Nonparametric Bayesian Statistics

Tamara Broderick
ITT Career Development Assistant Professor
Electrical Engineering & Computer Science
MIT

Nonparametric Bayes

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“Wikipedia phenomenon”

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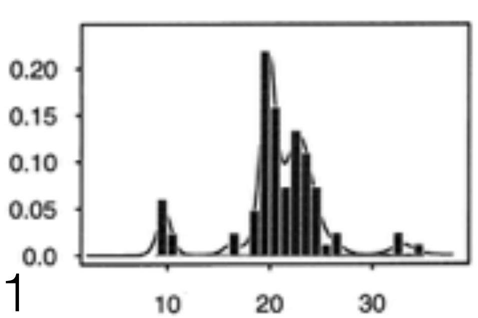

[Escobar, West 1995; Ghosal, et al 1999]

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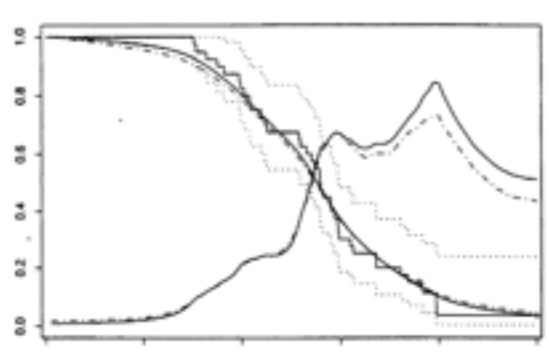

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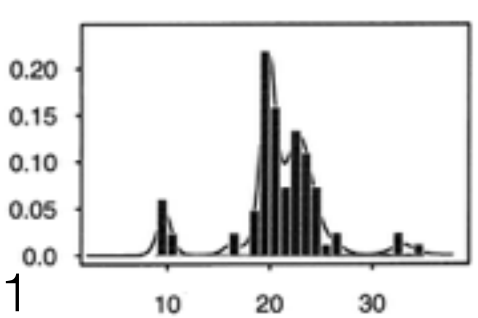

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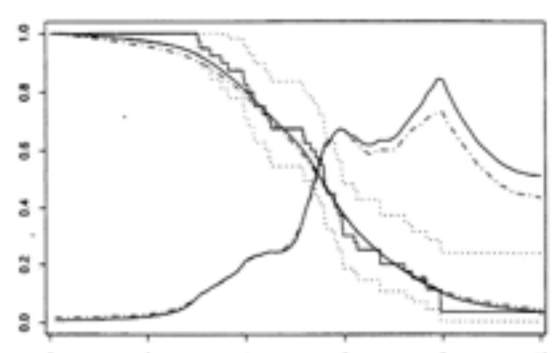

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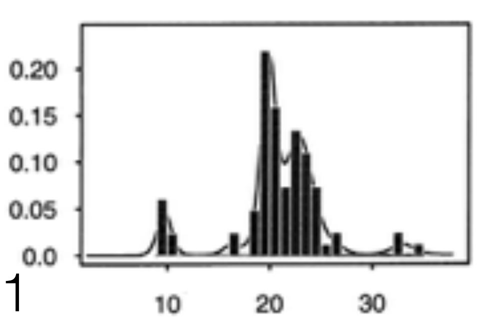

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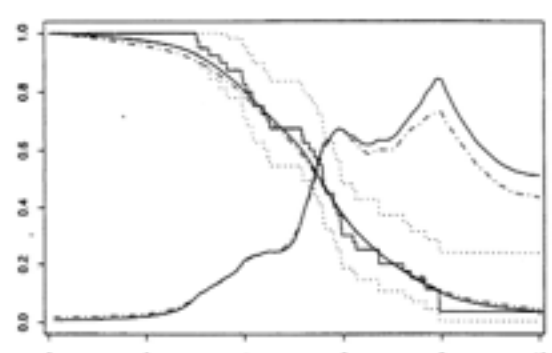

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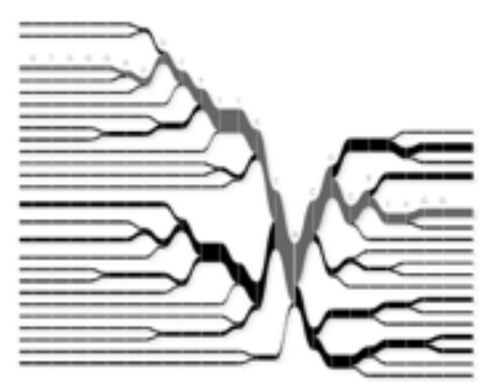
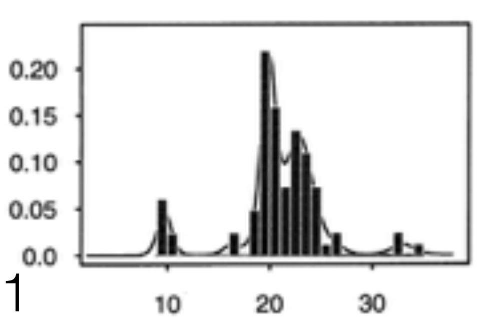


[Ewens, 1972; Hartl, Clark 2003]



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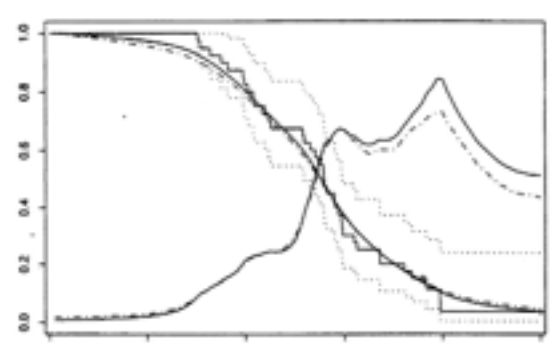

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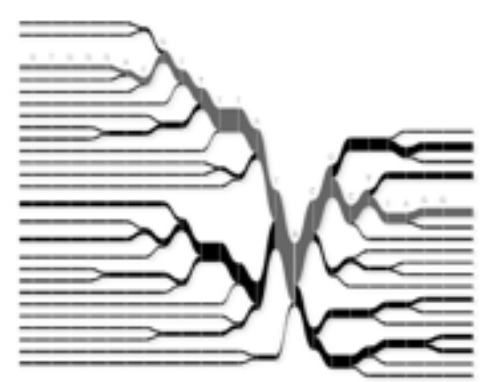

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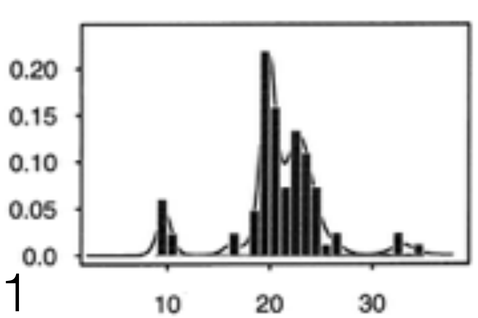

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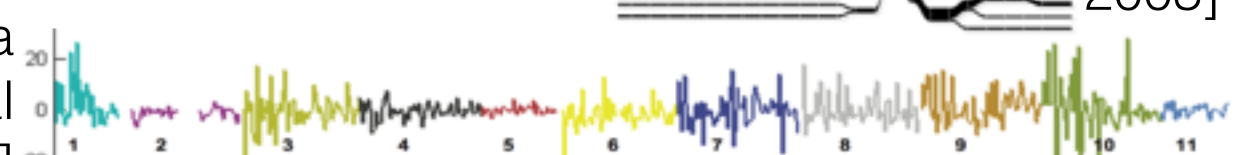

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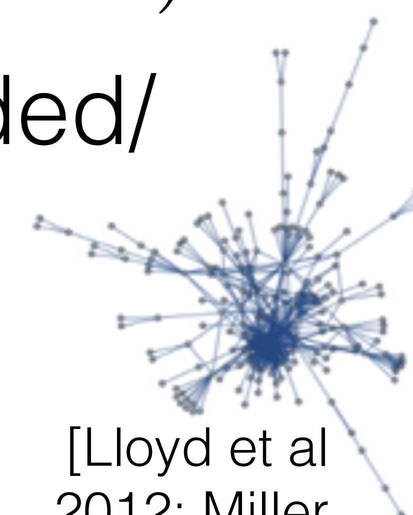

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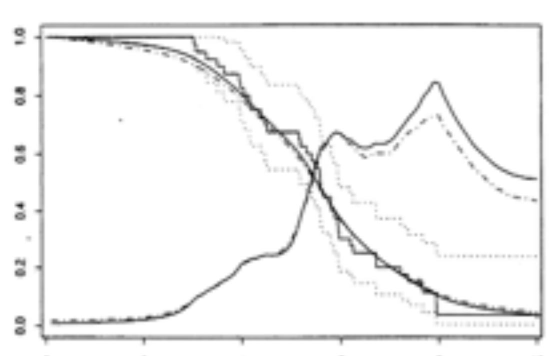

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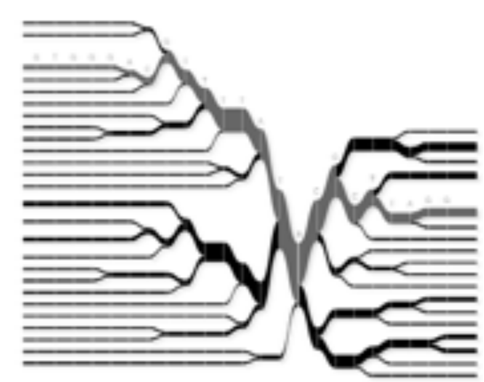

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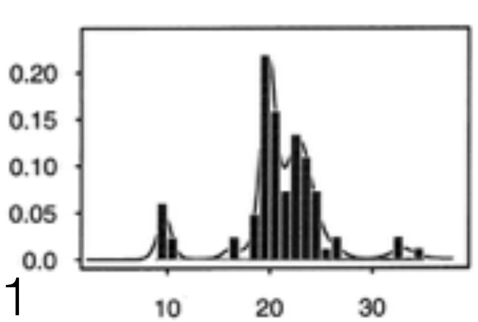

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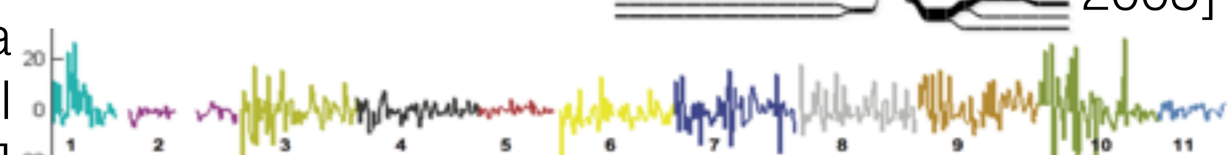

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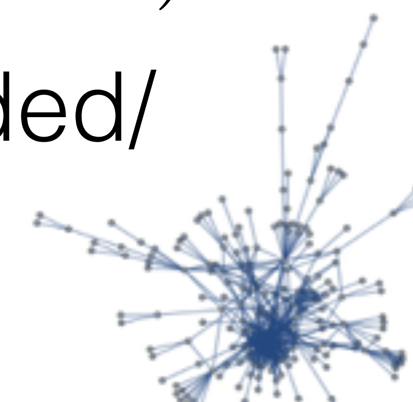

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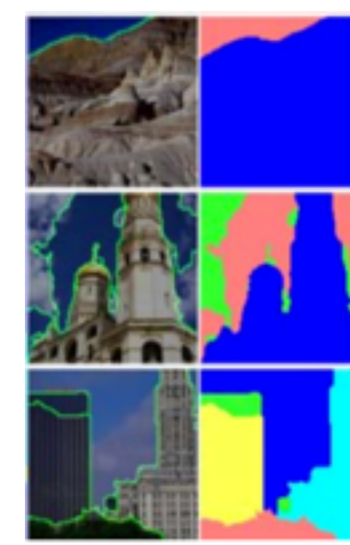
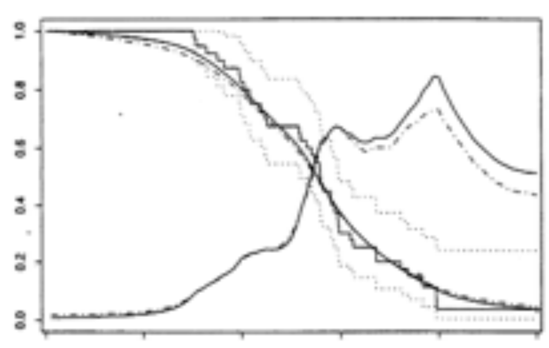


[Sudderth, Jordan 2009]



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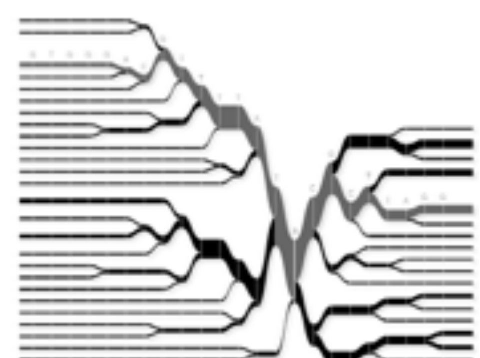

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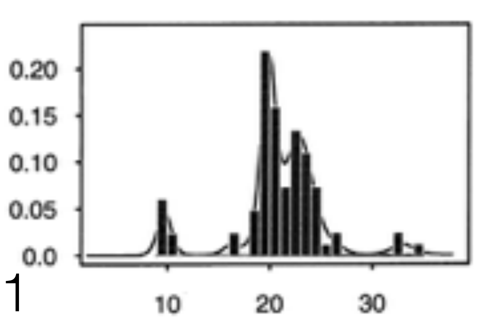

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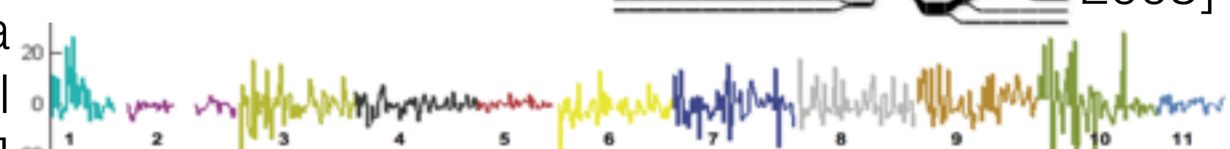

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[Hewitt, Savage 1955; Aldous 1983]

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- Dirichlet process

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  - Background for intuition

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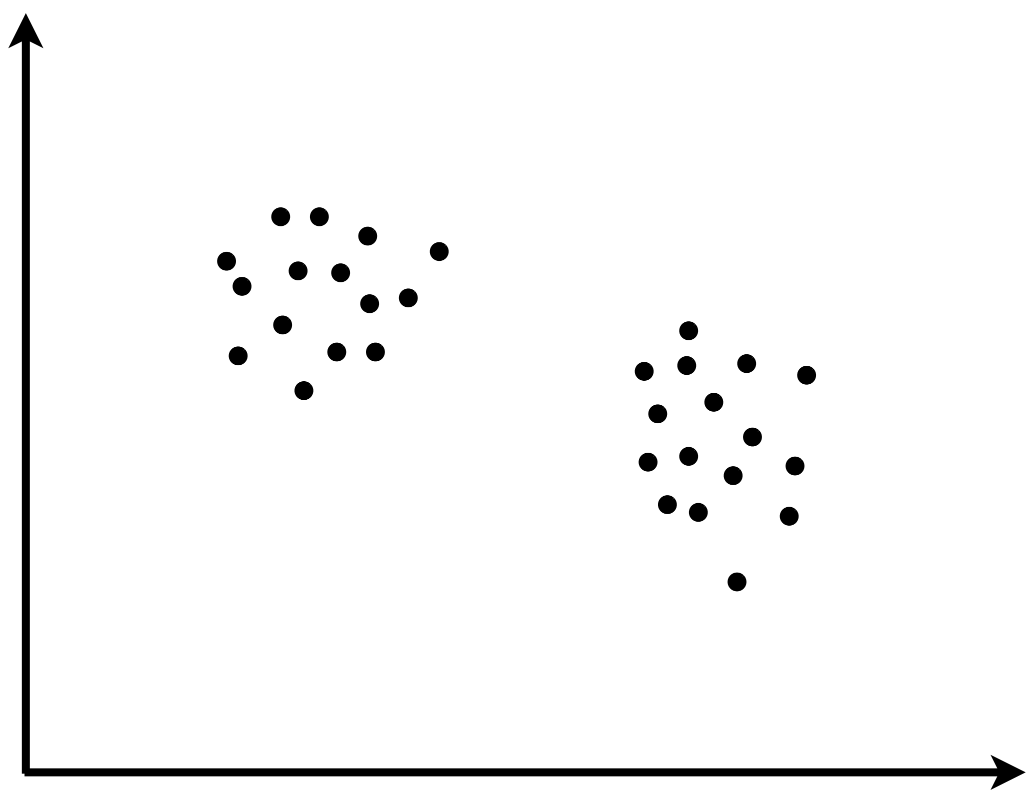
- Venture further into the wild world of Nonparametric Bayesian statistics

Generative model



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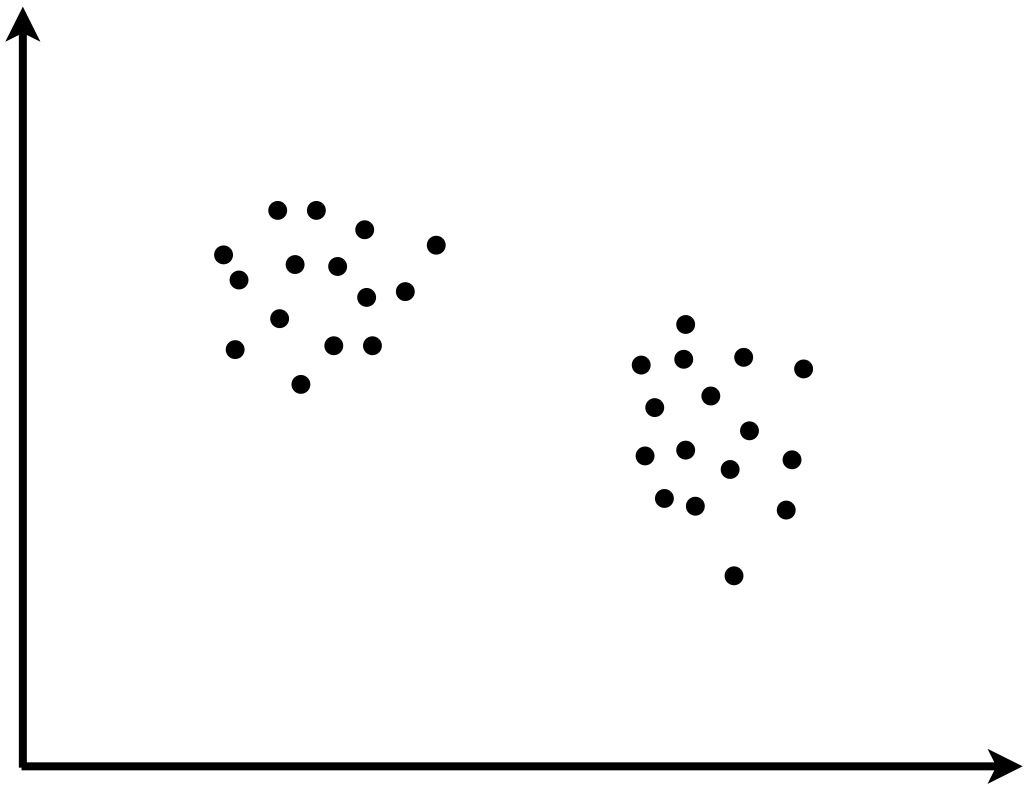

- Finite Gaussian mixture model (K=2 clusters)

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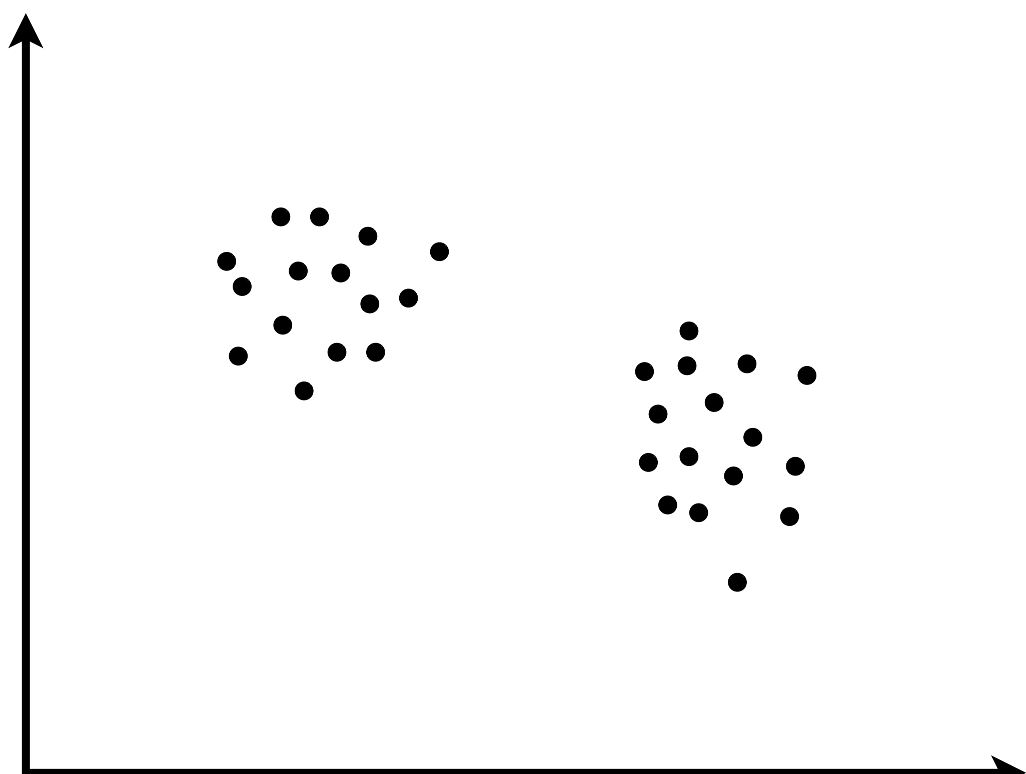

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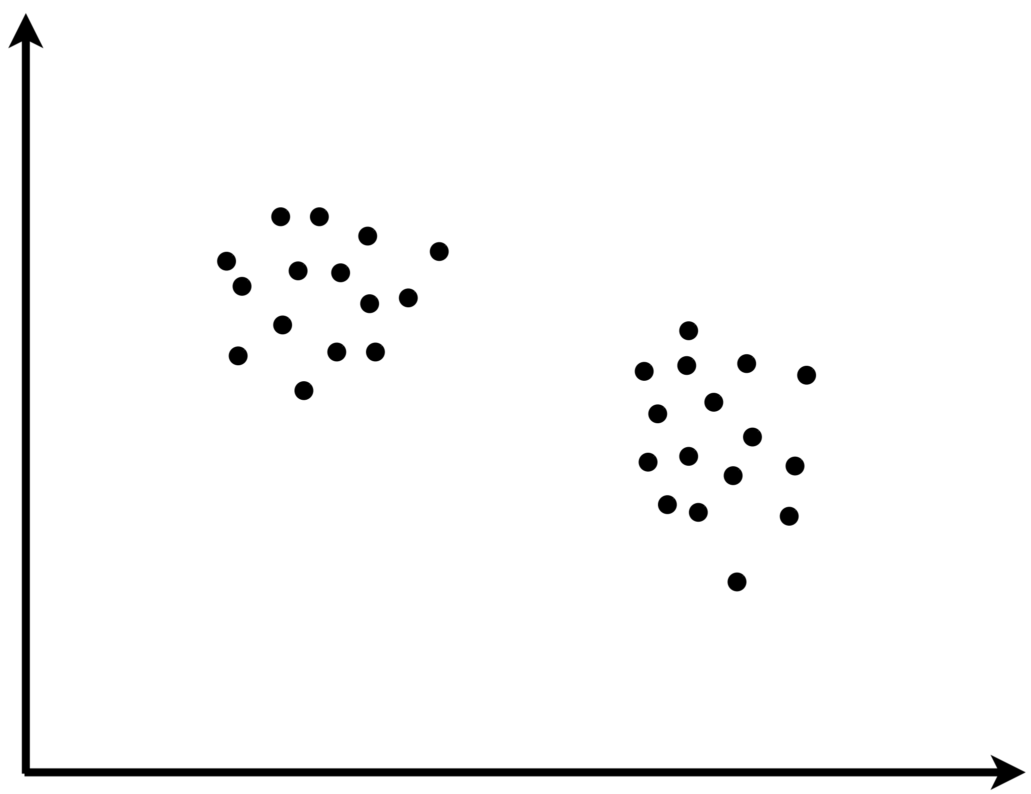

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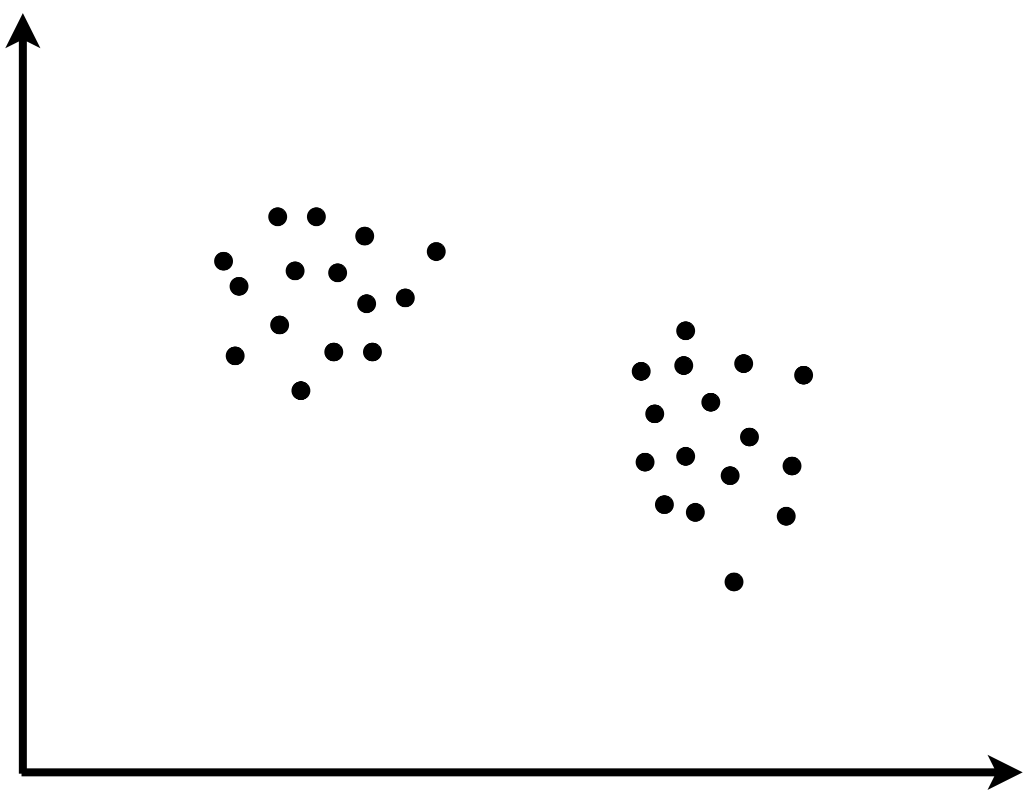

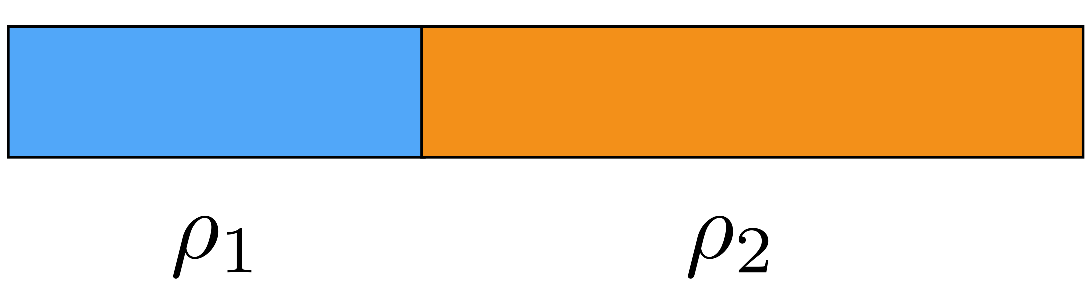

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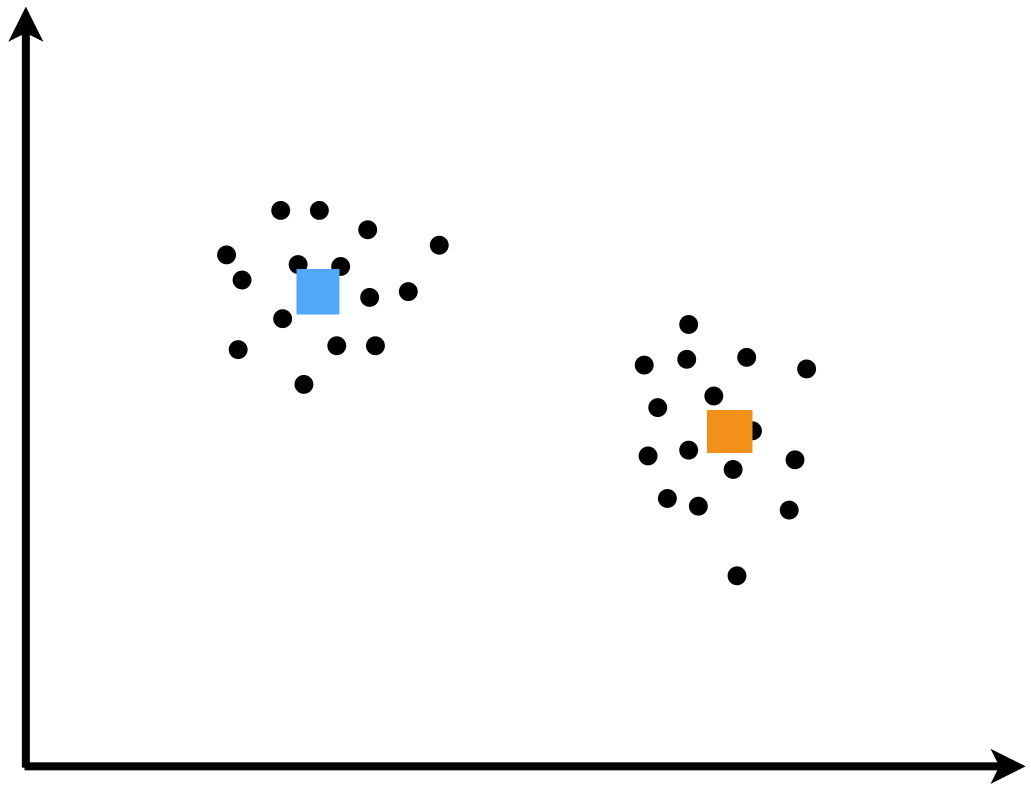

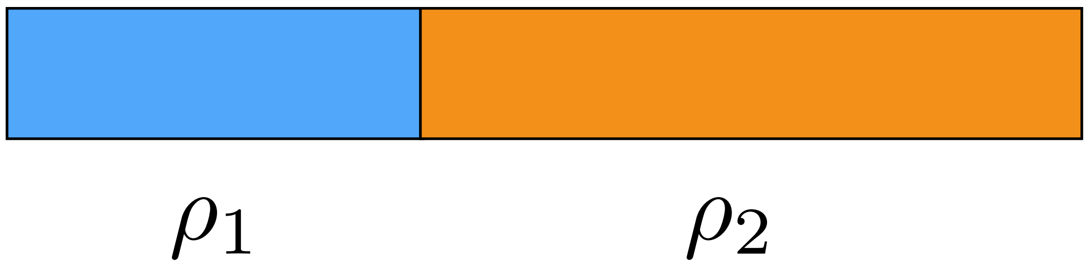

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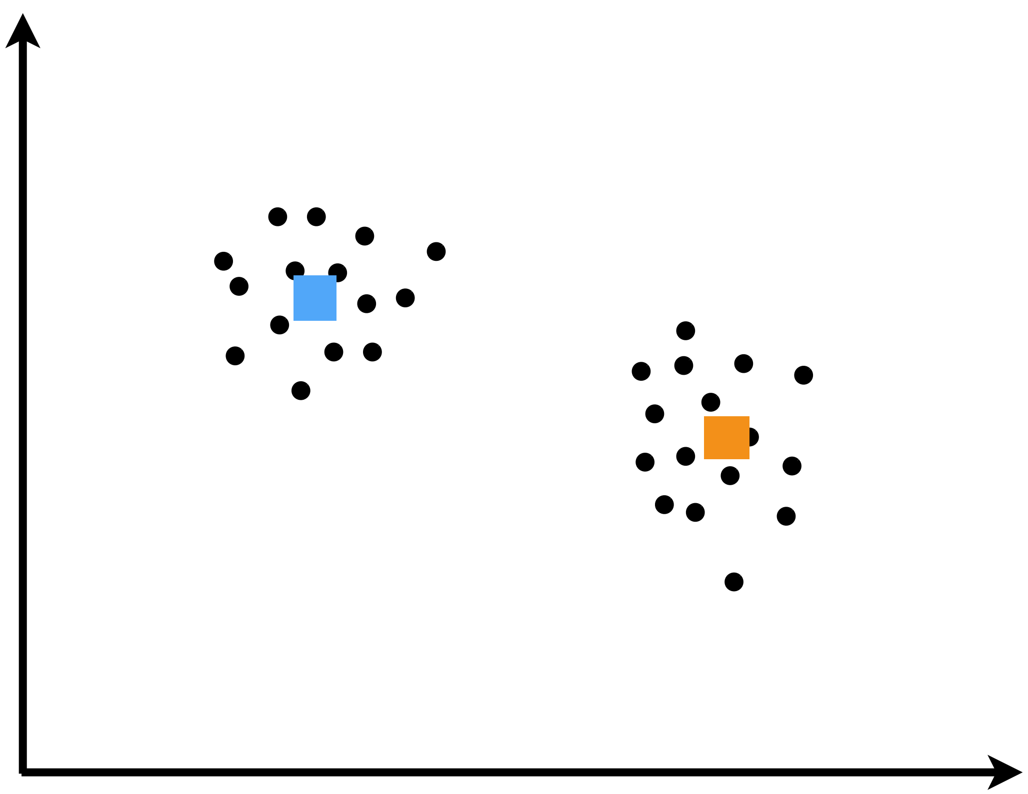

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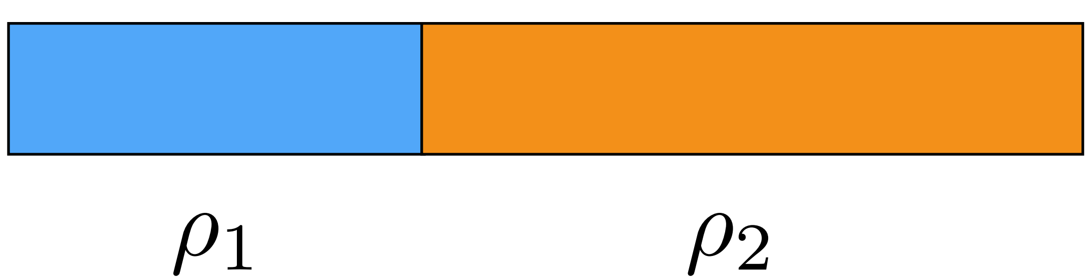

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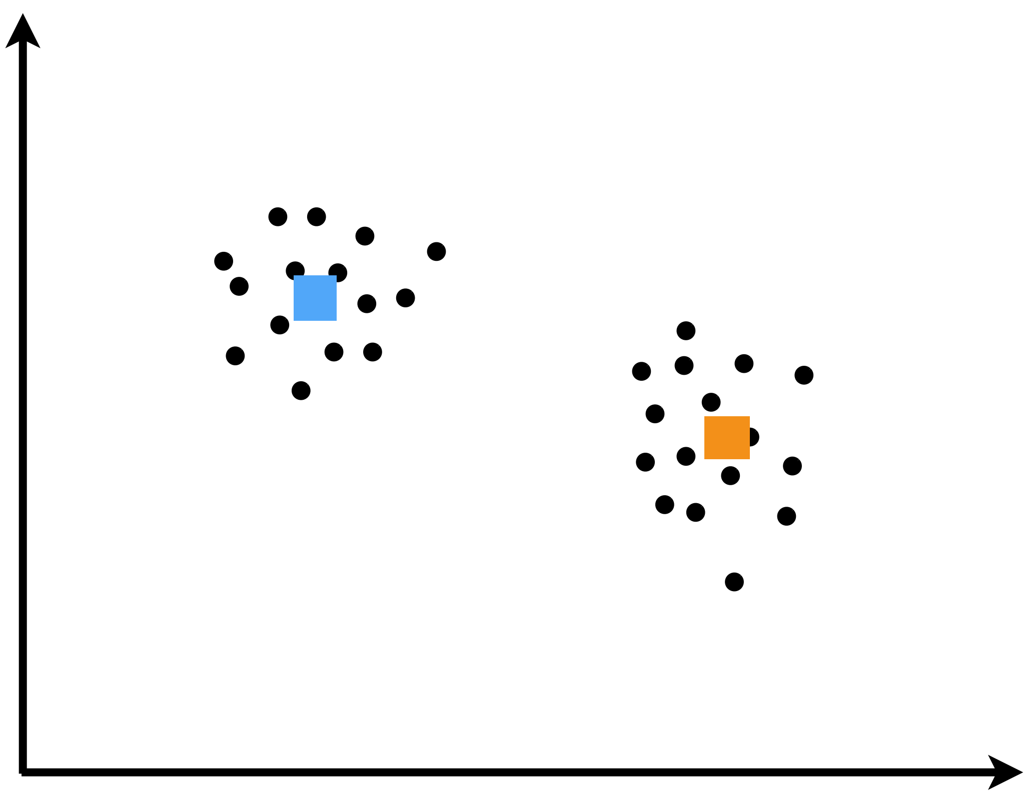

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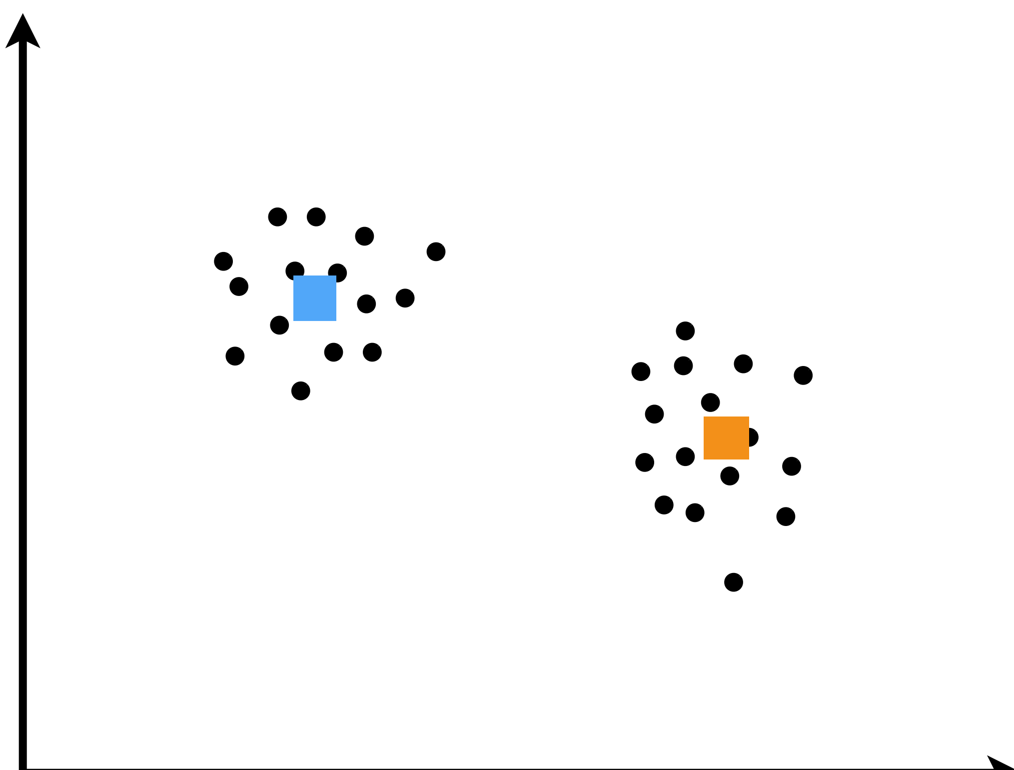

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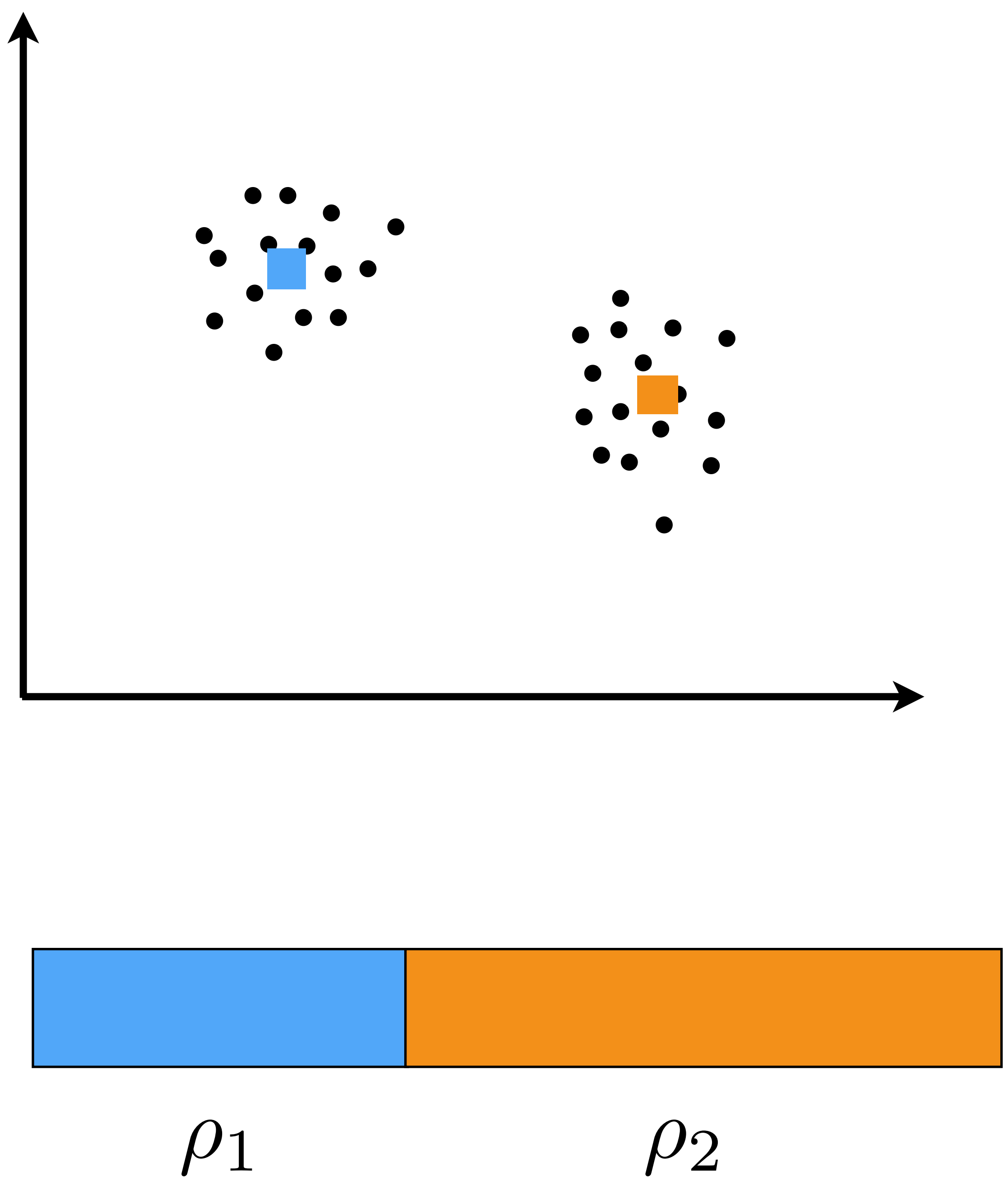

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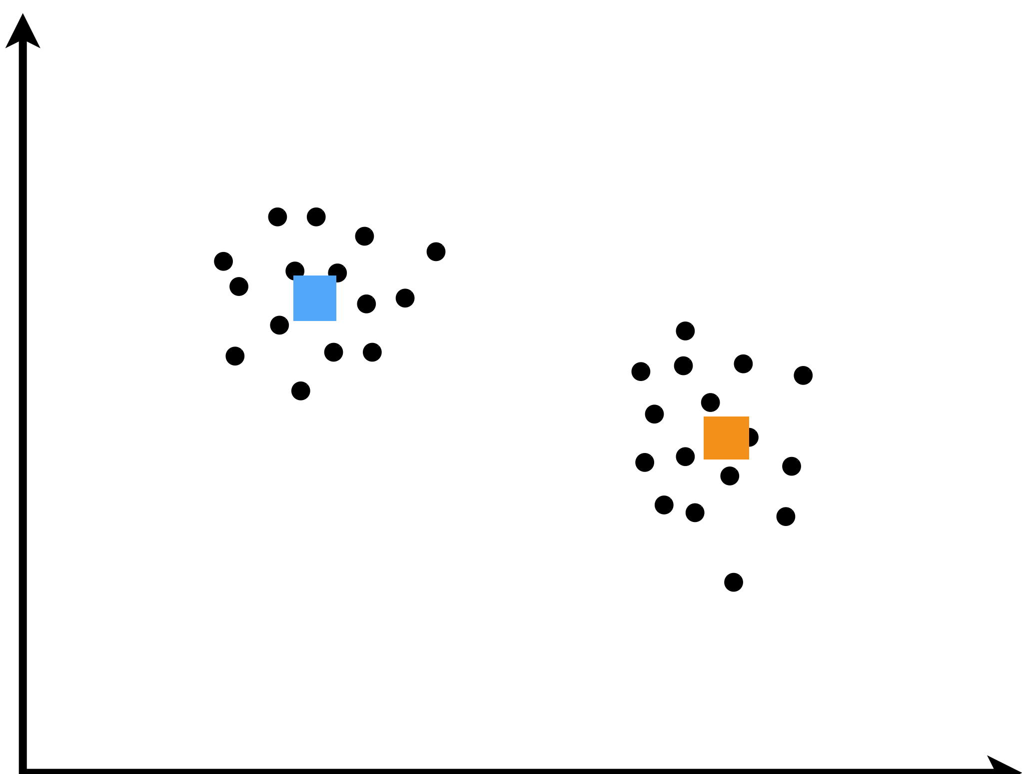

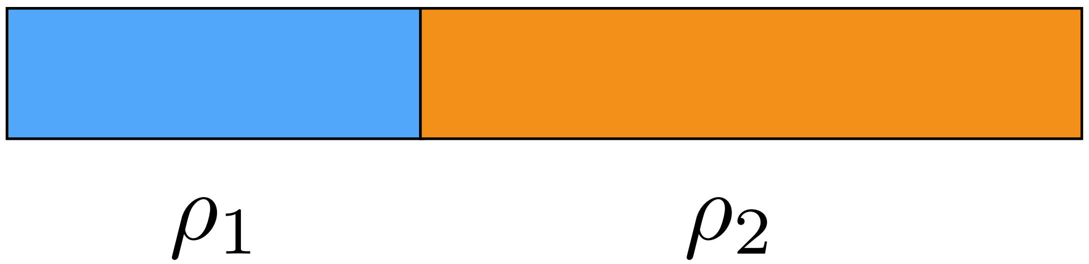

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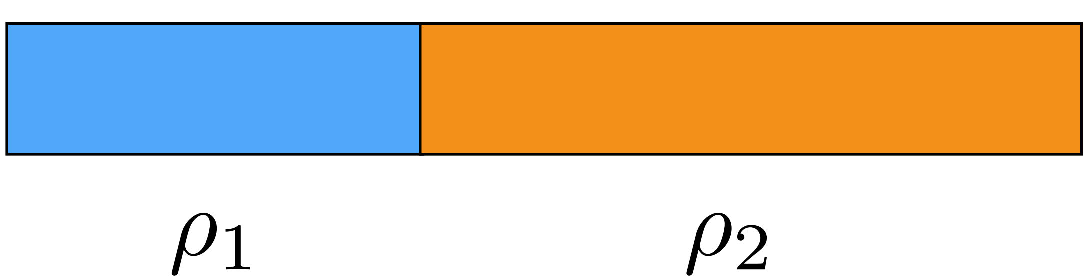

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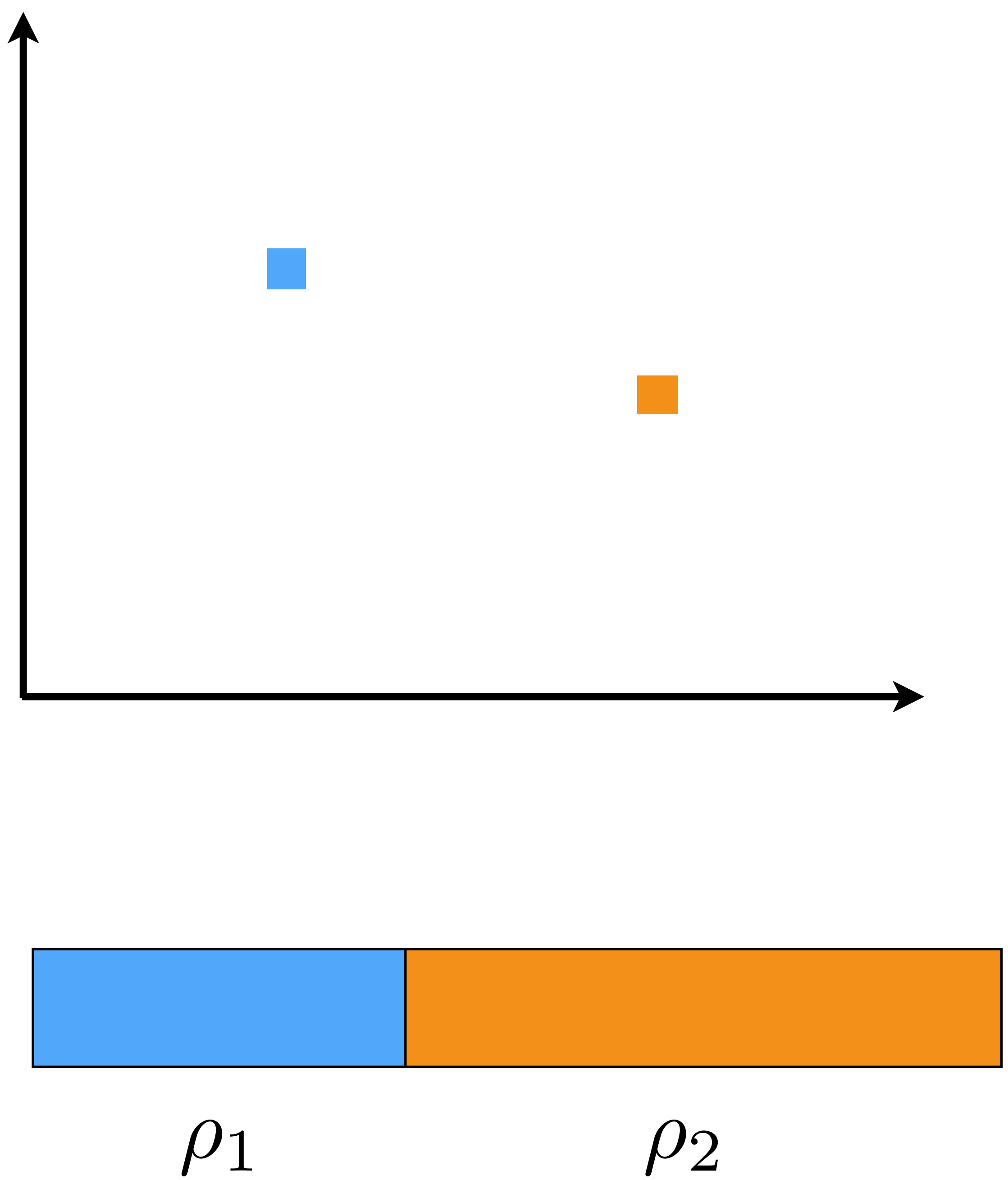

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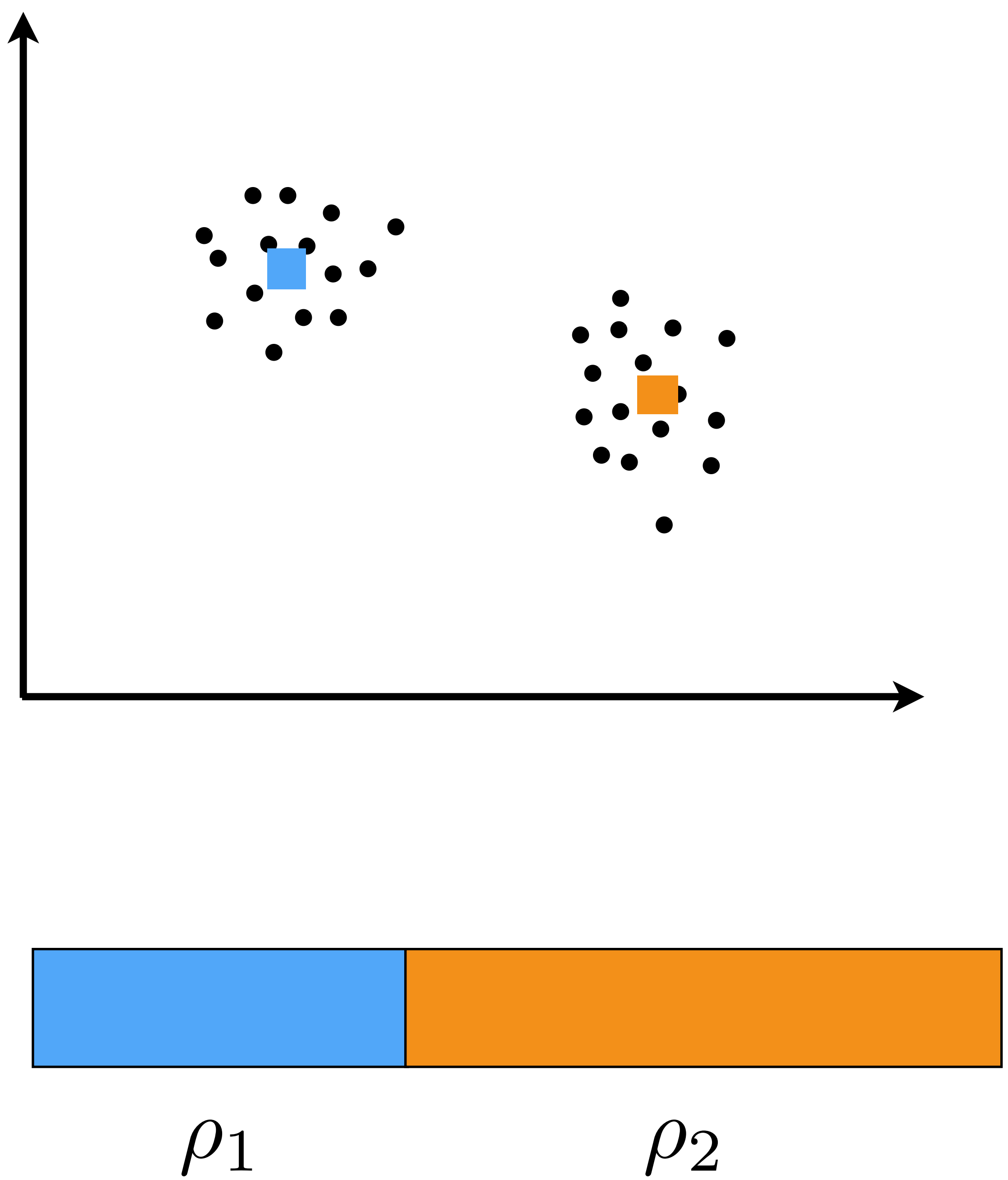

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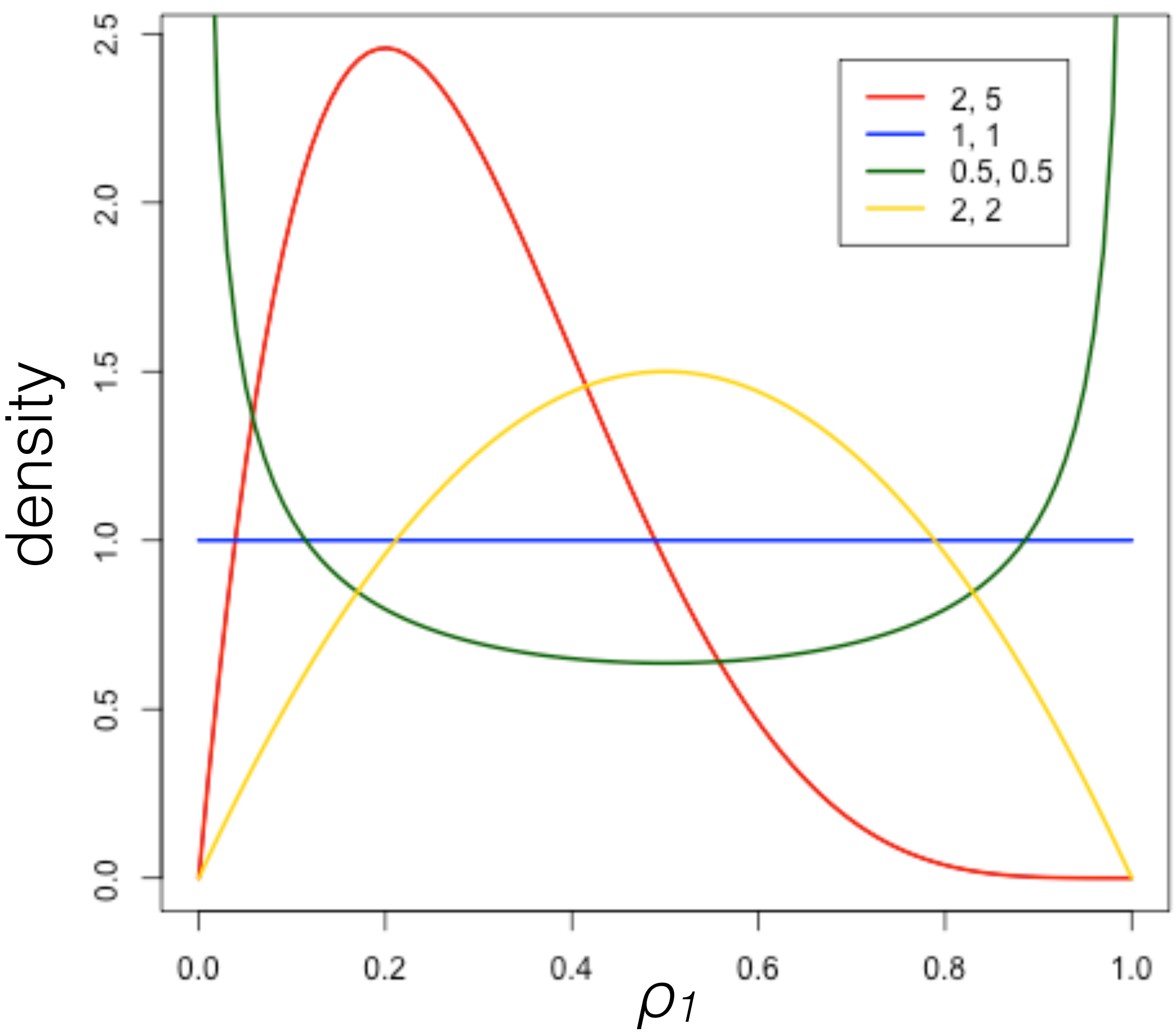

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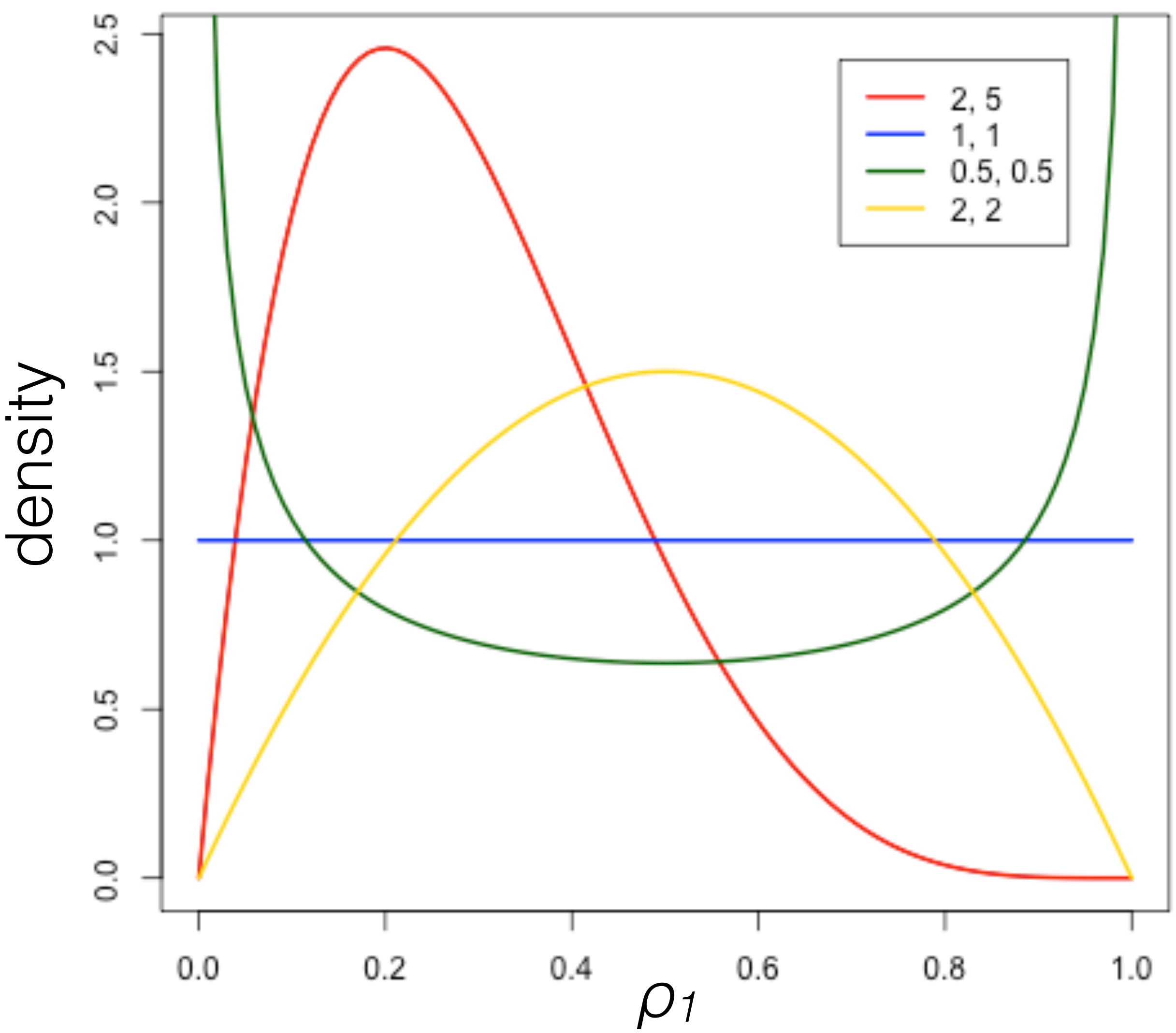

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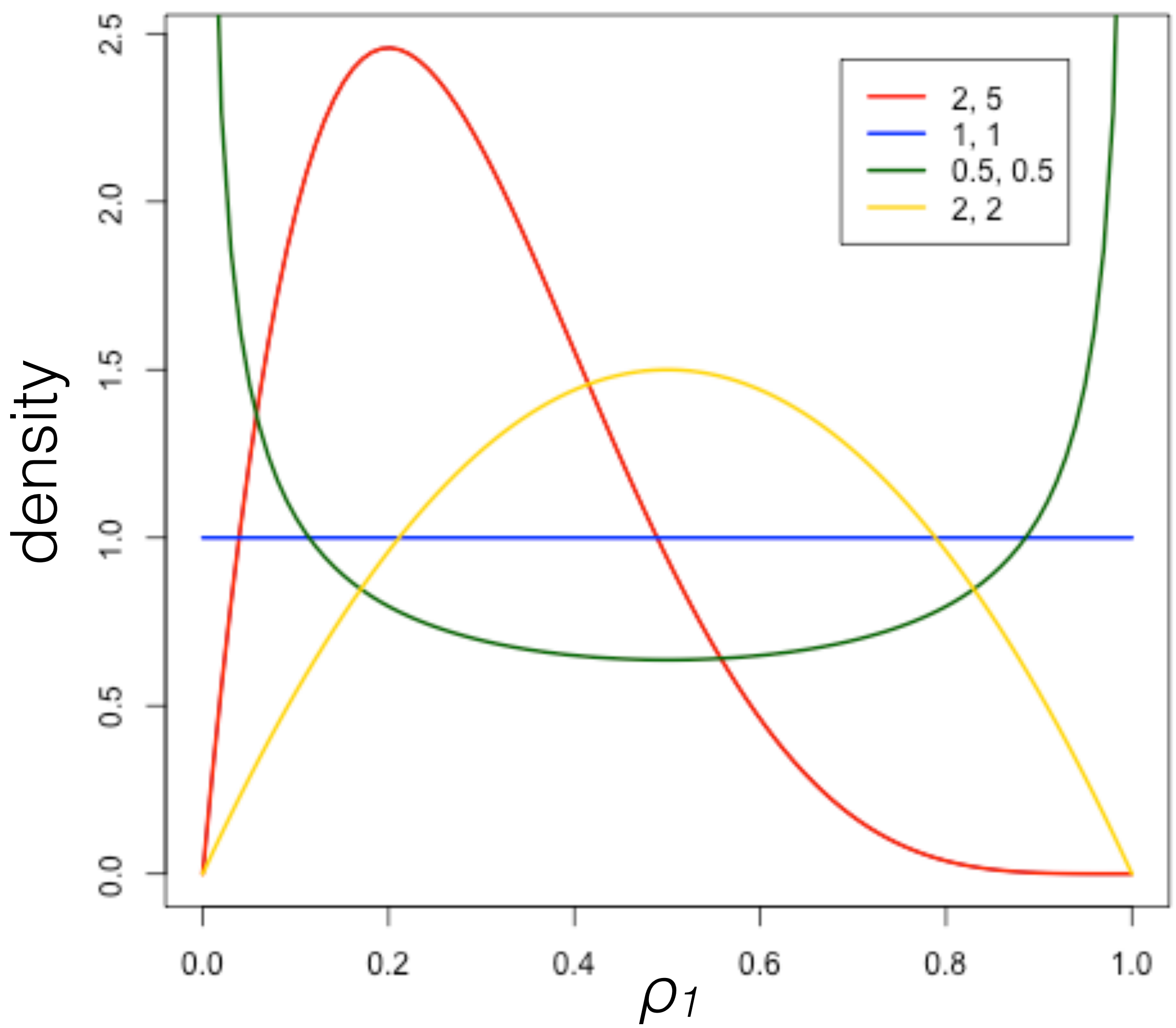

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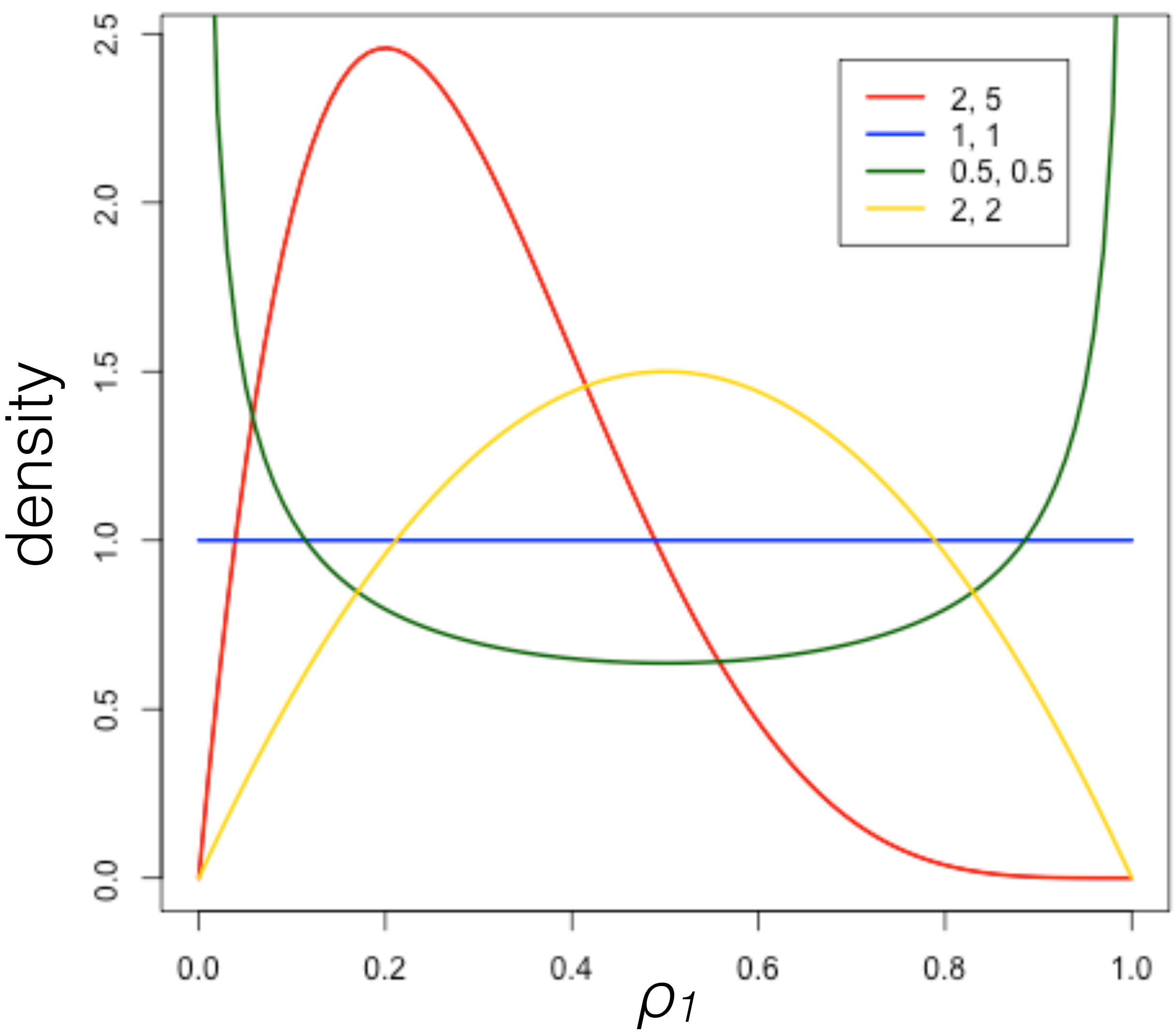

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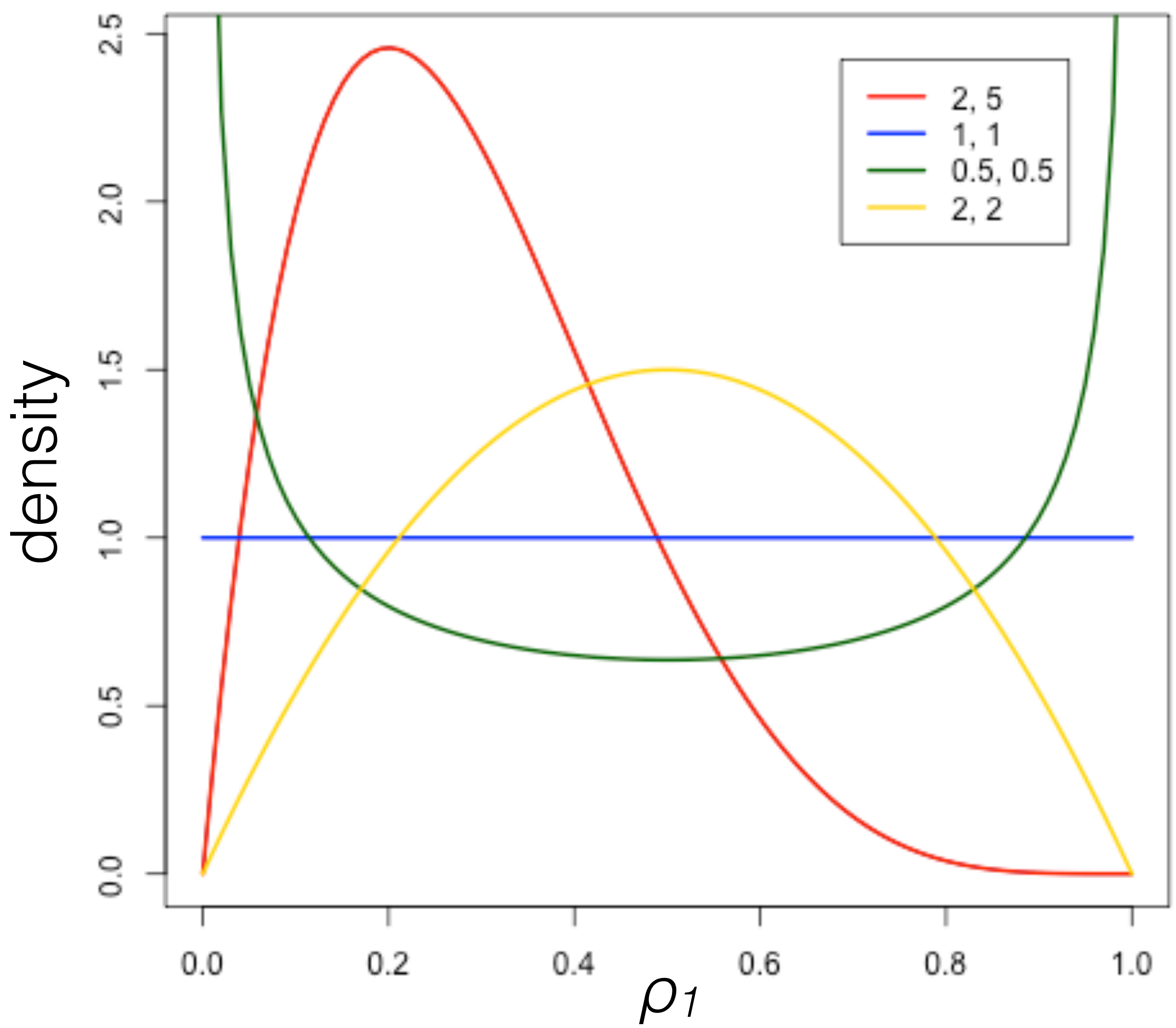

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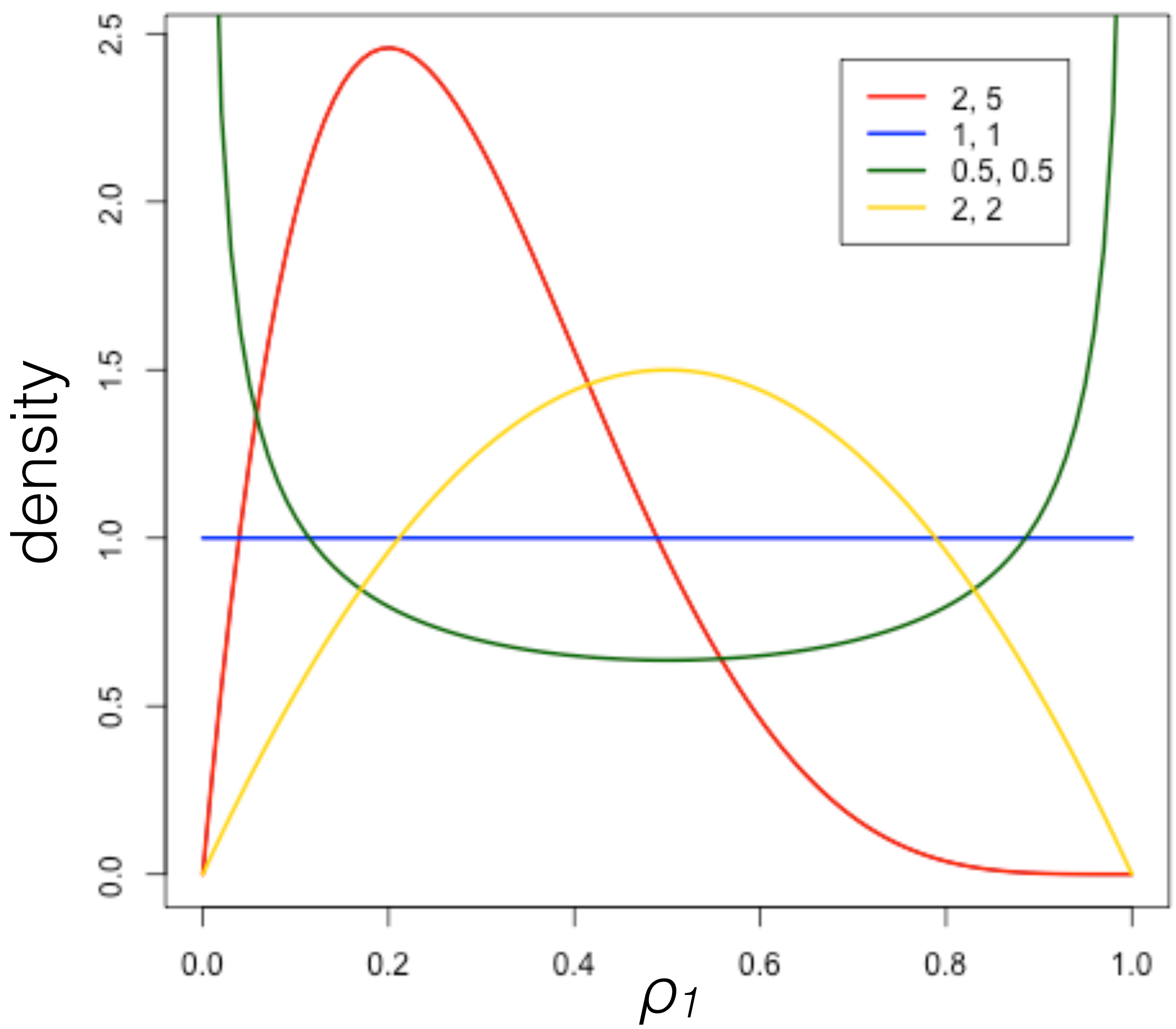

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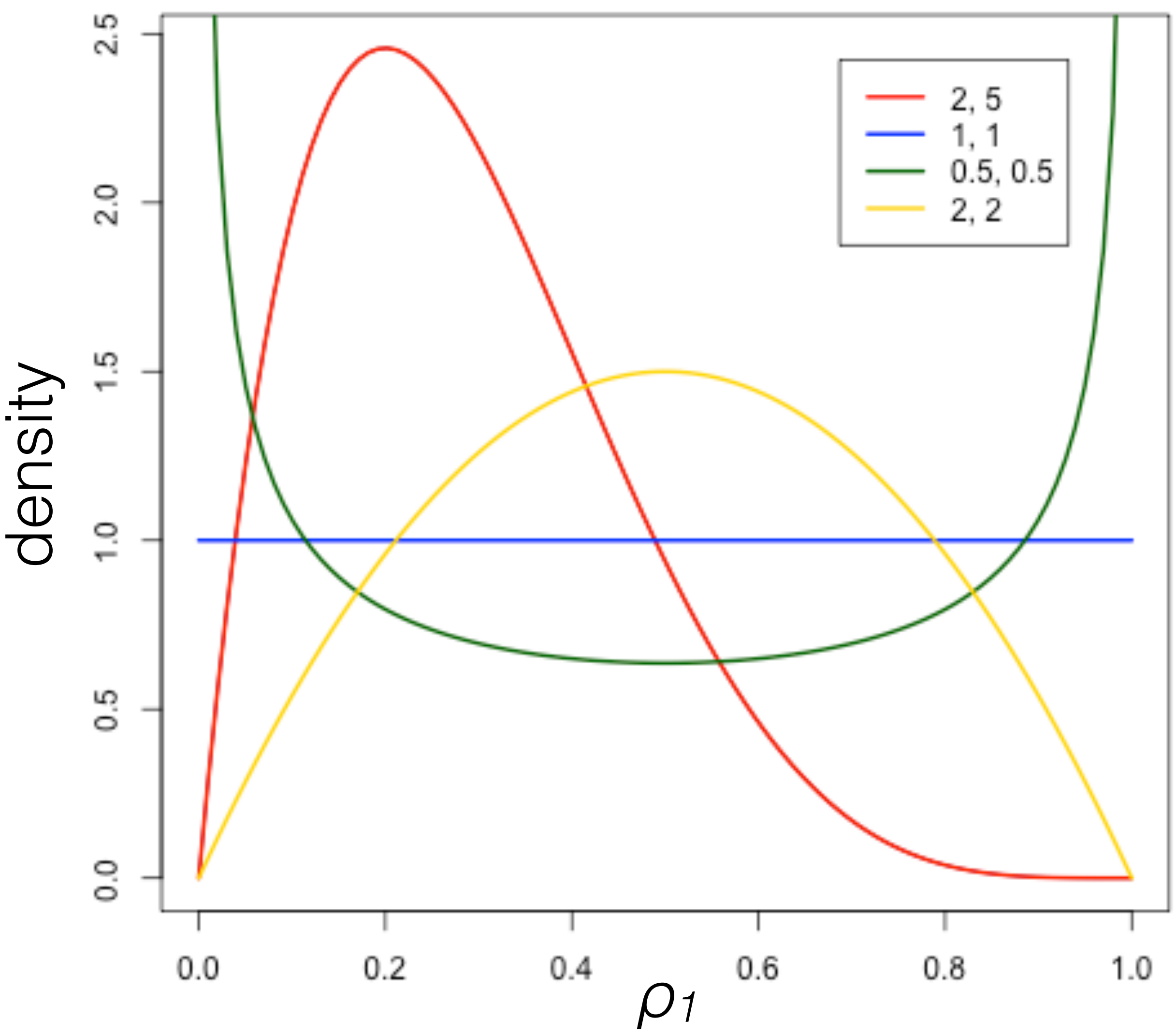

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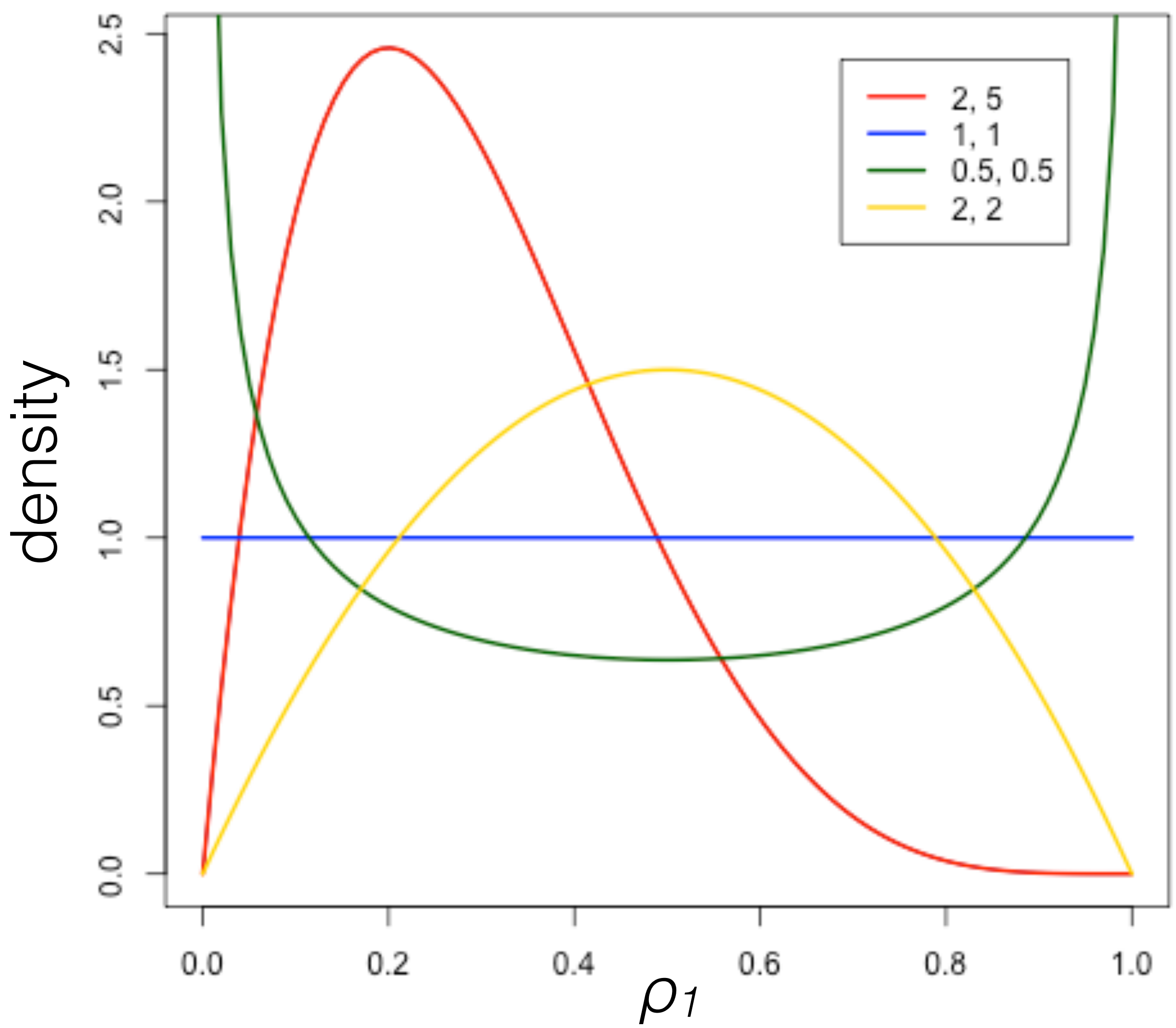

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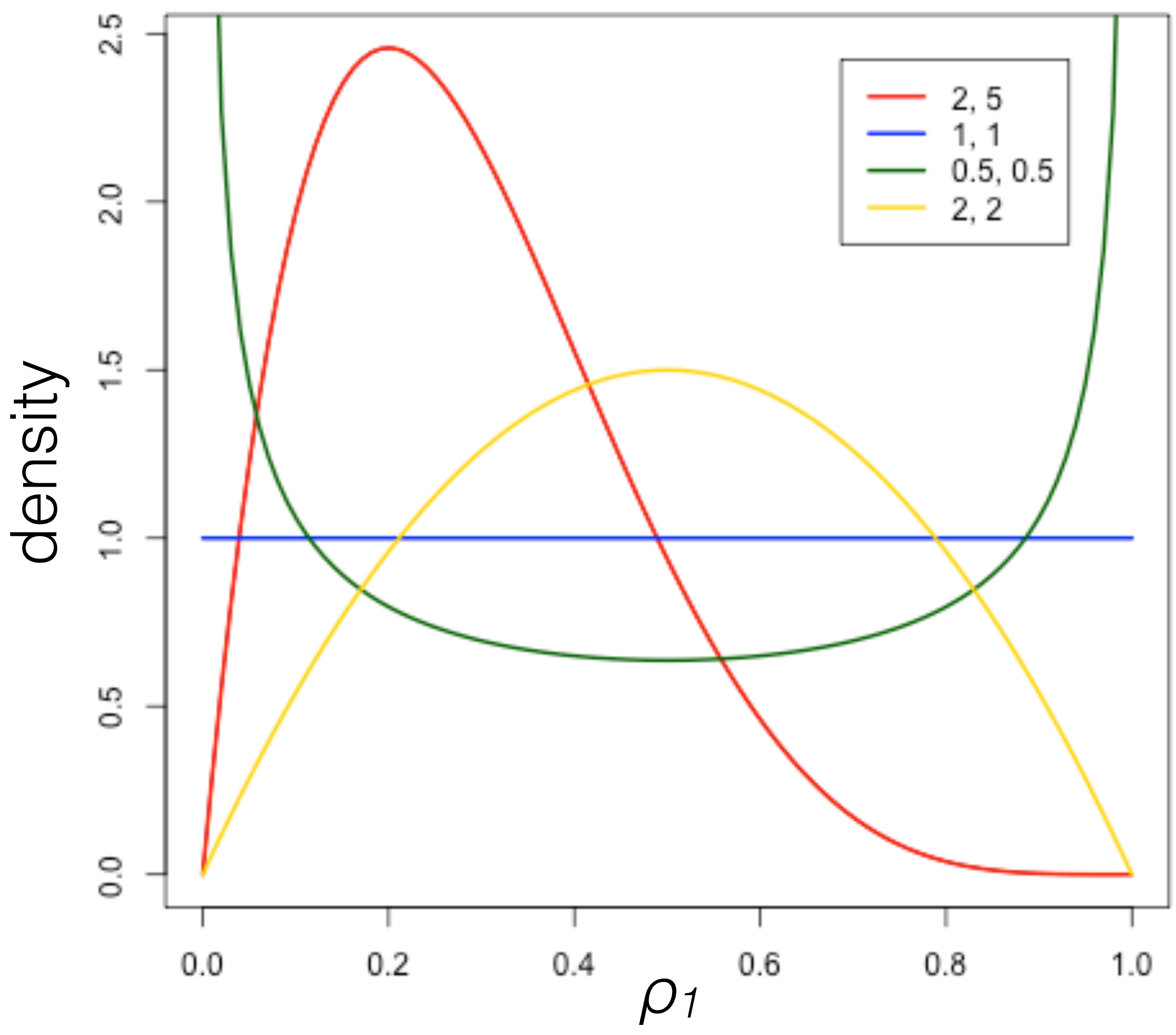

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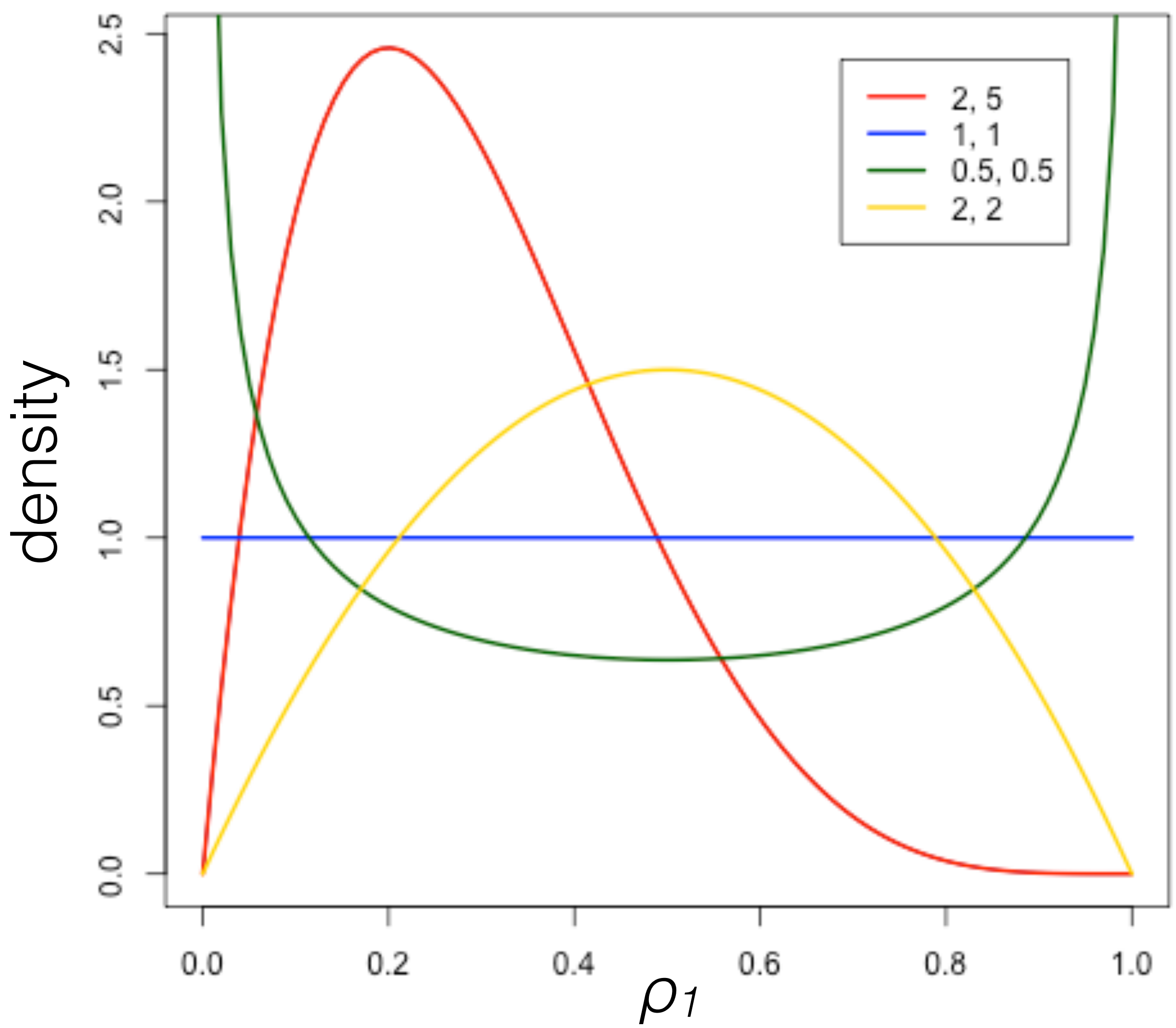

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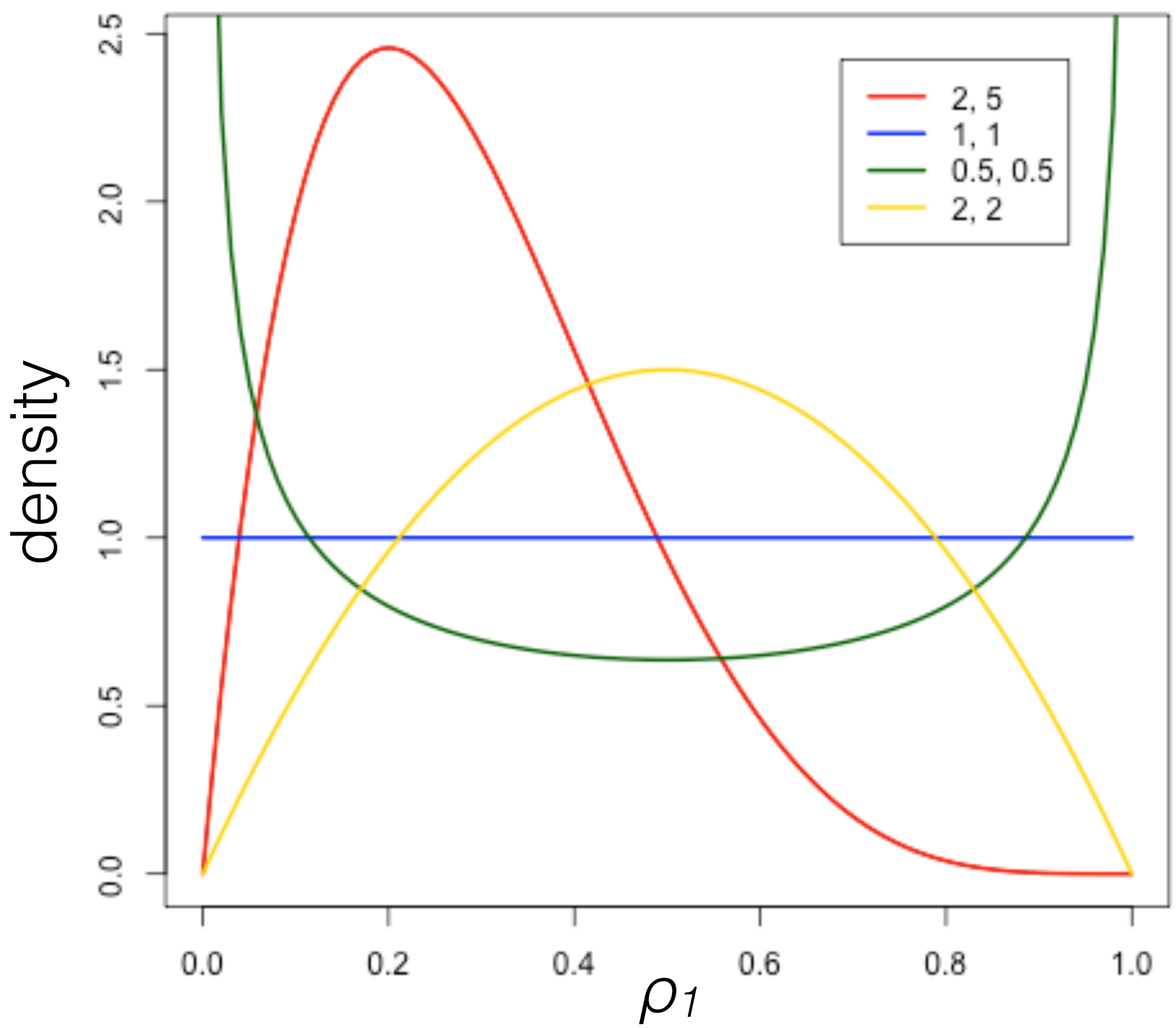

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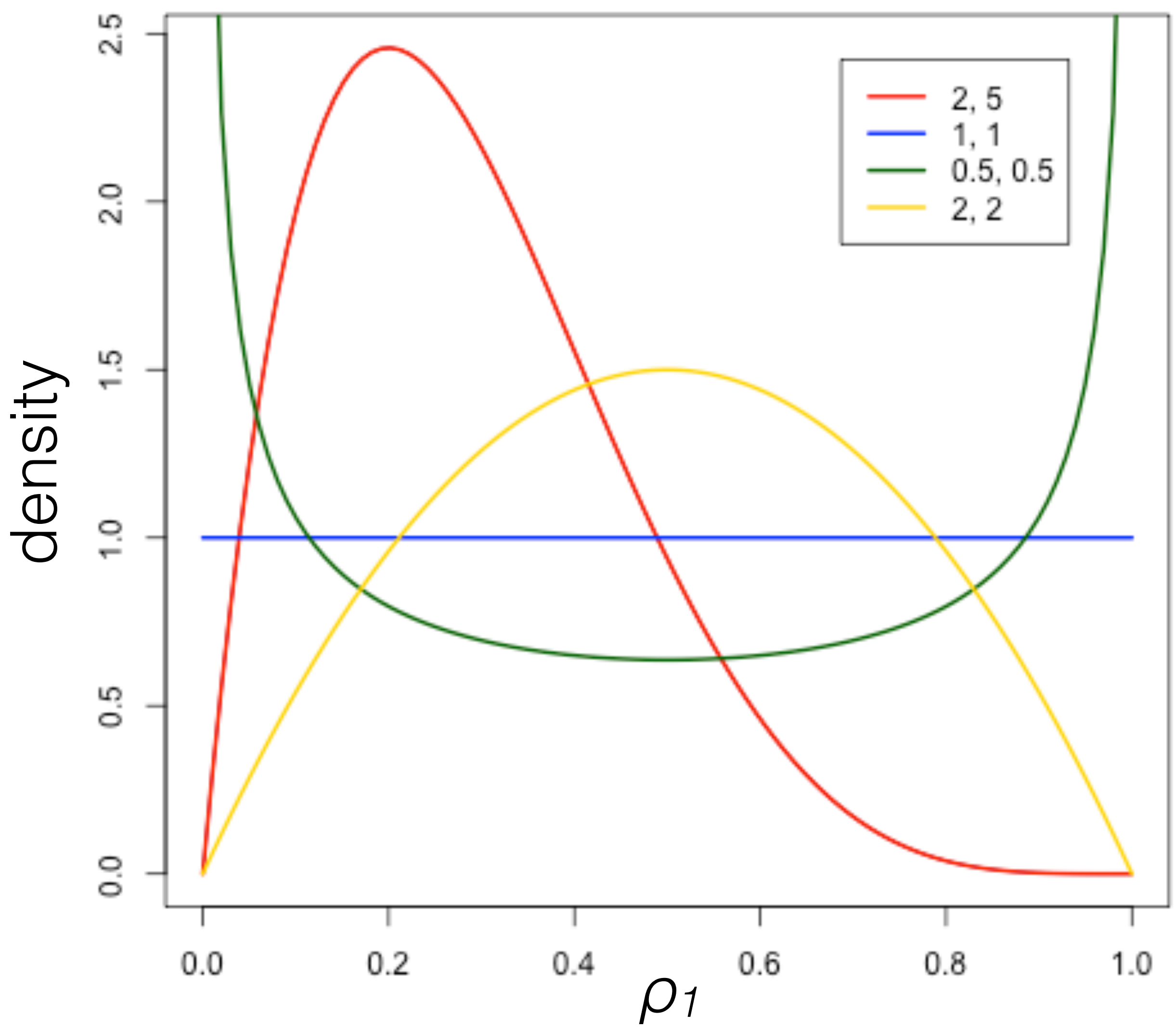

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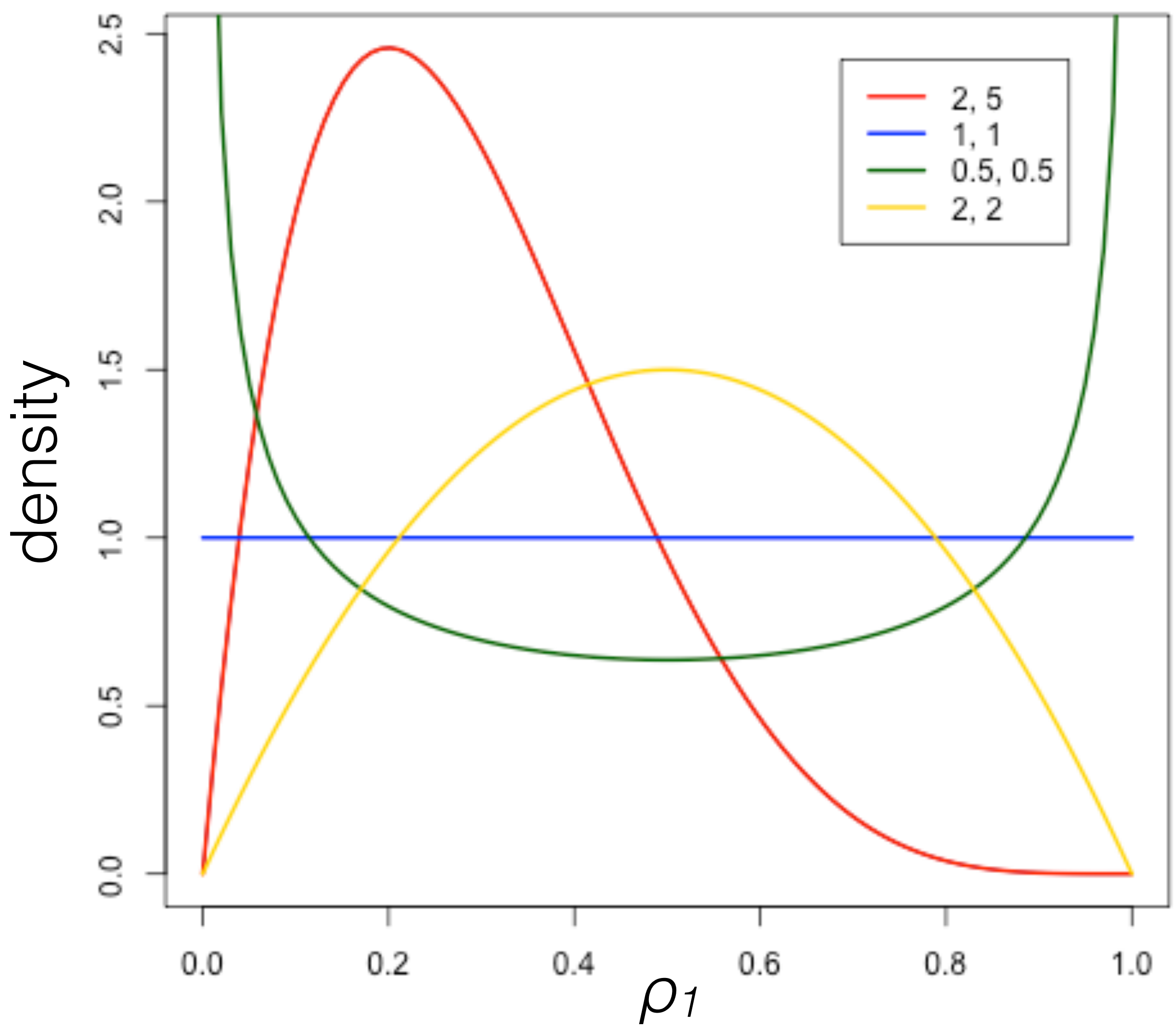

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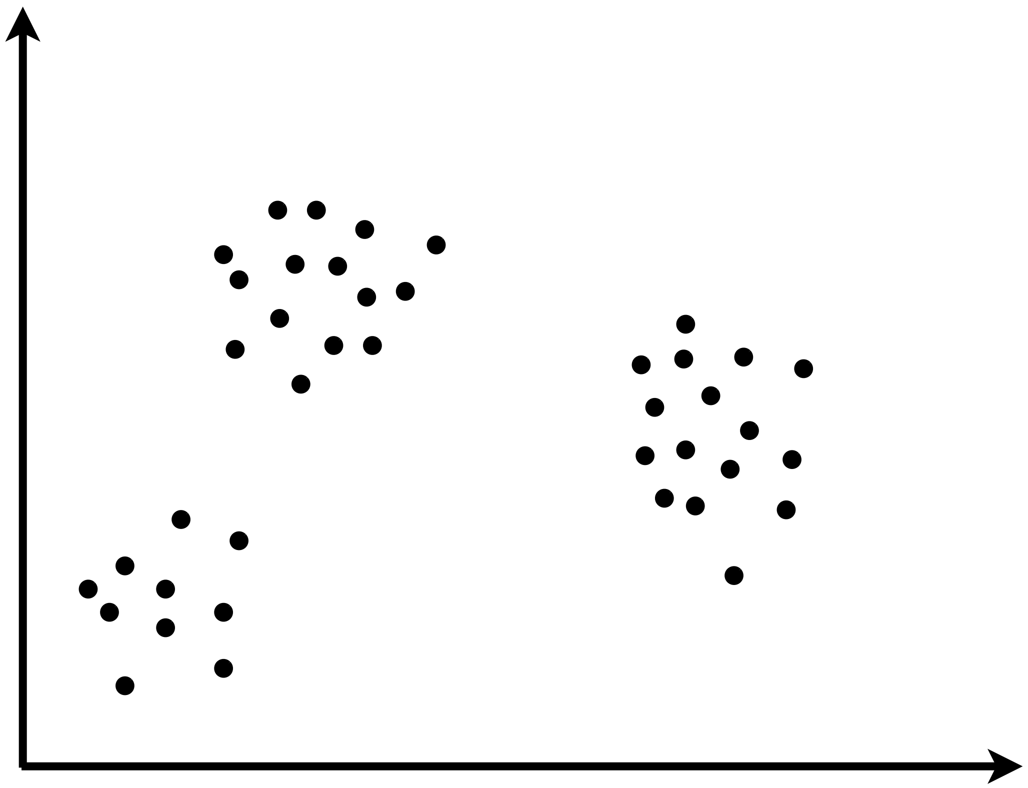

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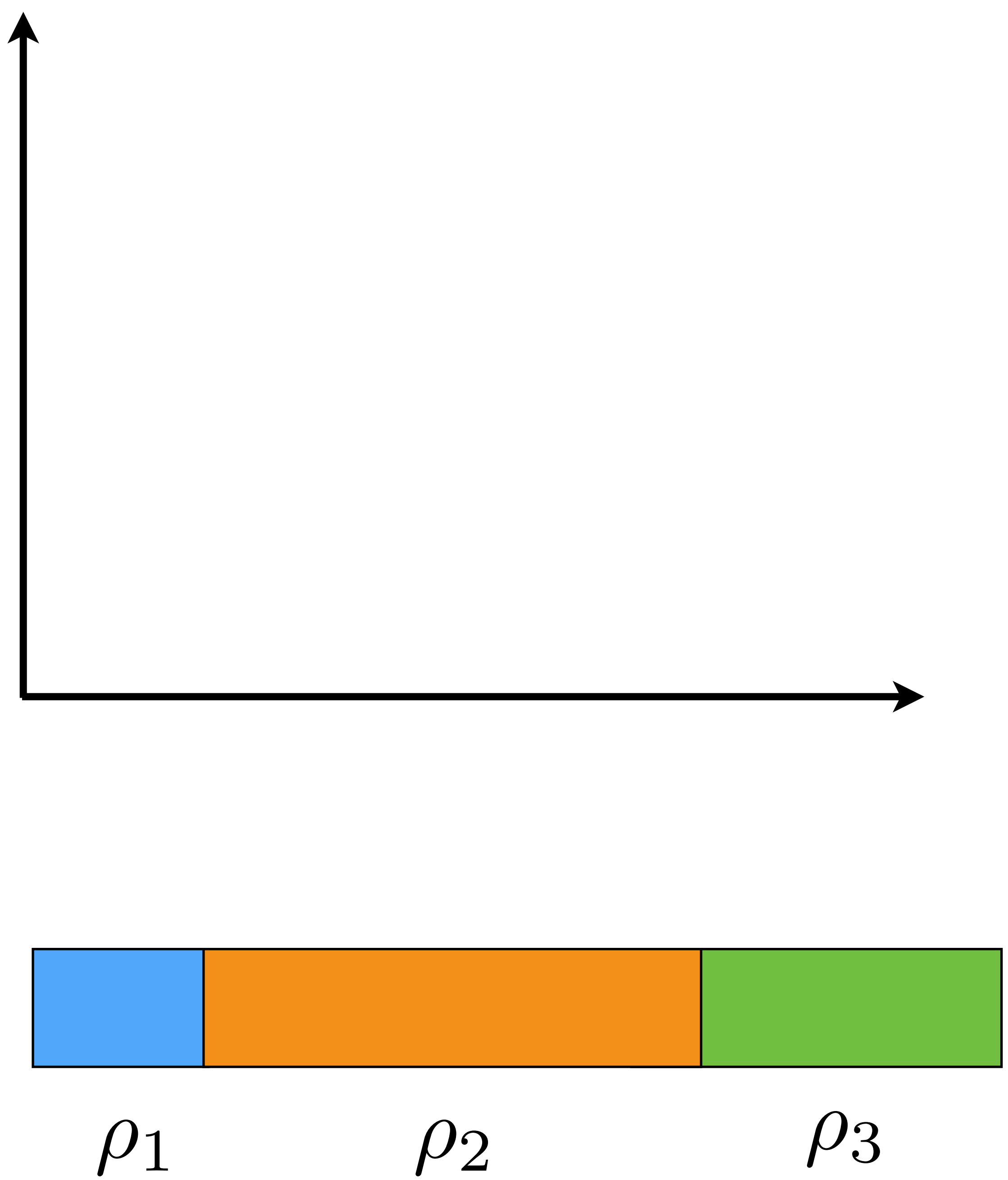

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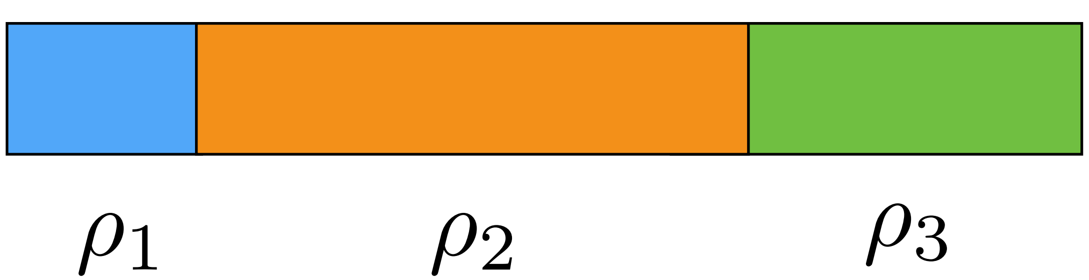

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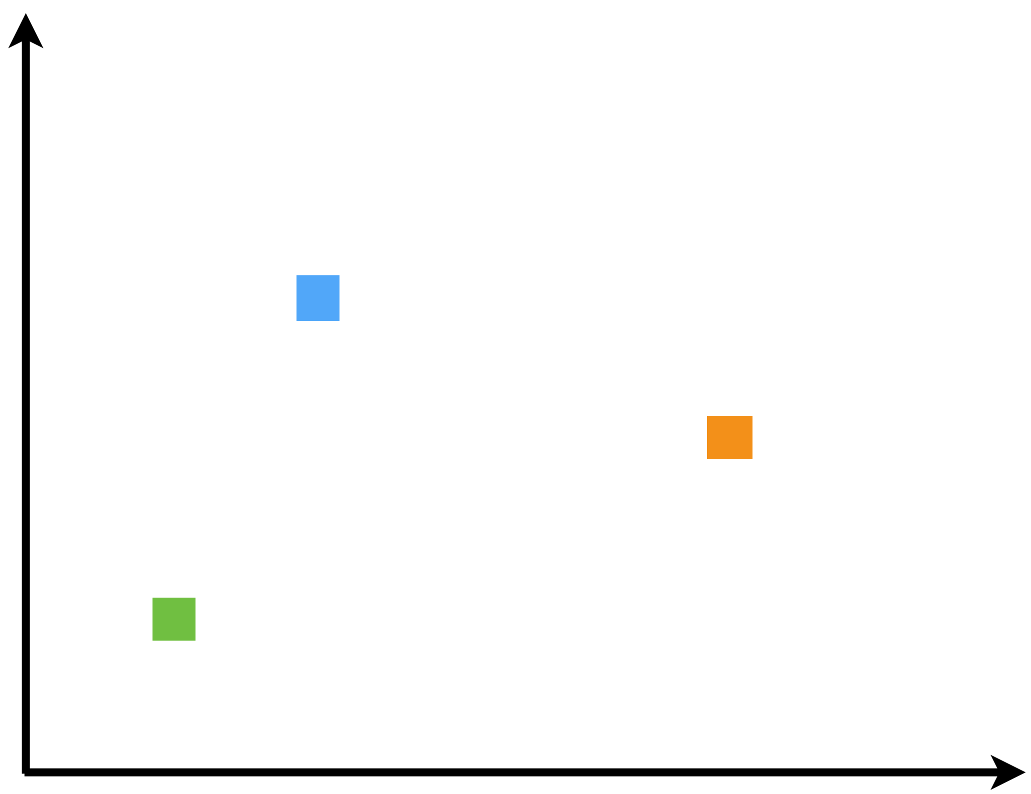

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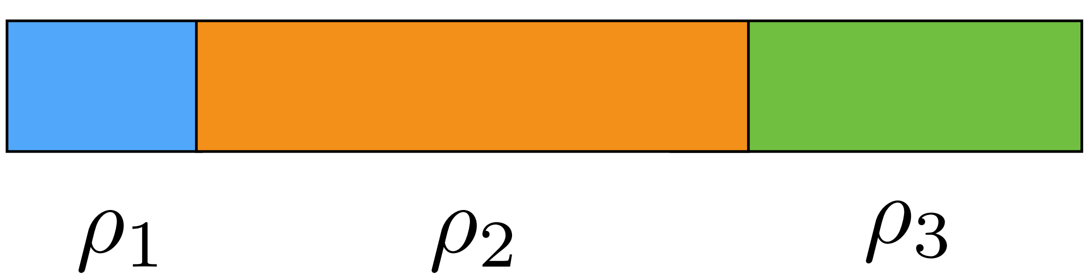

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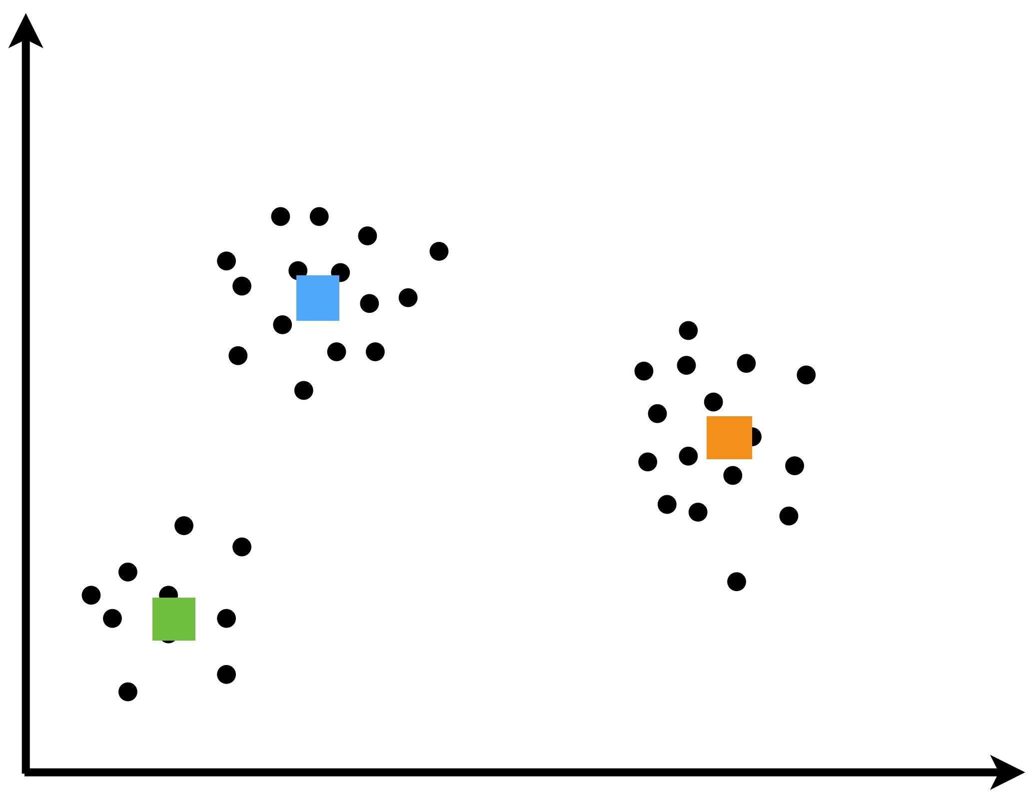

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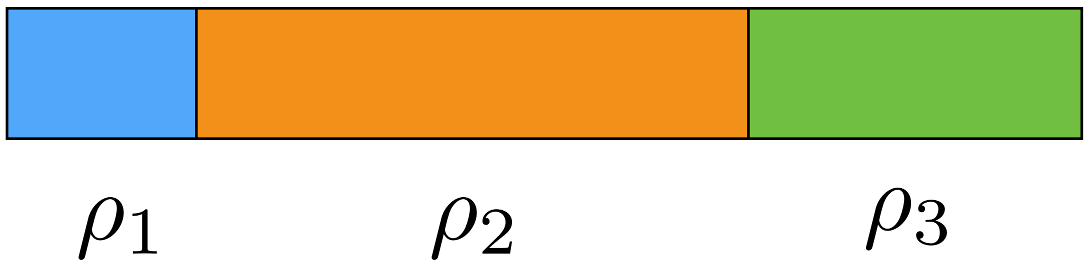
$$z_n \overset{iid}{\sim} \text{Categorical}(\rho_{1:K})$$

$$\mu_k \overset{iid}{\sim} \mathcal{N}(\mu_0, \Sigma_0)$$

$$x_n \overset{indep}{\sim} \mathcal{N}(\mu_{z_n}, \Sigma)$$



Dirichlet distribution review

$$\text{Dirichlet}(\rho_{1:K}|a_{1:K}) = \frac{\Gamma(\sum_{k=1}^{K} a_k)}{\prod_{k=1}^{K} \Gamma(a_k)} \prod_{k=1}^{K} \rho_k^{a_k - 1} \qquad a_k > 0$$

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$$a_k > 0$$

$$\rho_k \in (0, 1)$$

$$\sum_k \rho_k = 1$$

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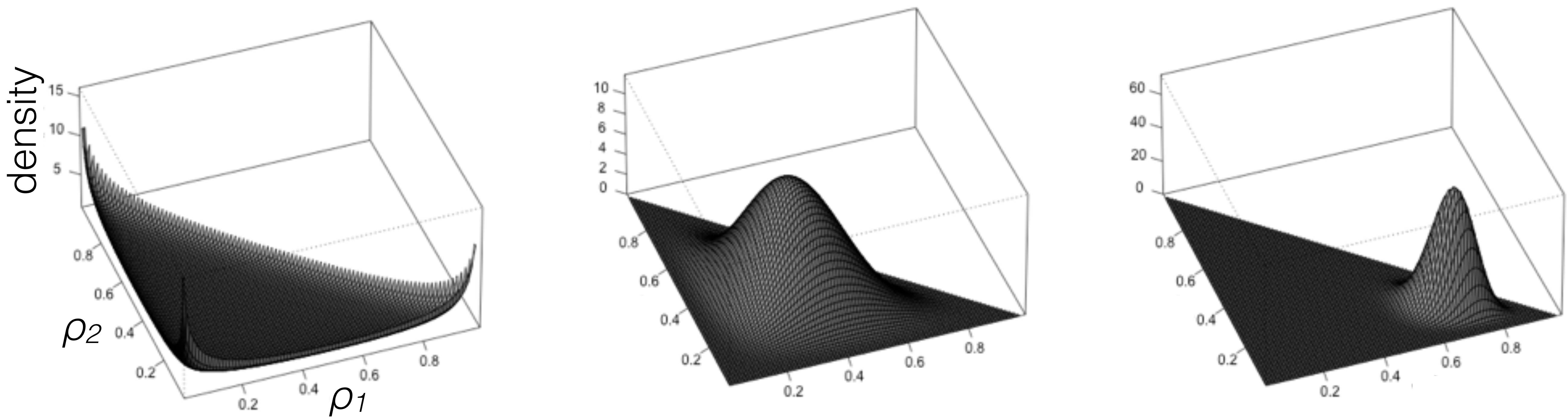
a = (0.5,0.5,0.5) a = (5,5,5) a = (40,10,10)



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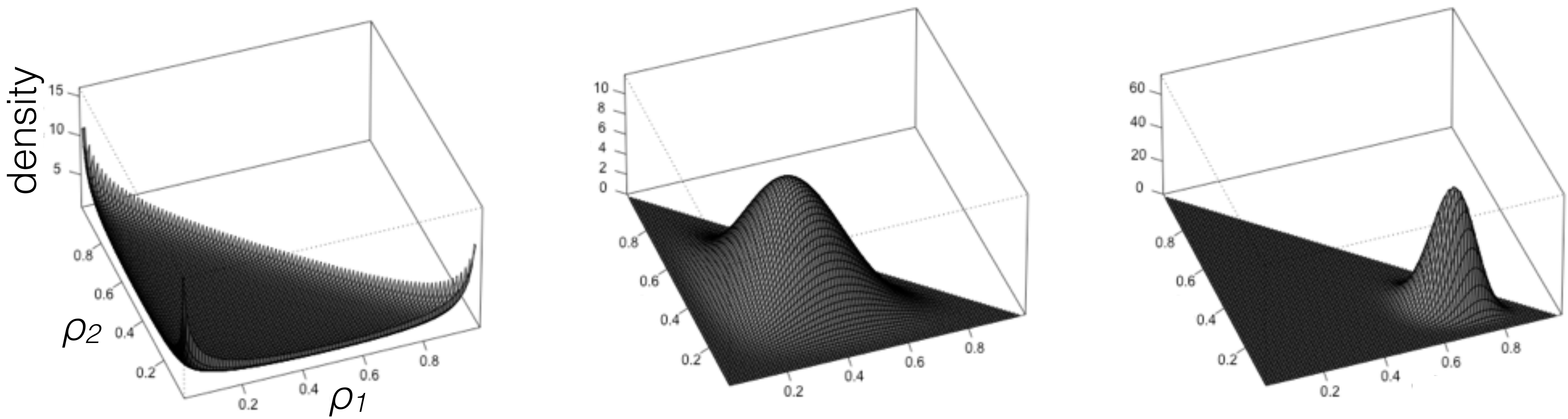

- What happens? $a = a_k = 1$

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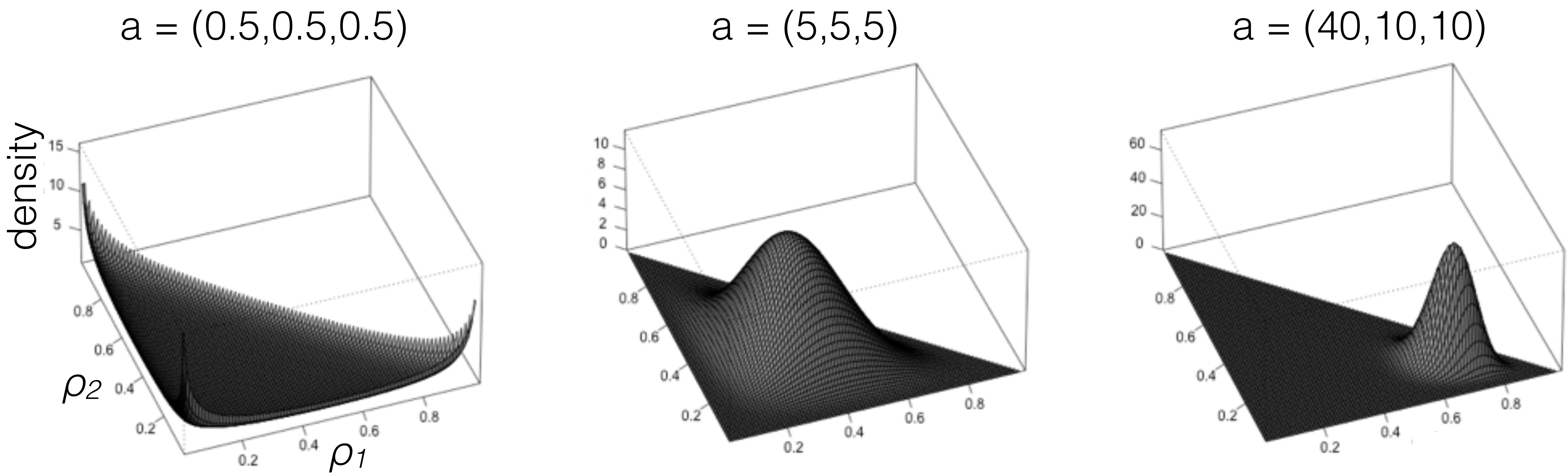

- What happens? $a = a_k = 1$ $\quad a = a_k \to 0$

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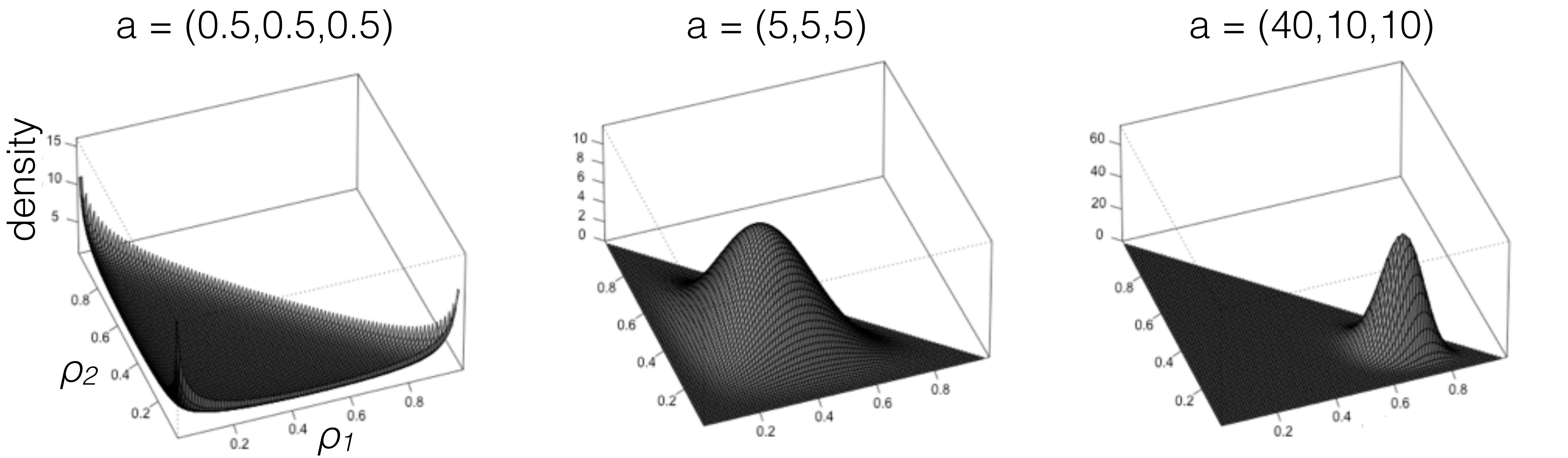

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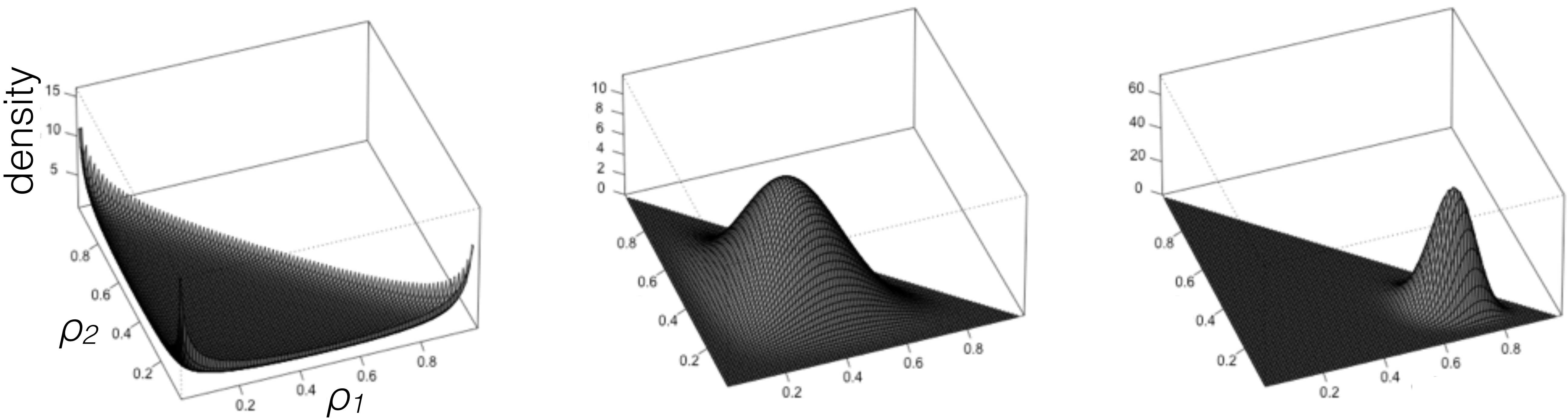

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[demo]

Dirichlet distribution review

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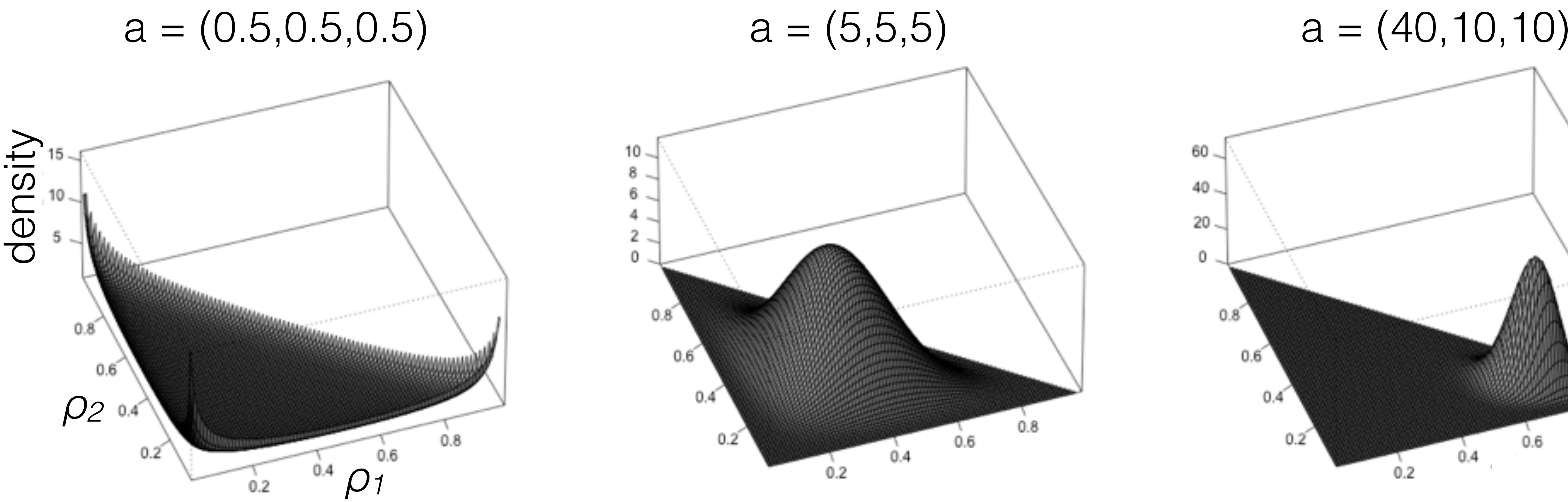

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- Dirichlet is conjugate to Categorical

[demo]

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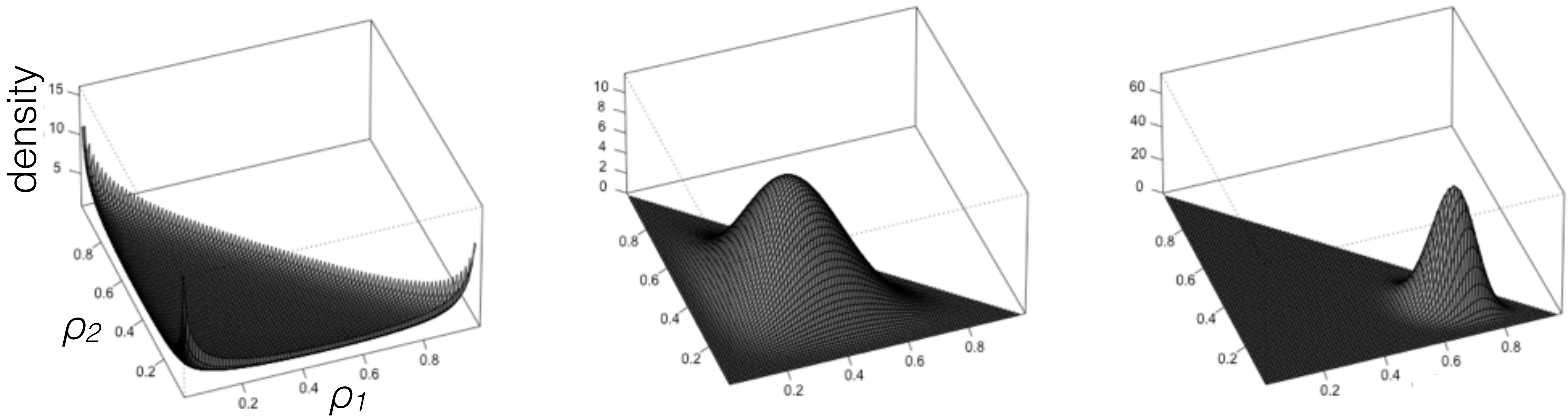

- What happens? $a = a_k = 1$ $a = a_k \to 0$ $a = a_k \to \infty$ [demo]
- Dirichlet is conjugate to Categorical

$\rho_{1:K} \sim \mathrm{Dirichlet}(a_{1:K}), z \sim \mathrm{Cat}(\rho_{1:K})$

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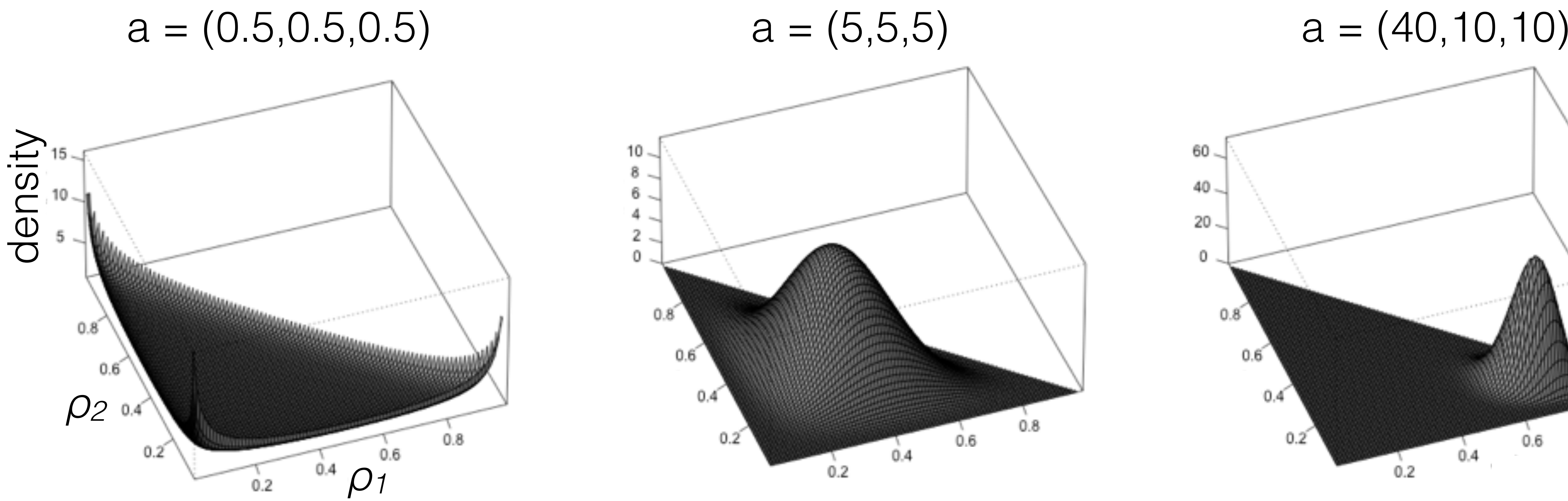

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[demo]

- Dirichlet is conjugate to Categorical

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$$\rho_{1:K}|z \stackrel{d}{=} \text{Dirichlet}(a'_{1:K}), a'_k = a_k + \mathbf{1}\{z = k\}$$

What if $K \gg N$?

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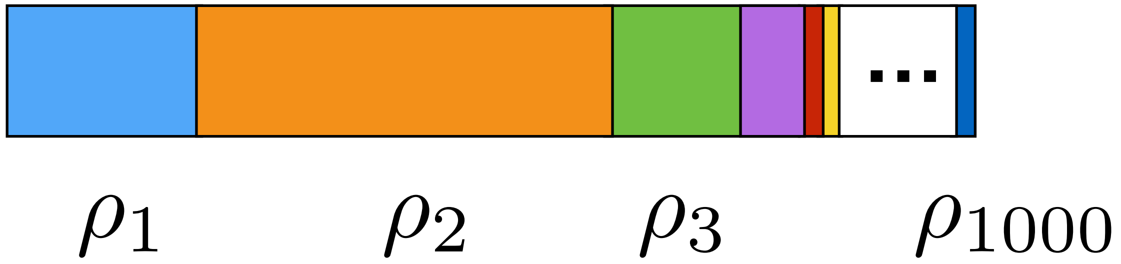
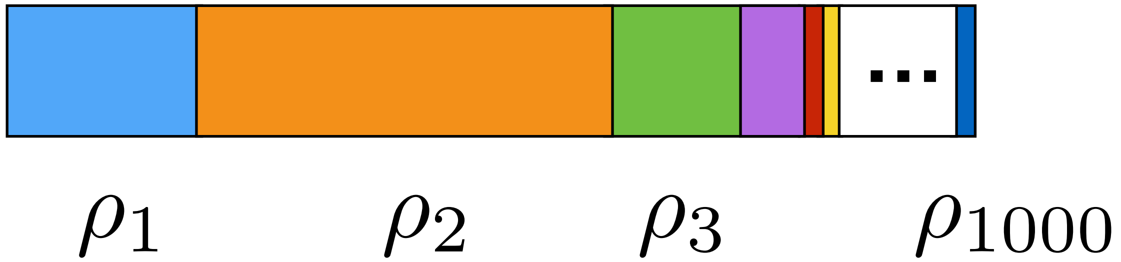


What if $K \gg N$?

- e.g. species sampling, topic modeling, groups on a social network, etc.



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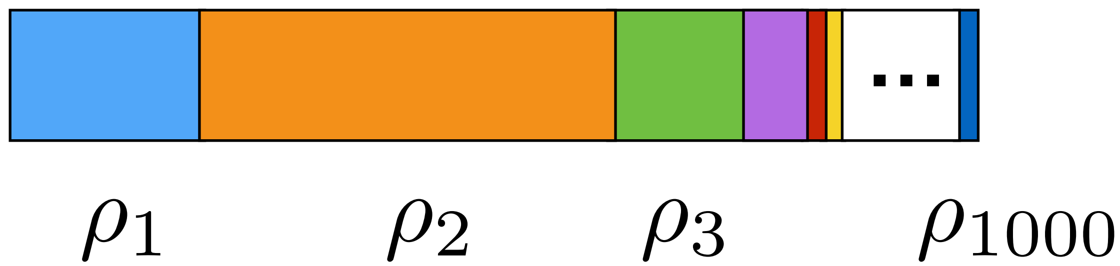

- Components: number of latent groups

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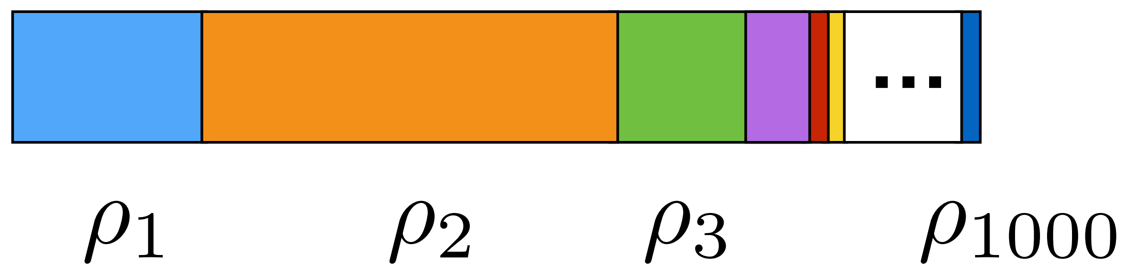

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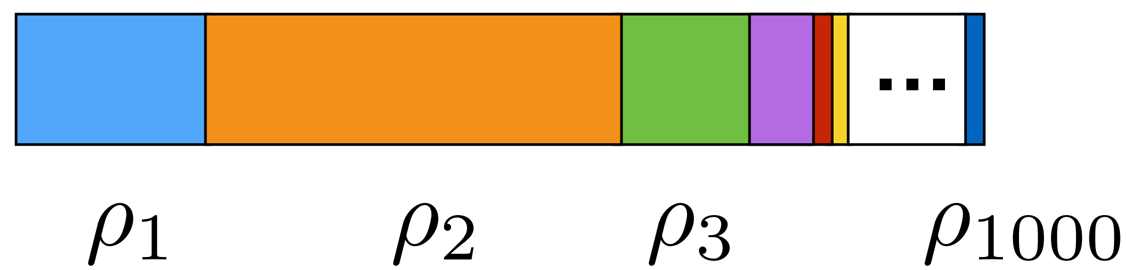

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- [demo 1, demo 2]

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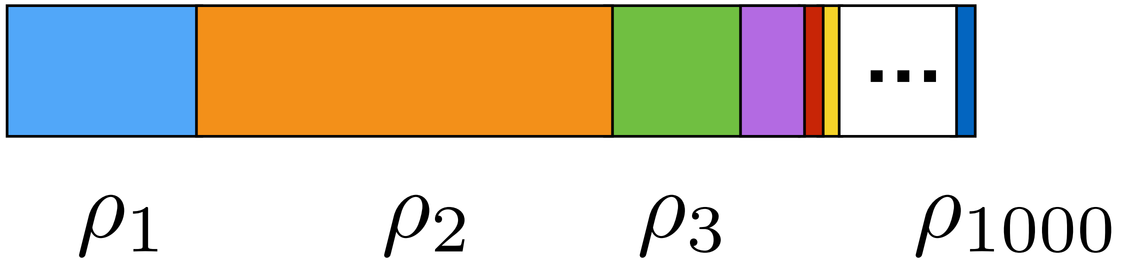

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- [demo 1, demo 2]
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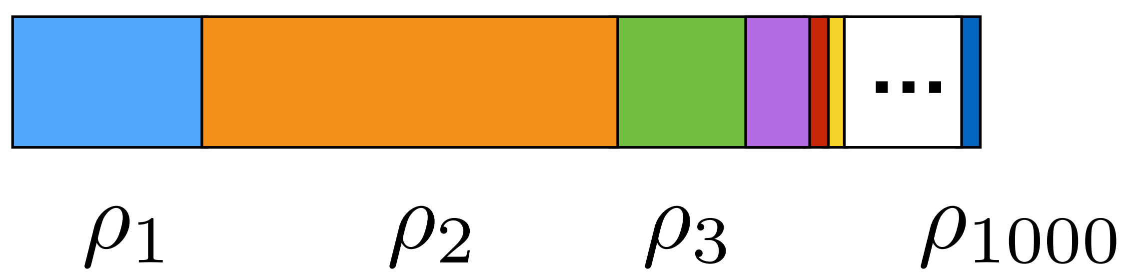

- Components: number of latent groups
- Clusters: number of components represented in the data
- [demo 1, demo 2]
- Number of clusters for N data points is $< K$ and random
- Number of clusters grows with N

- Here, difficult to choose finite K in advance (contrast with small K): don't know K, difficult to infer, streaming data

Choosing $K = \infty$

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- "Stick breaking"

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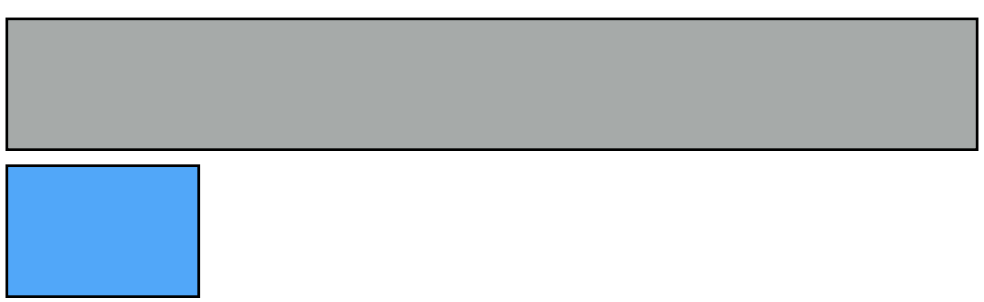

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$V_1 \sim \mathrm{Beta}(a_1, a_2 + a_3 + a_4)$ $\qquad$ $\rho_1 = V_1$

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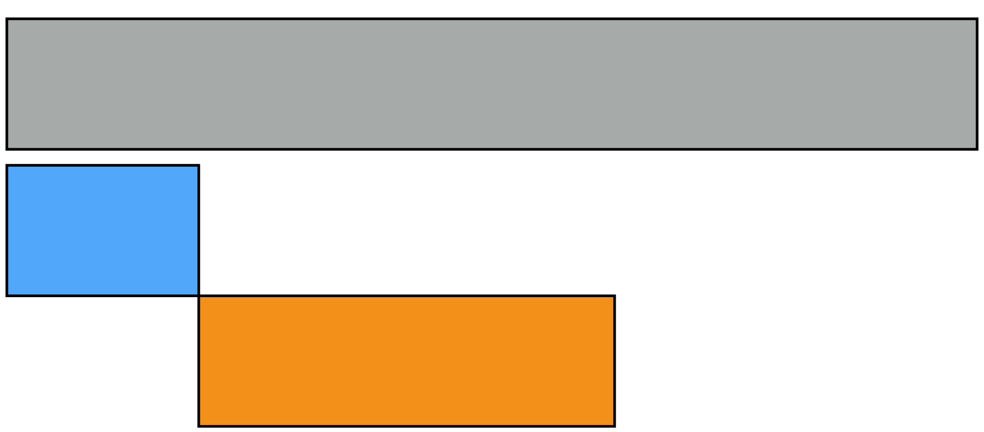

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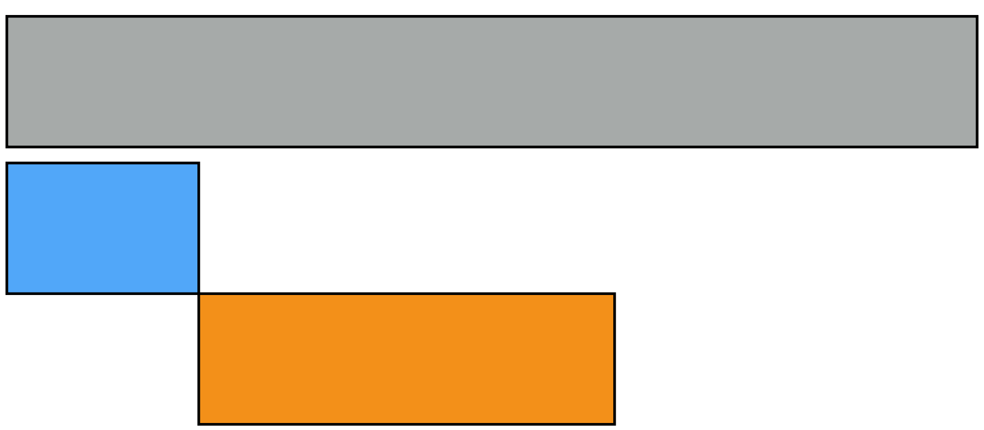

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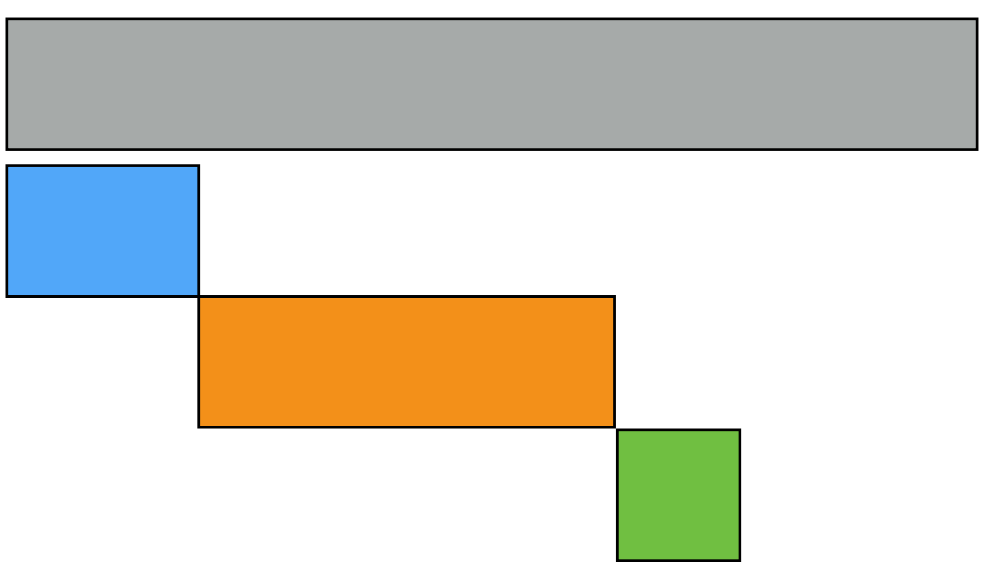

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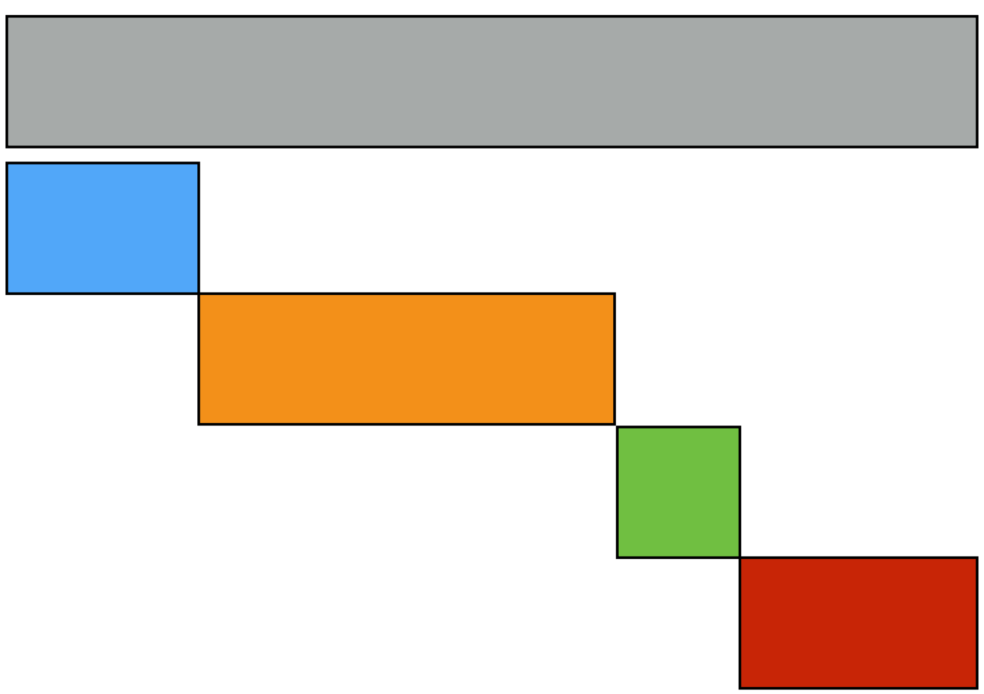

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$$\rho_4 = 1 - \sum_{k=1}^{3} \rho_k$$

Choosing $K = \infty$

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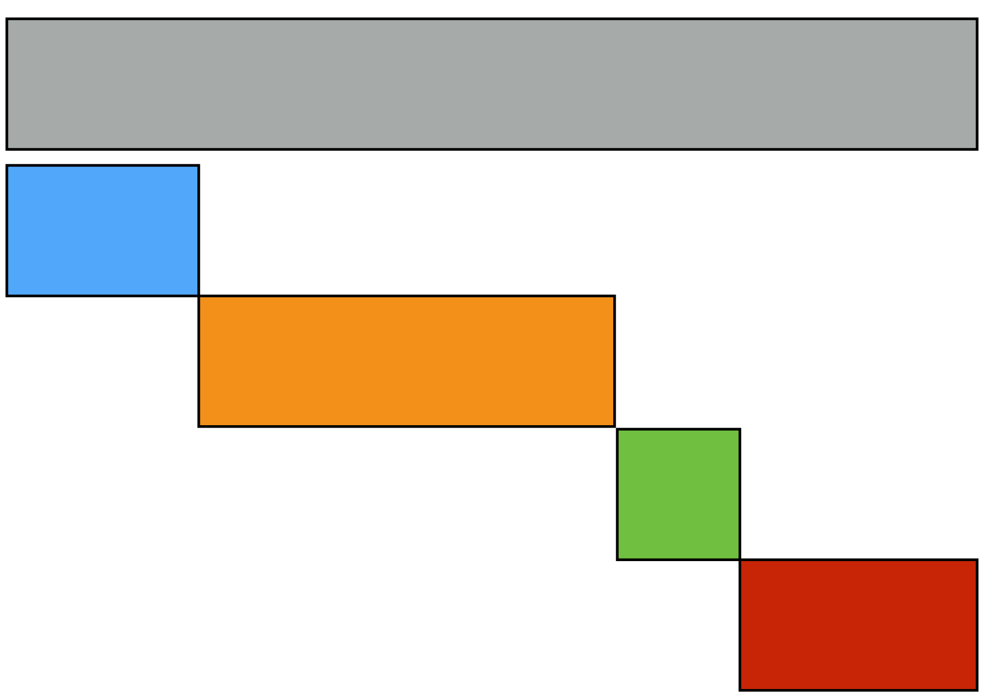

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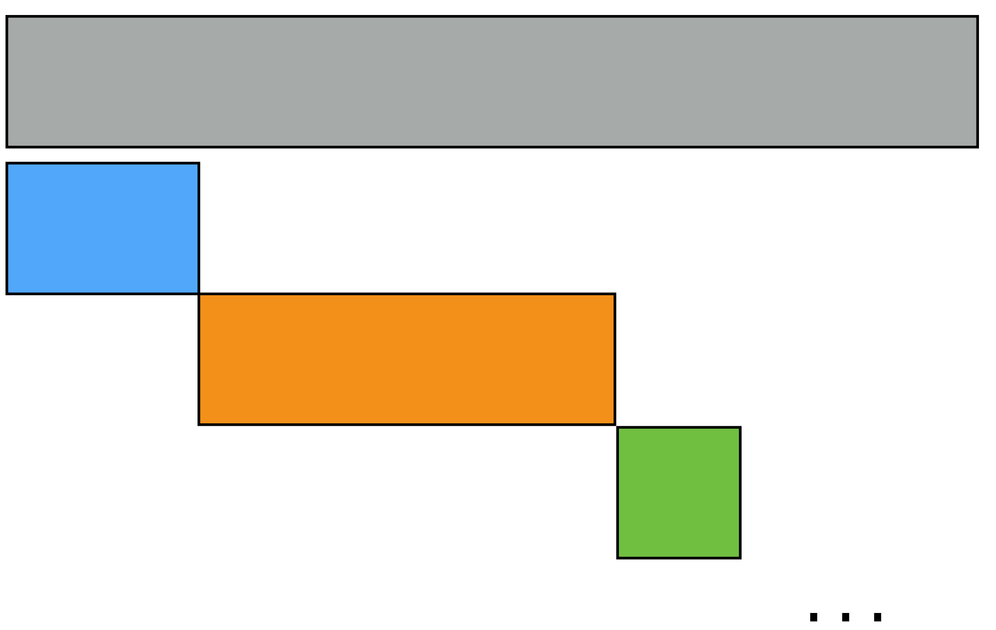

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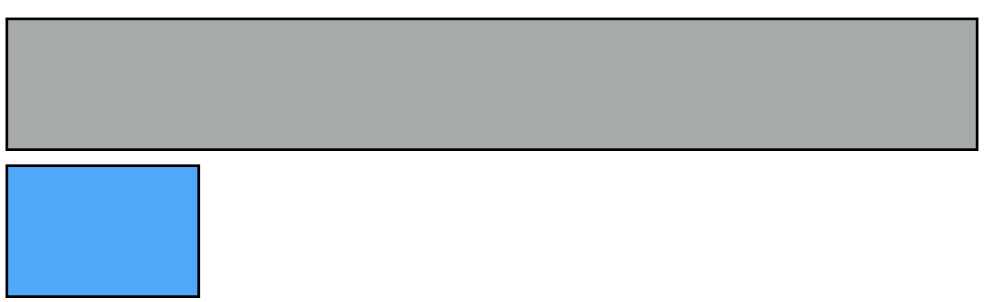

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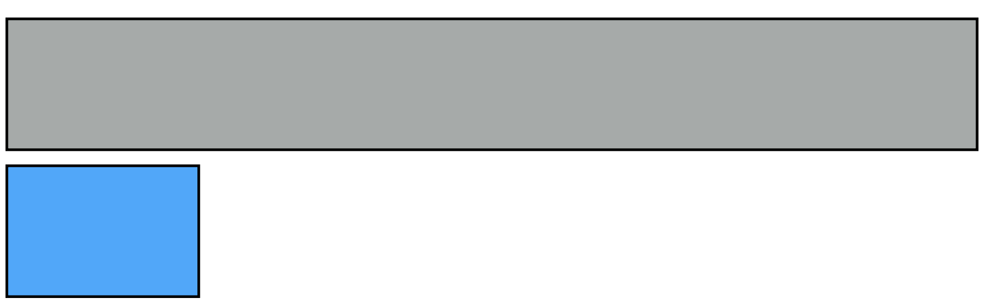

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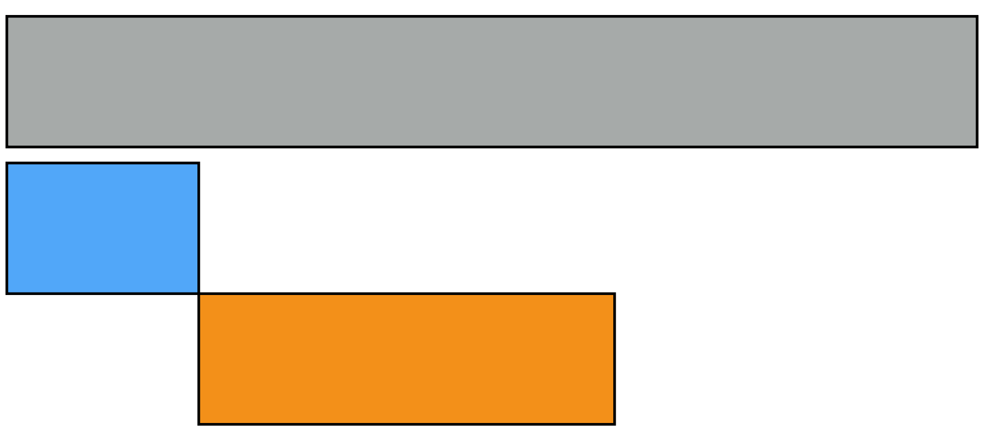

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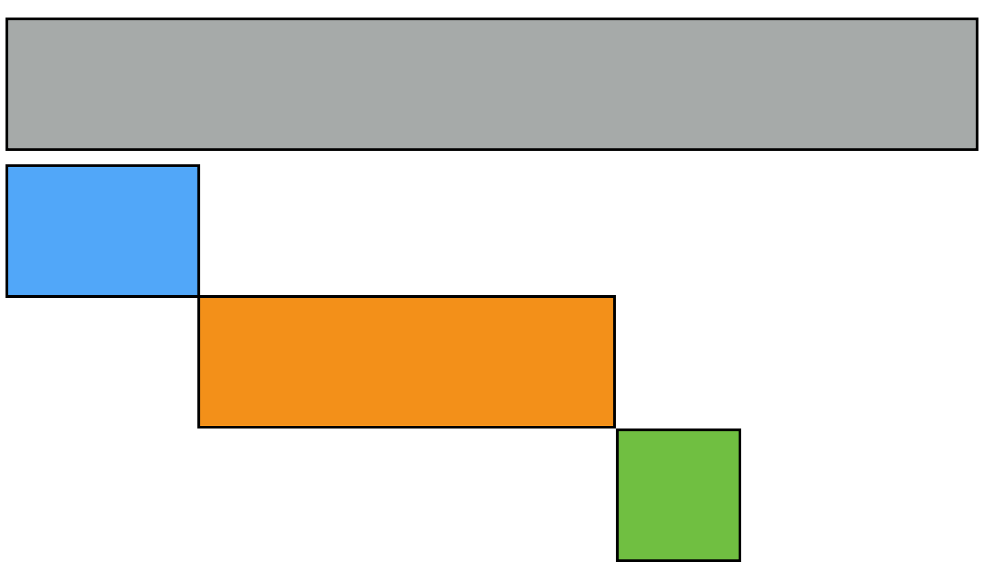

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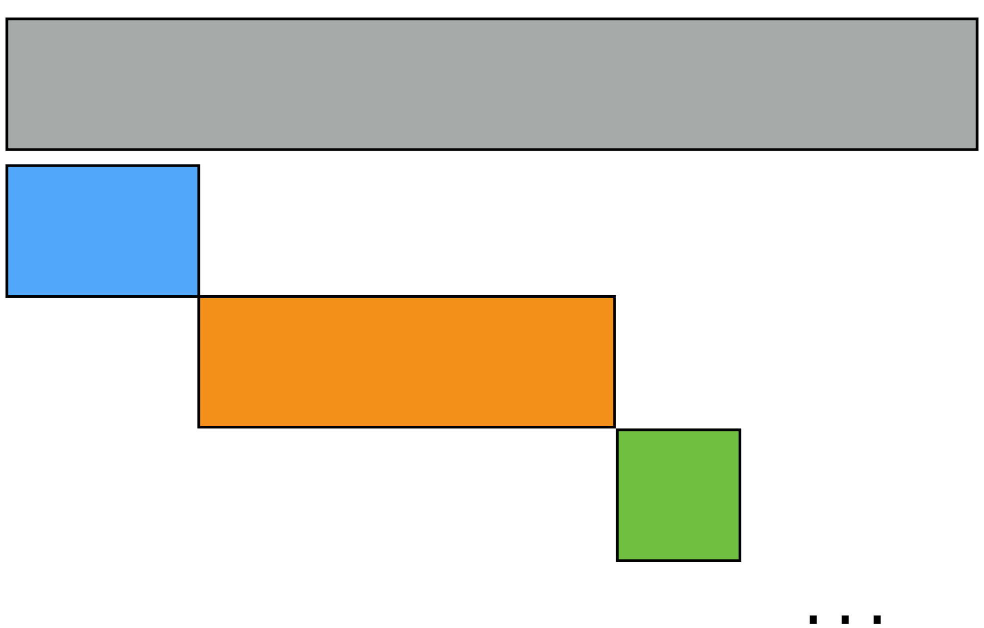

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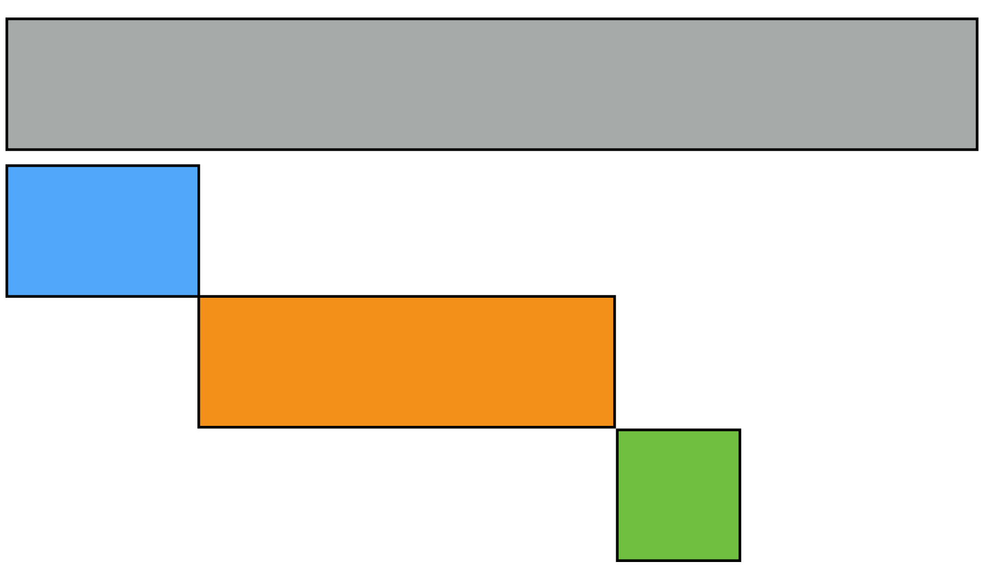

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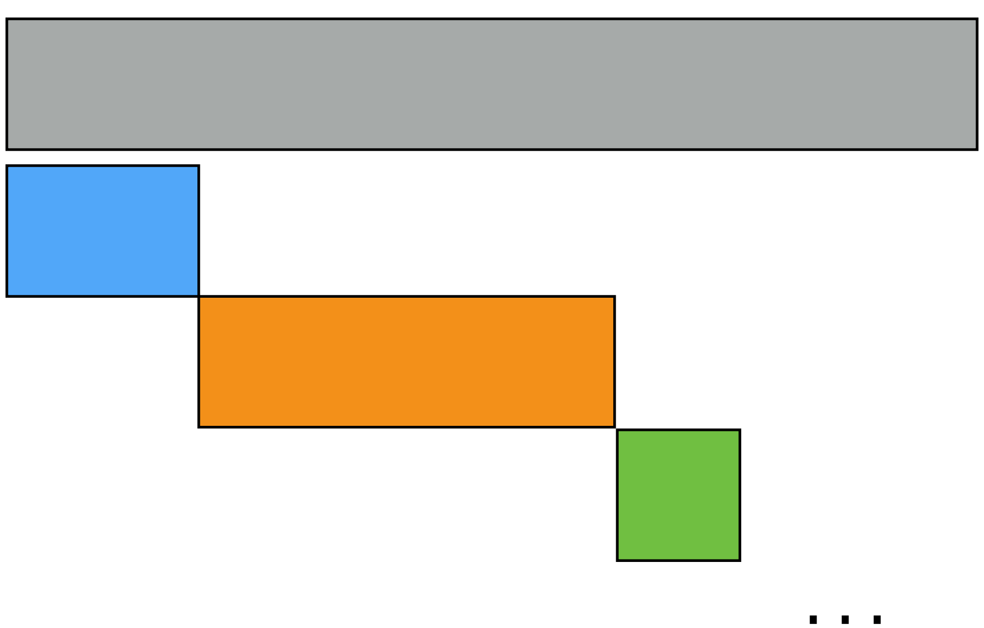

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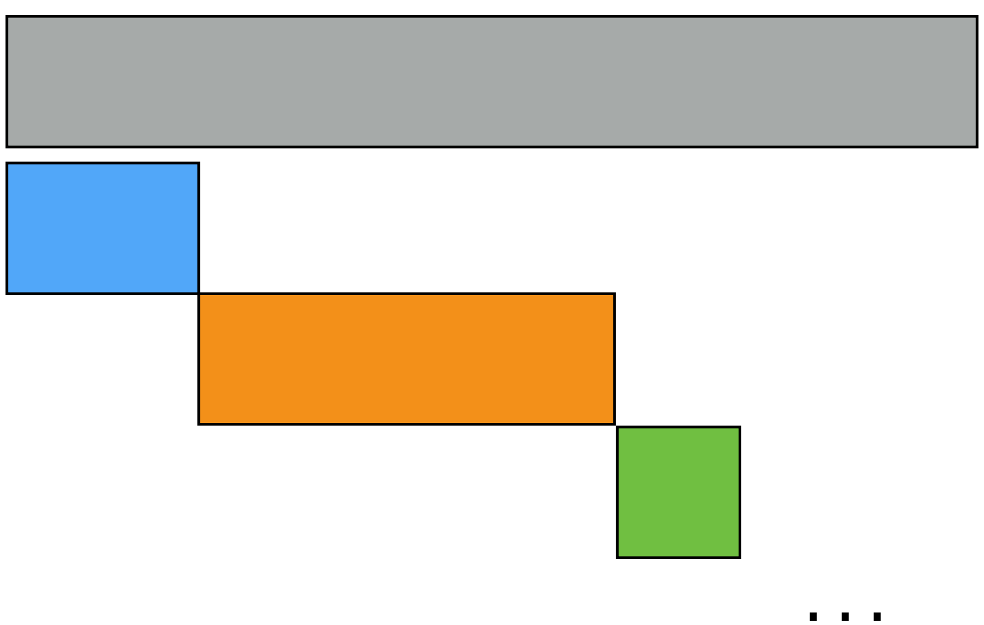

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[Ishwaran, James 2001]

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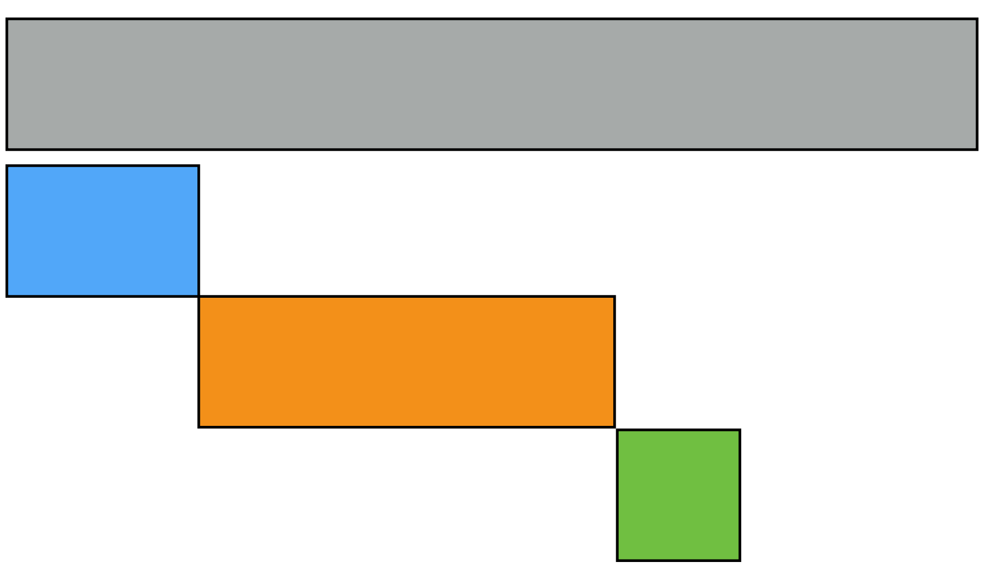

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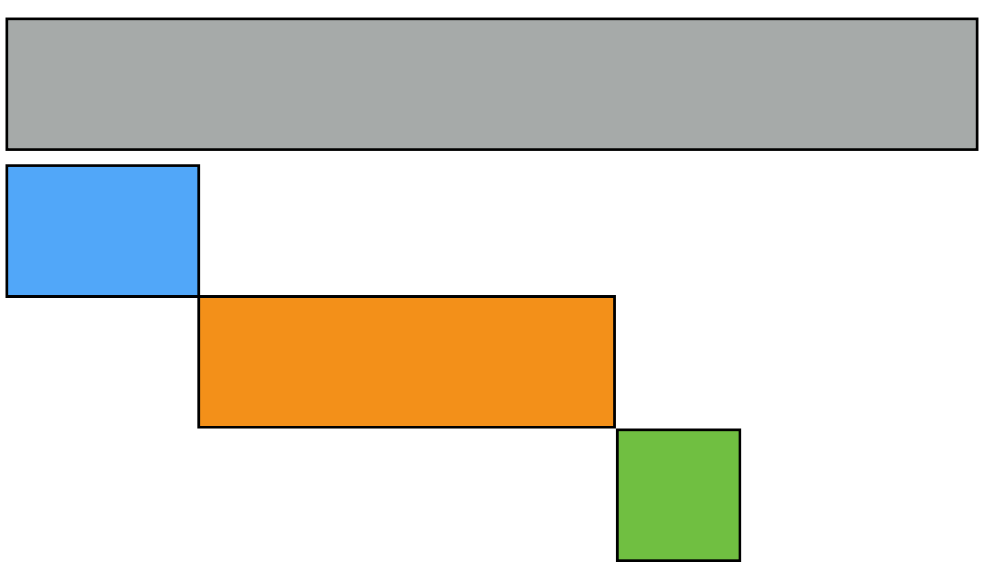

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 [McCloskey 1965; Engen 1975; Patil and Taillie 1977; Ewens 1987; Sethuraman 1994; Ishwaran, James 2001]

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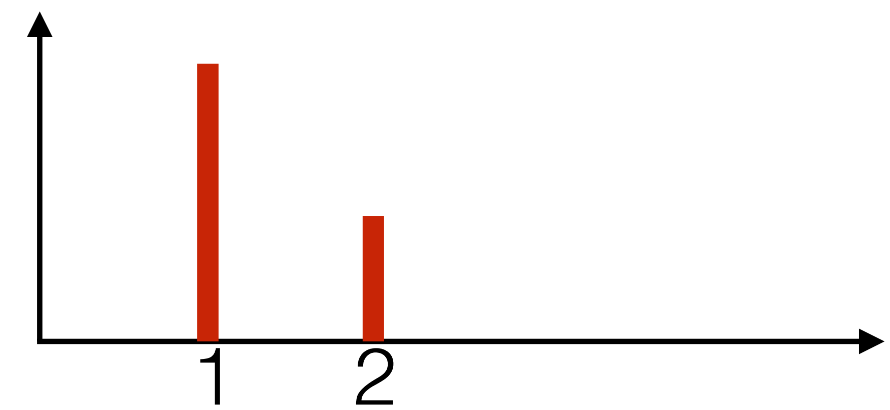

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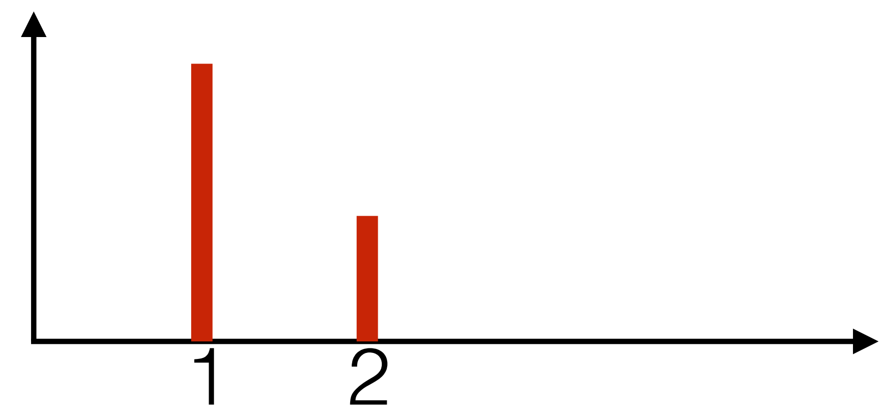

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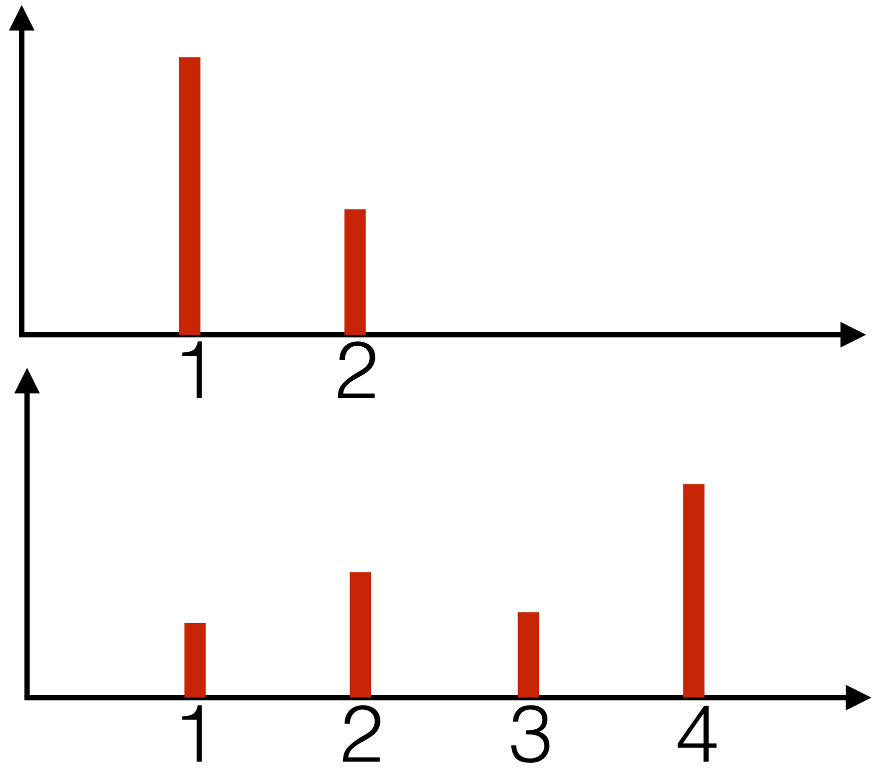

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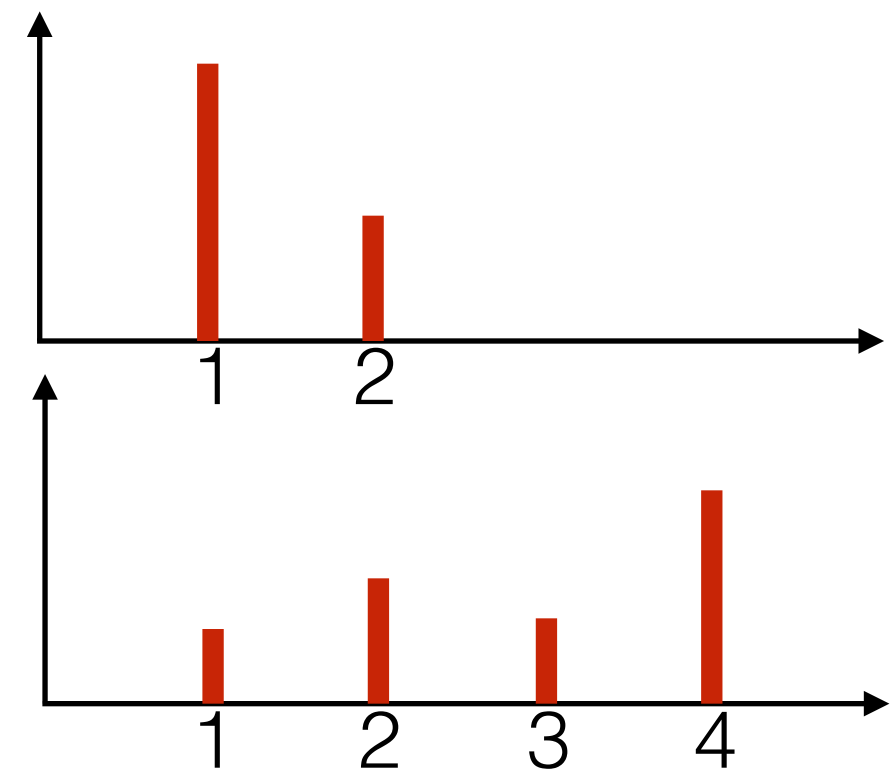

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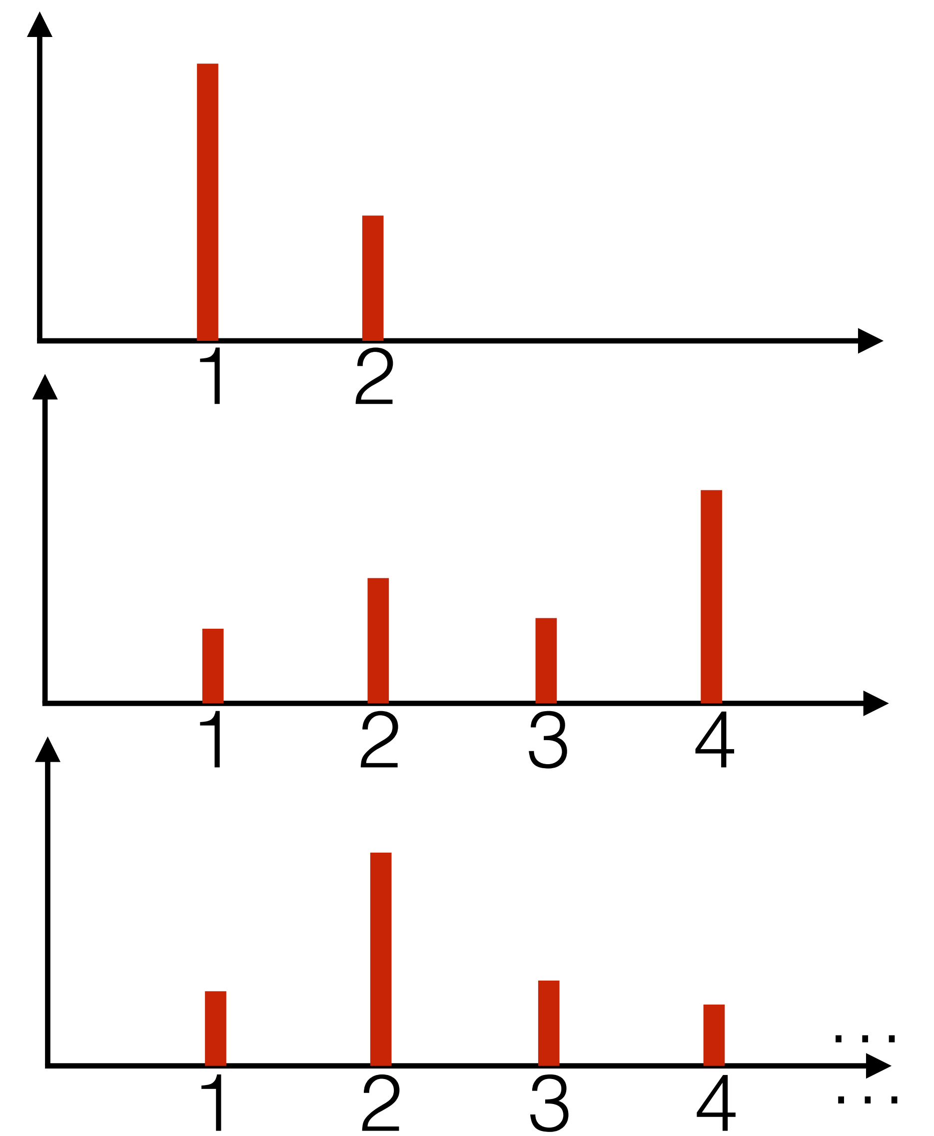

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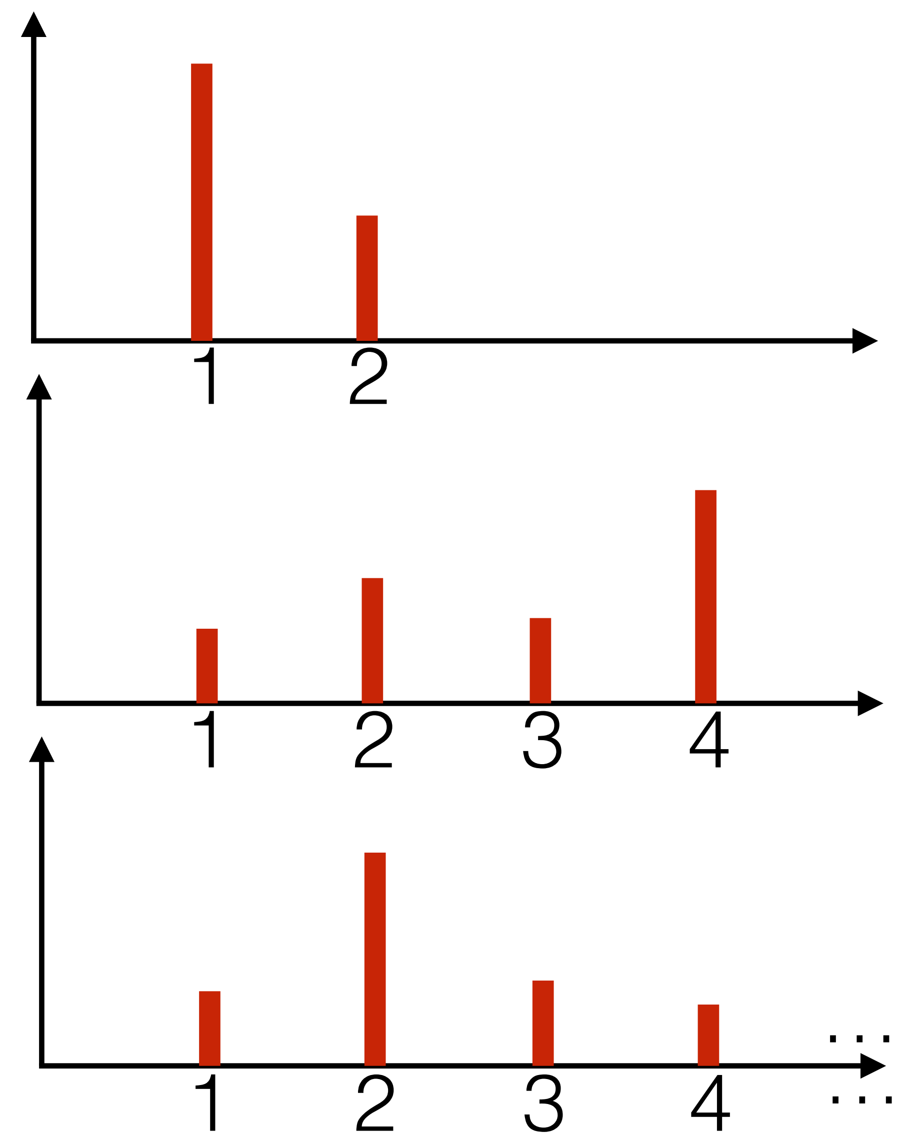

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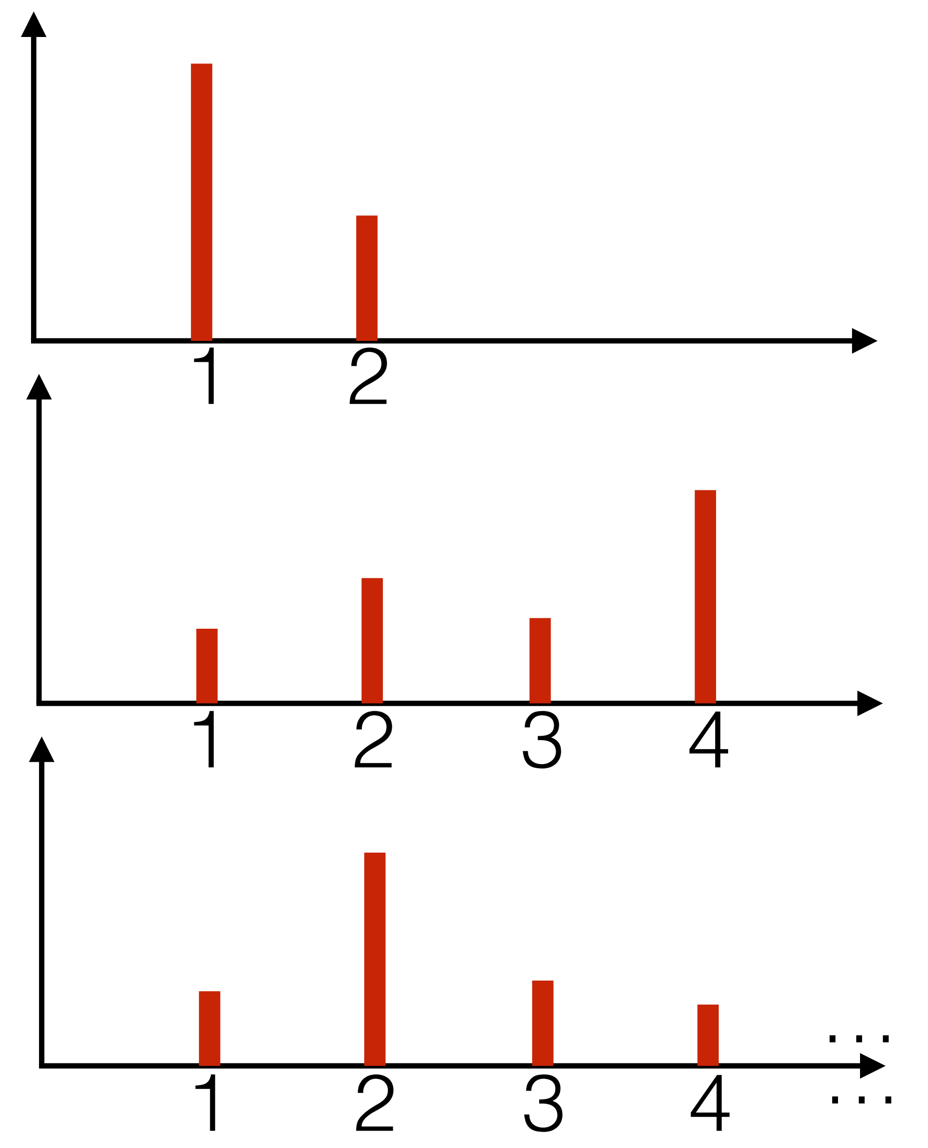

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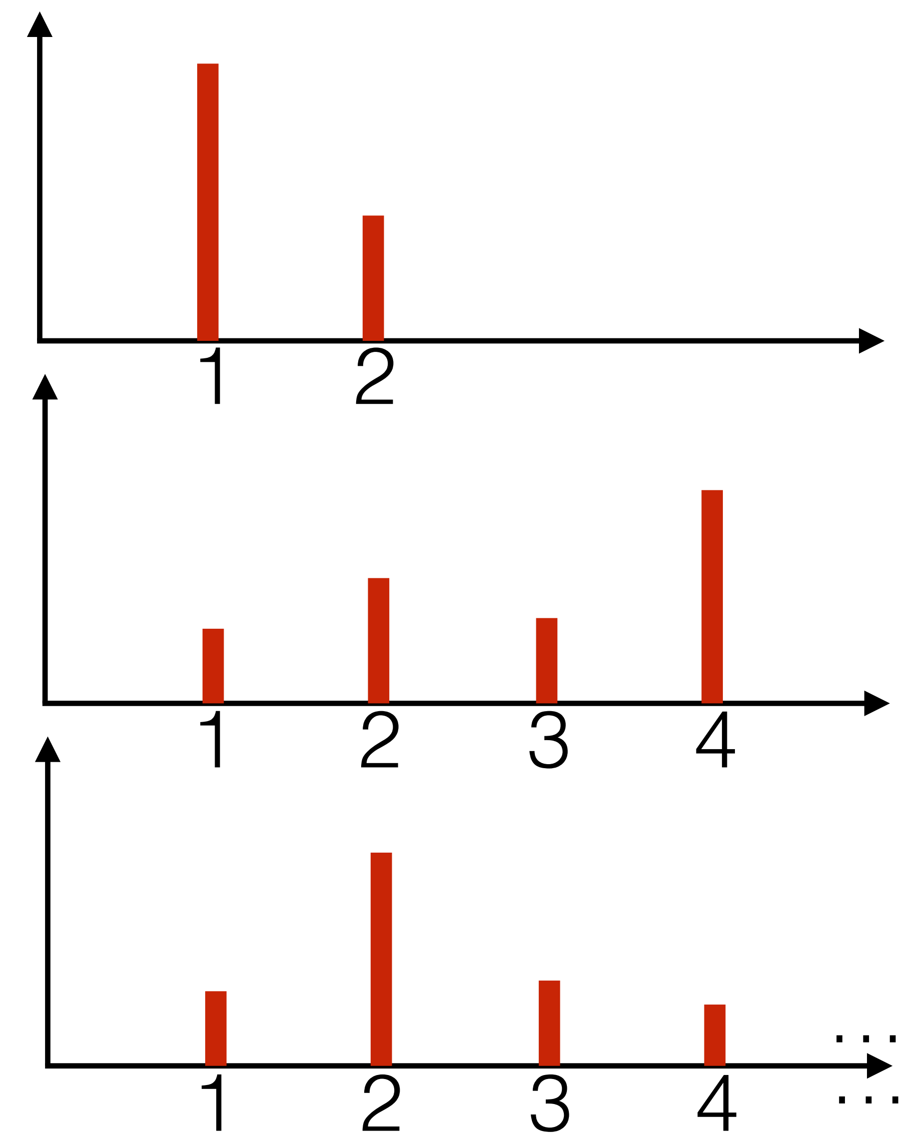

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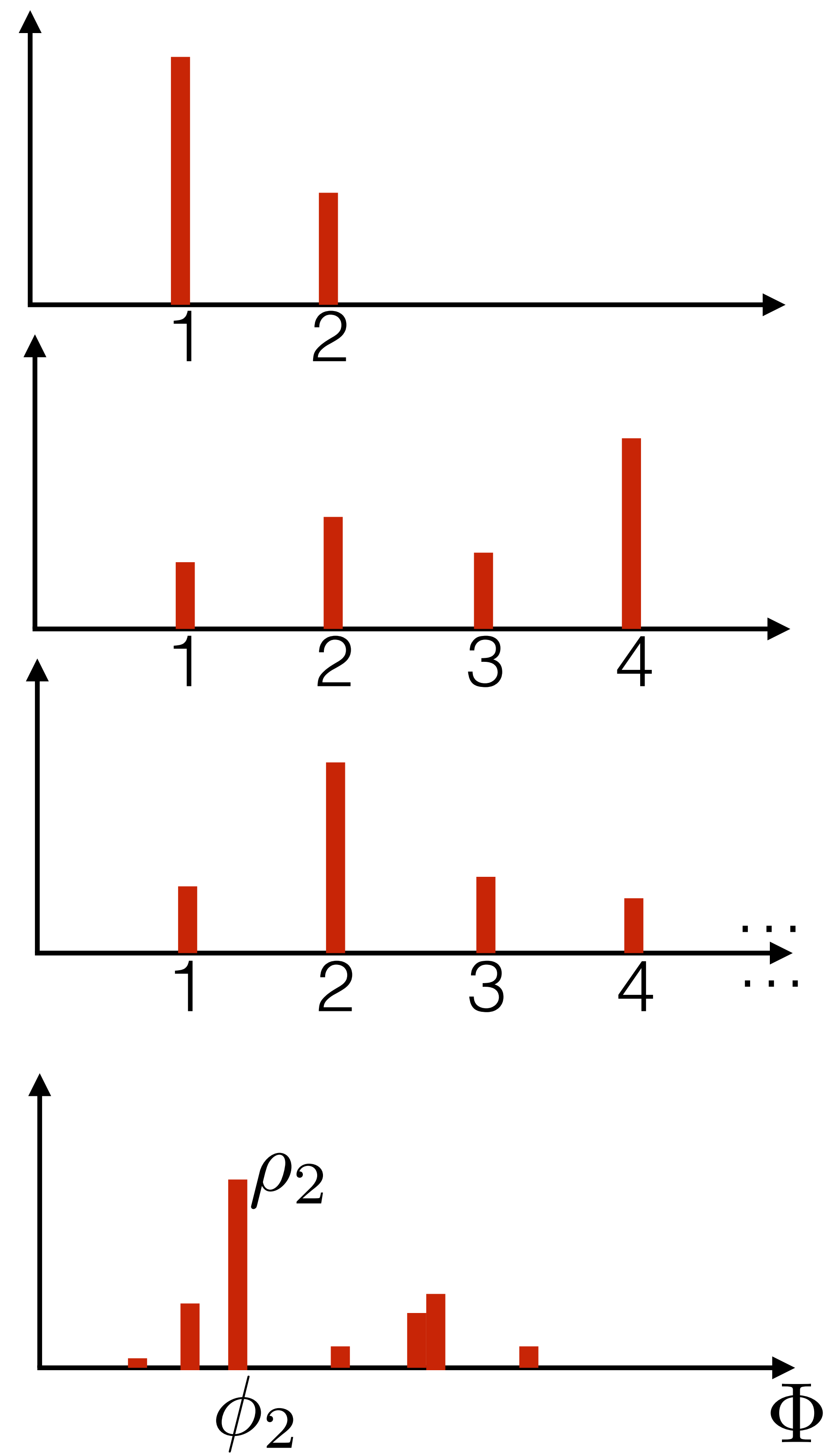

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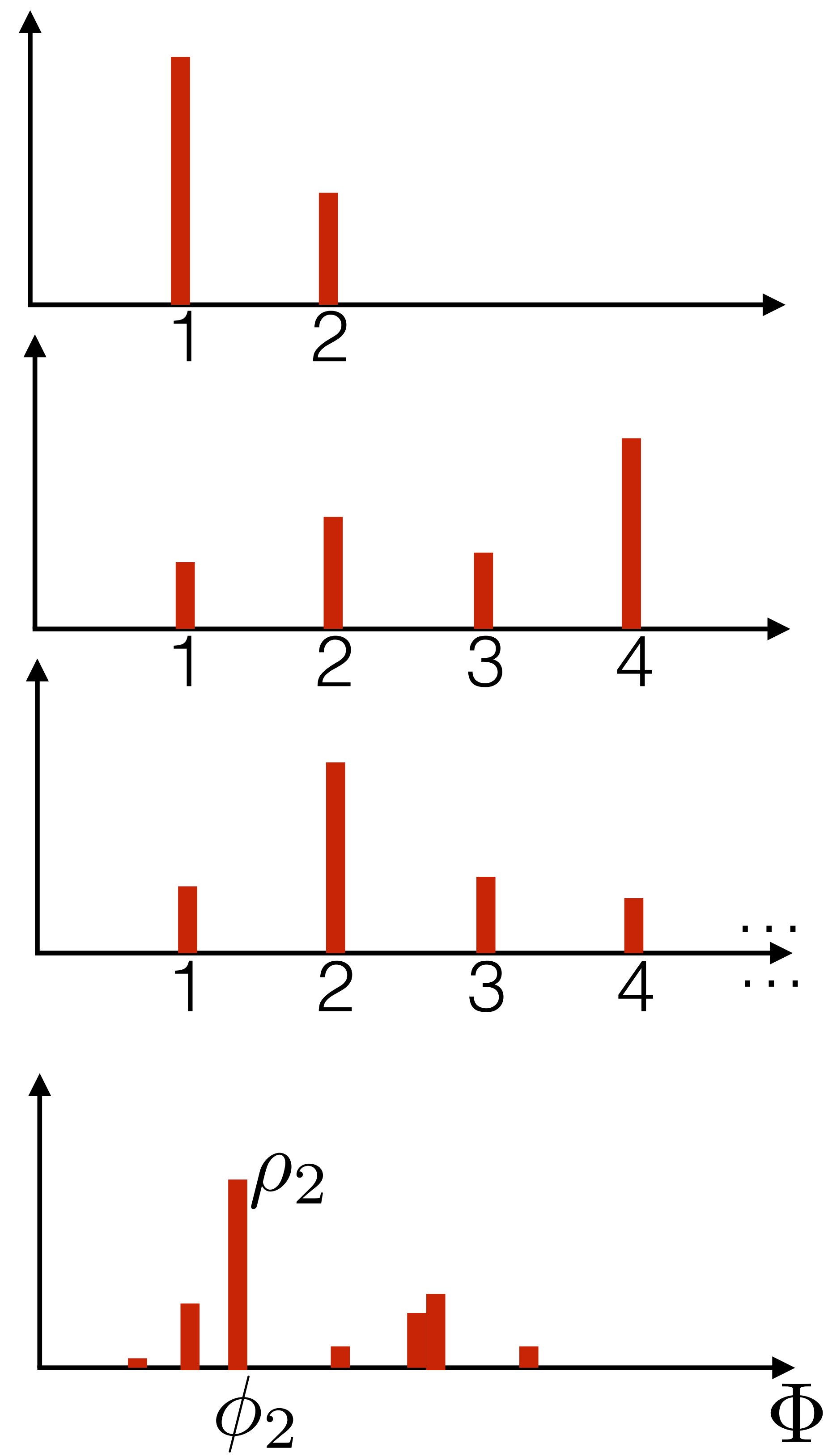

[Ferguson 1973]

Dirichlet process mixture model

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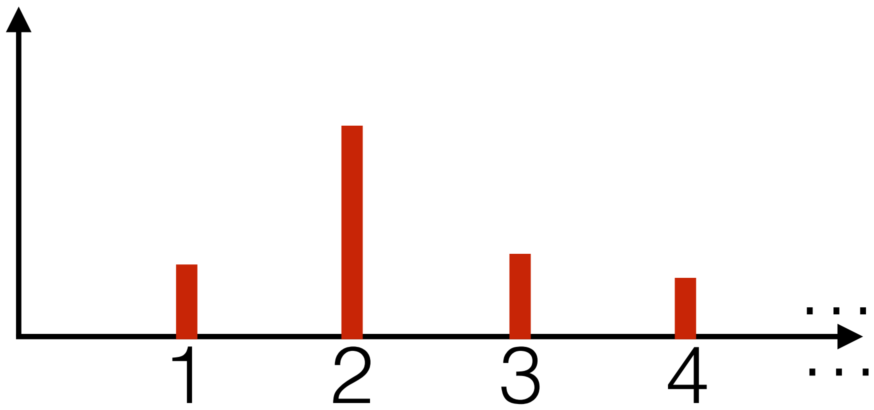

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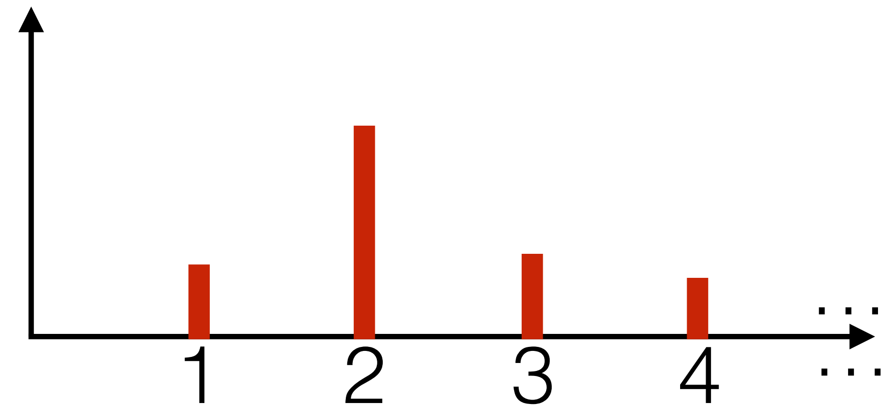

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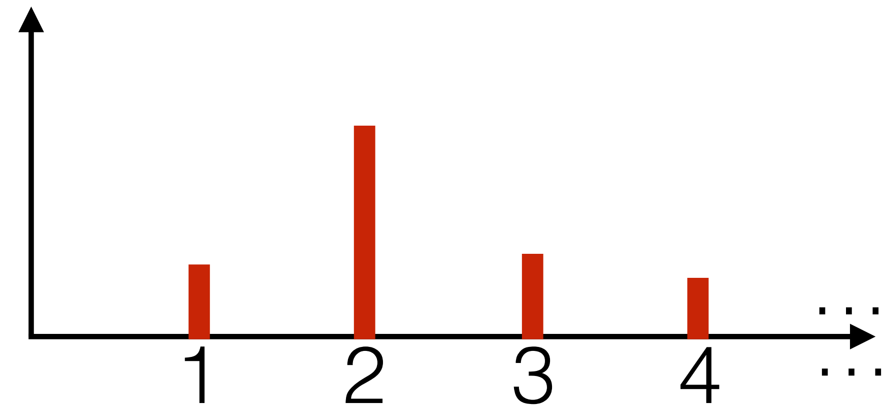

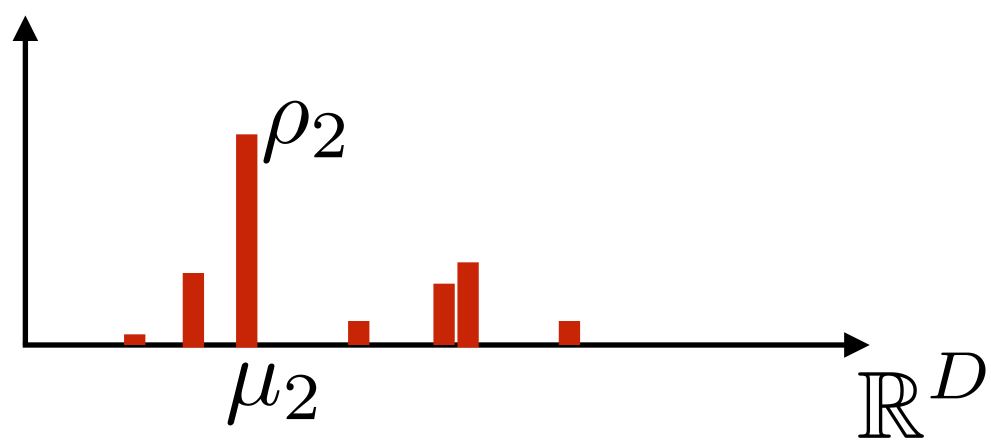

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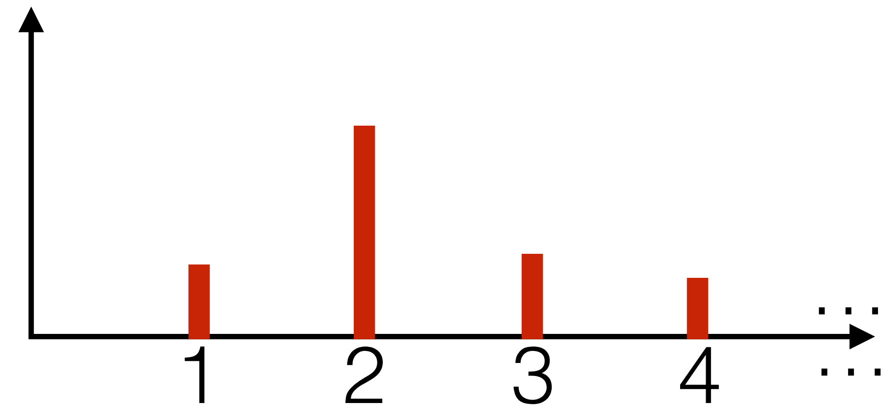

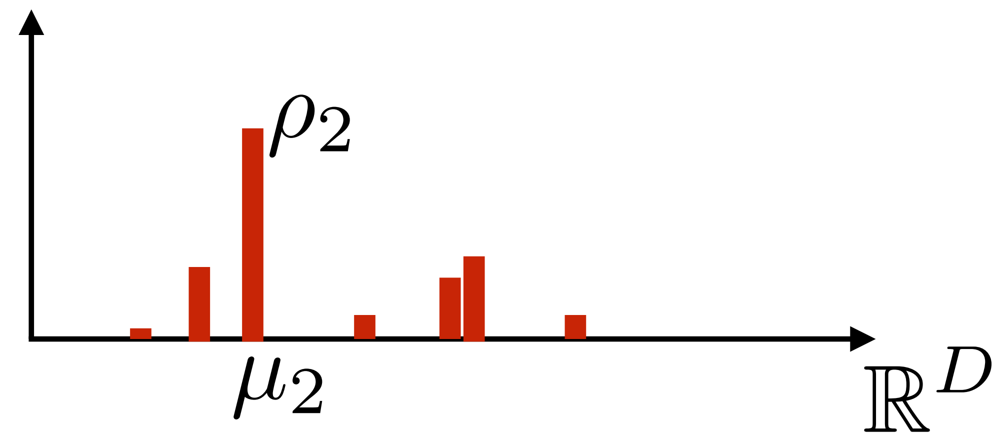

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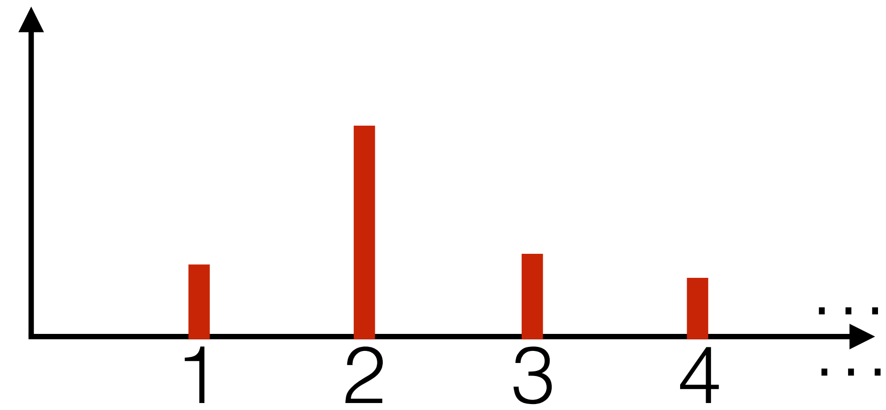

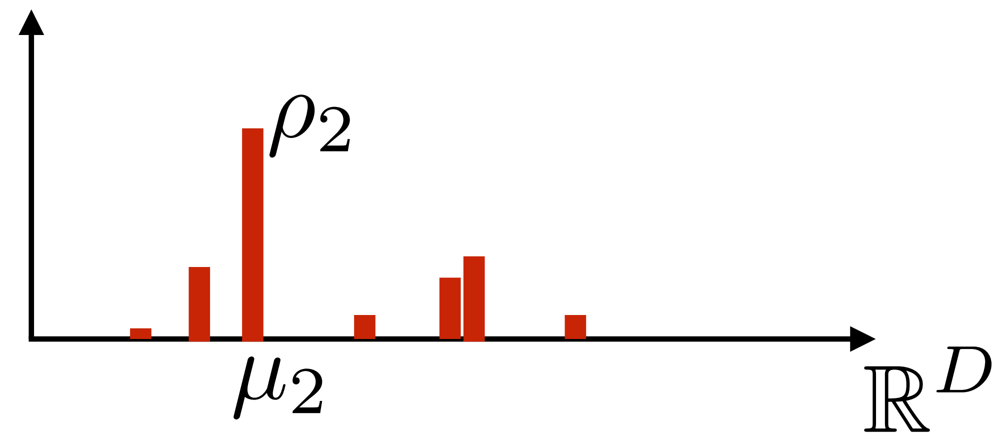

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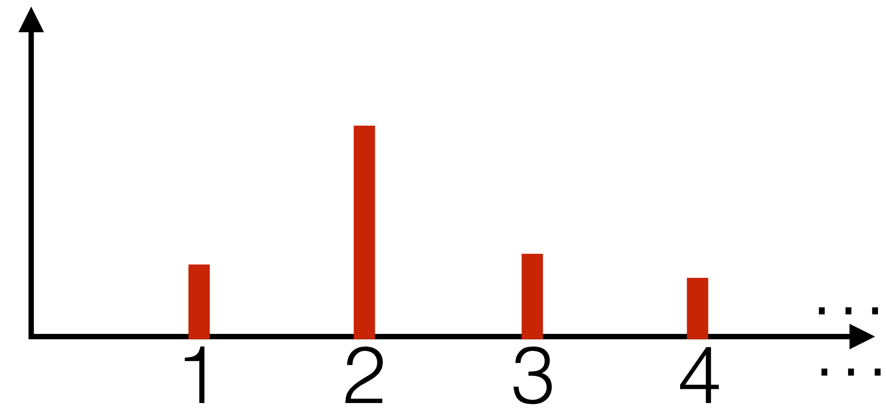

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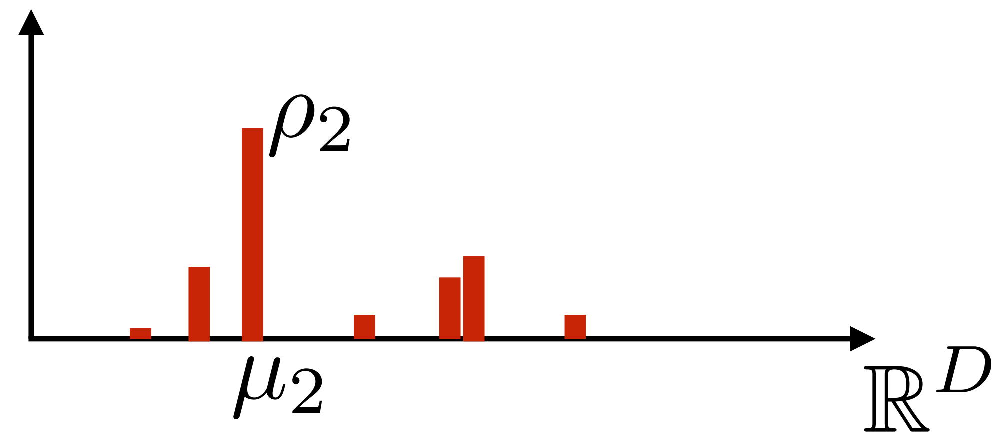

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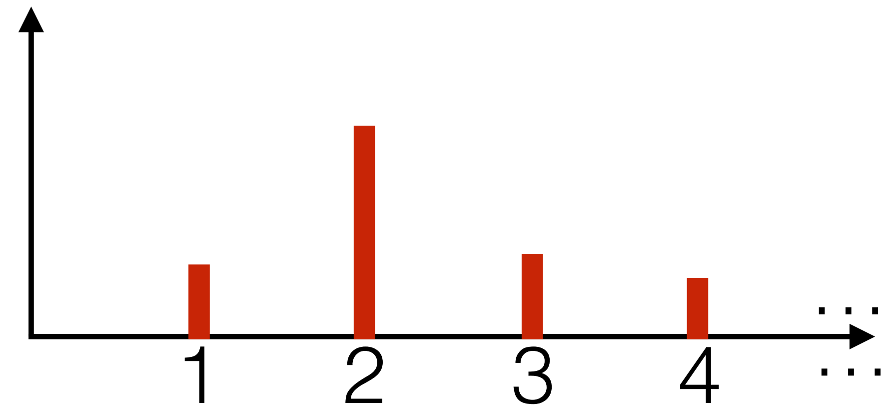

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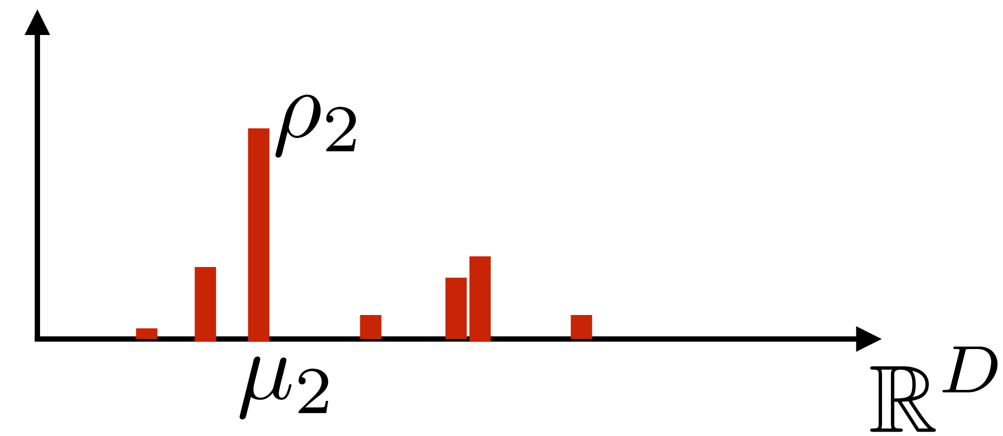

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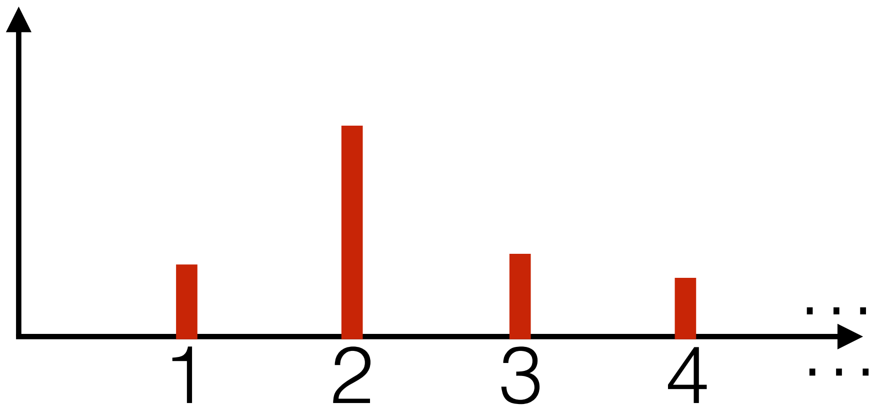

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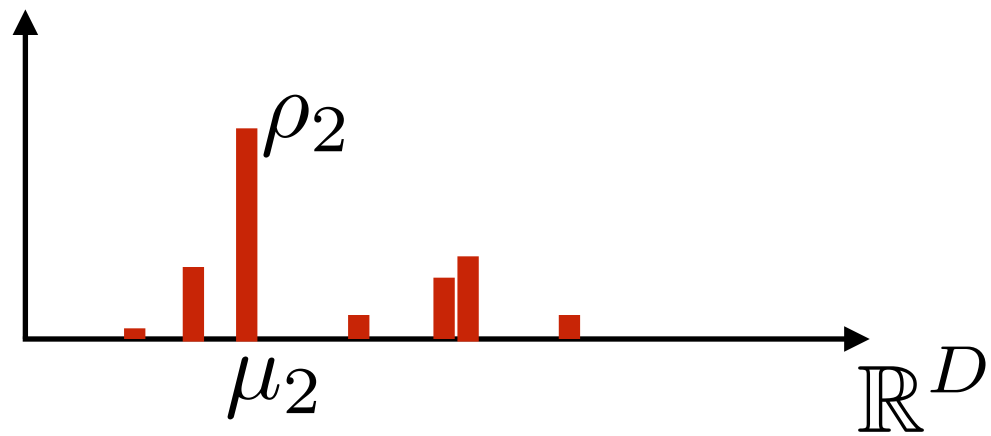

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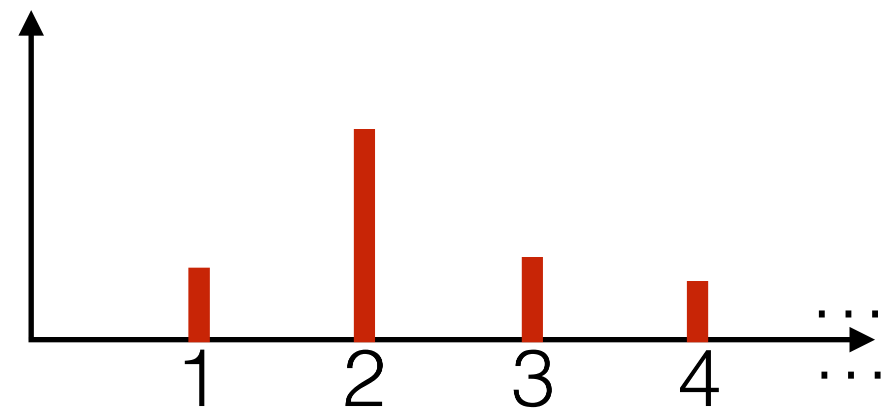

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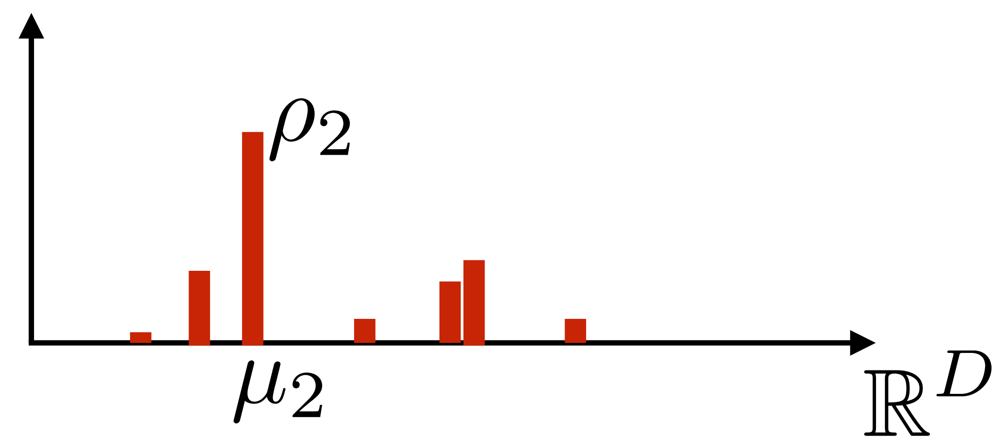

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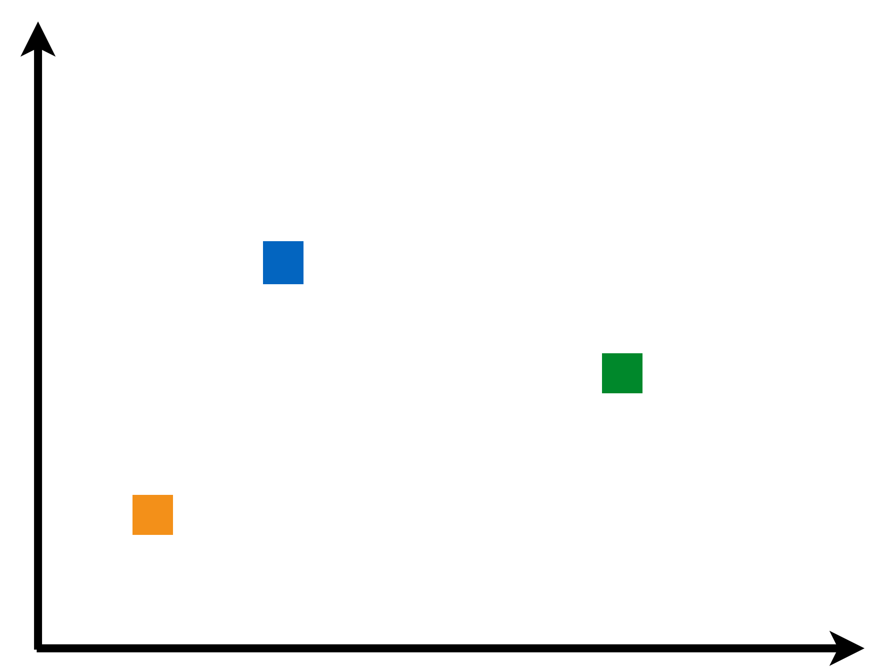

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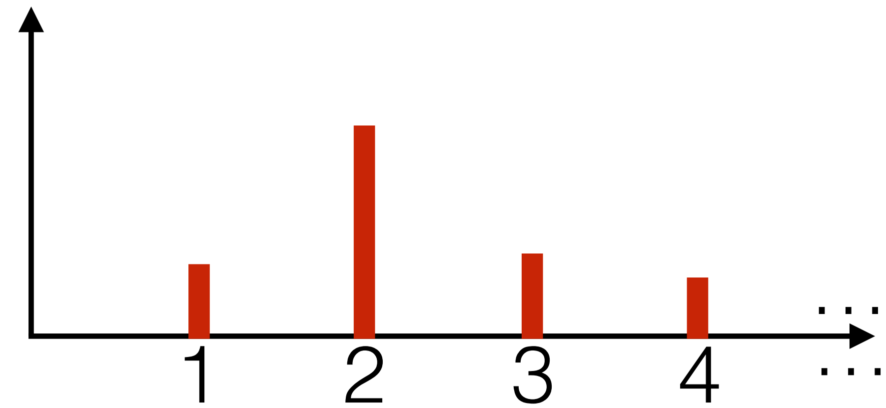

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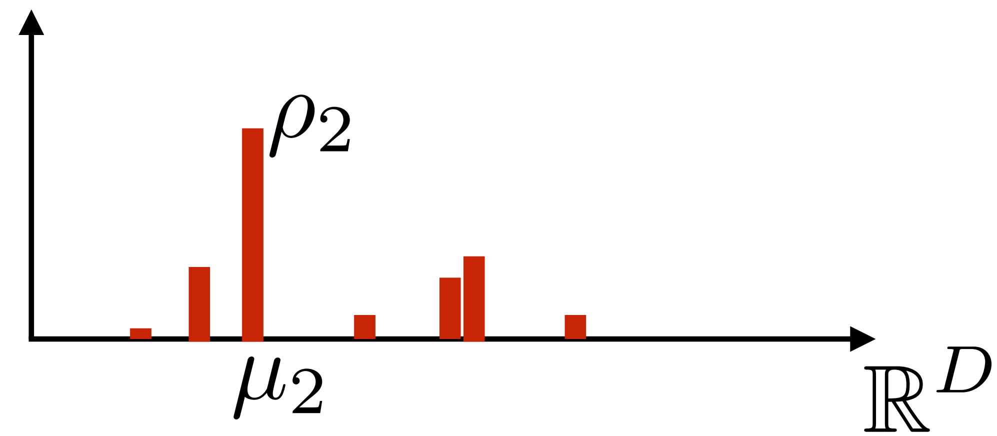

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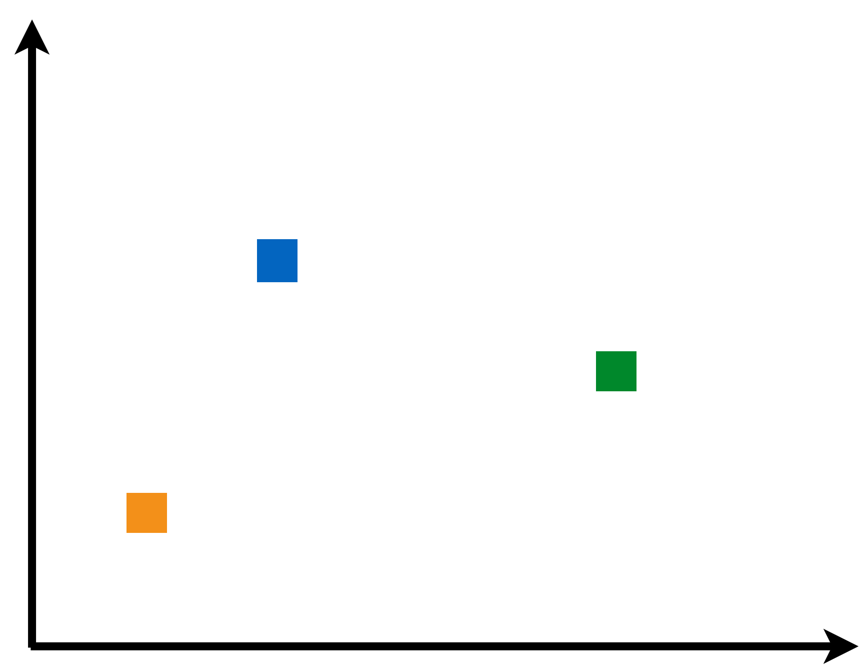

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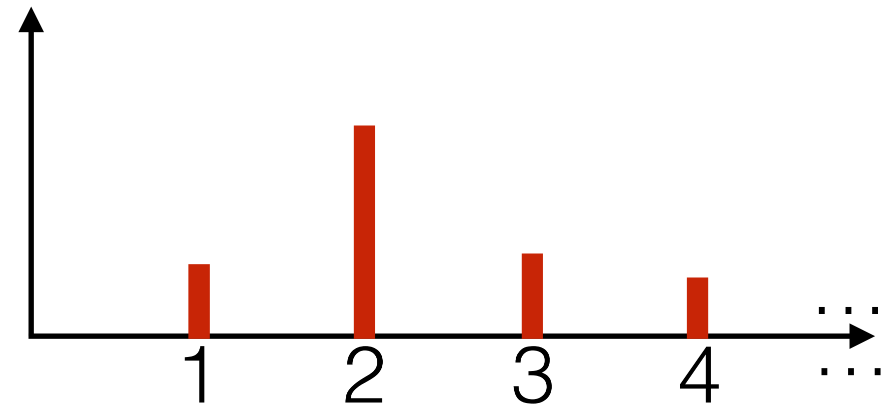

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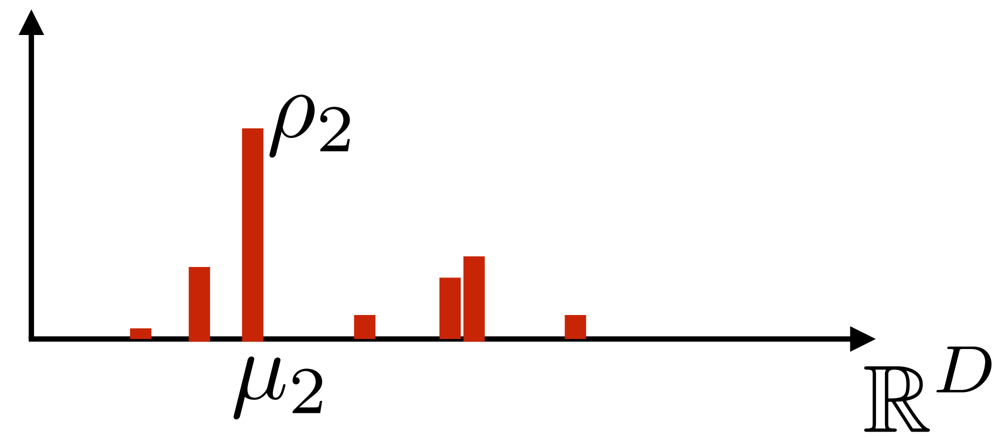

$$z_n \overset{iid}{\sim} \mathrm{Categorical}(\rho)$$

$$\mu_n^* = \mu_{z_n}$$

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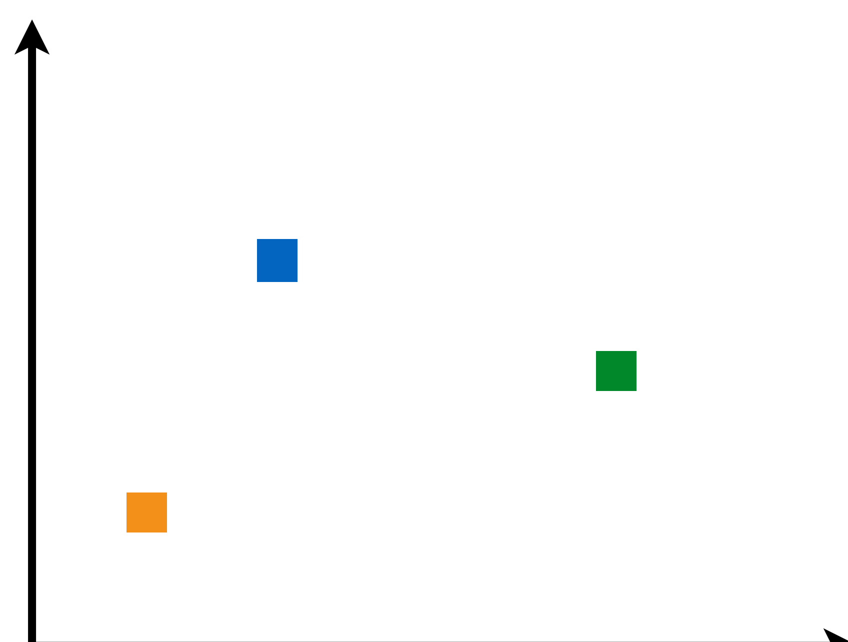
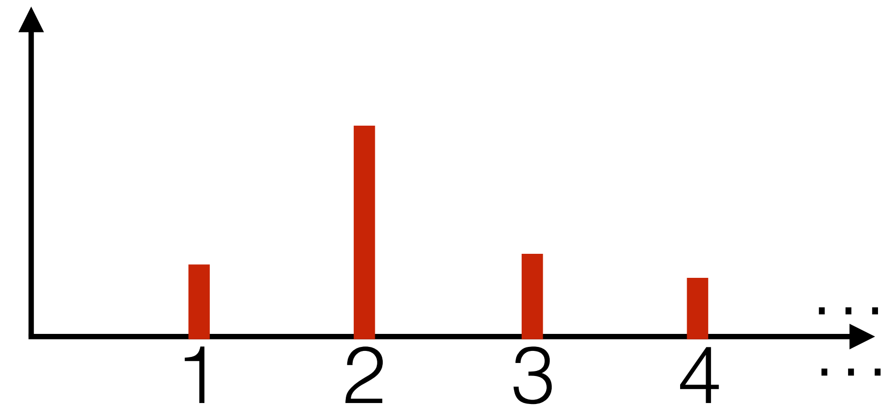


$$x_n \overset{indep}{\sim} \mathcal{N}(\mu_n^*, \Sigma)$$

Dirichlet process mixture model

- Gaussian mixture model



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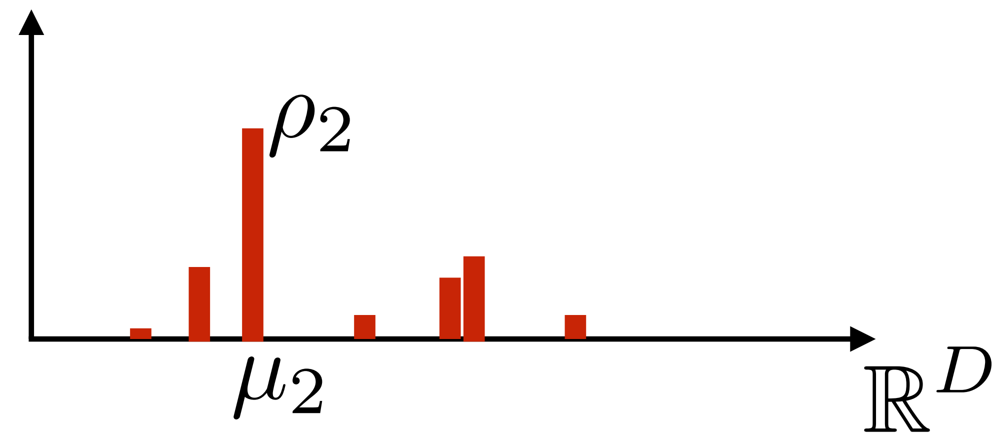

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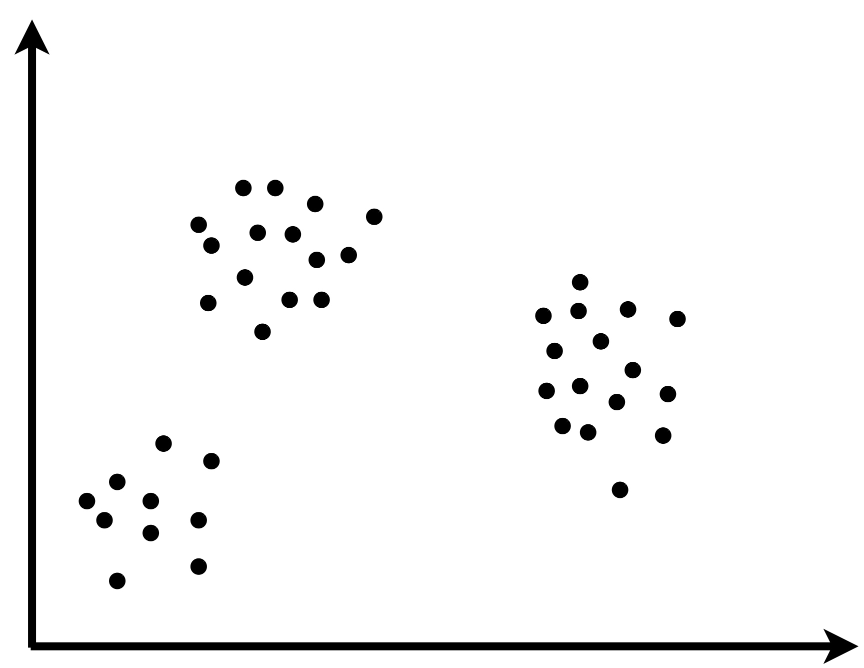

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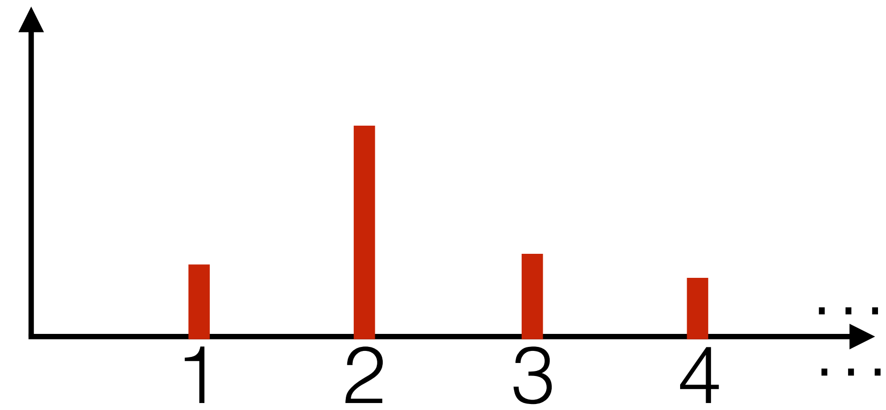
- Gaussian mixture model



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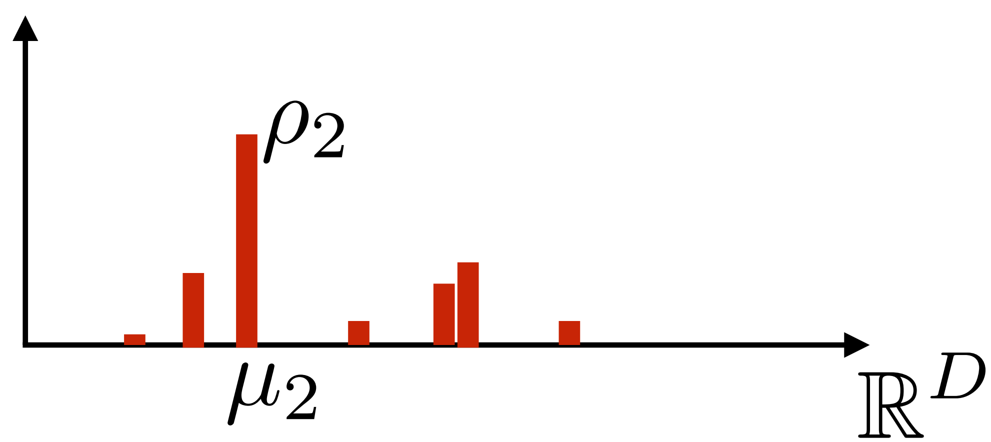

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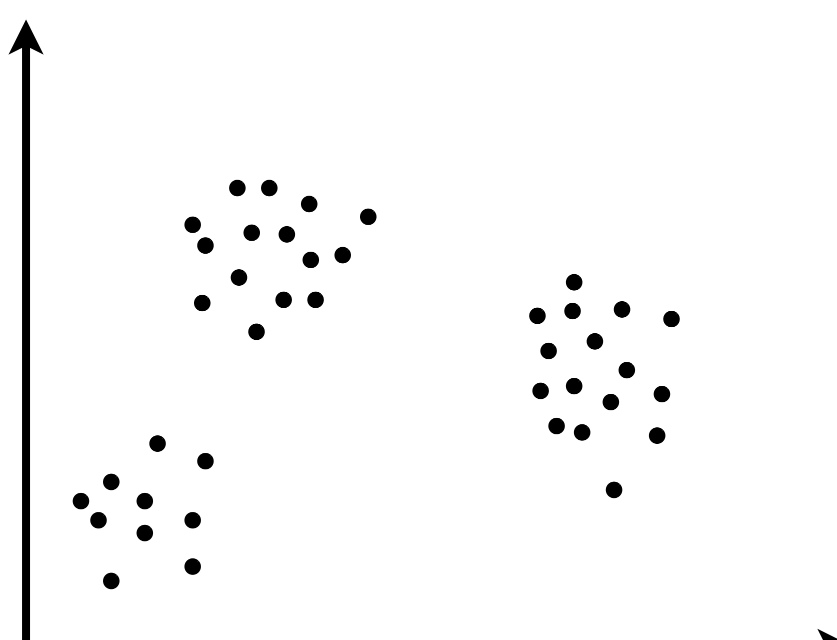

$$x_n \overset{indep}{\sim} \mathcal{N}(\mu_n^*, \Sigma)$$

[demo]

Dirichlet process mixture model

- More generally

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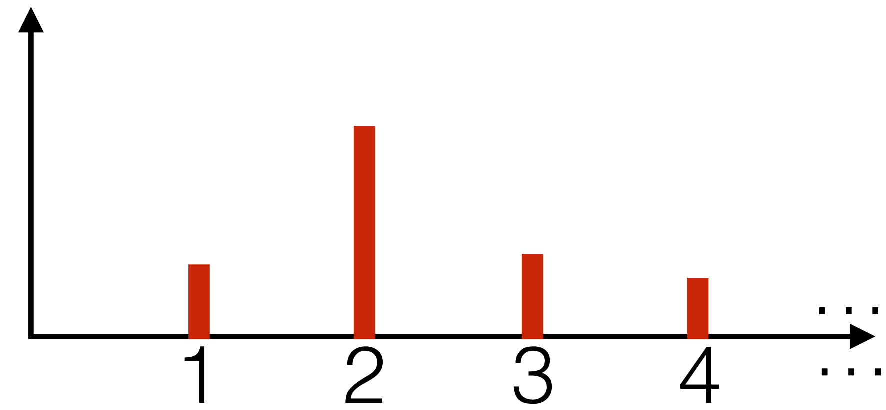

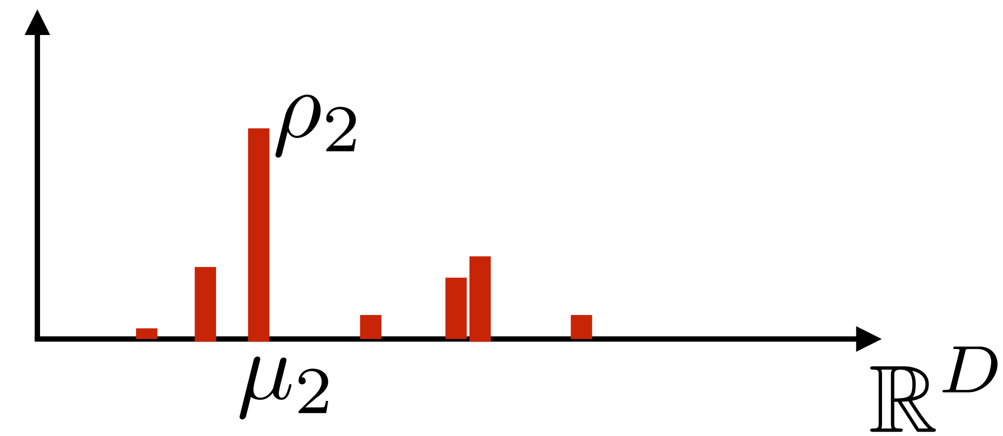

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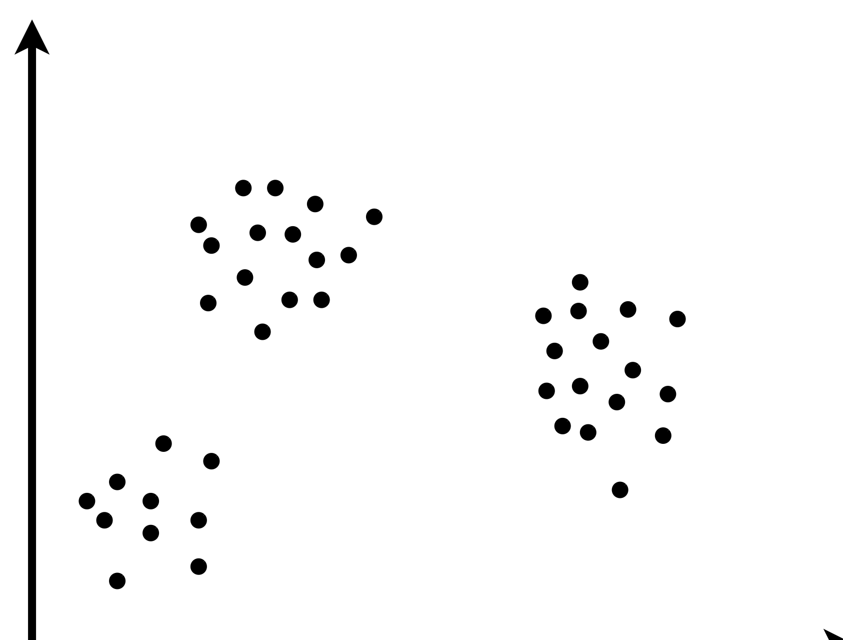

Dirichlet process mixture model

- More generally

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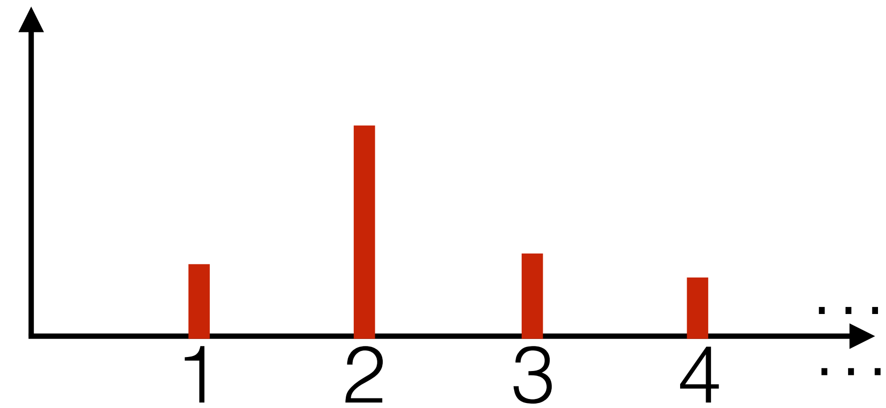

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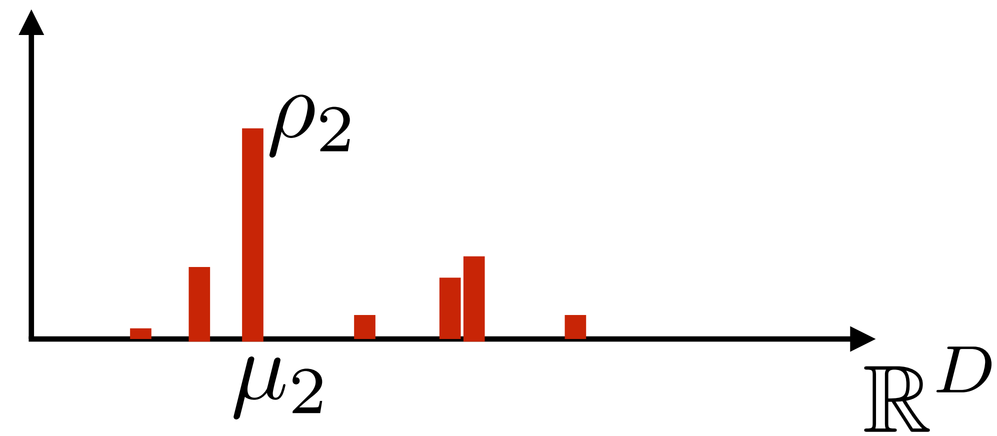

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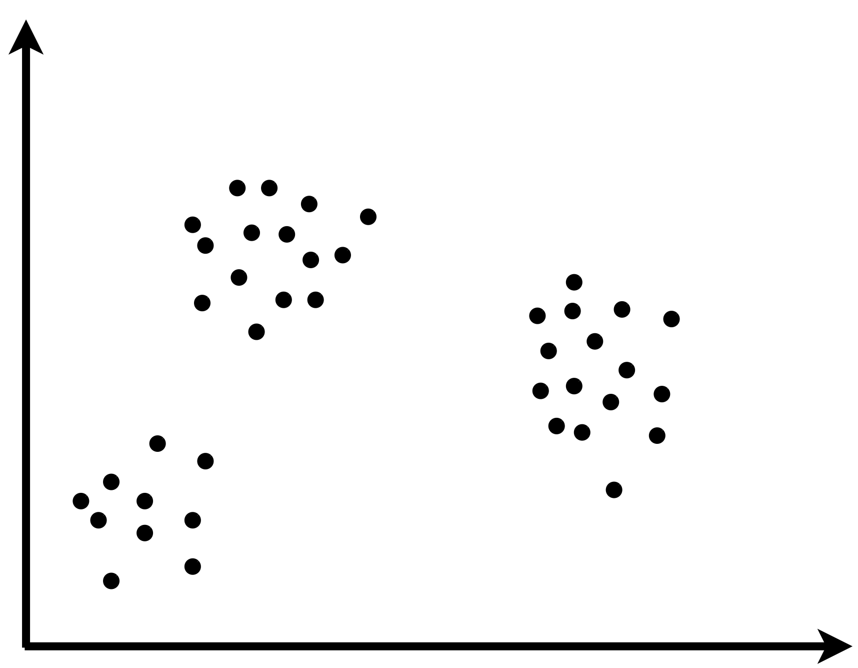

Dirichlet process mixture model

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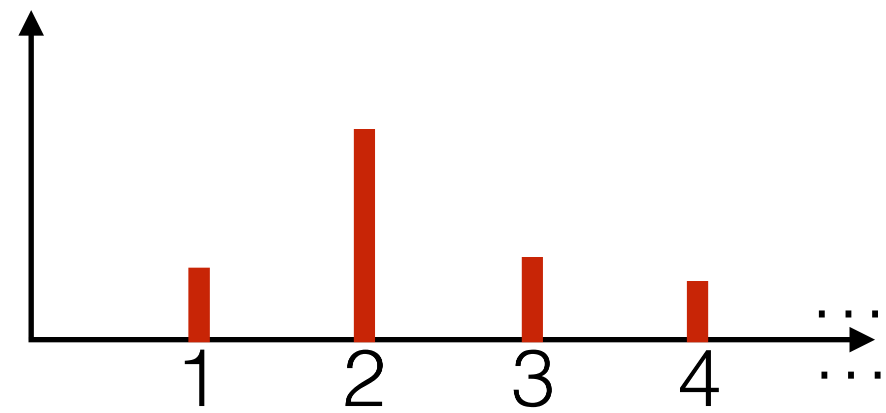

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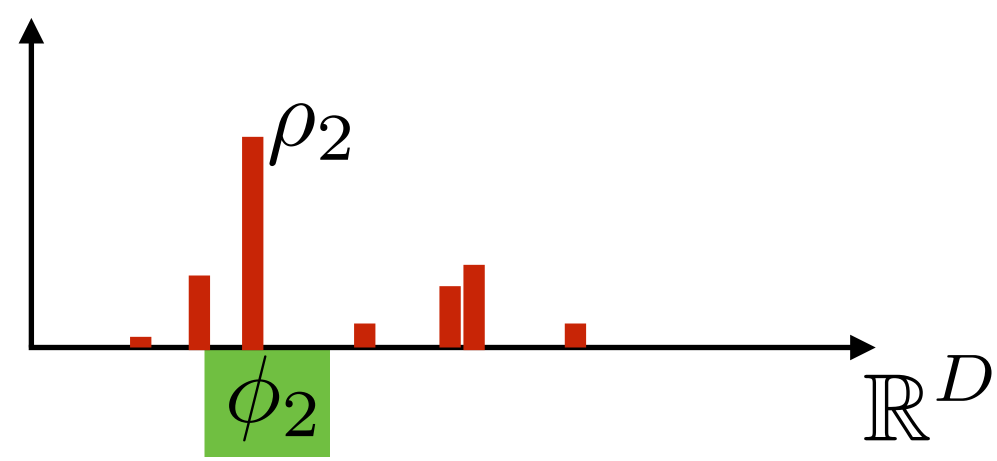

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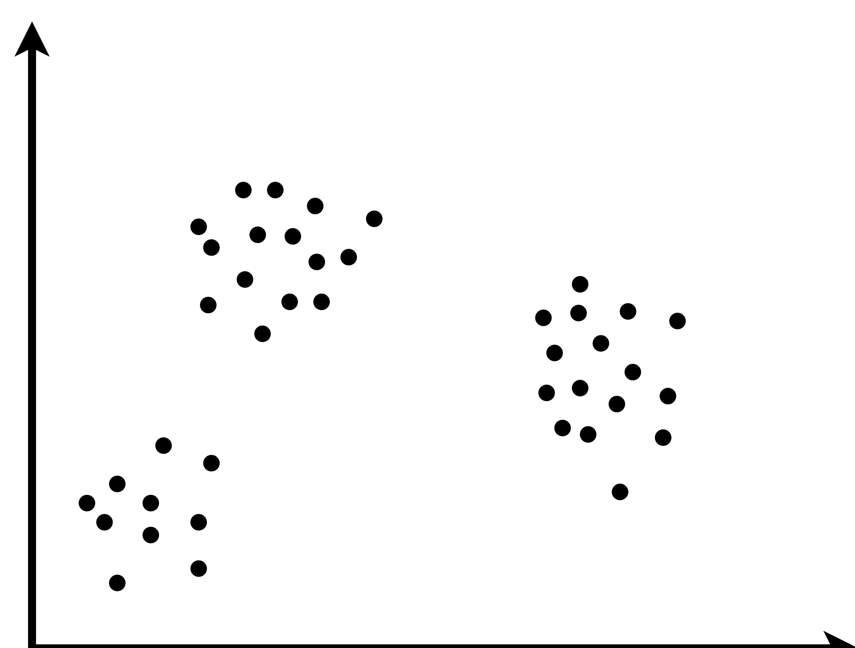

Dirichlet process mixture model

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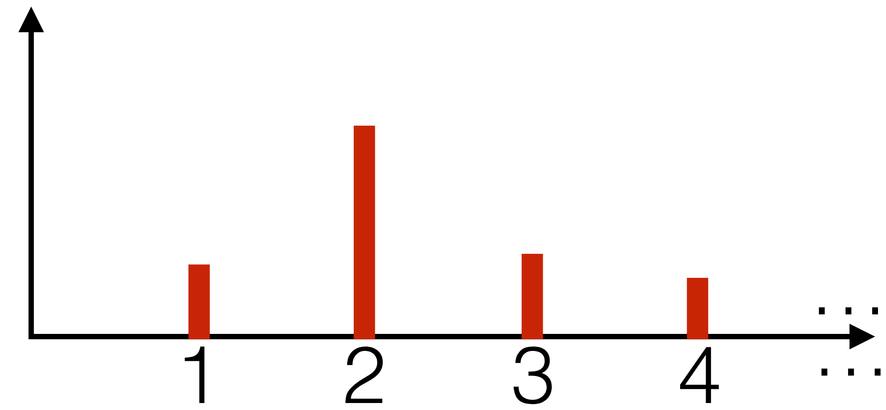

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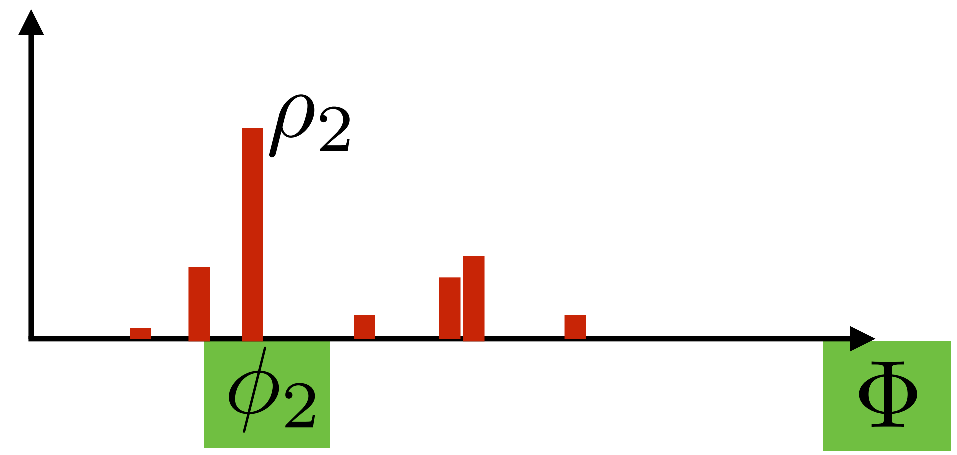

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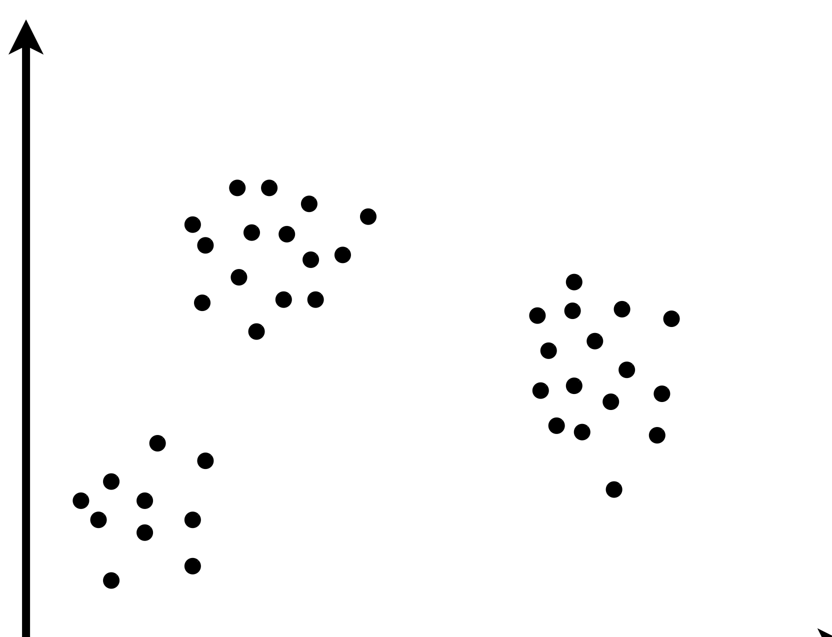

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Dirichlet process mixture model

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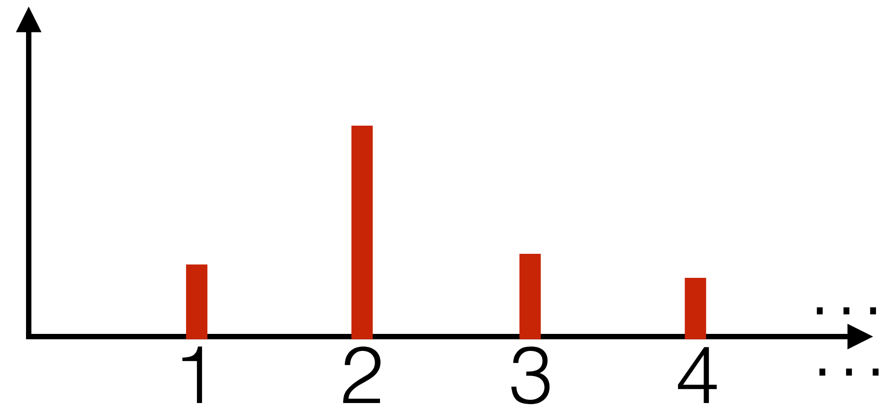

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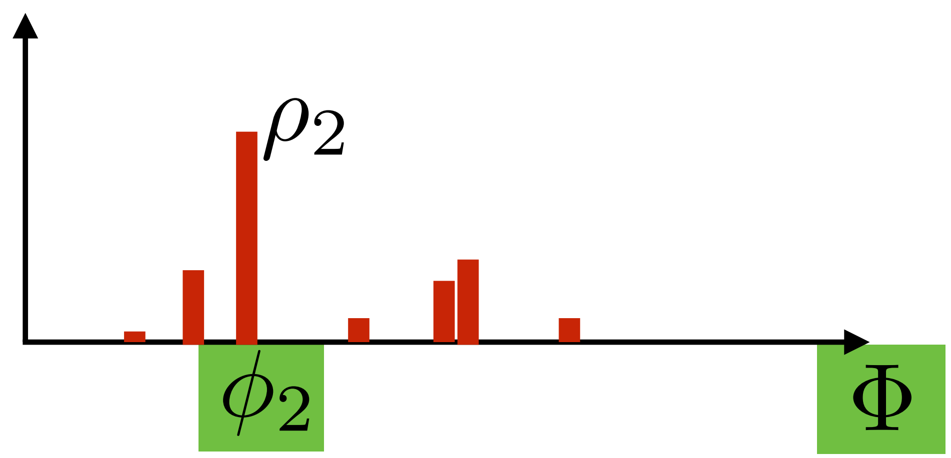

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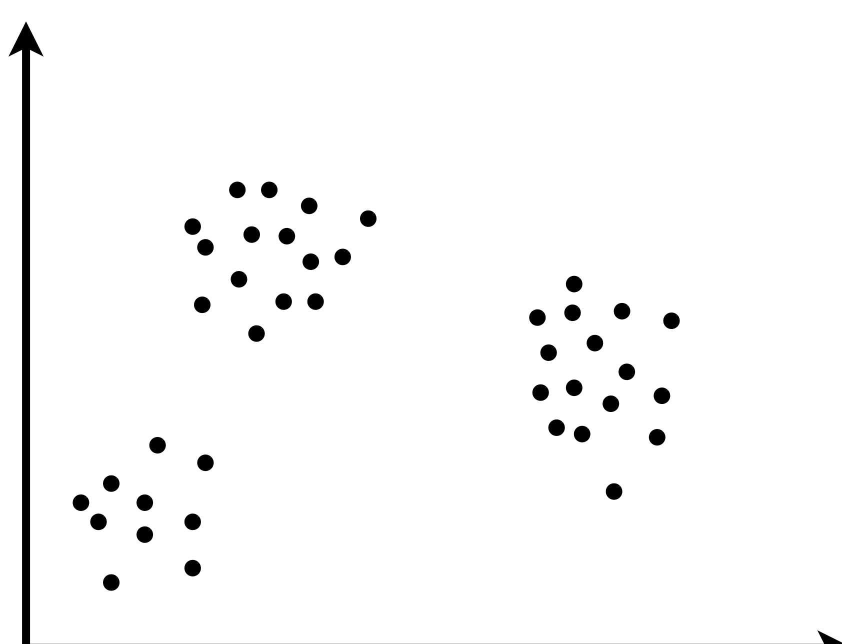

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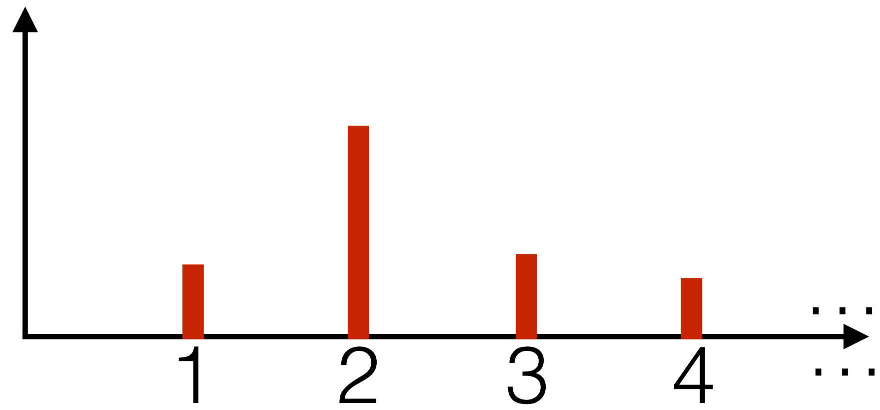

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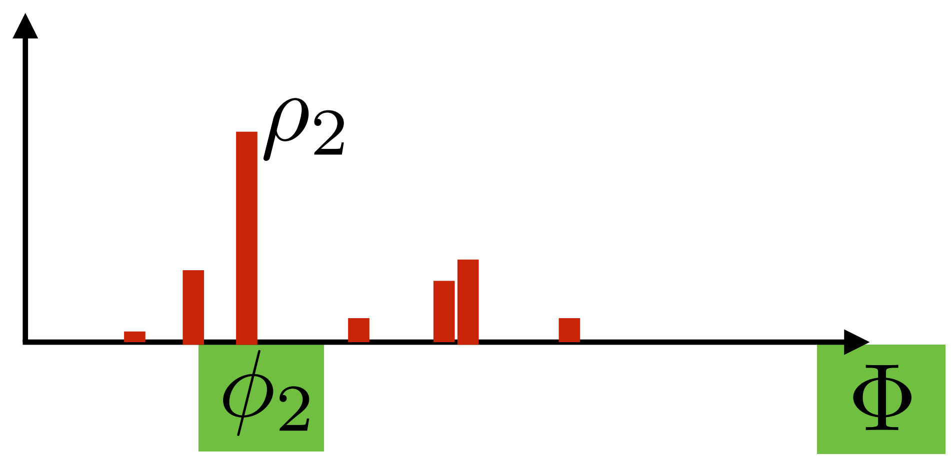

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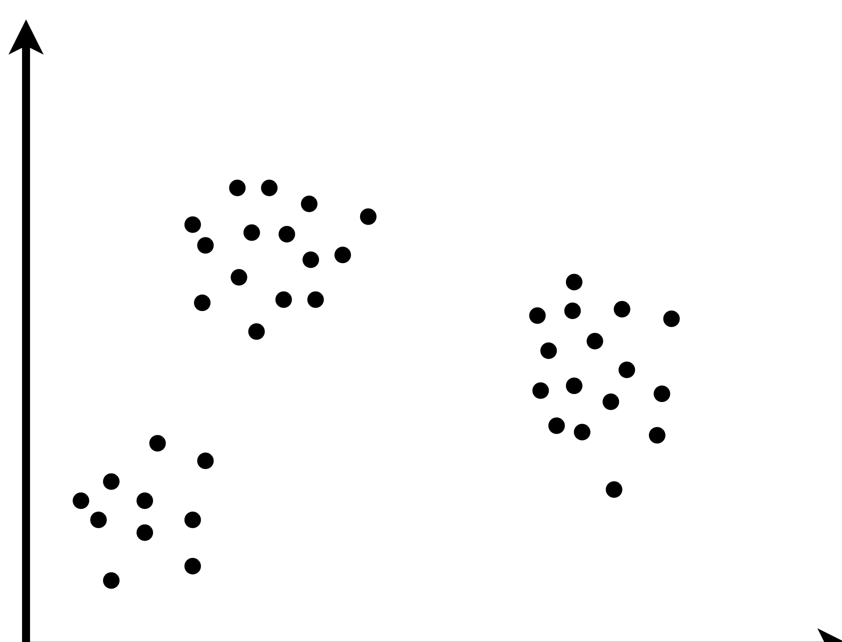

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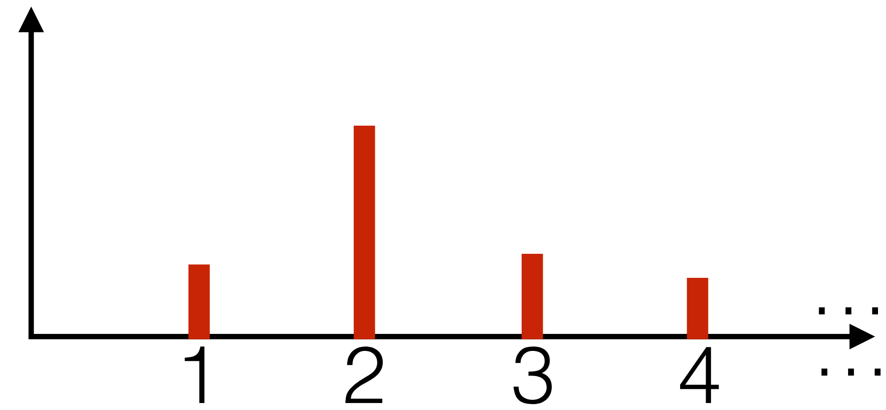

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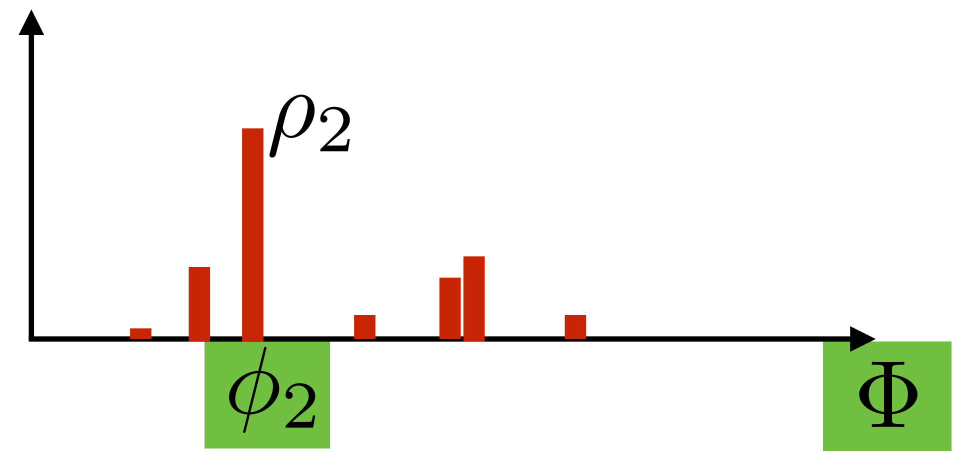

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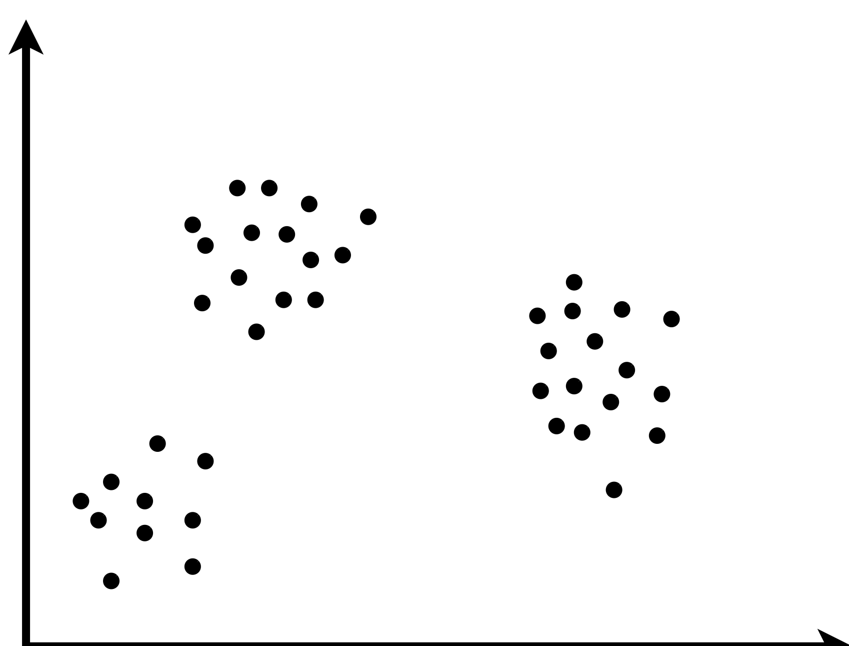

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Dirichlet process mixture model

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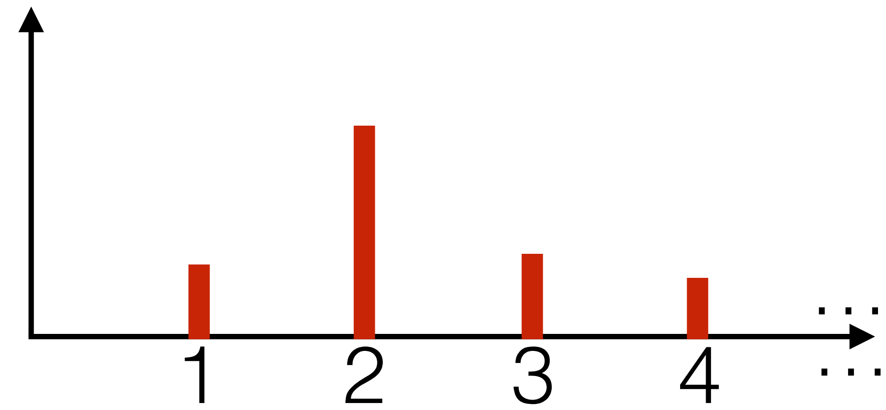

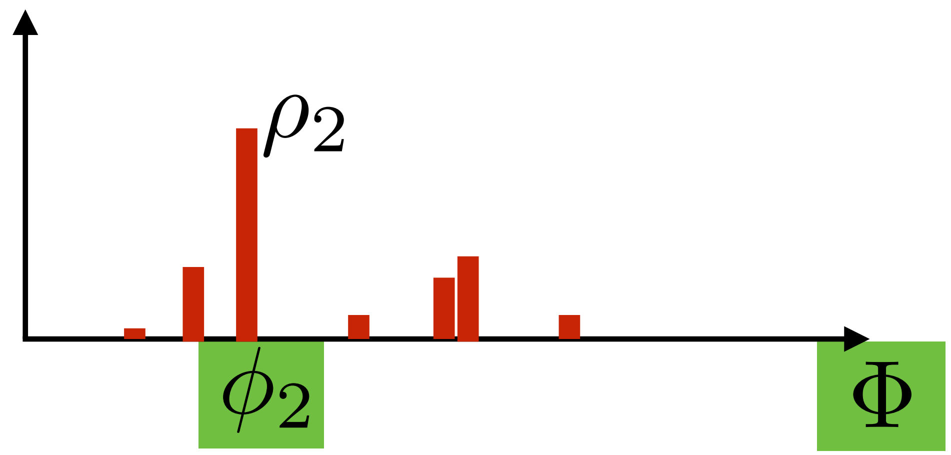

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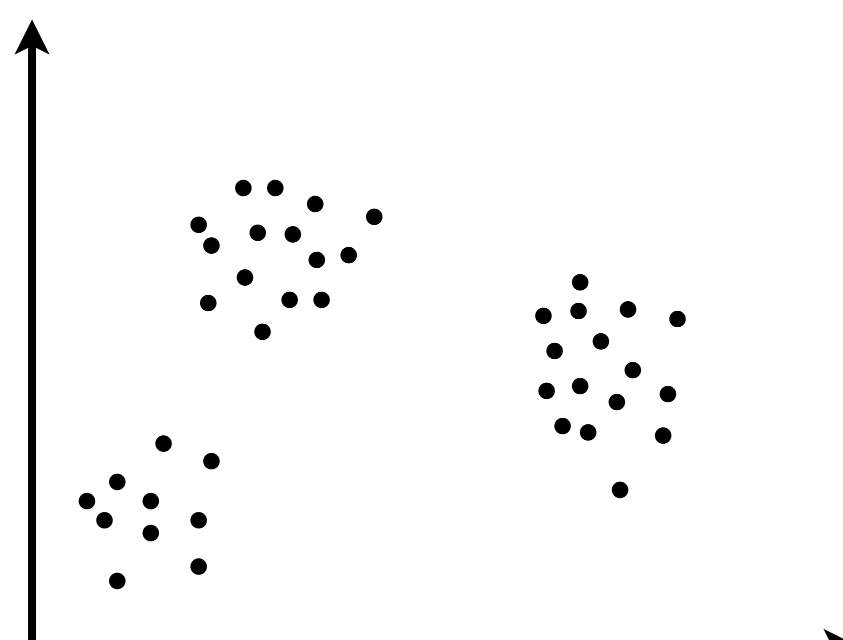

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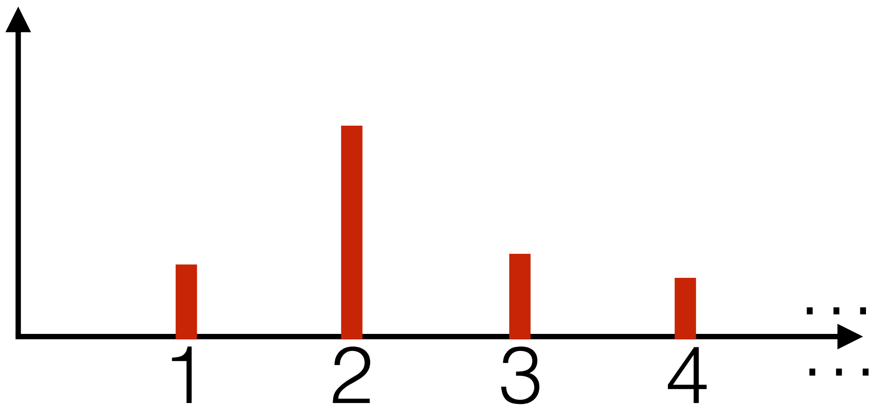

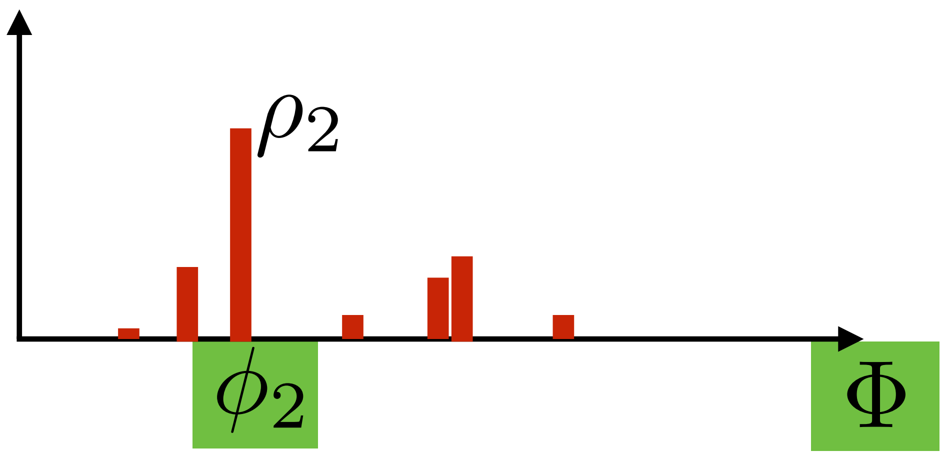

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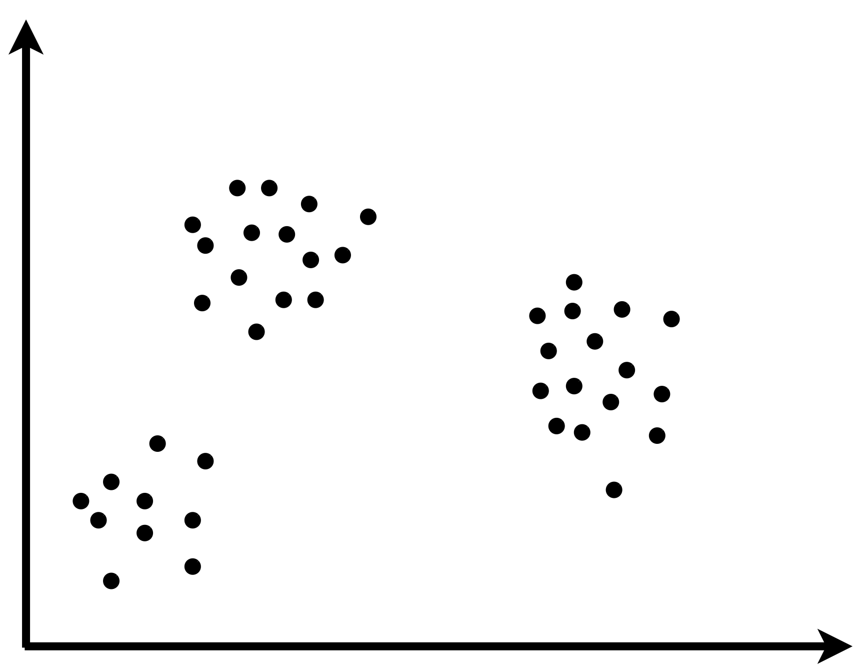
$$x_n \overset{indep}{\sim} F(\theta_n)$$



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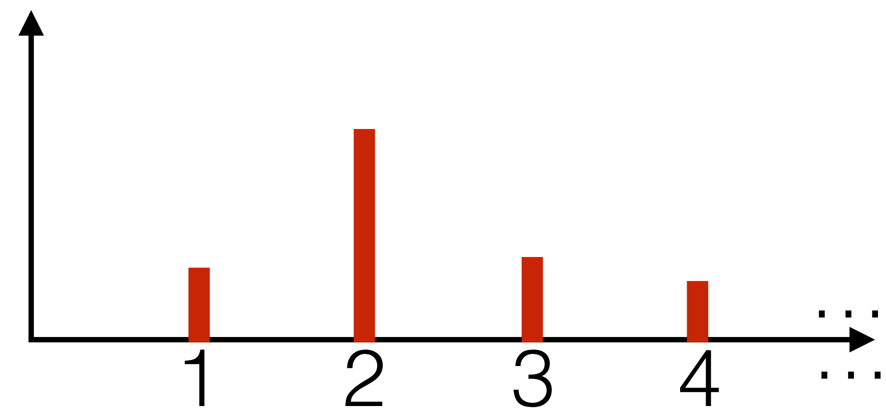

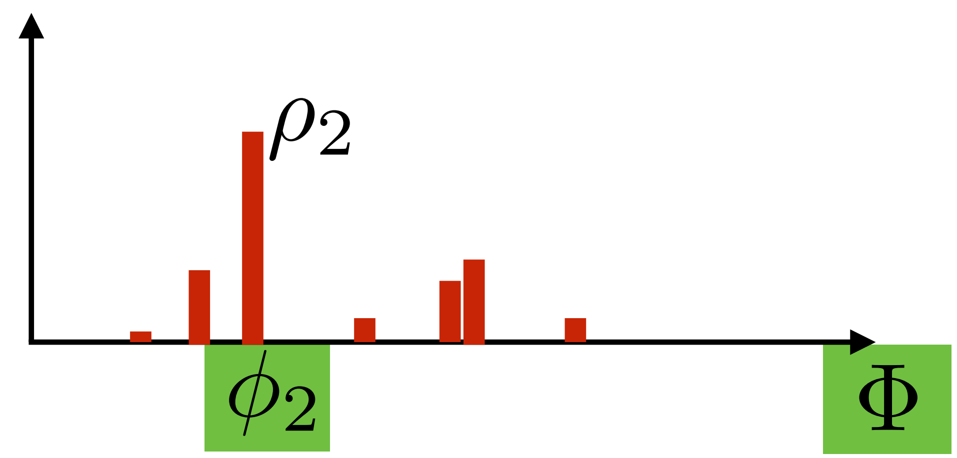

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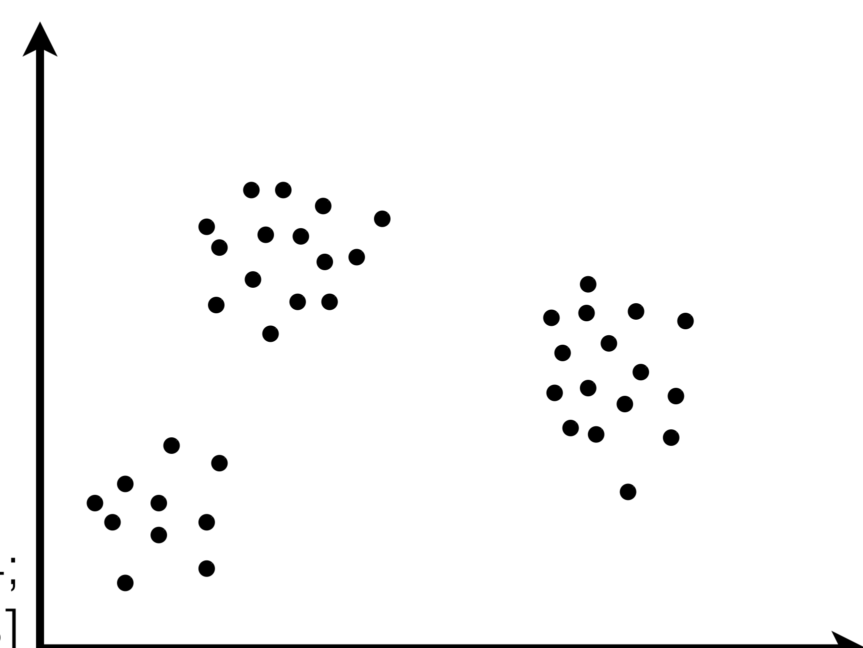
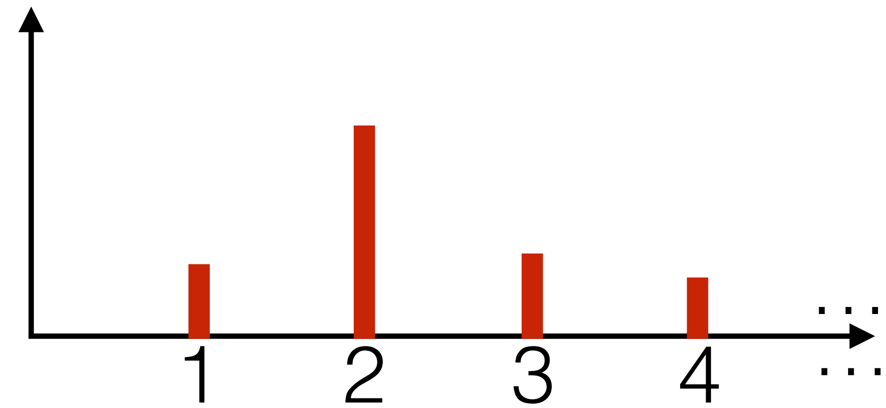
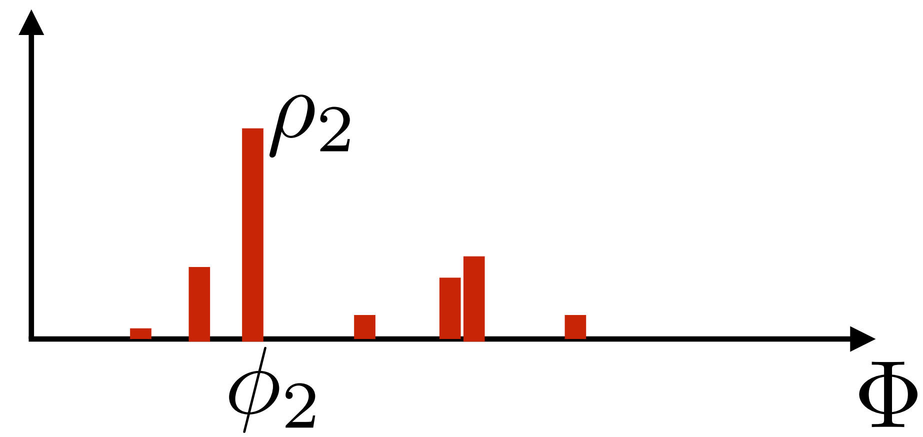
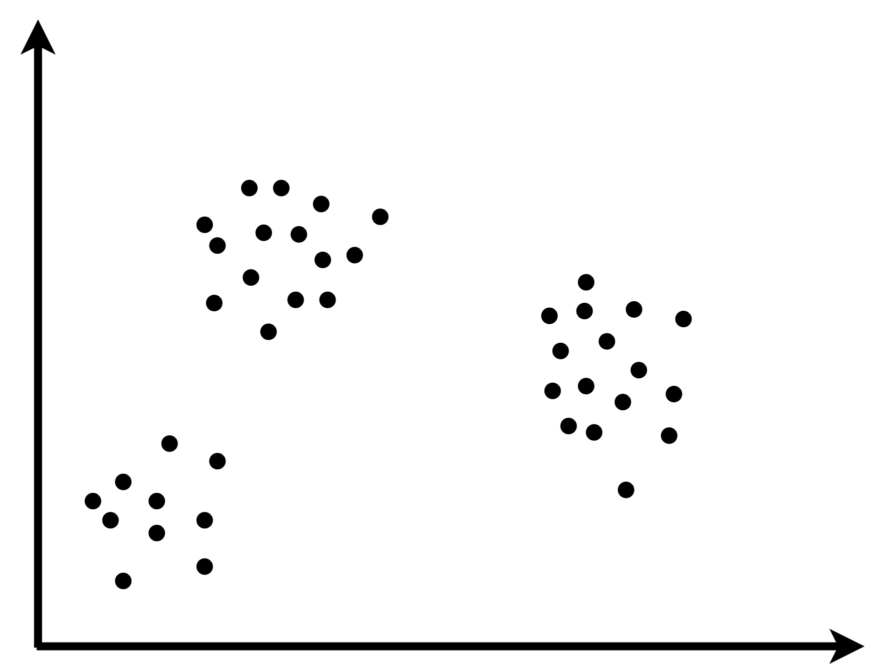


[Antoniak 1974; Ferguson 1983; West, Müller, Escobar 1994; Escobar, West 1995; MacEachern, Müller 1998]

DPMM Exercises

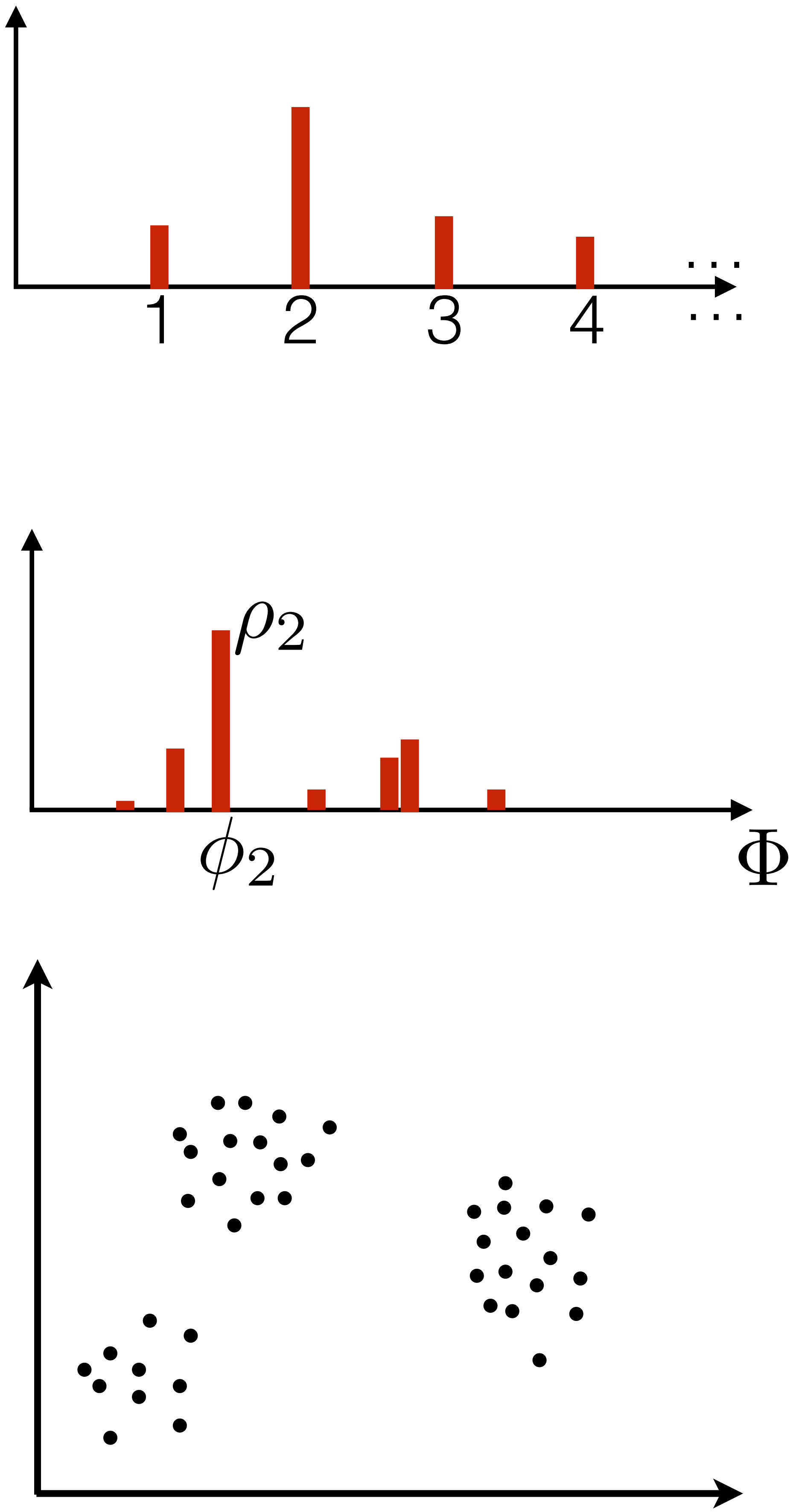






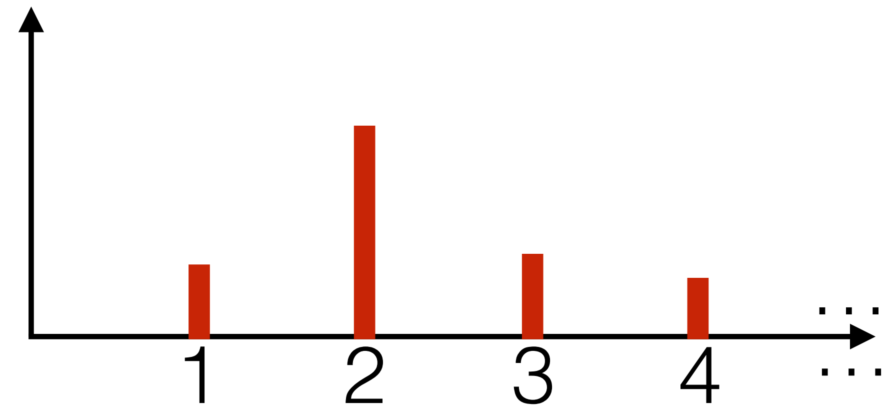
DPMM Exercises

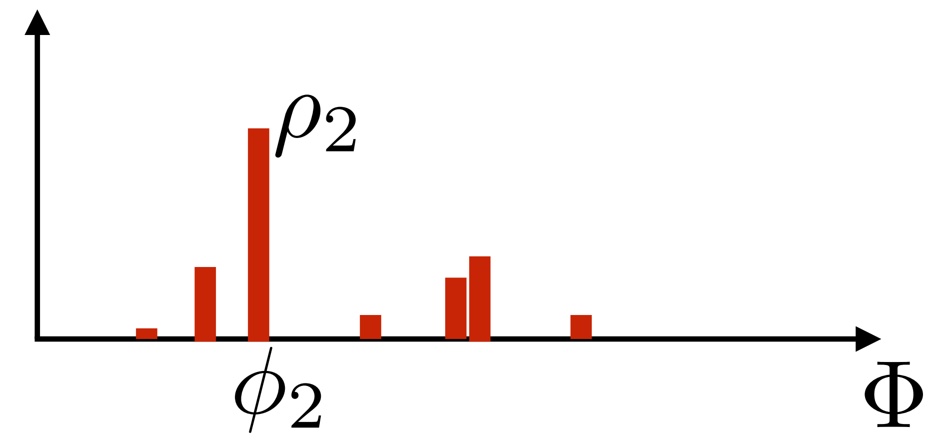
- Code your own DPMM simulator

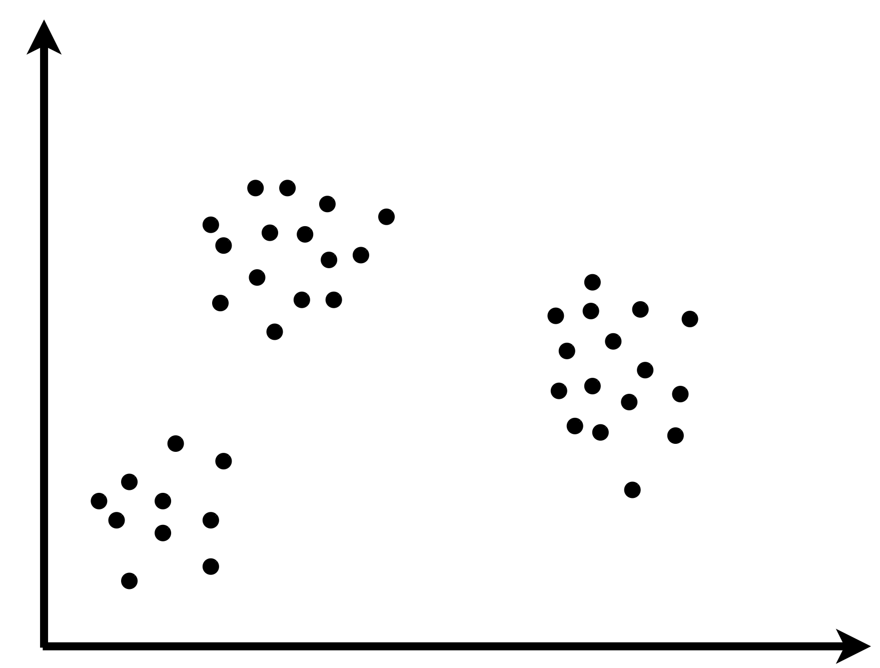


DPMM Exercises

- Code your own DPMM simulator
- How does the number of clusters vary with *N* ? (theory/simulations)

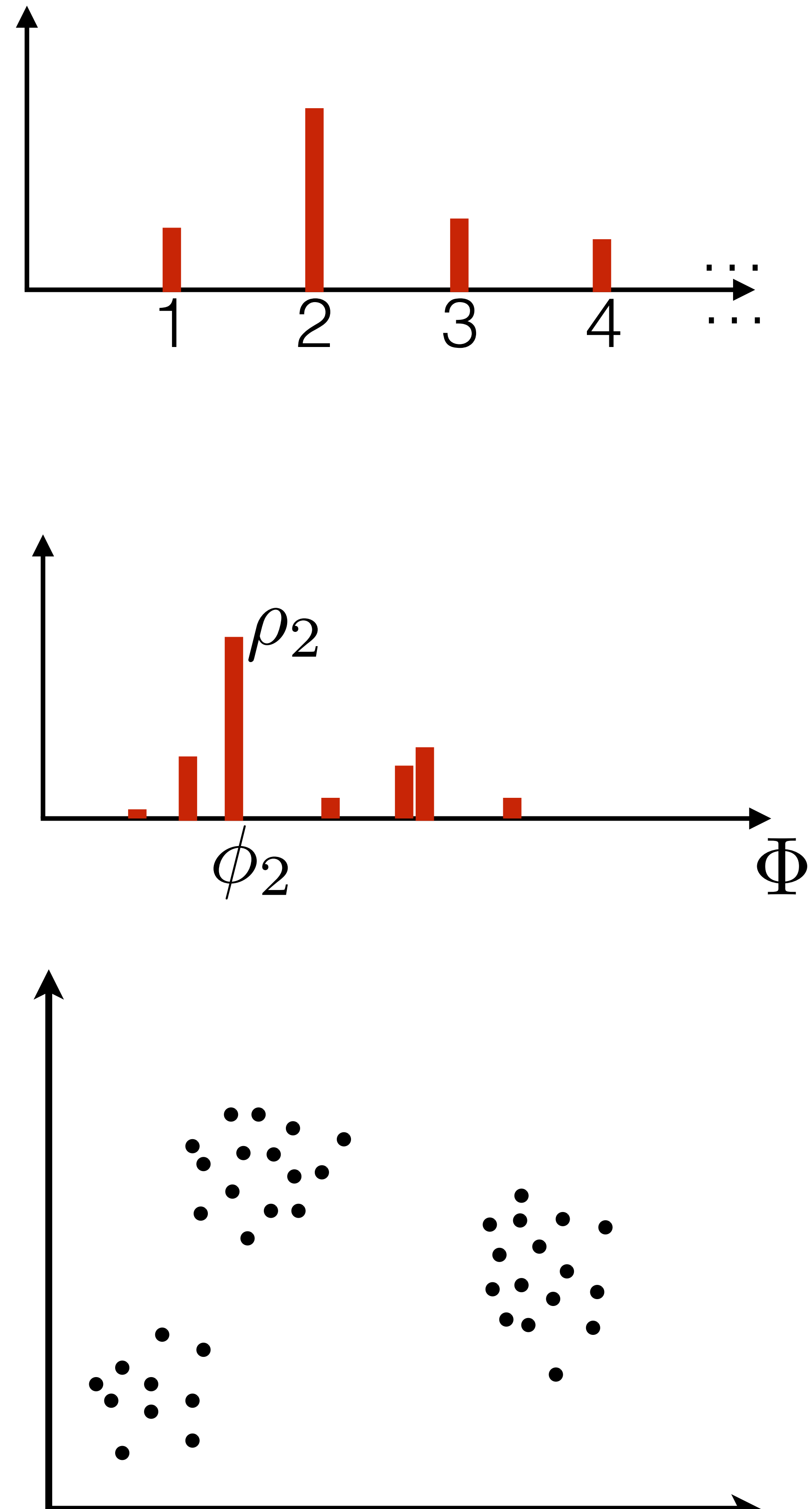






DPMM Exercises

- Code your own DPMM simulator
- How does the number of clusters vary with N? (theory/simulations)
- How does the number of clusters vary with α? (theory/simulations)



DPMM Exercises

- Code your own DPMM simulator
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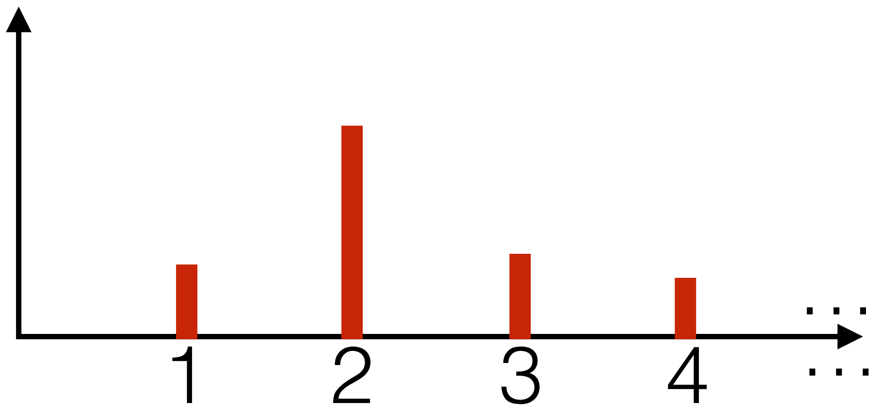
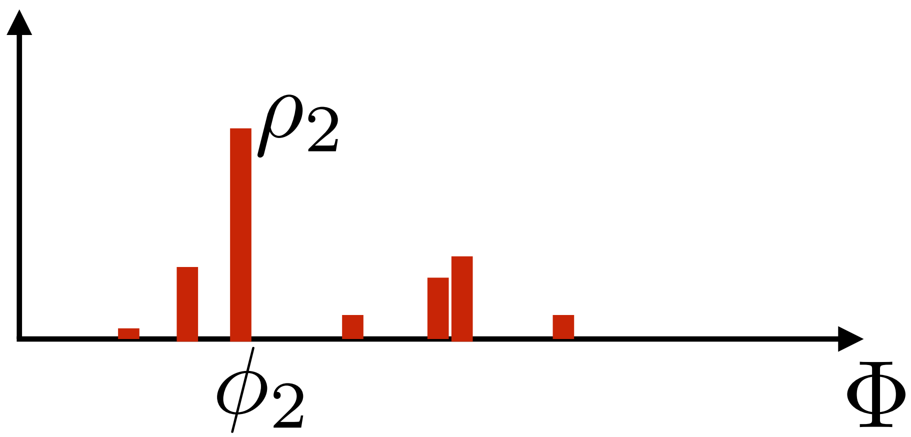
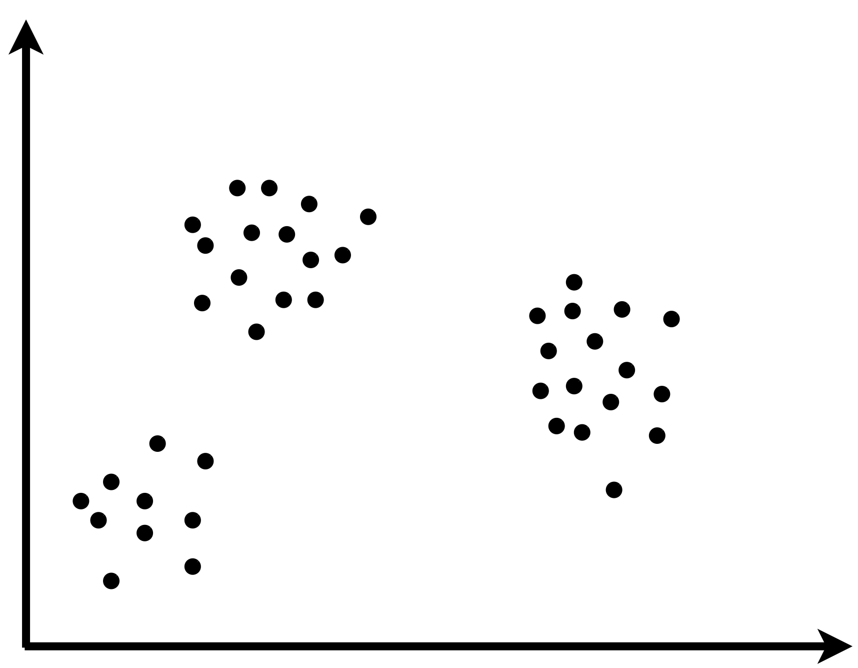
- How does the number of clusters vary with α? (theory/simulations)
- For fixed *N*, what is the distribution over # clusters?







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