Nonparametric Bayesian Statistics: Part II

Tamara Broderick
ITT Career Development Assistant Professor
Electrical Engineering & Computer Science
MIT

Recall: Part I

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- **Dirichlet process (DP) stick-breaking**

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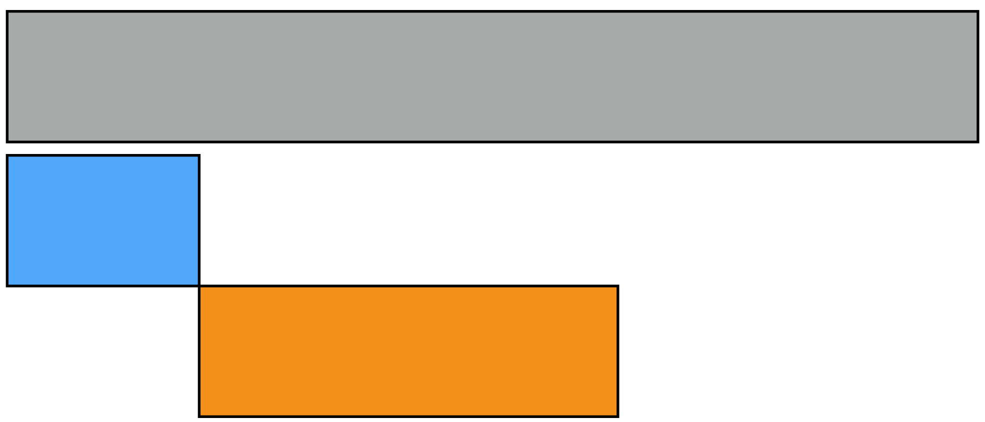

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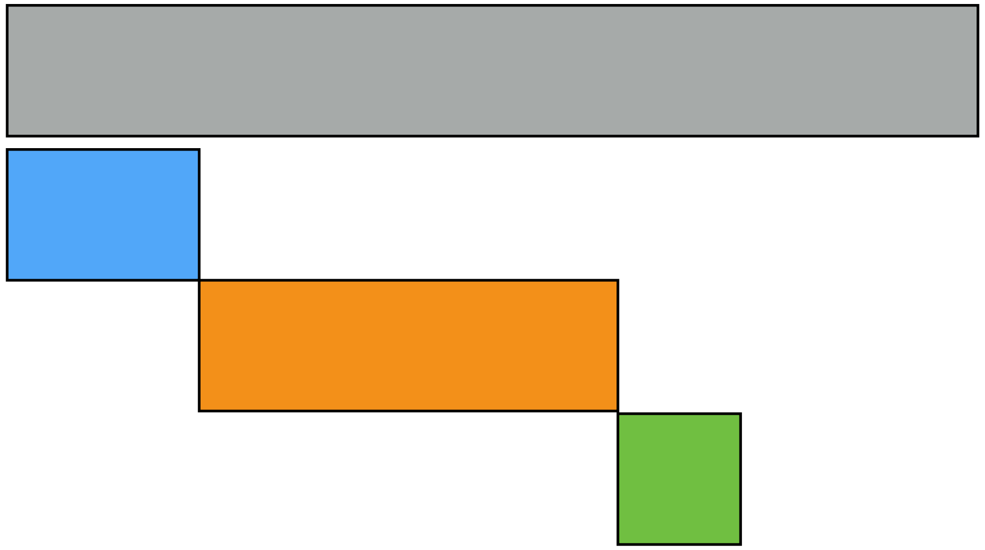

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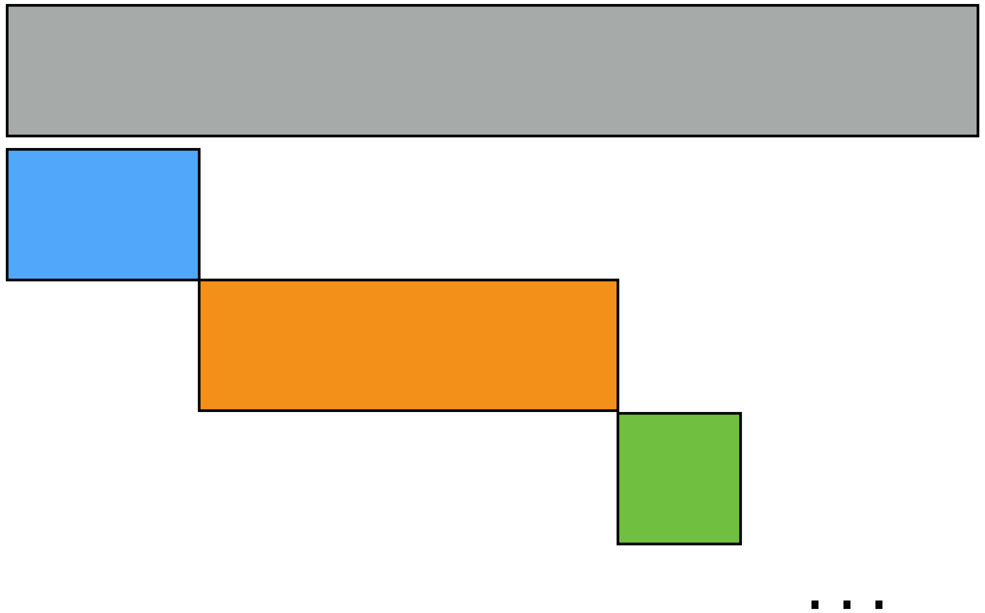

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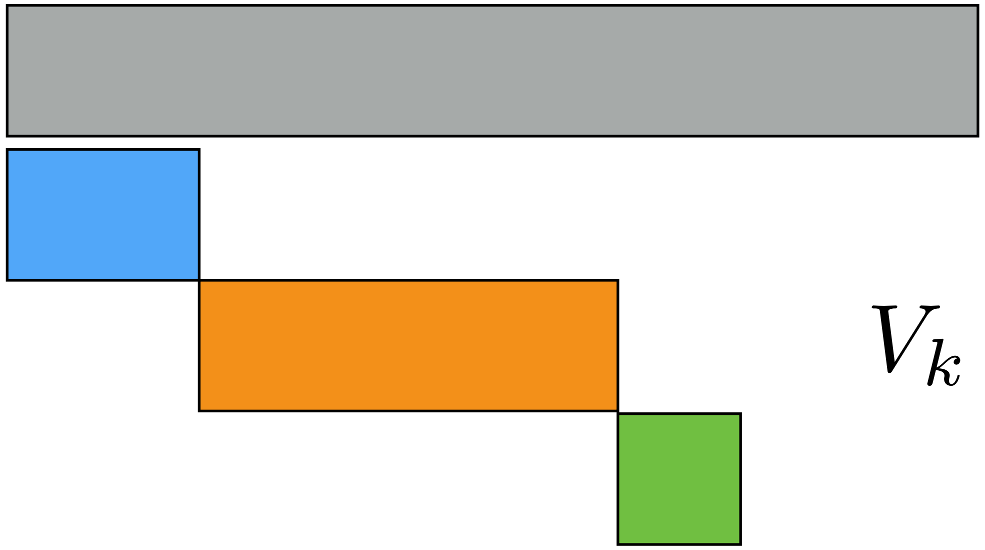

$$V_k \stackrel{iid}{\sim} \mathrm{Beta}(1, \alpha)$$

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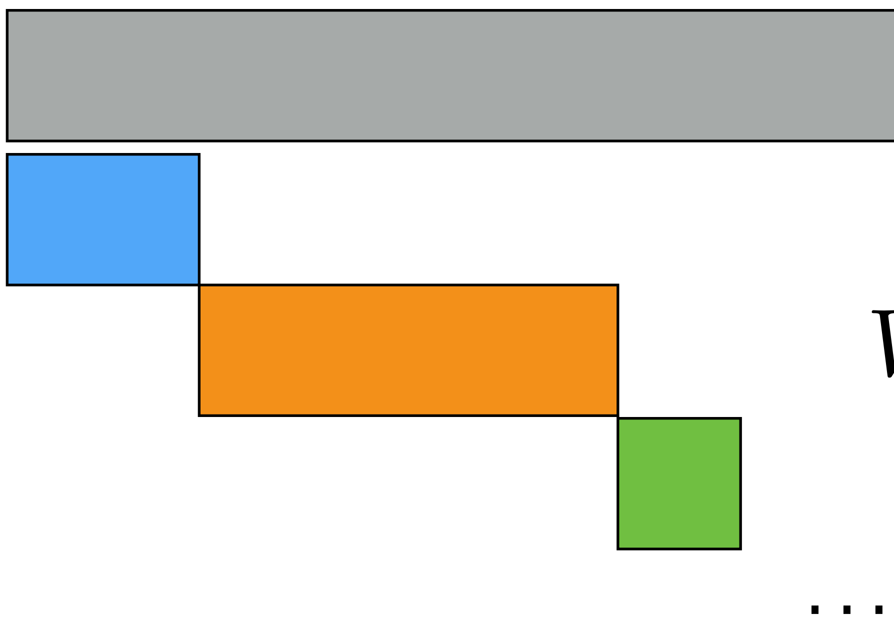

$$V_k \overset{iid}{\sim} \mathrm{Beta}(1, \alpha) \qquad \rho_k = \left[\prod_{j=1}^{k-1} (1 - V_j)\right] V_k$$

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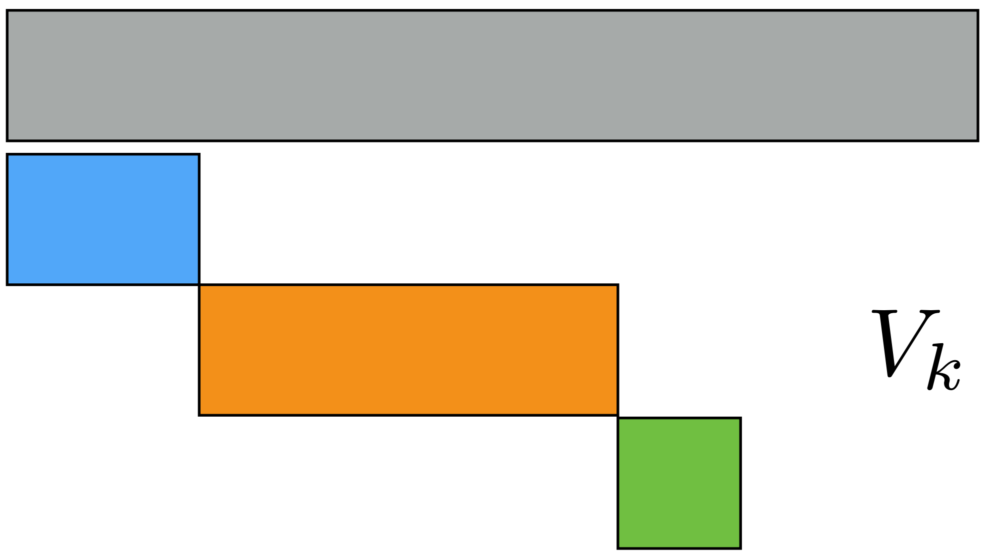

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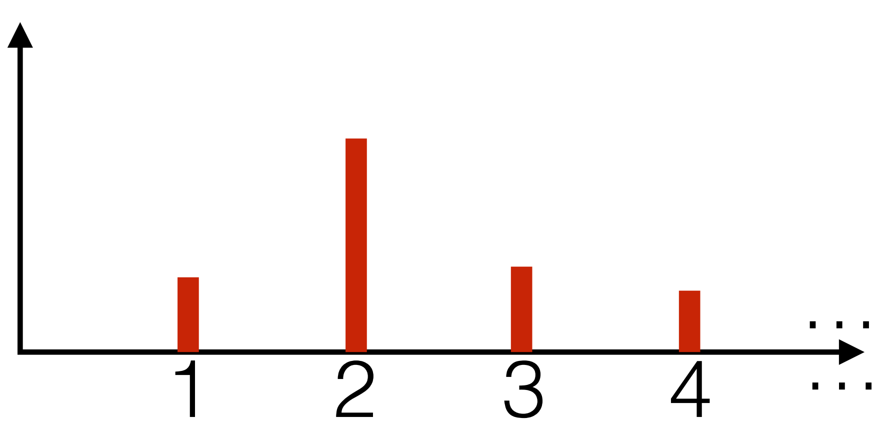

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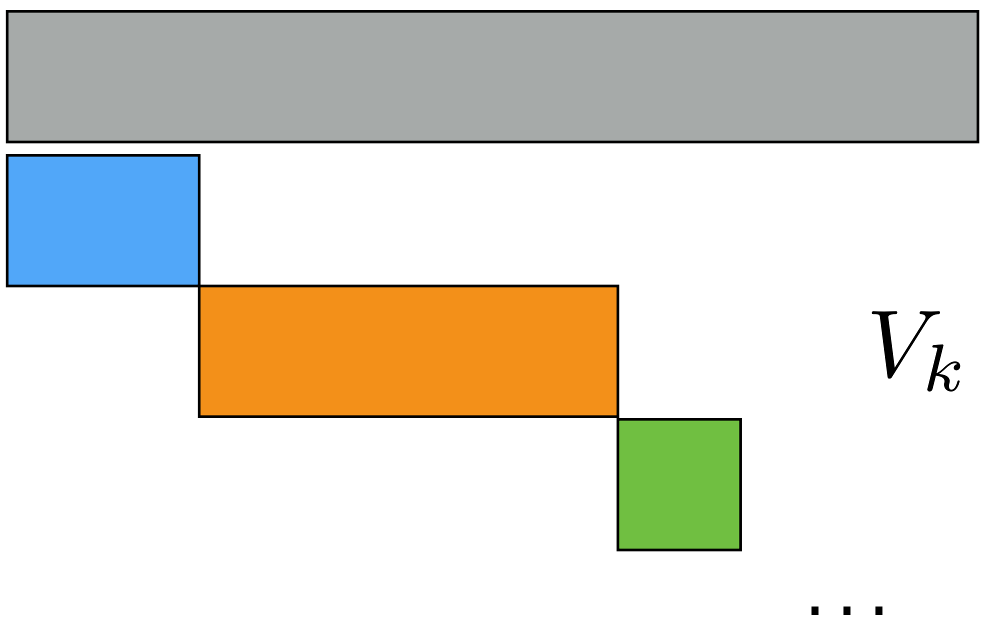

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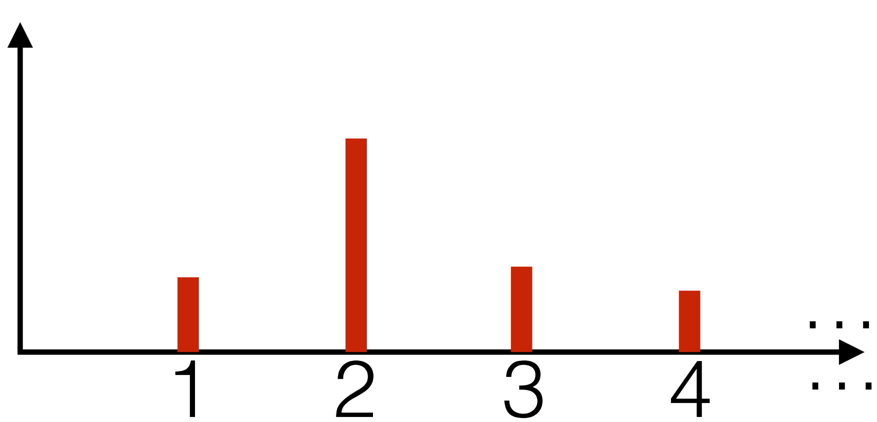

- Part of: DP mixture model

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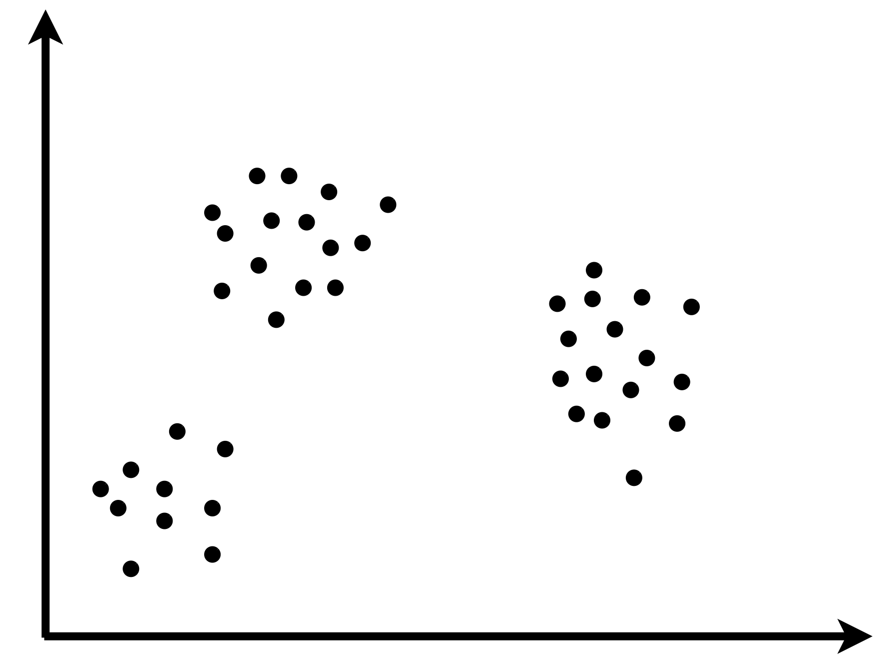
DP or not DP, that is the question



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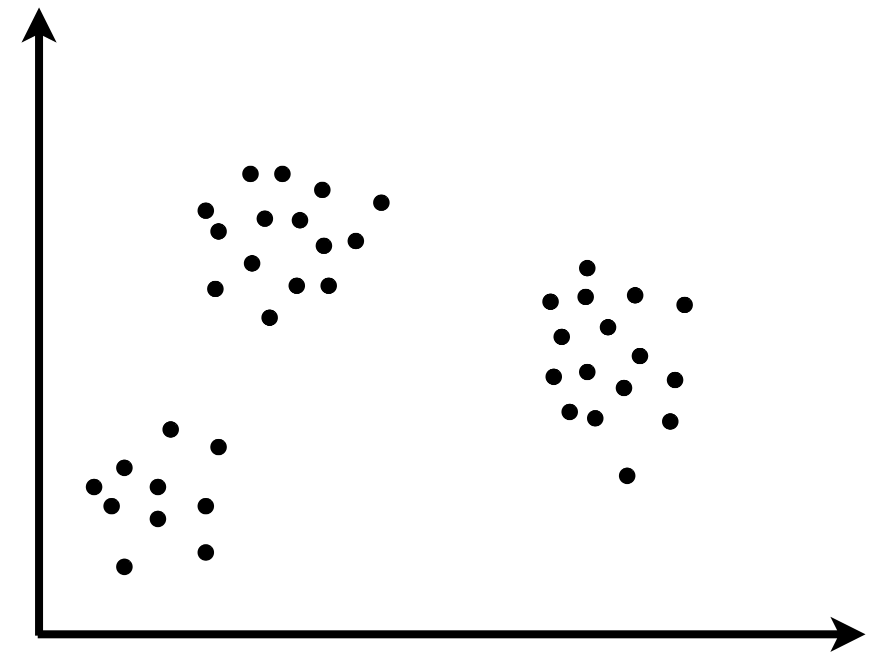

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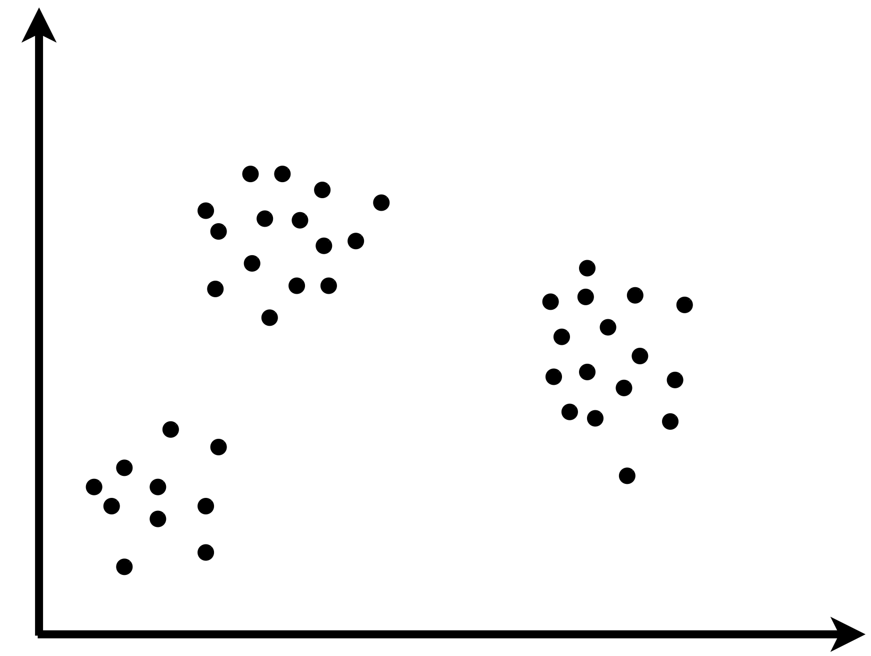
- GEM: 
- Compare to:



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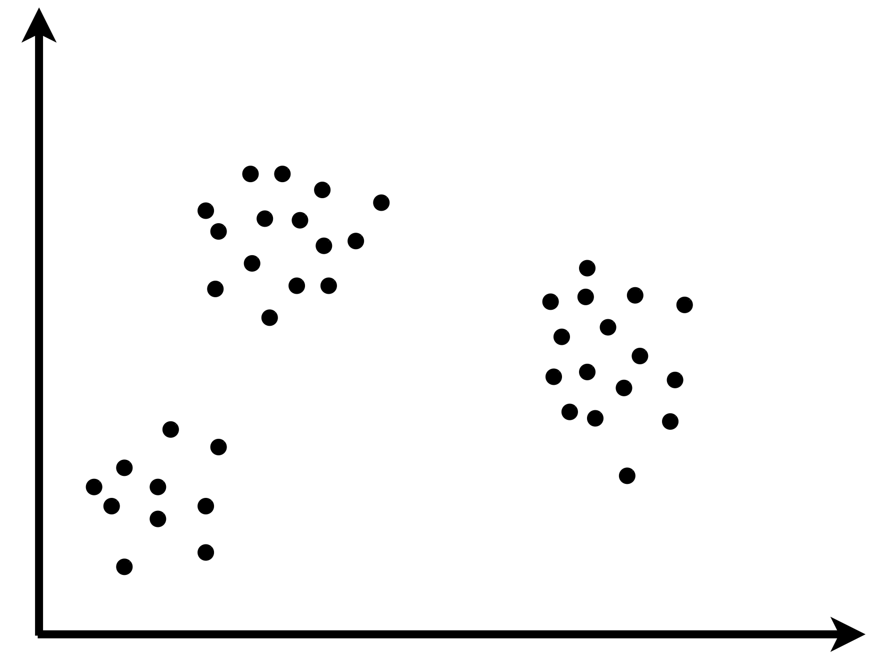
- GEM: 

- Compare to:
  - Finite (small *K*) mixture model



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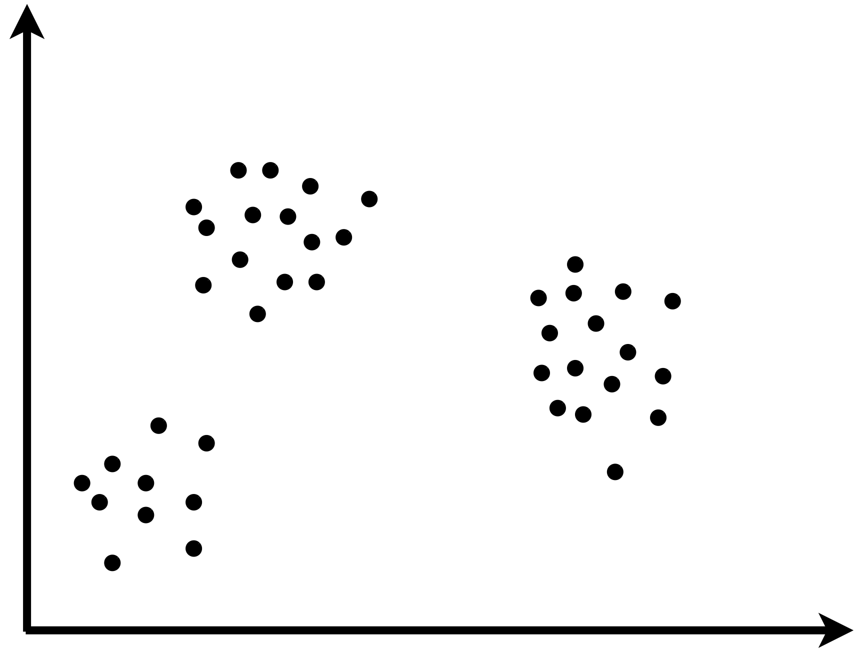

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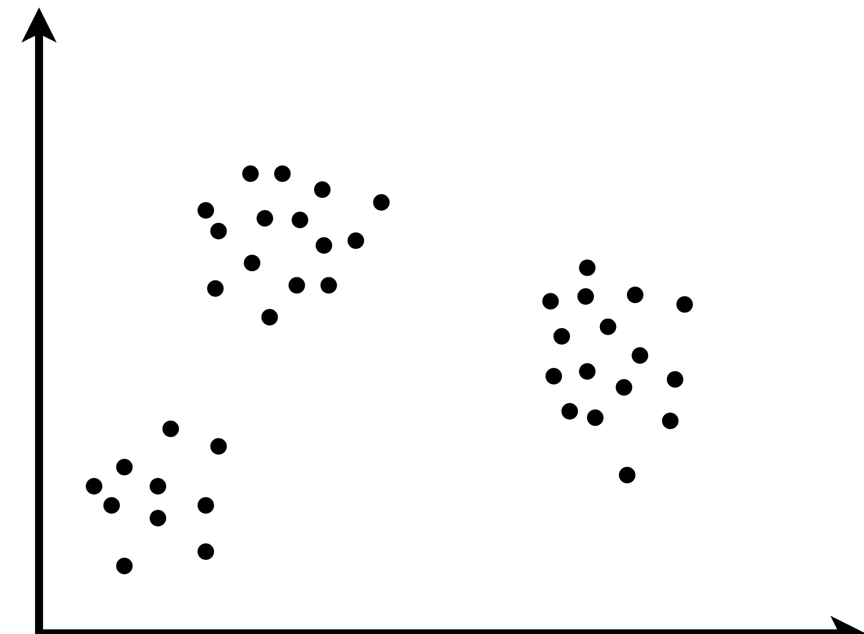
- GEM: 



- Compare to:
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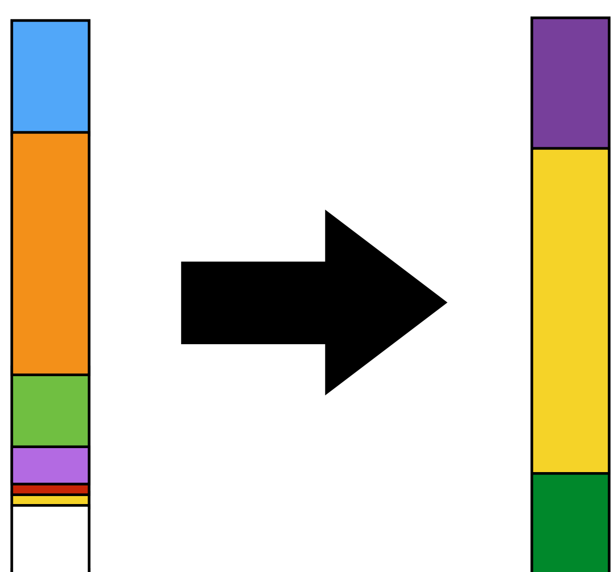
  - Finite (large *K*) mixture model

  - Time series



Nonparametric Bayes: Part II

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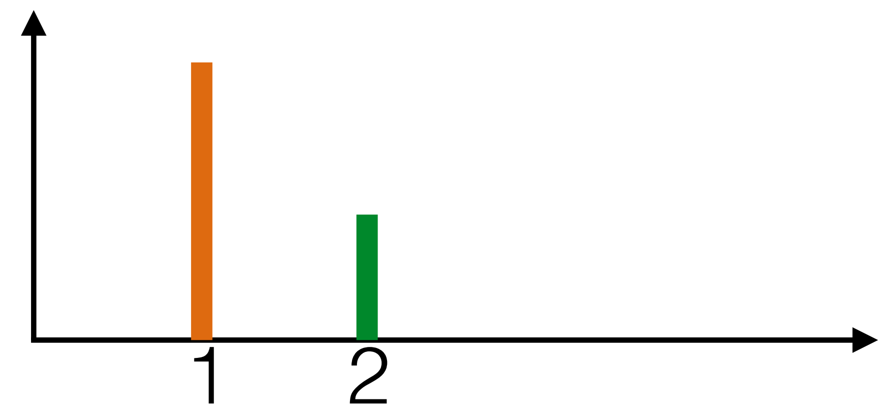
  - Avoid the infinity of parameters for inference
  - e.g. Chinese restaurant process

Marginal cluster assignments

$$\rho_1 \sim \mathrm{Beta}(a_1, a_2), z_n \stackrel{iid}{\sim} \mathrm{Cat}(\rho_1, \rho_2)$$



Marginal cluster assignments

- Integrate out the frequencies

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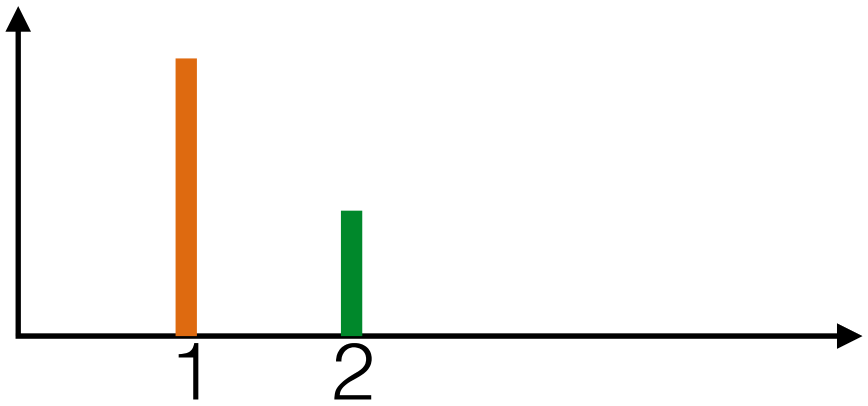

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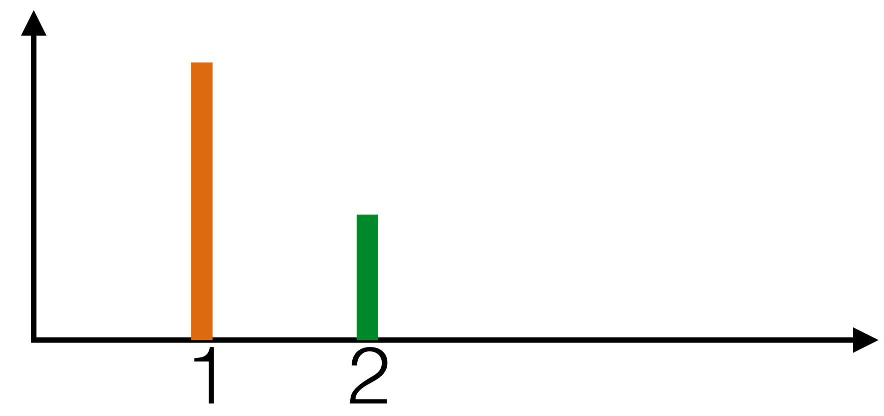

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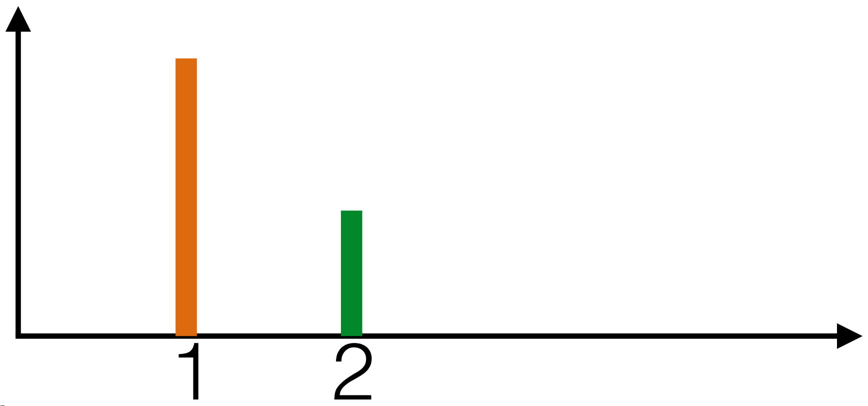

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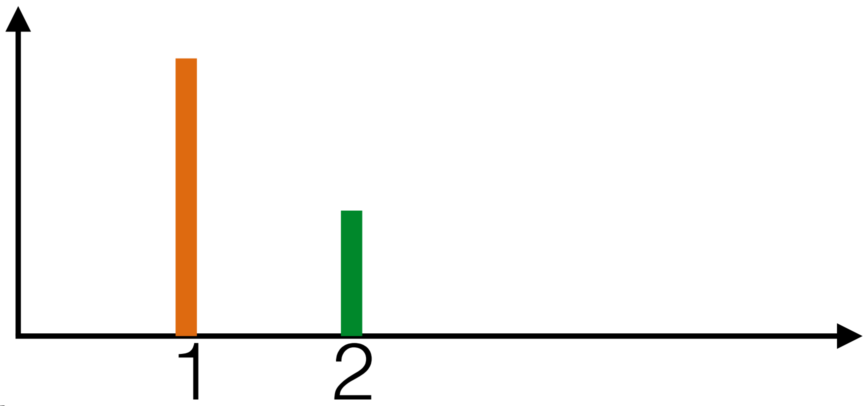

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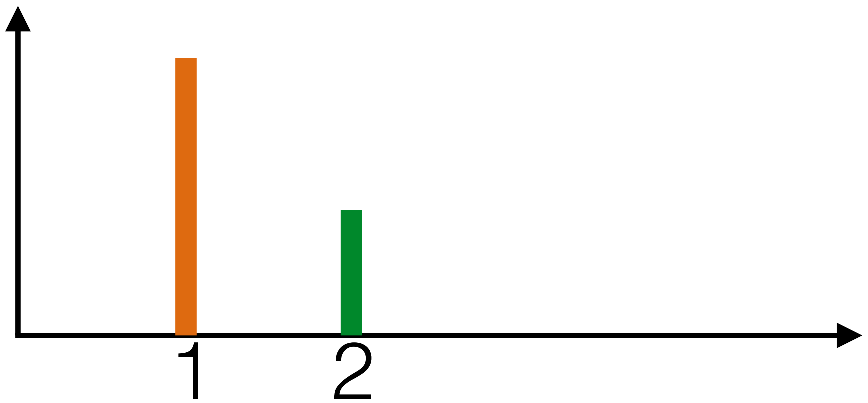

$$p(z_n = 1 | z_1, \ldots, z_{n-1})$$
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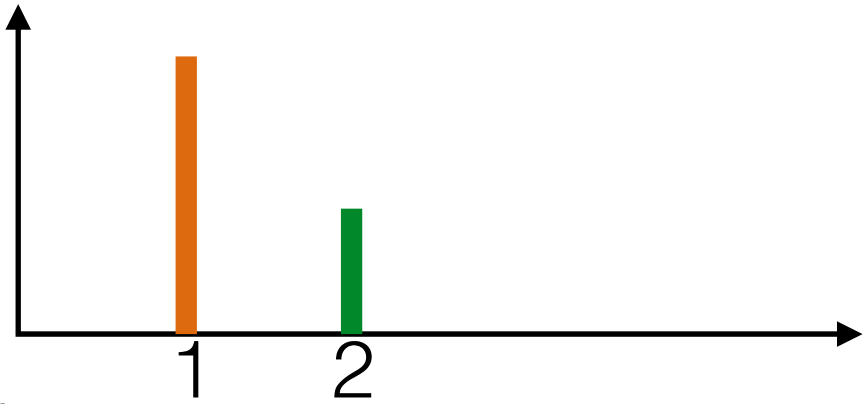

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$$= \int \rho_1 \frac{\Gamma(a_{1,n} + a_{2,n})}{\Gamma(a_{1,n})\Gamma(a_{2,n})} \rho_1^{a_{1,n}-1} (1-\rho_1)^{a_{2,n}-1} d\rho_1$$

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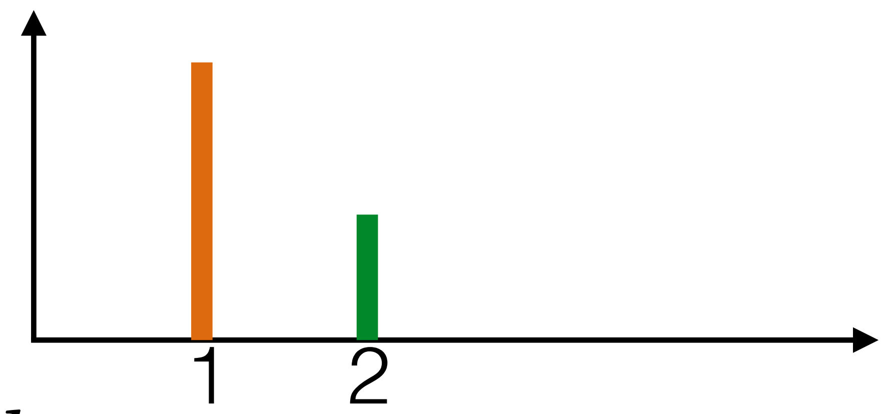

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$$= \frac{\Gamma(a_{1,n} + a_{2,n})}{\Gamma(a_{1,n})\Gamma(a_{2,n})} \frac{\Gamma(a_{1,n}+1)\Gamma(a_{2,n})}{\Gamma(a_{1,n} + a_{2,n} + 1)}$$

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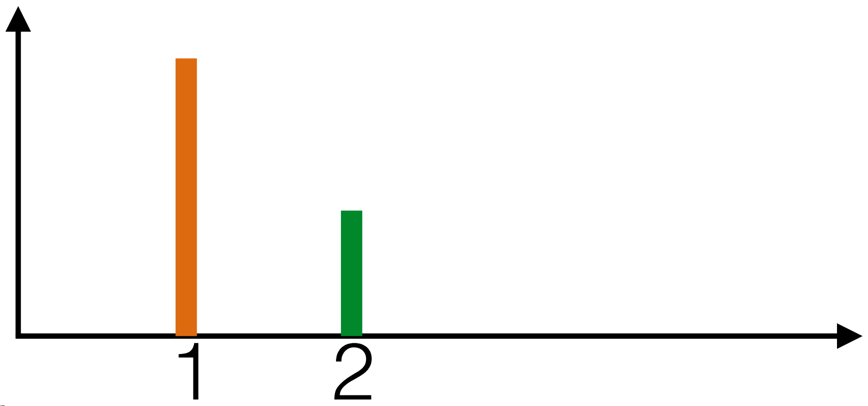

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$$\begin{aligned}
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&= \frac{\Gamma(a_{1,n} + a_{2,n})}{\Gamma(a_{1,n})\Gamma(a_{2,n})} \frac{\Gamma(a_{1,n} + 1)\Gamma(a_{2,n})}{\Gamma(a_{1,n} + a_{2,n} + 1)} \\
&= \frac{a_{1,n}}{a_{1,n} + a_{2,n}}
\end{aligned}$$

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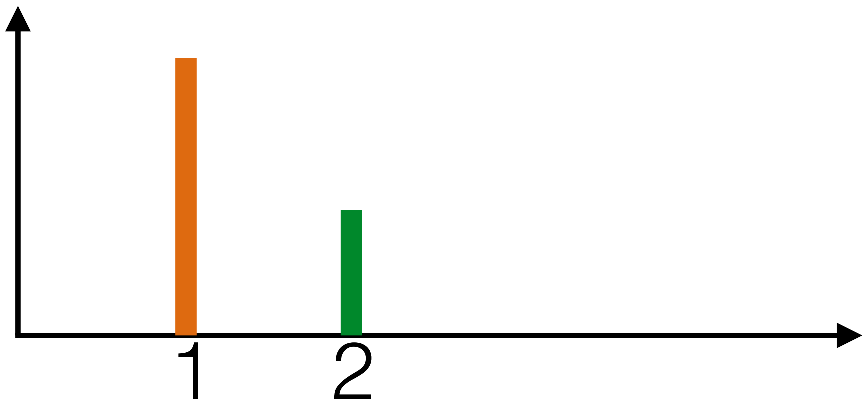

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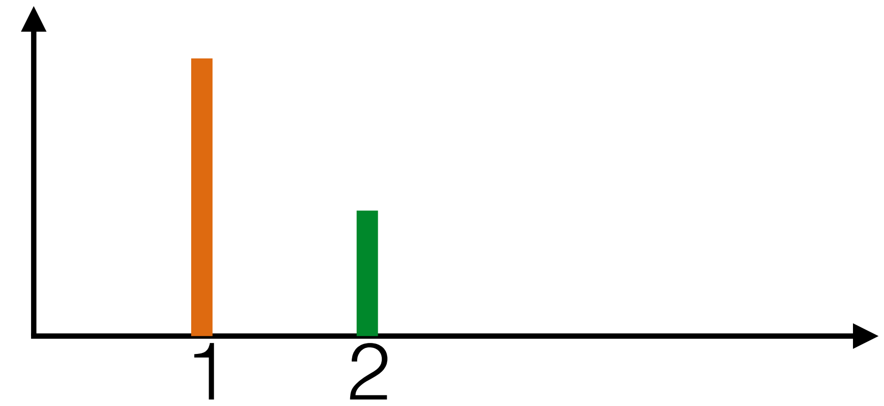

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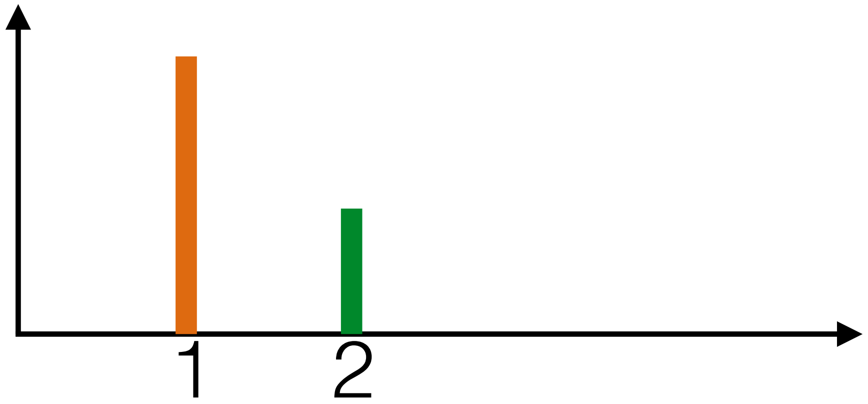

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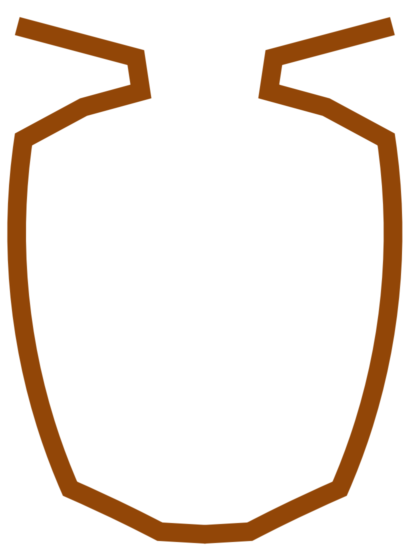

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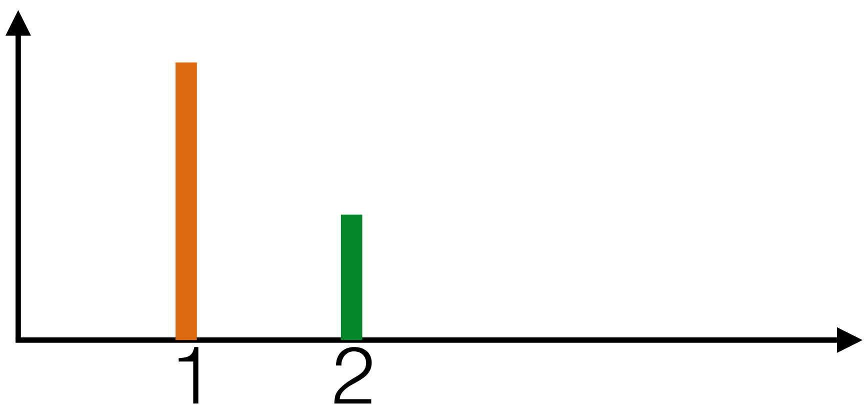

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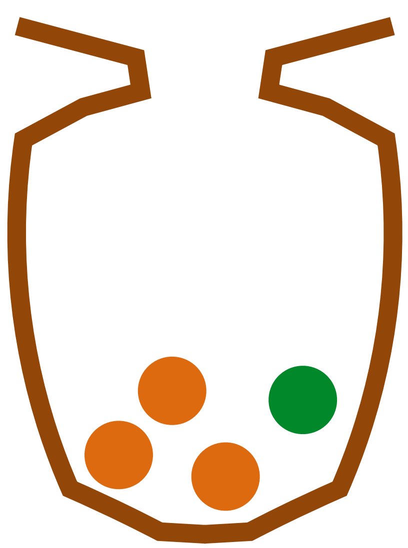

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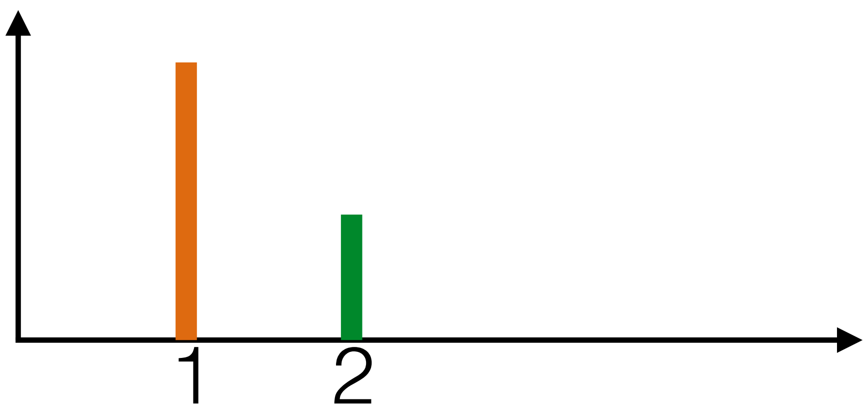

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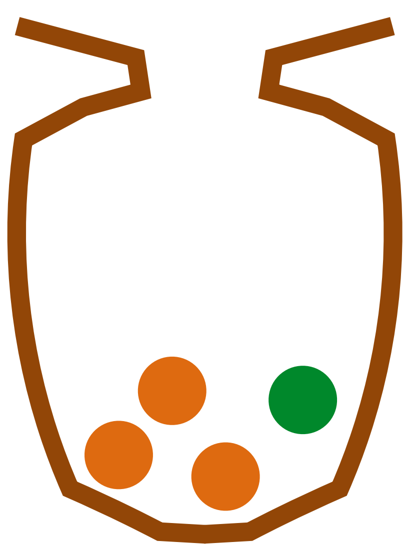
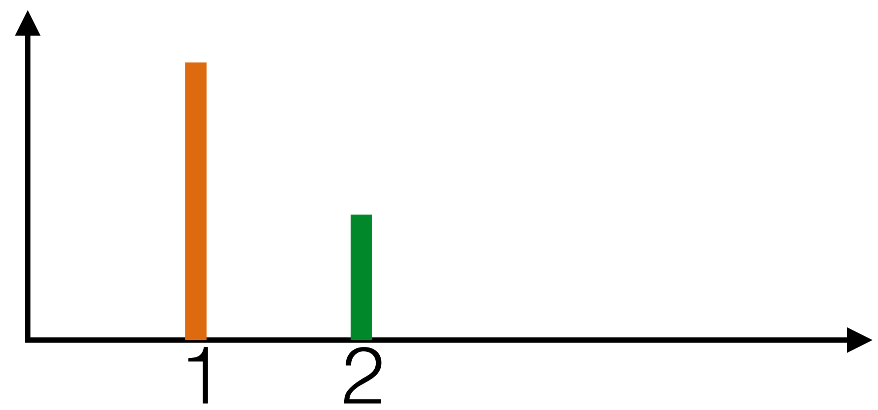
  - Choose any ball with equal probability



Marginal cluster assignments

- Integrate out the frequencies

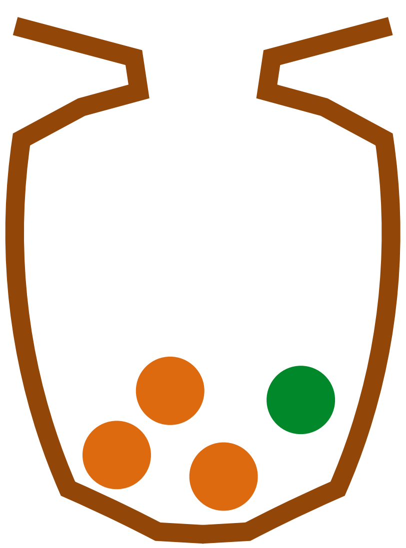


$$\rho_1 \sim \text{Beta}(a_1, a_2), z_n \overset{iid}{\sim} \text{Cat}(\rho_1, \rho_2)$$

$$p(z_n = 1 | z_1, \ldots, z_{n-1}) = \frac{a_{1,n}}{a_{1,n} + a_{2,n}}$$

$$a_{1,n} := a_1 + \sum_{m=1}^{n-1} \mathbf{1}\{z_m = 1\}, a_{2,n} = a_2 + \sum_{m=1}^{n-1} \mathbf{1}\{z_m = 2\}$$

- Pólya urn
  - Choose any ball with equal probability
  - Replace and add ball of same color



Marginal cluster assignments

- Integrate out the frequencies

$$\rho_1 \sim \mathrm{Beta}(a_1, a_2), z_n \stackrel{iid}{\sim} \mathrm{Cat}(\rho_1, \rho_2)$$

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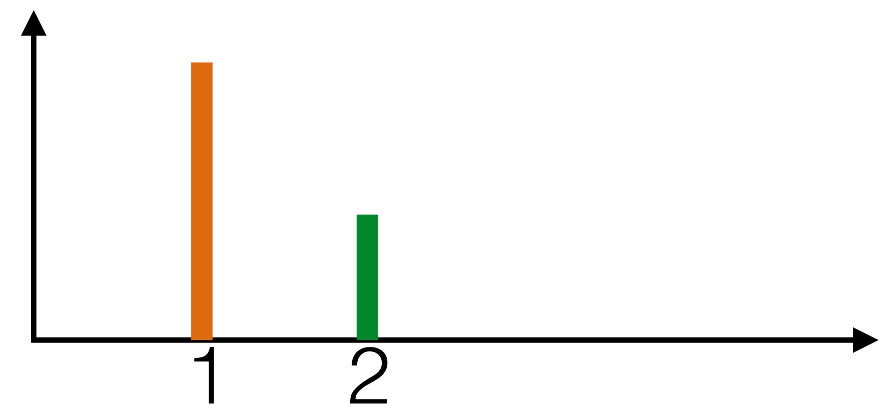

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- Pólya urn
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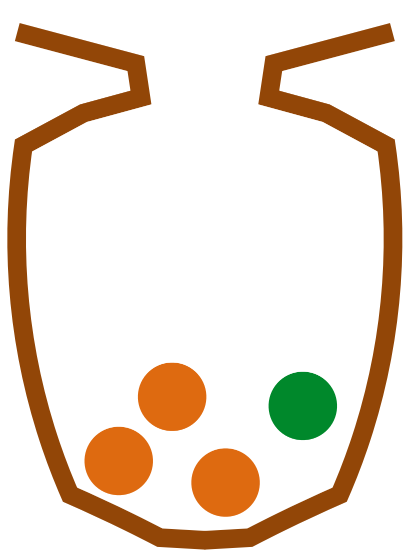


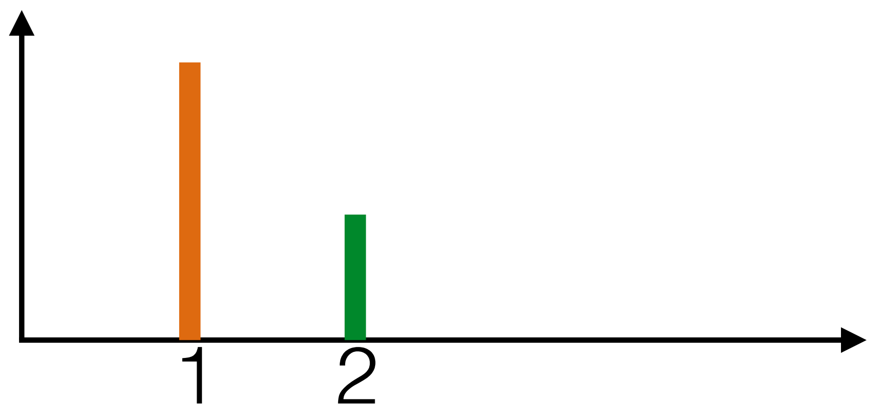
$$\lim_{n \to \infty} \frac{\#\ \text{orange}}{\#\ \text{total}}$$

Marginal cluster assignments

- Integrate out the frequencies

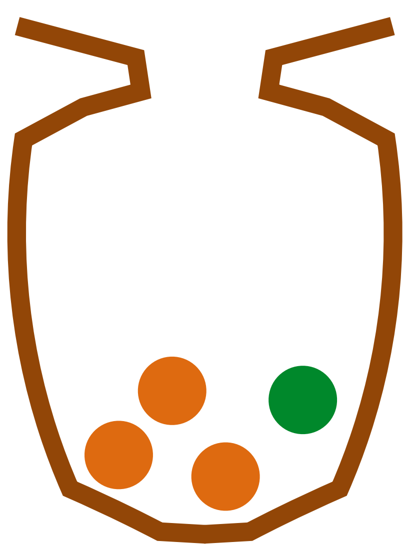
$$\rho_1 \sim \mathrm{Beta}(a_1, a_2),\, z_n \overset{iid}{\sim} \mathrm{Cat}(\rho_1, \rho_2)$$

$$p(z_n = 1 | z_1, \ldots, z_{n-1}) = \frac{a_{1,n}}{a_{1,n} + a_{2,n}}$$



$$a_{1,n} := a_1 + \sum_{m=1}^{n-1} \mathbf{1}\{z_m = 1\},\, a_{2,n} = a_2 + \sum_{m=1}^{n-1} \mathbf{1}\{z_m = 2\}$$

- Pólya urn
  - Choose any ball with equal probability
  - Replace and add ball of same color



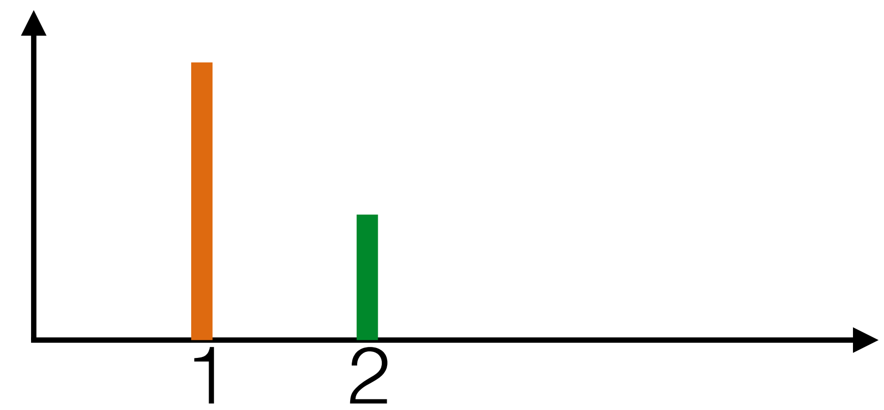
$$\lim_{n \to \infty} \frac{\#\ \text{orange}}{\#\ \text{total}} = \rho_{\text{orange}}$$

Marginal cluster assignments

- Integrate out the frequencies

$$\rho_1 \sim \mathrm{Beta}(a_1, a_2),\, z_n \overset{iid}{\sim} \mathrm{Cat}(\rho_1, \rho_2)$$

$$p(z_n = 1 | z_1, \ldots, z_{n-1}) = \frac{a_{1,n}}{a_{1,n} + a_{2,n}}$$



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- Pólya urn
  - Choose any ball with equal probability
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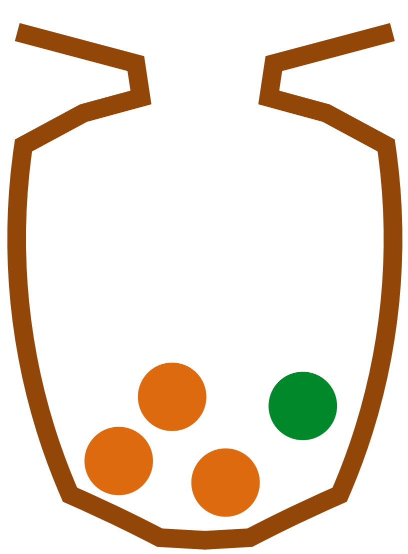

$$\lim_{n \to \infty} \frac{\#\ \mathrm{orange}}{\#\ \mathrm{total}} = \rho_{\mathrm{orange}} \overset{d}{=} \mathrm{Beta}(a_{\mathrm{orange}}, a_{\mathrm{green}})$$

Marginal cluster assignments

- Integrate out the frequencies

$$\rho_1 \sim \text{Beta}(a_1, a_2),\, z_n \stackrel{iid}{\sim} \text{Cat}(\rho_1, \rho_2)$$

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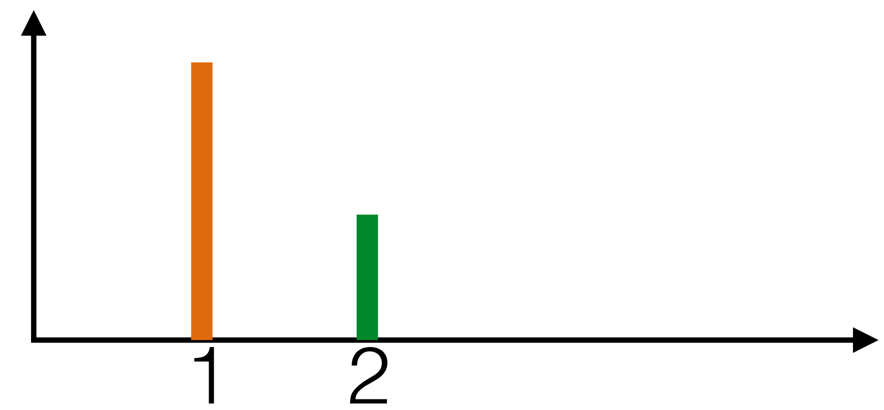

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  - Choose any ball with equal probability
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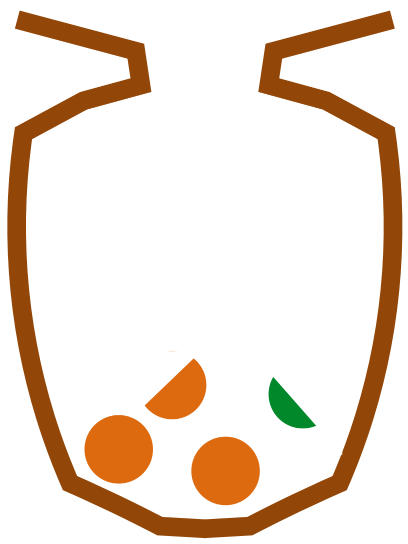
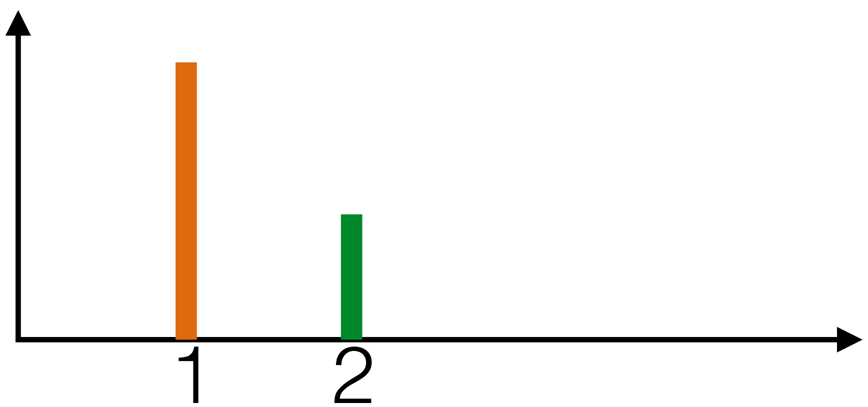


$$\lim_{n\to\infty} \frac{\#\text{ orange}}{\#\text{ total}} = \rho_{\text{orange}} \stackrel{d}{=} \text{Beta}(a_{\text{orange}}, a_{\text{green}})$$

Marginal cluster assignments

- Integrate out the frequencies



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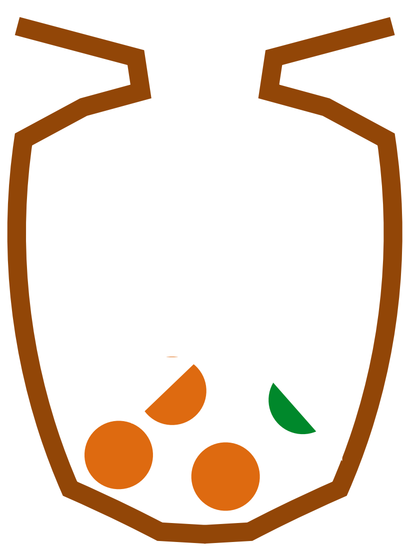
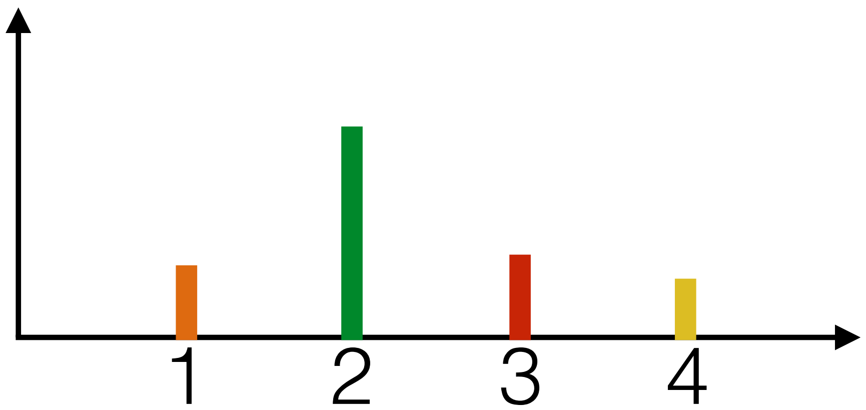
  - Choose any ball with prob proportional to its mass
  - Replace and add ball of same color



$$\lim_{n\to\infty} \frac{\#\ \text{orange}}{\#\ \text{total}} = \rho_{\text{orange}} \overset{d}{=} \mathrm{Beta}(a_{\text{orange}}, a_{\text{green}})$$

Marginal cluster assignments

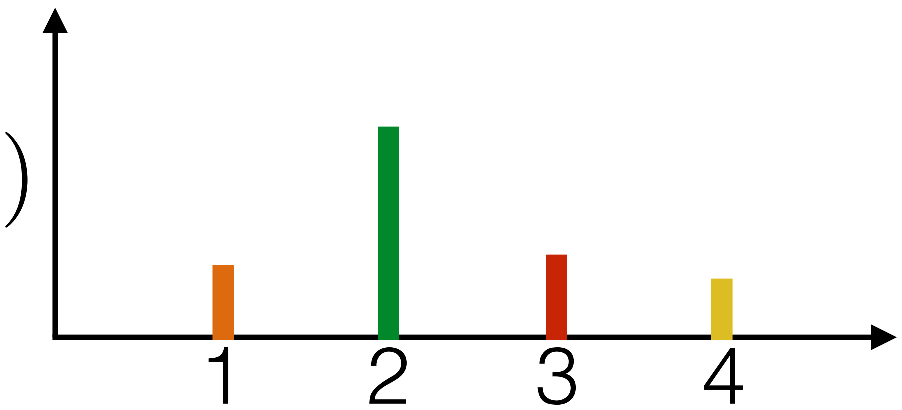
- Integrate out the frequencies



Marginal cluster assignments

- Integrate out the frequencies

$\rho_{1:K} \sim \mathrm{Dirichlet}(a_{1:K}), z_n \overset{iid}{\sim} \mathrm{Cat}(\rho_{1:K})$

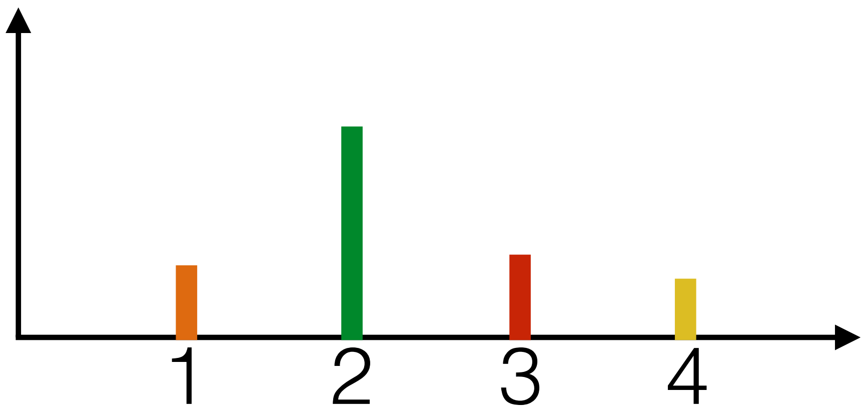


Marginal cluster assignments

- Integrate out the frequencies

$$\rho_{1:K} \sim \mathrm{Dirichlet}(a_{1:K}), z_n \overset{iid}{\sim} \mathrm{Cat}(\rho_{1:K})$$

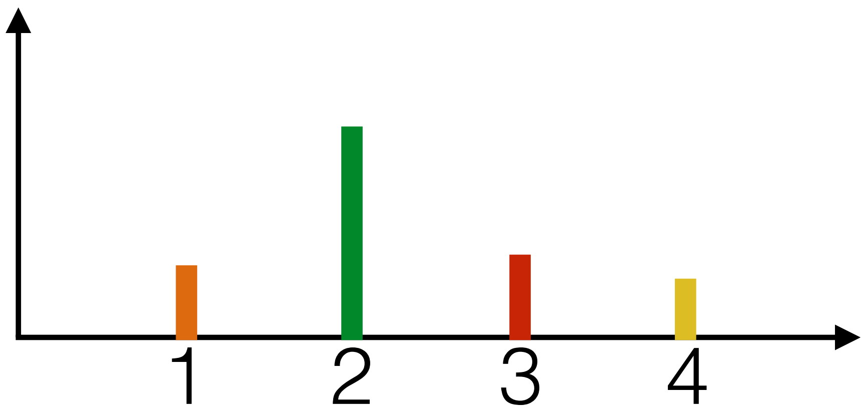
$$p(z_n = k | z_1, \ldots, z_{n-1}) = \frac{a_{k,n}}{\sum_{j=1}^{K} a_{j,n}}$$



Marginal cluster assignments

- Integrate out the frequencies

$$\rho_{1:K} \sim \mathrm{Dirichlet}(a_{1:K}), z_n \overset{iid}{\sim} \mathrm{Cat}(\rho_{1:K})$$



$$p(z_n = k | z_1, \ldots, z_{n-1}) = \frac{a_{k,n}}{\sum_{j=1}^{K} a_{j,n}}$$

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Marginal cluster assignments

- Integrate out the frequencies

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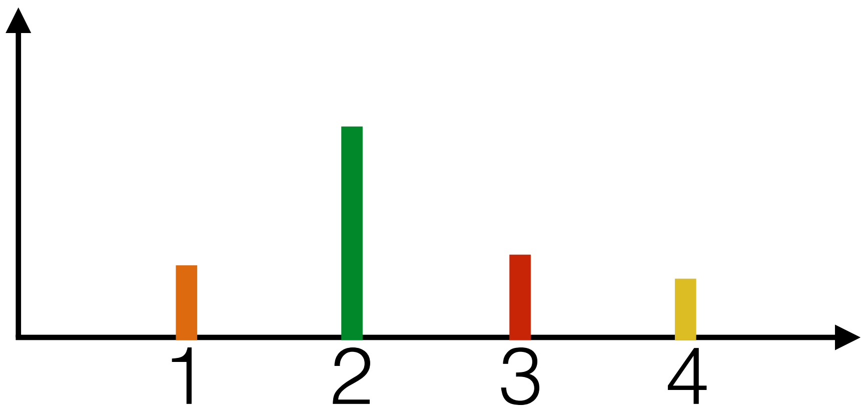

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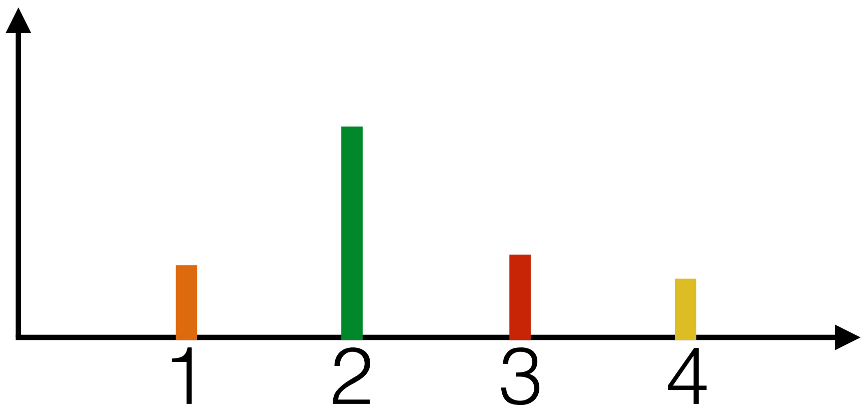
- multivariate Pólya urn

Marginal cluster assignments

- Integrate out the frequencies

$$\rho_{1:K} \sim \text{Dirichlet}(a_{1:K}), z_n \overset{iid}{\sim} \text{Cat}(\rho_{1:K})$$



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- Integrate out the frequencies

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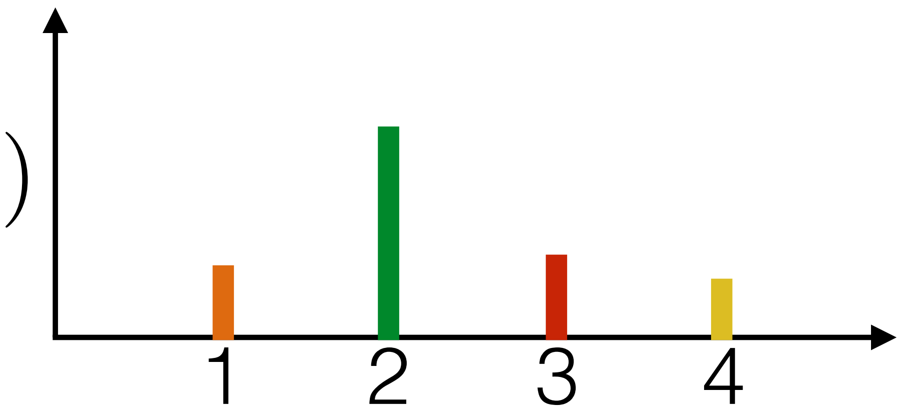

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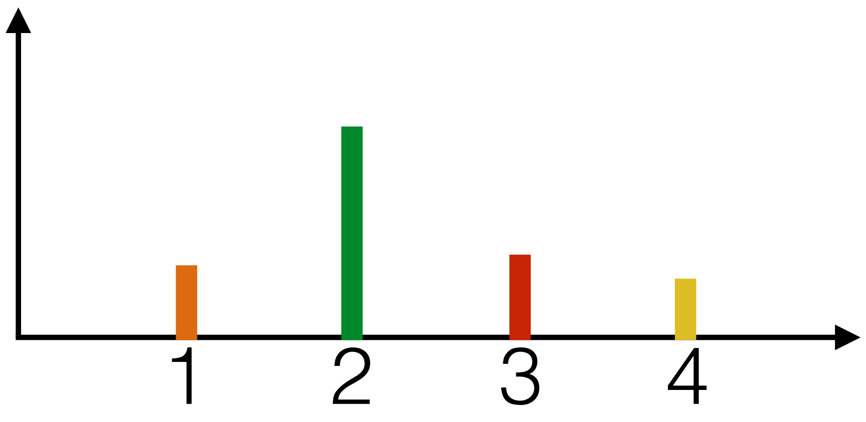
- multivariate Pólya urn
  - Choose any ball with prob proportional to its mass

Marginal cluster assignments

- Integrate out the frequencies

$$\rho_{1:K} \sim \text{Dirichlet}(a_{1:K}),\, z_n \stackrel{iid}{\sim} \text{Cat}(\rho_{1:K})$$



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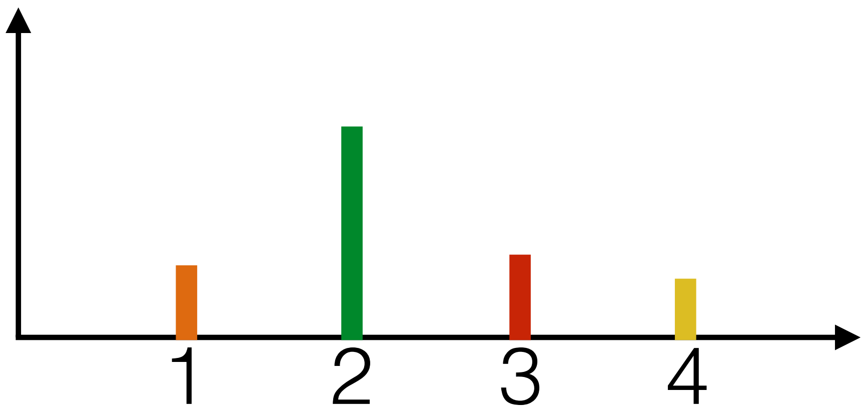
- multivariate Pólya urn
  - Choose any ball with prob proportional to its mass
  - Replace and add ball of same color

Marginal cluster assignments

- Integrate out the frequencies

$$\rho_{1:K} \sim \text{Dirichlet}(a_{1:K}),\ z_n \overset{iid}{\sim} \text{Cat}(\rho_{1:K})$$



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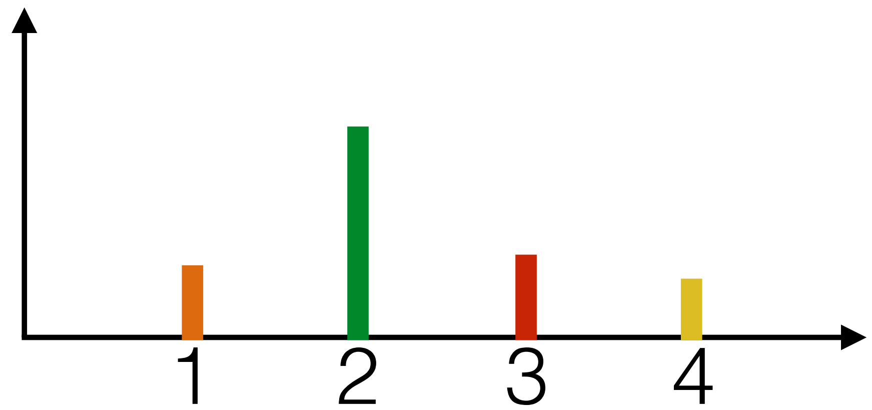
- multivariate Pólya urn
  - Choose any ball with prob proportional to its mass
  - Replace and add ball of same color

$$\lim_{n \to \infty} \frac{(\#\ \text{orange},\ \#\ \text{green},\ \#\ \text{red},\ \#\ \text{yellow})}{\#\ \text{total}}$$

Marginal cluster assignments

- Integrate out the frequencies

$$\rho_{1:K} \sim \text{Dirichlet}(a_{1:K}),\ z_n \overset{iid}{\sim} \text{Cat}(\rho_{1:K})$$



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- multivariate Pólya urn
  - Choose any ball with prob proportional to its mass
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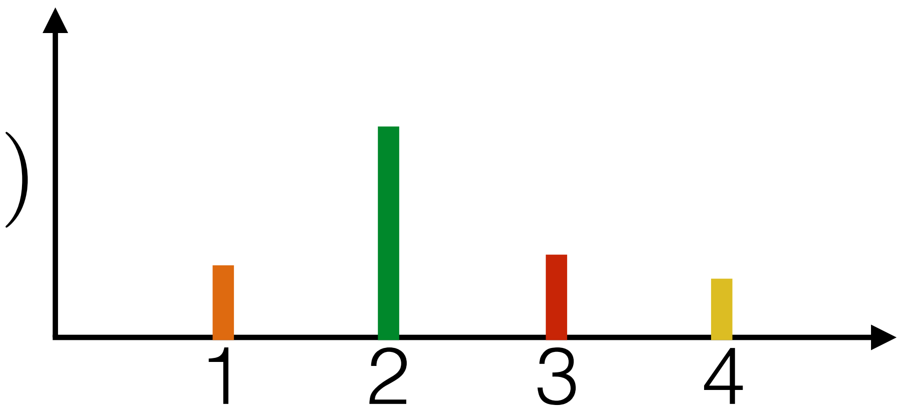
$$\lim_{n \to \infty} \frac{(\#\ \text{orange},\ \#\ \text{green},\ \#\ \text{red},\ \#\ \text{yellow})}{\#\ \text{total}}$$

$$\to (\rho_{\text{orange}}, \rho_{\text{green}}, \rho_{\text{red}}, \rho_{\text{yellow}})$$

Marginal cluster assignments

- Integrate out the frequencies



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- multivariate Pólya urn
  - Choose any ball with prob proportional to its mass
  - Replace and add ball of same color

$$\lim_{n \to \infty} \frac{(\#\text{ orange}, \#\text{ green}, \#\text{ red}, \#\text{ yellow})}{\#\text{ total}}$$

$$\to (\rho_{\text{orange}}, \rho_{\text{green}}, \rho_{\text{red}}, \rho_{\text{yellow}})$$

$$\overset{d}{=} \text{Dirichlet}(a_{\text{orange}}, a_{\text{green}}, a_{\text{red}}, a_{\text{yellow}})$$

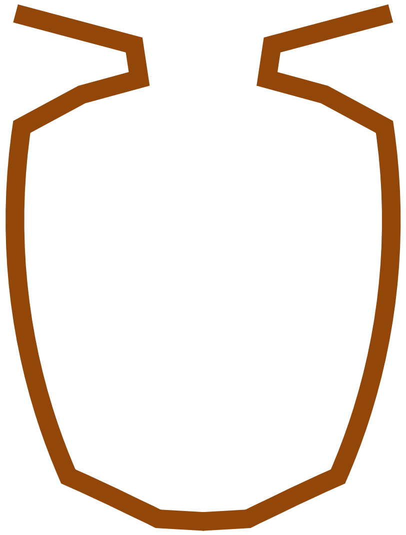
Marginal cluster assignments

- Hoppe urn / Blackwell-MacQueen urn

[Blackwell, MacQueen 1973; Hoppe 1984]

Marginal cluster assignments

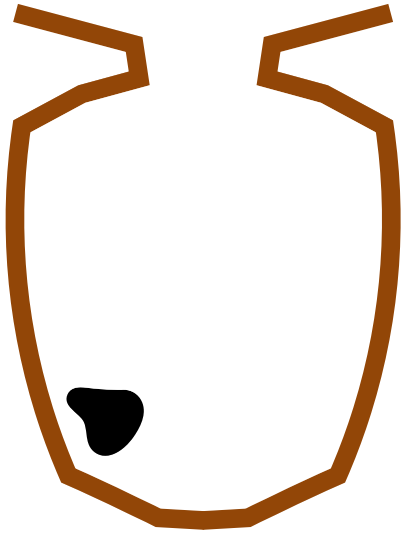
- Hoppe urn / Blackwell-MacQueen urn



[Blackwell, MacQueen 1973; Hoppe 1984]

Marginal cluster assignments

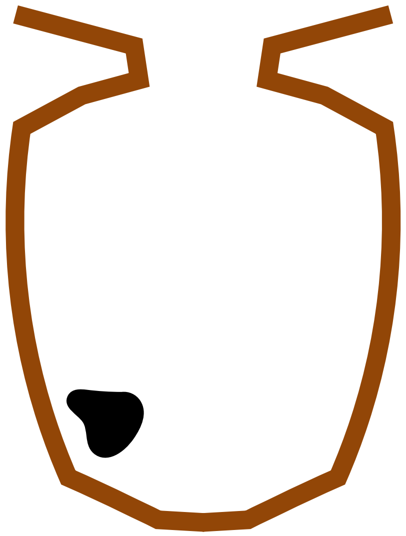
- Hoppe urn / Blackwell-MacQueen urn



[Blackwell, MacQueen 1973; Hoppe 1984]

Marginal cluster assignments

- Hoppe urn / Blackwell-MacQueen urn

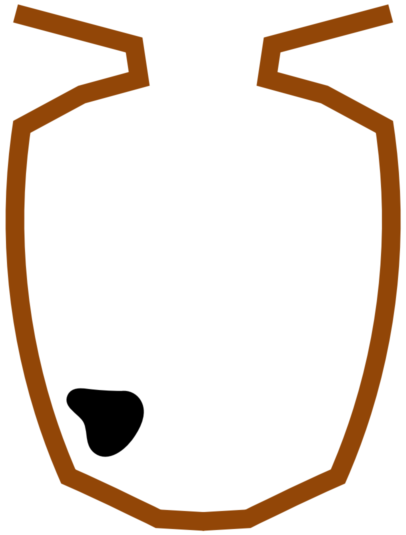


  - Choose ball with prob proportional to its mass

[Blackwell, MacQueen 1973; Hoppe 1984]

Marginal cluster assignments

- Hoppe urn / Blackwell-MacQueen urn

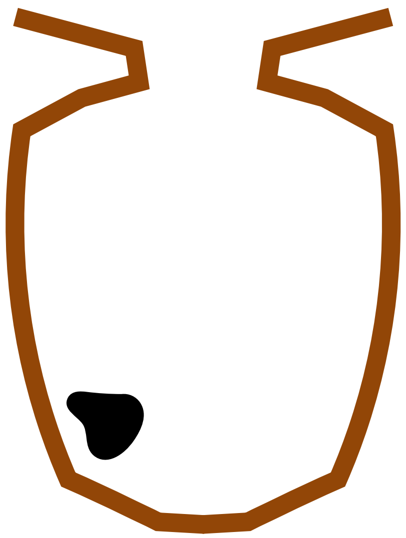


- Choose ball with prob proportional to its mass
  - If black, replace and add ball of new color

[Blackwell, MacQueen 1973; Hoppe 1984]

Marginal cluster assignments

- Hoppe urn / Blackwell-MacQueen urn



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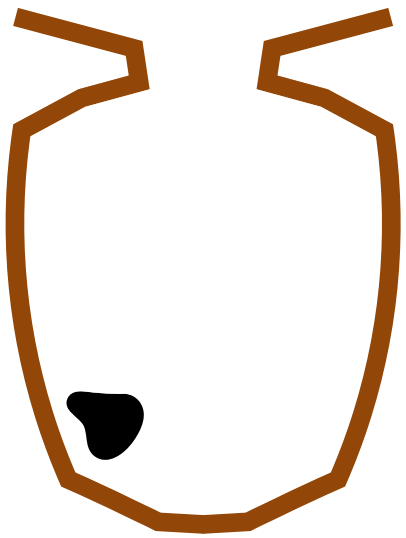
[Blackwell, MacQueen 1973; Hoppe 1984]

Marginal cluster assignments

- Hoppe urn / Blackwell-MacQueen urn



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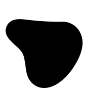
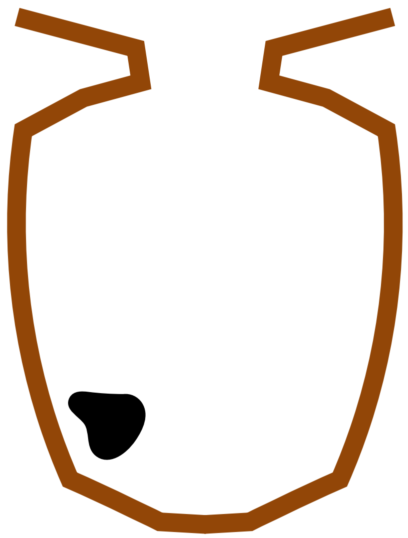
Step 0



[Blackwell, MacQueen 1973; Hoppe 1984]

Marginal cluster assignments

- Hoppe urn / Blackwell-MacQueen urn



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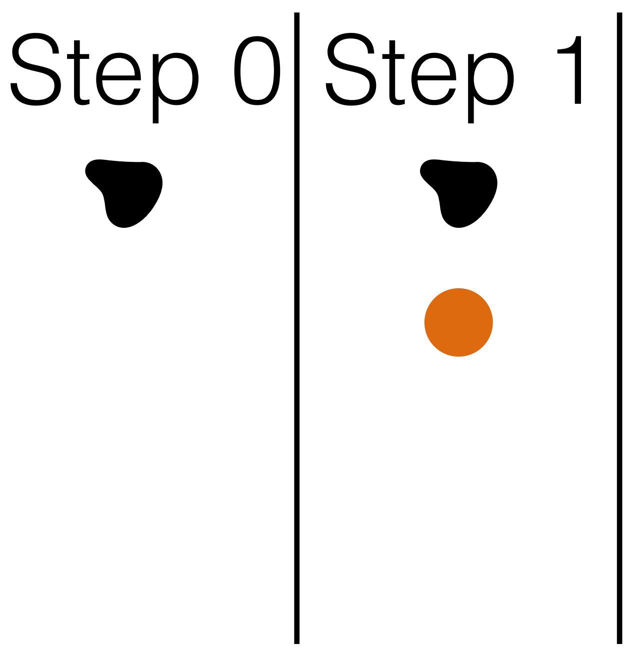
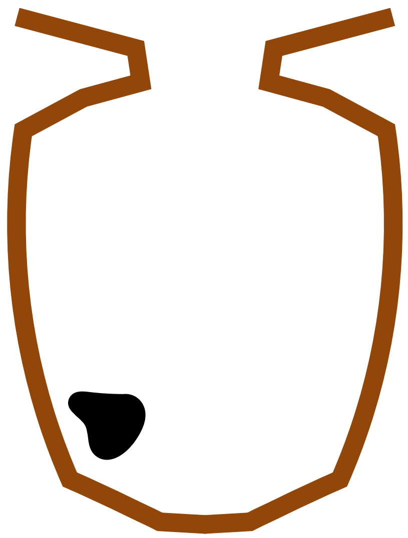


[Blackwell, MacQueen 1973; Hoppe 1984]

Marginal cluster assignments

- Hoppe urn / Blackwell-MacQueen urn



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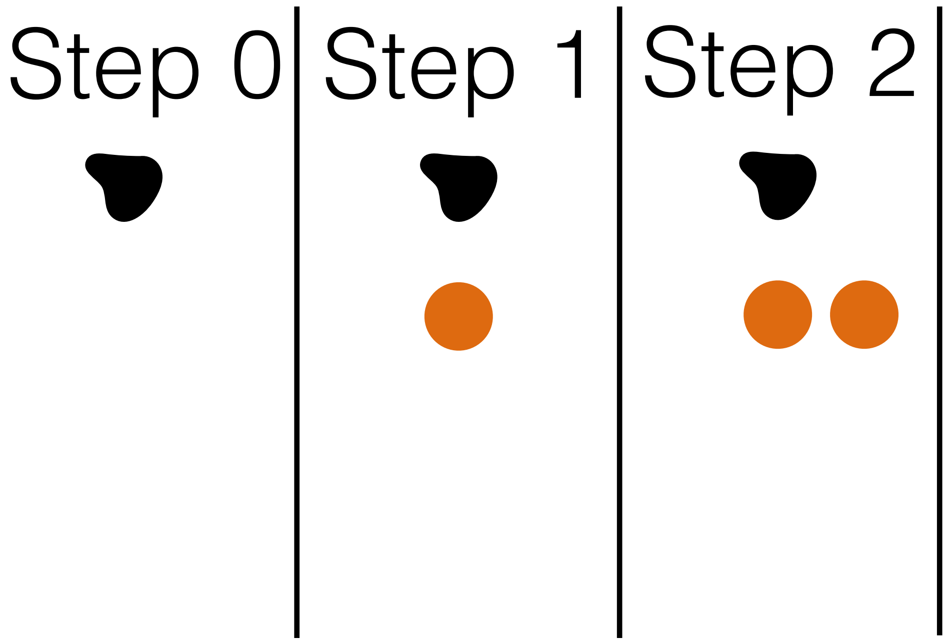
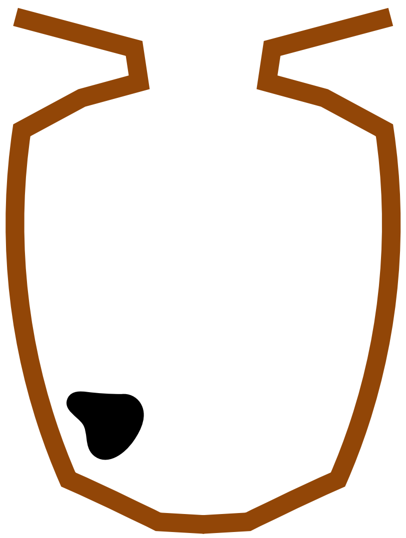


[Blackwell, MacQueen 1973; Hoppe 1984]

Marginal cluster assignments

- Hoppe urn / Blackwell-MacQueen urn



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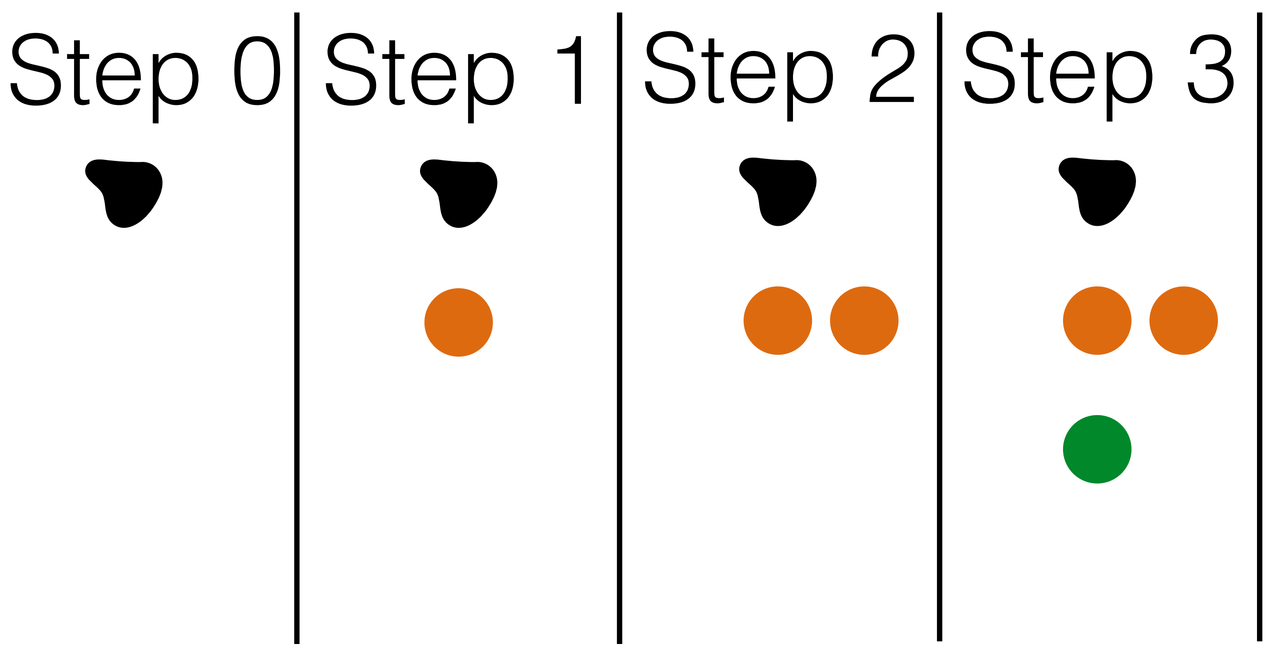

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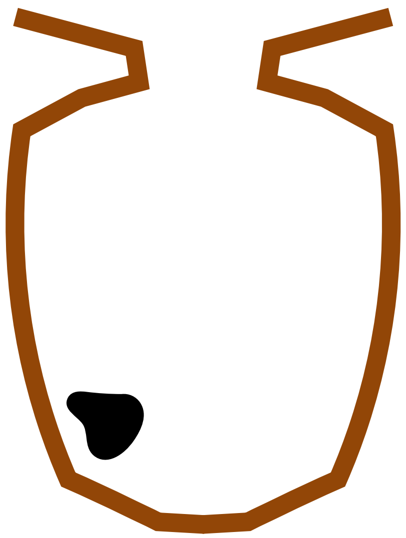
Marginal cluster assignments

- Hoppe urn / Blackwell-MacQueen urn



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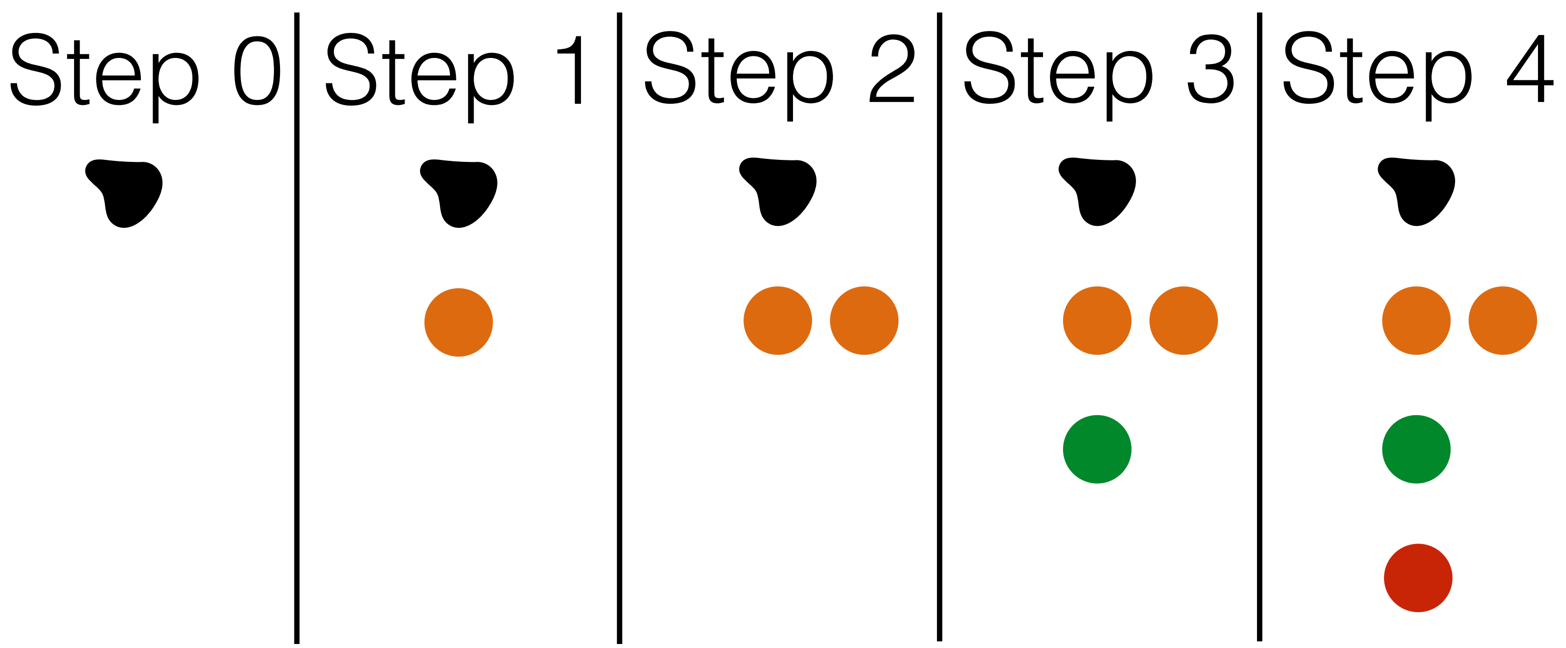

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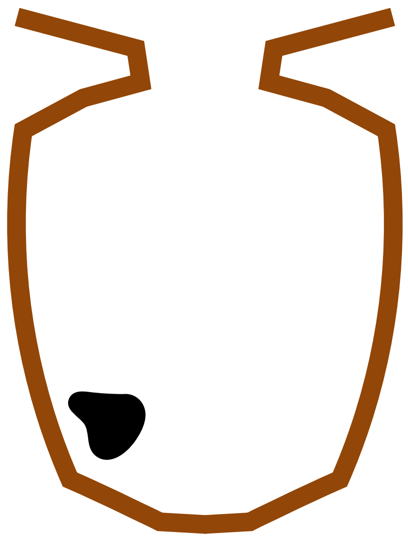
Marginal cluster assignments

- Hoppe urn / Blackwell-MacQueen urn



  - Choose ball with prob proportional to its mass
    - If black, replace and add ball of new color
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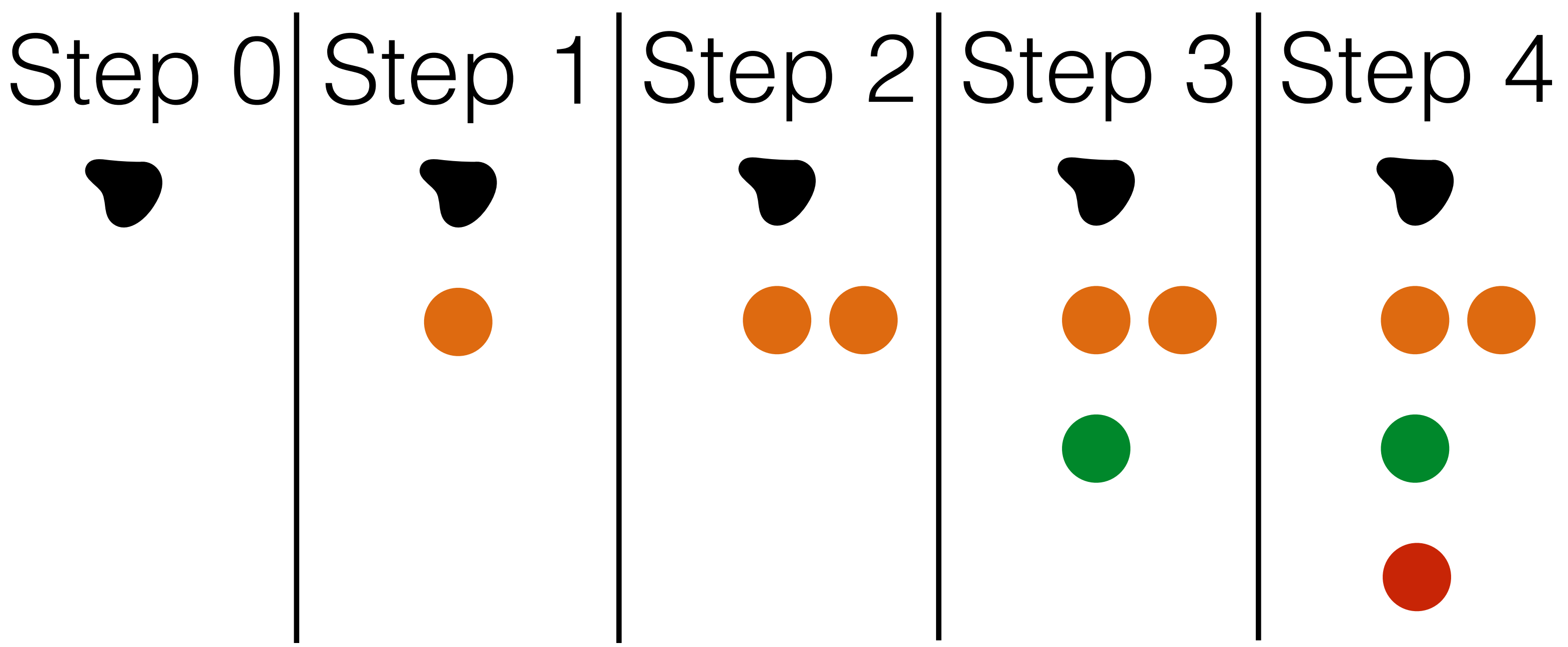
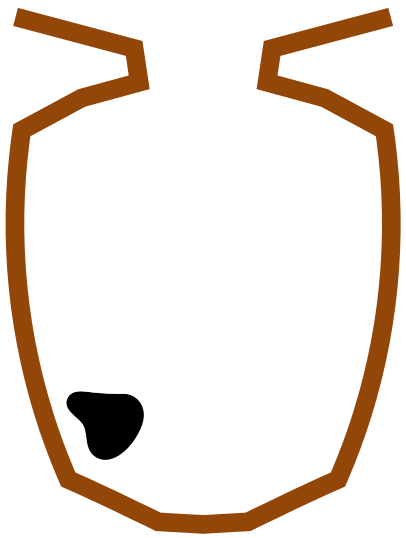


$$(\#\text{orange}, \#\text{other}) = \text{PolyaUrn}(1, \alpha)$$

Marginal cluster assignments

- Hoppe urn / Blackwell-MacQueen urn



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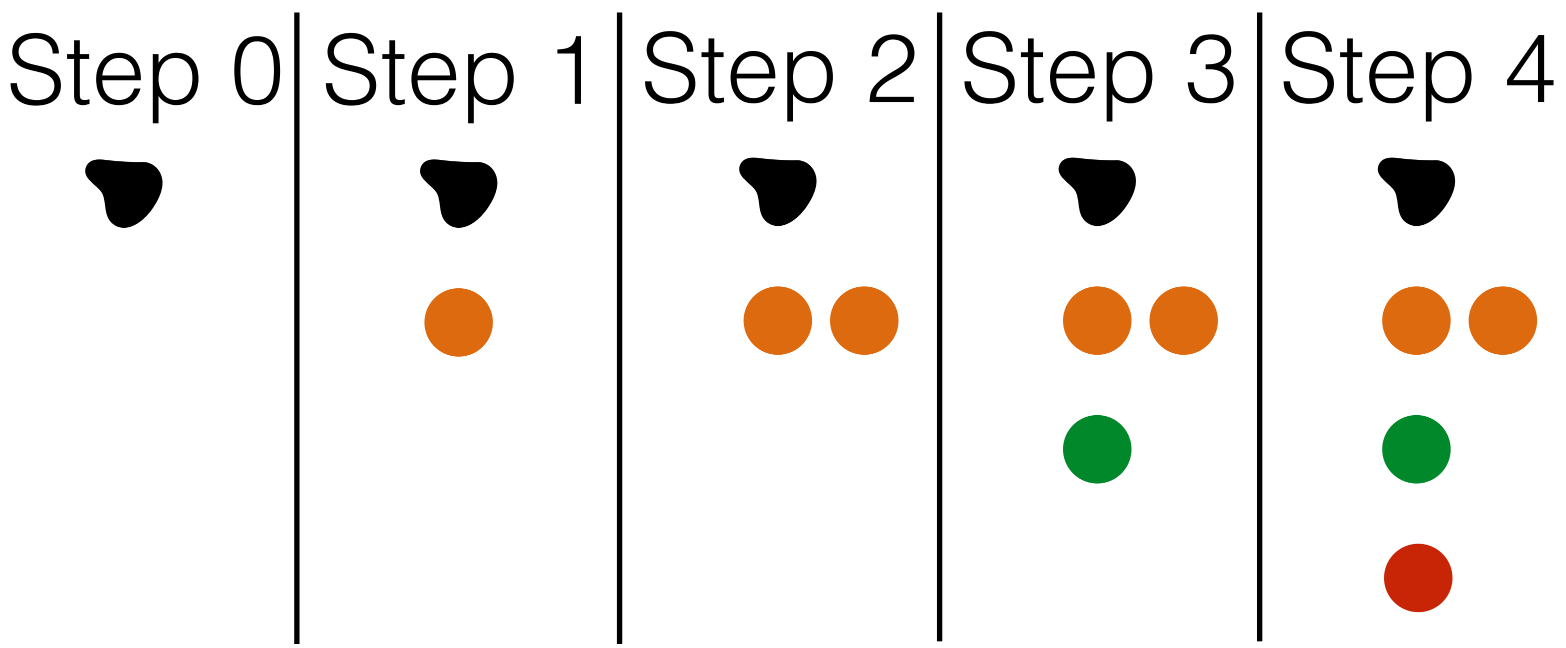
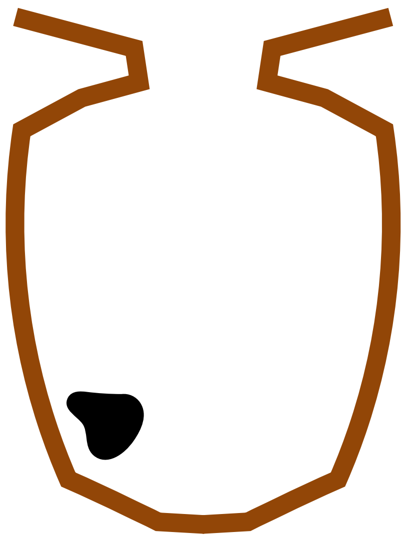


$$(\#\text{orange}, \#\text{other}) = \text{PolyaUrn}(1, \alpha)$$

- not orange: $(\#\text{green}, \#\text{other}) = \text{PolyaUrn}(1, \alpha)$

Marginal cluster assignments

- Hoppe urn / Blackwell-MacQueen urn



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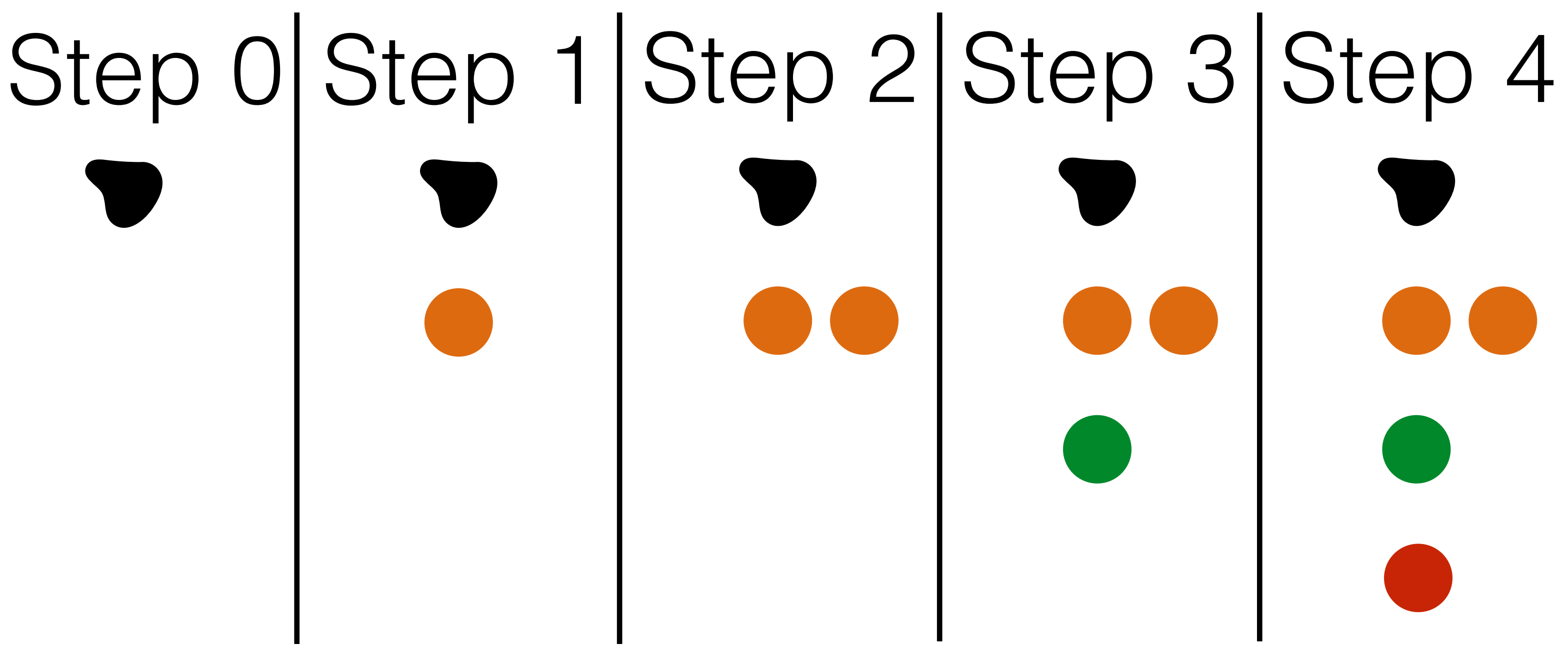

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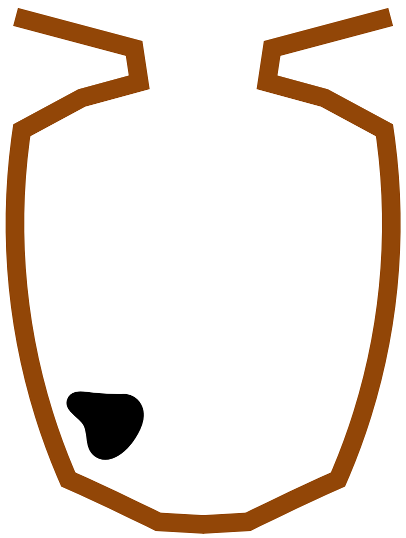
- not orange: $(\#\text{green}, \#\text{other}) = \text{PolyaUrn}(1, \alpha)$
- not orange, green: $(\#\text{red}, \#\text{other}) = \text{PolyaUrn}(1, \alpha)$

Marginal cluster assignments

- Hoppe urn / Blackwell-MacQueen urn



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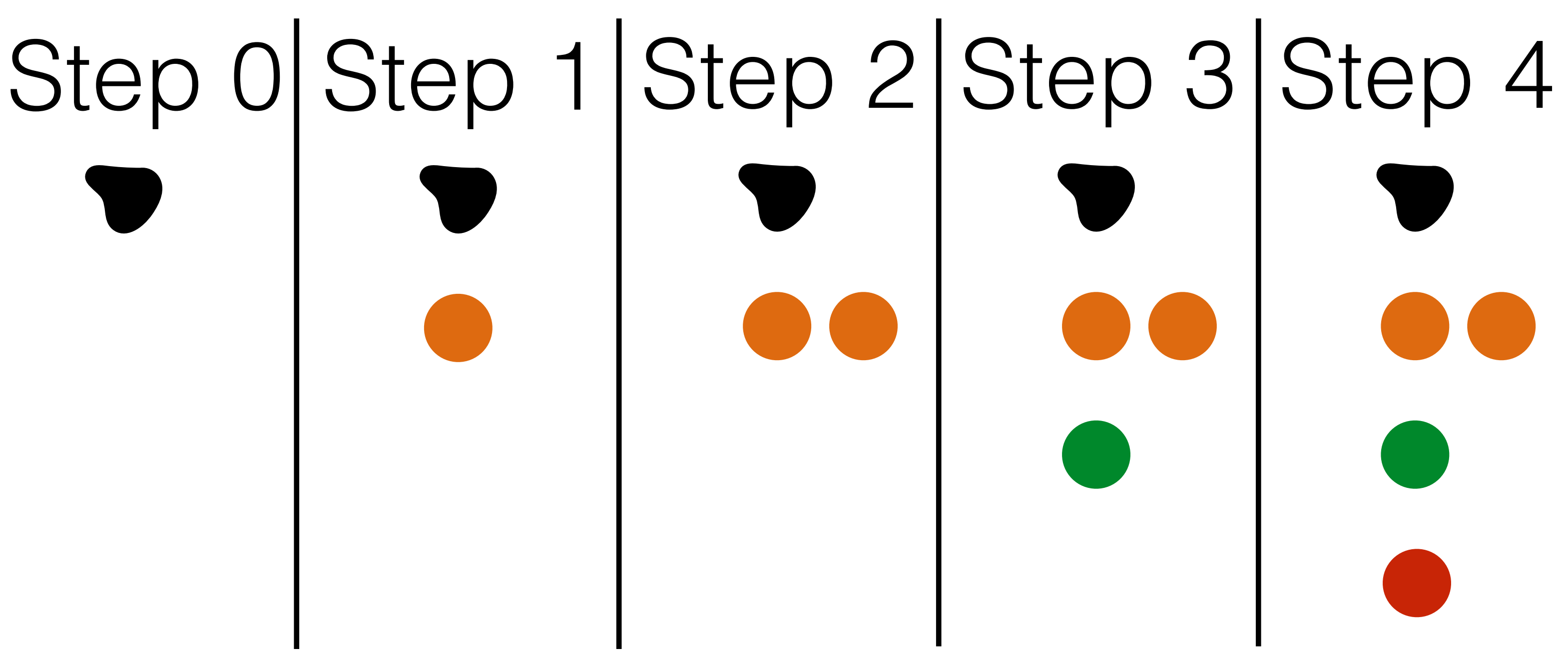

$V_k \overset{iid}{\sim} \text{Beta}(1, \alpha)$

$(\#\text{orange}, \#\text{other}) = \text{PolyaUrn}(1, \alpha)$

- not orange: $(\#\text{green}, \#\text{other}) = \text{PolyaUrn}(1, \alpha)$
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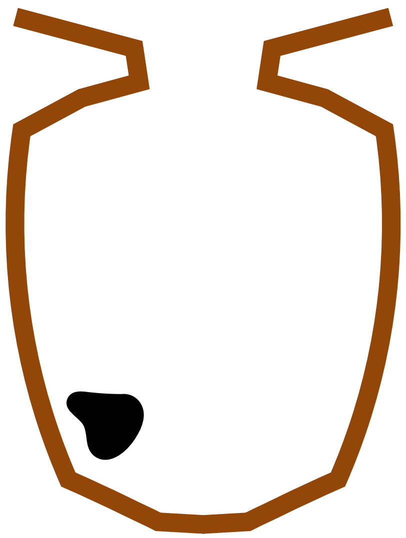
Marginal cluster assignments

- Hoppe urn / Blackwell-MacQueen urn



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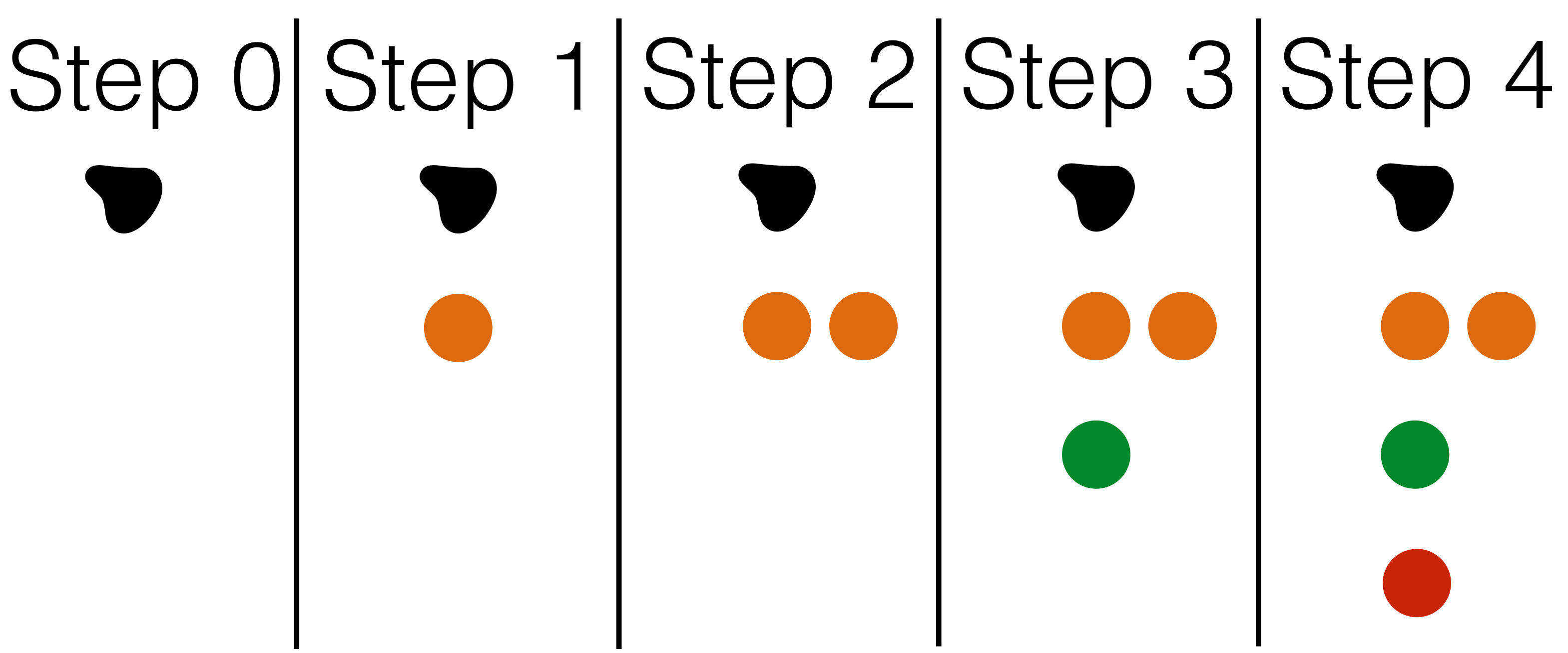

$$V_k \stackrel{iid}{\sim} \mathrm{Beta}(1, \alpha)$$

$$\rho_1 = V_1$$

$$(\#\mathrm{orange}, \#\mathrm{other}) = \mathrm{PolyaUrn}(1, \alpha)$$

- not orange: $(\#\mathrm{green}, \#\mathrm{other}) = \mathrm{PolyaUrn}(1, \alpha)$
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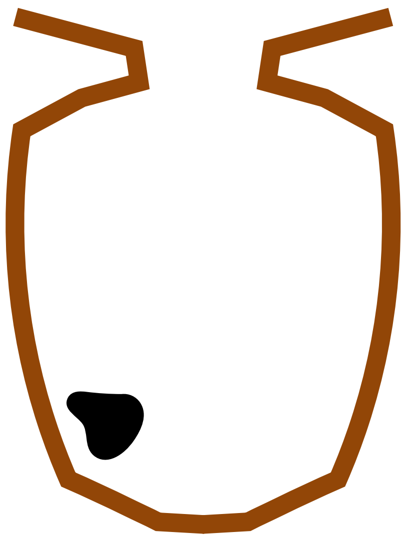
Marginal cluster assignments

- Hoppe urn / Blackwell-MacQueen urn



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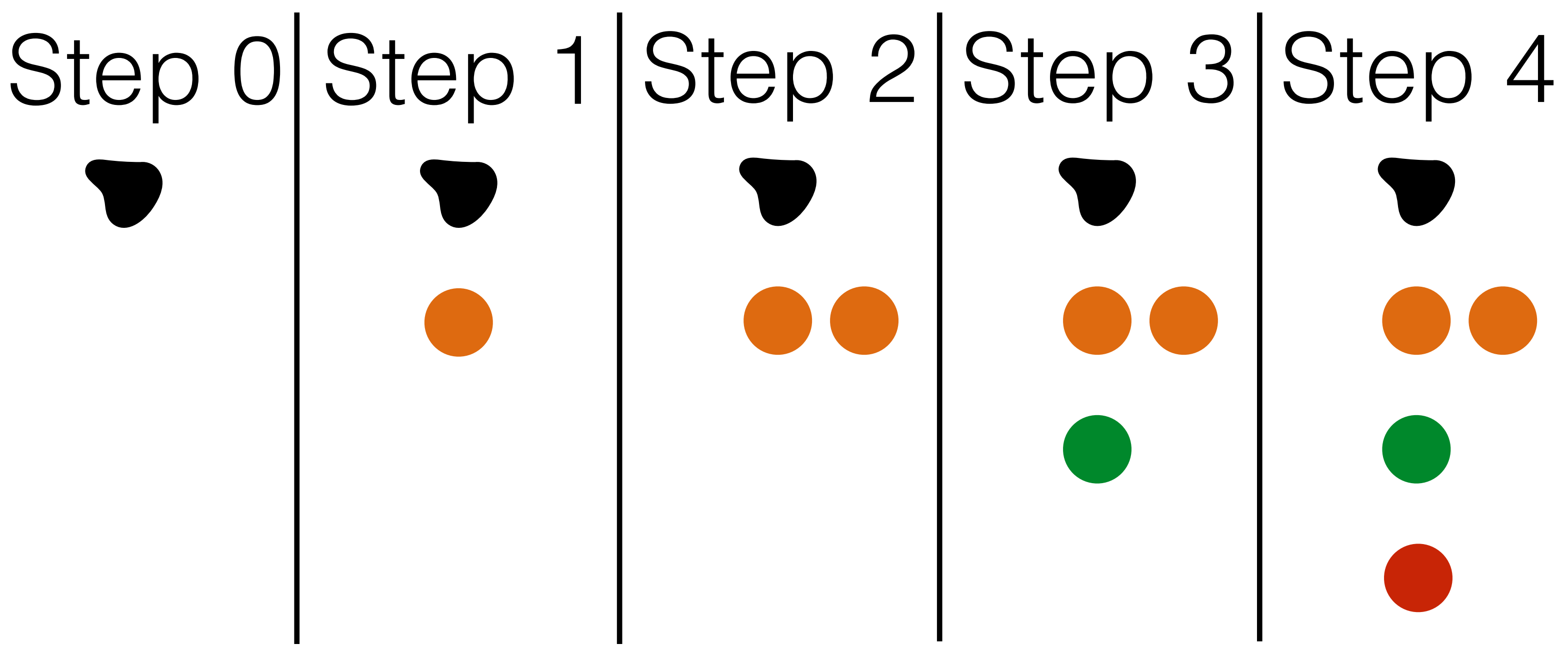

$$V_k \stackrel{iid}{\sim} \mathrm{Beta}(1, \alpha)$$

$$\rho_1 = V_1$$

$$\rho_2 = (1 - V_1)V_2$$

$$(\#\text{orange}, \#\text{other}) = \mathrm{PolyaUrn}(1, \alpha)$$

- not orange: $(\#\text{green}, \#\text{other}) = \mathrm{PolyaUrn}(1, \alpha)$
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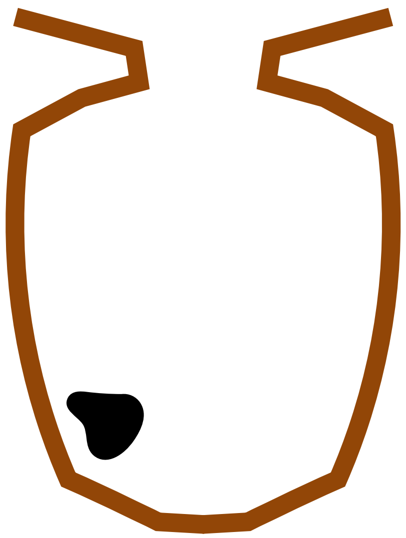
Marginal cluster assignments

- Hoppe urn / Blackwell-MacQueen urn



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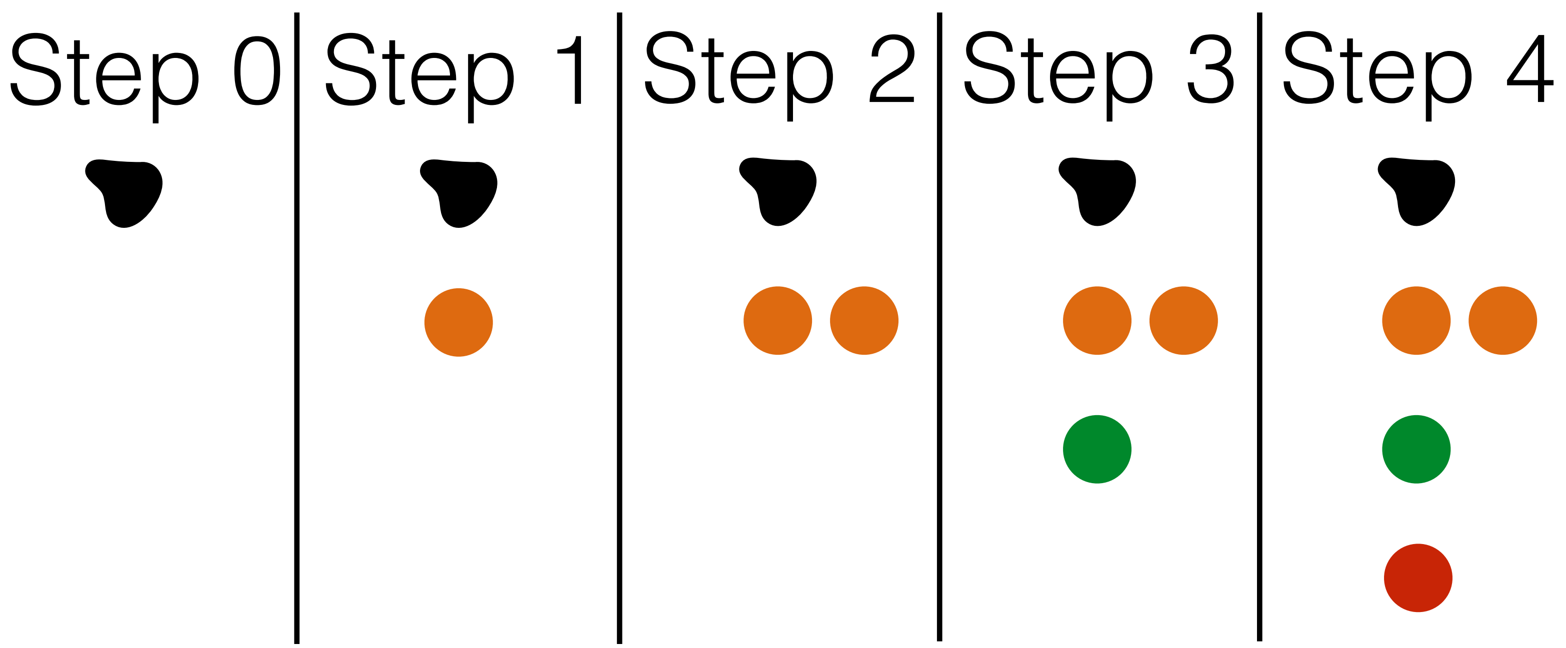

$$V_k \overset{iid}{\sim} \mathrm{Beta}(1, \alpha)$$

$$\rho_1 = V_1$$

$$\rho_2 = (1 - V_1)V_2$$

$$\rho_3 = [\prod_{k=1}^{2}(1 - V_k)]V_3$$

$$(\#\text{orange}, \#\text{other}) = \mathrm{PolyaUrn}(1, \alpha)$$

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Chinese restaurant process

Chinese restaurant process

- Same thing we just did

Chinese restaurant process

- Same thing we just did
- Each customer walks into the restaurant

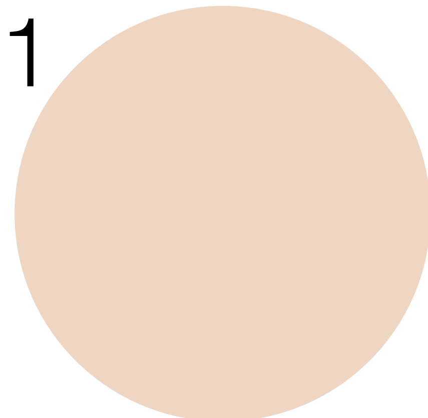
Chinese restaurant process

- Same thing we just did
- Each customer walks into the restaurant
  - Sits at existing table with prob proportional to # people there

Chinese restaurant process

- Same thing we just did
- Each customer walks into the restaurant
  - Sits at existing table with prob proportional to # people there
  - Forms new table with prob proportional to α

Chinese restaurant process



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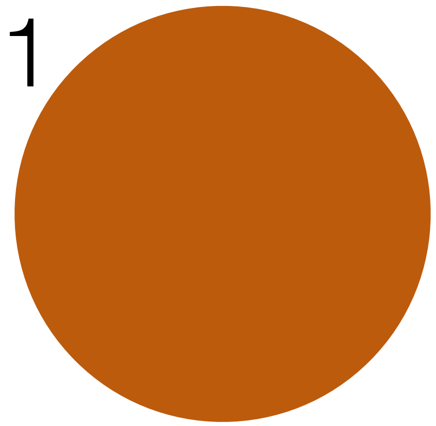
Chinese restaurant process



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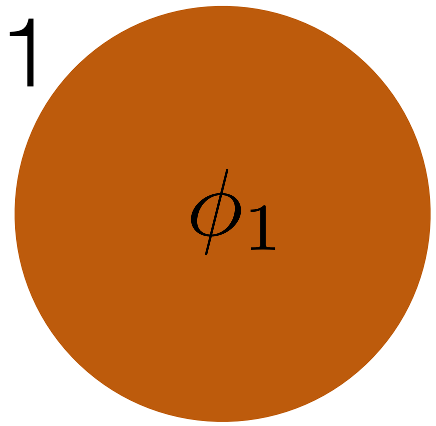
Chinese restaurant process



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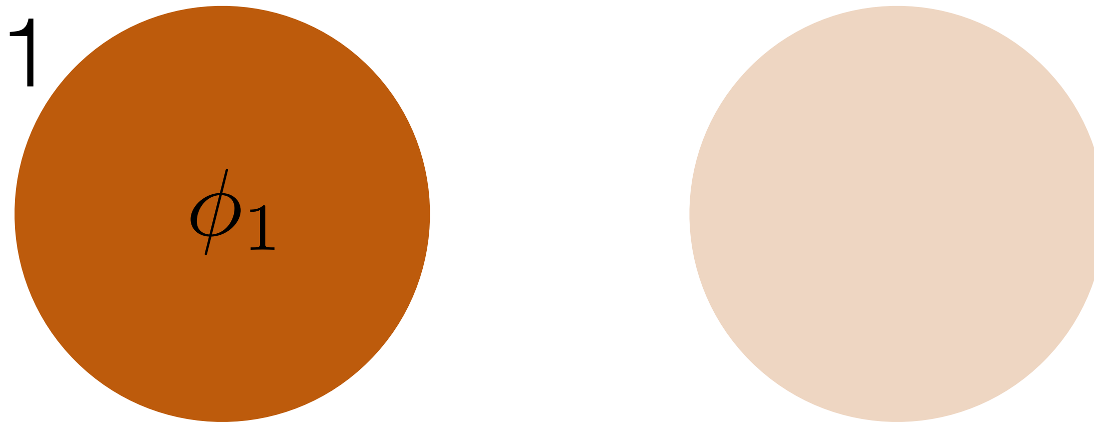
Chinese restaurant process



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Chinese restaurant process

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Chinese restaurant process

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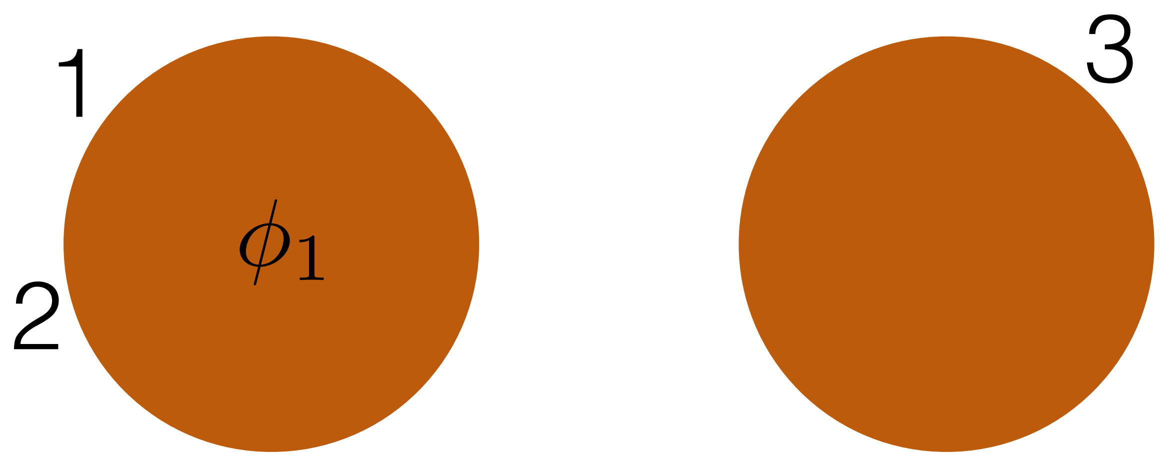
Chinese restaurant process



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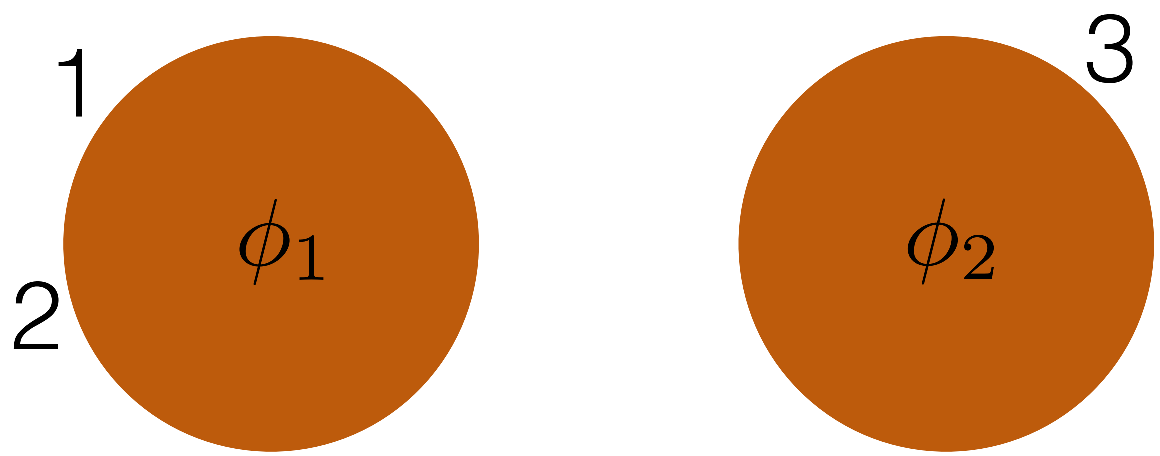
Chinese restaurant process



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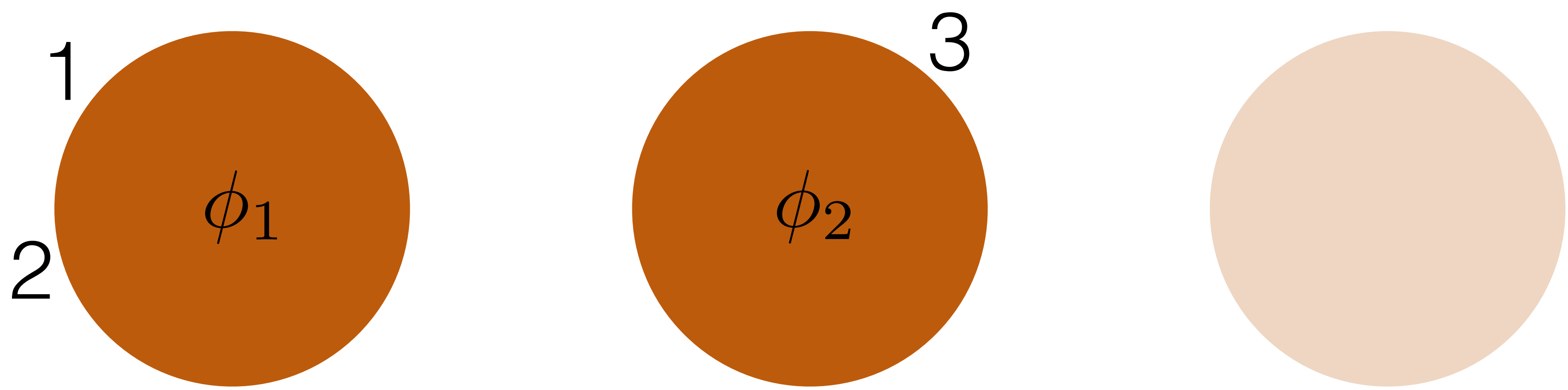
Chinese restaurant process



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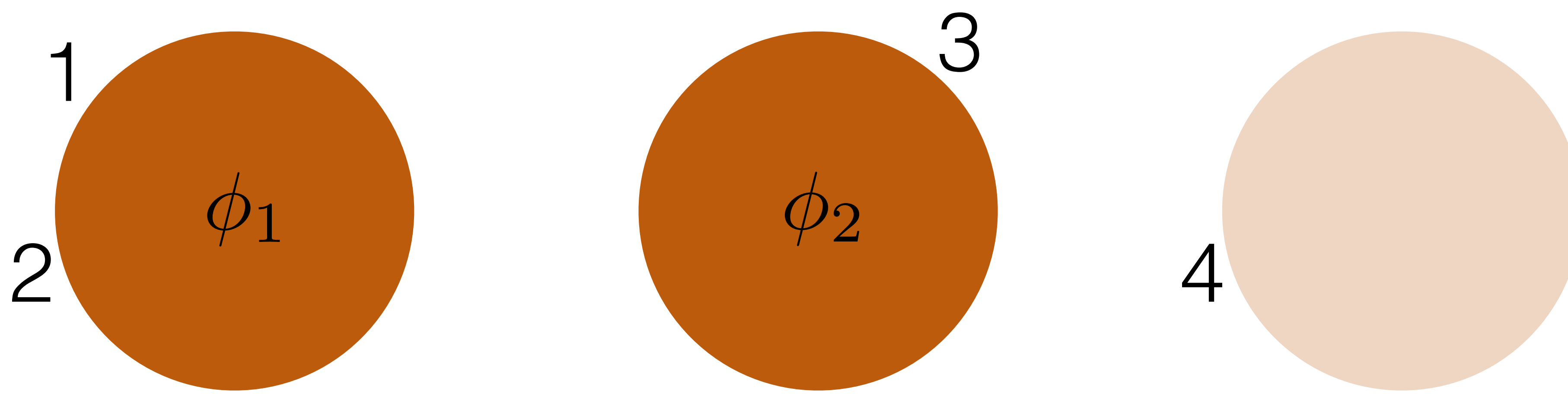
Chinese restaurant process



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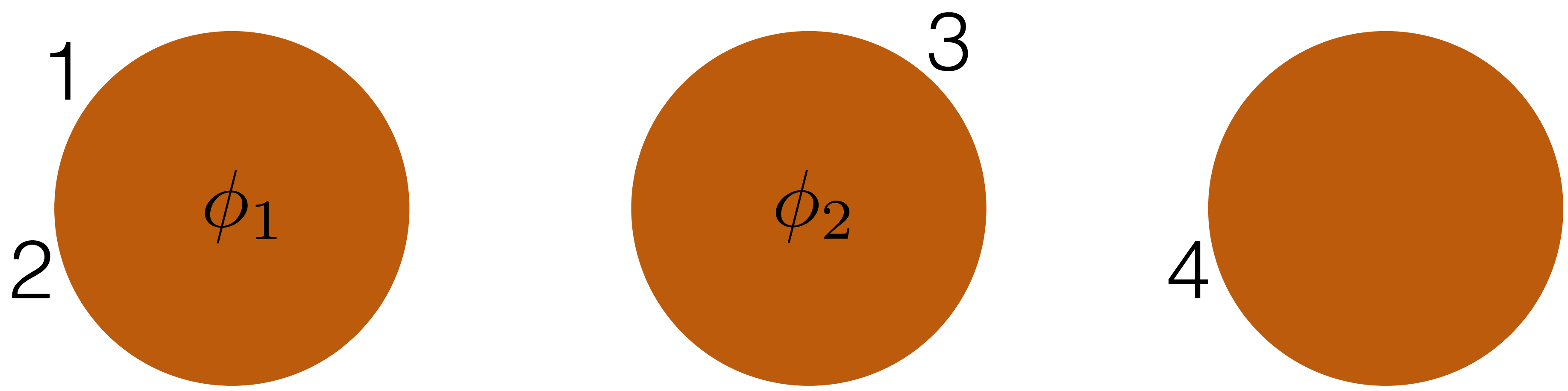
Chinese restaurant process



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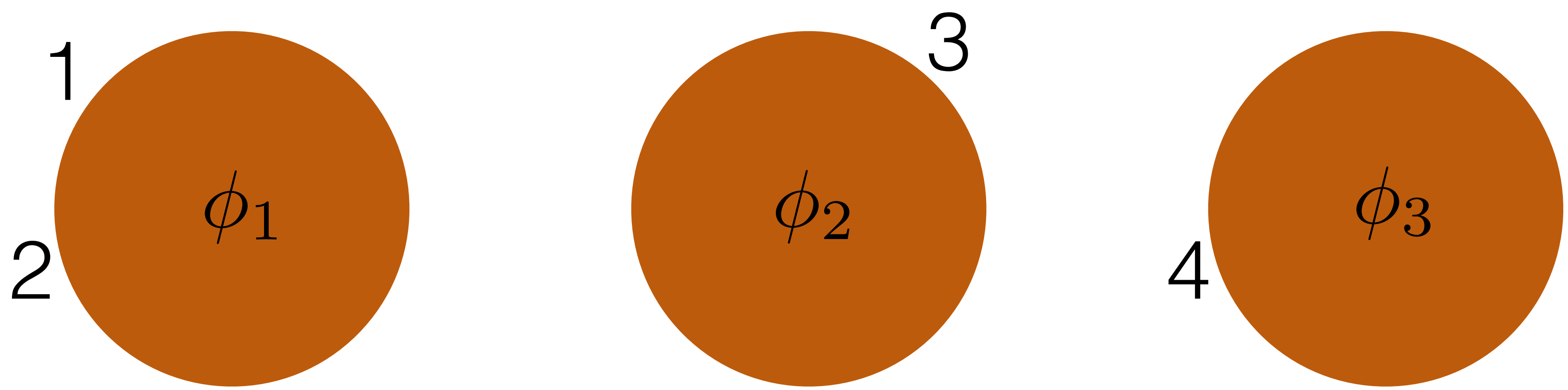
Chinese restaurant process



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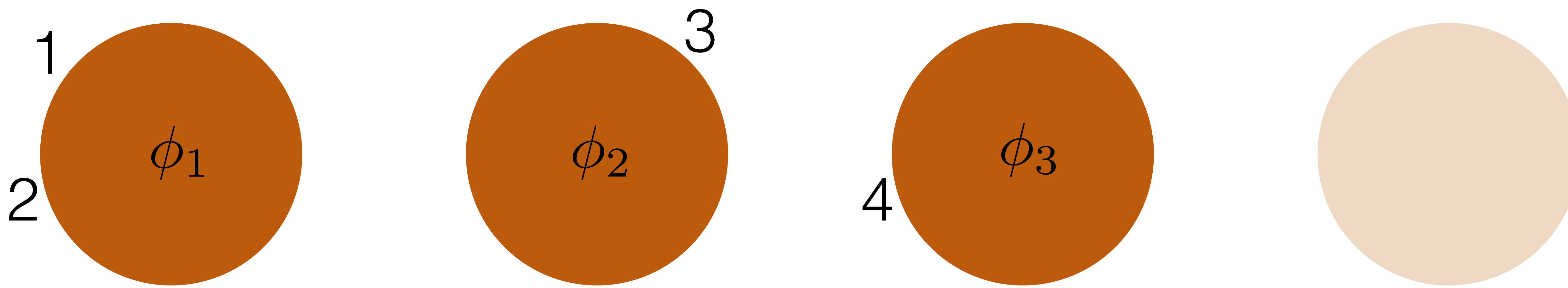
Chinese restaurant process



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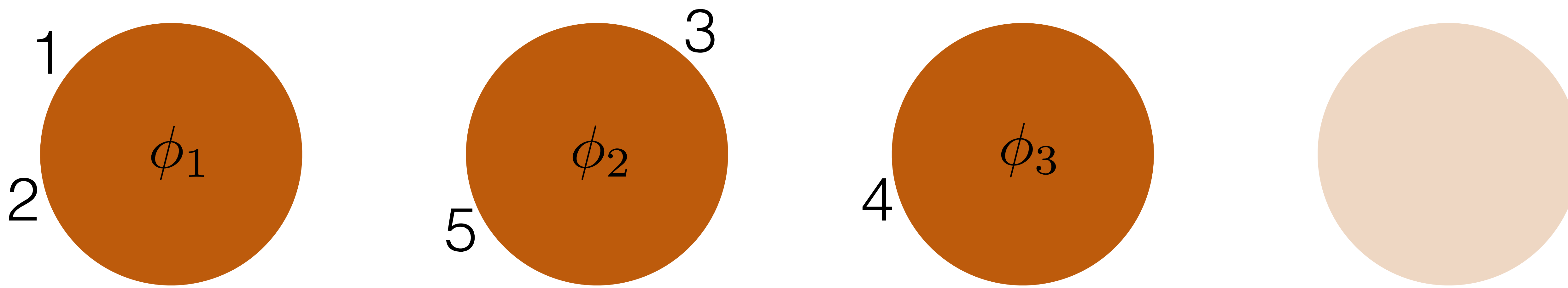
Chinese restaurant process



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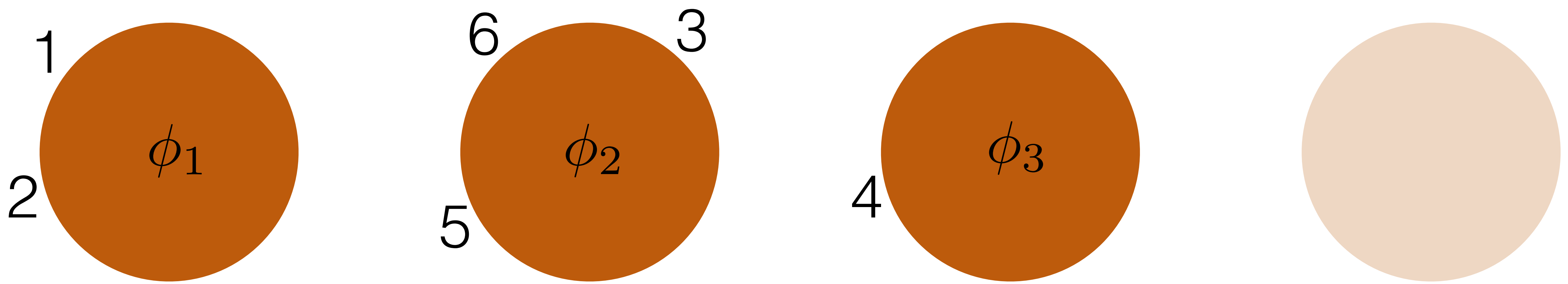
Chinese restaurant process



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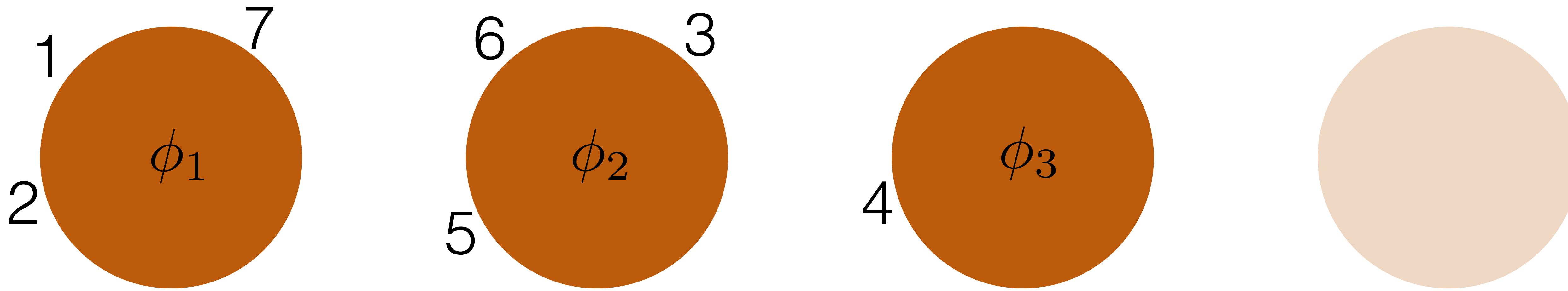
Chinese restaurant process



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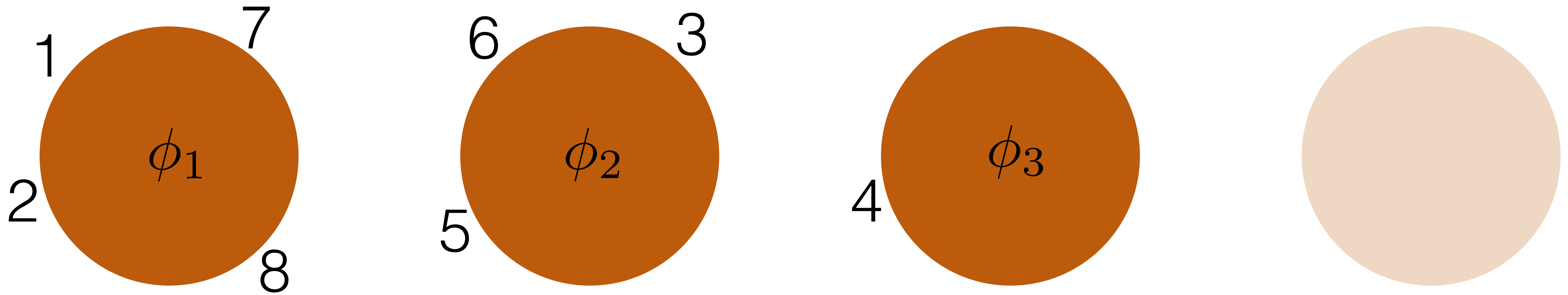
Chinese restaurant process



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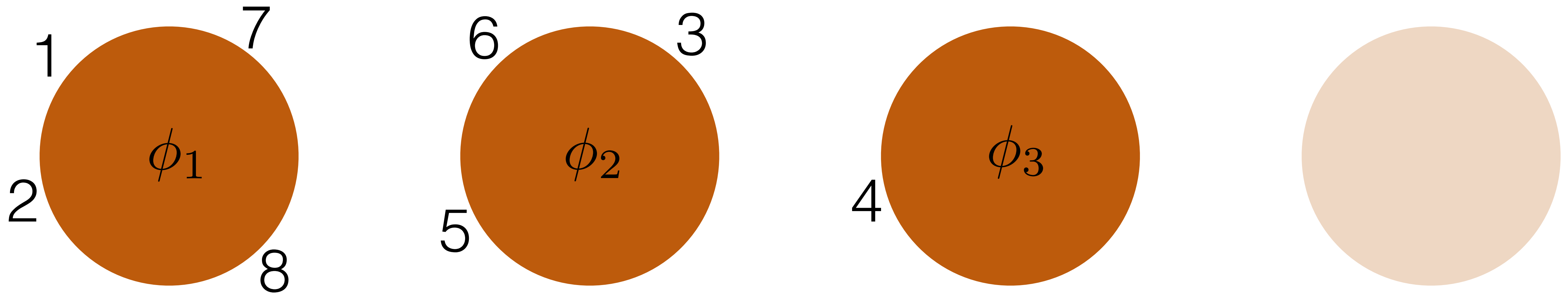
Chinese restaurant process



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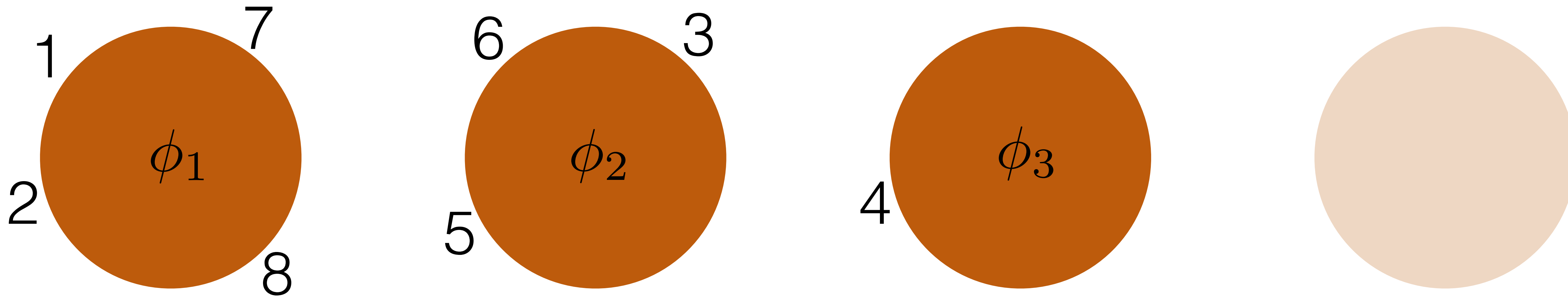
[Aldous 1983]

Chinese restaurant process



- Same thing we just did
- Each customer walks into the restaurant
  - Sits at existing table with prob proportional to # people there
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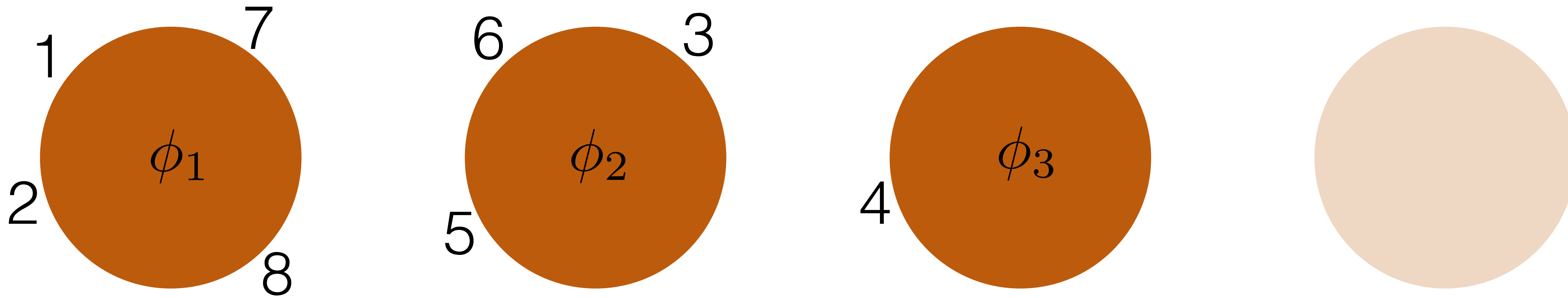
- Marginal for the Categorical likelihood with GEM prior

Chinese restaurant process



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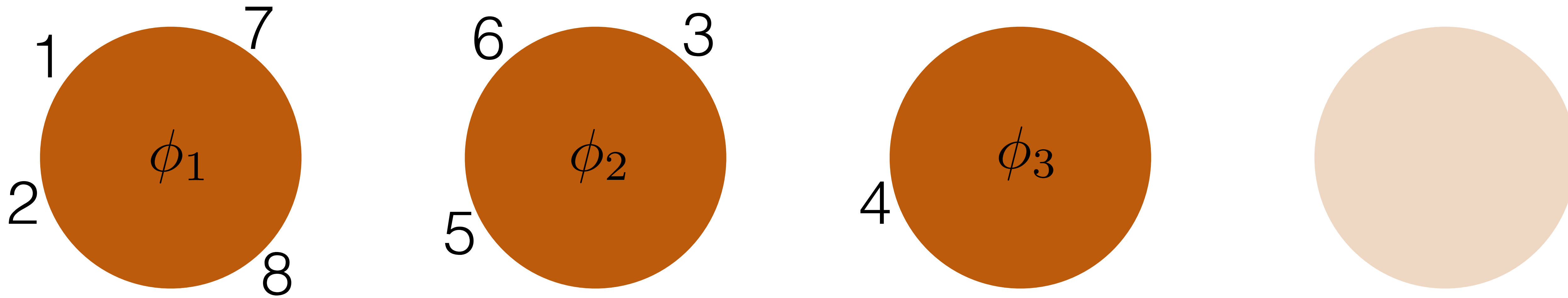
- Marginal for the Categorical likelihood with GEM prior

$$z_1 = z_2 = z_7 = z_8 = 1, z_3 = z_5 = z_6 = 2, z_4 = 3$$

Chinese restaurant process



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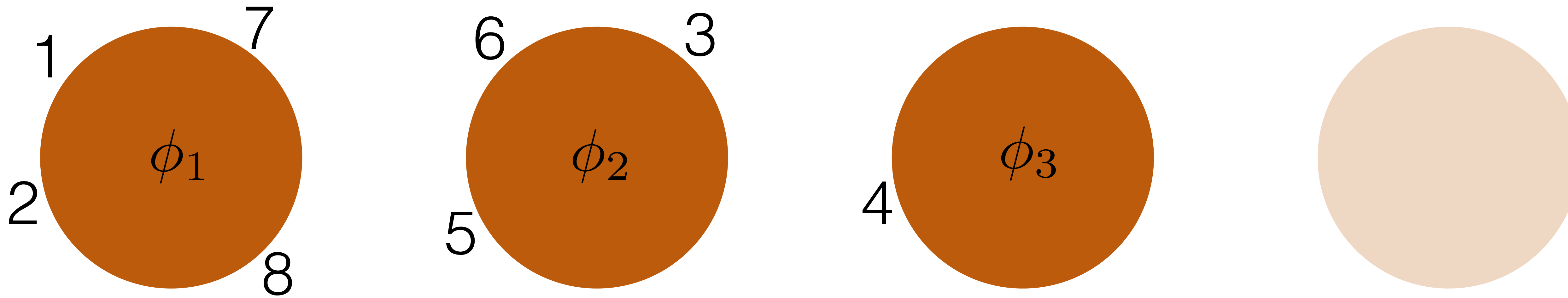
- Marginal for the Categorical likelihood with GEM prior

$$z_1 = z_2 = z_7 = z_8 = 1, z_3 = z_5 = z_6 = 2, z_4 = 3$$
$$\Rightarrow \Pi_8 = \{\{1, 2, 7, 8\}, \{3, 5, 6\}, \{4\}\}$$

Chinese restaurant process



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- Marginal for the Categorical likelihood with GEM prior

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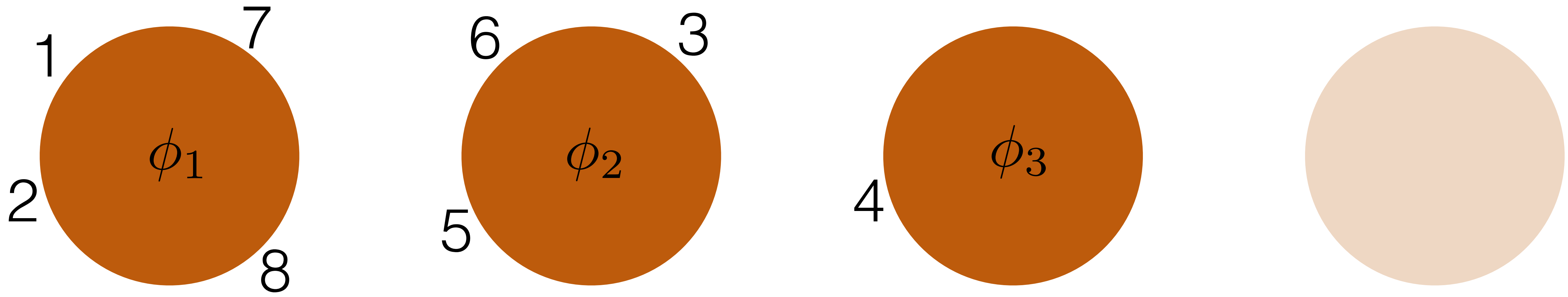
$$\Rightarrow \Pi_8 = \{\{1,2,7,8\},\{3,5,6\},\{4\}\}$$

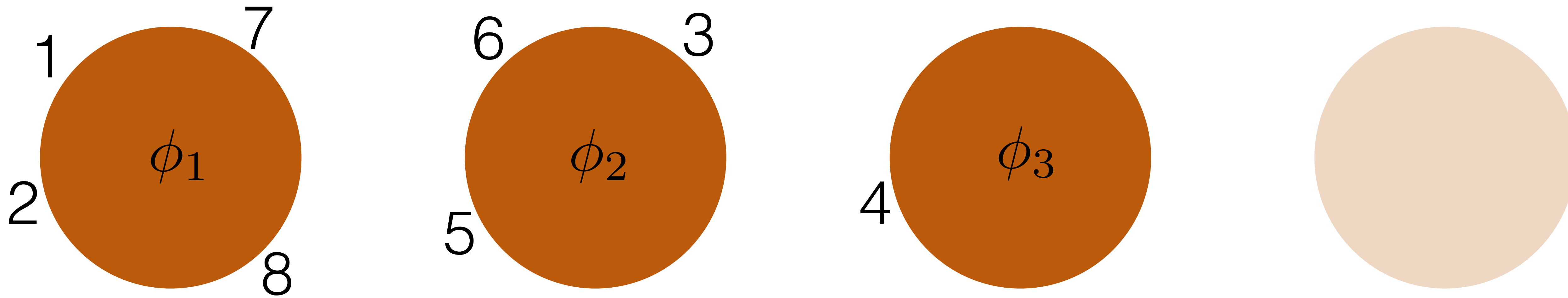
- *Partition of [8]*: set of mutually exclusive & exhaustive sets of $[8] = \{1, \ldots, 8\}$

Chinese restaurant process



- Probability of this seating:

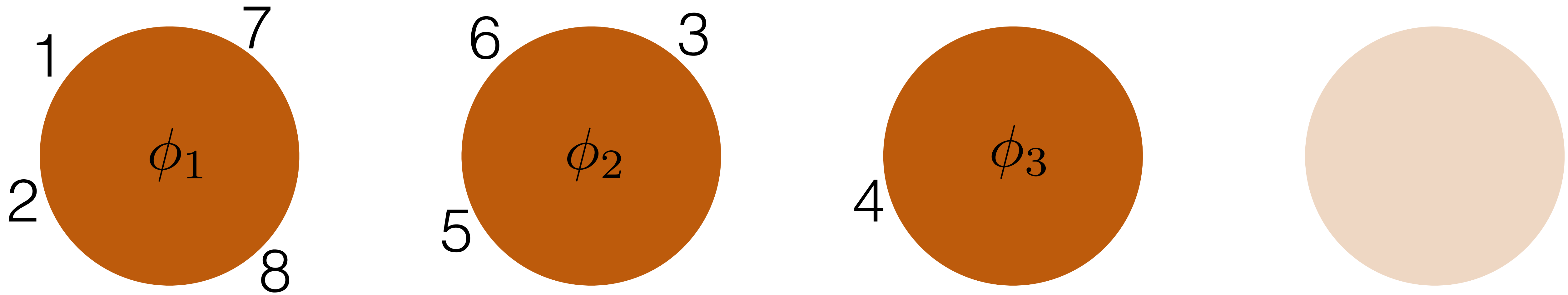
Chinese restaurant process



- Probability of this seating:

$$\frac{\alpha}{\alpha}$$

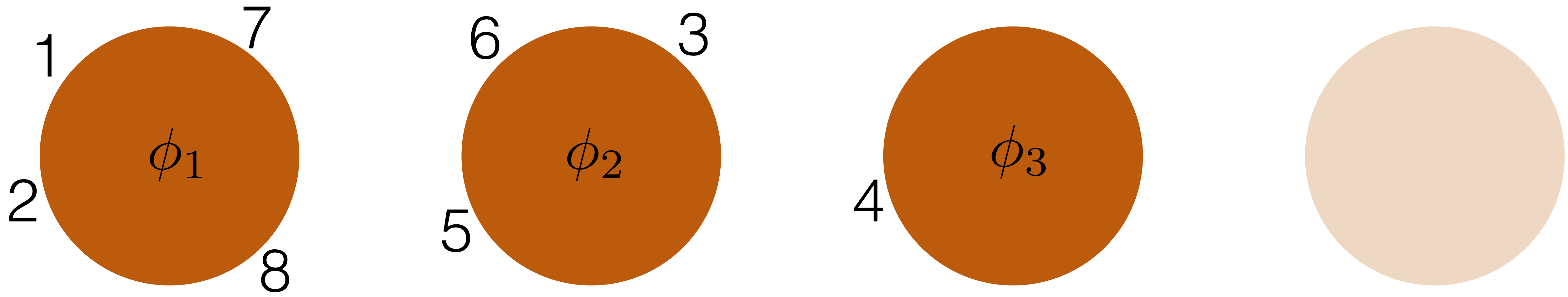
Chinese restaurant process



- Probability of this seating:

$$\frac{\alpha}{\alpha} \cdot \frac{1}{\alpha + 1}$$

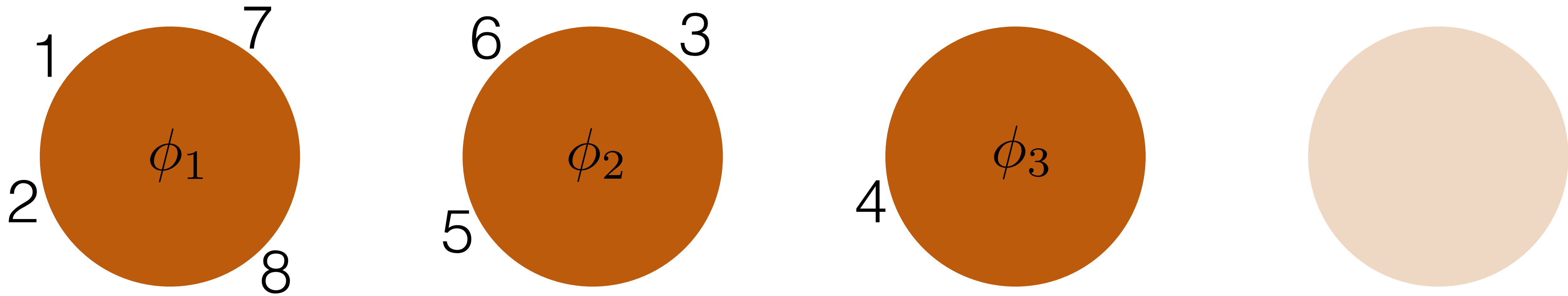
Chinese restaurant process



- Probability of this seating:

$$\frac{\alpha}{\alpha} \cdot \frac{1}{\alpha+1} \cdot \frac{\alpha}{\alpha+2}$$

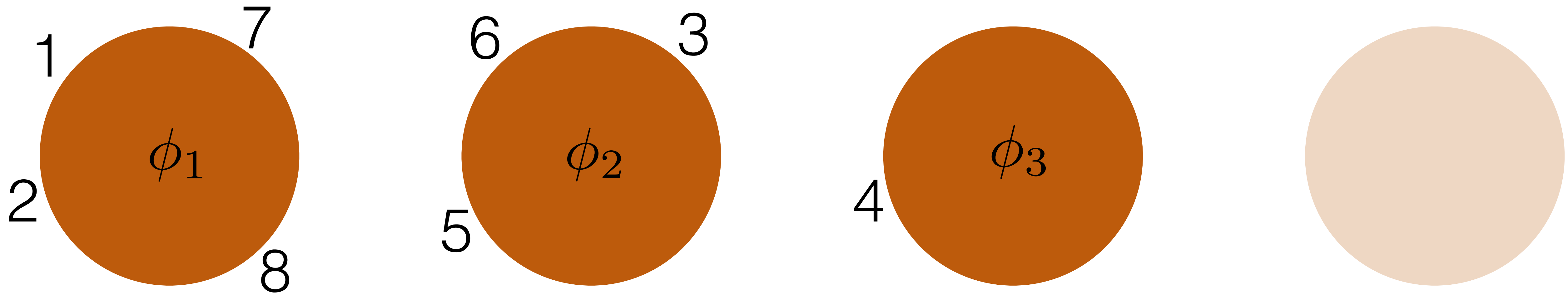
Chinese restaurant process



- Probability of this seating:

$$\frac{\alpha}{\alpha} \cdot \frac{1}{\alpha+1} \cdot \frac{\alpha}{\alpha+2} \cdot \frac{\alpha}{\alpha+3}$$

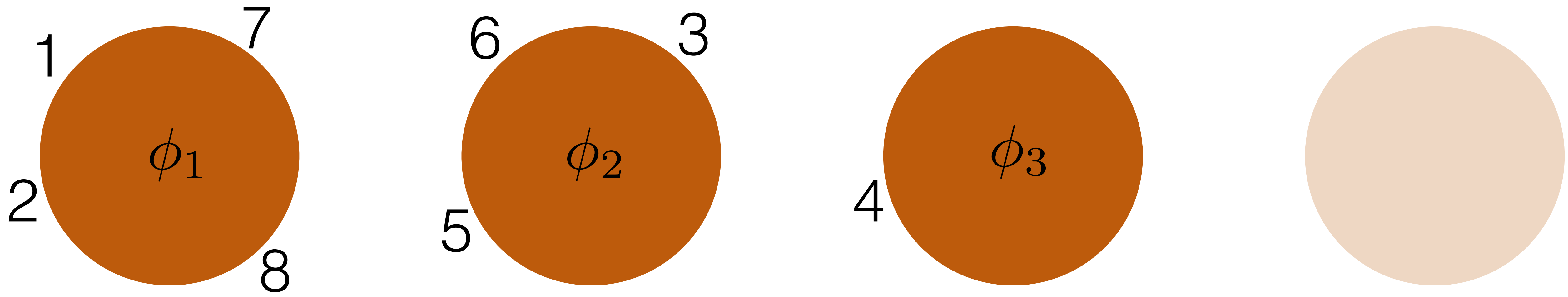
Chinese restaurant process



- Probability of this seating:

$$\frac{\alpha}{\alpha} \cdot \frac{1}{\alpha+1} \cdot \frac{\alpha}{\alpha+2} \cdot \frac{\alpha}{\alpha+3} \cdot \frac{1}{\alpha+4}$$

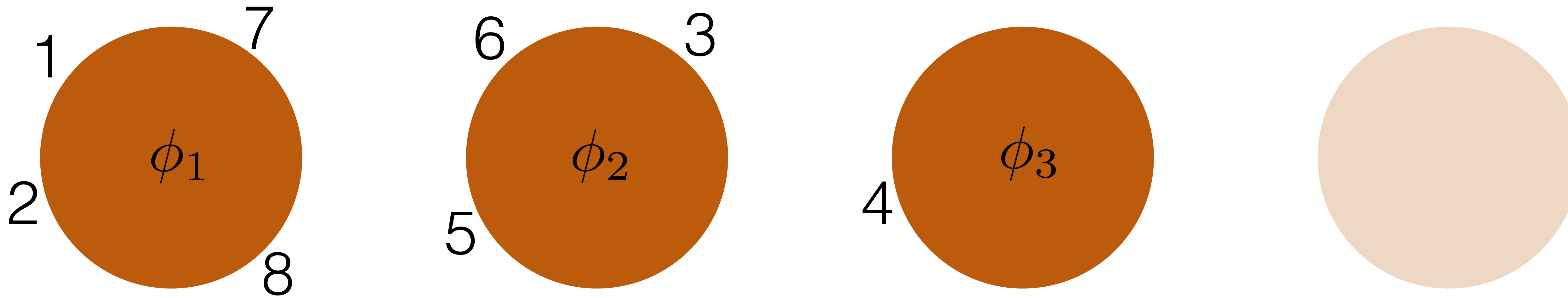
Chinese restaurant process



- Probability of this seating:

$$\frac{\alpha}{\alpha} \cdot \frac{1}{\alpha+1} \cdot \frac{\alpha}{\alpha+2} \cdot \frac{\alpha}{\alpha+3} \cdot \frac{1}{\alpha+4} \cdot \frac{2}{\alpha+5}$$

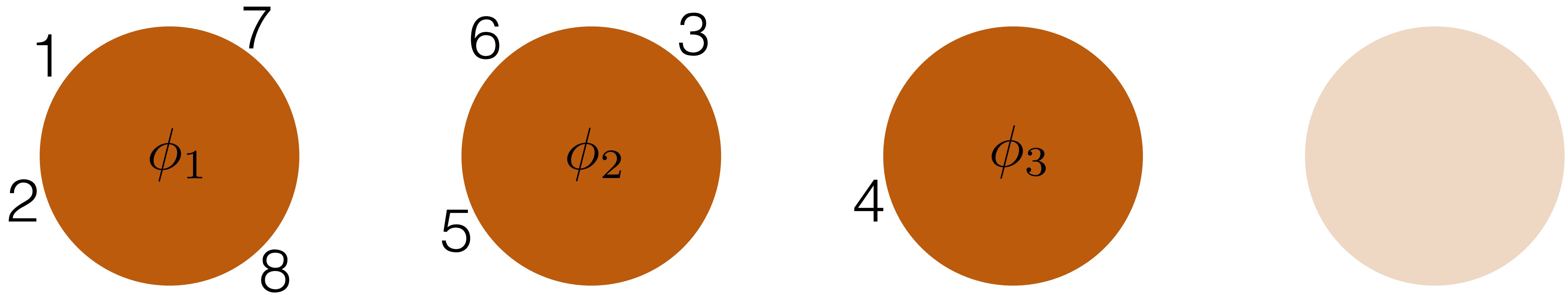
Chinese restaurant process



- Probability of this seating:

$$\frac{\alpha}{\alpha} \cdot \frac{1}{\alpha + 1} \cdot \frac{\alpha}{\alpha + 2} \cdot \frac{\alpha}{\alpha + 3} \cdot \frac{1}{\alpha + 4} \cdot \frac{2}{\alpha + 5} \cdot \frac{2}{\alpha + 6}$$

Chinese restaurant process



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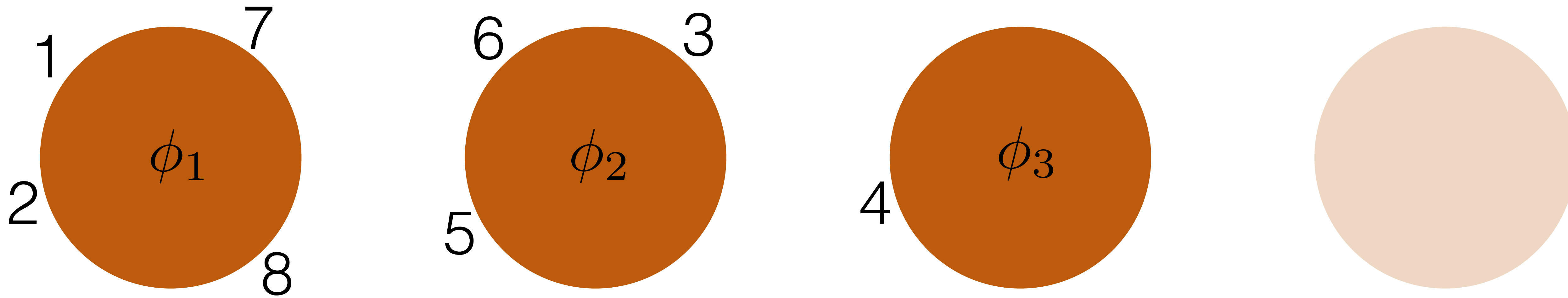
$$\frac{\alpha}{\alpha} \cdot \frac{1}{\alpha+1} \cdot \frac{\alpha}{\alpha+2} \cdot \frac{\alpha}{\alpha+3} \cdot \frac{1}{\alpha+4} \cdot \frac{2}{\alpha+5} \cdot \frac{2}{\alpha+6} \cdot \frac{3}{\alpha+7}$$

Chinese restaurant process

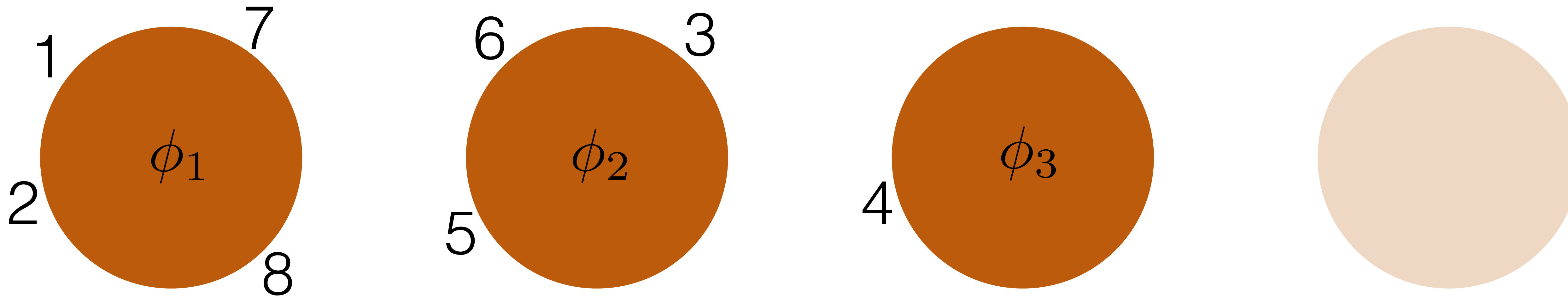


- Probability of this seating:

$$\frac{\alpha}{\alpha} \cdot \frac{1}{\alpha+1} \cdot \frac{\alpha}{\alpha+2} \cdot \frac{\alpha}{\alpha+3} \cdot \frac{1}{\alpha+4} \cdot \frac{2}{\alpha+5} \cdot \frac{2}{\alpha+6} \cdot \frac{3}{\alpha+7}$$

- Probability of *N* customers (K_N tables, n_k at table *k*):

Chinese restaurant process



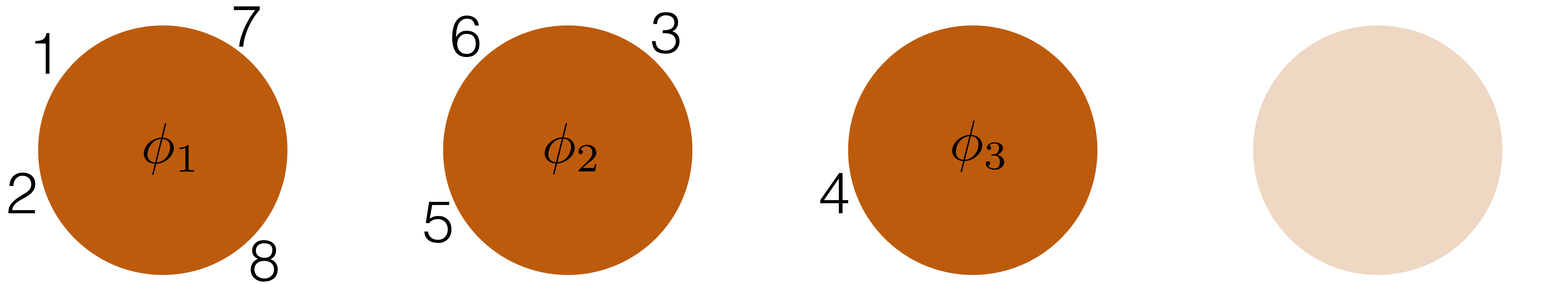
- Probability of this seating:

$$\frac{\alpha}{\alpha} \cdot \frac{1}{\alpha+1} \cdot \frac{\alpha}{\alpha+2} \cdot \frac{\alpha}{\alpha+3} \cdot \frac{1}{\alpha+4} \cdot \frac{2}{\alpha+5} \cdot \frac{2}{\alpha+6} \cdot \frac{3}{\alpha+7}$$

- Probability of *N* customers (K_N tables, n_k at table *k*):

$$\frac{}{\alpha \cdots (\alpha + N - 1)}$$

Chinese restaurant process



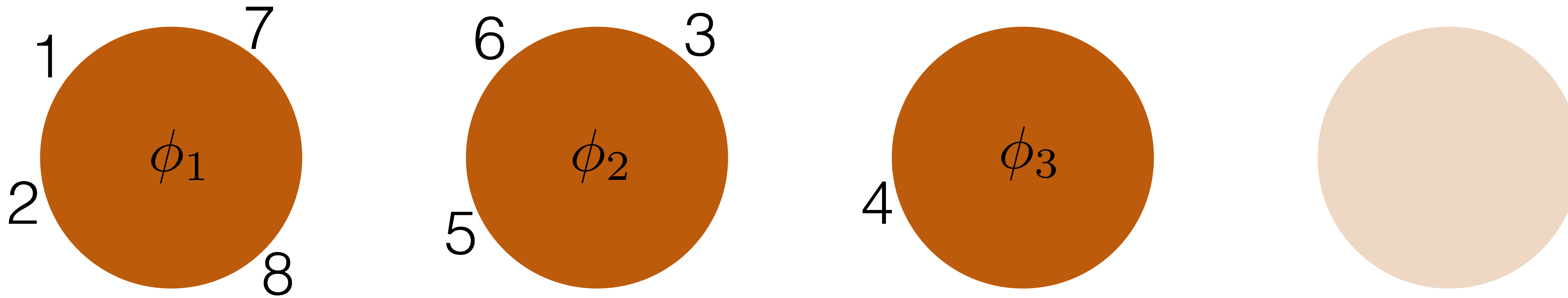
- Probability of this seating:

$$\frac{\alpha}{\alpha} \cdot \frac{1}{\alpha+1} \cdot \frac{\alpha}{\alpha+2} \cdot \frac{\alpha}{\alpha+3} \cdot \frac{1}{\alpha+4} \cdot \frac{2}{\alpha+5} \cdot \frac{2}{\alpha+6} \cdot \frac{3}{\alpha+7}$$

- Probability of *N* customers (K_N tables, n_k at table *k*):

$$\frac{\alpha^{K_N}}{\alpha \cdots (\alpha+N-1)}$$

Chinese restaurant process



- Probability of this seating:

$$\frac{\alpha}{\alpha} \cdot \frac{1}{\alpha+1} \cdot \frac{\alpha}{\alpha+2} \cdot \frac{\alpha}{\alpha+3} \cdot \frac{1}{\alpha+4} \cdot \frac{2}{\alpha+5} \cdot \frac{2}{\alpha+6} \cdot \frac{3}{\alpha+7}$$

- Probability of *N* customers (K_N tables, n_k at table *k*):

$$\frac{\alpha^{K_N} \prod_{k=1}^{K_N} (n_k - 1)!}{\alpha \cdots (\alpha + N - 1)}$$

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