Nonparametric Bayesian Methods: Models, Algorithms, and Applications (Part 2)

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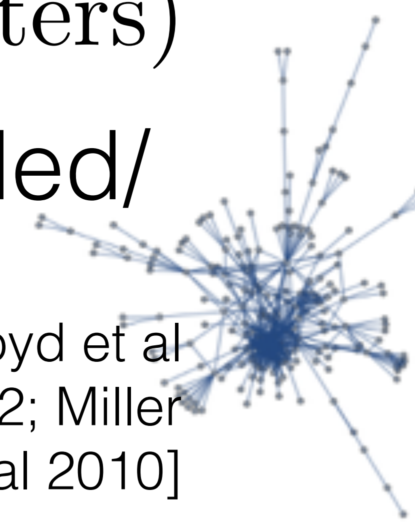
ITT Career Development Assistant Professor
Electrical Engineering & Computer Science
MIT

Nonparametric Bayes

- Bayesian methods that are not parametric
- Bayesian

$$\mathbb{P}(\text{parameters}|\text{data}) \propto \mathbb{P}(\text{data}|\text{parameters})\mathbb{P}(\text{parameters})$$

- Not parametric (i.e. not finite parameter, unbounded/ growing/infinite number of parameters)



[Lloyd et al 2012; Miller et al 2010]

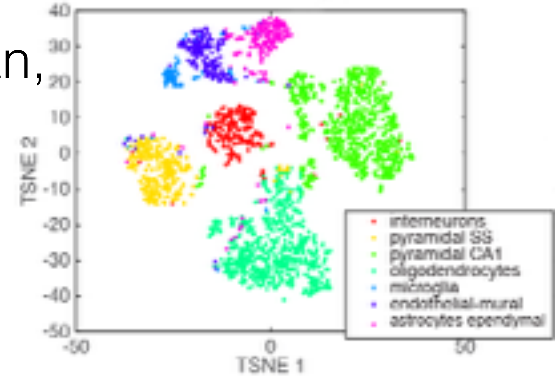
[Ed Bowlby, NOAA]

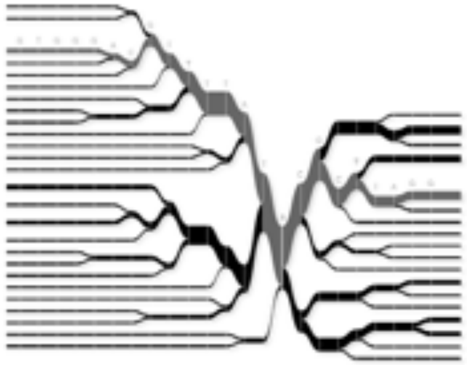
[Fox et al 2014]

[MIT xPRO]

[Lan et al 2015]

[Prabhakaran, Azizi, Carr, Pe'er 2016]





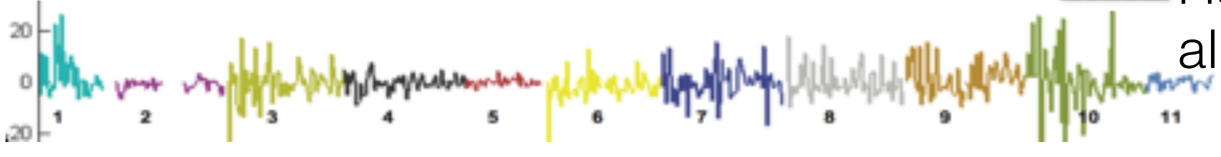
[Ewens 1972; Hartl, Clark 2003; Harris et al 2015]

[Xu et al 2015; Cassidy et al 2015]

[ESO/ L. Calçada/ M. Kornmesser 2017]

[Del Pozzo et al 2017, 2018]

[Saria et al 2010]



Roadmap

[slides, code: www.tamarabroderick.com/tutorials.html]

- Example problem: clustering
- Example NPBayes model: Dirichlet process
- Chinese restaurant process
- Inference
- Venture further into the wild world of Nonparametric Bayes

- Big questions
  - Why NPBayes?
  - What does an infinite/growing number of parameters really mean (in NPBayes)?
  - Why is NPBayes challenging but practical?

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- Here, difficult to choose finite K in advance (contrast with small K): don't know K, difficult to infer, streaming data
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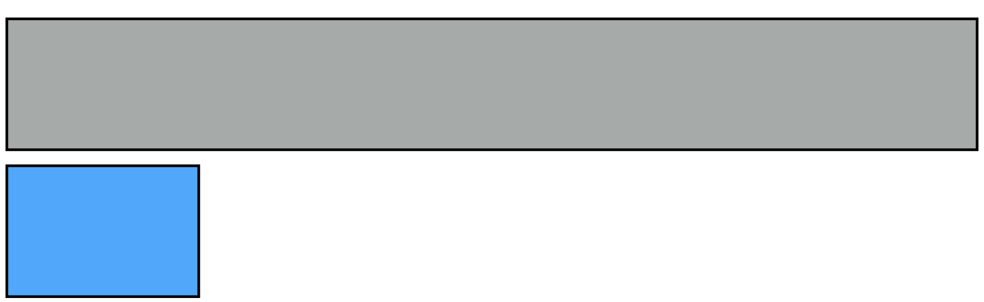

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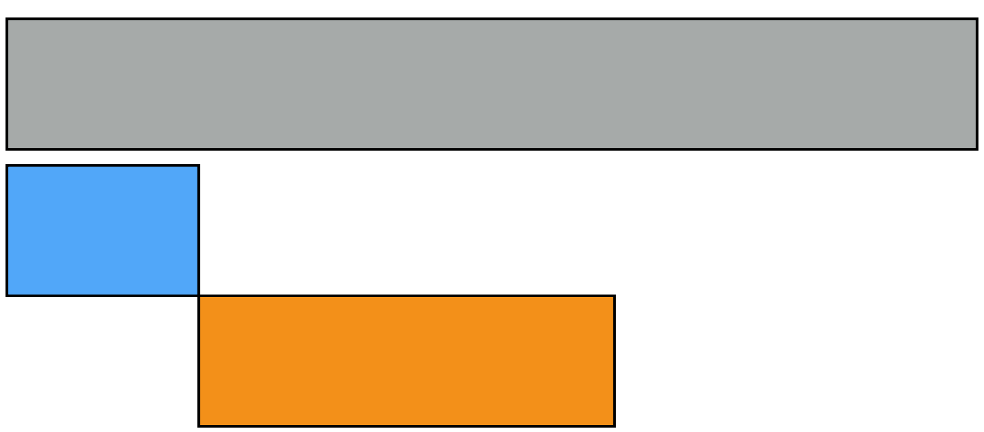

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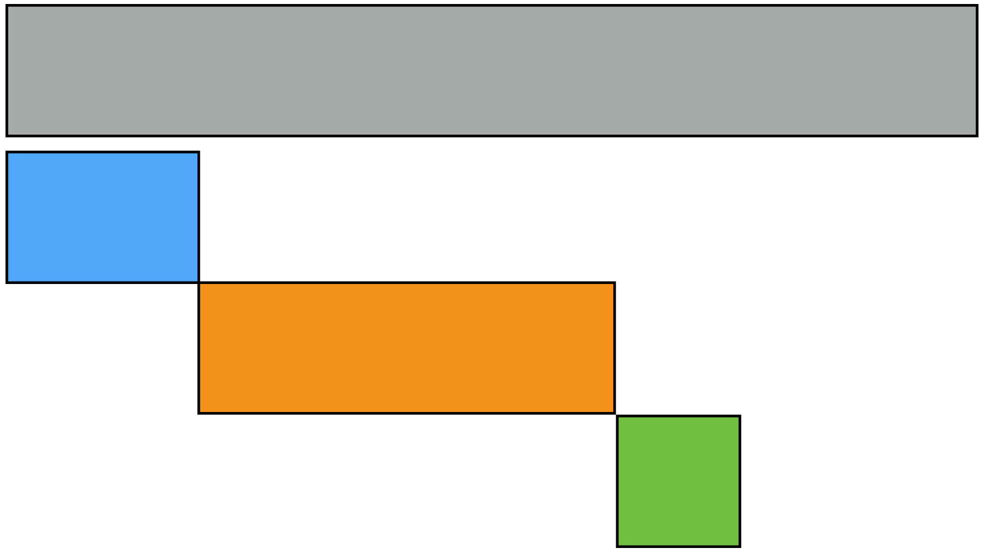

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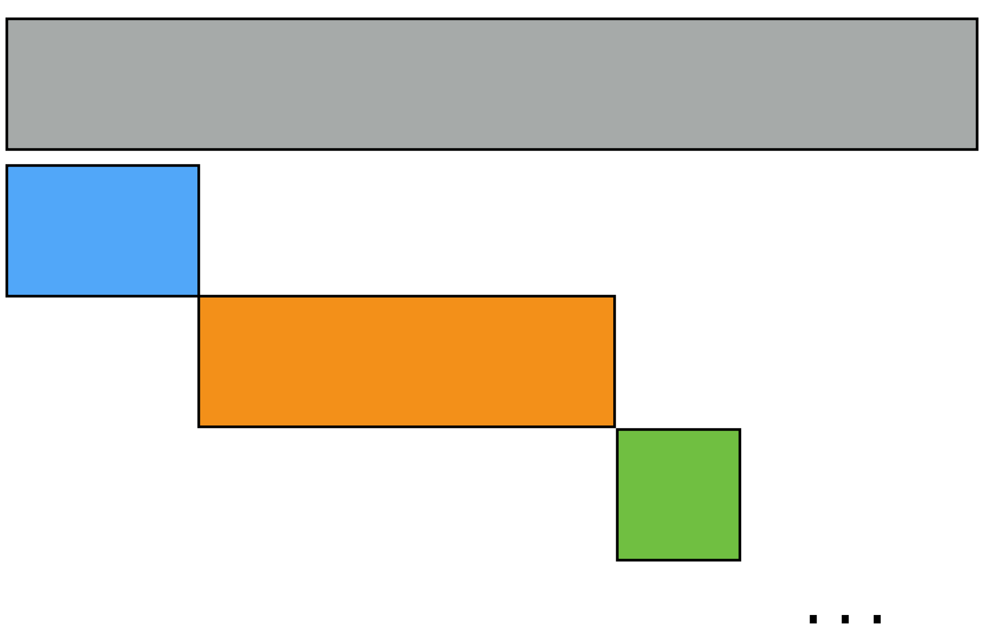

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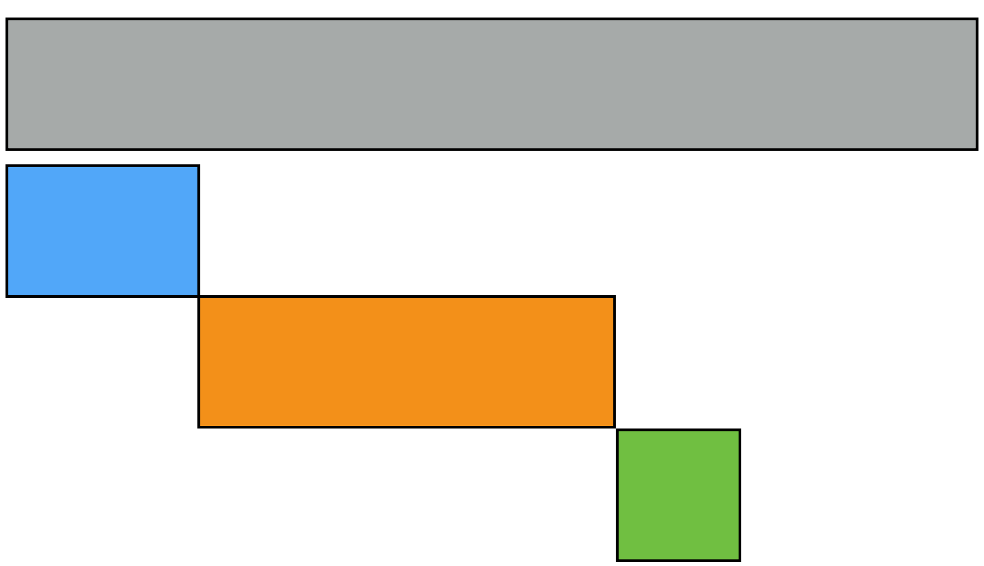

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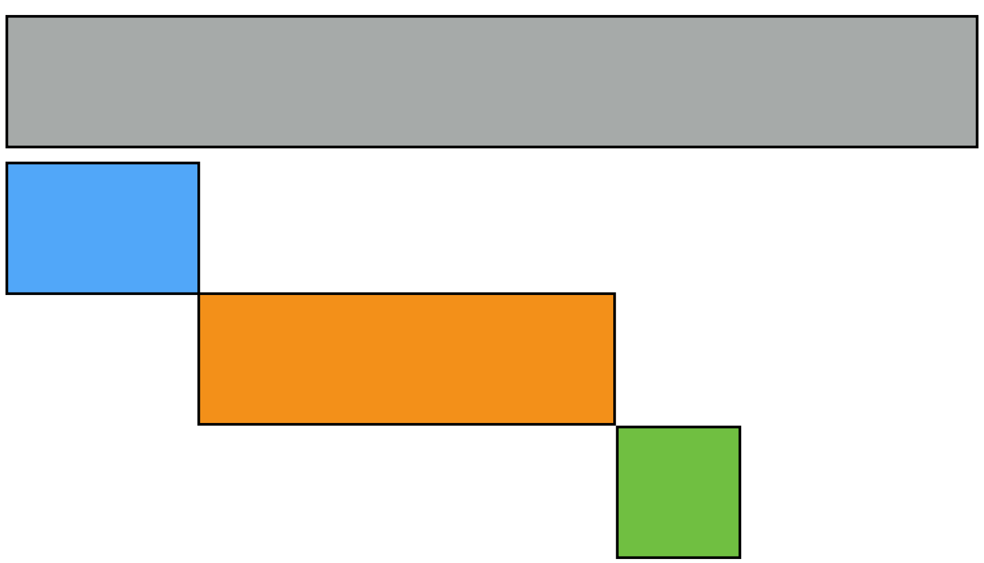

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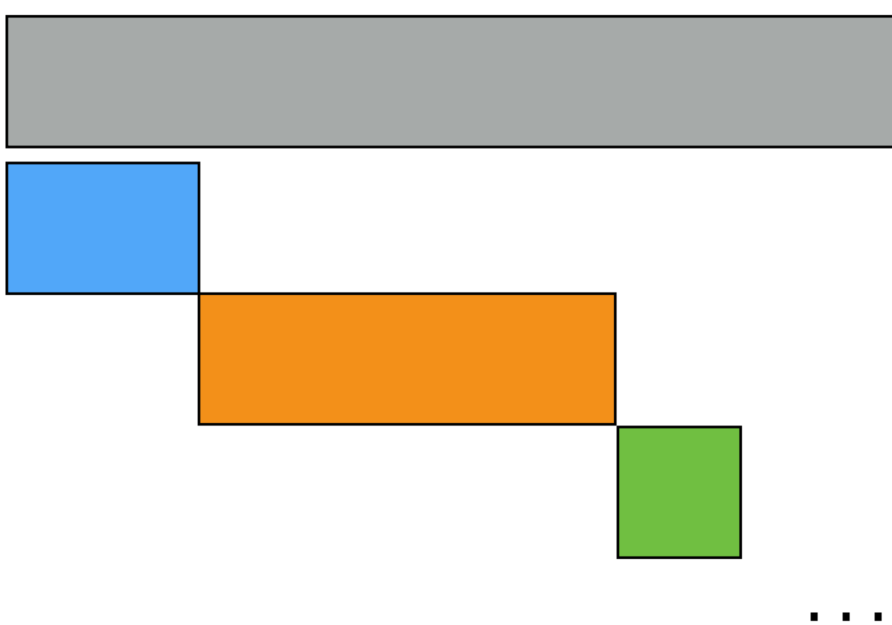

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[van der Vaart, Ghosal 2017]

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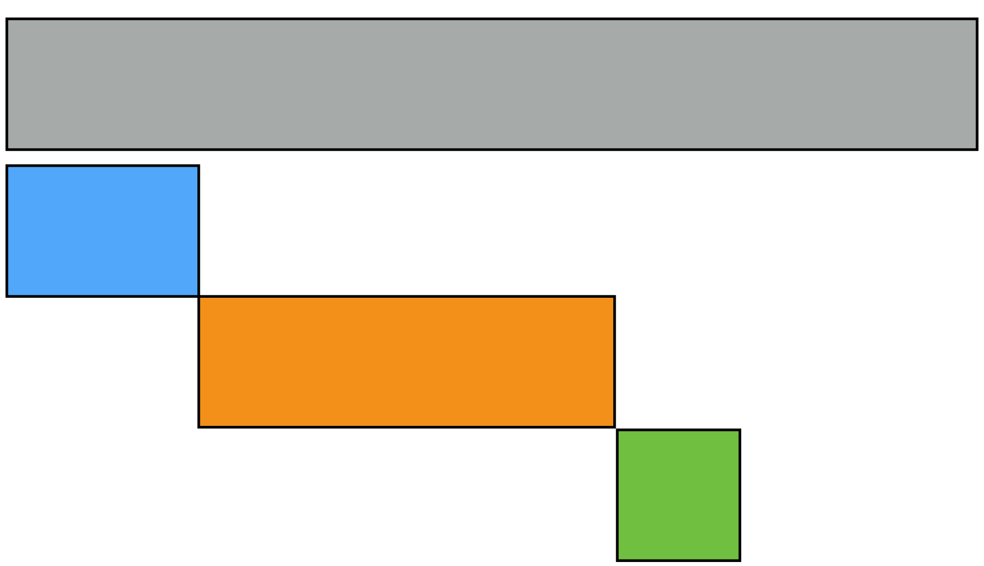

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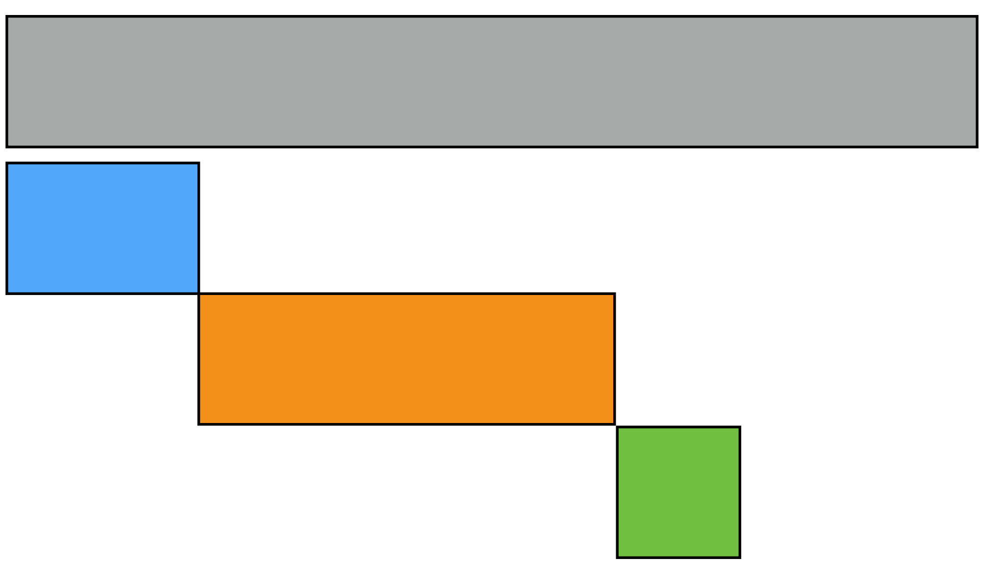

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[McCloskey 1965; Engen 1975; Patil and Taillie 1977; Ewens 1987; Sethuraman 1994; van der Vaart, Ghosal 2017]

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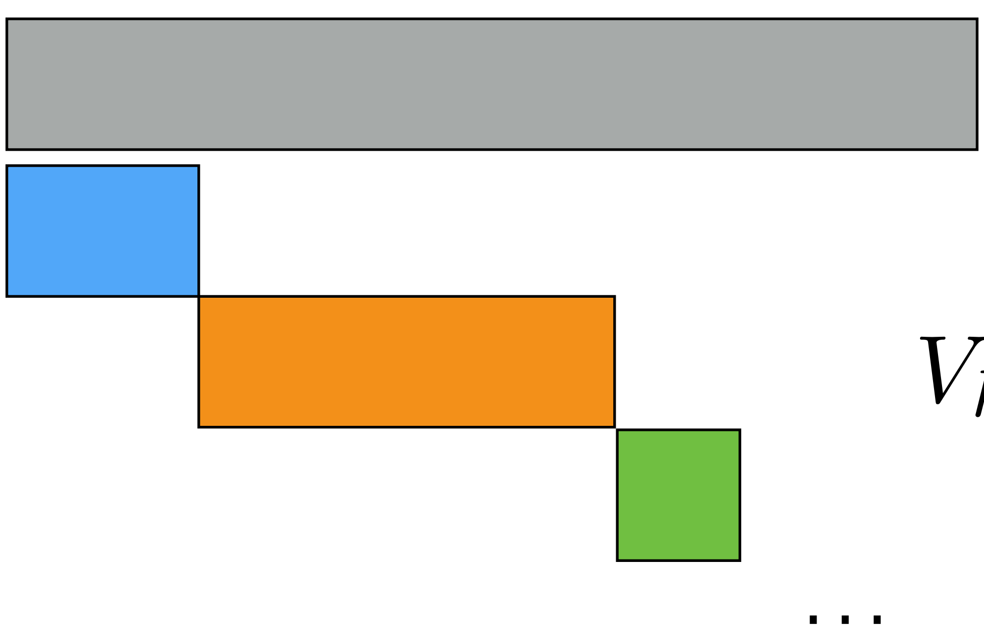

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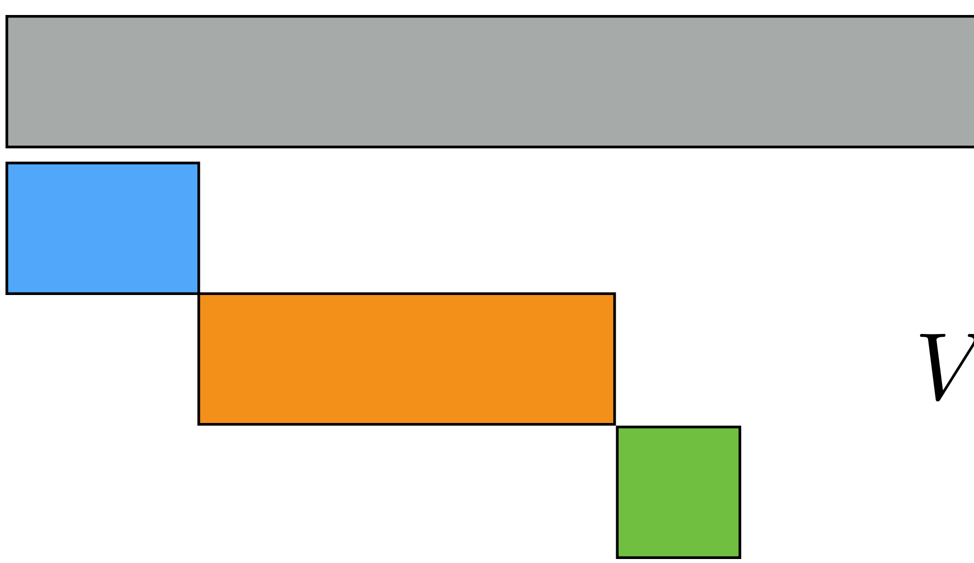

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[demo]

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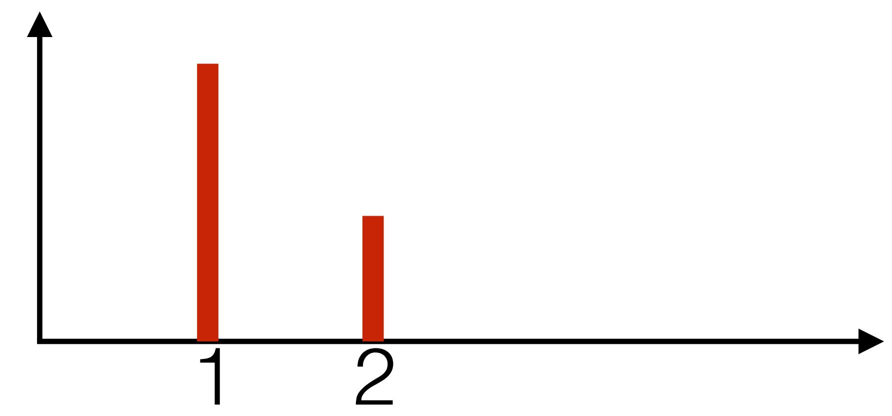

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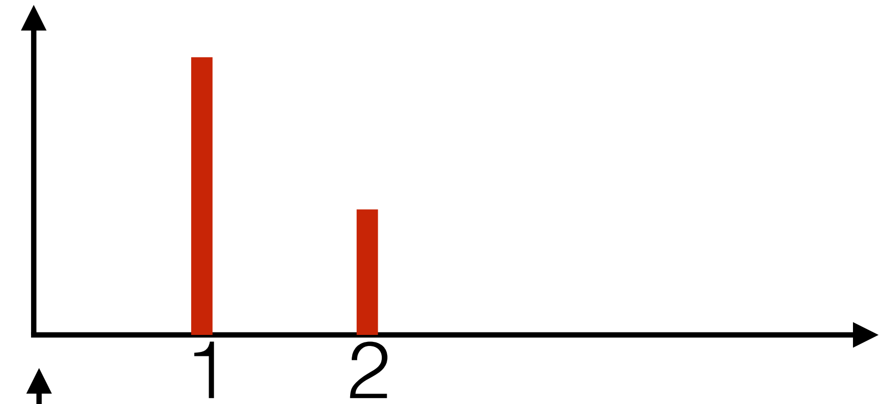

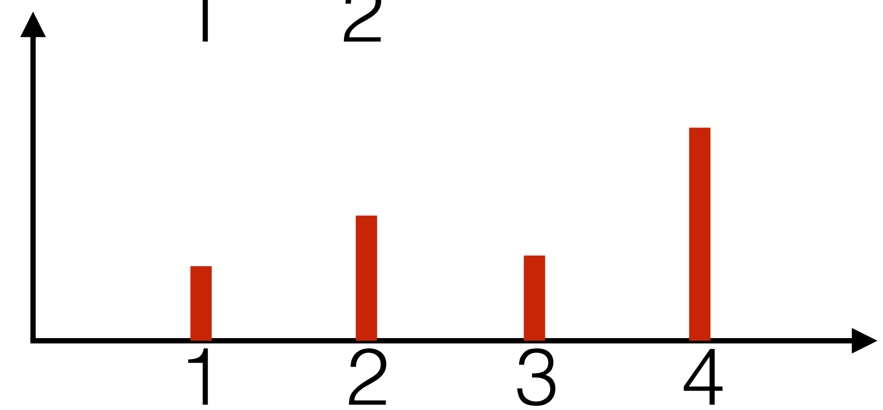

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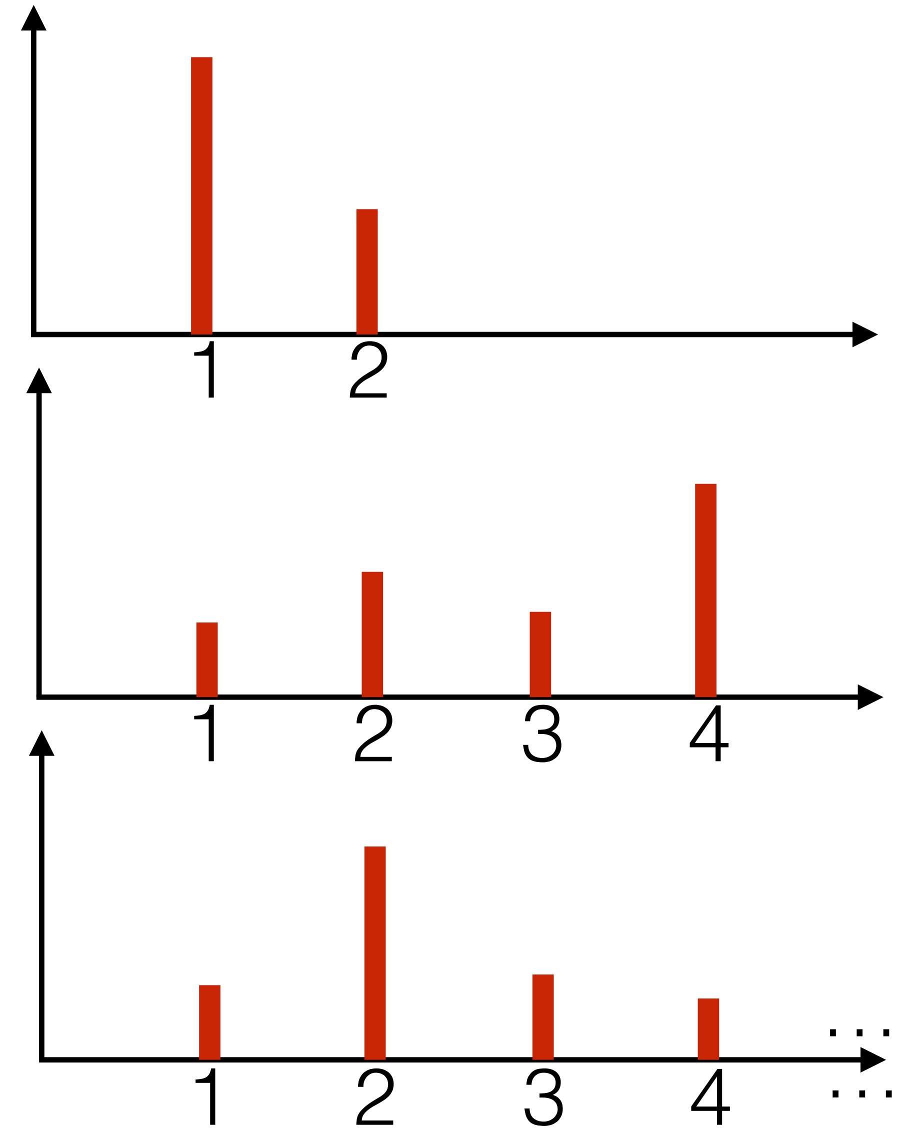

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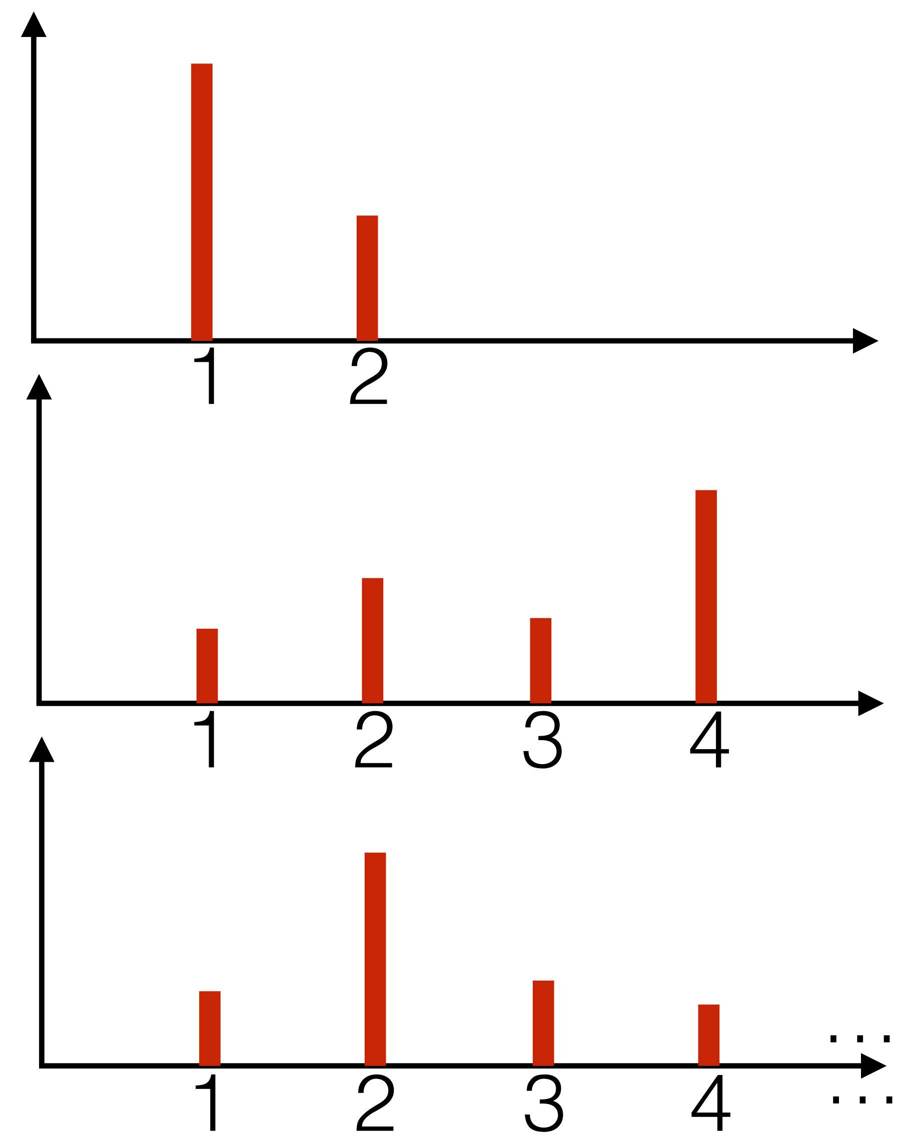

- Infinity of parameters: components
- Growing number of parameters: clusters

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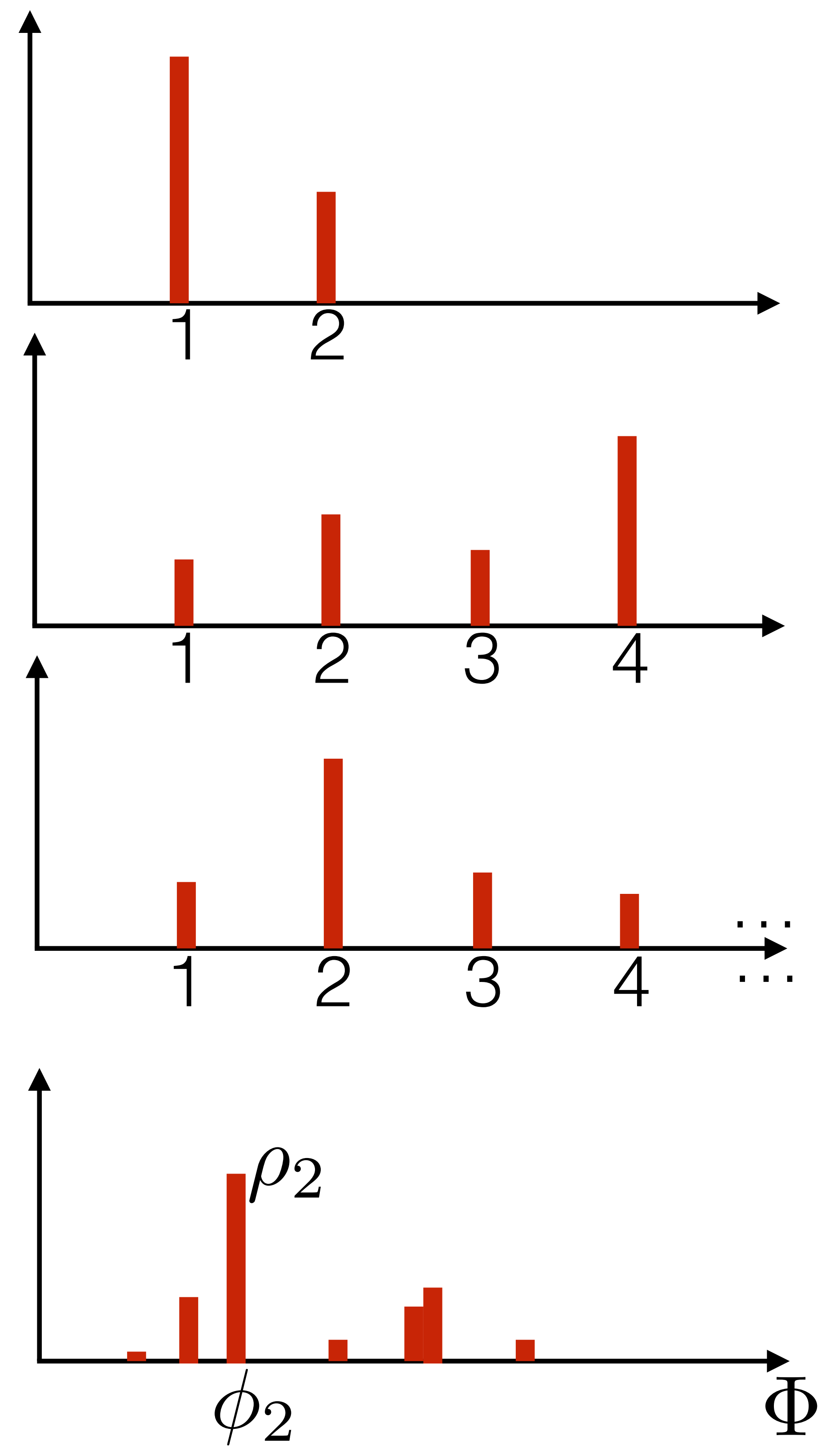

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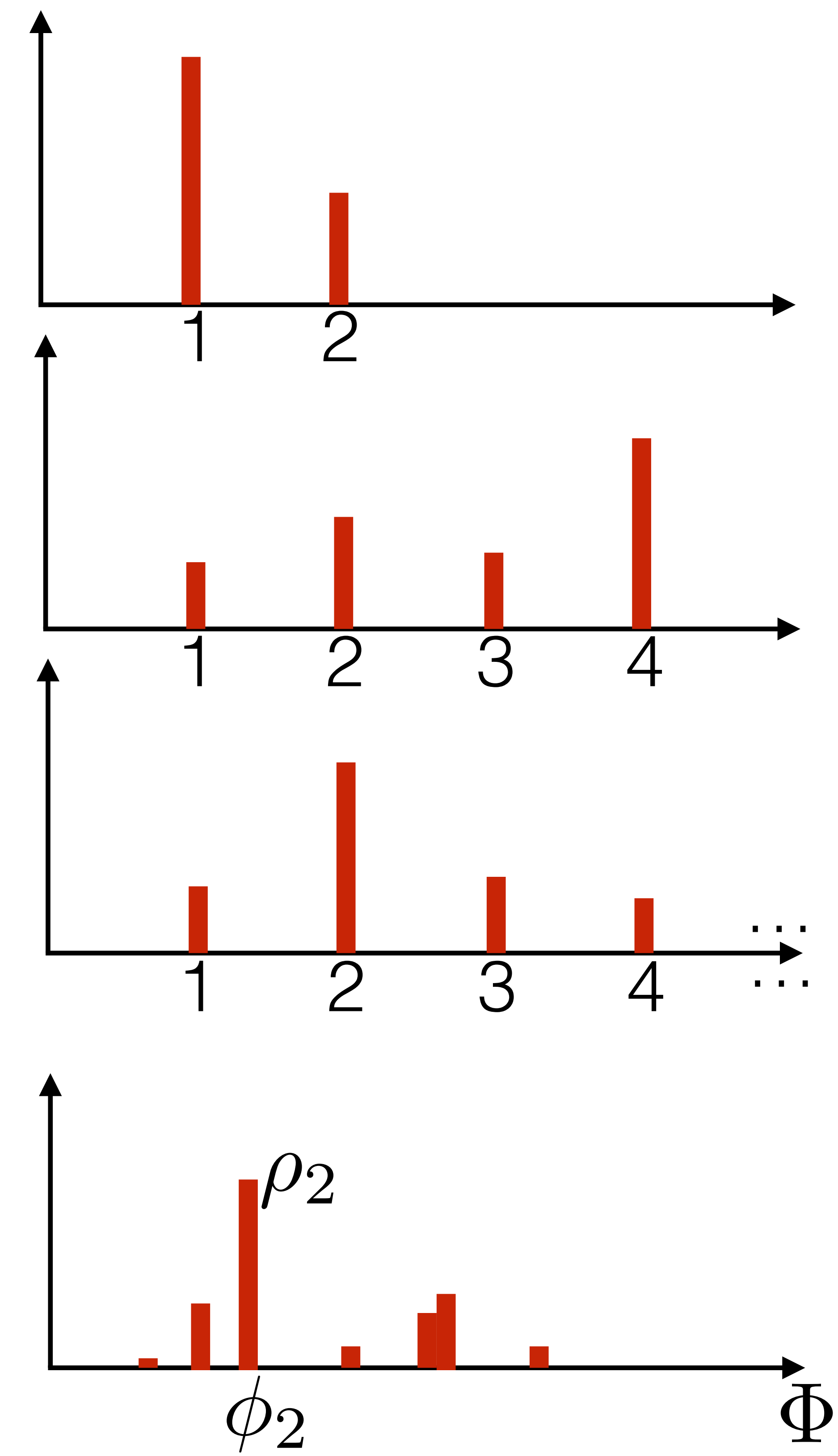

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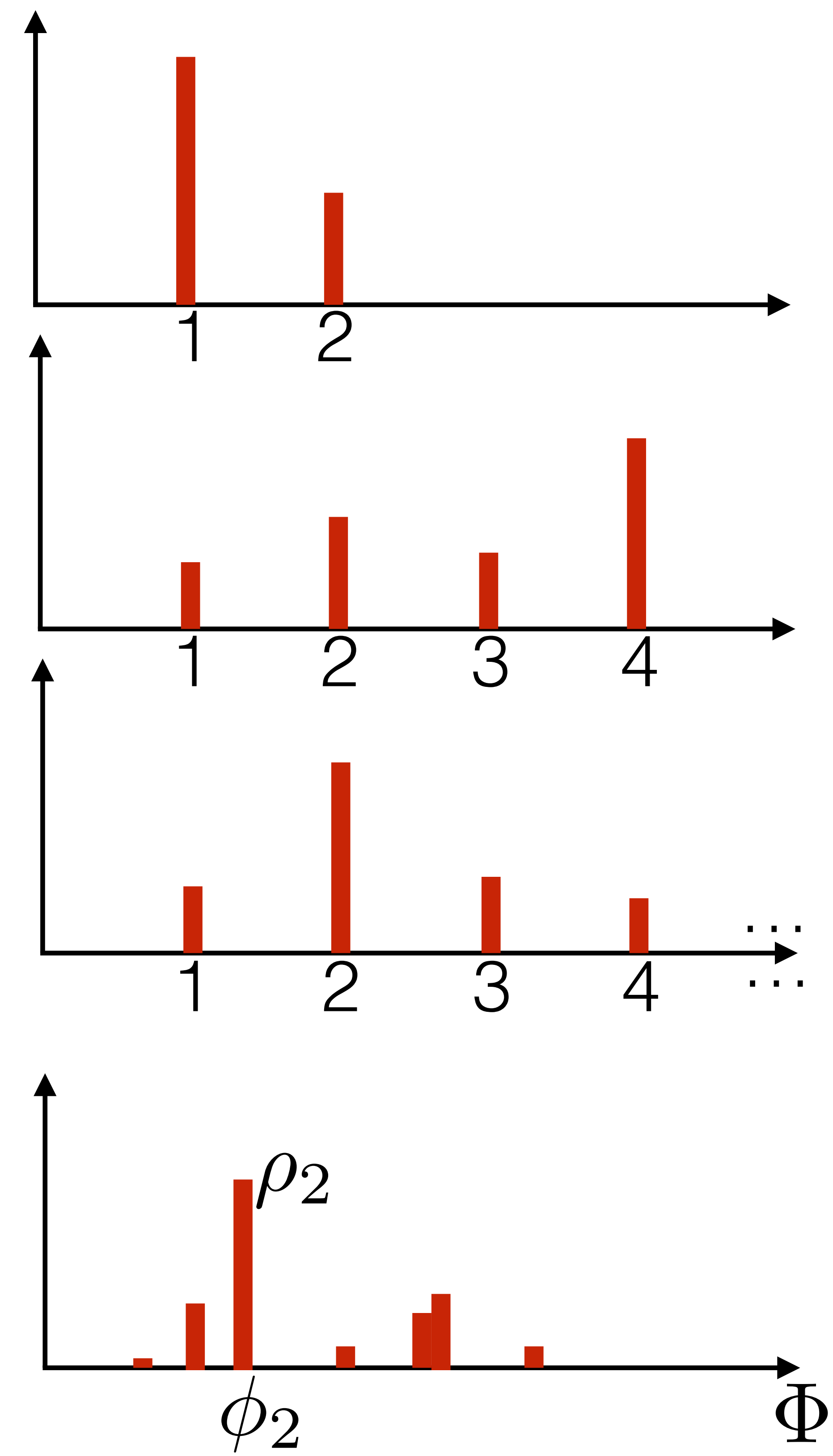

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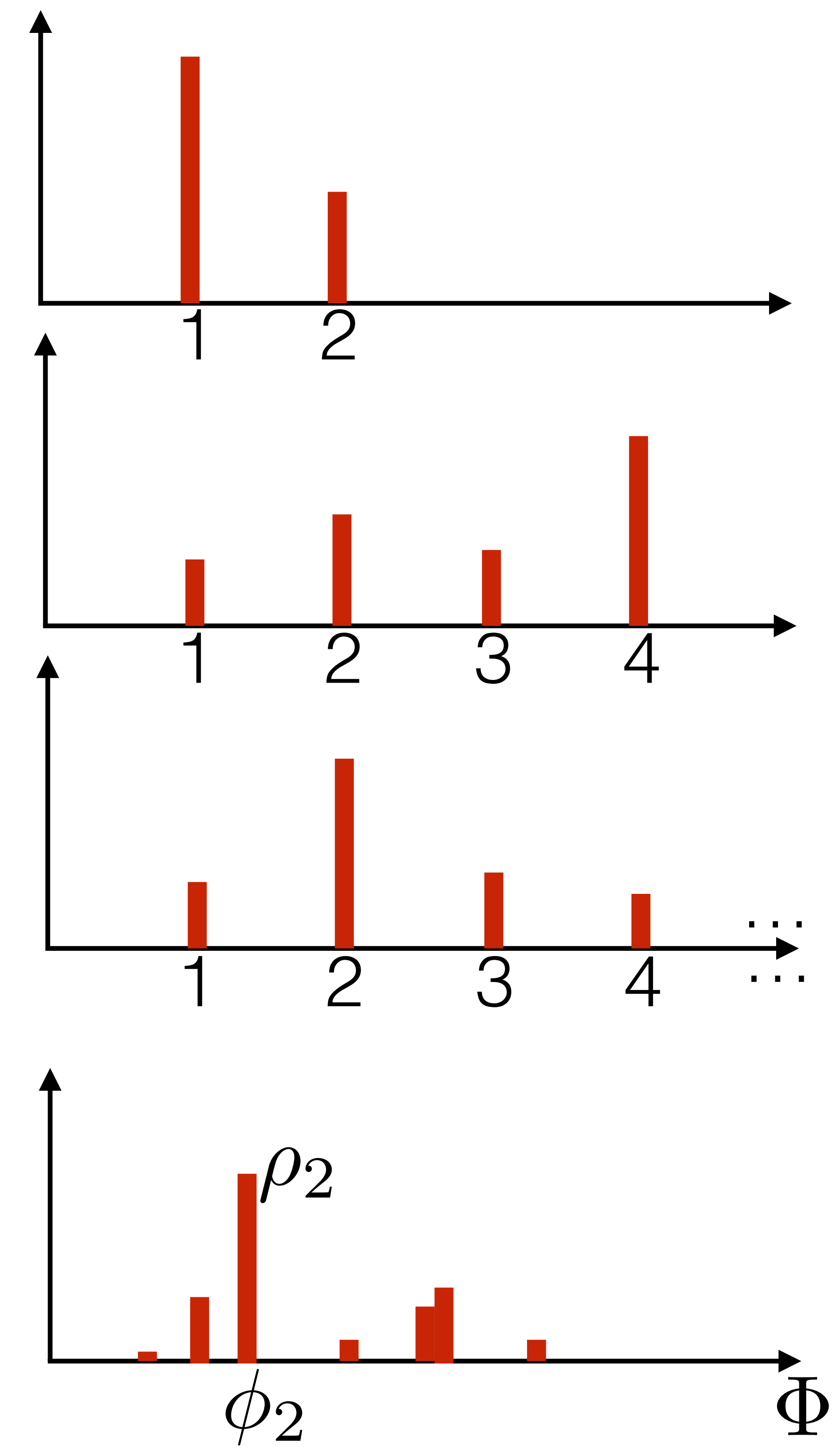

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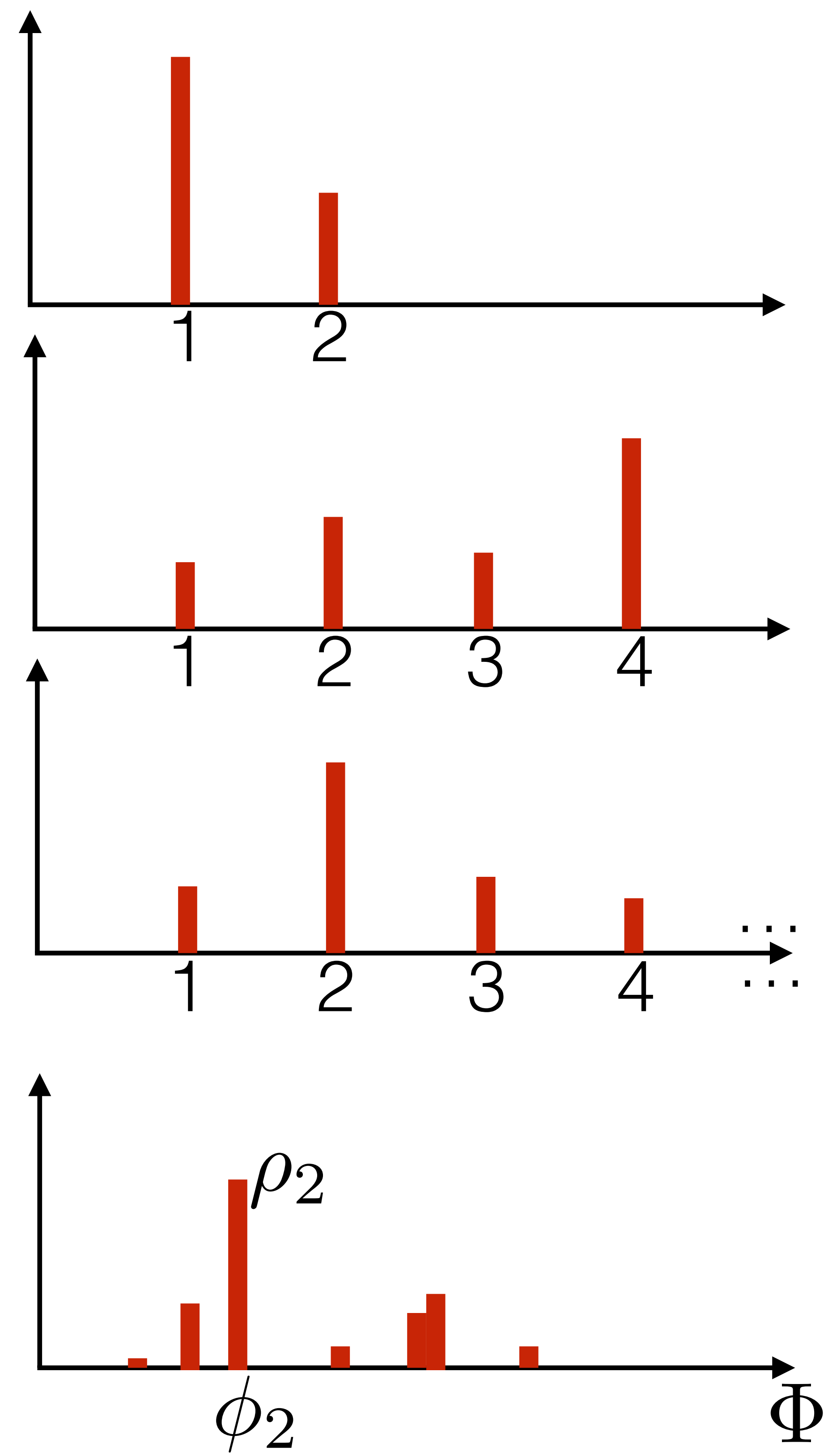

[Ferguson 1973]

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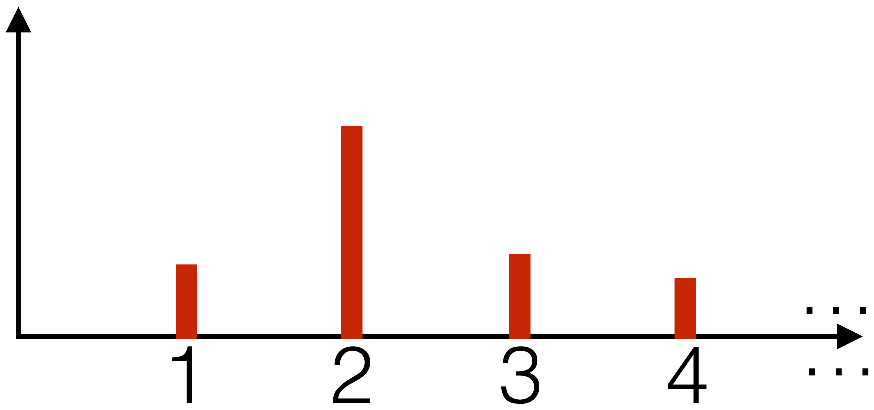

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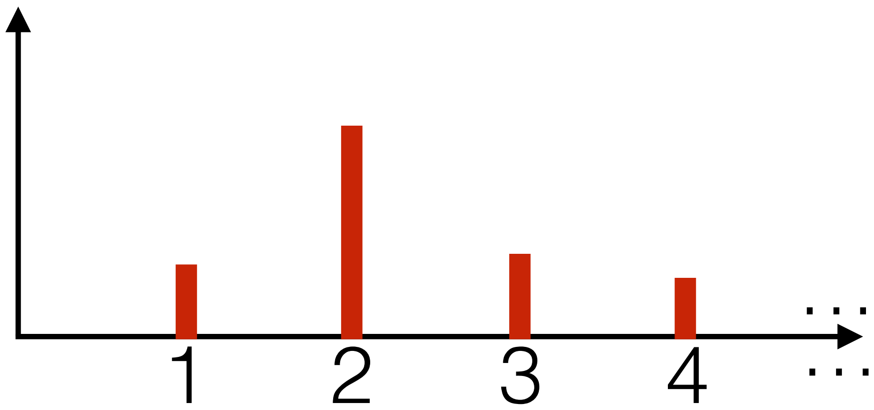

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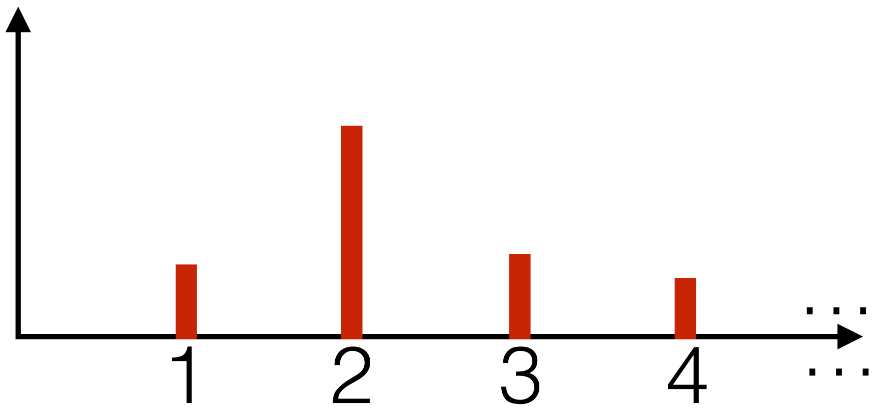

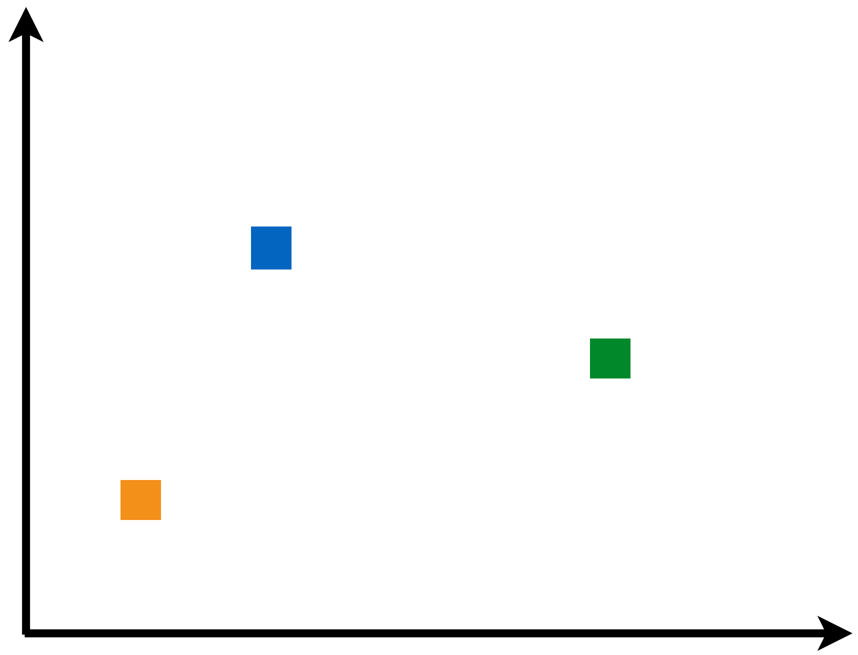

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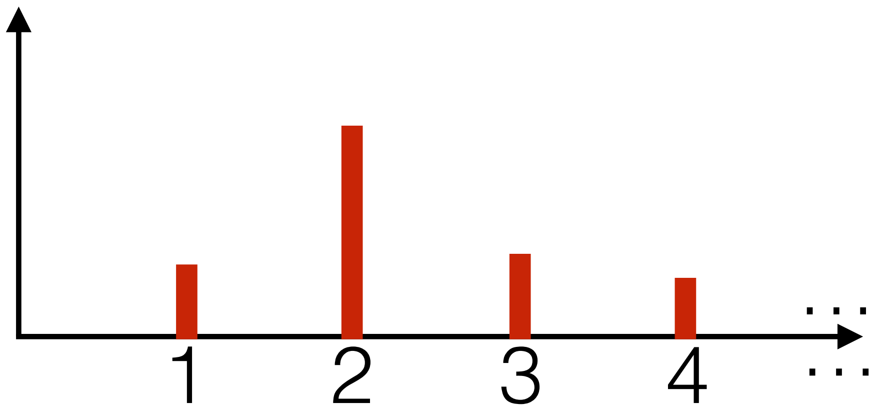

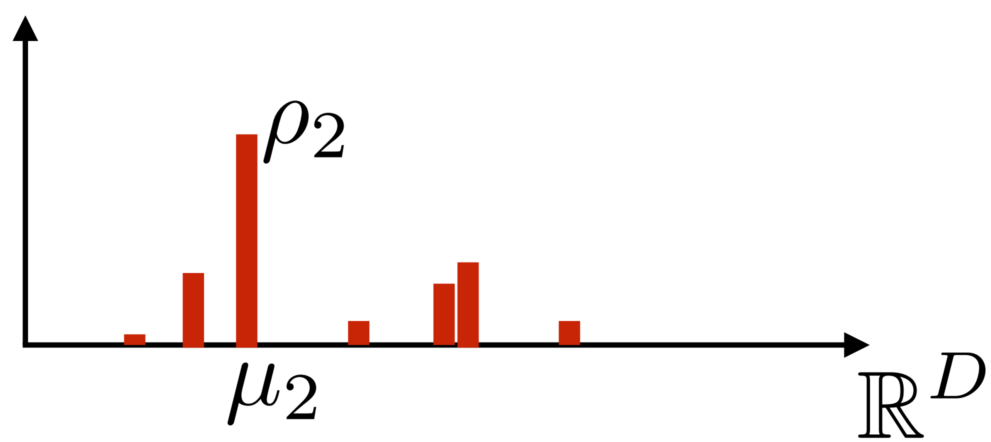

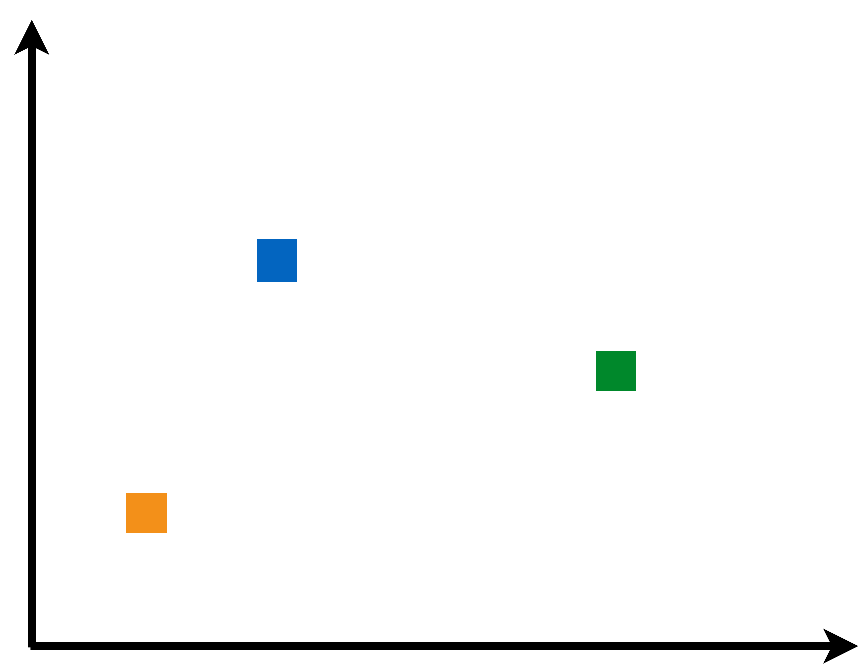

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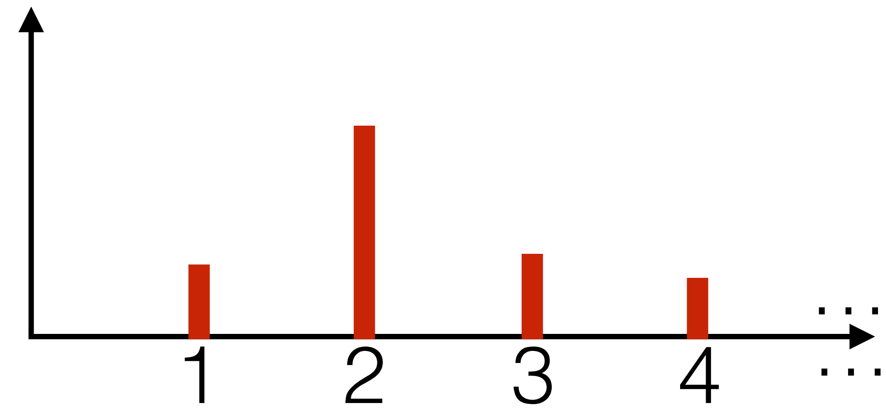

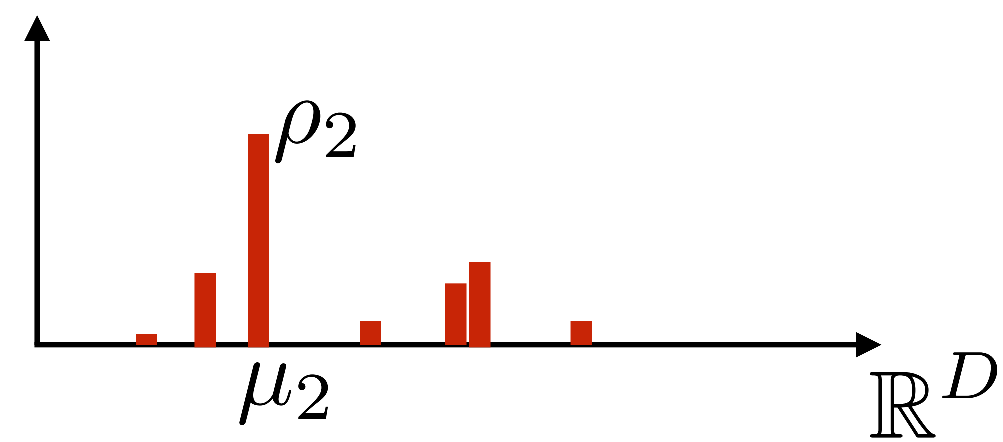

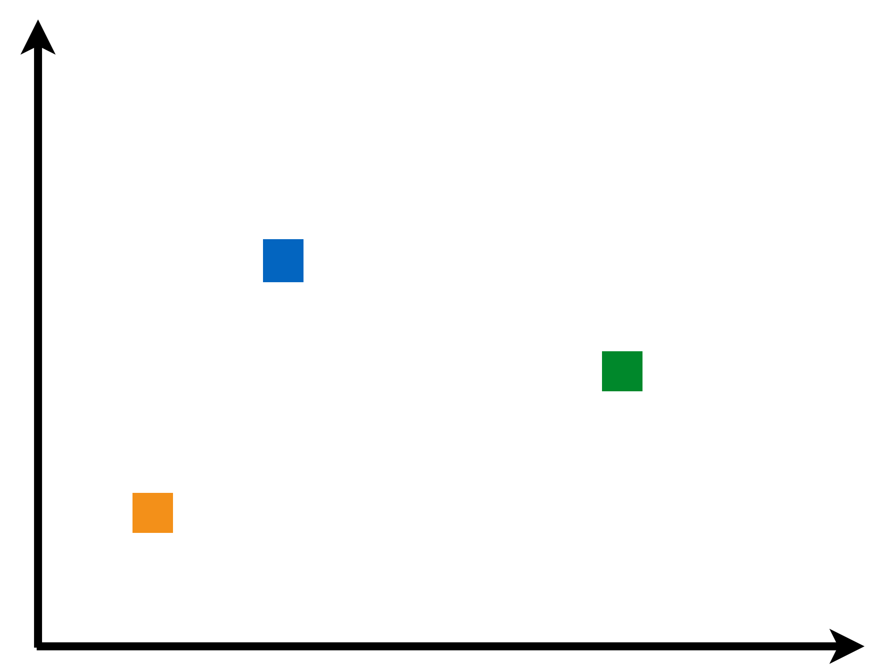

Dirichlet process mixture model

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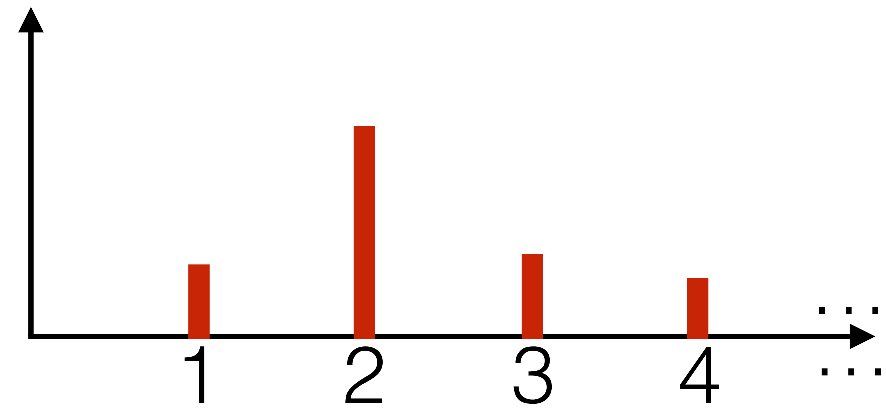

$$\rho = (\rho_1, \rho_2, \ldots) \sim \mathrm{GEM}(\alpha)$$

$$\mu_k \overset{iid}{\sim} \mathcal{N}(\mu_0, \Sigma_0), k = 1, 2, \ldots$$

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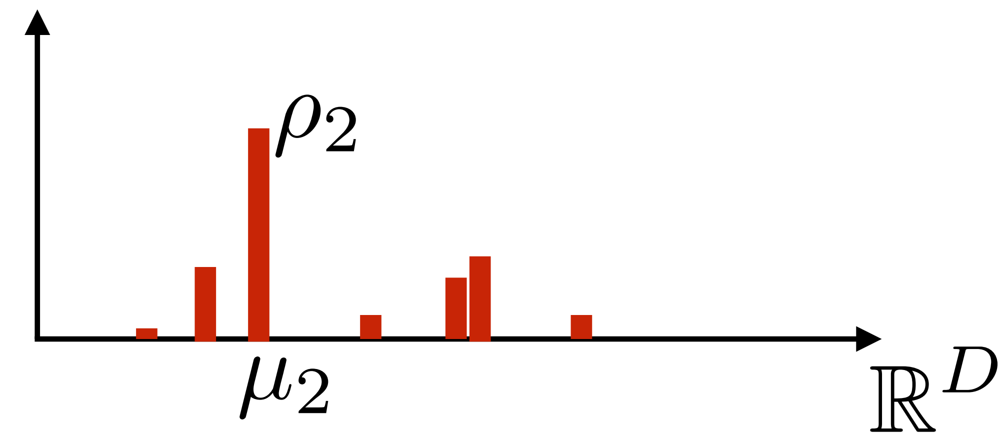

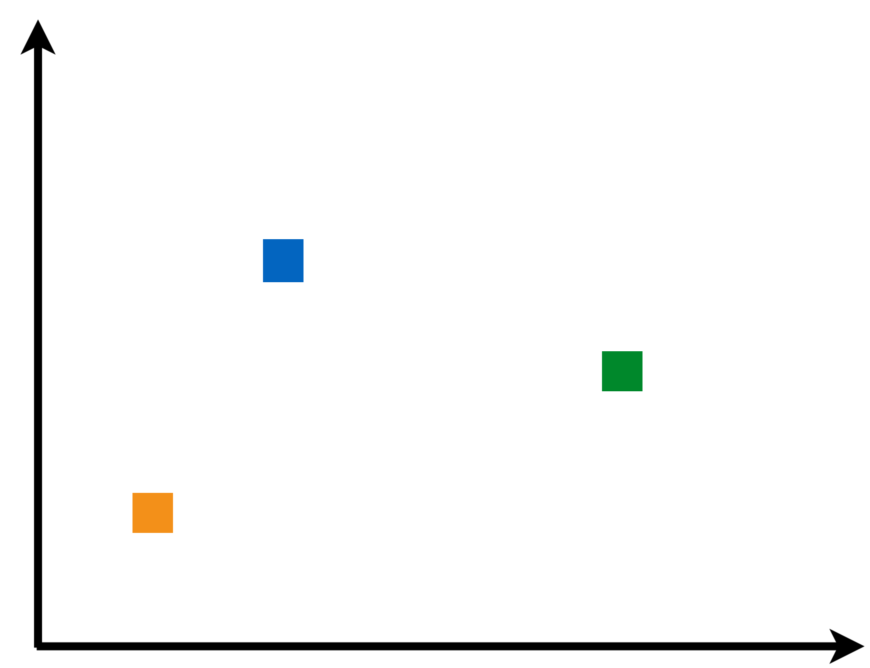

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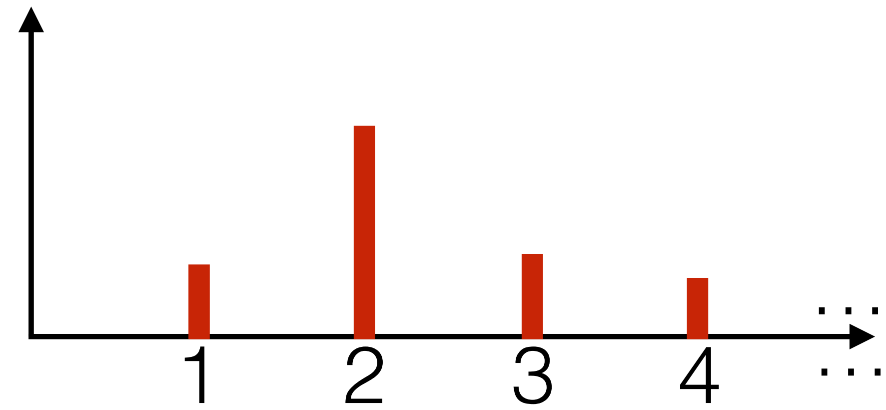

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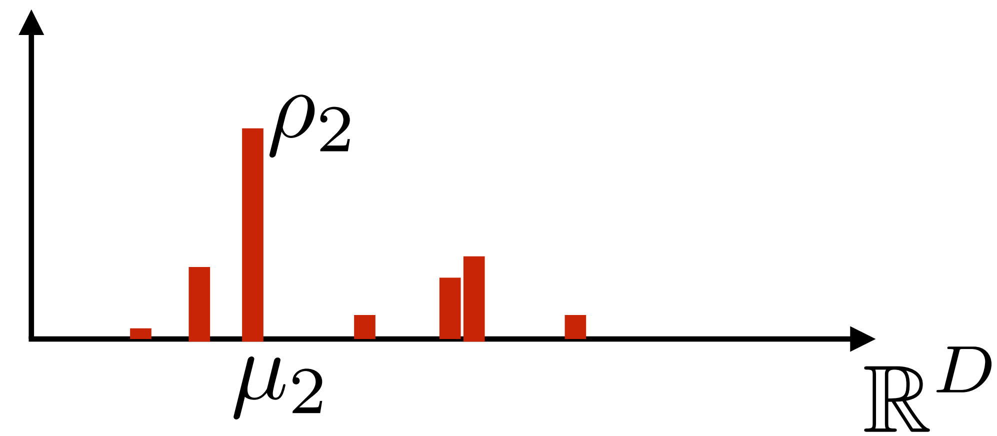
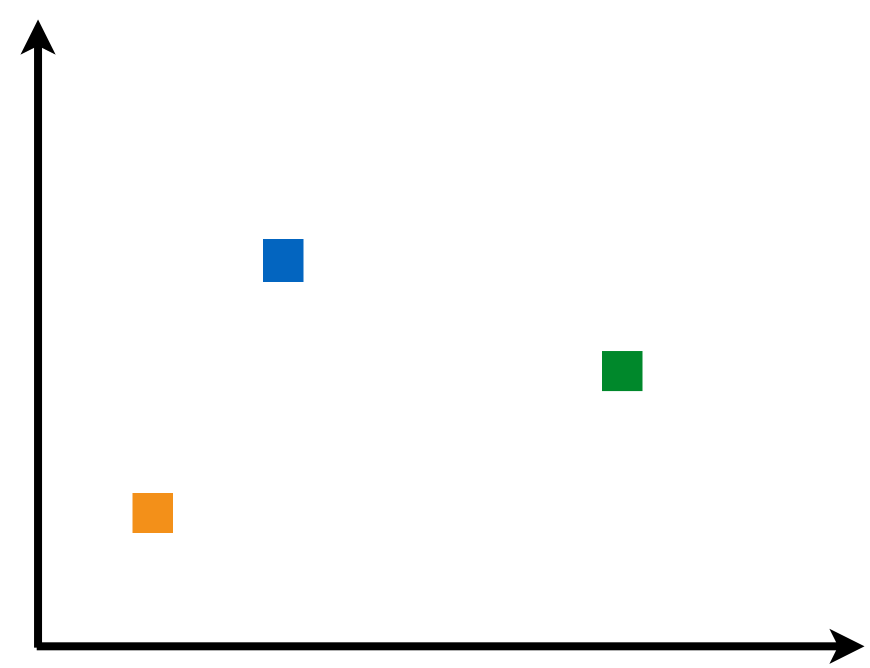


$$z_n \overset{iid}{\sim} \text{Categorical}(\rho)$$



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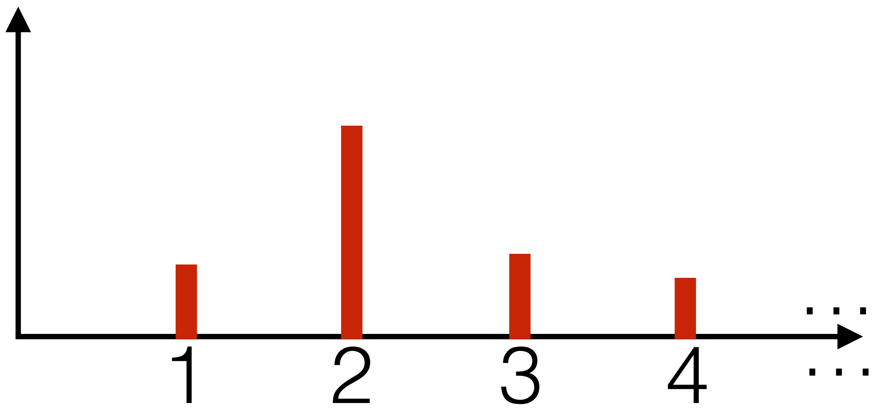

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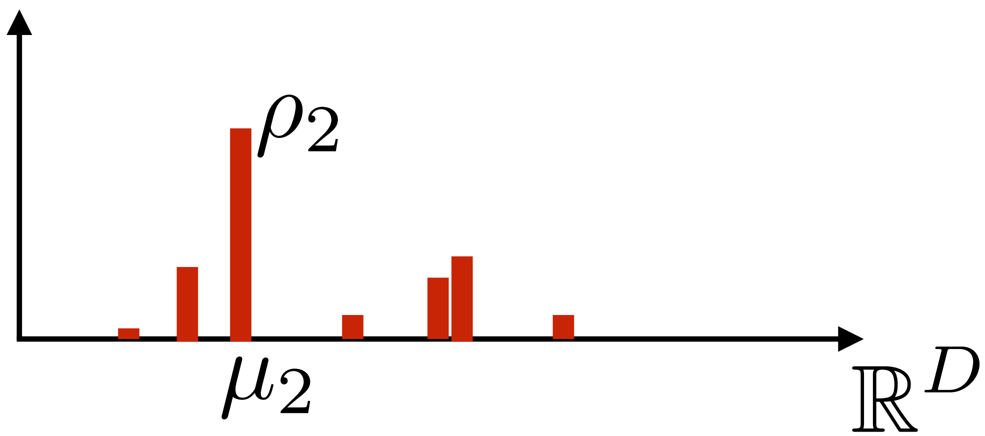

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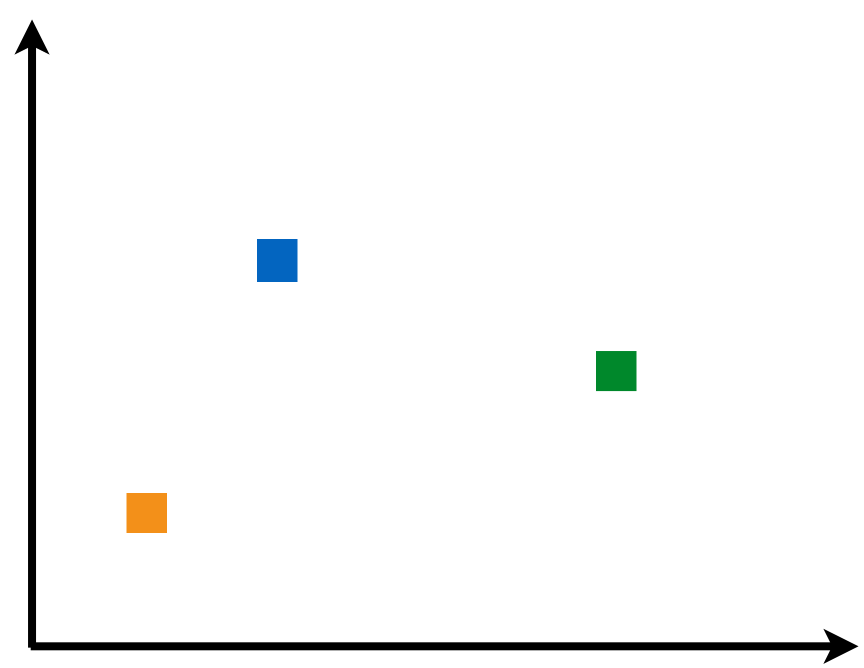

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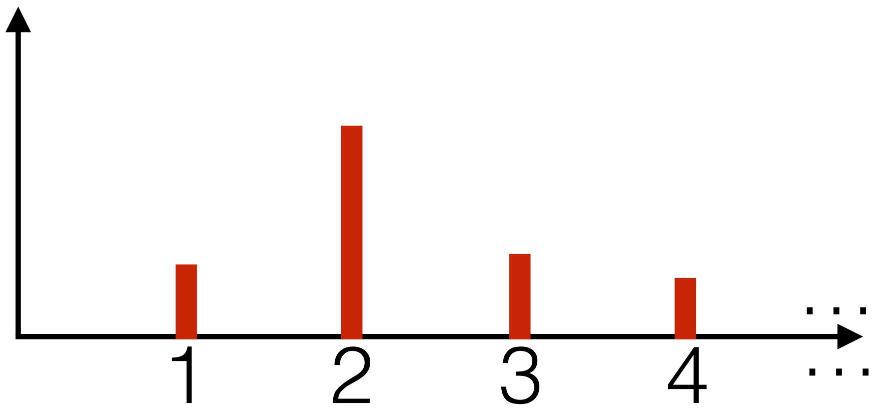

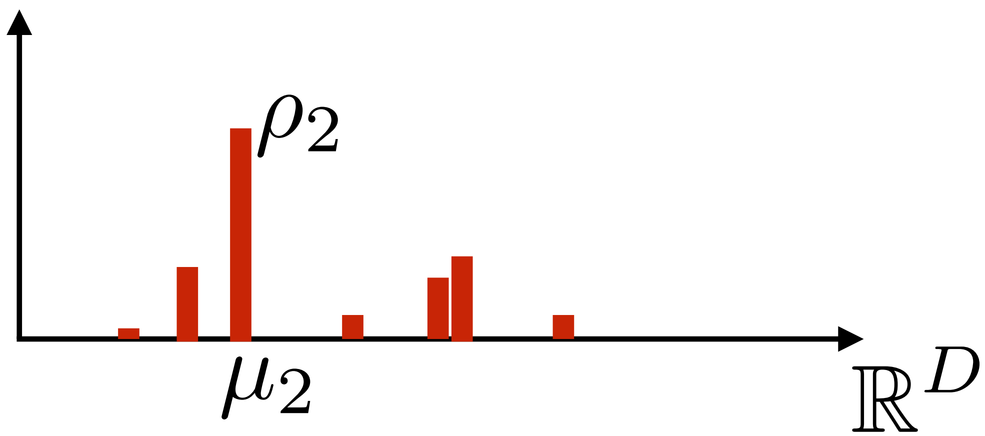

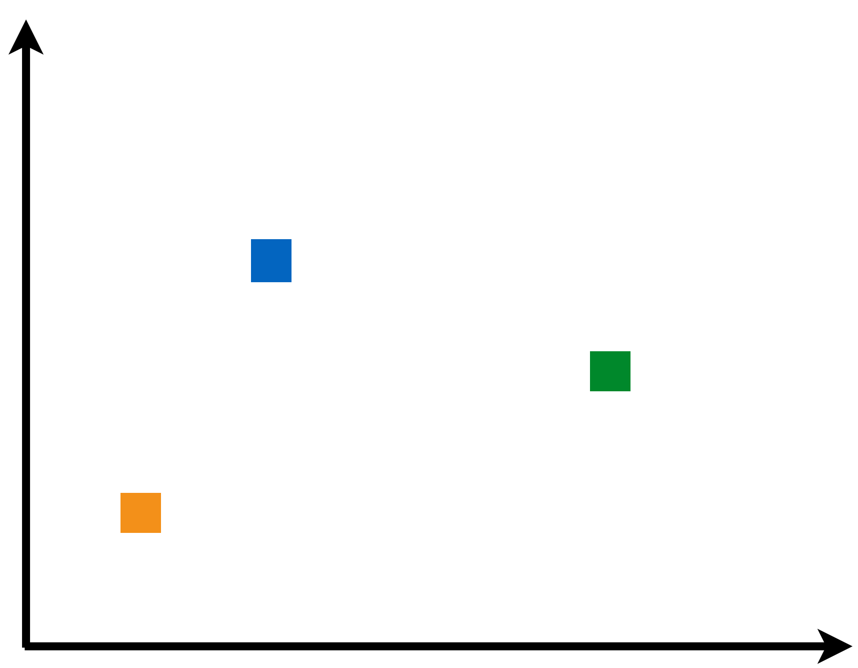

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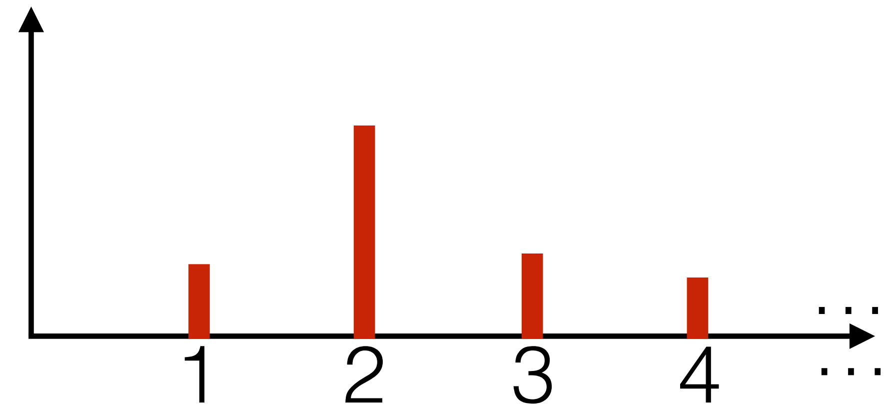

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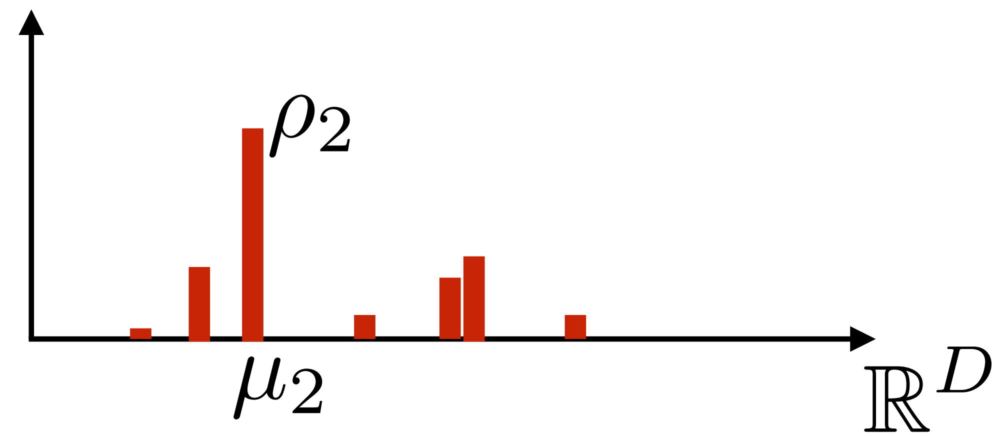

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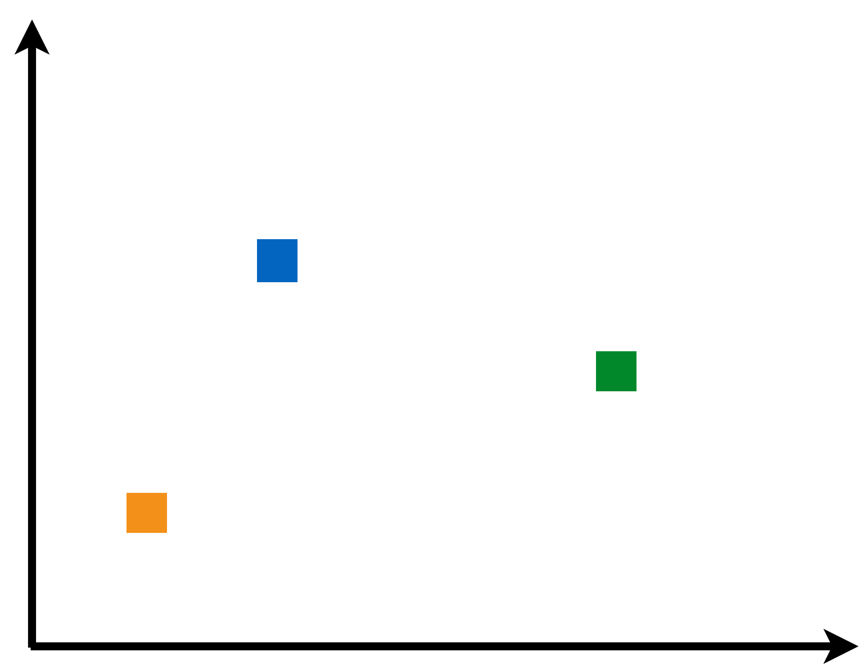

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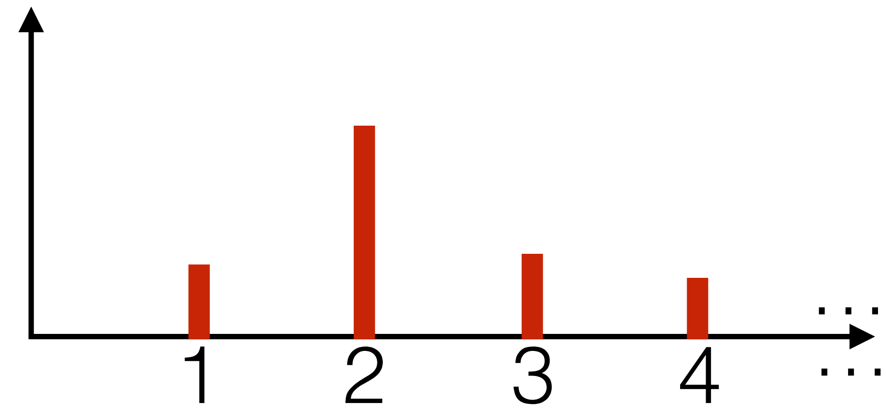

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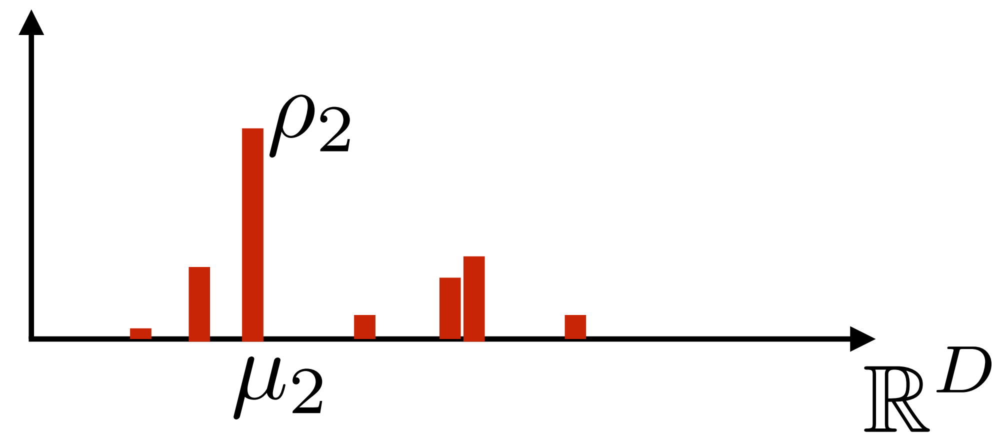

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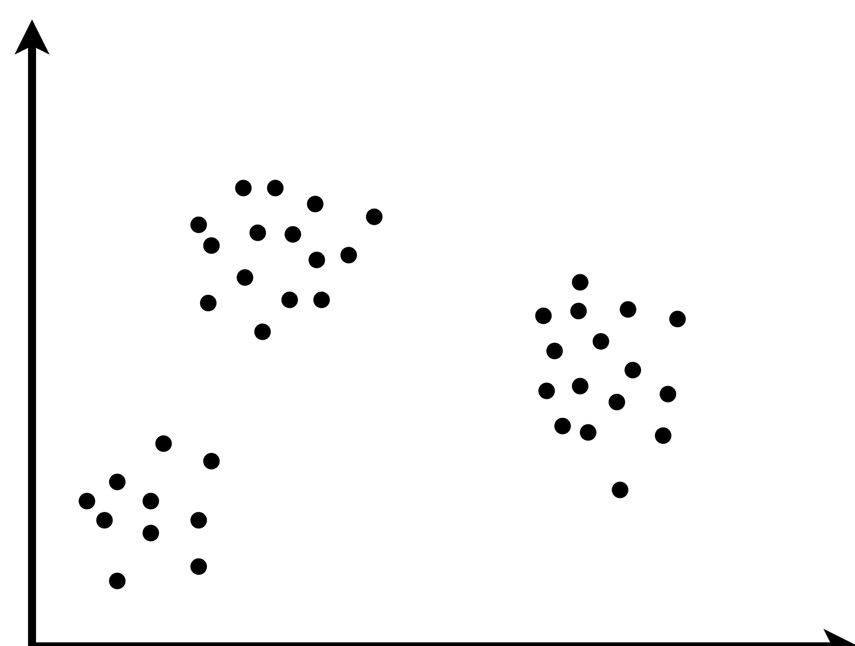

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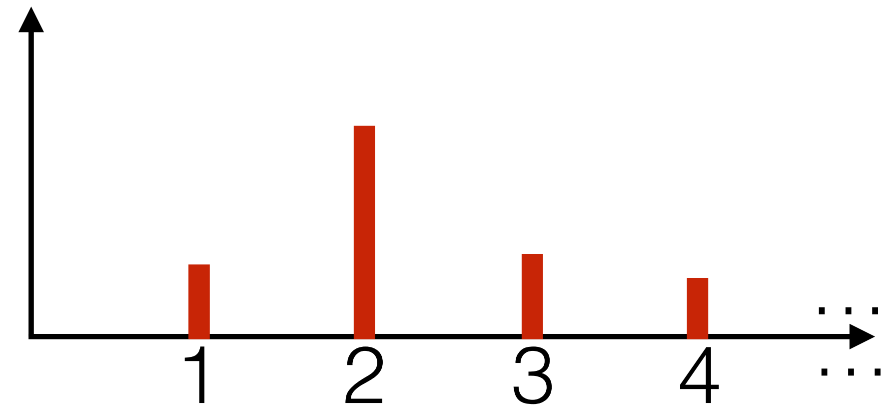

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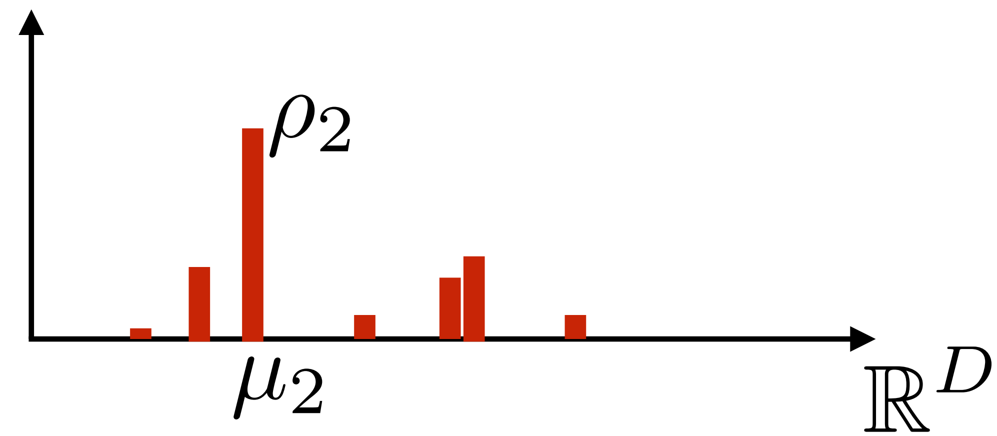

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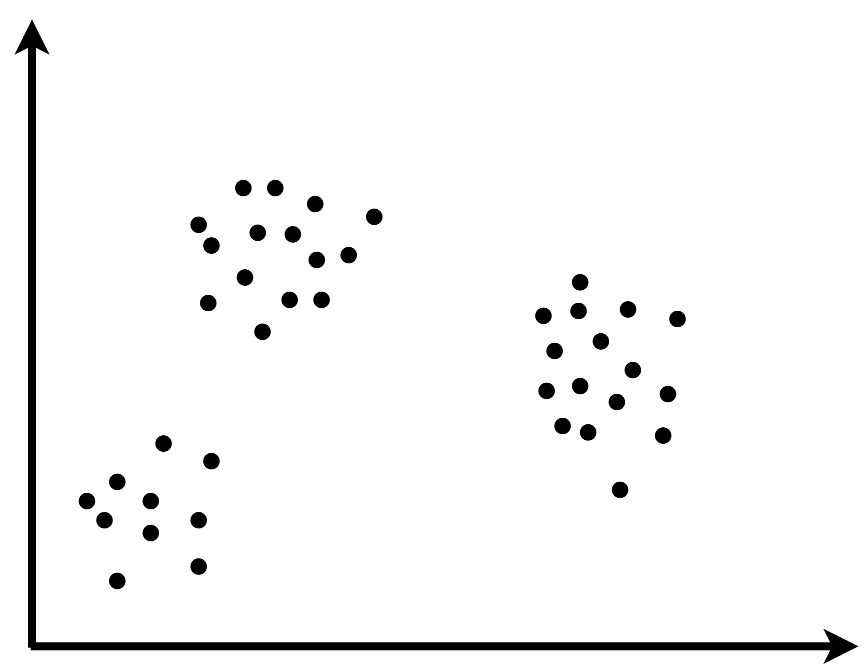

$$x_n \stackrel{indep}{\sim} \mathcal{N}(\mu_n^*, \Sigma)$$

[demo]

Dirichlet process mixture model

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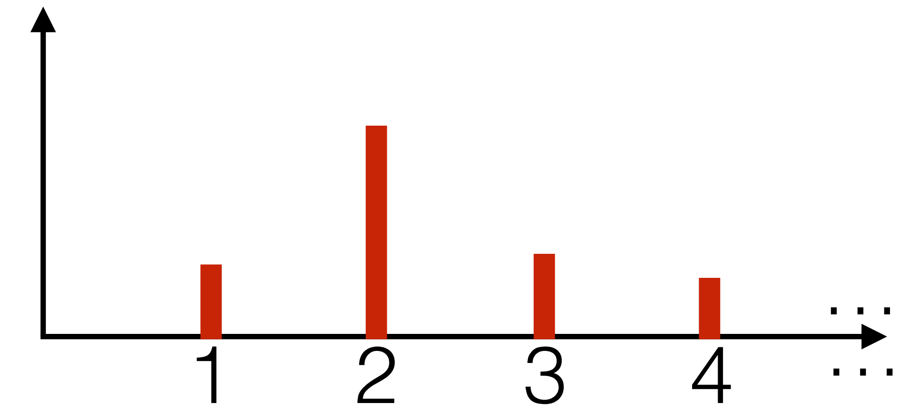

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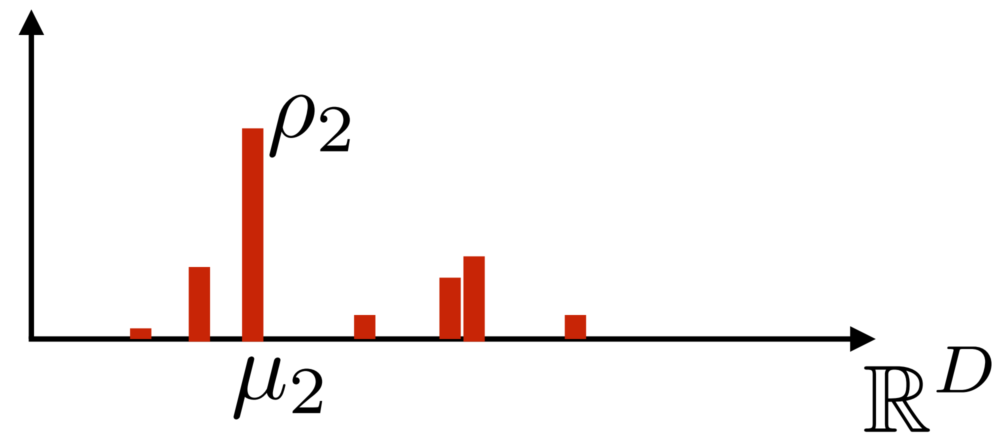

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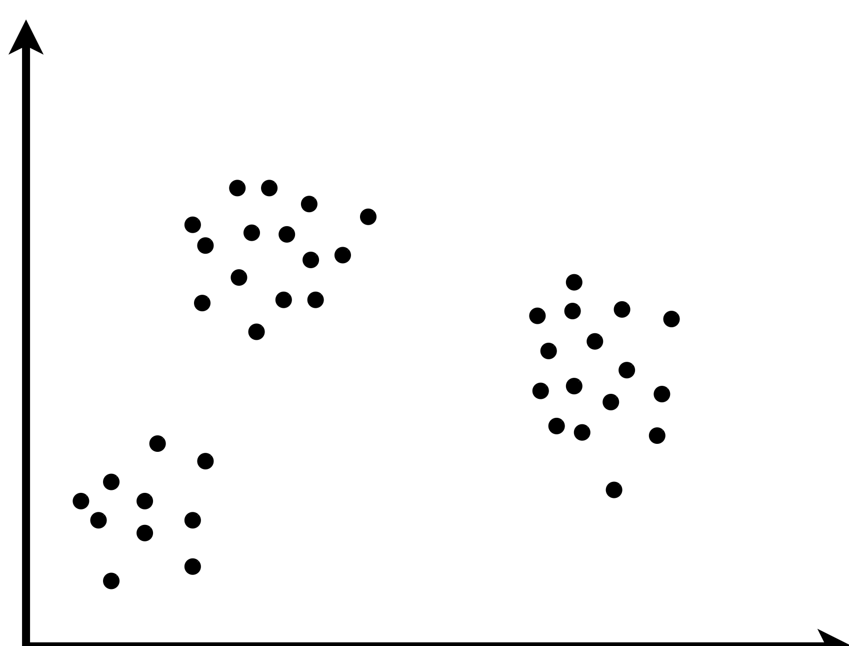

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Dirichlet process mixture model

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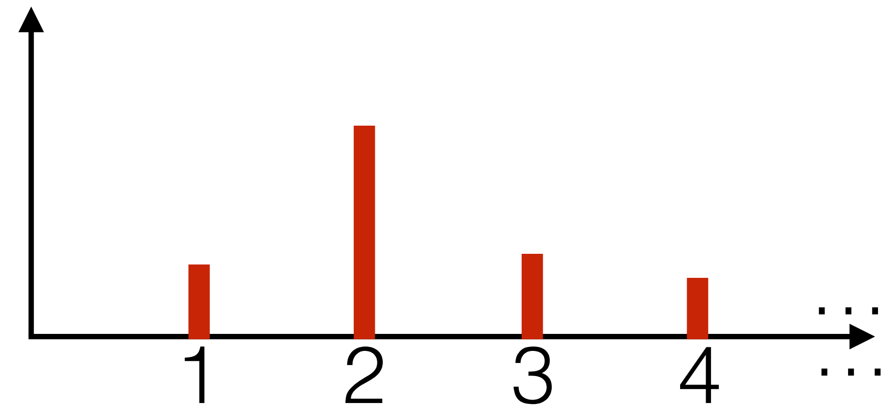

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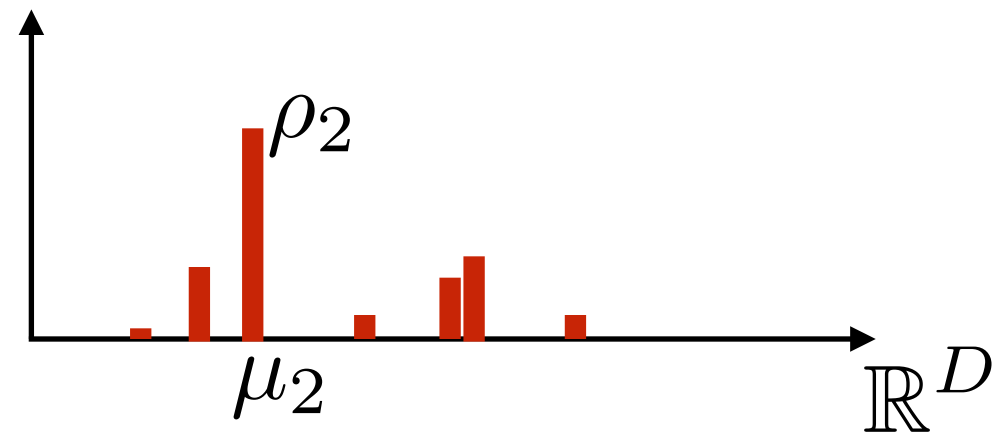

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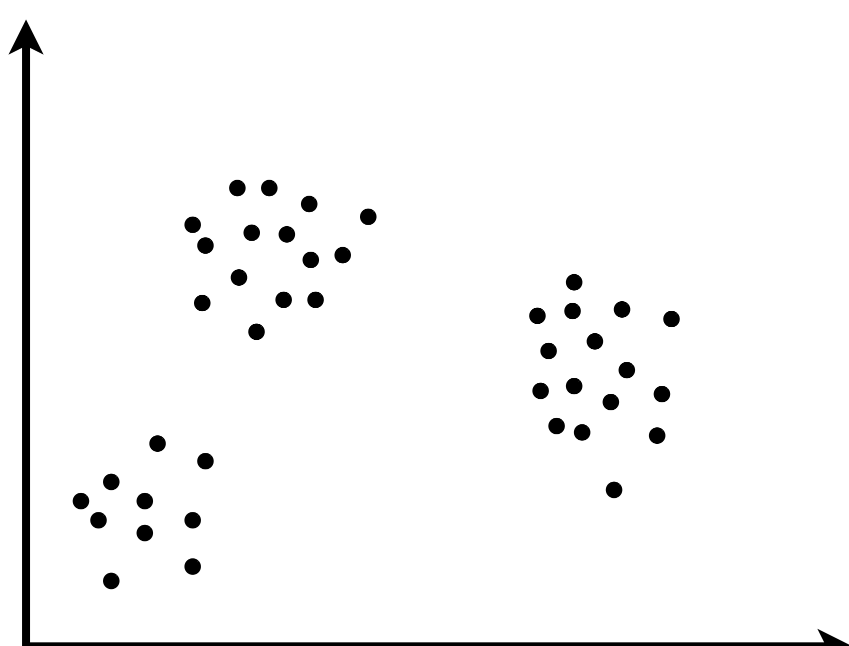

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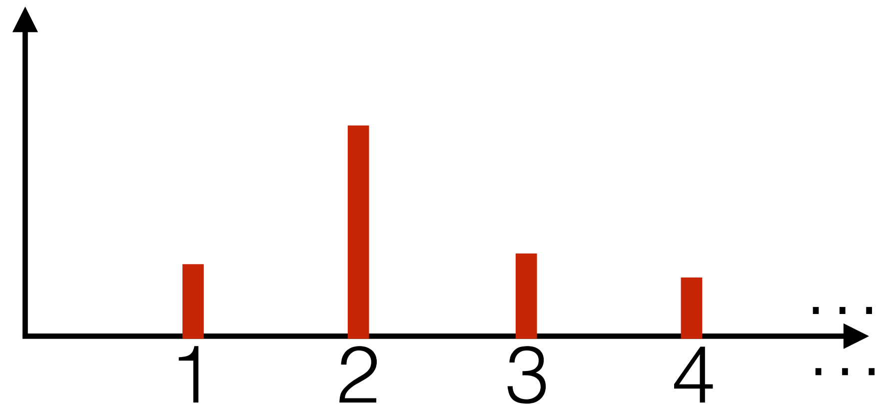

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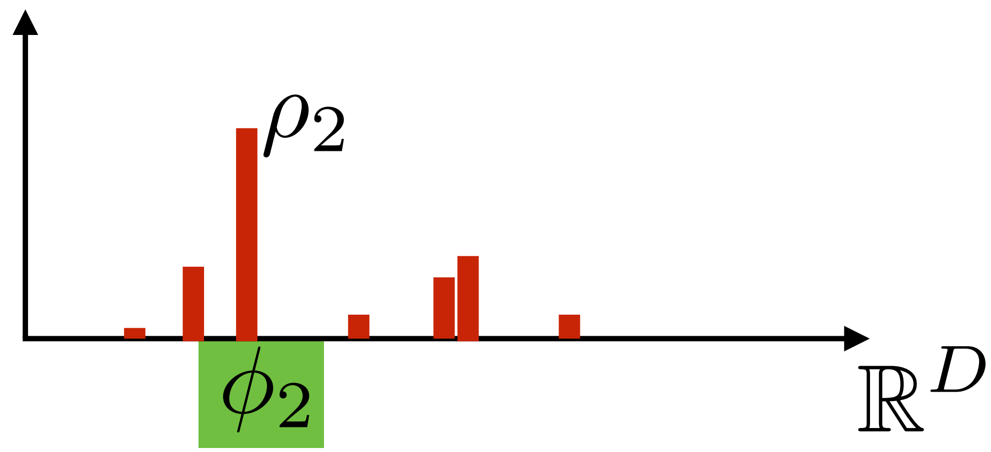

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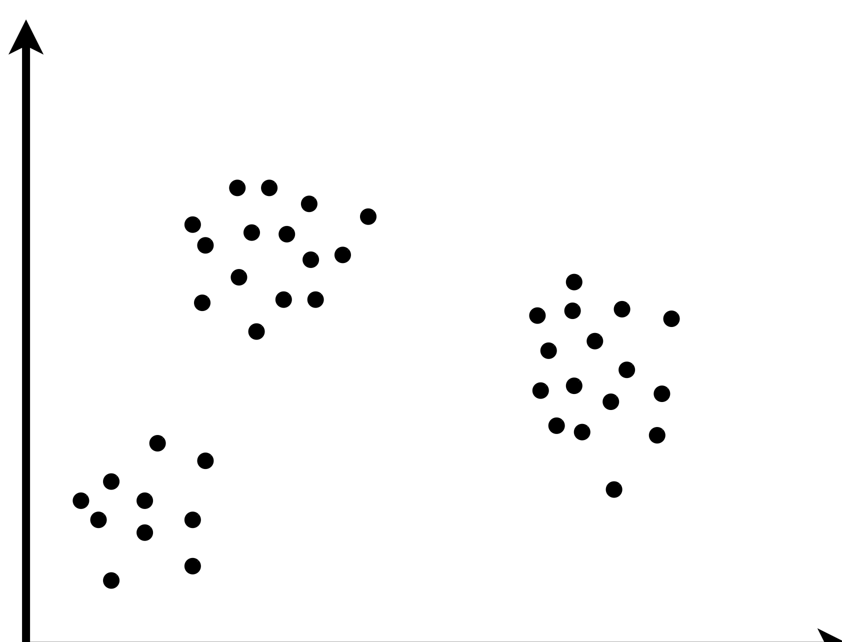

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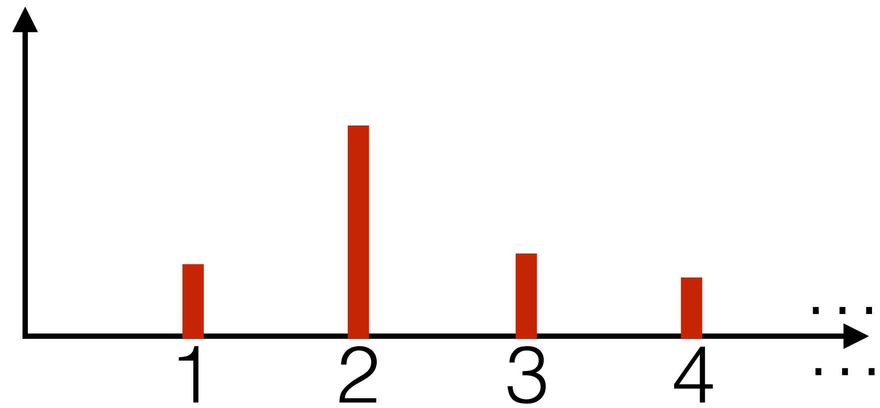

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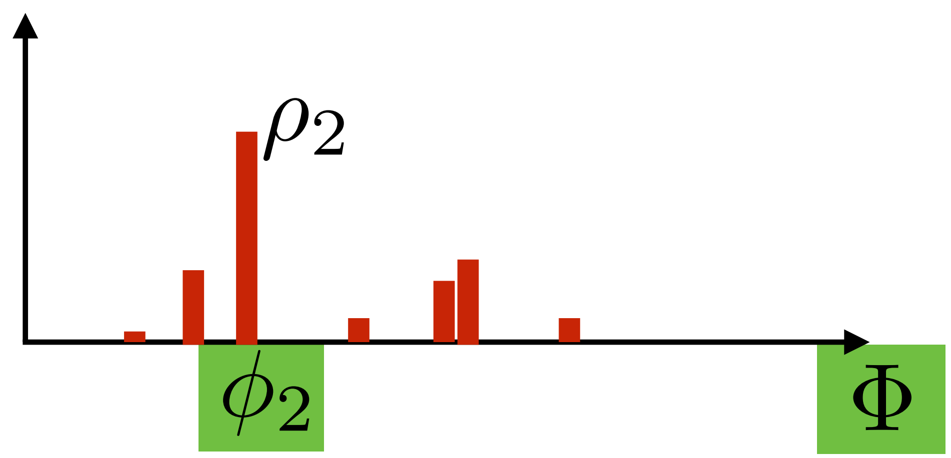

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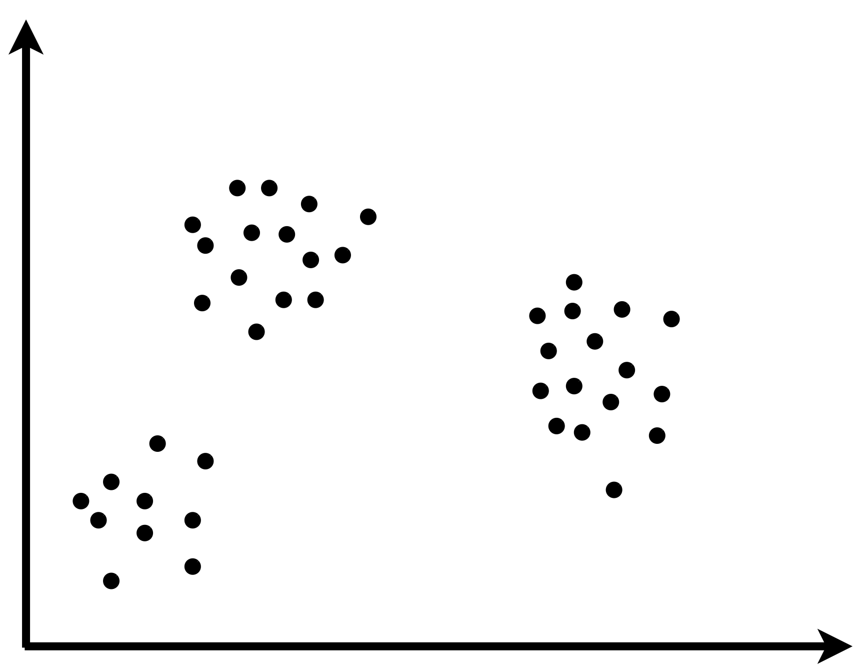

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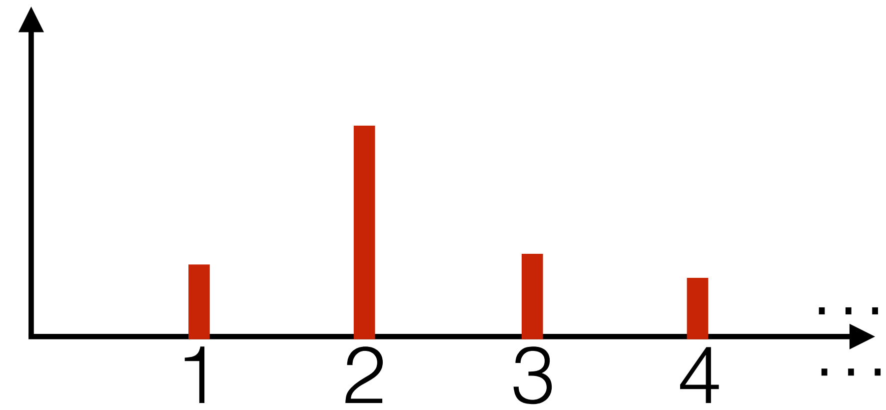

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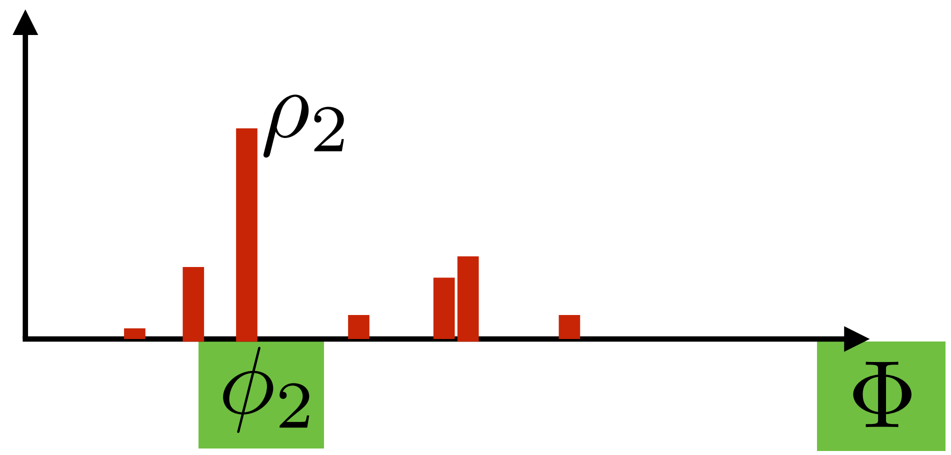

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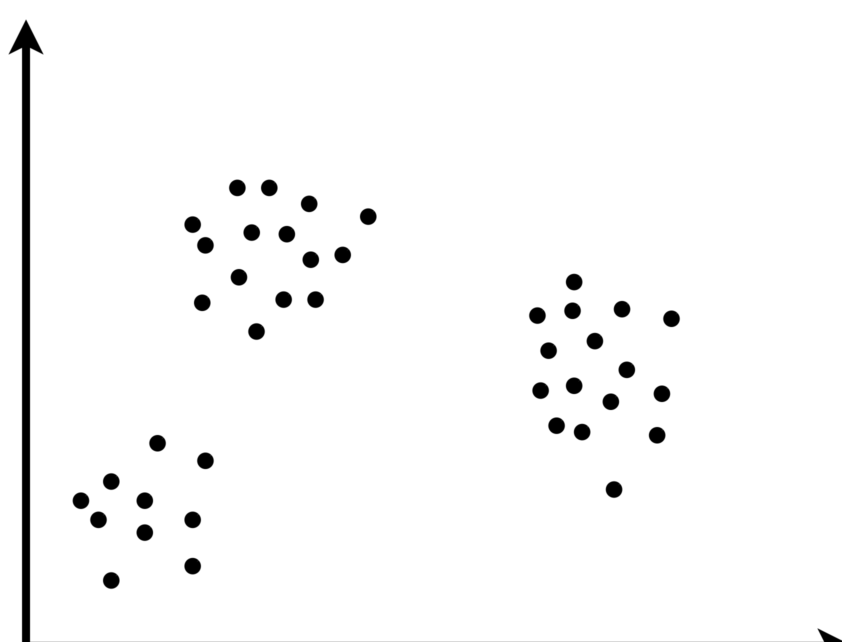

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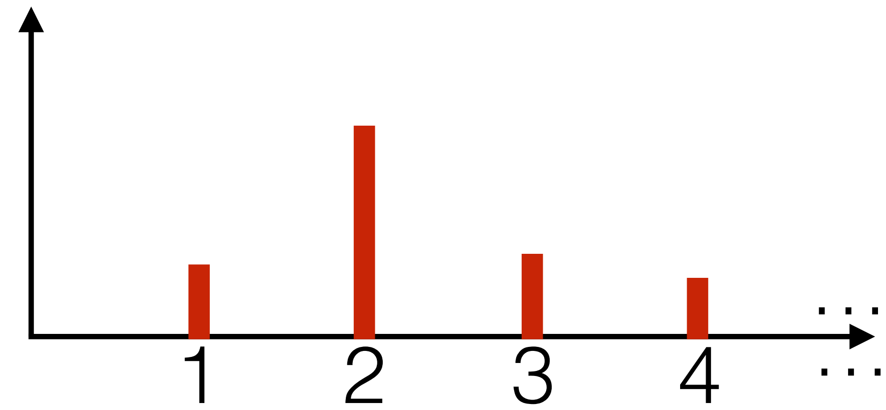

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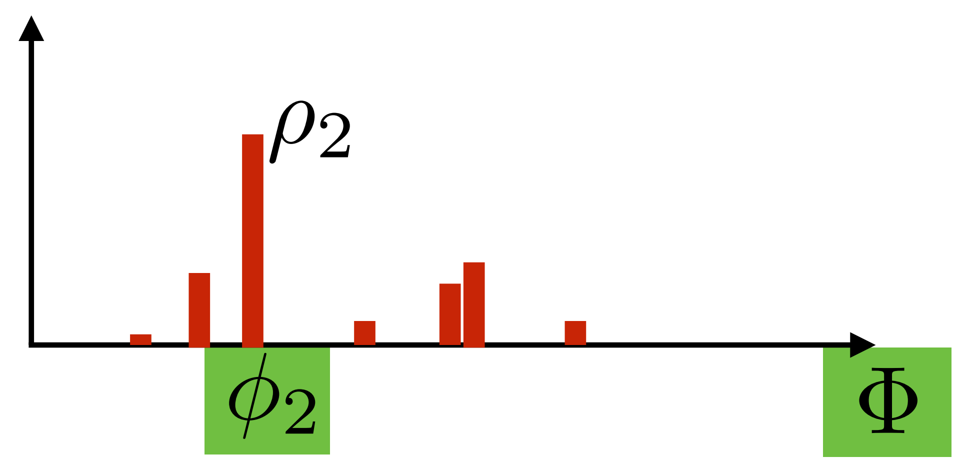

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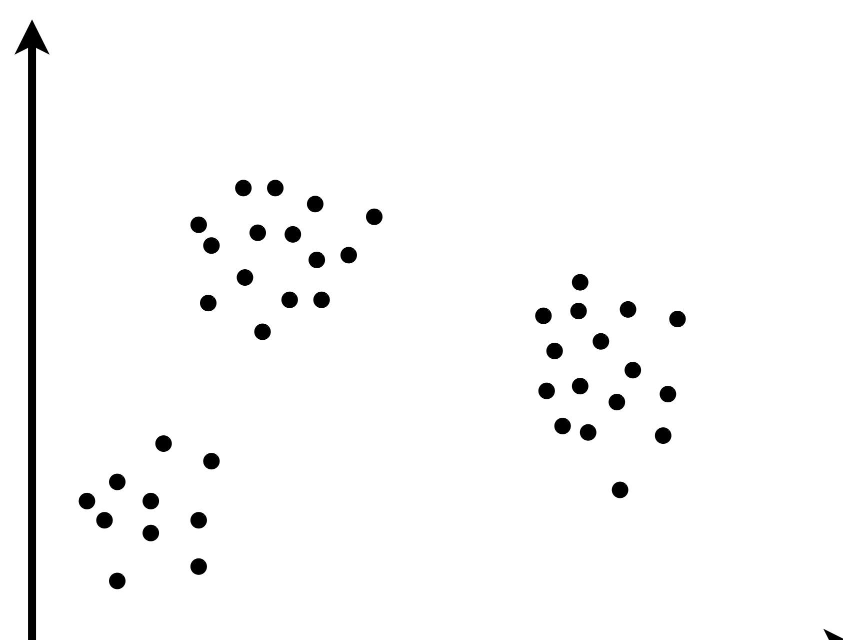

Dirichlet process mixture model

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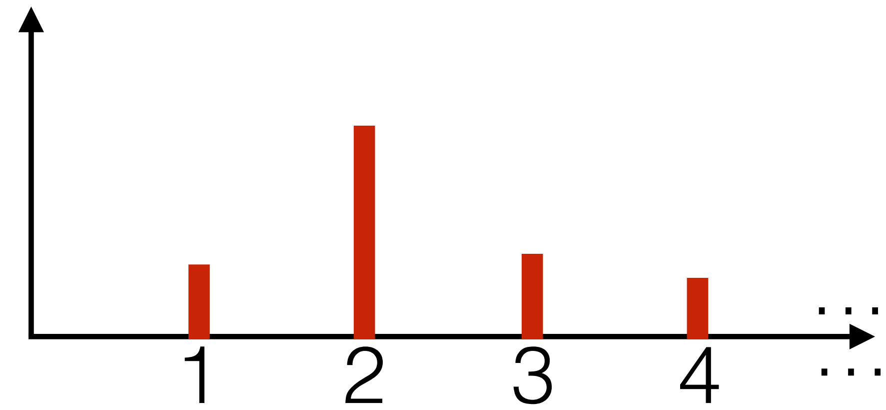

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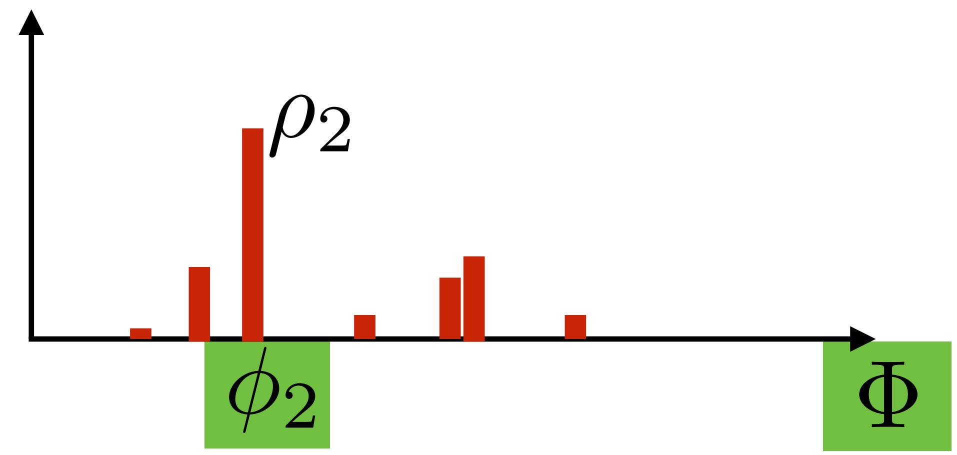

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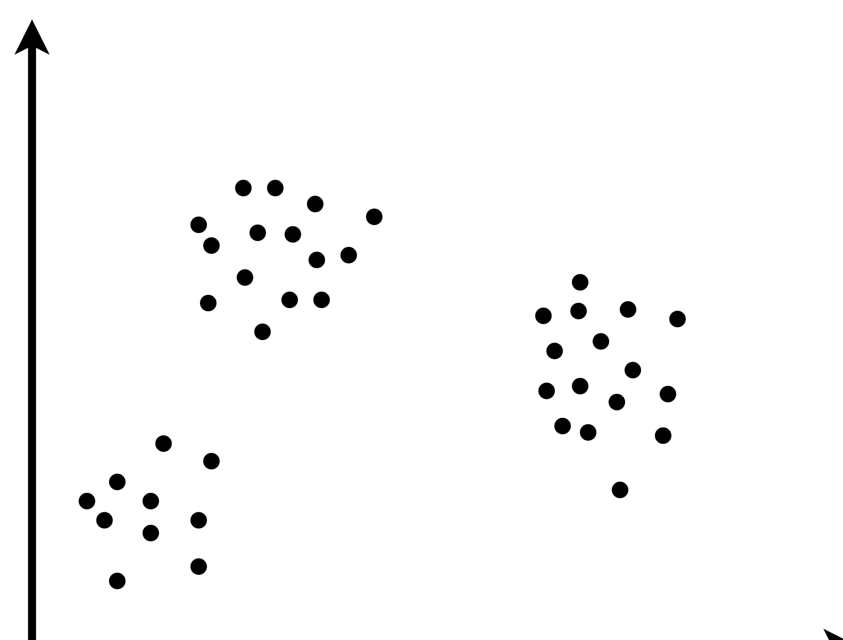

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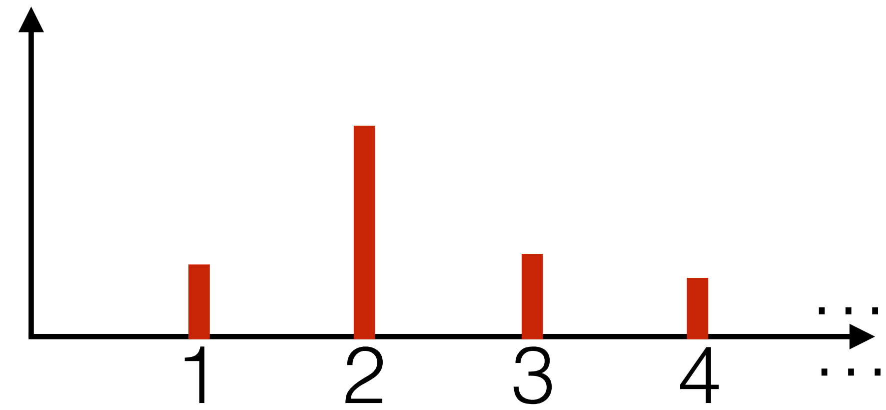

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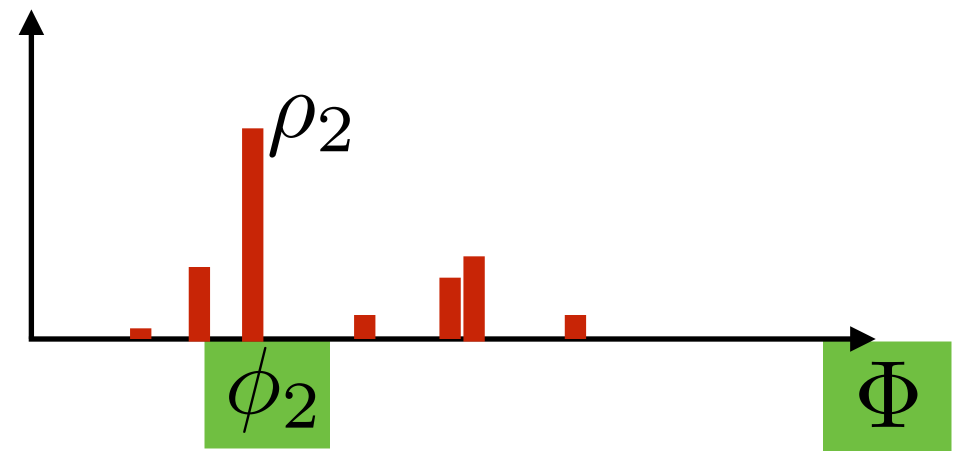

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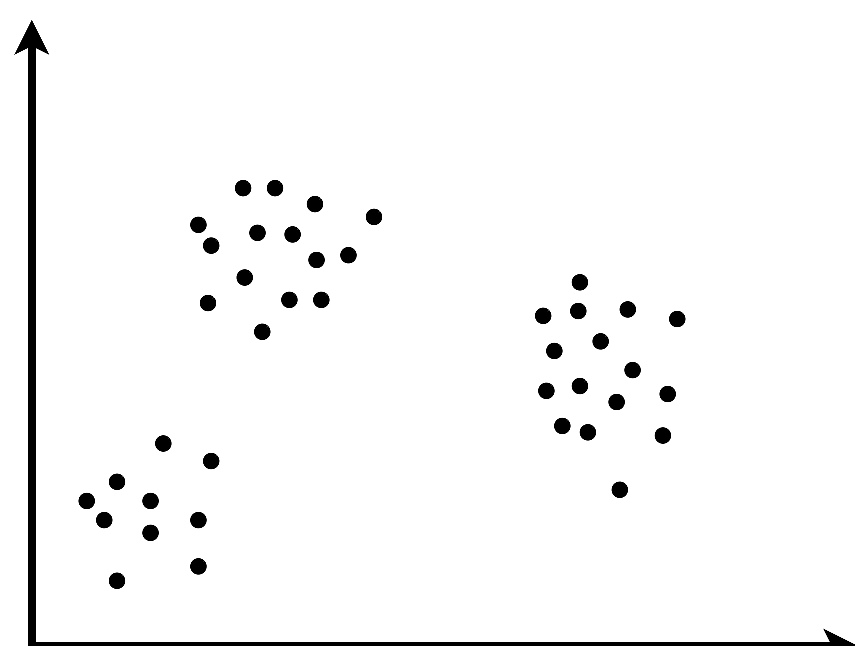

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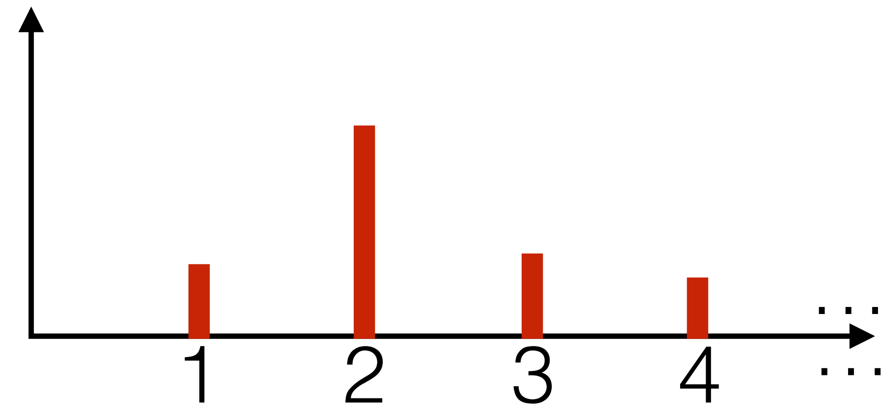

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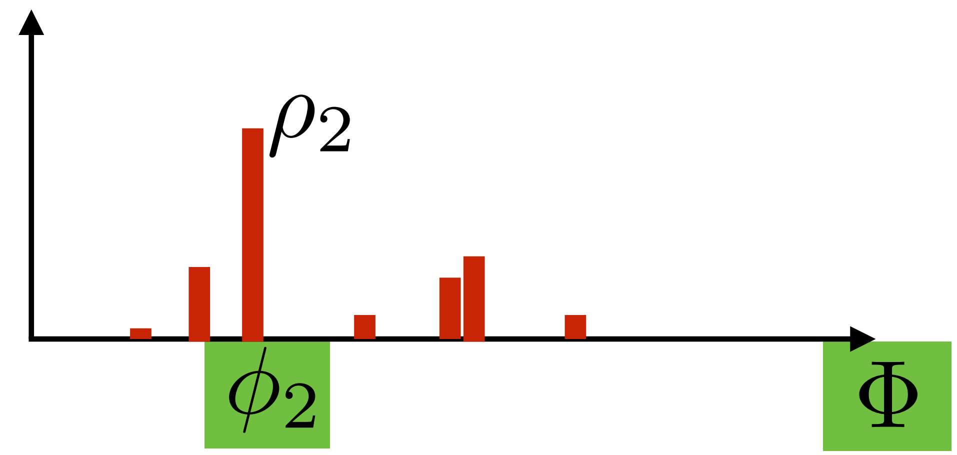

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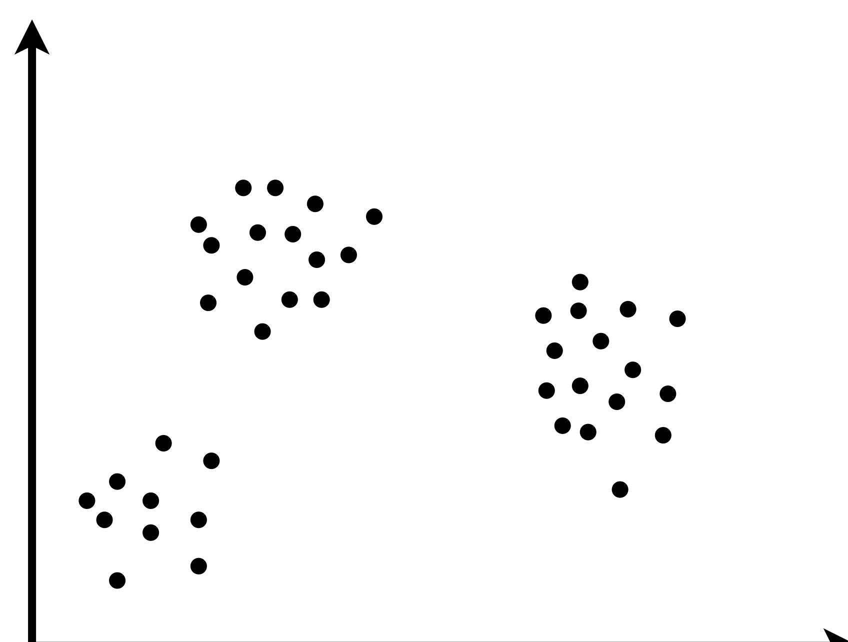
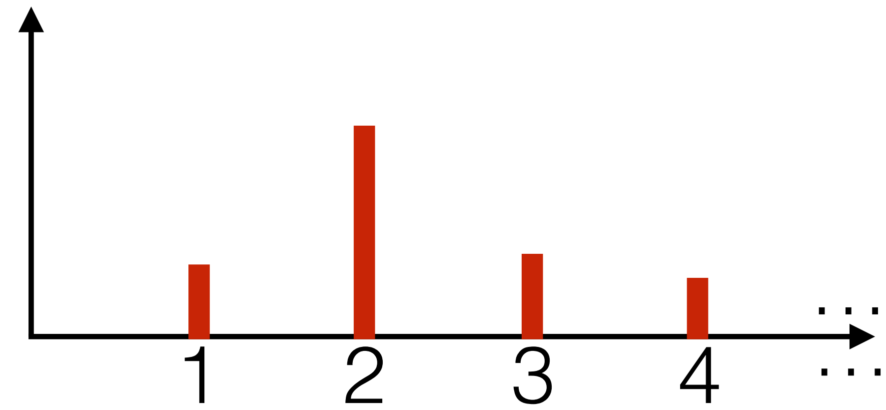


$$x_n \overset{indep}{\sim} F(\theta_n)$$

Dirichlet process mixture model

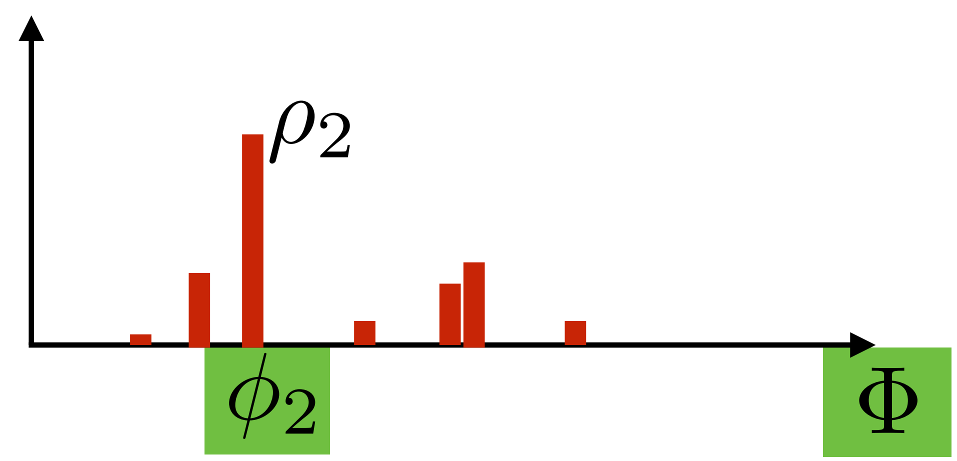
- More generally

$$\rho = (\rho_1, \rho_2, \ldots) \sim \mathrm{GEM}(\alpha)$$



$$\phi_k \stackrel{iid}{\sim} G_0 \qquad k = 1, 2, \ldots$$

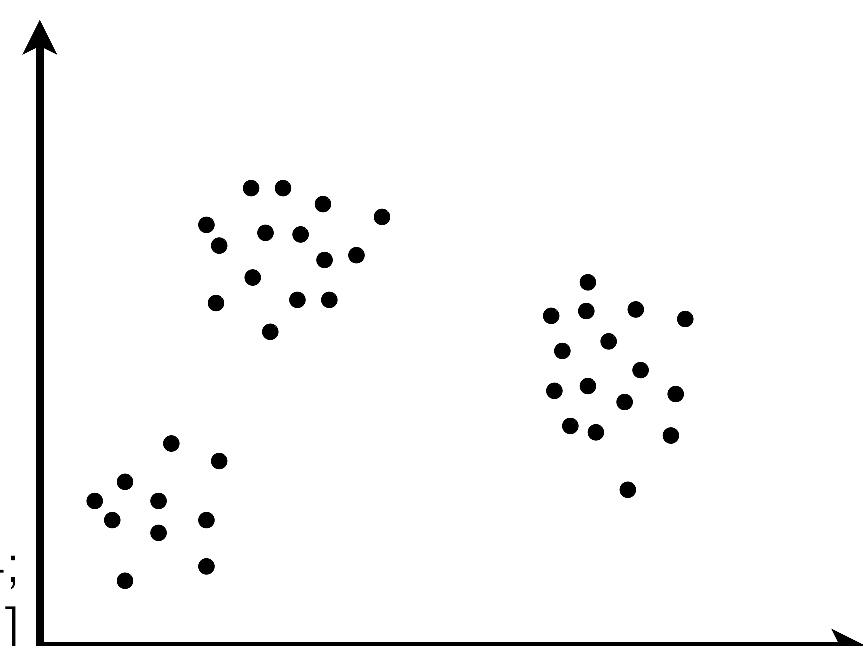
- i.e. $G = \sum_{k=1}^{\infty} \rho_k \delta_{\phi_k} \stackrel{d}{=} \mathrm{DP}(\alpha, G_0)$



$$z_n \stackrel{iid}{\sim} \mathrm{Categorical}(\rho)$$

$$\theta_n = \phi_{z_n}$$

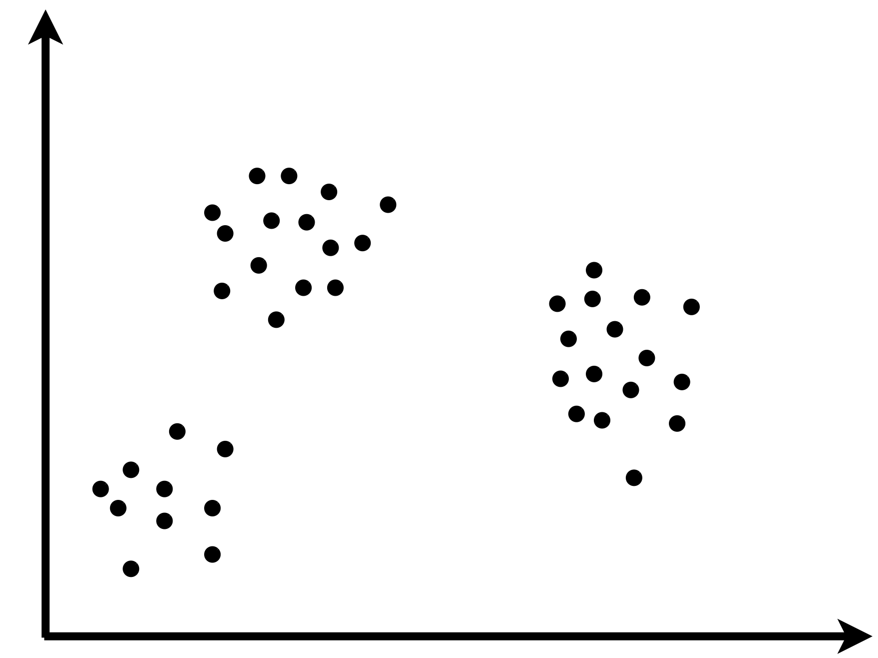
- i.e. $\theta_n \stackrel{iid}{\sim} G$



$$x_n \stackrel{indep}{\sim} F(\theta_n)$$

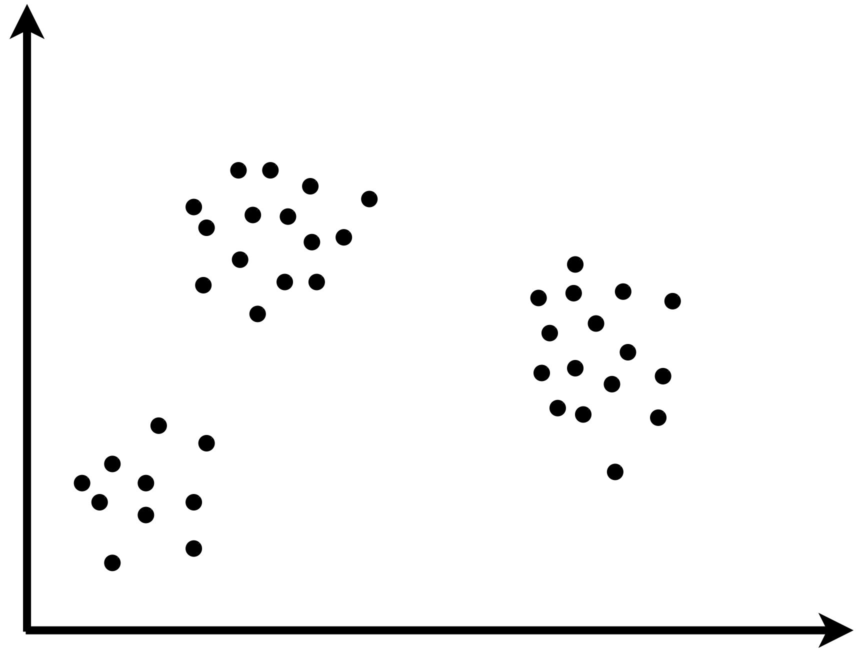
[Antoniak 1974; Ferguson 1983; West, Müller, Escobar 1994; Escobar, West 1995; MacEachern, Müller 1998]

DP or not DP, that is the question



DP or not DP, that is the question

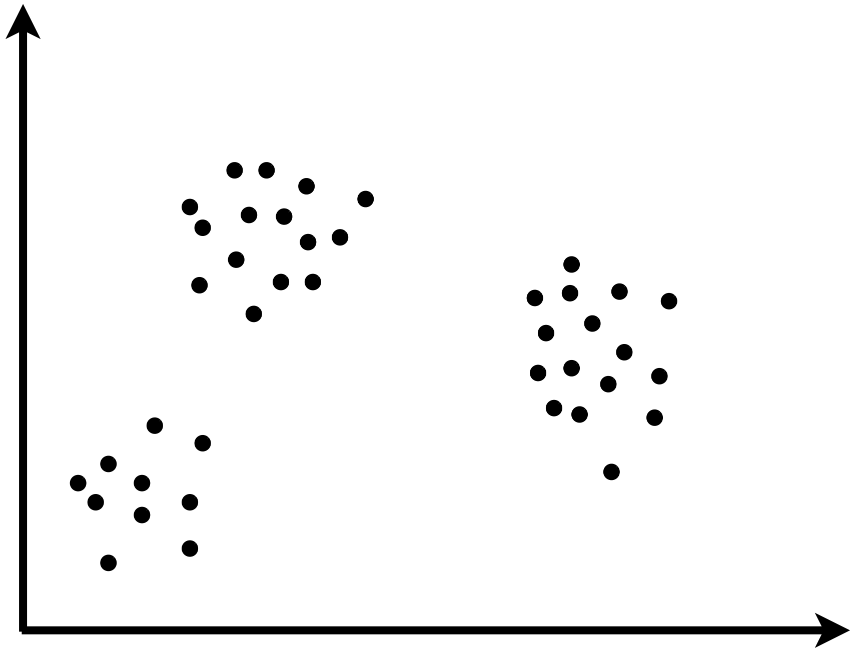
- GEM: 



DP or not DP, that is the question

- GEM: 

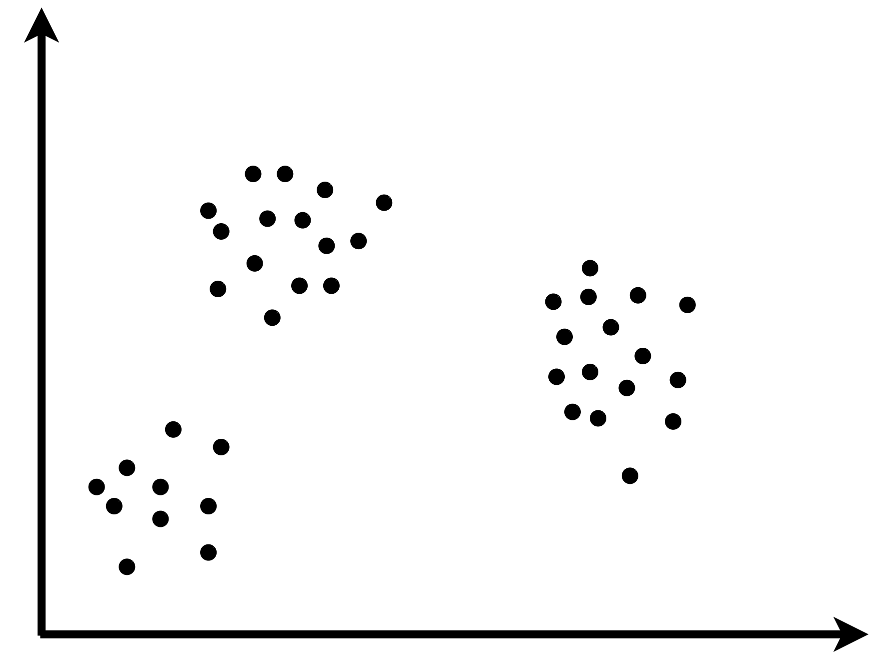
- Compare to:



DP or not DP, that is the question

- GEM: 

- Compare to:
  - Finite (small *K*) mixture model



DP or not DP, that is the question

- GEM: 
- Compare to:
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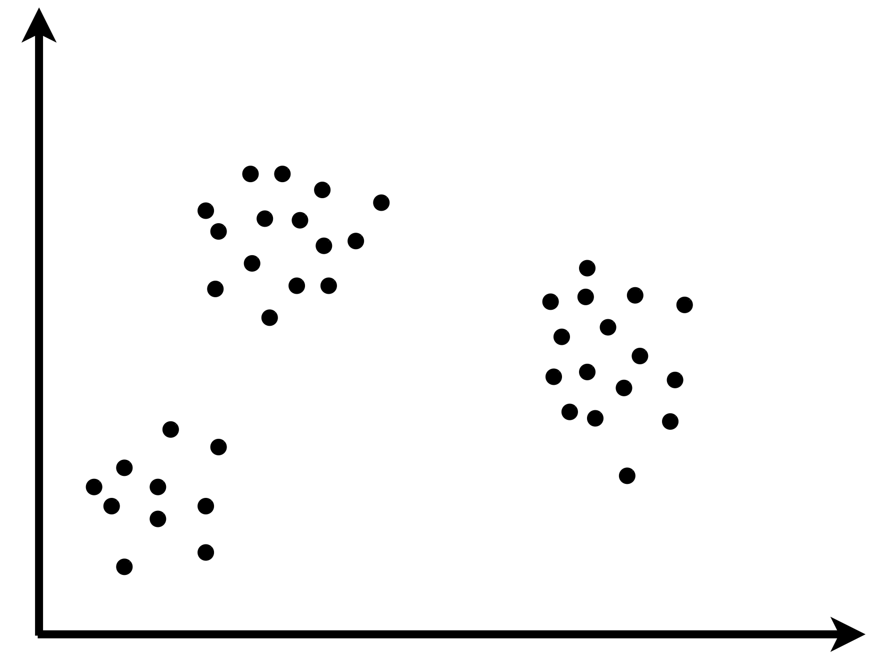
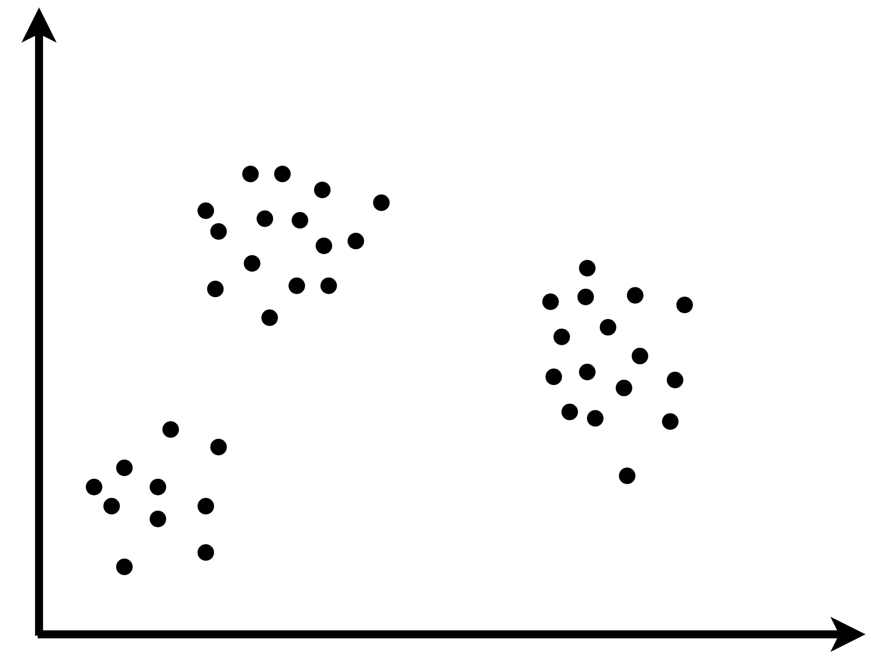


  - Finite (large *K*) mixture model

DP or not DP, that is the question

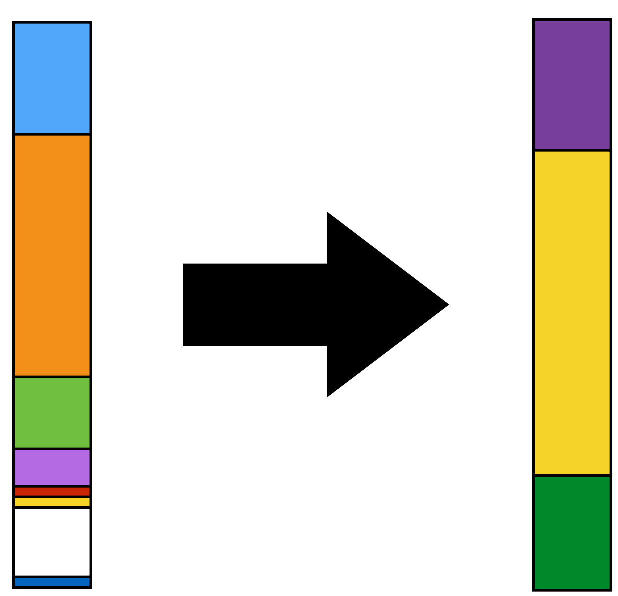
- GEM: 



- Compare to:
  - Finite (small *K*) mixture model

  - Finite (large *K*) mixture model

  - Time series



Roadmap

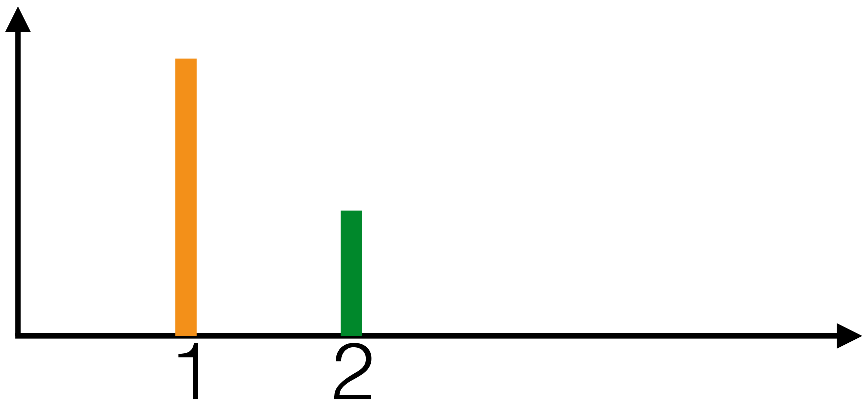
- Example problem: clustering
- Example NPBayes model: Dirichlet process
- Chinese restaurant process
- Inference
- Venture further into the wild world of Nonparametric Bayes

- Big questions
  - Why NPBayes? Learn more as acquire more data
  - What does an infinite/growing number of parameters really mean (in NPBayes)? Components vs. clusters; latent vs. realized
  - Why is NPBayes challenging but practical? Infinite dimensional parameter; more on this today!

Marginal cluster assignments

Marginal cluster assignments

$$\rho_1 \sim \text{Beta}(a_1, a_2), z_n \overset{iid}{\sim} \text{Cat}(\rho_1, \rho_2)$$



Marginal cluster assignments

- Integrate out the frequencies

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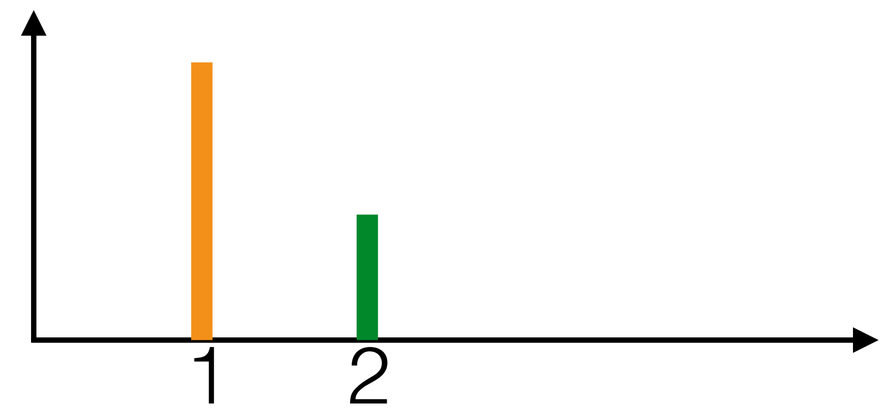

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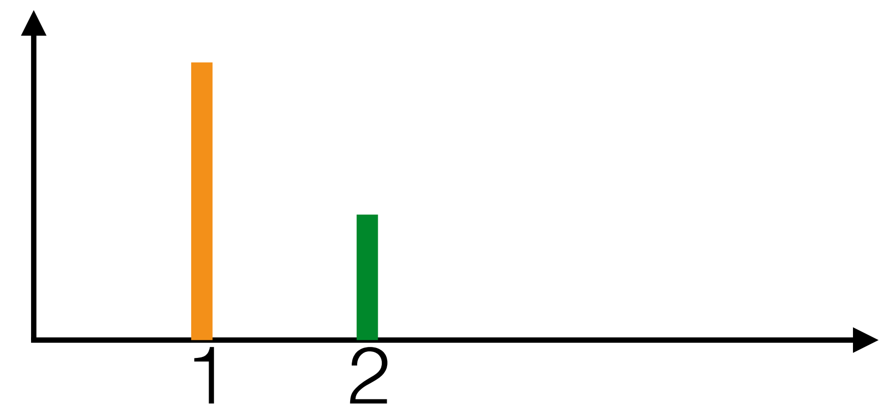

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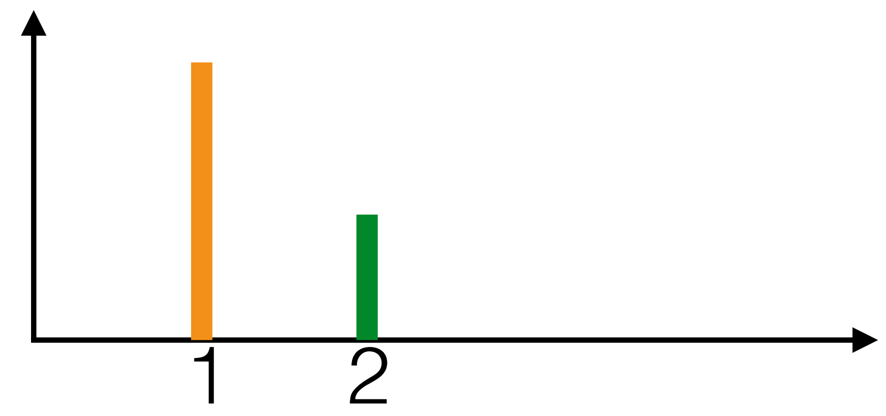
$$p(z_n = 1 | z_1, \ldots, z_{n-1})$$

$$= \int p(z_n = 1, \rho_1 | z_1, \ldots, z_{n-1}) d\rho_1$$



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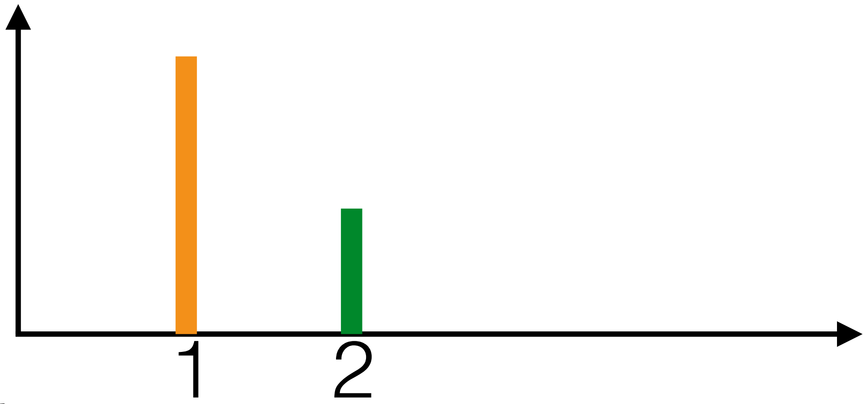

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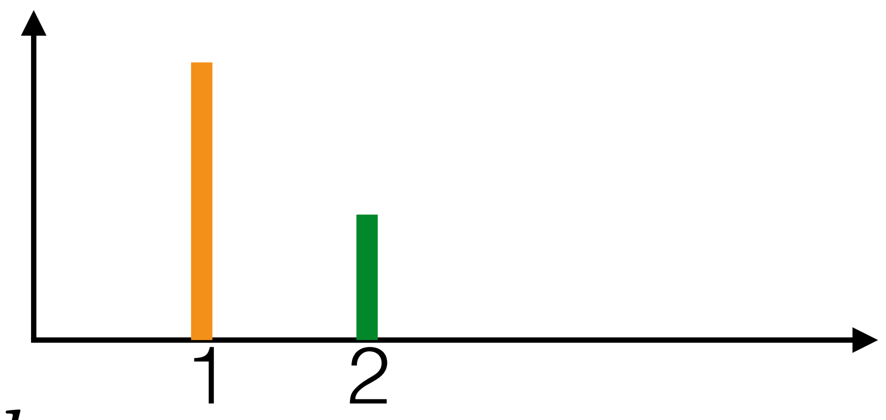

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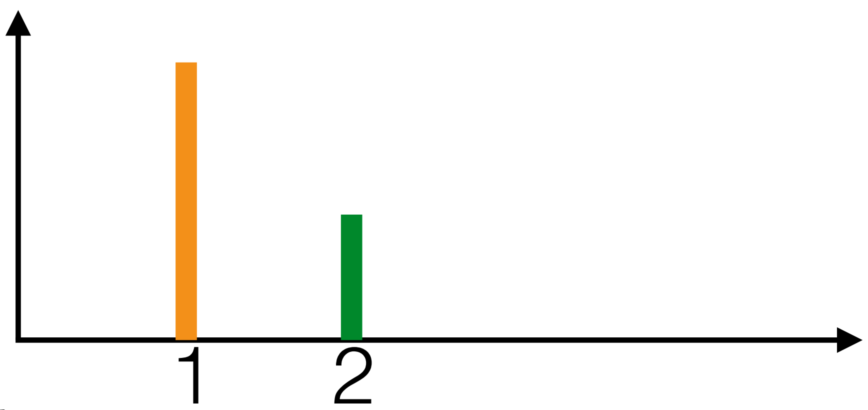

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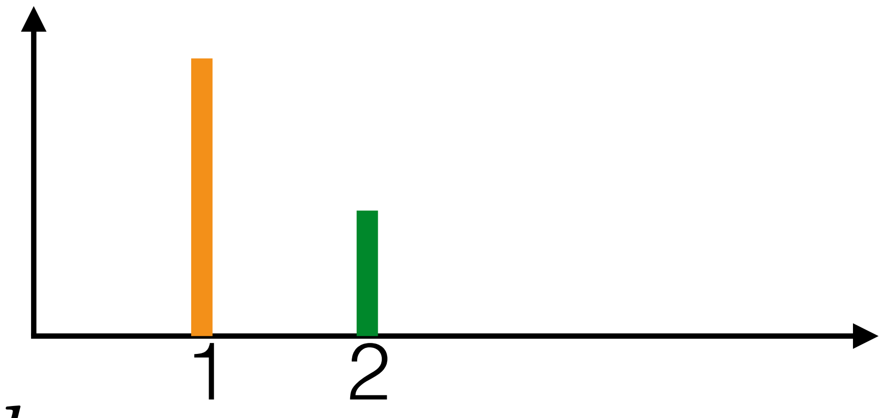

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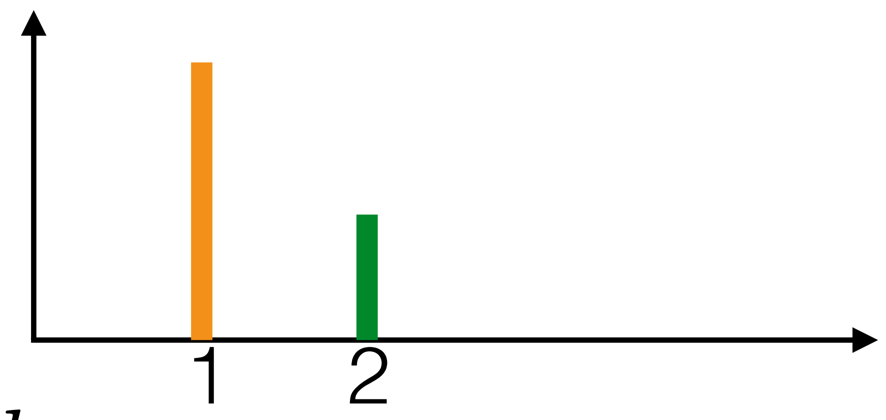
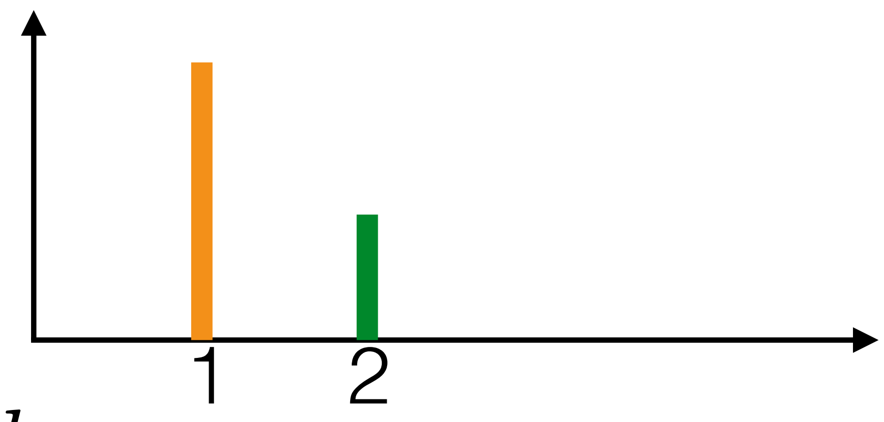


Marginal cluster assignments

- Integrate out the frequencies



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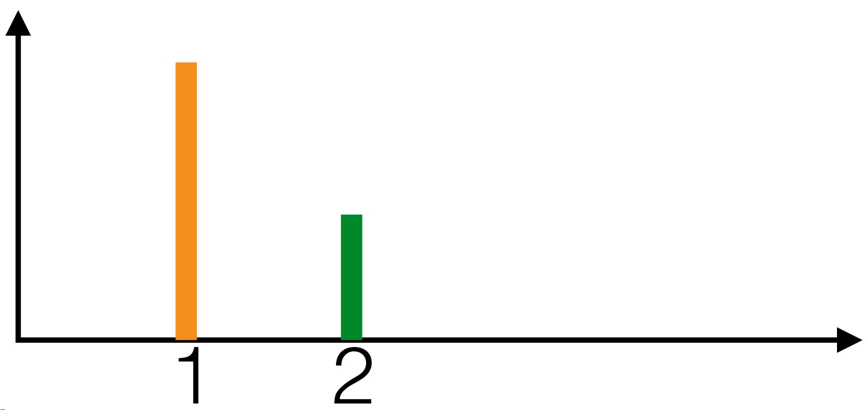
Marginal cluster assignments

- Integrate out the frequencies



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Marginal cluster assignments

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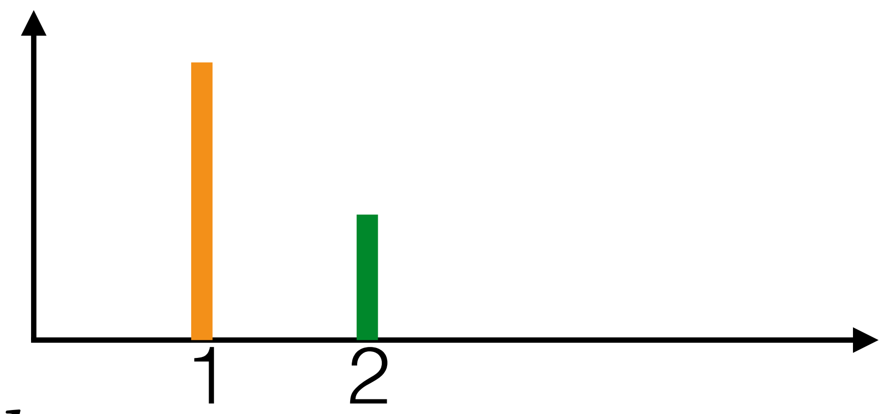

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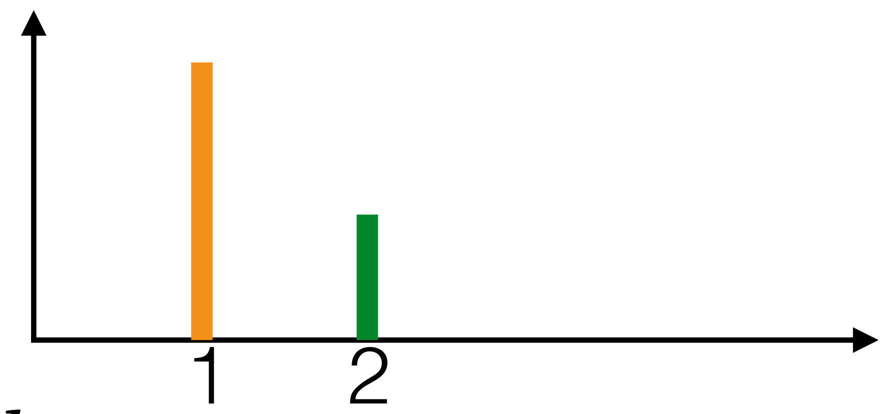
$$= \int \rho_1 \frac{\Gamma(a_{1,n} + a_{2,n})}{\Gamma(a_{1,n})\Gamma(a_{2,n})} \rho_1^{a_{1,n}-1} (1-\rho_1)^{a_{2,n}-1} d\rho_1$$

Marginal cluster assignments

- Integrate out the frequencies



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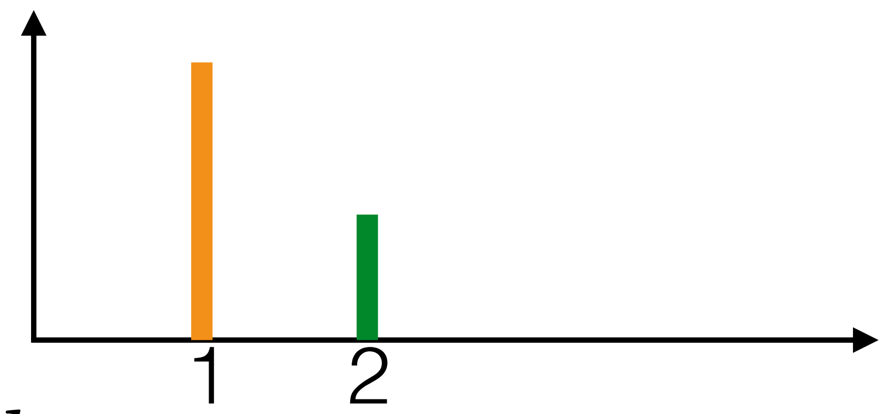
$$= \frac{\Gamma(a_{1,n} + a_{2,n})}{\Gamma(a_{1,n})\Gamma(a_{2,n})} \frac{\Gamma(a_{1,n} + 1)\Gamma(a_{2,n})}{\Gamma(a_{1,n} + a_{2,n} + 1)}$$

Marginal cluster assignments

- Integrate out the frequencies



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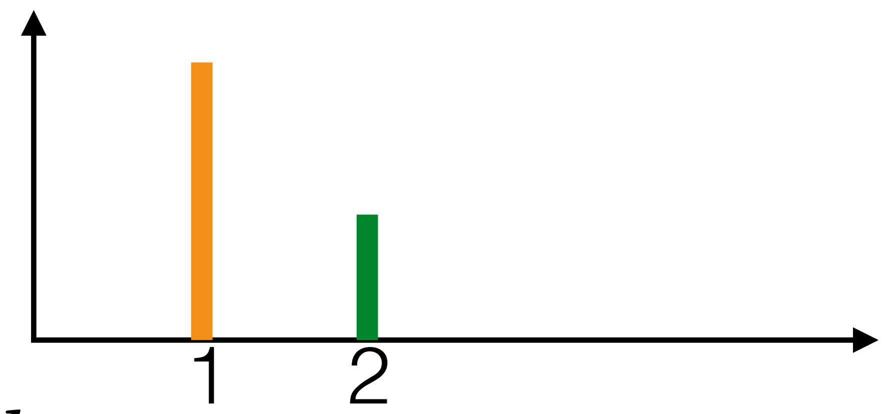
$$= \frac{\Gamma(a_{1,n} + a_{2,n})}{\Gamma(a_{1,n})\Gamma(a_{2,n})} \frac{\Gamma(a_{1,n}+1)\Gamma(a_{2,n})}{\Gamma(a_{1,n} + a_{2,n} + 1)}$$

Recall
$$\Gamma(x+1) = x\Gamma(x)$$

Marginal cluster assignments

- Integrate out the frequencies



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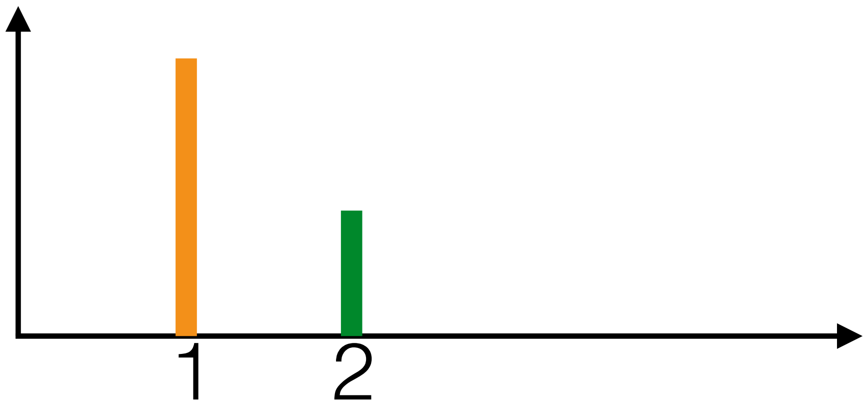

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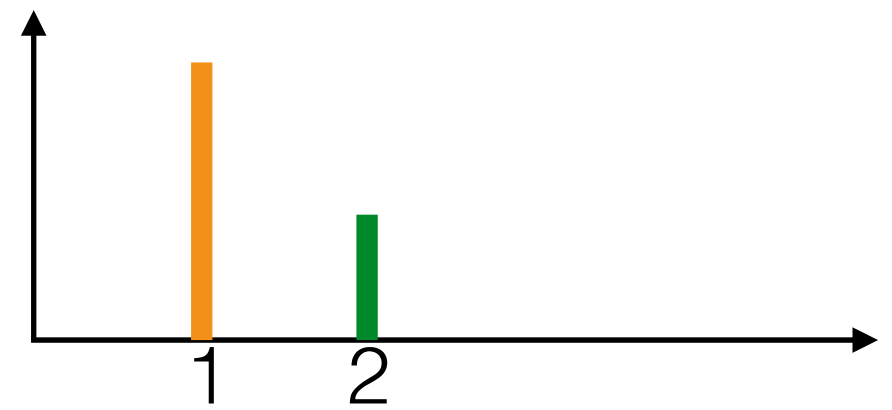

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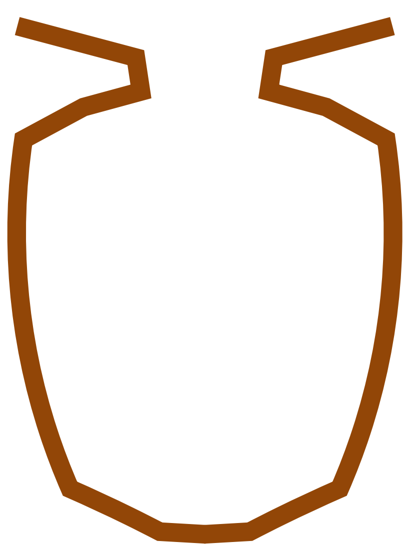

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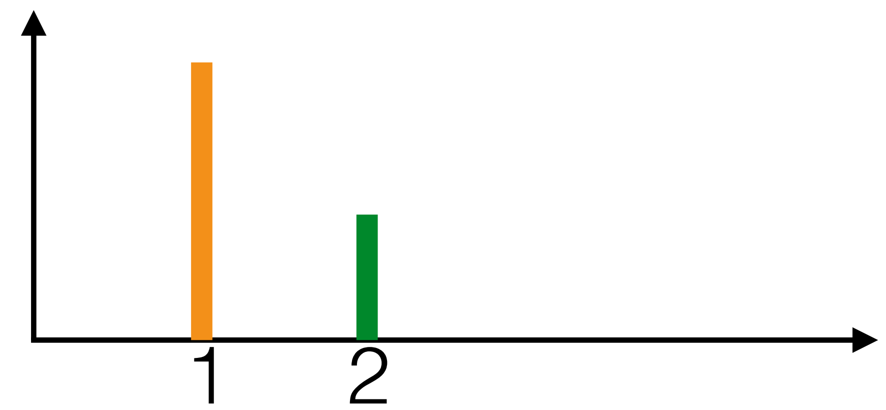

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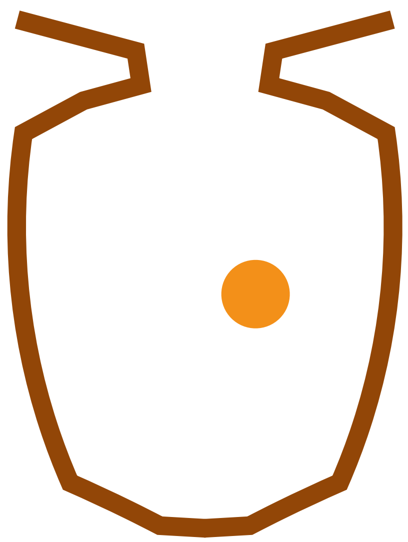

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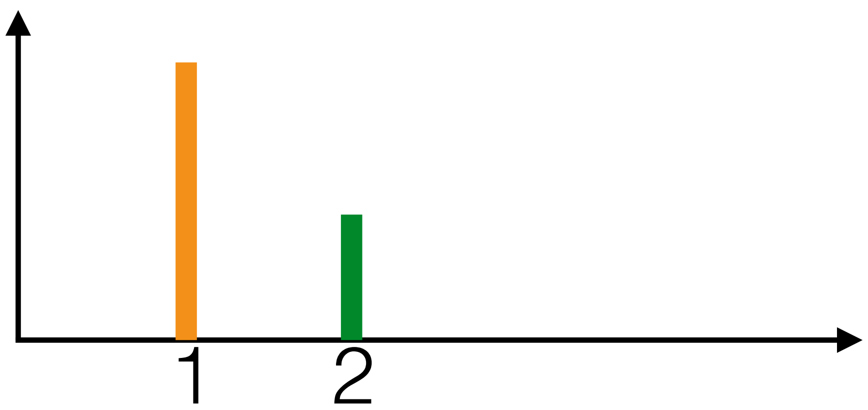

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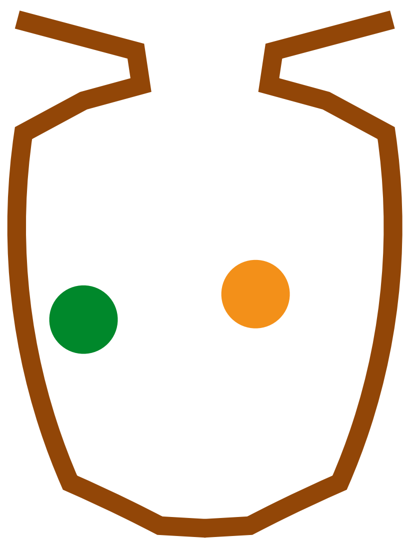

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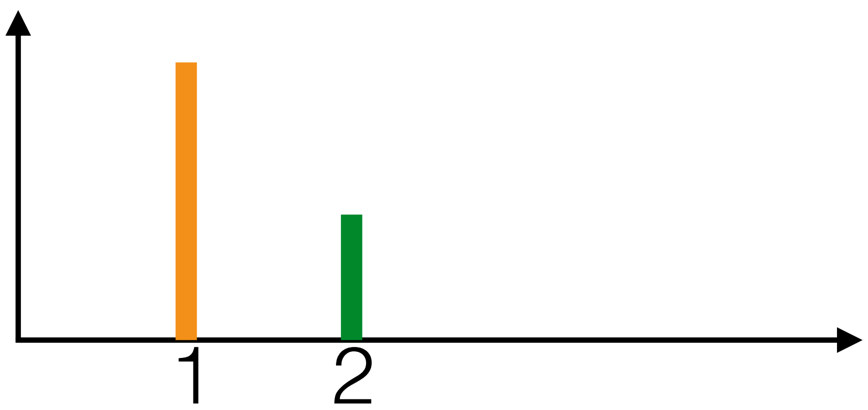

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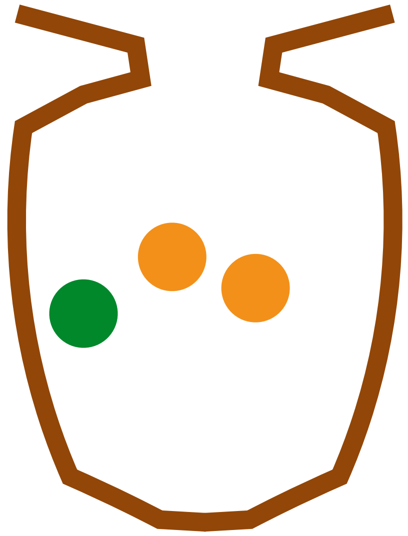
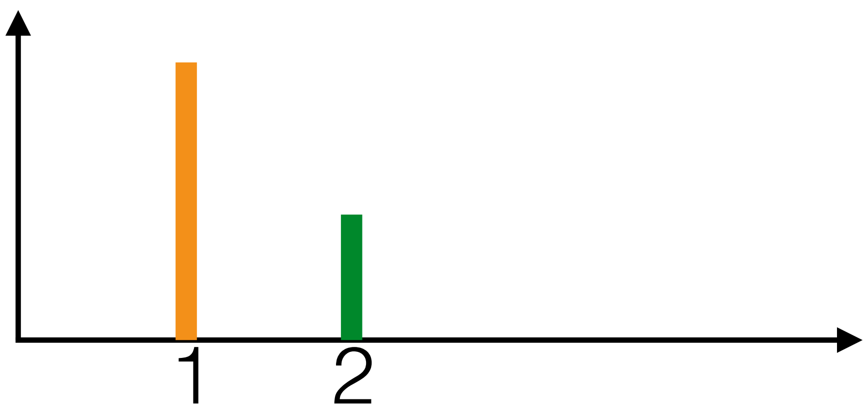


Marginal cluster assignments

- Integrate out the frequencies

$$\rho_1 \sim \text{Beta}(a_1, a_2), z_n \overset{iid}{\sim} \text{Cat}(\rho_1, \rho_2)$$

$$p(z_n = 1 | z_1, \ldots, z_{n-1}) = \frac{a_{1,n}}{a_{1,n} + a_{2,n}}$$



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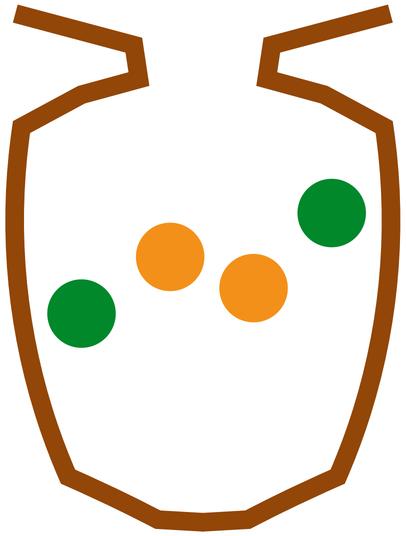

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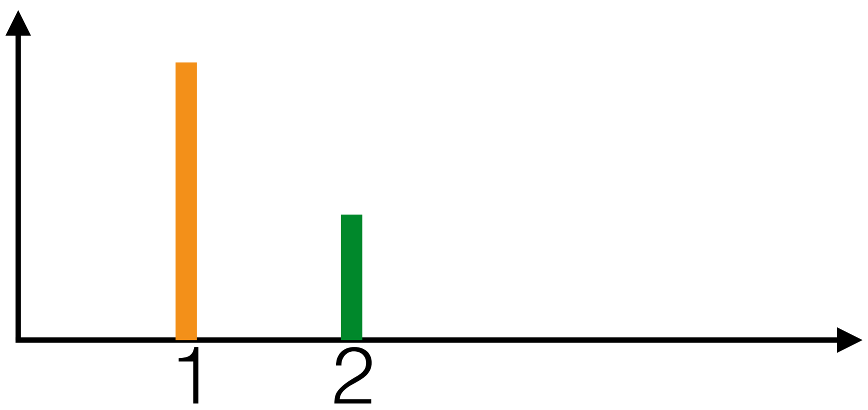

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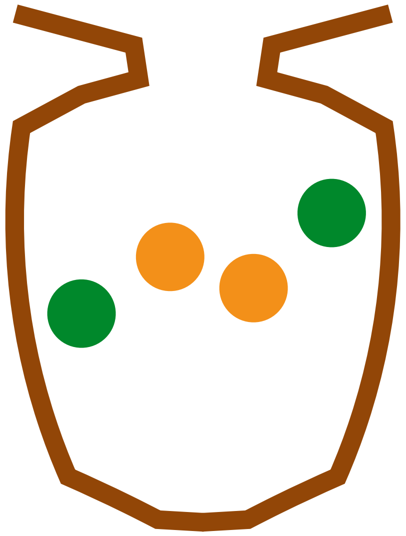

$$\lim_{n \to \infty} \frac{\#\ \text{orange}}{\#\ \text{total}}$$

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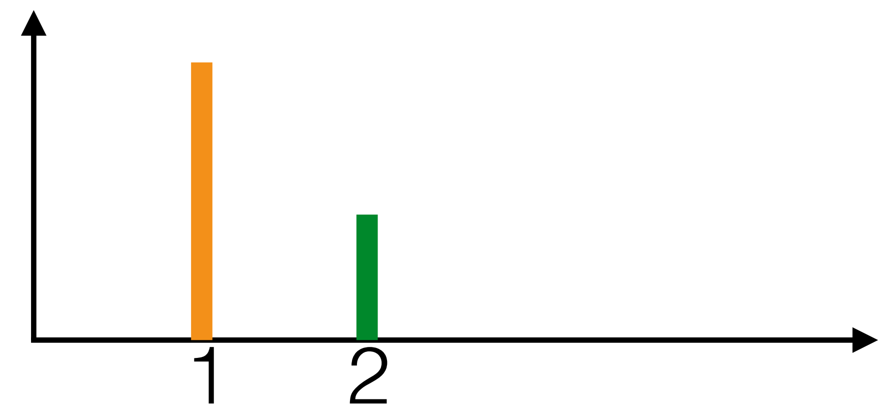

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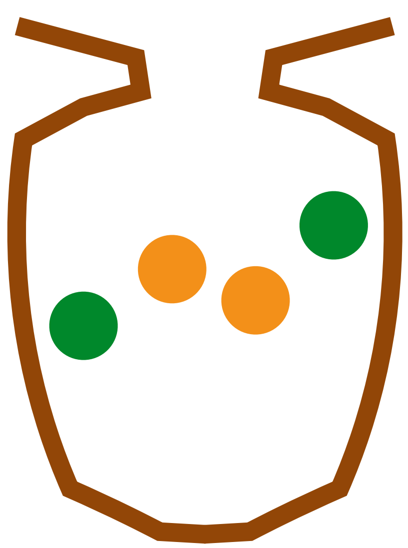

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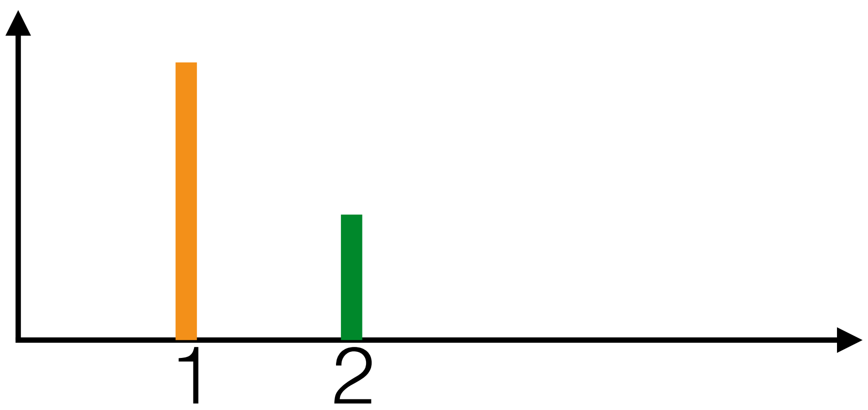

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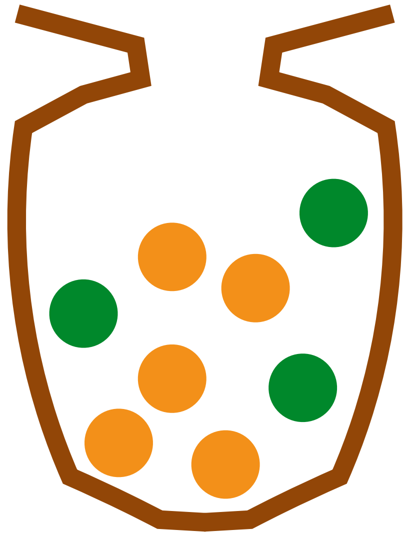

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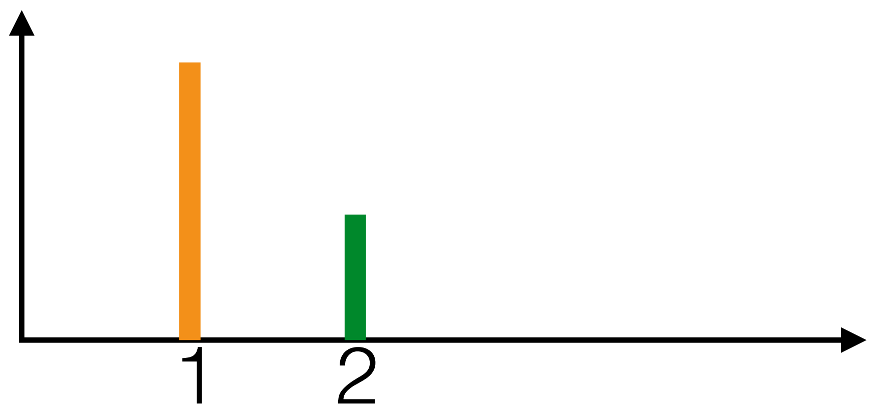

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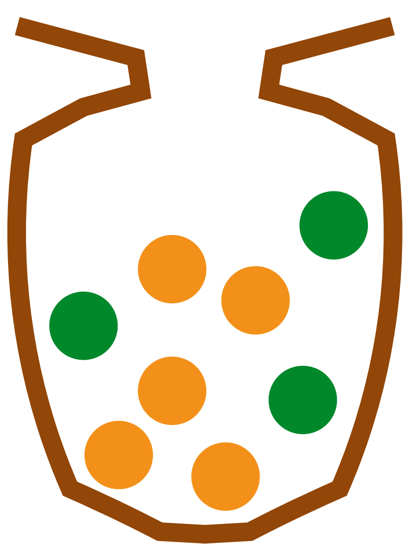

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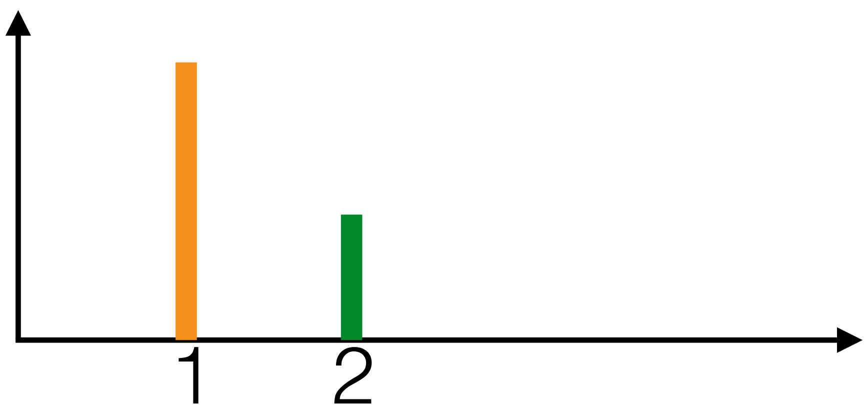

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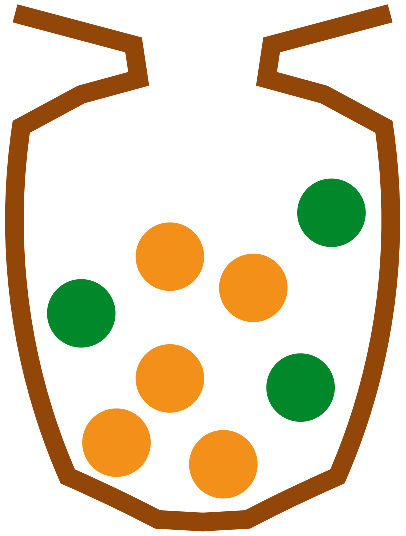

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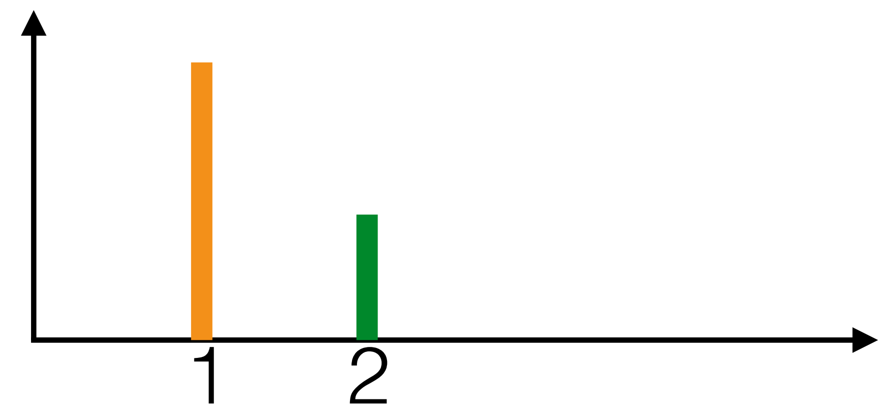

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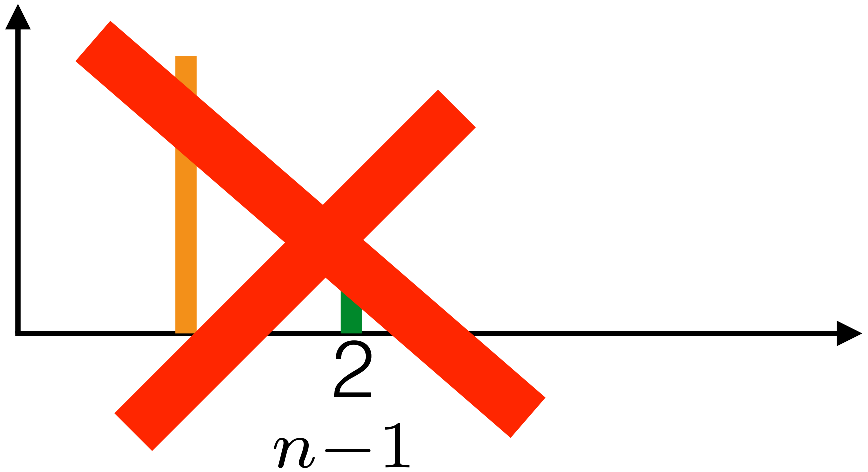

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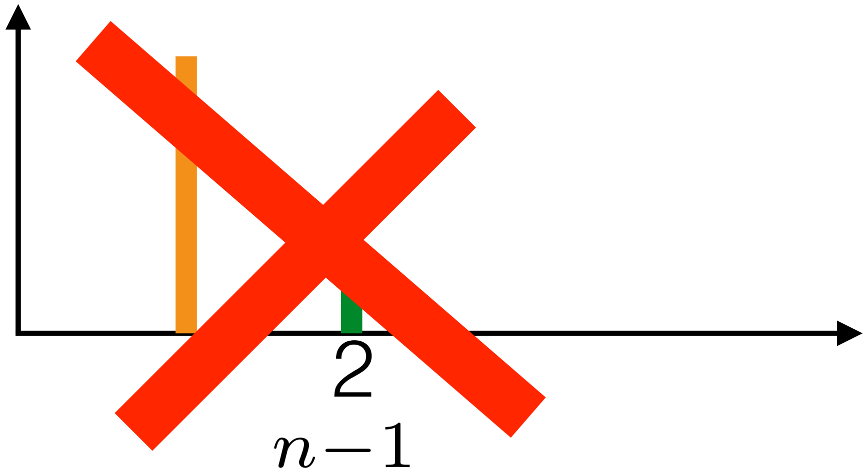

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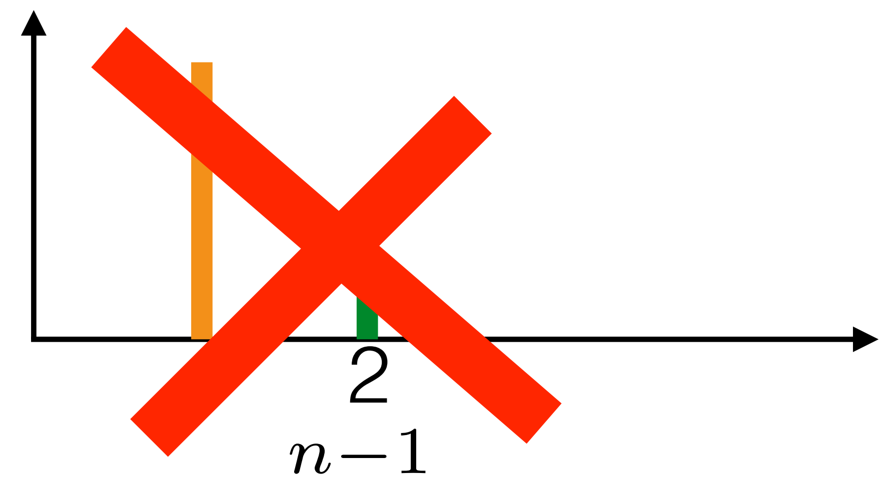

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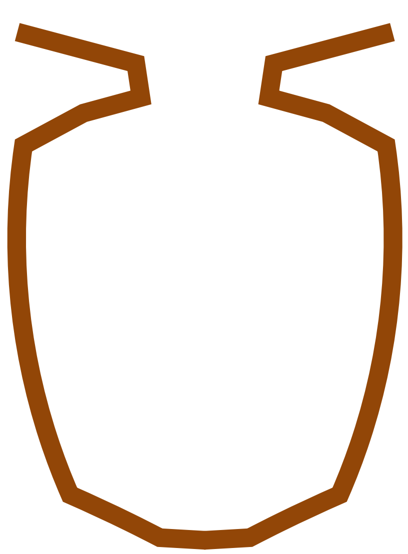

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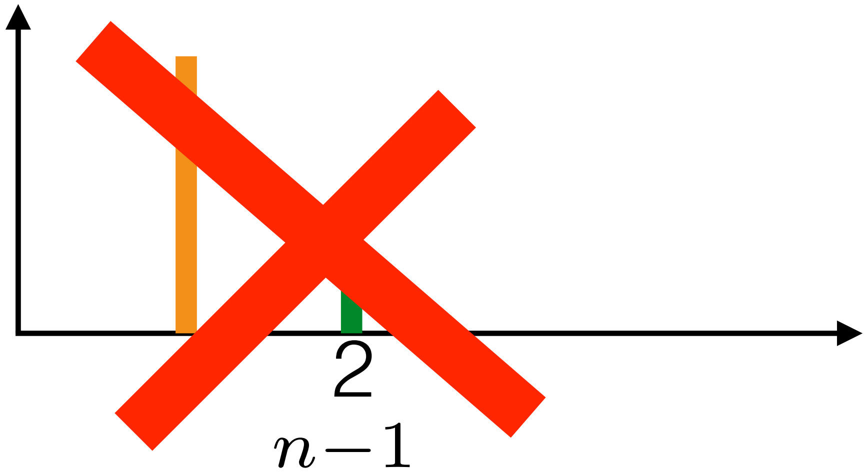

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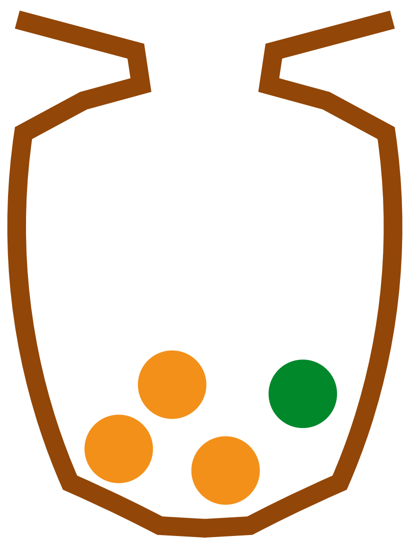

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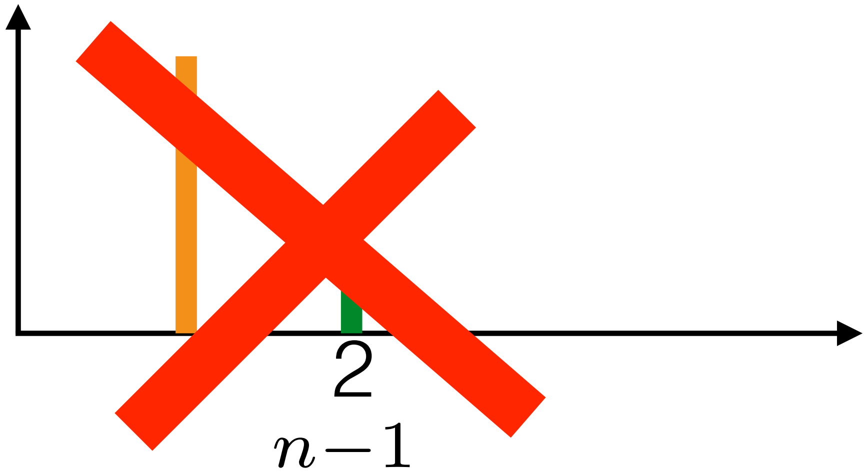

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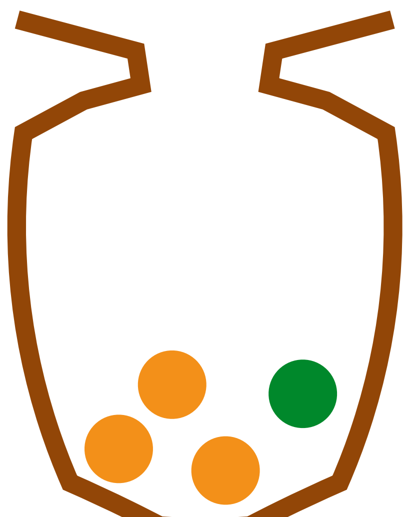

Marginal cluster assignments

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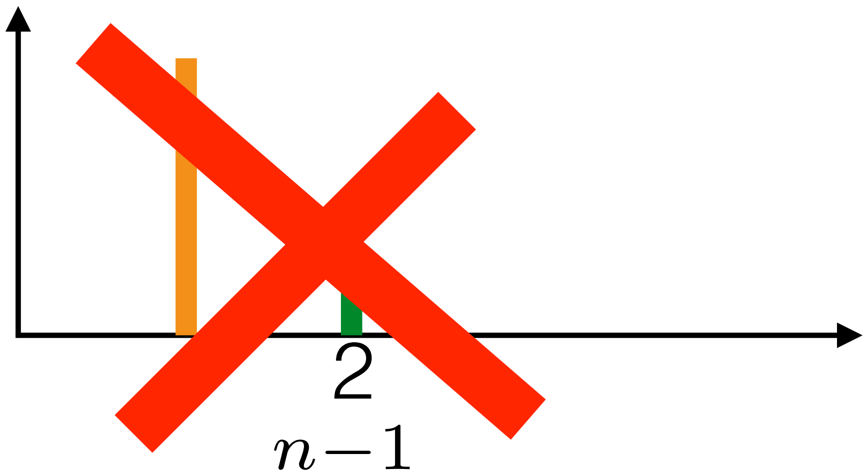

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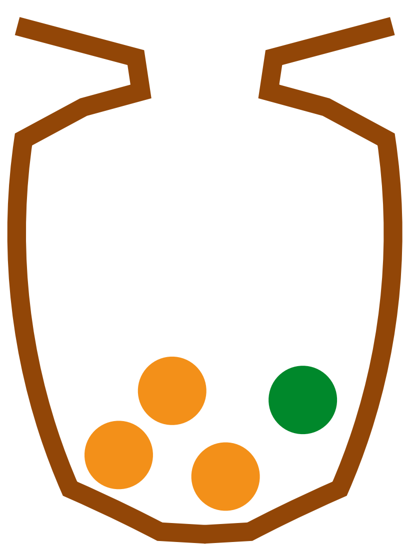
- Pólya urn
  - Choose any ball with equal probability
  - Replace and add ball of same color



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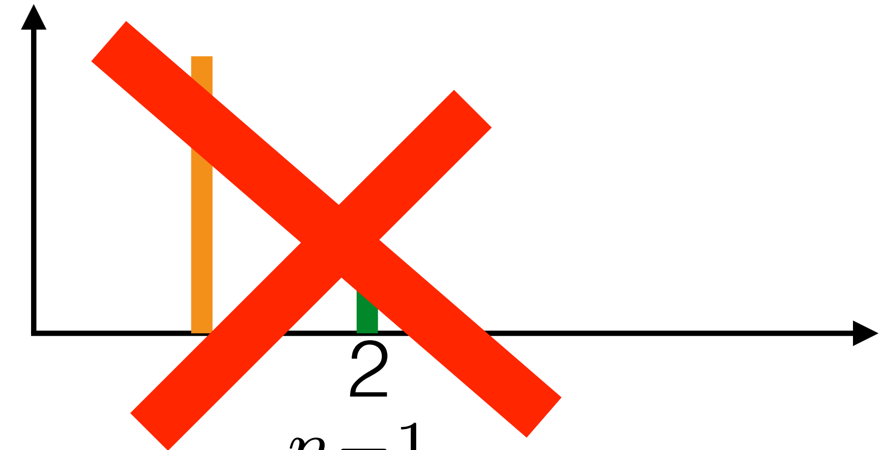

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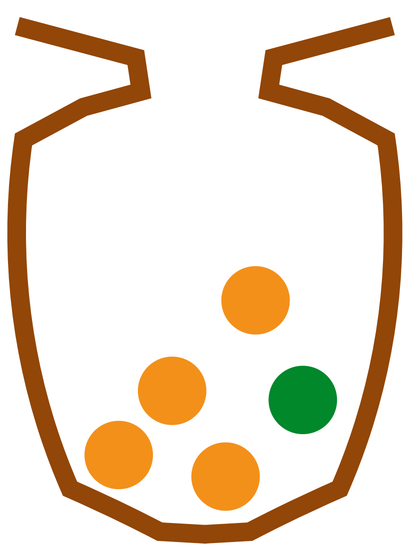

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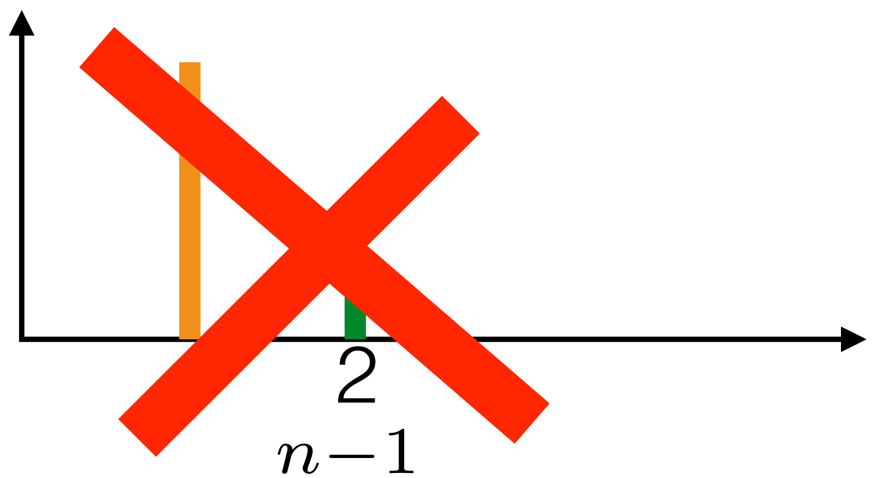

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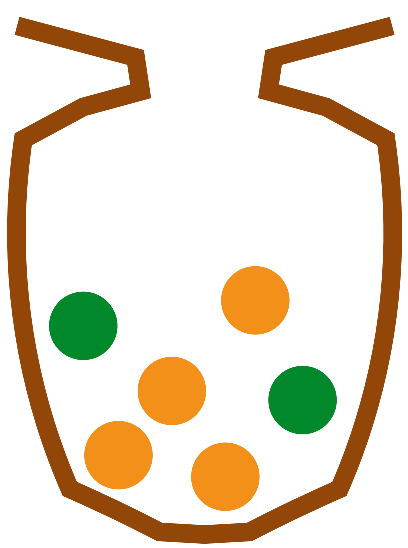

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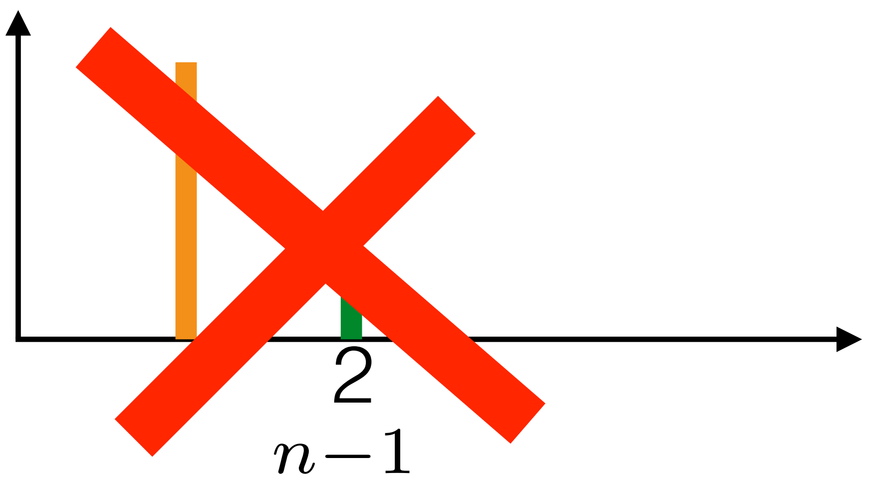

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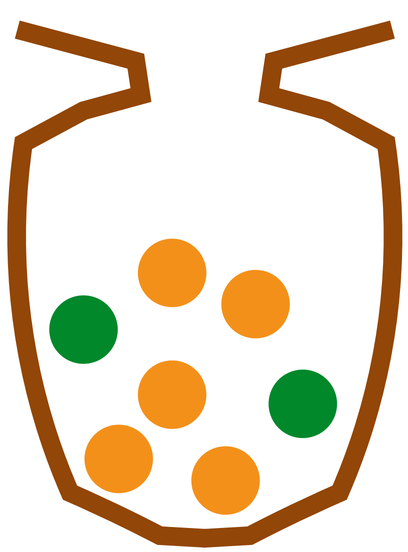

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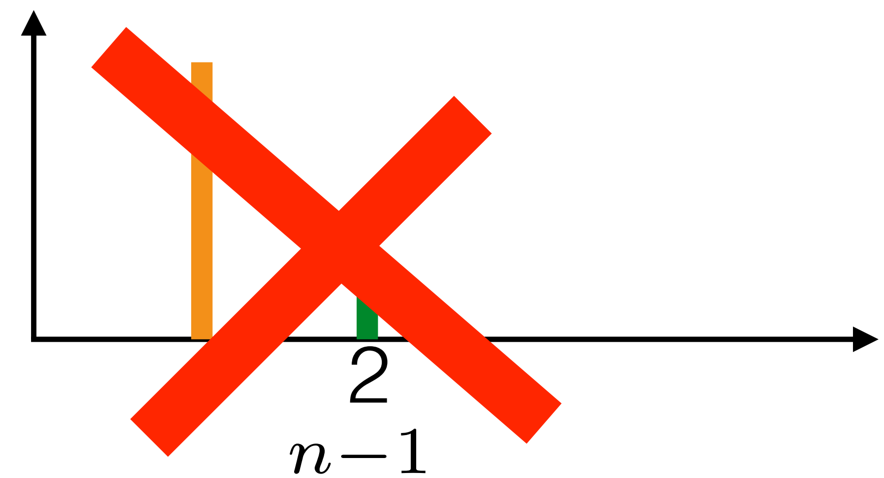

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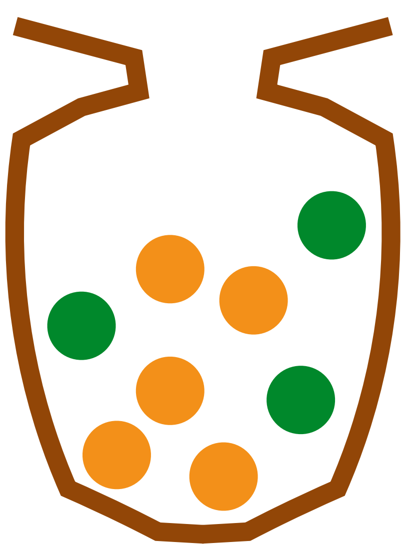

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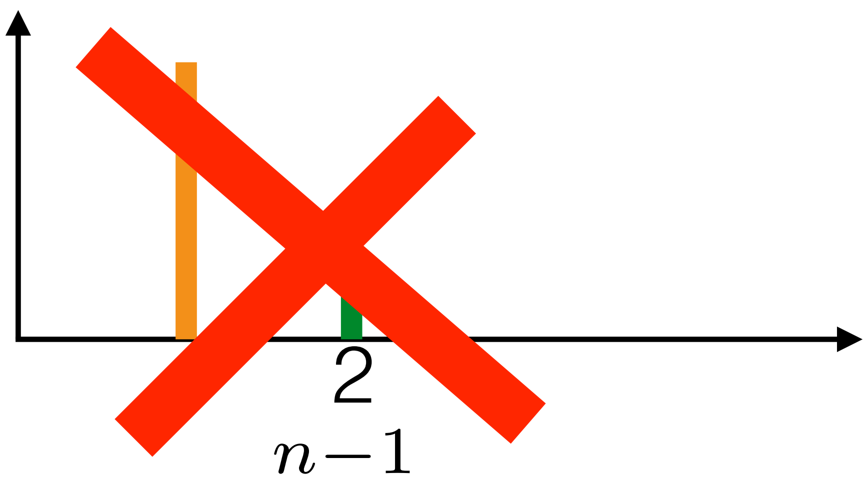

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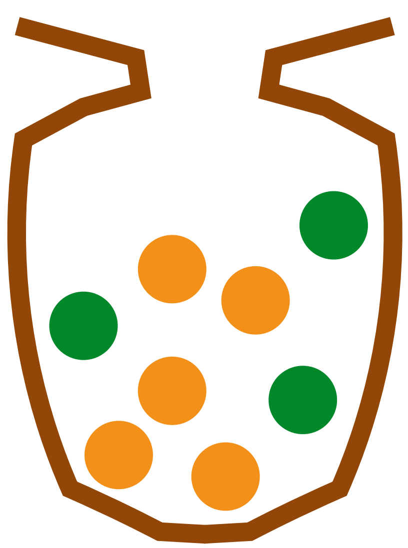

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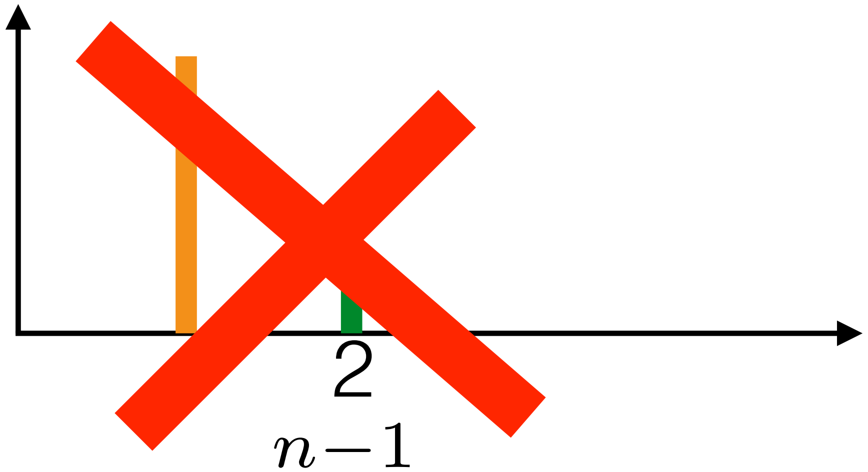

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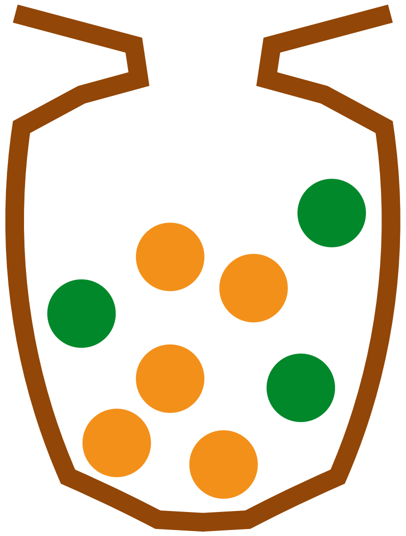

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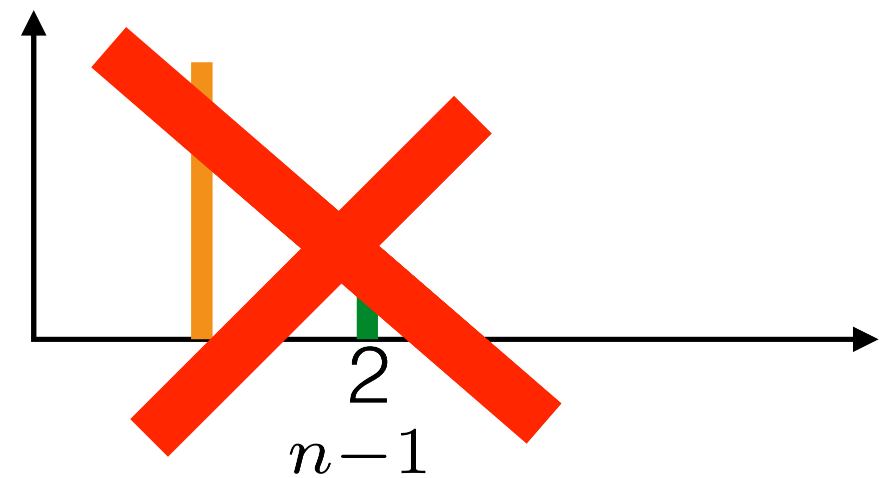

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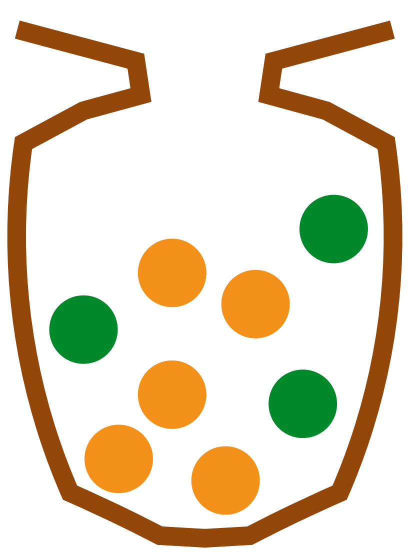


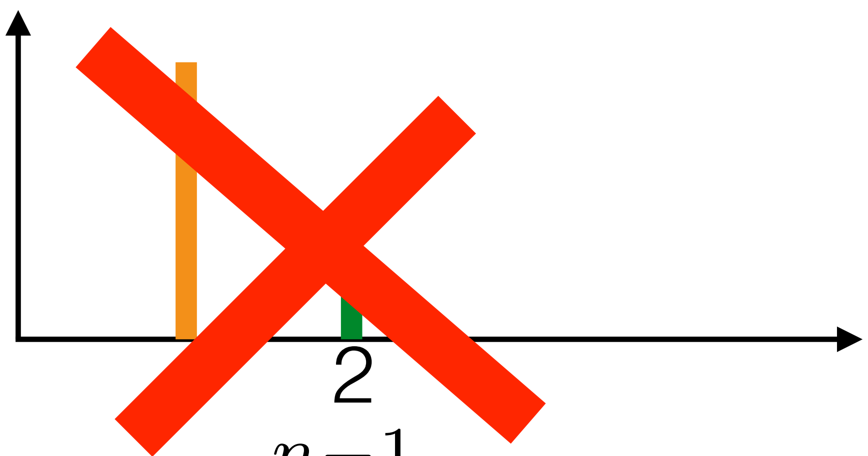
$$\lim_{n \to \infty} \frac{\#\ \text{orange}}{\#\ \text{total}} = \rho_{\text{orange}} \overset{d}{=} \mathrm{Beta}(a_{\text{orange}}, a_{\text{green}})$$

Marginal cluster assignments

- Integrate out the frequencies

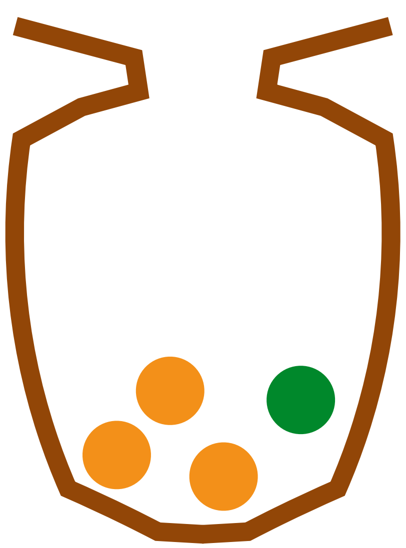
$$\rho_1 \sim \text{Beta}(a_1, a_2),\, z_n \overset{iid}{\sim} \text{Cat}(\rho_1, \rho_2)$$

$$p(z_n = 1 | z_1, \ldots, z_{n-1}) = \frac{a_{1,n}}{a_{1,n} + a_{2,n}}$$



$$a_{1,n} := a_1 + \sum_{m=1}^{n-1} \mathbf{1}\{z_m = 1\},\, a_{2,n} = a_2 + \sum_{m=1}^{n-1} \mathbf{1}\{z_m = 2\}$$

- Pólya urn
  - Choose any ball with equal probability
  - Replace and add ball of same color



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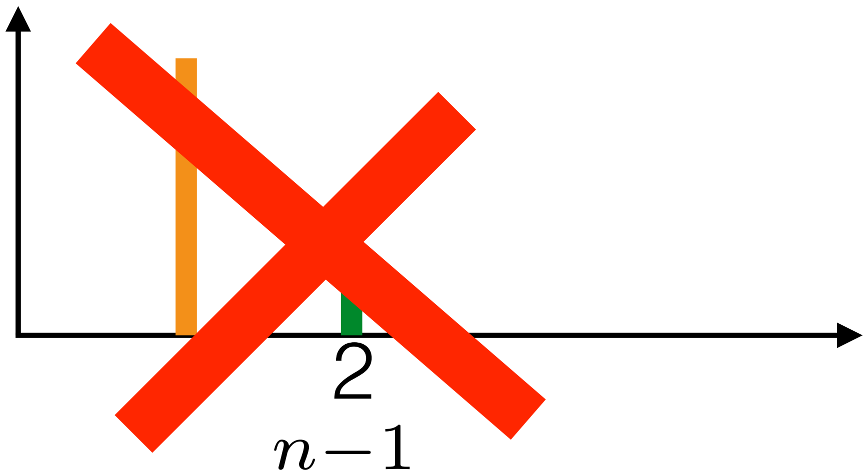

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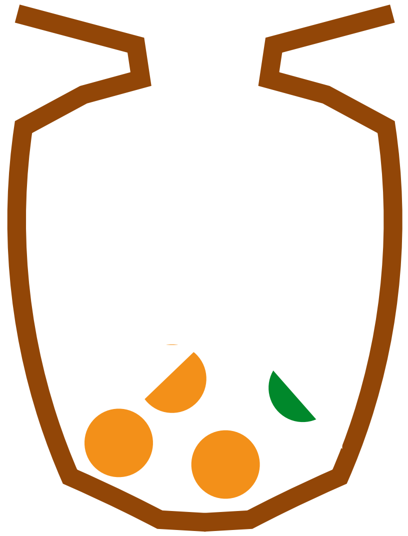

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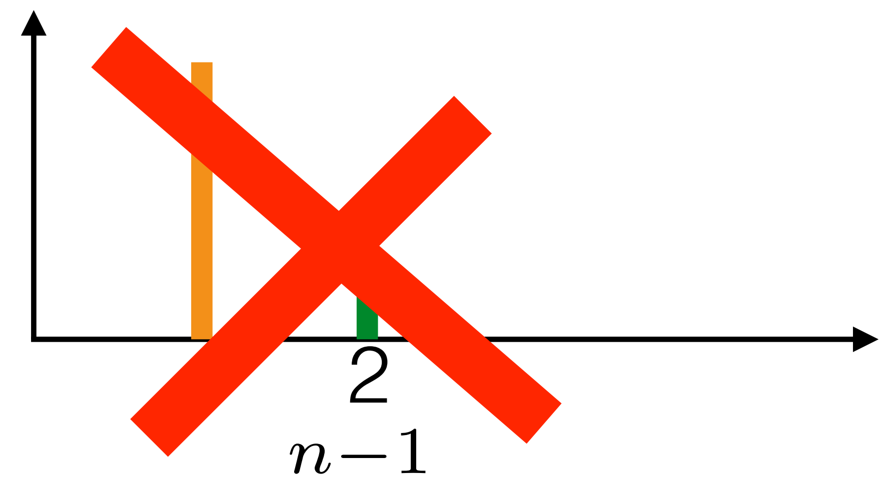

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- Pólya urn
  - Choose any ball with prob proportional to its mass
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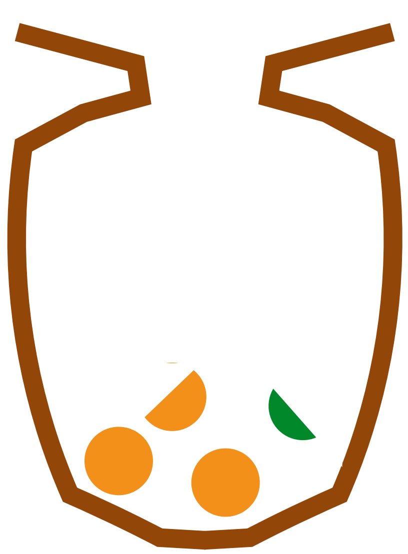

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Marginal cluster assignments

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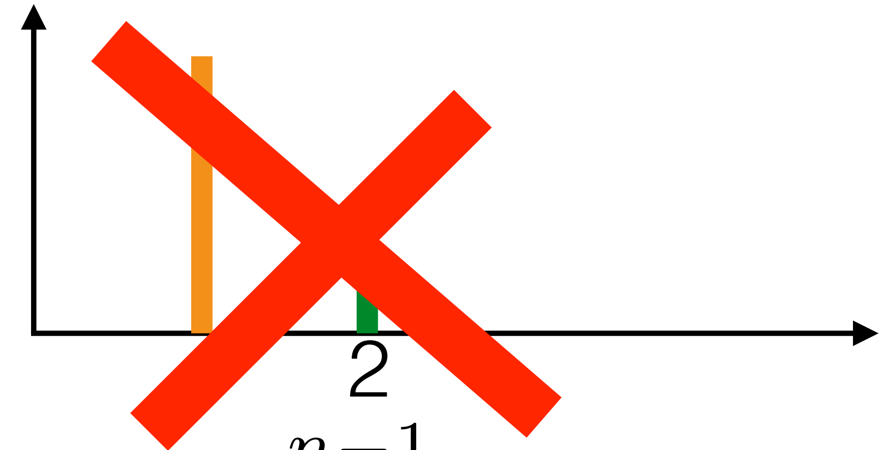

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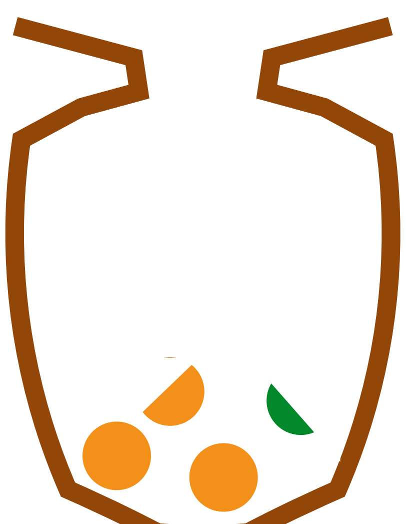
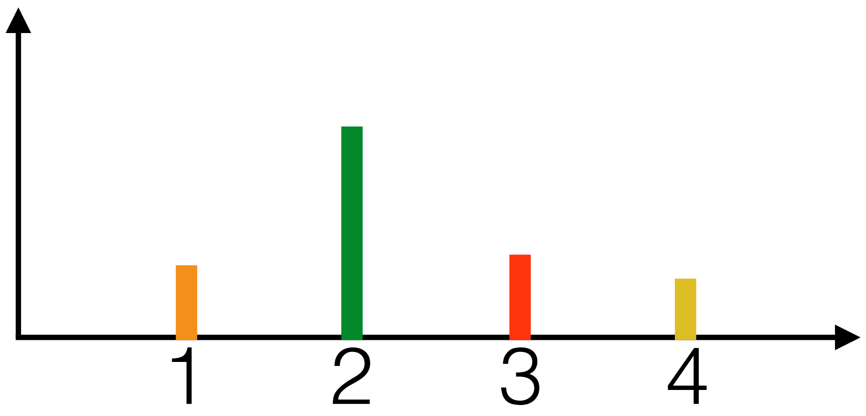


$$\lim_{n \to \infty} \frac{\#\ \text{orange}}{\#\ \text{total}} = \rho_{\text{orange}} \overset{d}{=} \text{Beta}(a_{\text{orange}}, a_{\text{green}})$$

$$\text{PolyaUrn}(a_{\text{orange}}, a_{\text{green}})$$

Marginal cluster assignments

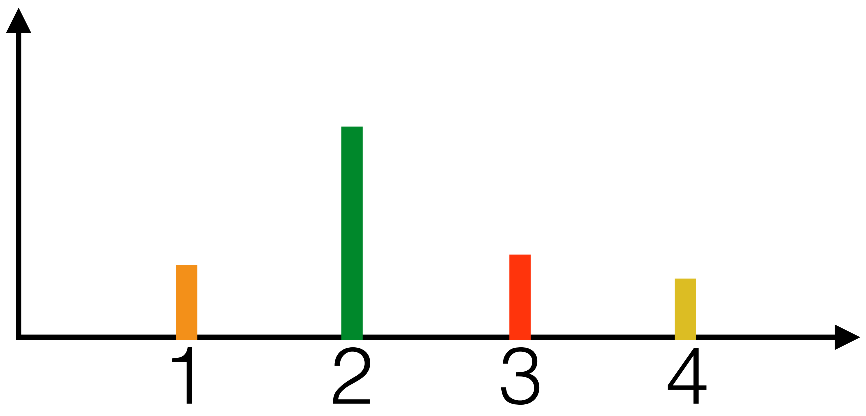
- Integrate out the frequencies



Marginal cluster assignments

- Integrate out the frequencies

$$\rho_{1:K} \sim \text{Dirichlet}(a_{1:K}), z_n \stackrel{iid}{\sim} \text{Cat}(\rho_{1:K})$$



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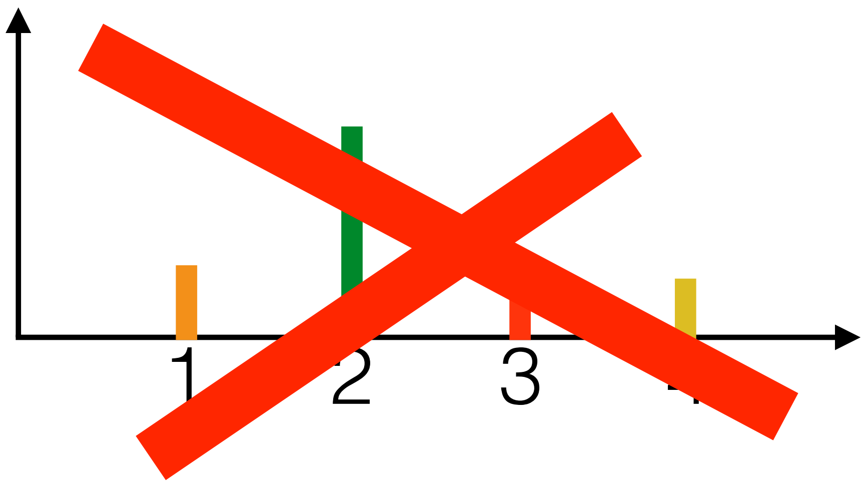

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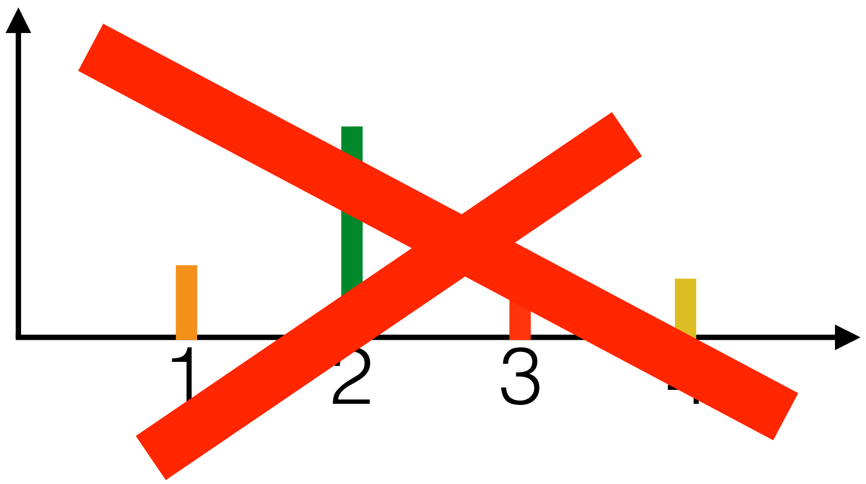

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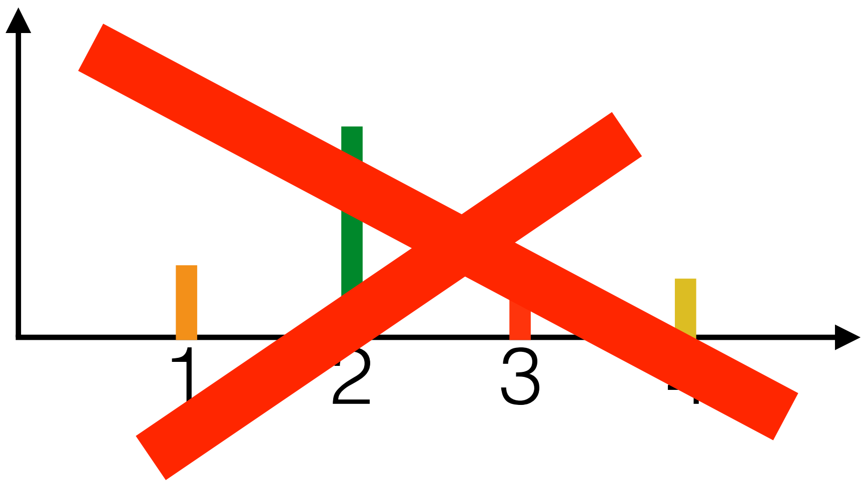

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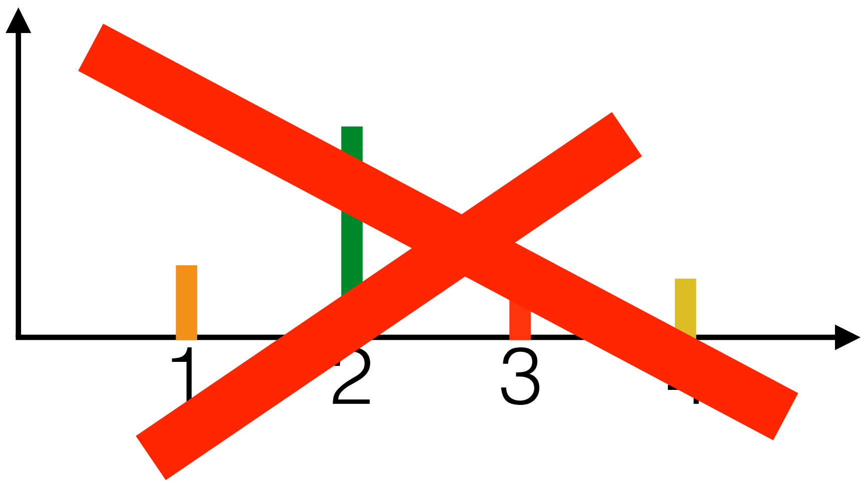

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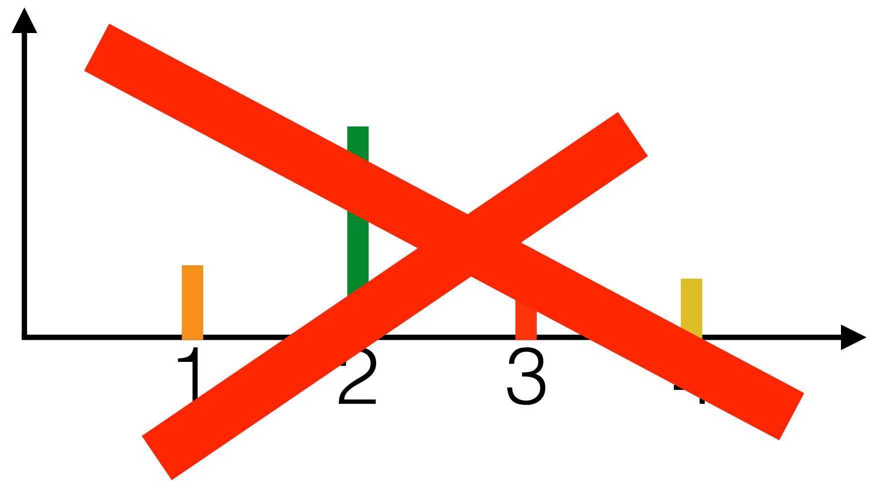
- multivariate Pólya urn

Marginal cluster assignments

- Integrate out the frequencies



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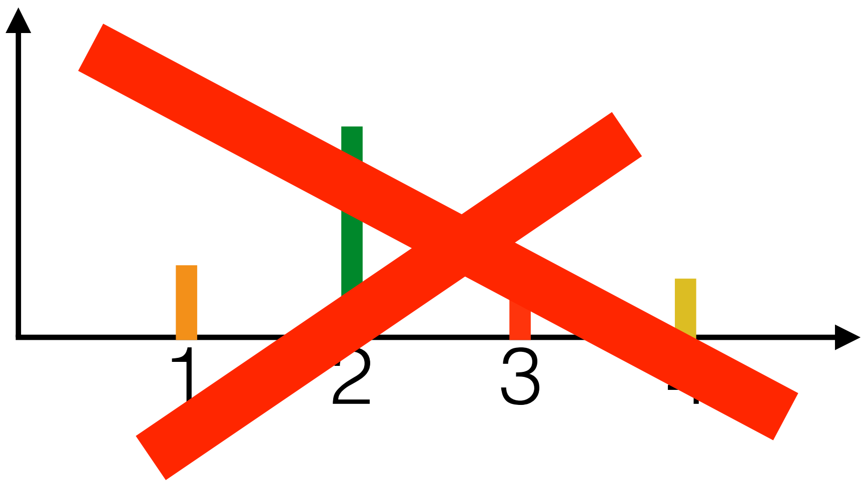

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  - Choose any ball with prob proportional to its mass

Marginal cluster assignments

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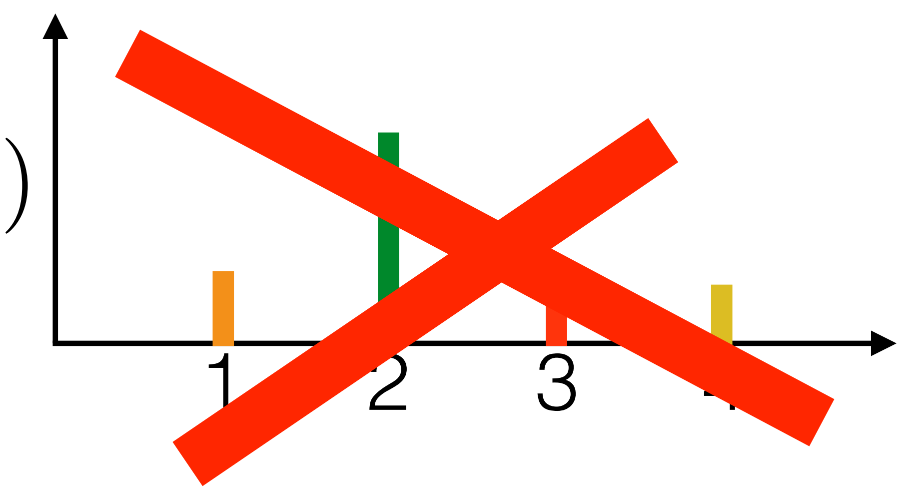

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Marginal cluster assignments

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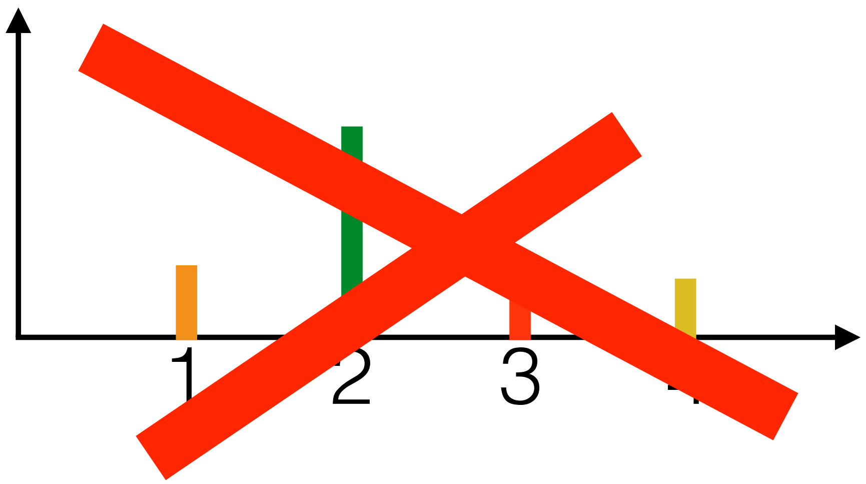

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- multivariate Pólya urn
  - Choose any ball with prob proportional to its mass
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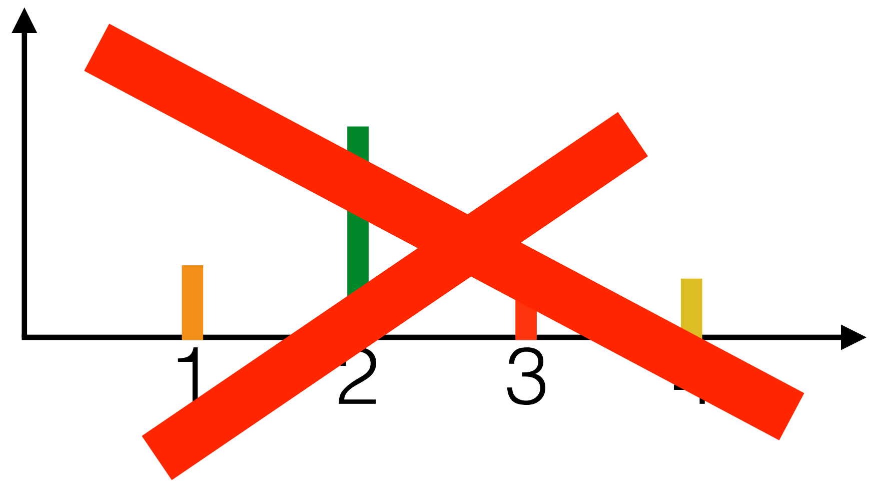
$$\lim_{n \to \infty} \frac{(\#\text{ orange, }\#\text{ green, }\#\text{ red, }\#\text{ yellow})}{\#\text{ total}}$$

Marginal cluster assignments

- Integrate out the frequencies

$$\rho_{1:K} \sim \text{Dirichlet}(a_{1:K}),\, z_n \overset{iid}{\sim} \text{Cat}(\rho_{1:K})$$



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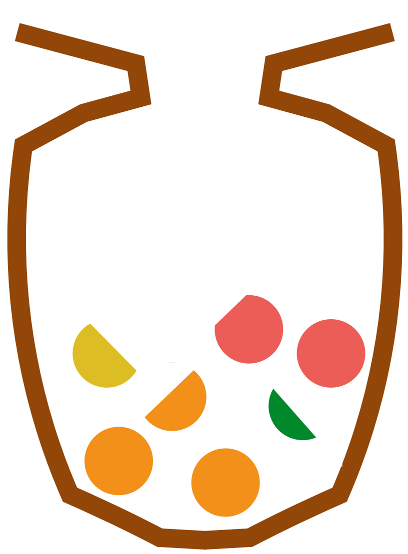


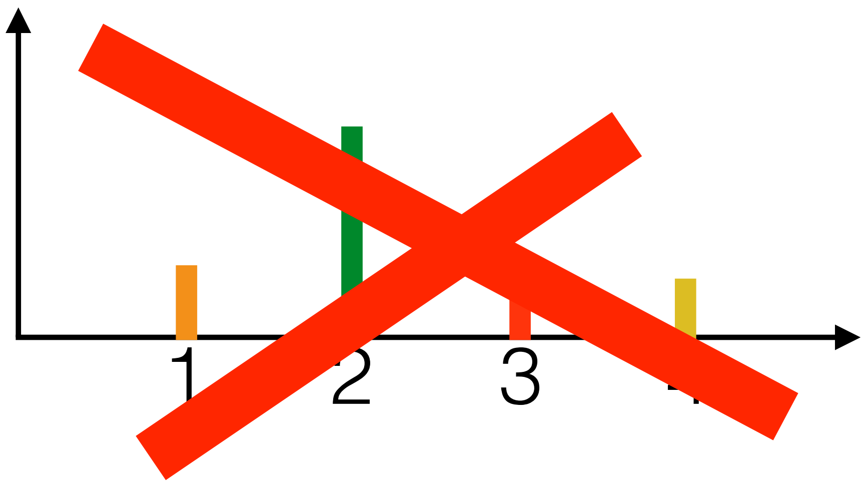
$$\lim_{n \to \infty} \frac{(\#\text{ orange}, \#\text{ green}, \#\text{ red}, \#\text{ yellow})}{\#\text{ total}}$$

$$\to (\rho_{\text{orange}}, \rho_{\text{green}}, \rho_{\text{red}}, \rho_{\text{yellow}})$$

Marginal cluster assignments

- Integrate out the frequencies

$$\rho_{1:K} \sim \text{Dirichlet}(a_{1:K}),\ z_n \overset{iid}{\sim} \text{Cat}(\rho_{1:K})$$



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$$\to (\rho_{\text{orange}}, \rho_{\text{green}}, \rho_{\text{red}}, \rho_{\text{yellow}})$$

$$\overset{d}{=} \text{Dirichlet}(a_{\text{orange}}, a_{\text{green}}, a_{\text{red}}, a_{\text{yellow}})$$

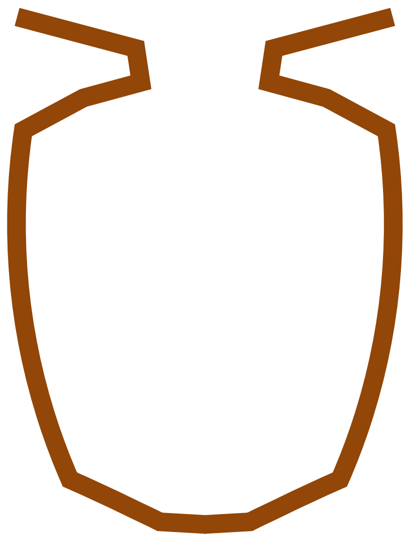
Marginal cluster assignments

- Hoppe urn / Blackwell-MacQueen urn

[Blackwell, MacQueen 1973; Hoppe 1984]

Marginal cluster assignments

- Hoppe urn / Blackwell-MacQueen urn



[Blackwell, MacQueen 1973; Hoppe 1984]

Marginal cluster assignments

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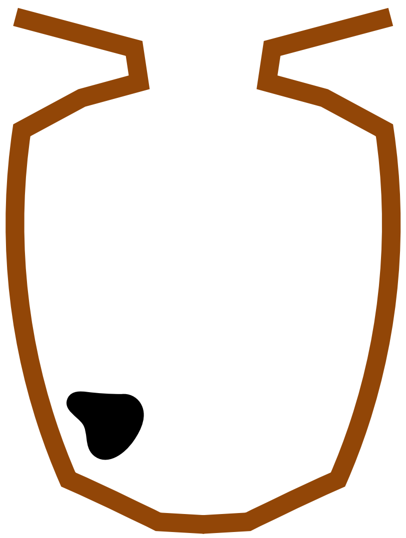
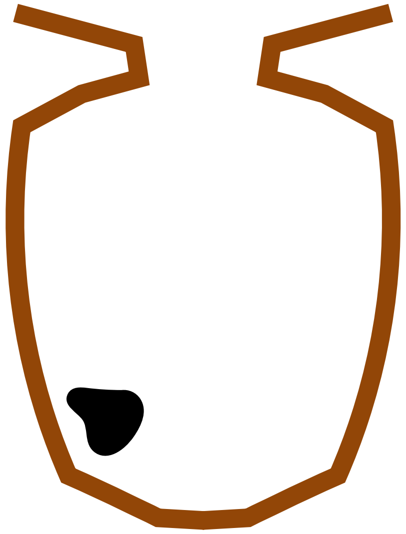


[Blackwell, MacQueen 1973; Hoppe 1984]

Marginal cluster assignments

- Hoppe urn / Blackwell-MacQueen urn

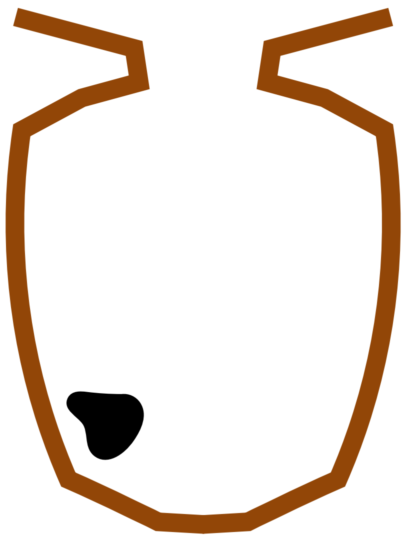


  - Choose ball with prob proportional to its mass

[Blackwell, MacQueen 1973; Hoppe 1984]

Marginal cluster assignments

- Hoppe urn / Blackwell-MacQueen urn

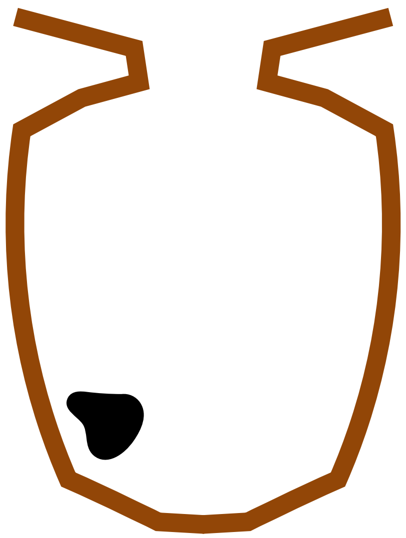


- Choose ball with prob proportional to its mass
  - If black, replace and add ball of new color

[Blackwell, MacQueen 1973; Hoppe 1984]

Marginal cluster assignments

- Hoppe urn / Blackwell-MacQueen urn



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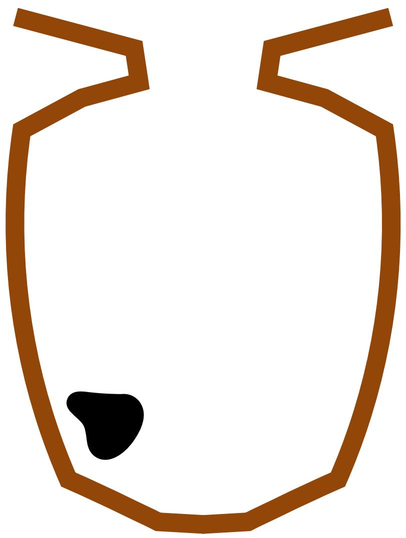

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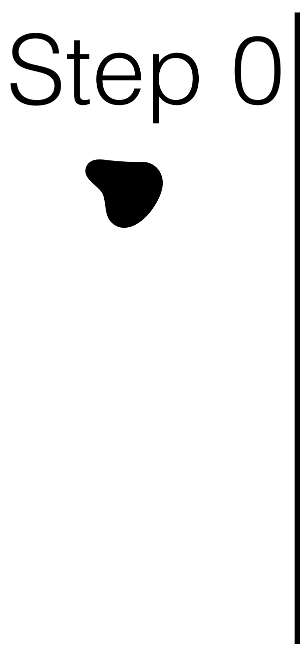
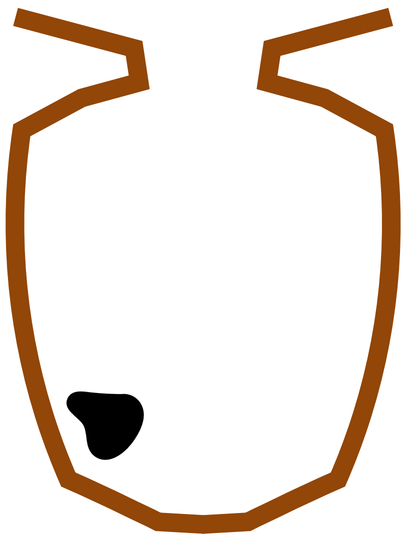


[Blackwell, MacQueen 1973; Hoppe 1984]

Marginal cluster assignments

- Hoppe urn / Blackwell-MacQueen urn



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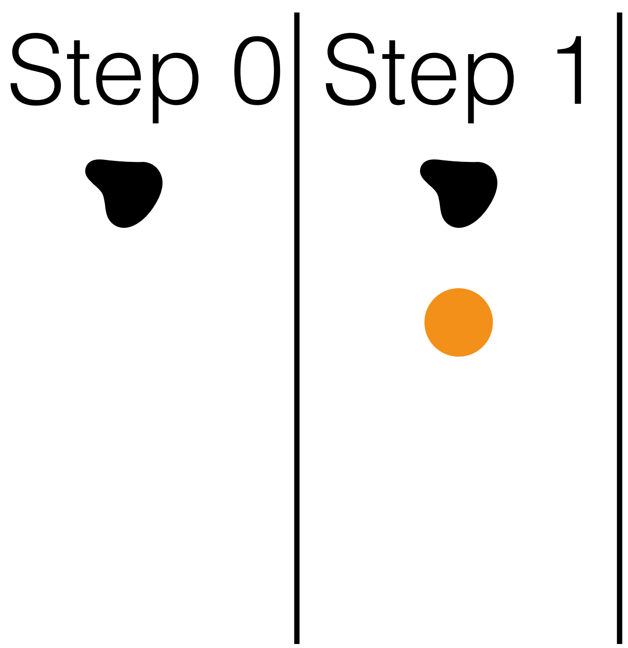
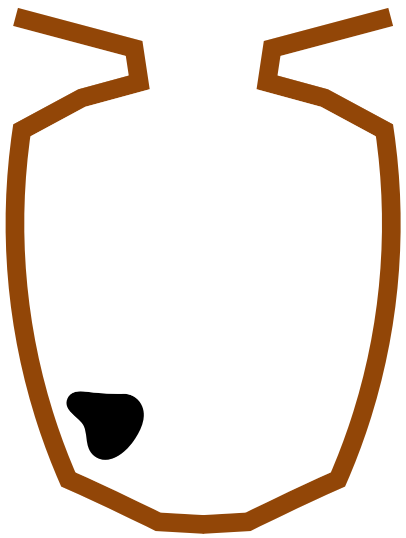


[Blackwell, MacQueen 1973; Hoppe 1984]

Marginal cluster assignments

- Hoppe urn / Blackwell-MacQueen urn



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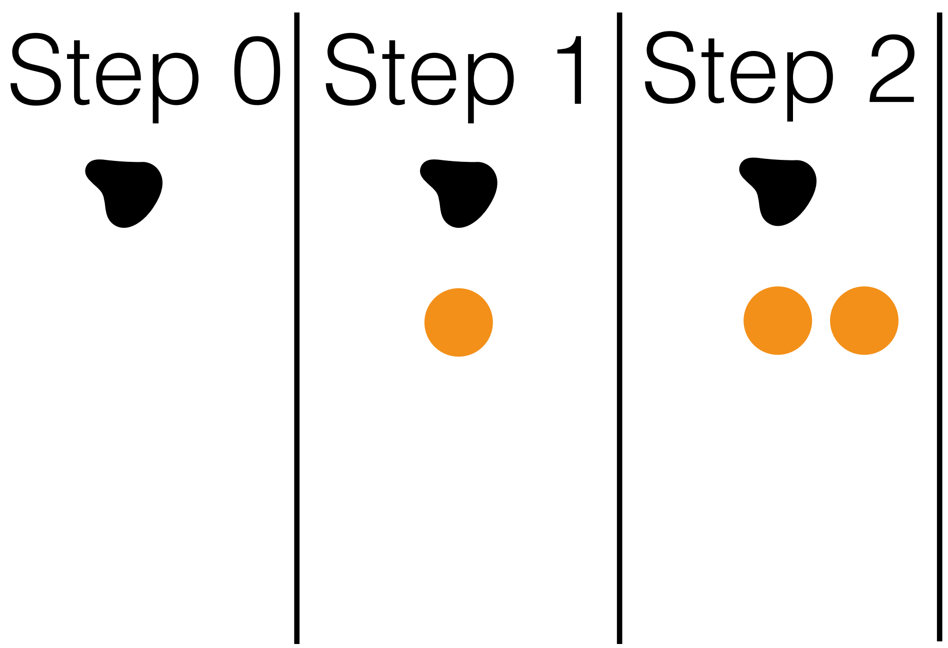
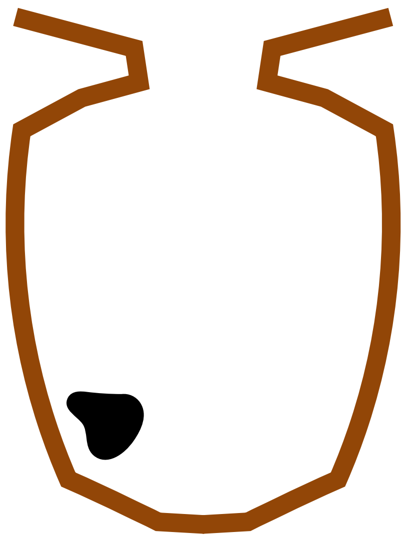


[Blackwell, MacQueen 1973; Hoppe 1984]

Marginal cluster assignments

- Hoppe urn / Blackwell-MacQueen urn



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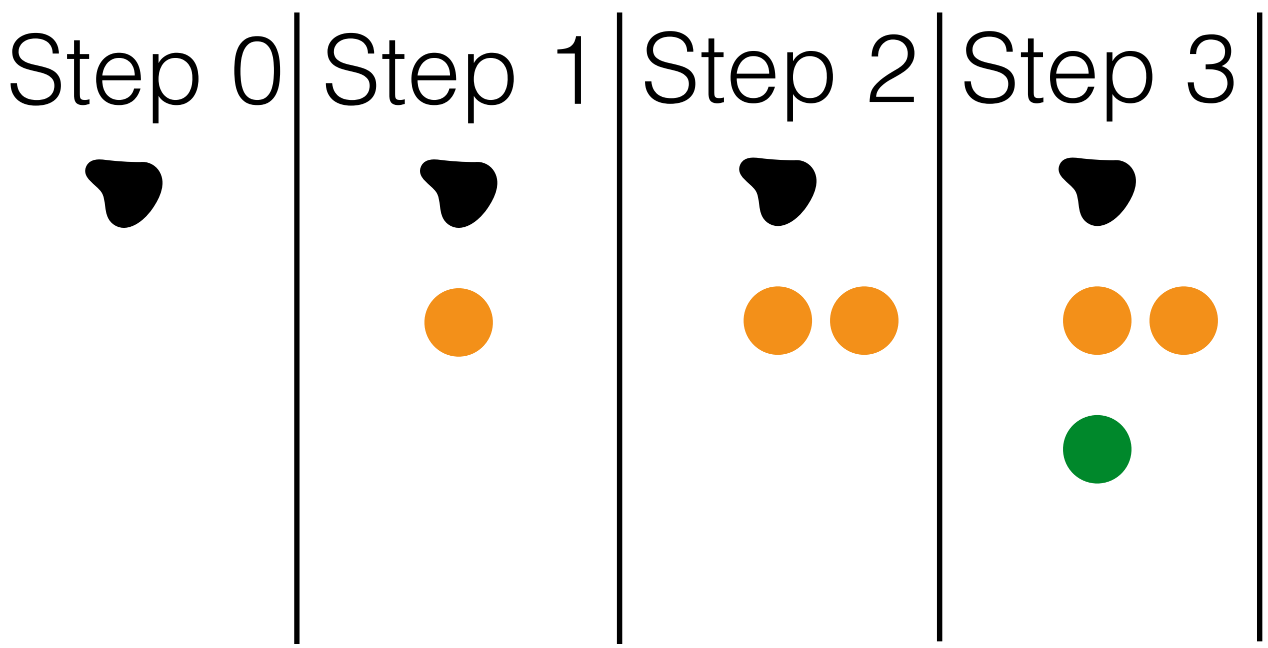

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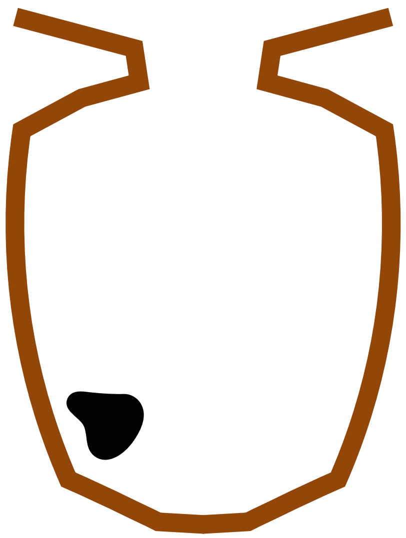
Marginal cluster assignments

- Hoppe urn / Blackwell-MacQueen urn



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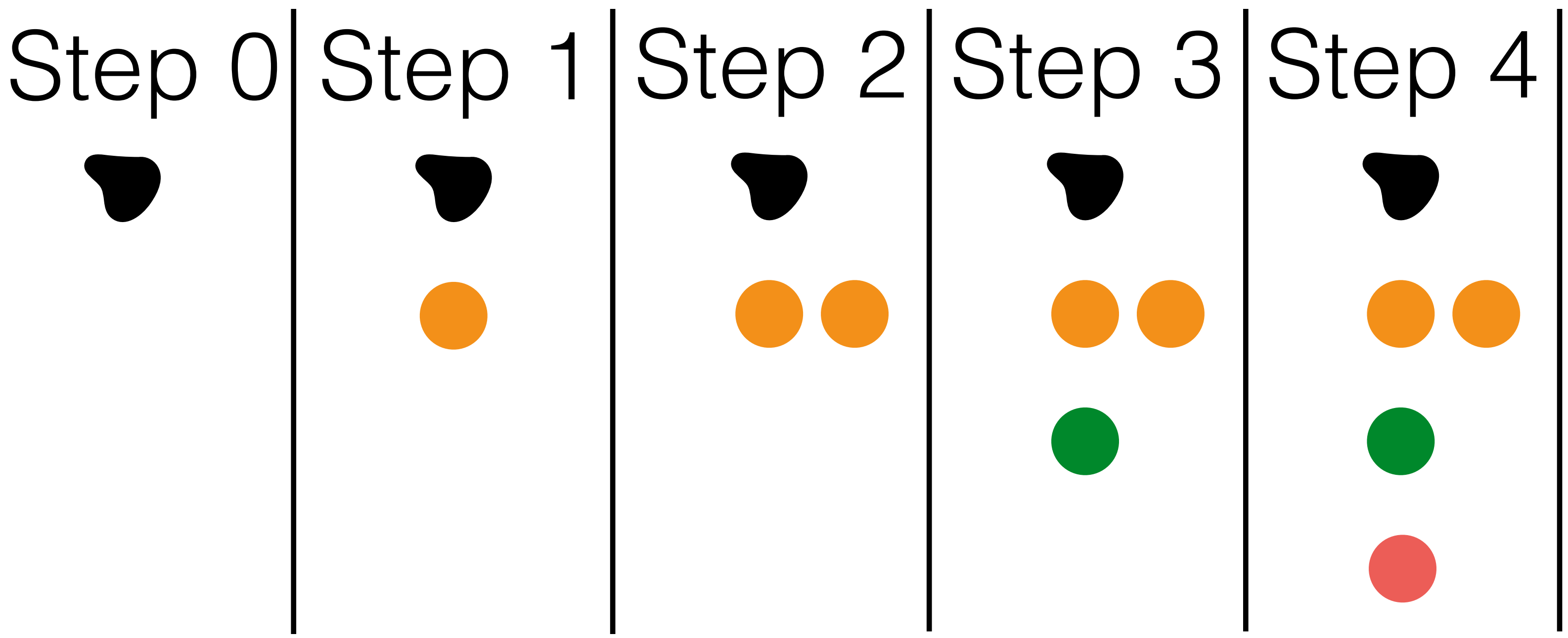
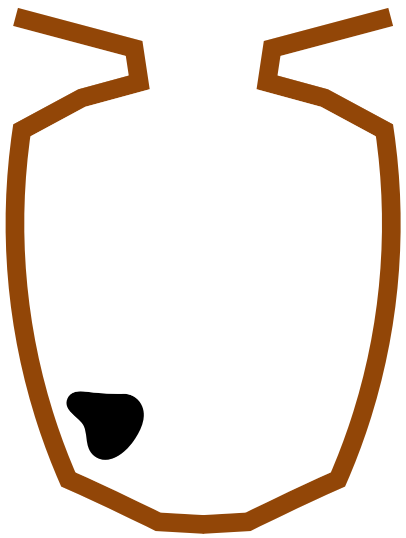


[Blackwell, MacQueen 1973; Hoppe 1984]

Marginal cluster assignments

- Hoppe urn / Blackwell-MacQueen urn



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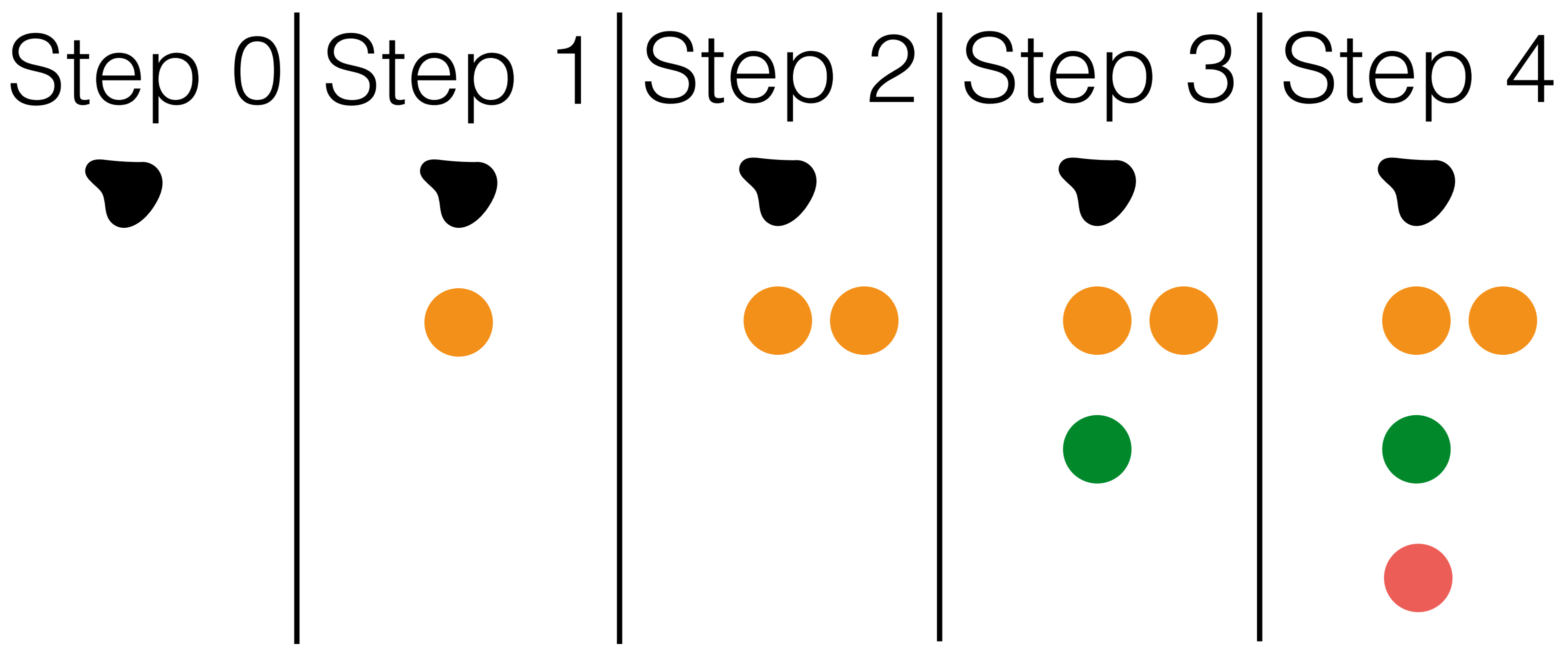
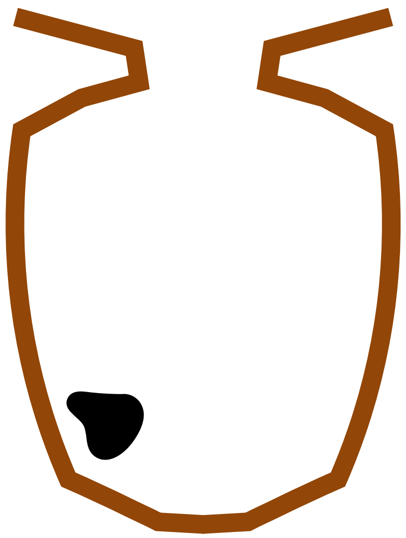


$(\#\text{orange}, \#\text{other})$

Marginal cluster assignments

- Hoppe urn / Blackwell-MacQueen urn



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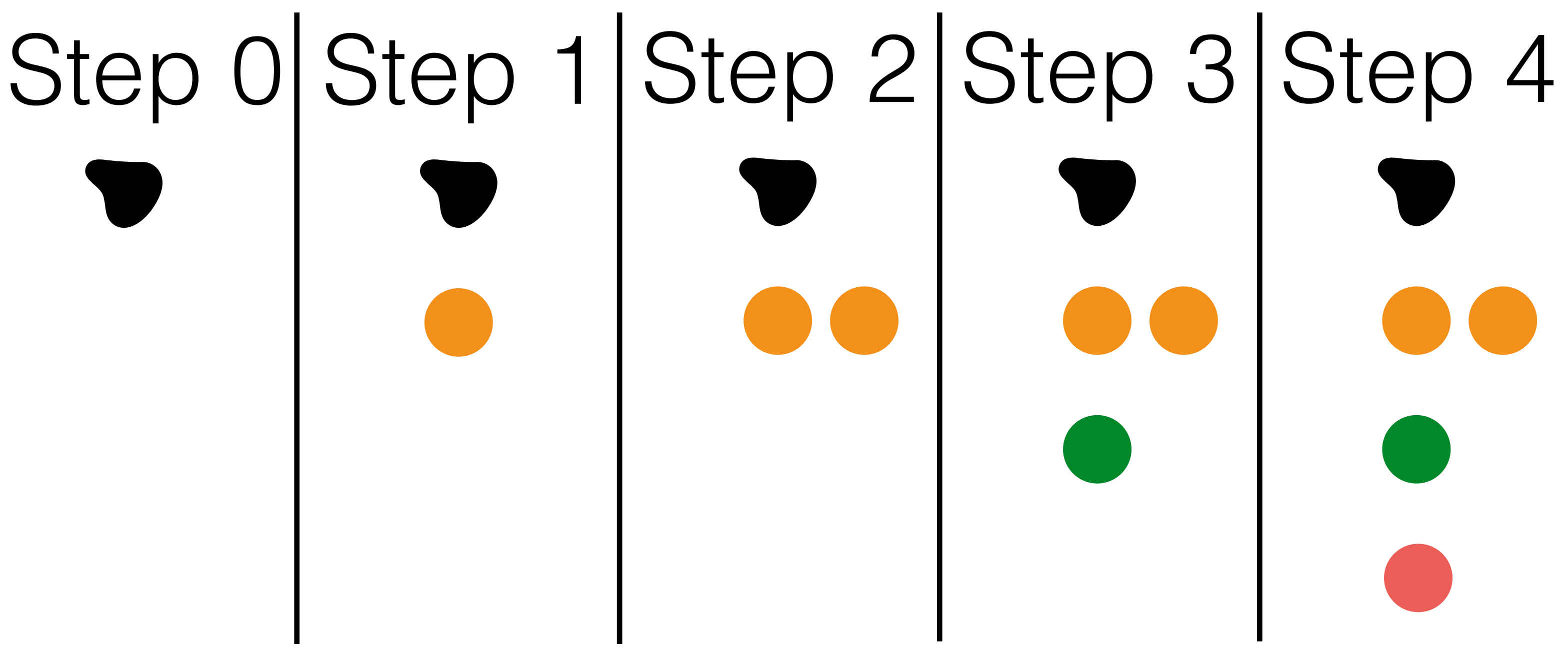
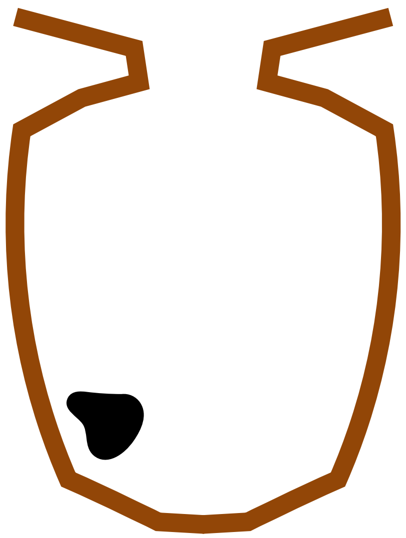


$$(\#\text{orange}, \#\text{other}) = \text{PolyaUrn}(1, \alpha)$$

Marginal cluster assignments

- Hoppe urn / Blackwell-MacQueen urn



- Choose ball with prob proportional to its mass
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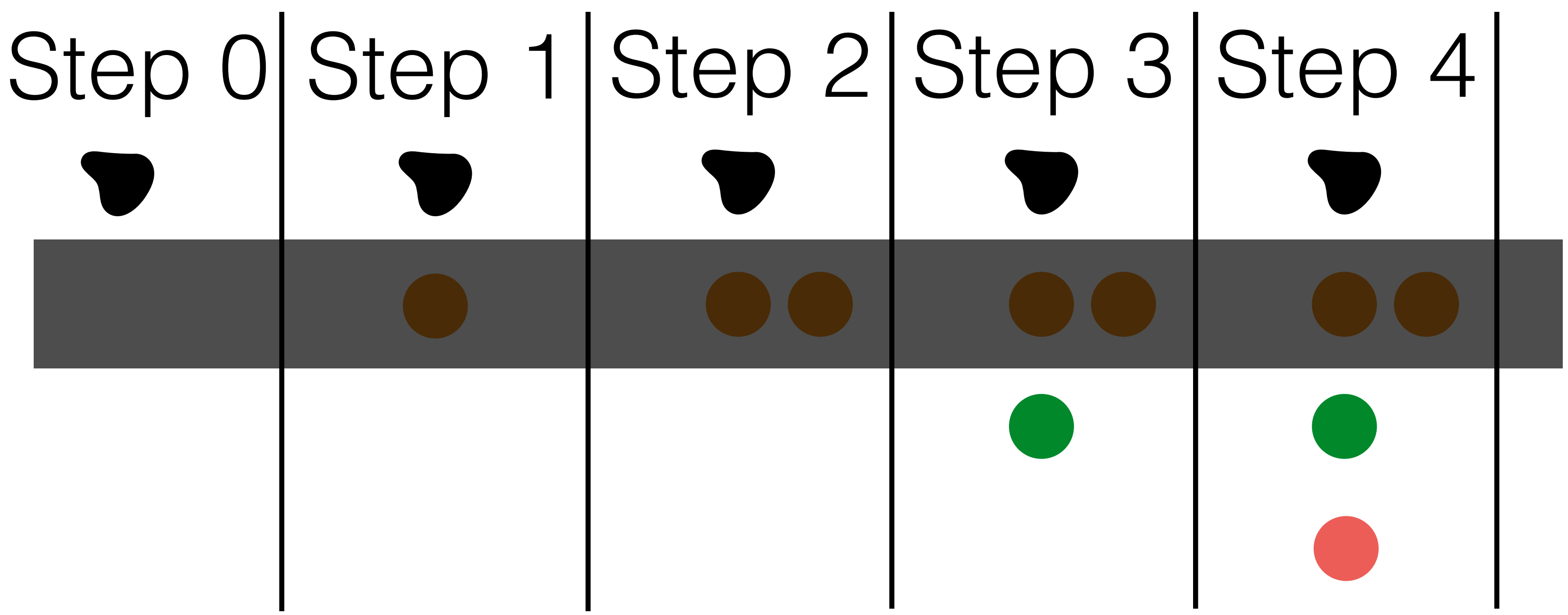

$$(\#\text{orange}, \#\text{other}) = \text{PolyaUrn}(1, \alpha)$$

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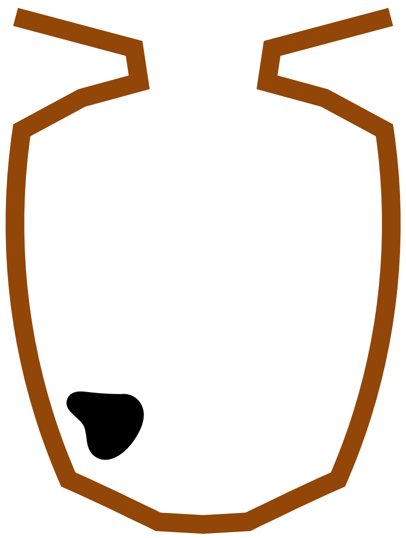

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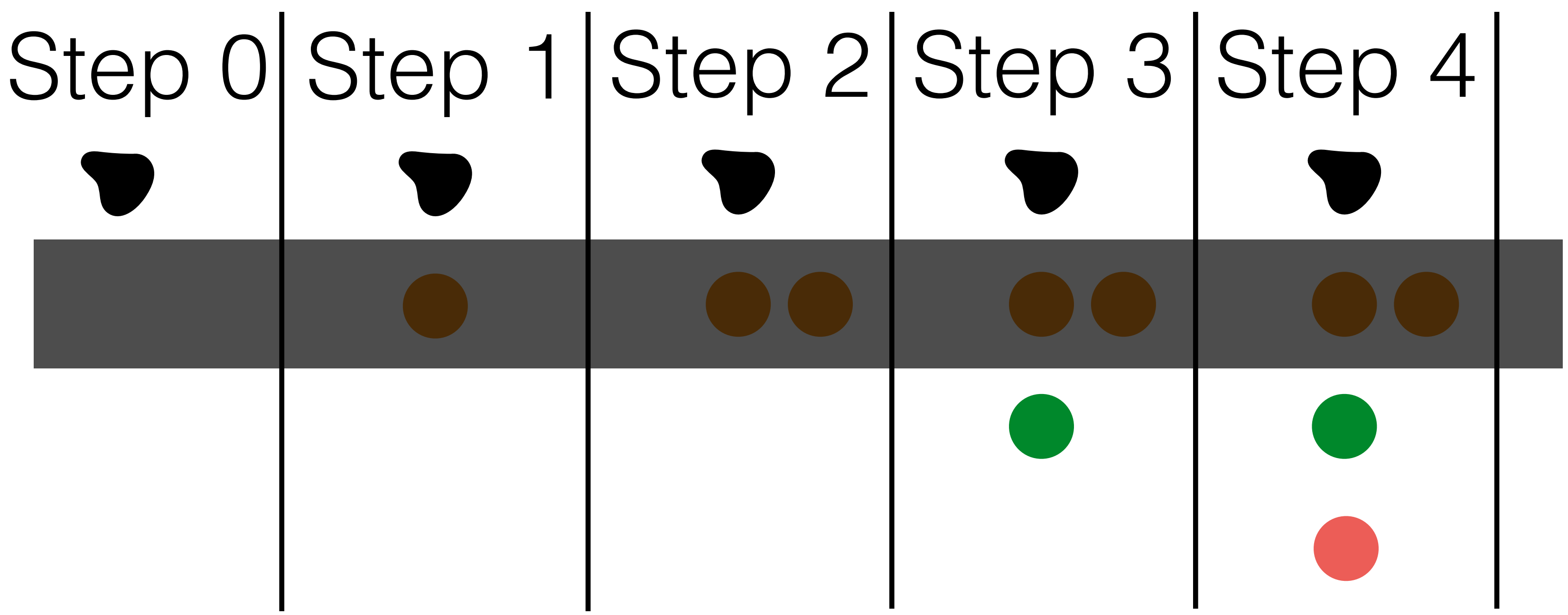

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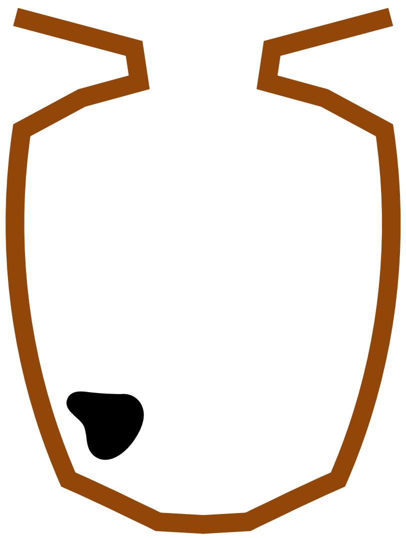

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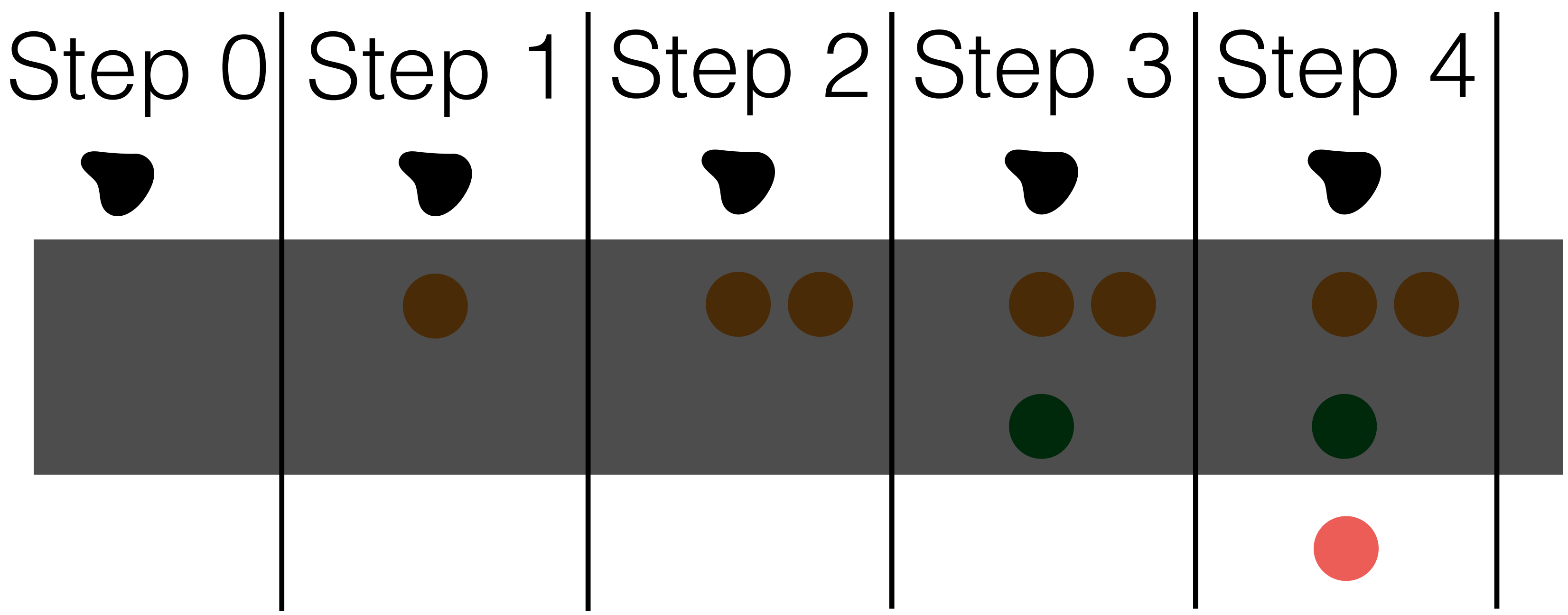

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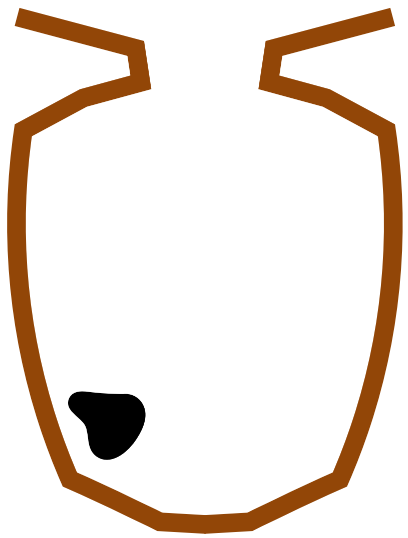

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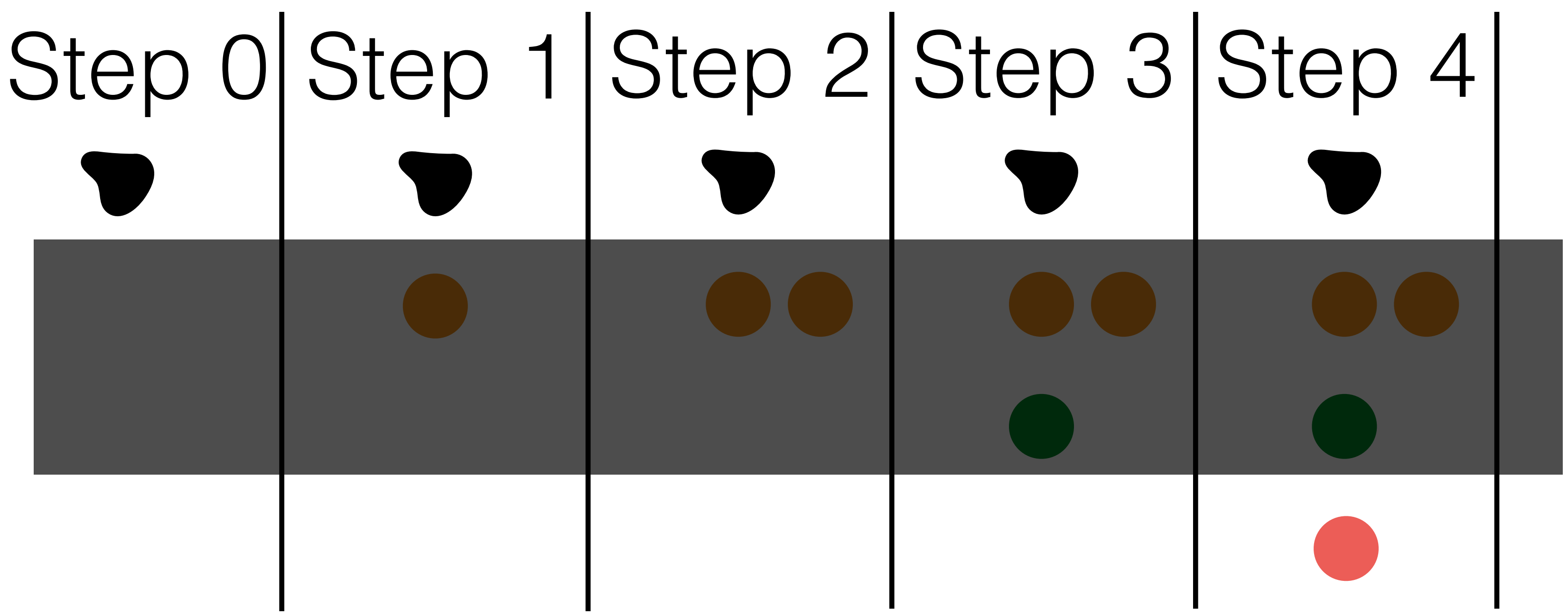

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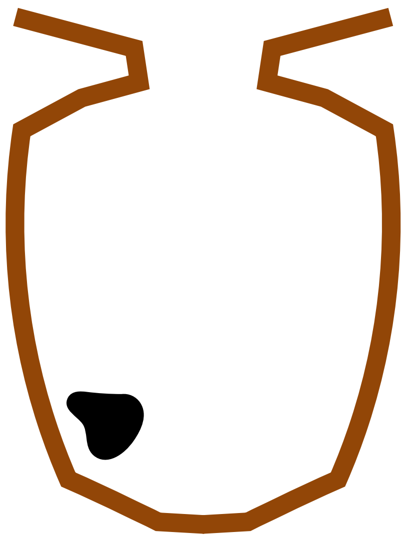

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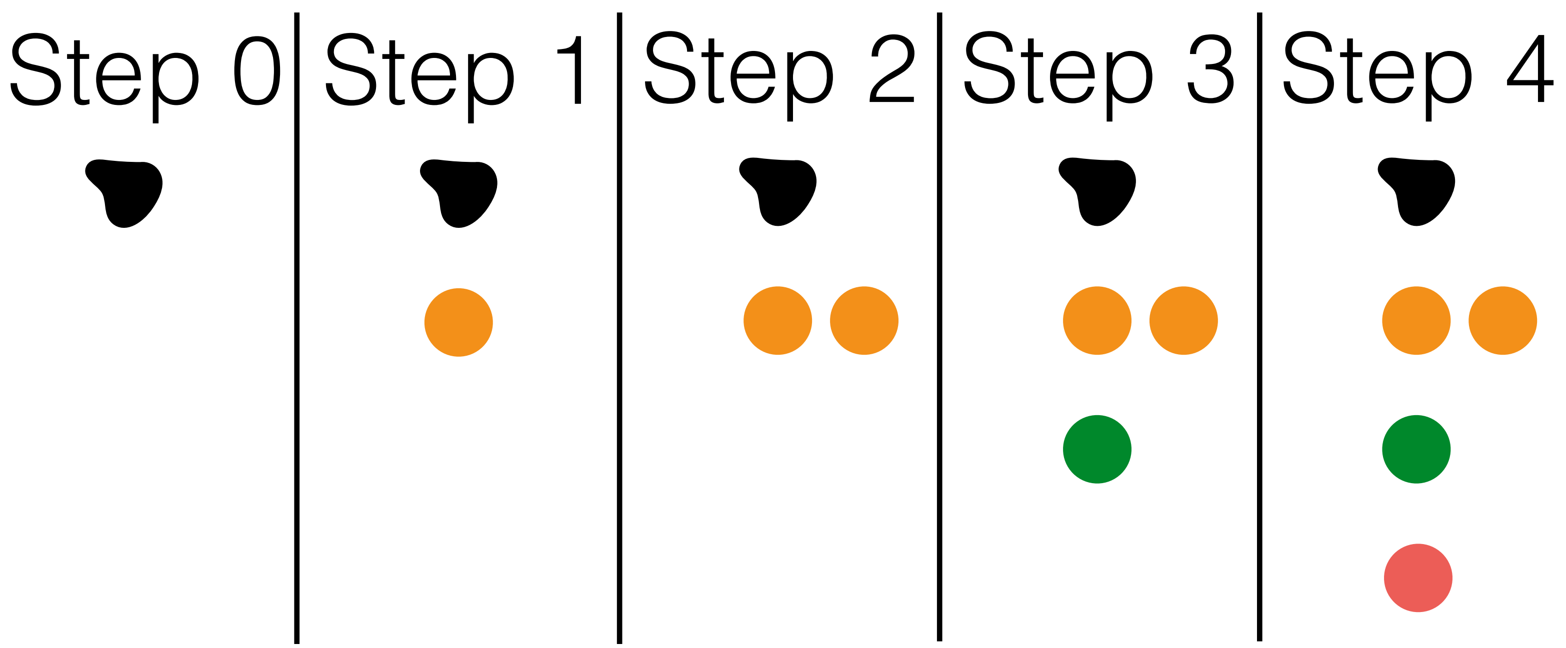

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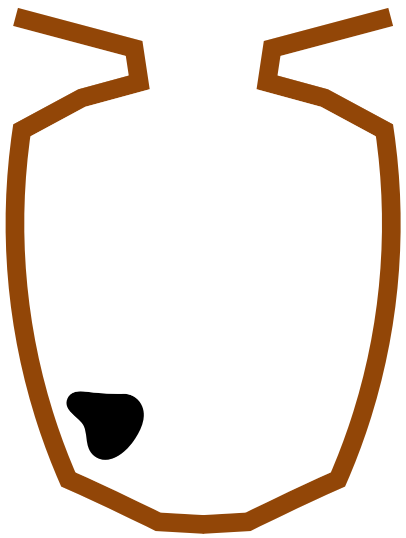

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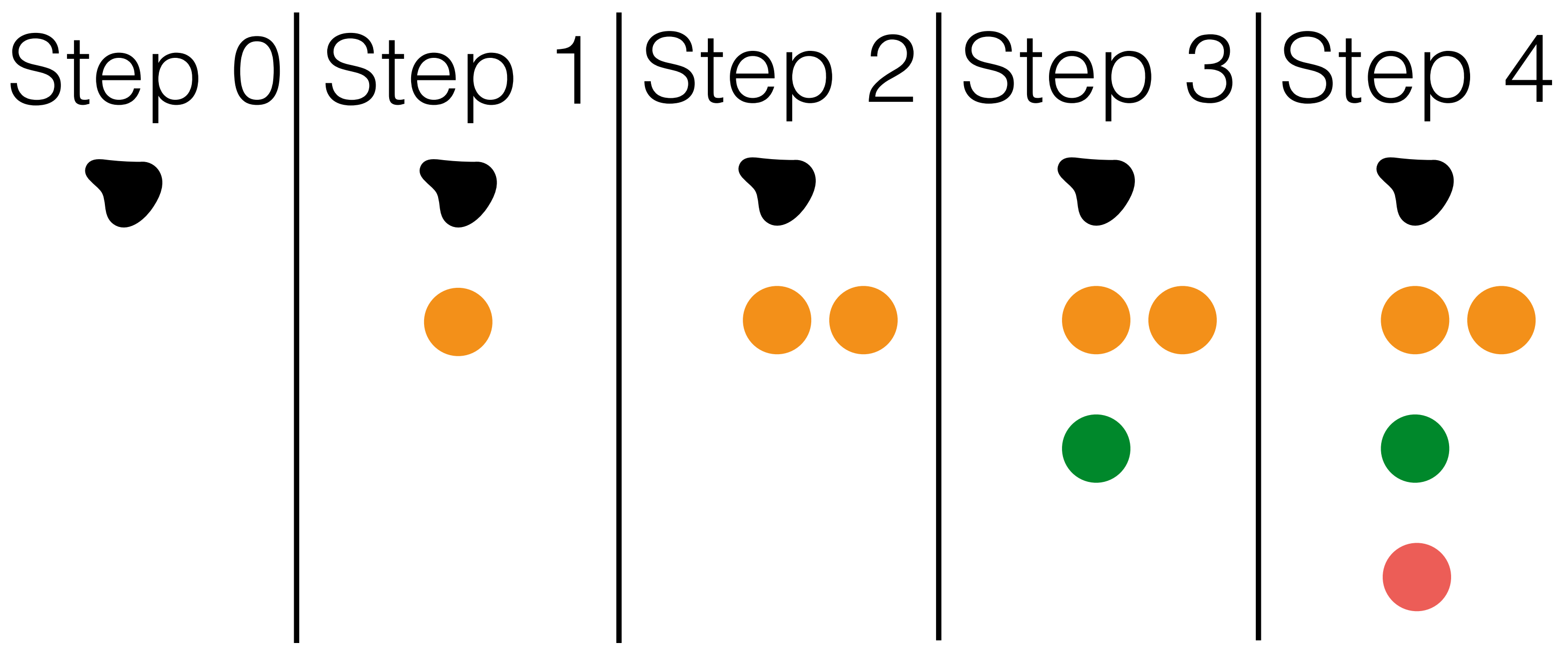

$$V_k \overset{iid}{\sim} \text{Beta}(1, \alpha)$$

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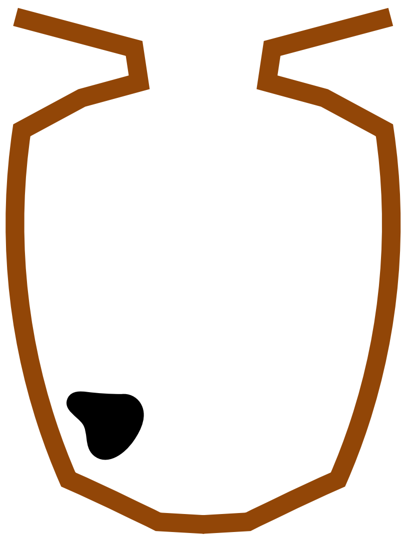

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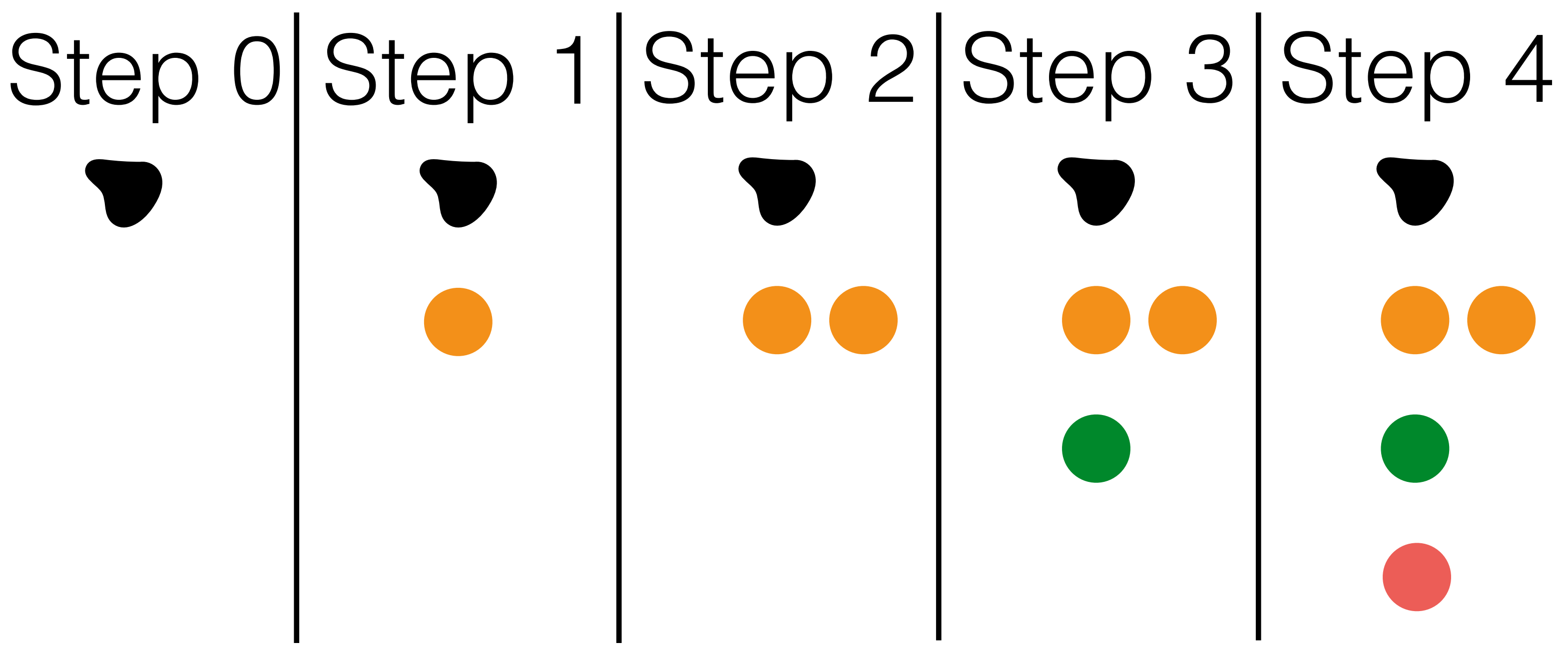

$$V_k \overset{iid}{\sim} \mathrm{Beta}(1, \alpha)$$

$$\rho_1 = V_1$$

$$(\#\text{orange}, \#\text{other}) = \text{PolyaUrn}(1, \alpha)$$

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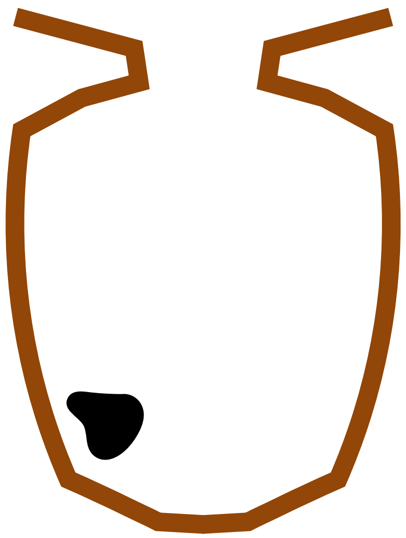

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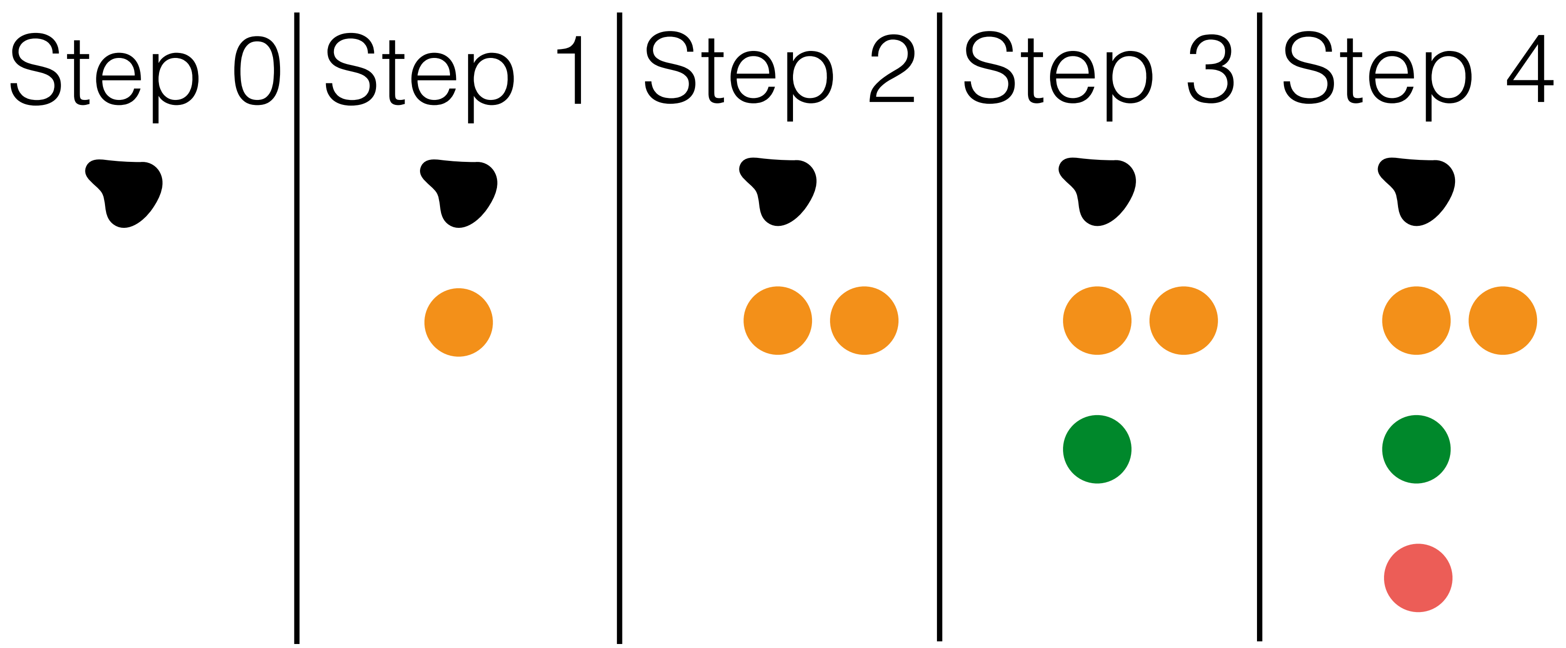

$$V_k \overset{iid}{\sim} \text{Beta}(1, \alpha)$$

$$\rho_1 = V_1$$

$$\rho_2 = (1 - V_1)V_2$$

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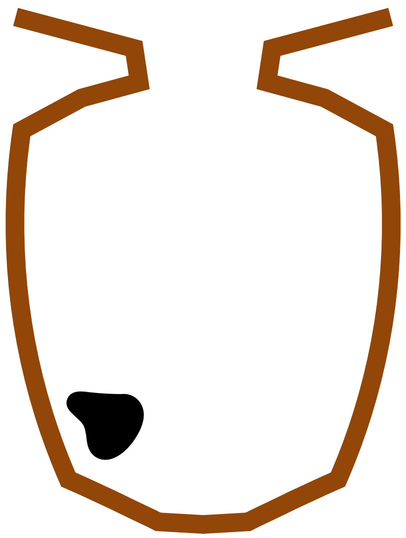

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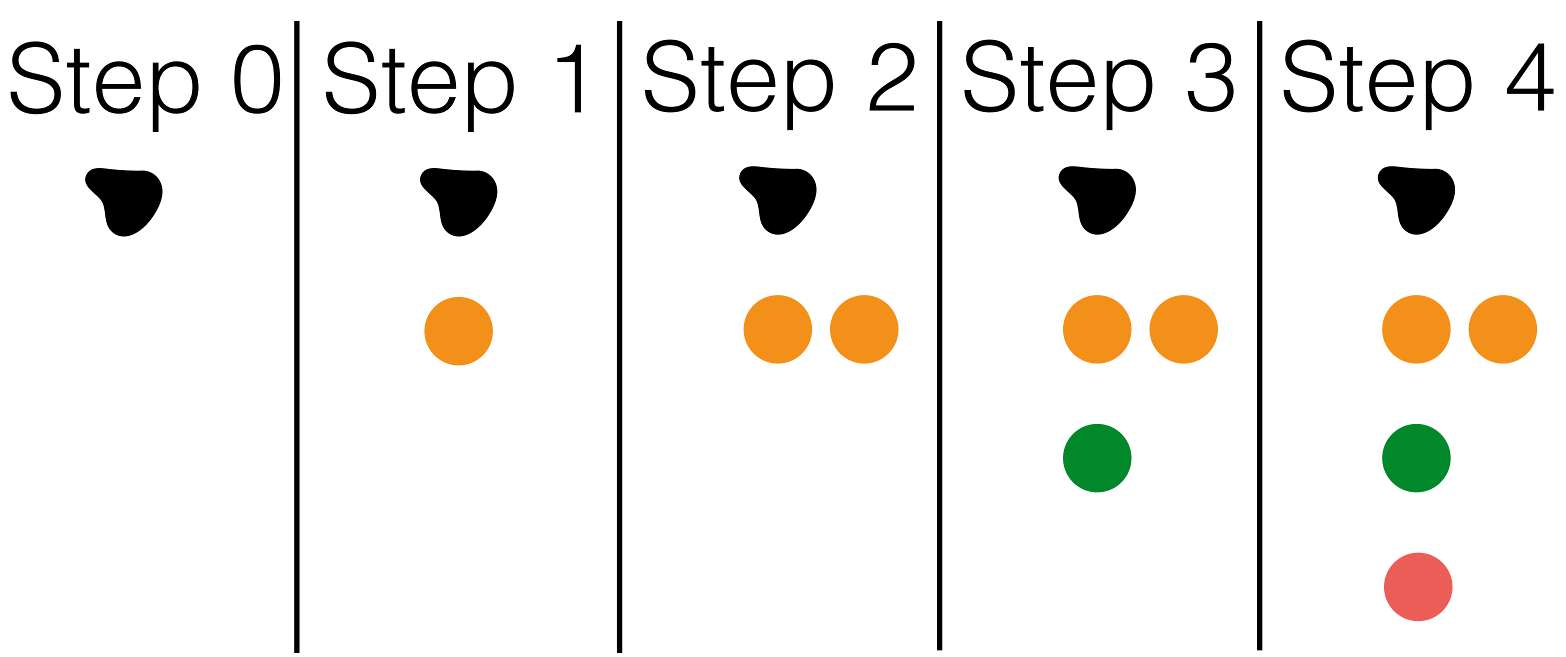

$$V_k \stackrel{iid}{\sim} \text{Beta}(1, \alpha)$$

$$\rho_1 = V_1$$

$$\rho_2 = (1 - V_1)V_2$$

$$\rho_3 = [\prod_{k=1}^{2}(1 - V_k)]V_3$$

$$(\#\text{orange}, \#\text{other}) = \text{PolyaUrn}(1, \alpha)$$

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Exercises

[slides, code: www.tamarabroderick.com/tutorials.html]

- Review Gibbs sampling.
- Derive the Dirichlet-Categorical marginal.
- What are the advantages and disadvantages of the DP and urn representations?
- Can you find a formula for the expected # clusters from a Hoppe-urn(α) after *N* data points? What happens as $N \to \infty$
- Code a Hoppe/Blackwell-MacQueen urn simulator. Examine the empirical distribution of the # clusters after *N* customers.
- Code a GEM & Categorical simulator. Compare your two simulators.

References

A full reference list is provided at the end of the “Part 3” slides.