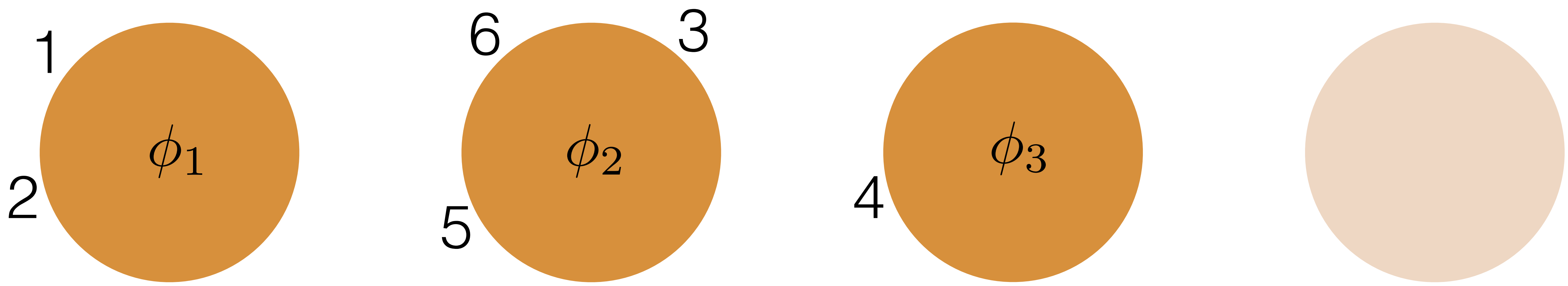
Chinese restaurant process



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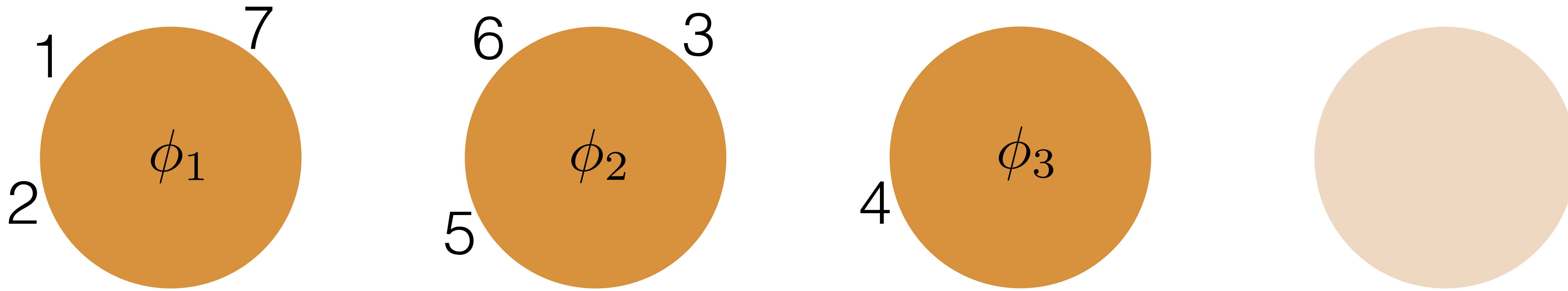
Chinese restaurant process



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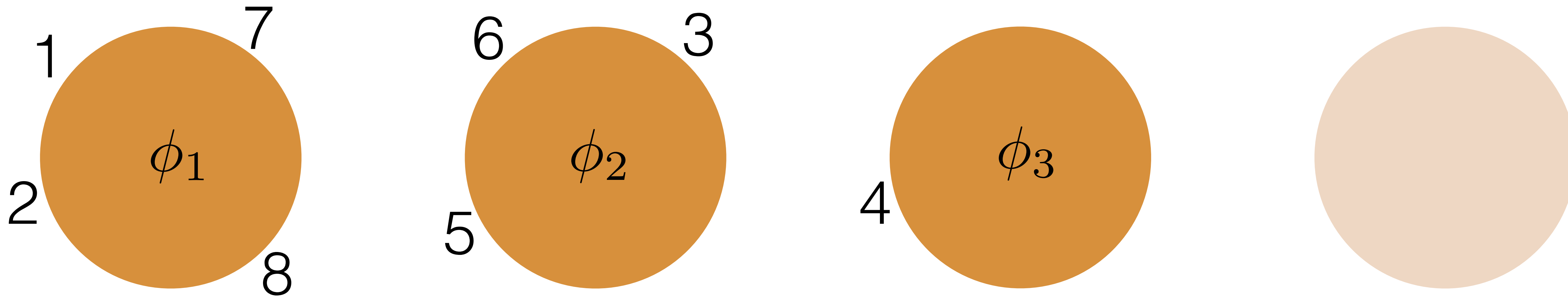
Chinese restaurant process



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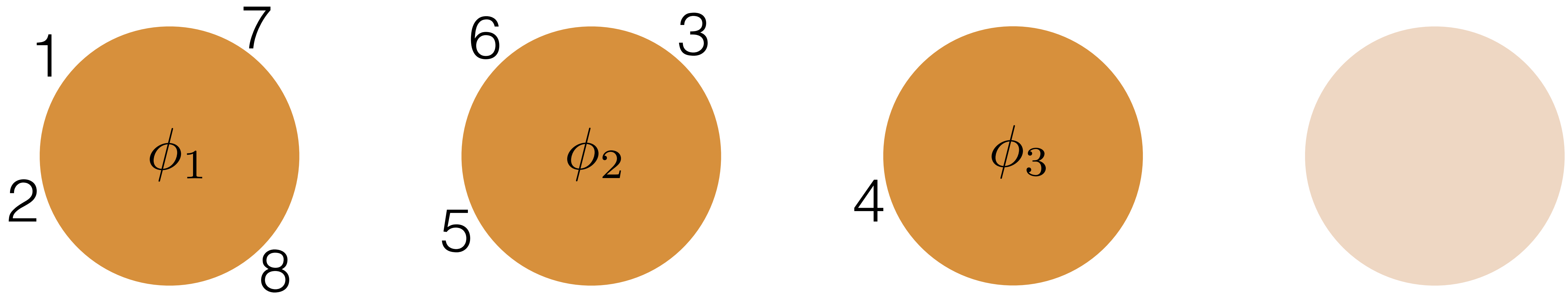
Chinese restaurant process



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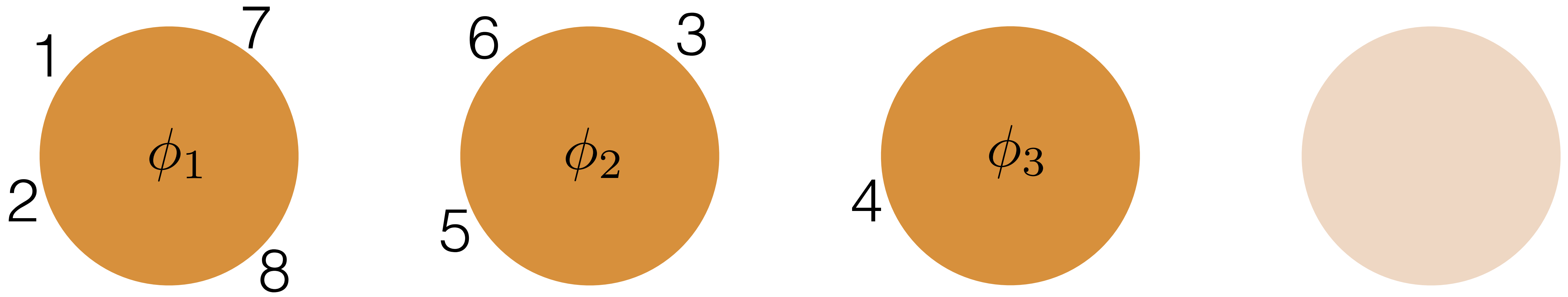
[Aldous 1983]

Chinese restaurant process



- Same thing we just did
- Each customer walks into the restaurant
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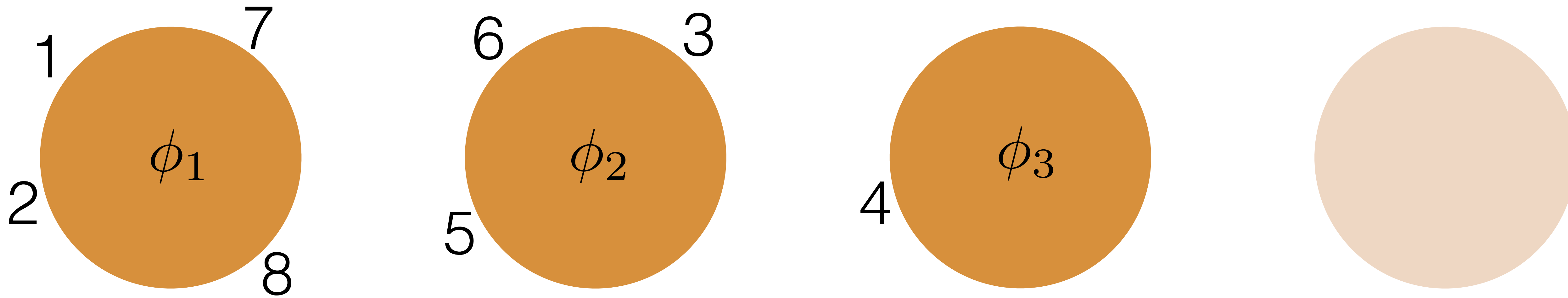
- Marginal for the Categorical likelihood with GEM prior

Chinese restaurant process



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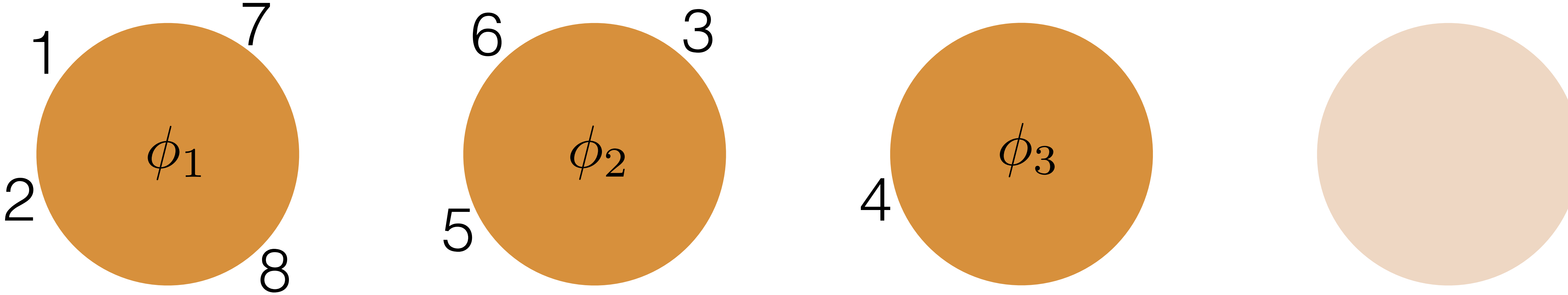
- Marginal for the Categorical likelihood with GEM prior

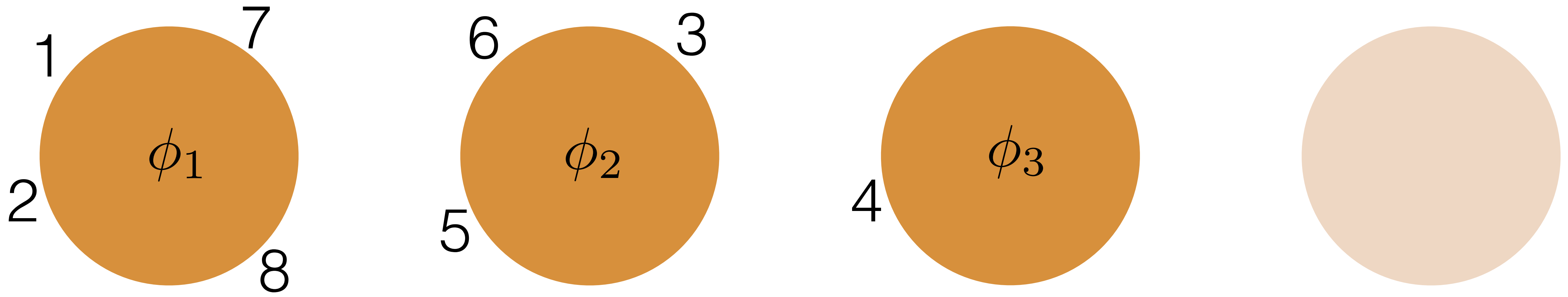
So far: Dirichlet process, Chinese restaurant process

- Infinity of parameters, growing number of parameters

Exercises

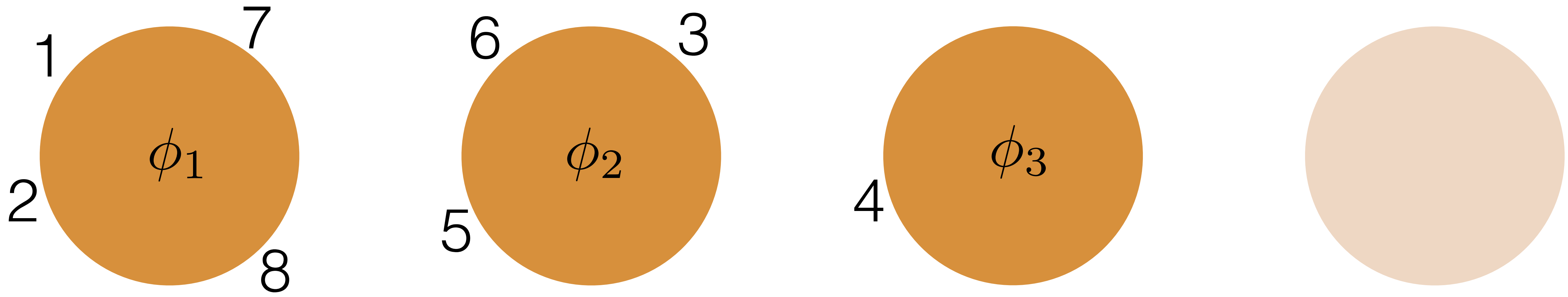


Exercises



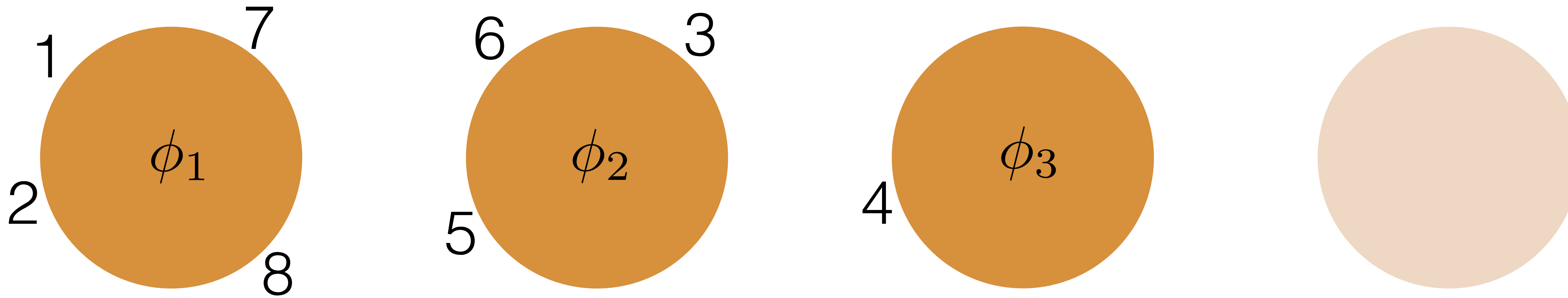
- Review Gibbs sampling

Exercises



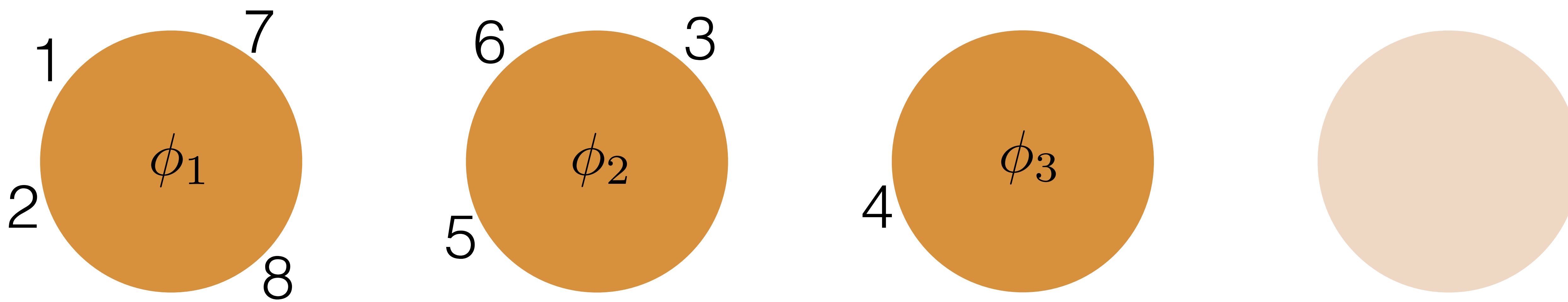
- Review Gibbs sampling
- What are the advantages and disadvantages of the DP and CRP representations?

Exercises



- Review Gibbs sampling
- What are the advantages and disadvantages of the DP and CRP representations?
- What is the expected number of clusters generated by a CRP(α) after N data points?

Exercises



- Review Gibbs sampling
- What are the advantages and disadvantages of the DP and CRP representations?
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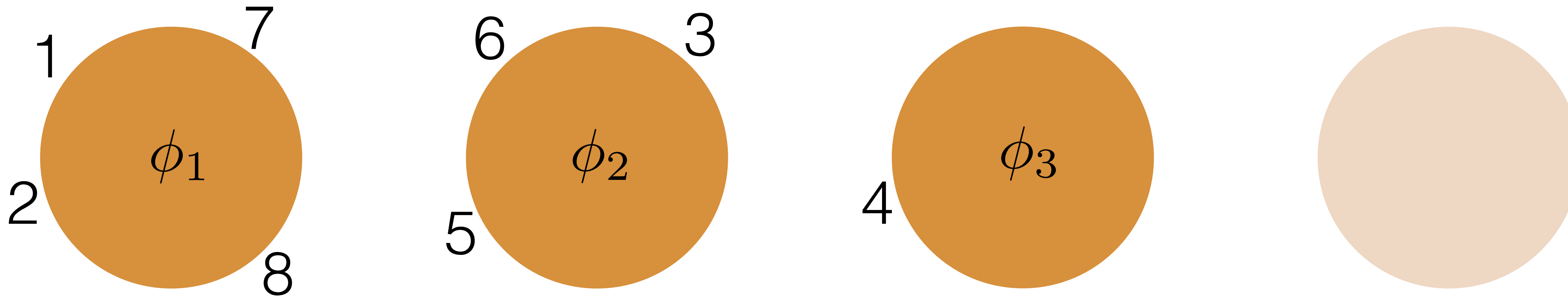
Exercises



- Review Gibbs sampling
- What are the advantages and disadvantages of the DP and CRP representations?
- What is the expected number of clusters generated by a CRP(α) after *N* data points?
- What do you think about the answer to the previous question when it comes to real-life data modeling?
- Code a CRP sampler. Examine the empirical distribution of the number of clusters after *N* customers.

References

A full reference list is provided at the end of the “Part III” slides.