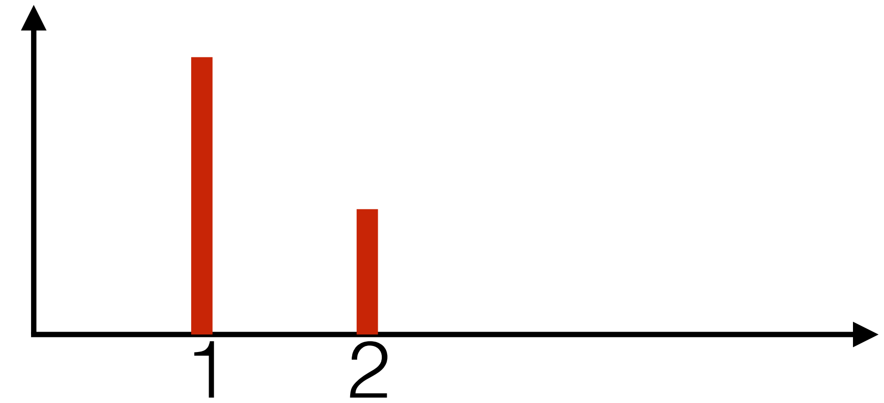
Nonparametric Bayesian Statistics: Part II

Tamara Broderick
ITT Career Development Assistant Professor
Electrical Engineering & Computer Science
MIT

Distributions

Distributions

- Beta → random distribution over $1, 2$



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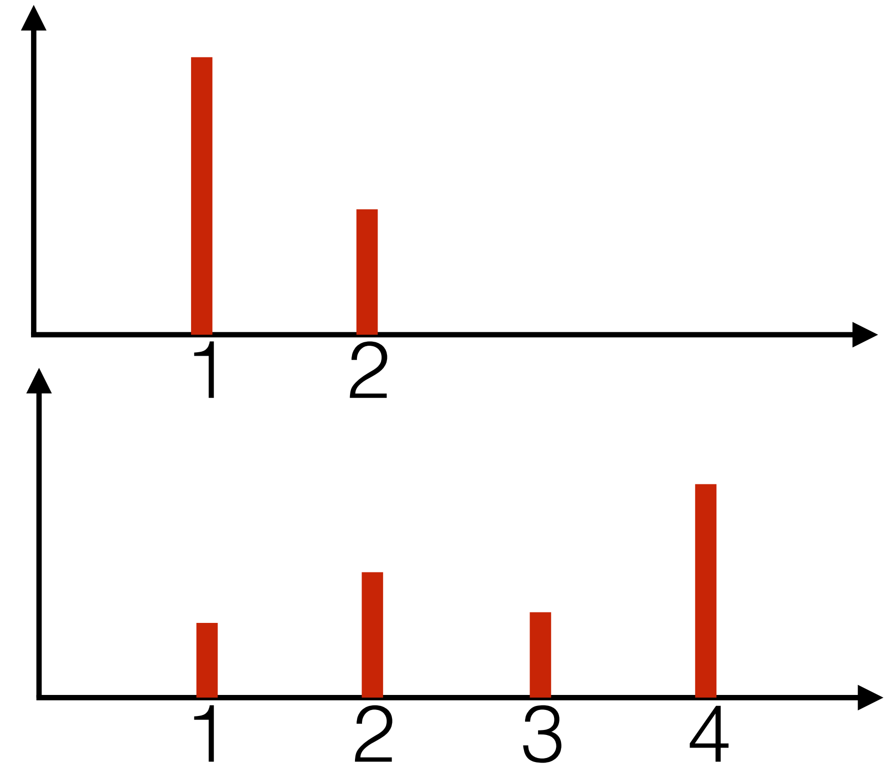

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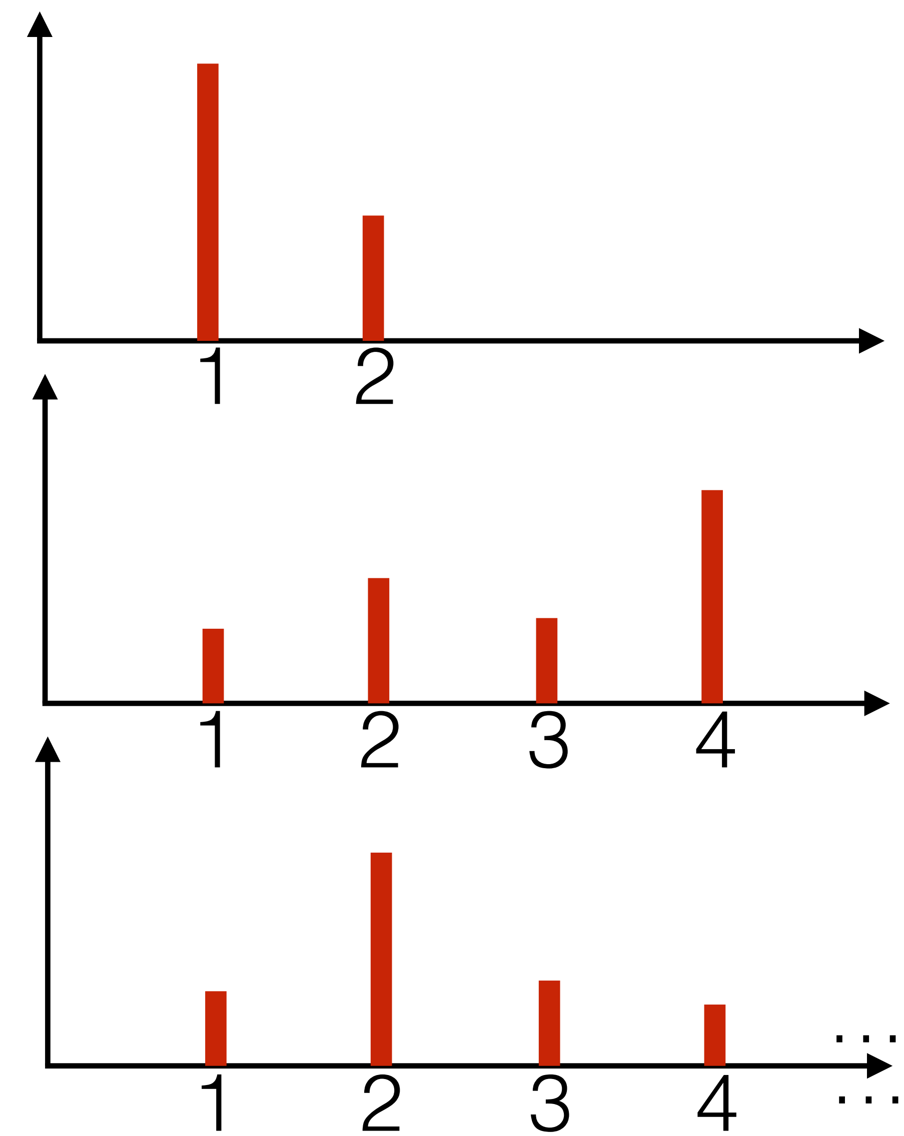

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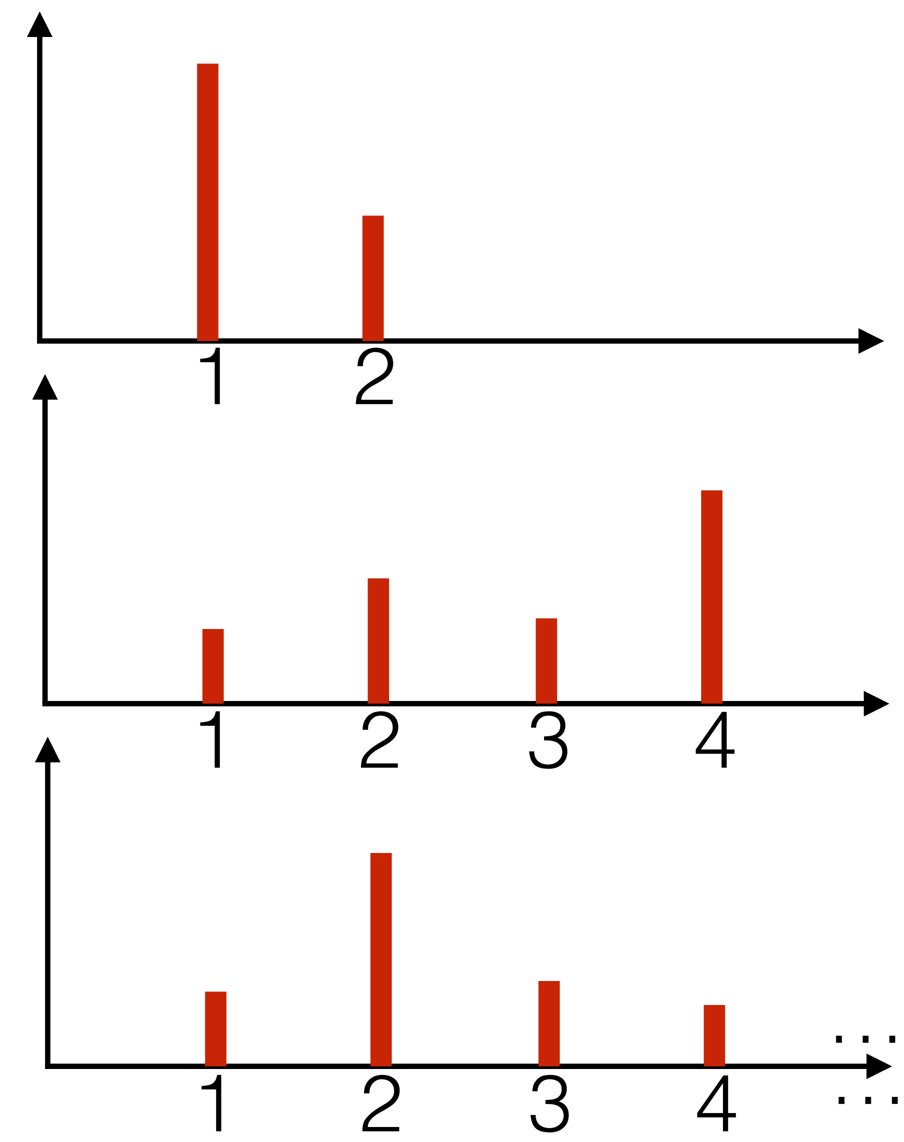

- Infinity of parameters: components
- Growing number of parameters: clusters

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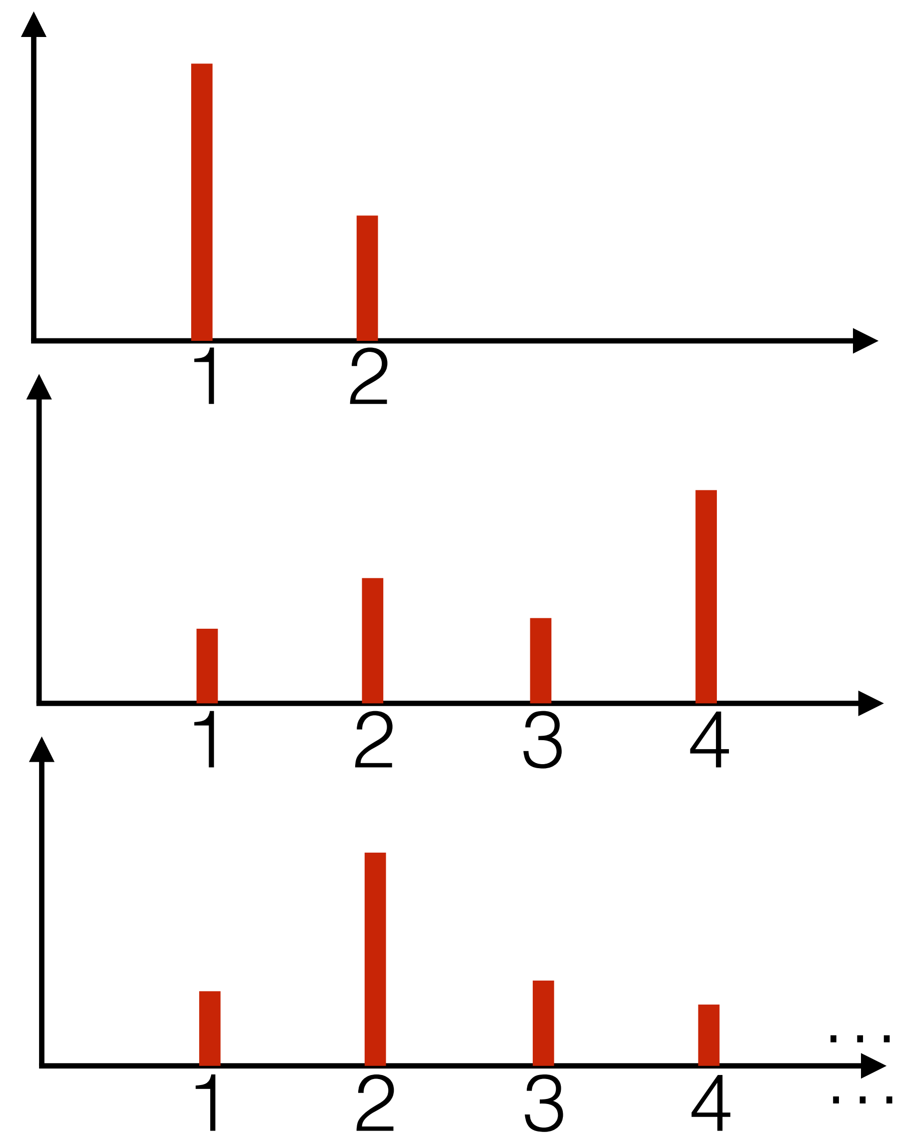

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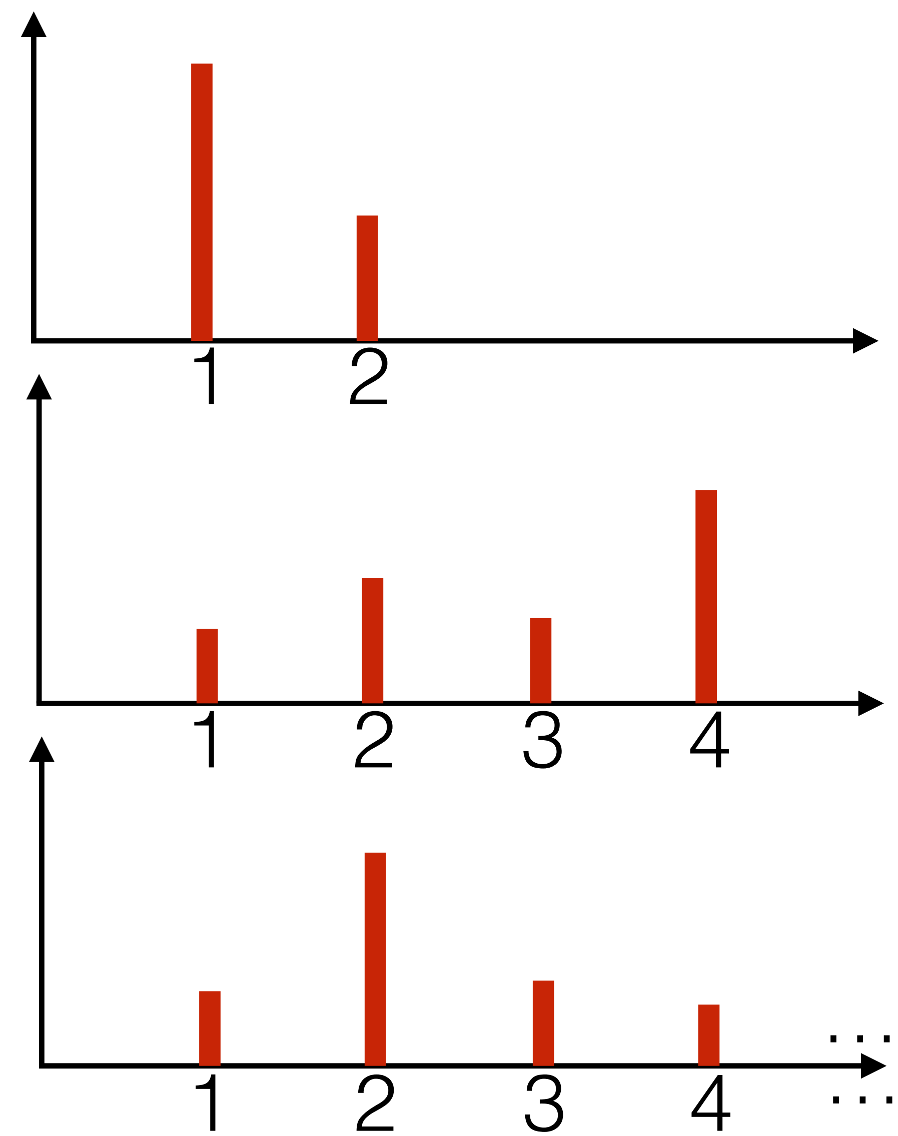

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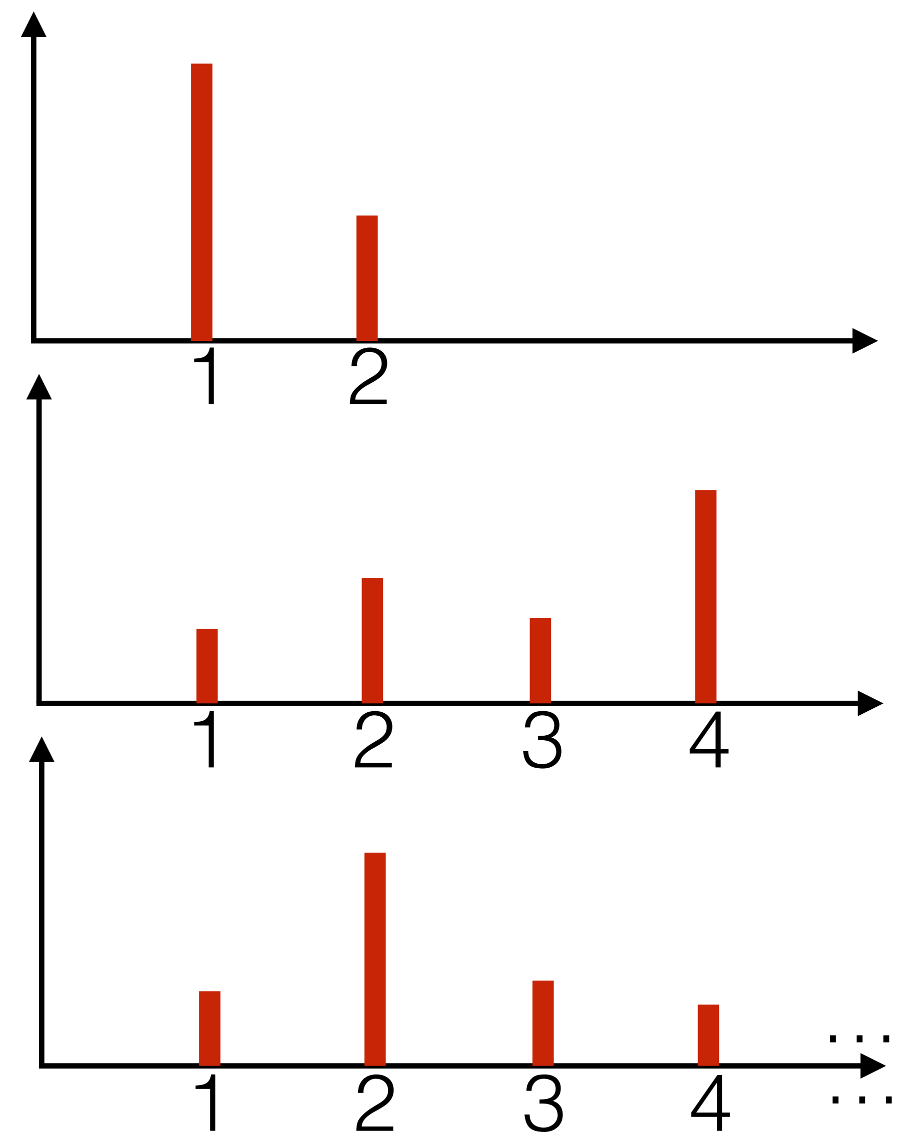

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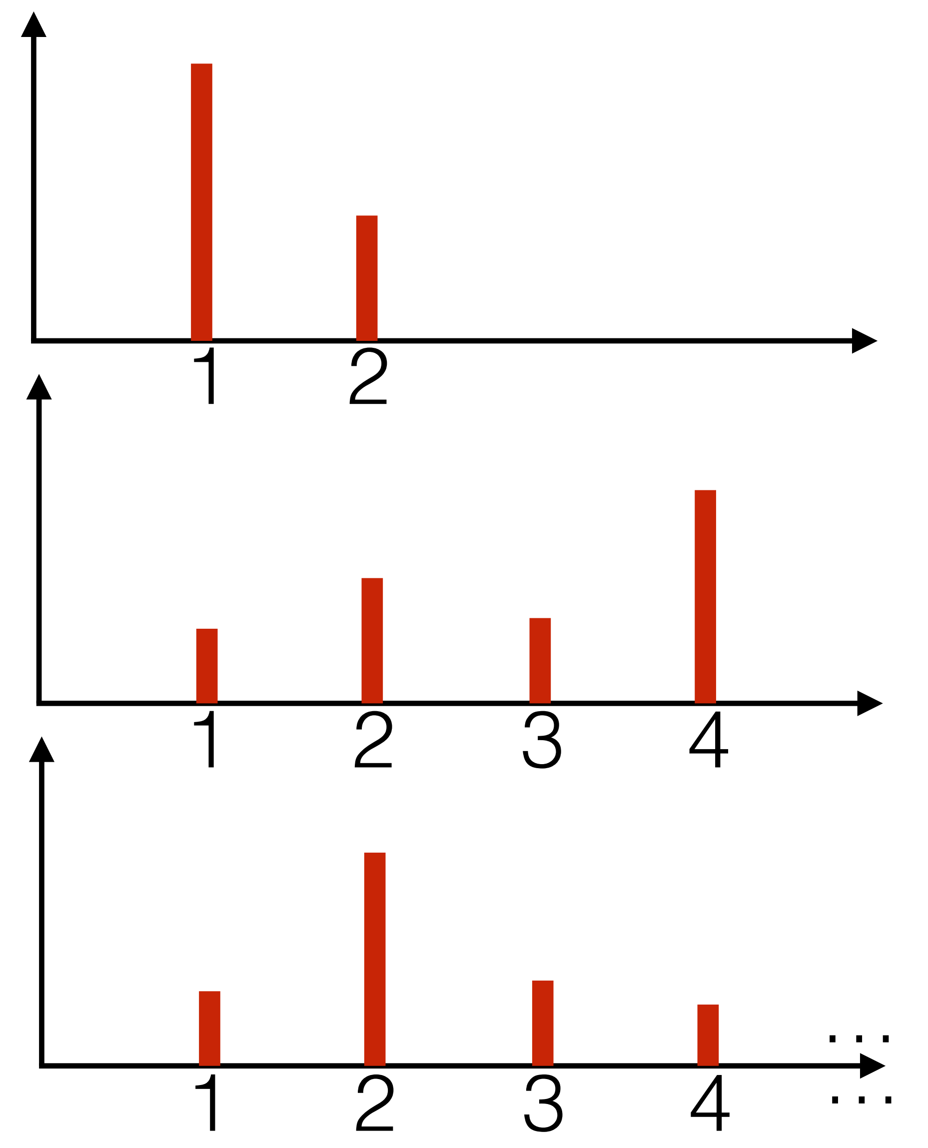

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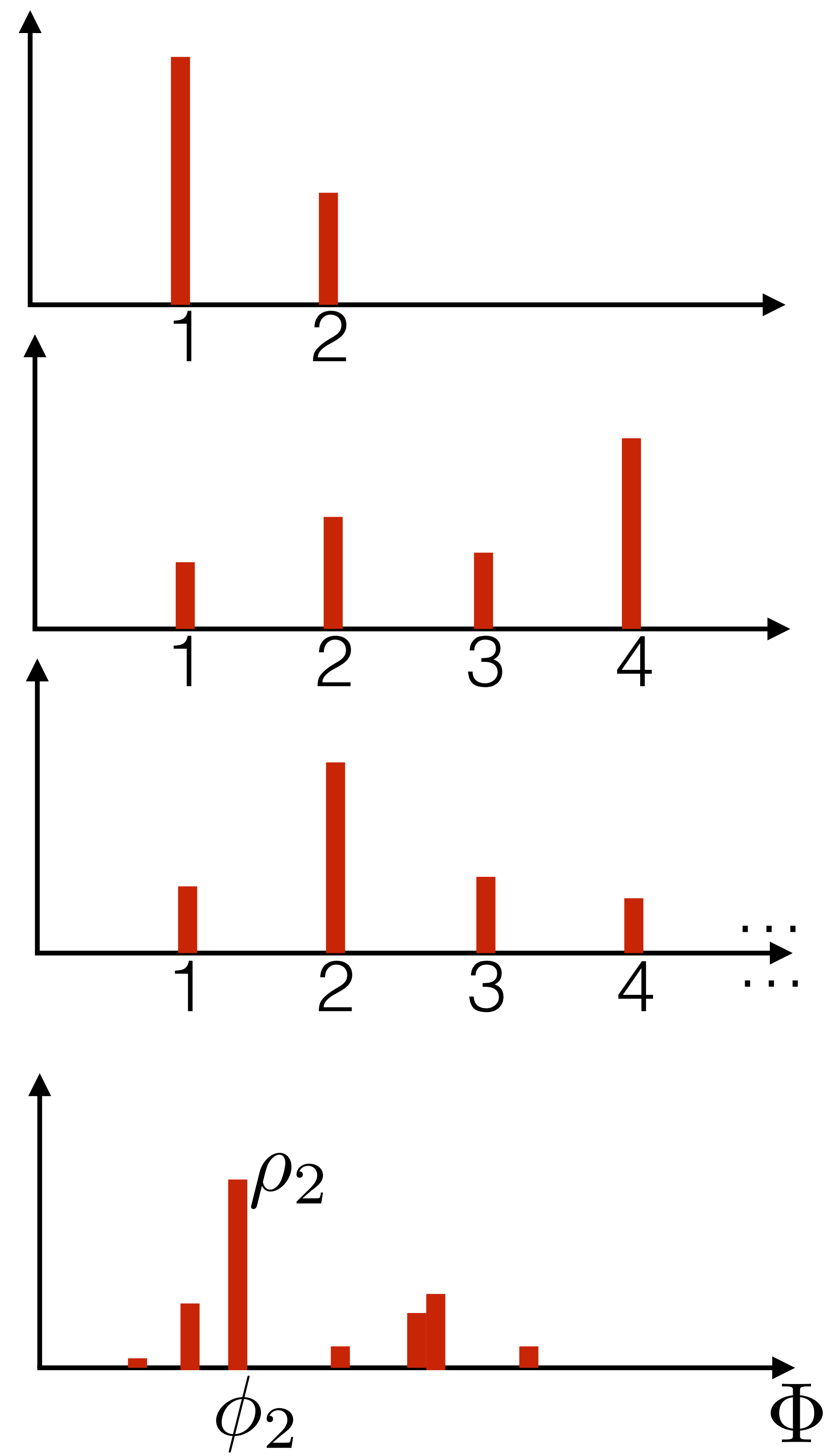

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- **Dirichlet process →** random distribution over Φ:

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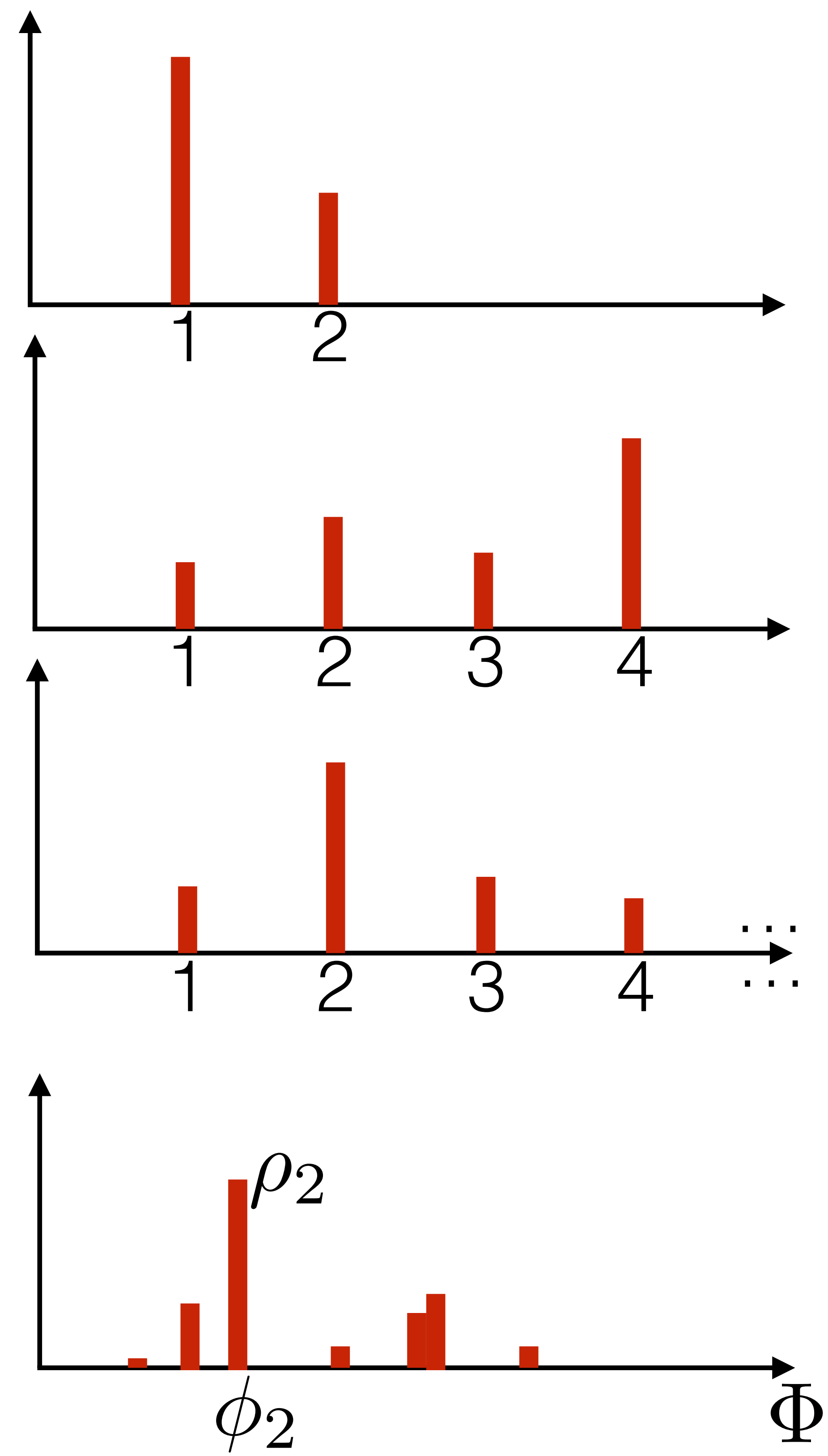

[Ferguson 1973]

Dirichlet process mixture model

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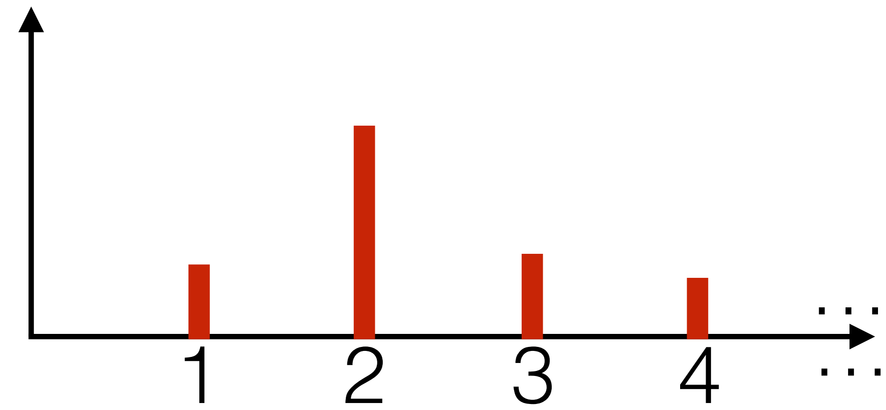

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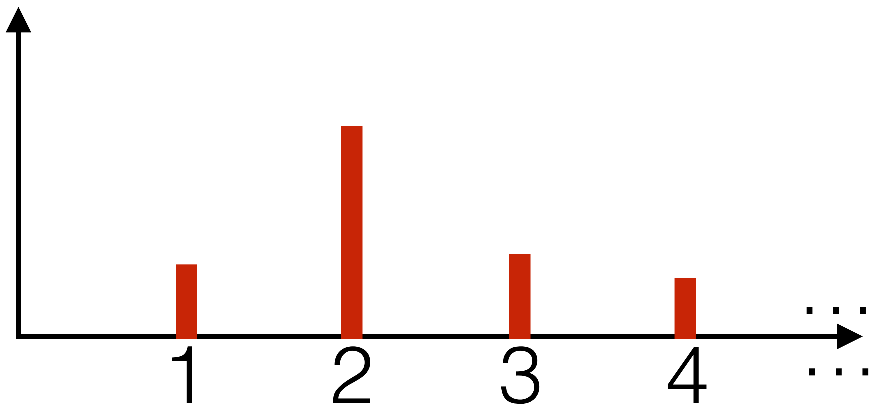

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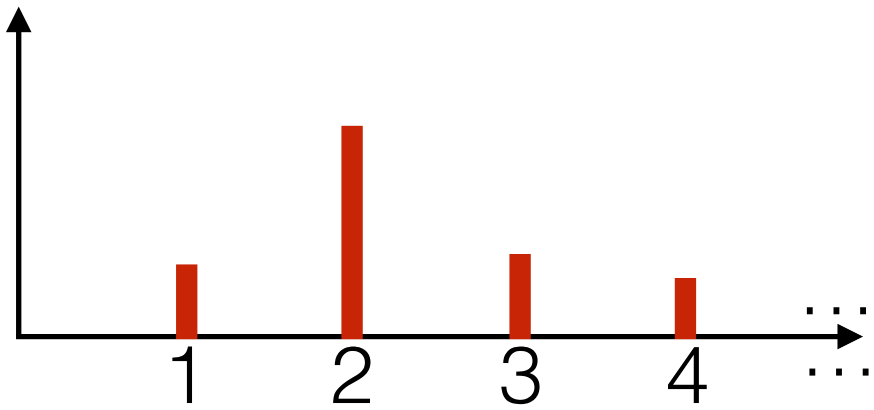

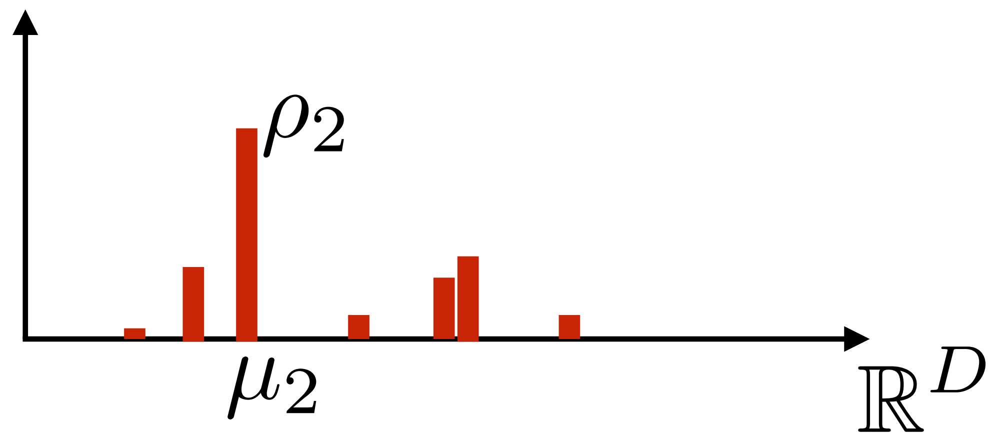

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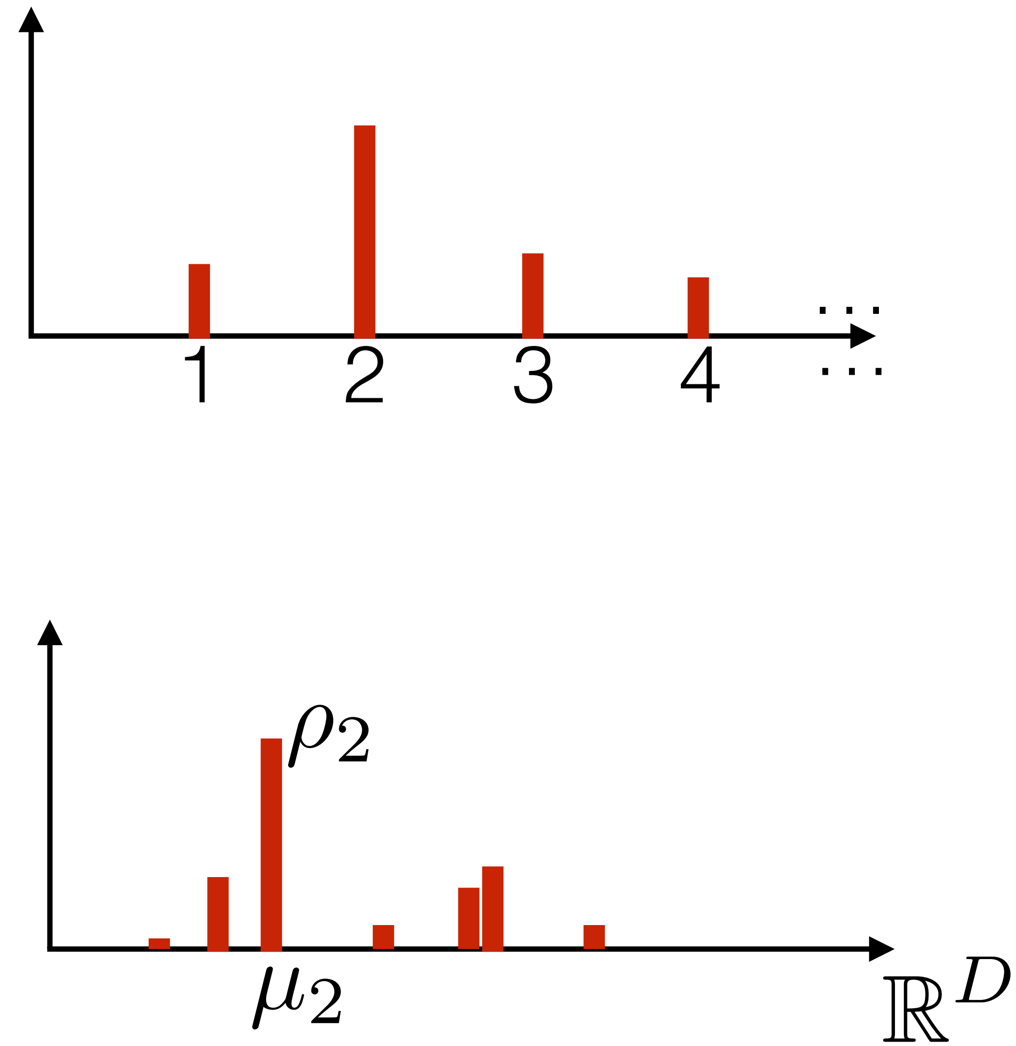

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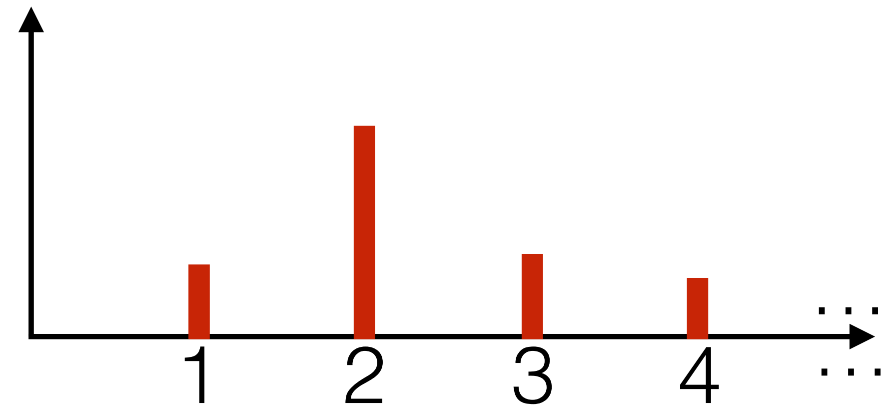

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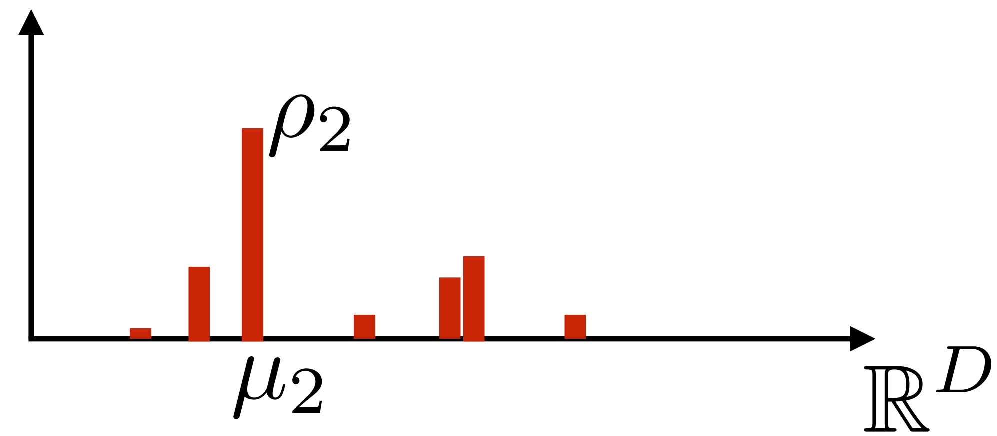

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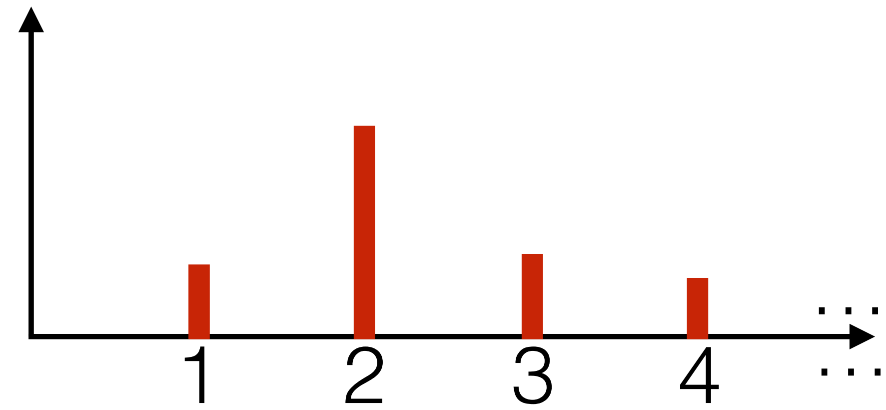

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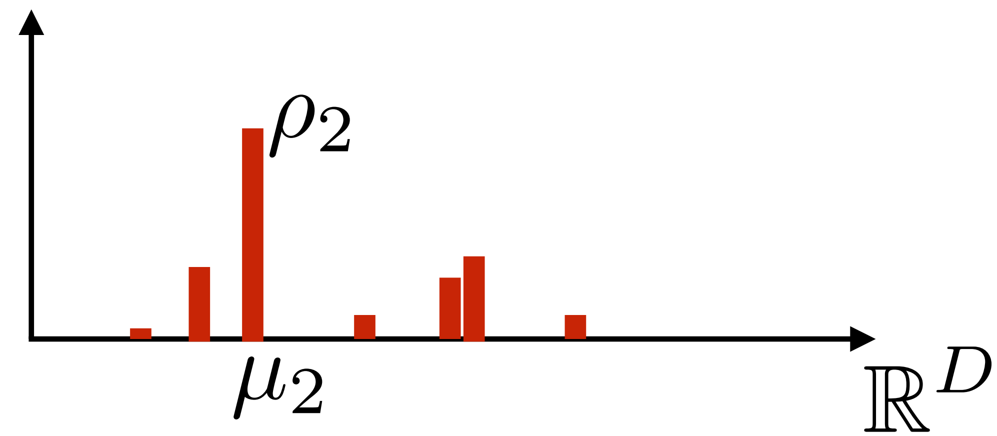

$$z_n \overset{iid}{\sim} \mathrm{Categorical}(\rho)$$

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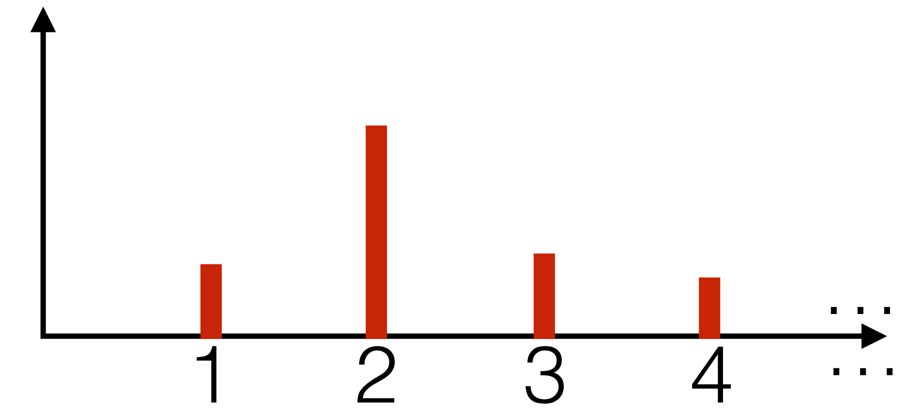

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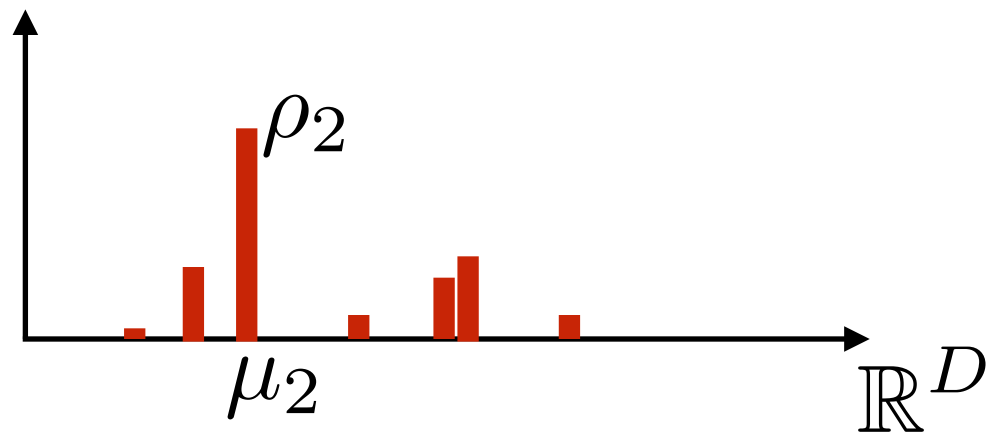

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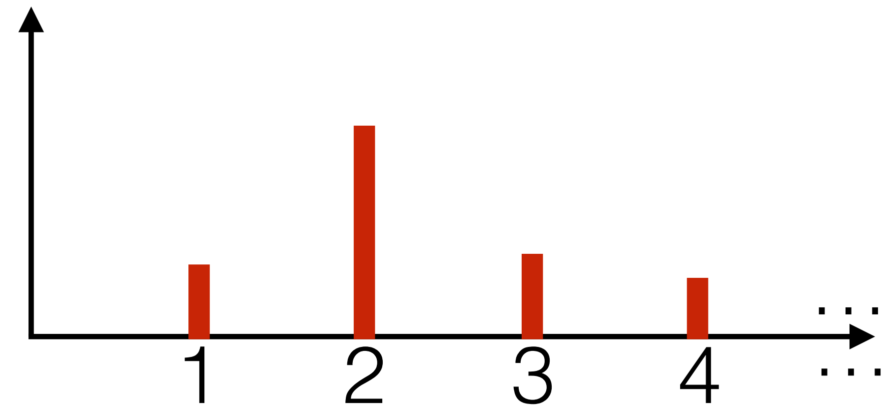

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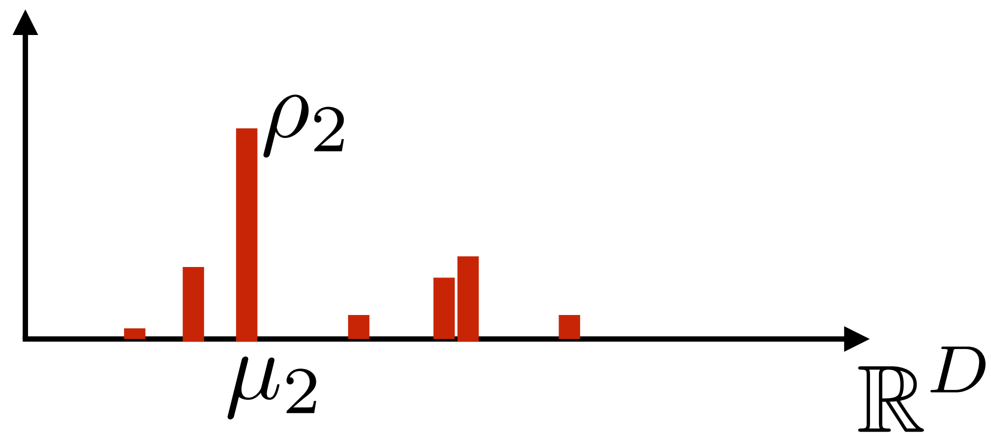

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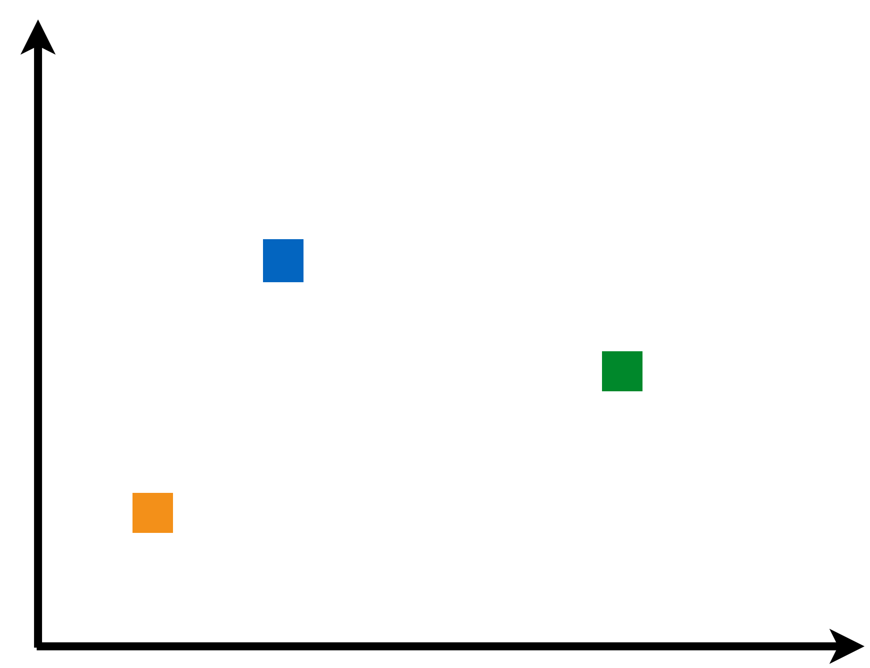

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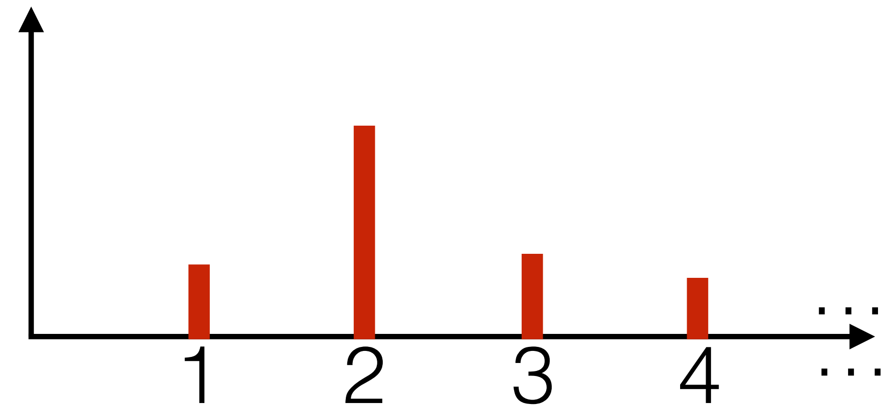

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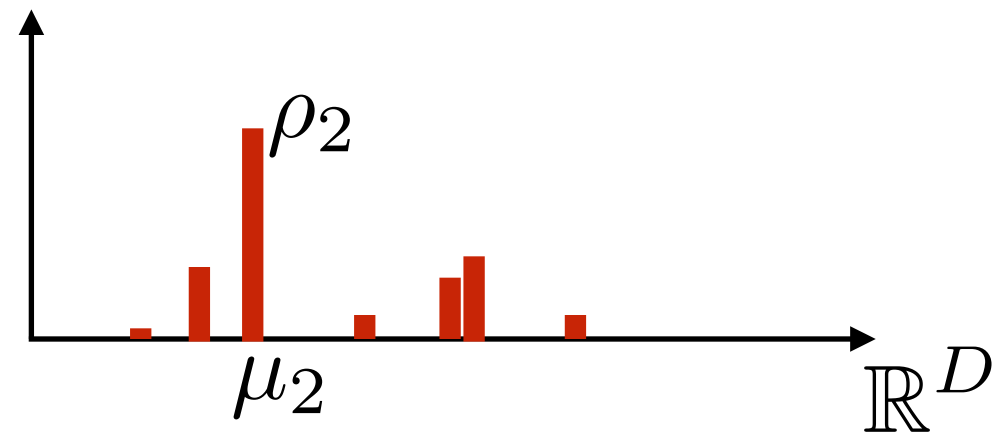

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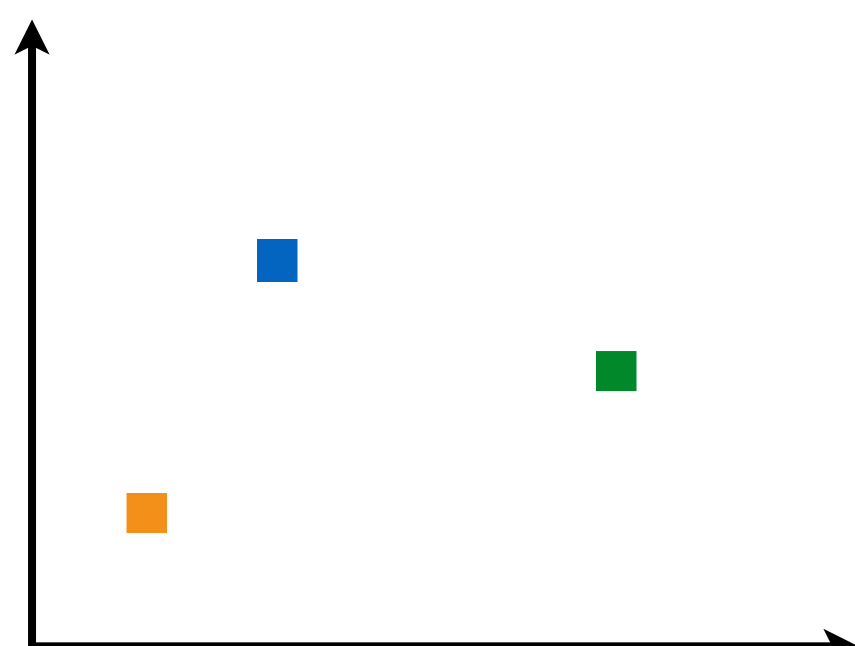

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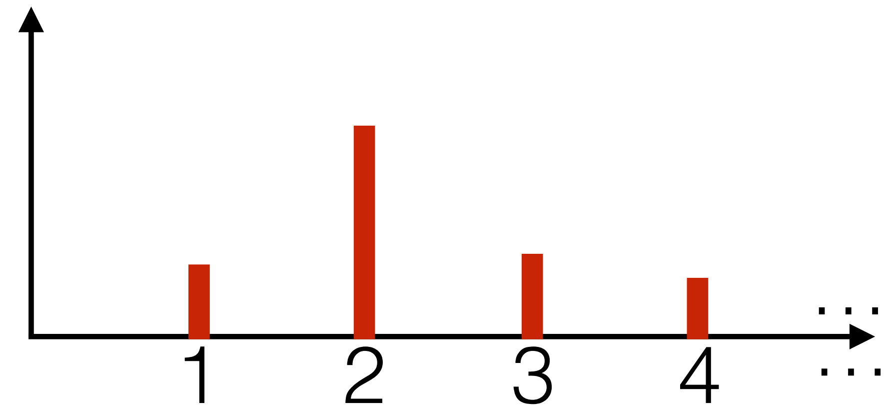

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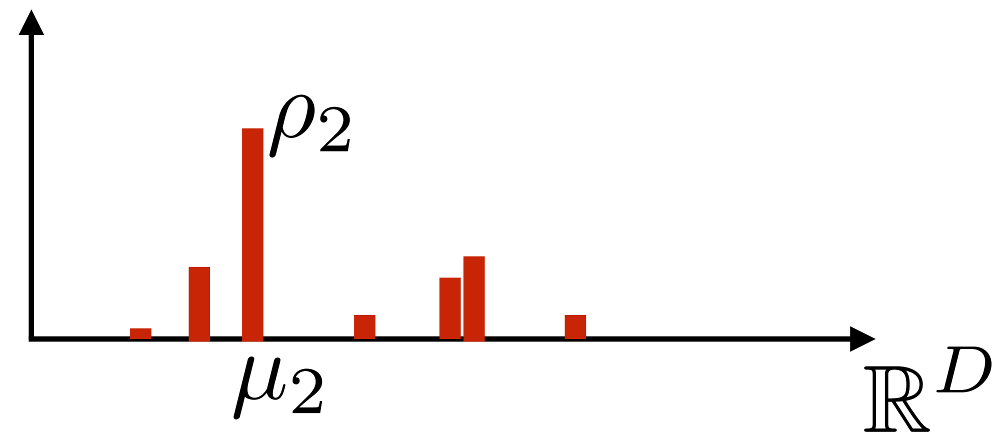

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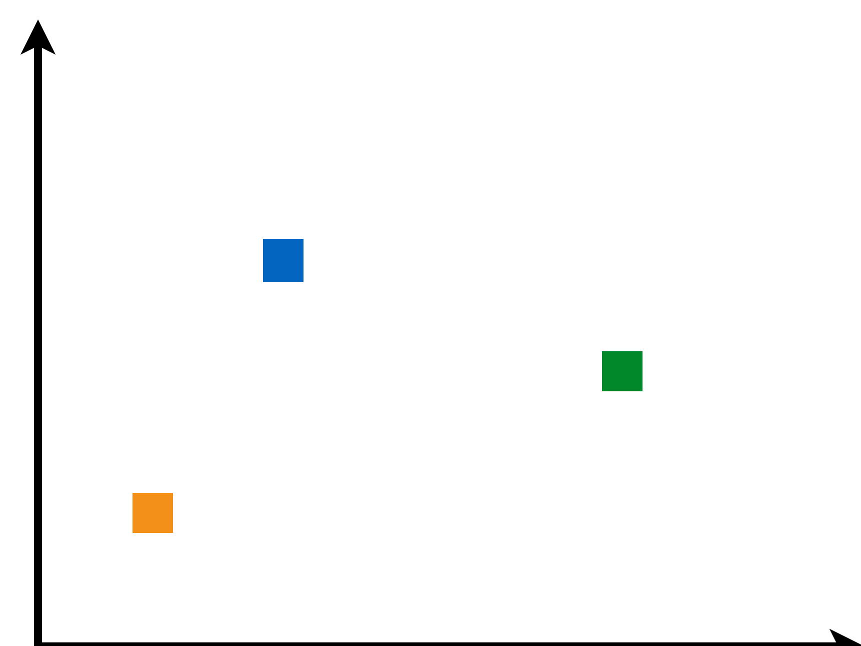

$$x_n \overset{indep}{\sim} \mathcal{N}(\mu_n^*, \Sigma)$$

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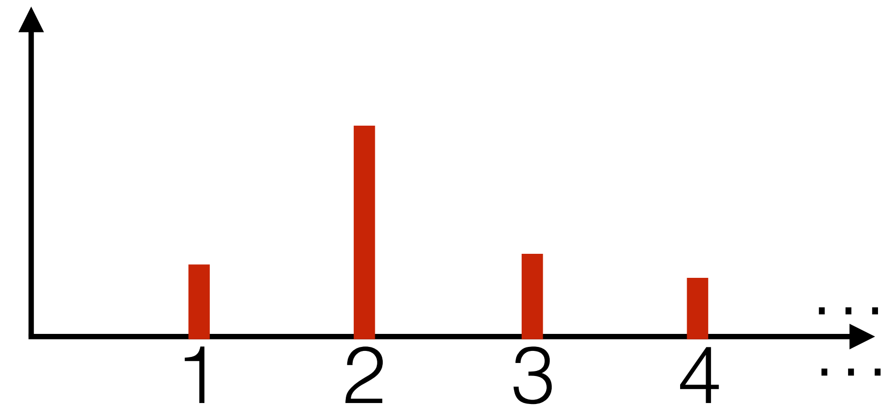

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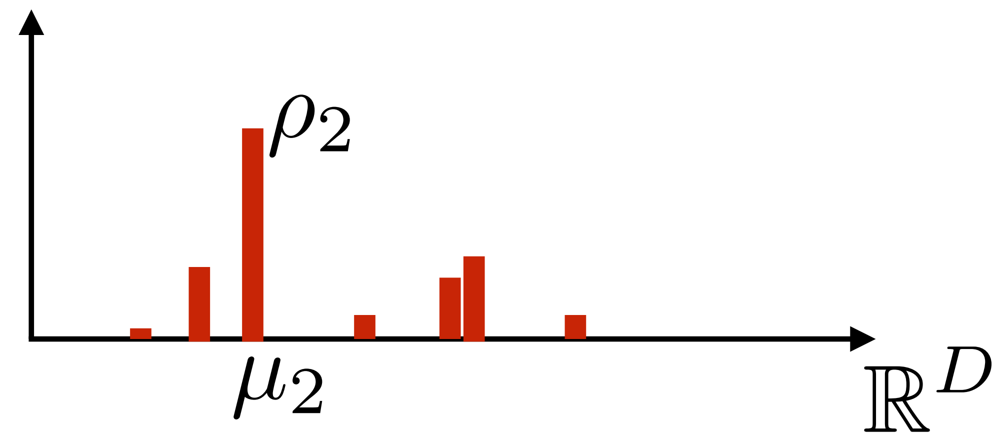

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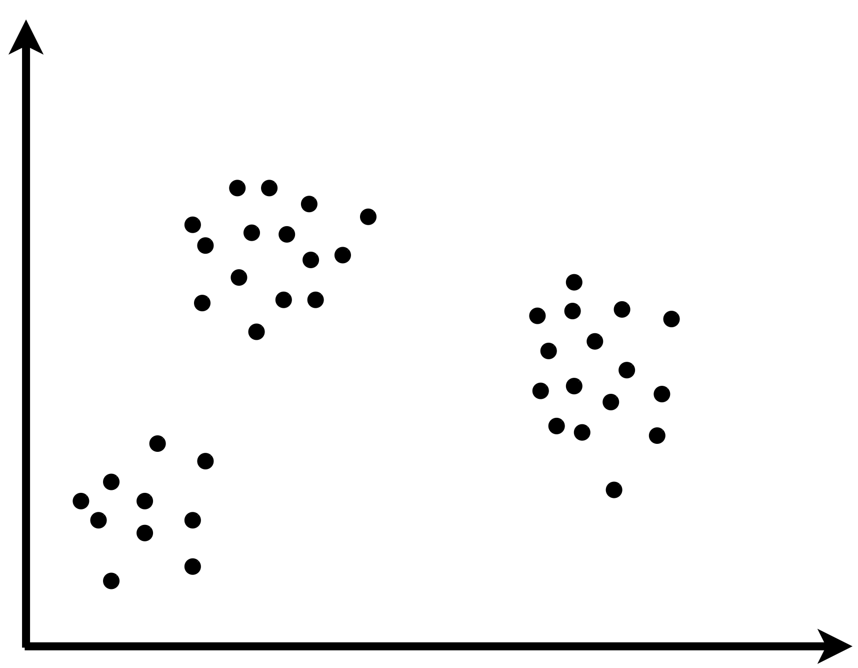

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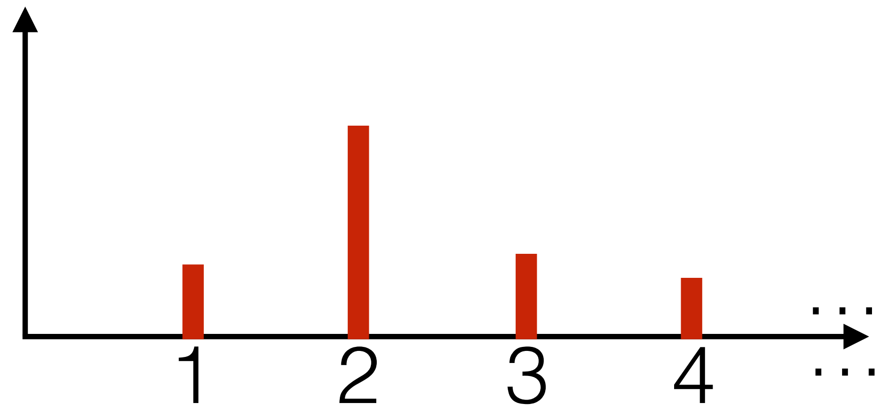

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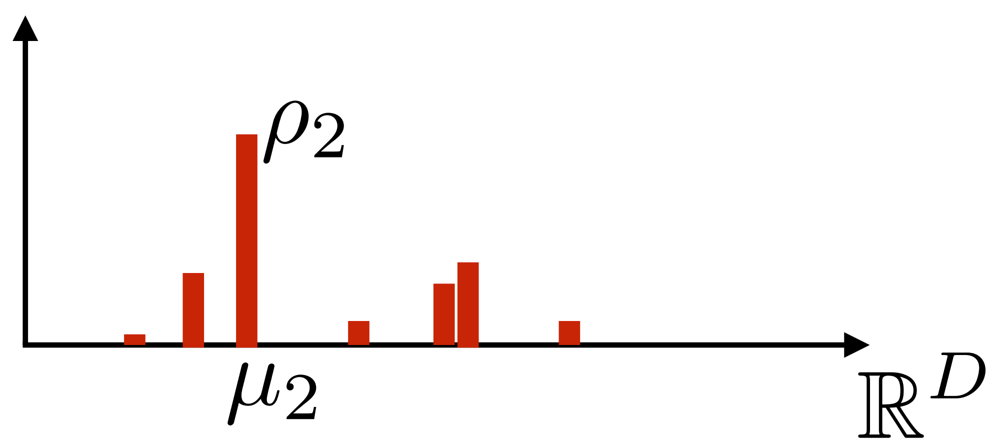

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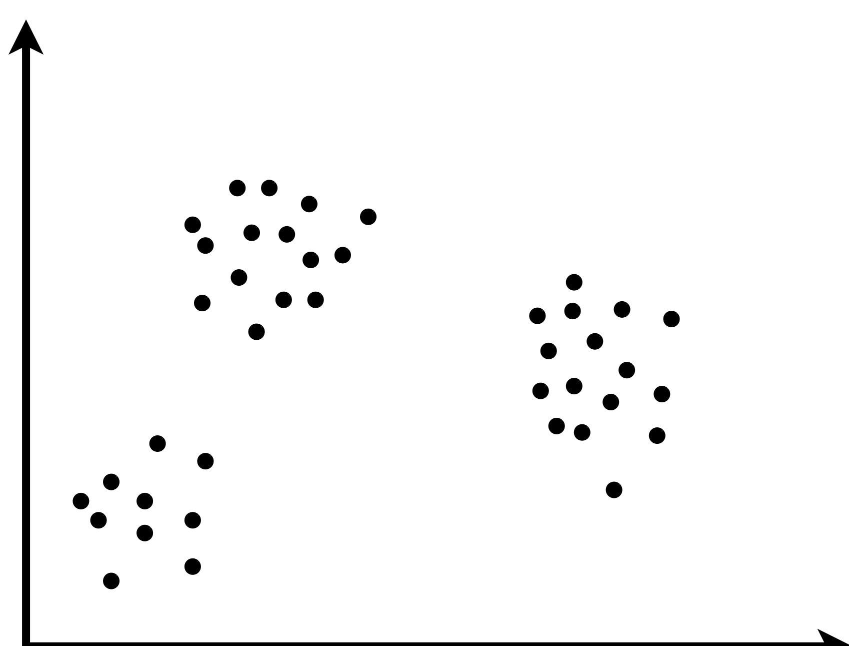

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[demo]

Dirichlet process mixture model

- More generally

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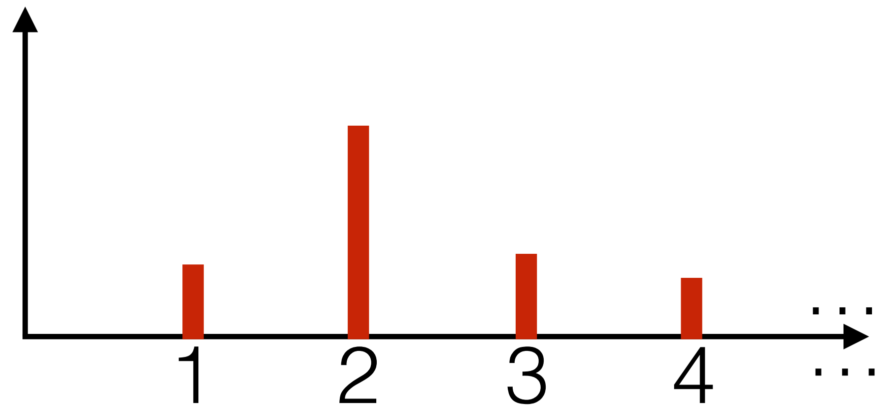

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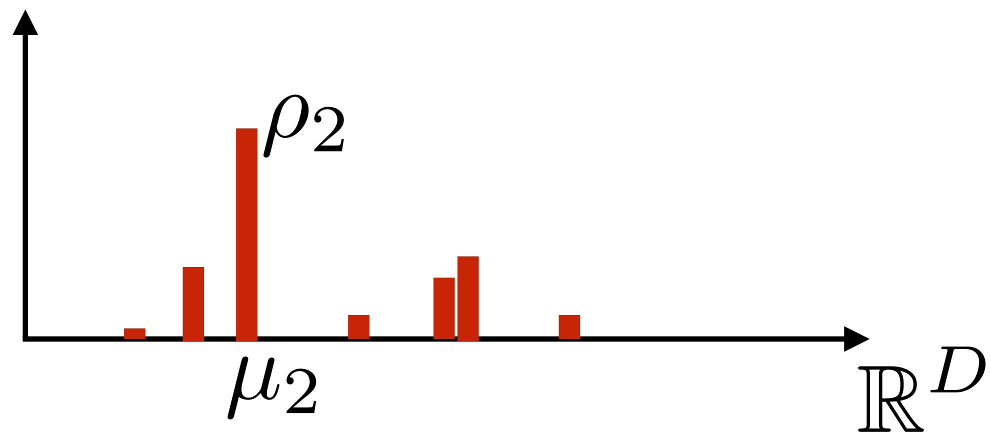

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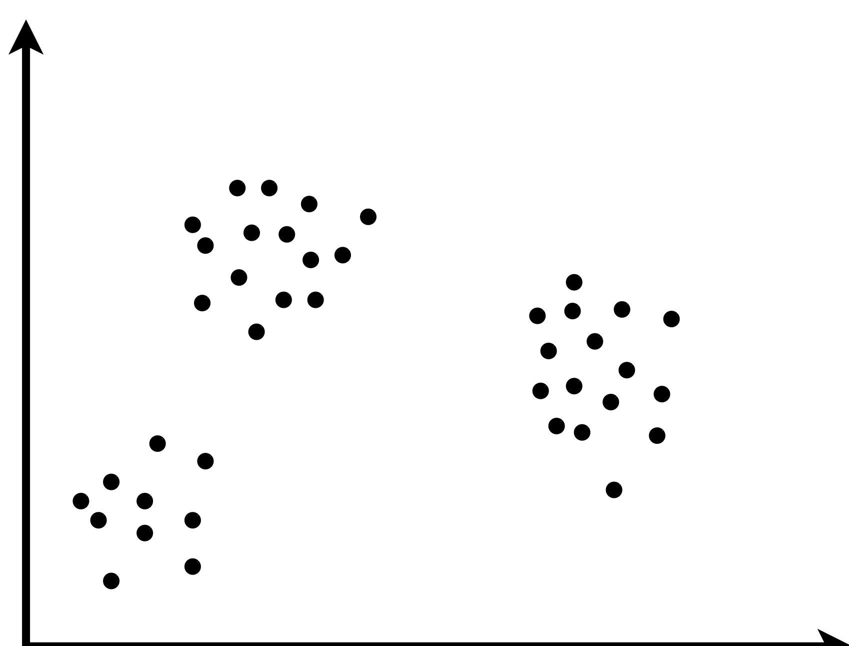

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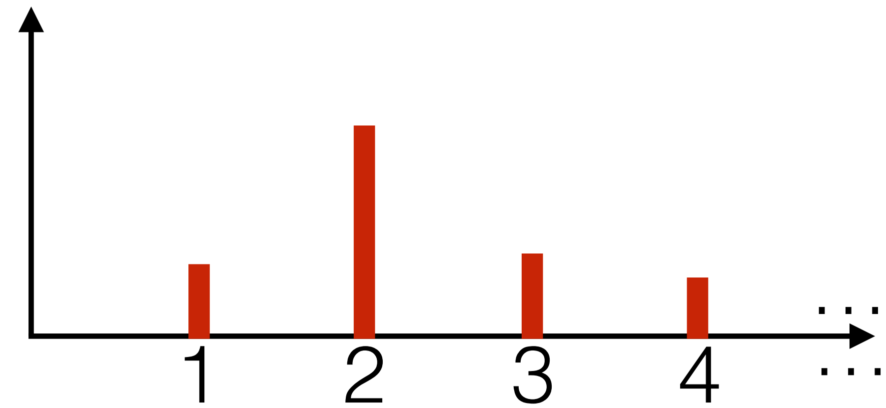

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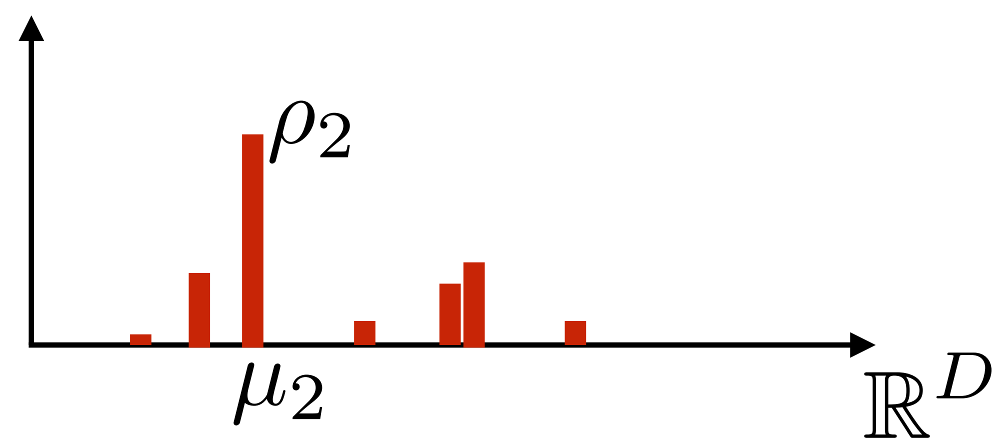

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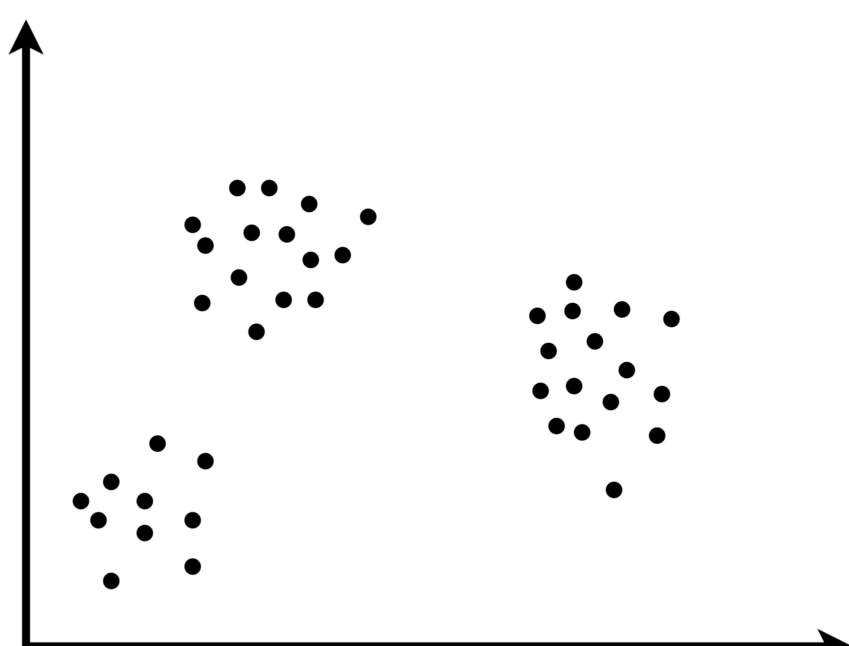

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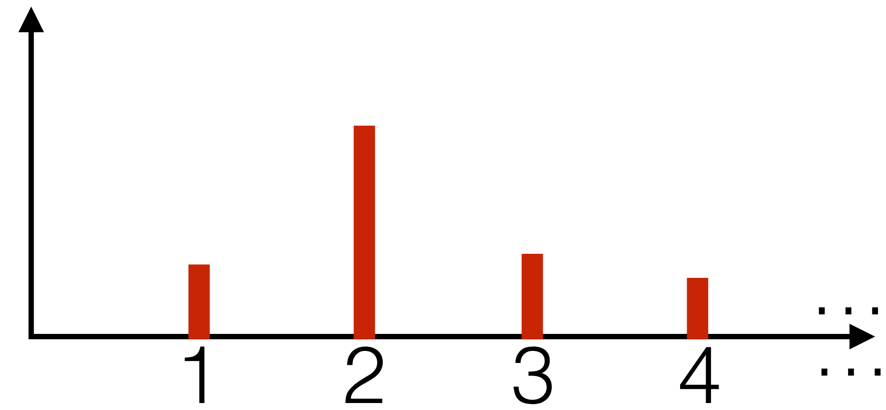

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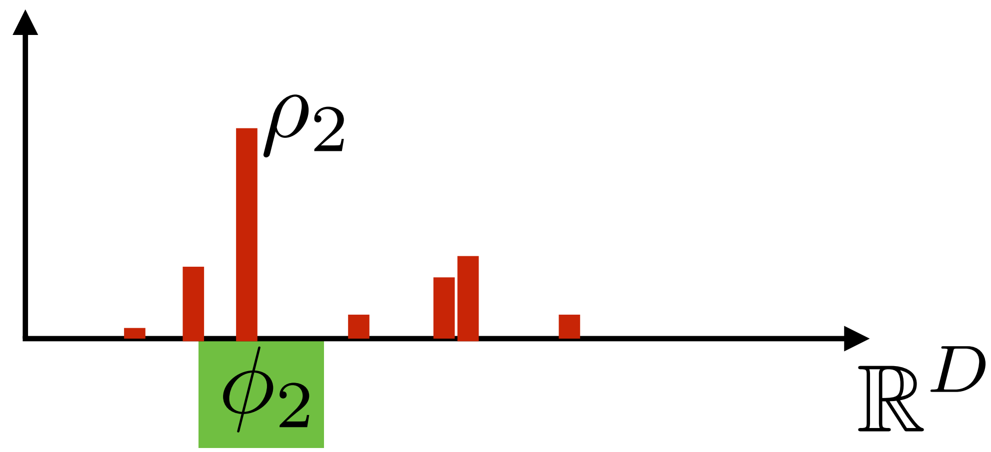

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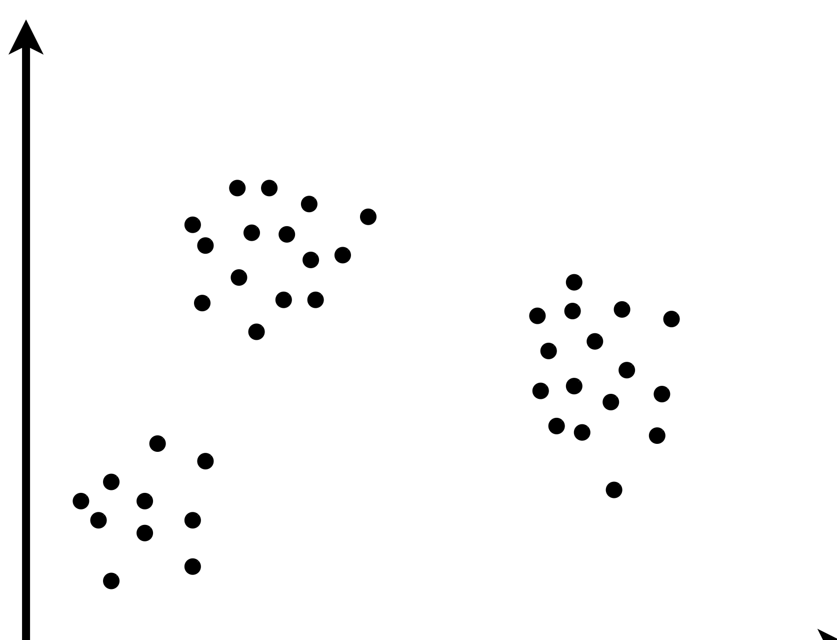

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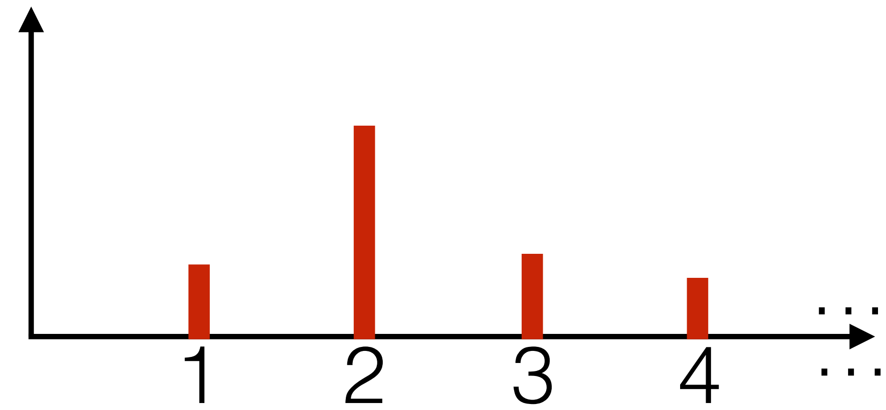
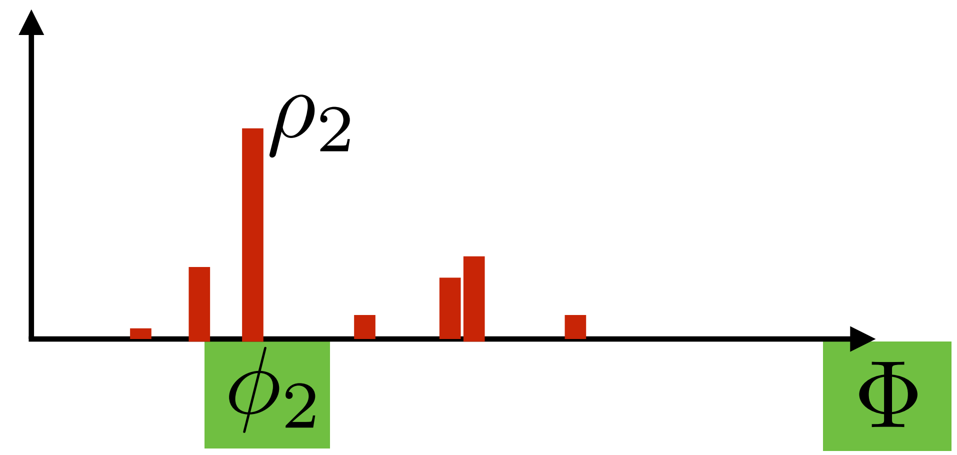


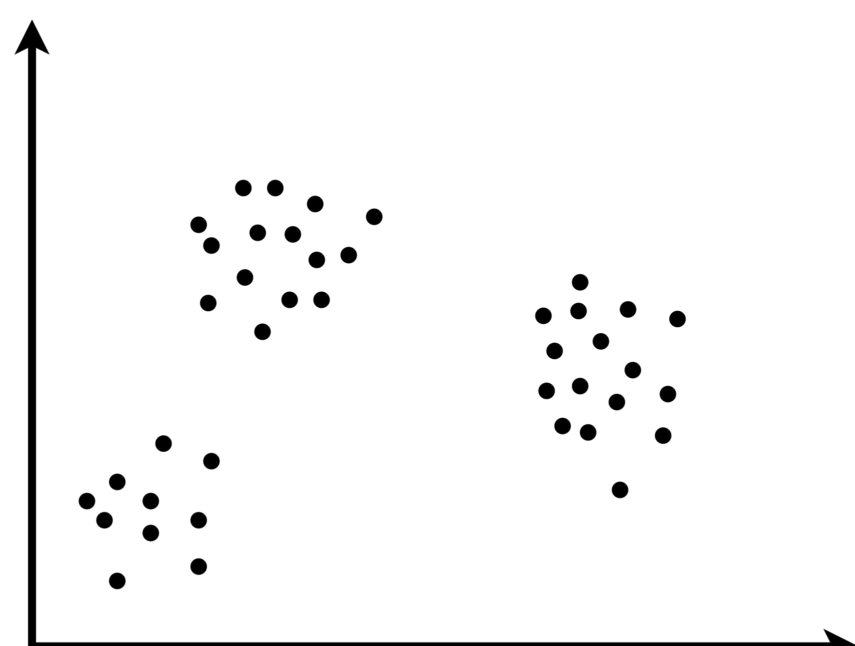
- i.e. $G = \sum_{k=1}^{\infty} \rho_k \delta_{\mu_k} \overset{d}{=} \text{DP}(\alpha, \mathcal{N}(\mu_0, \Sigma_0))$



$$z_n \overset{iid}{\sim} \text{Categorical}(\rho)$$

$$\mu_n^* = \mu_{z_n}$$

- i.e. $\mu_n^* \overset{iid}{\sim} G$



$$x_n \overset{indep}{\sim} \mathcal{N}(\mu_n^*, \Sigma)$$

Dirichlet process mixture model

- More generally

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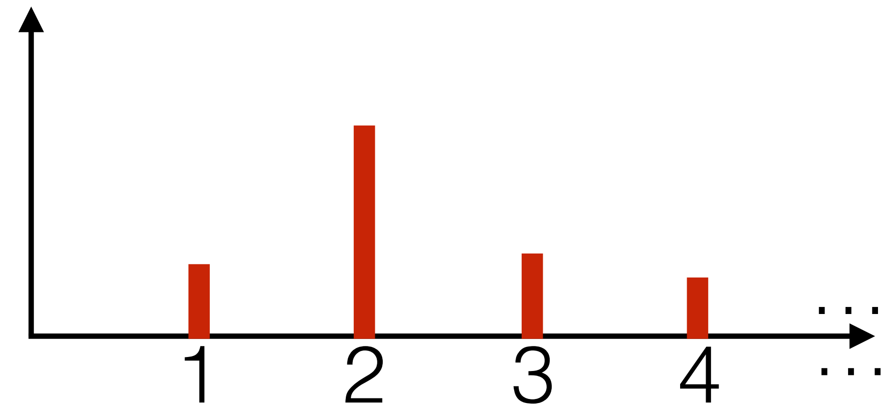

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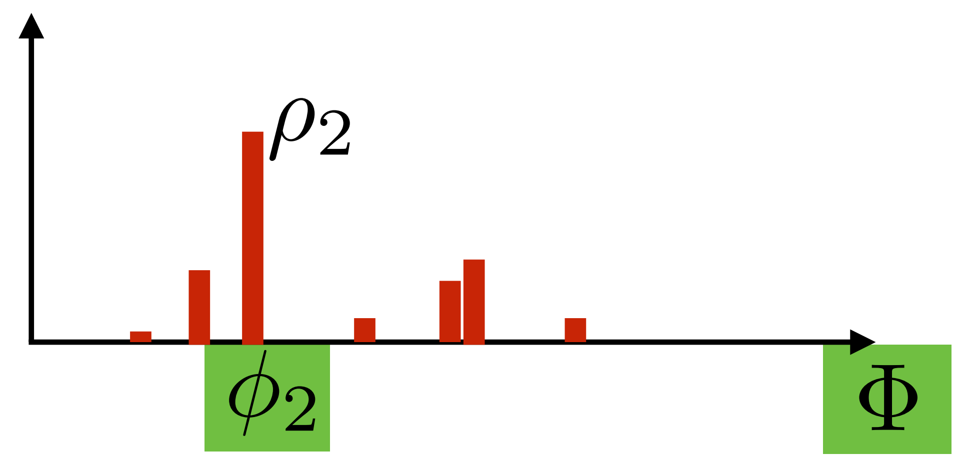

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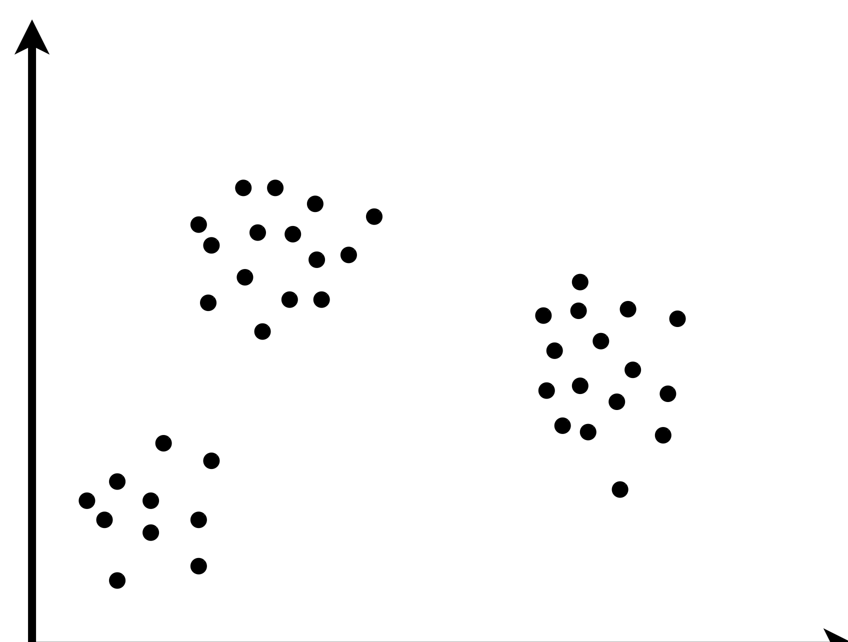

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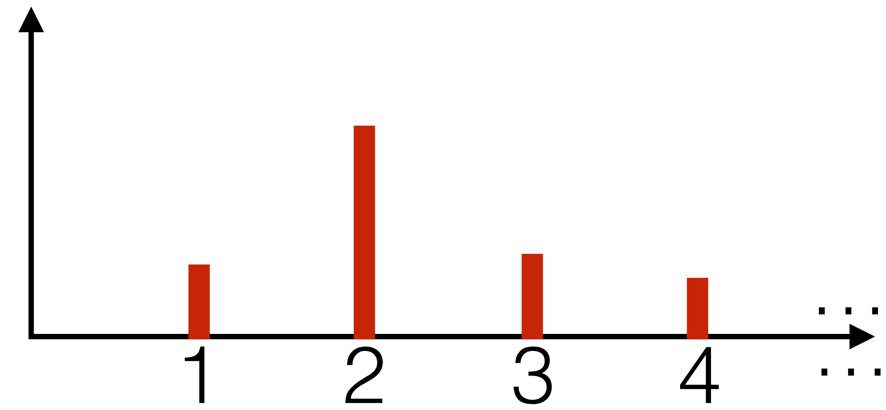

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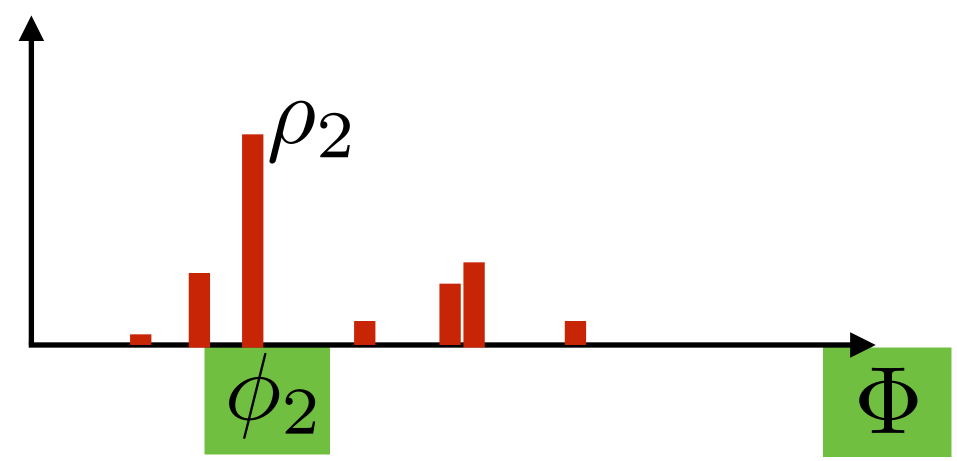

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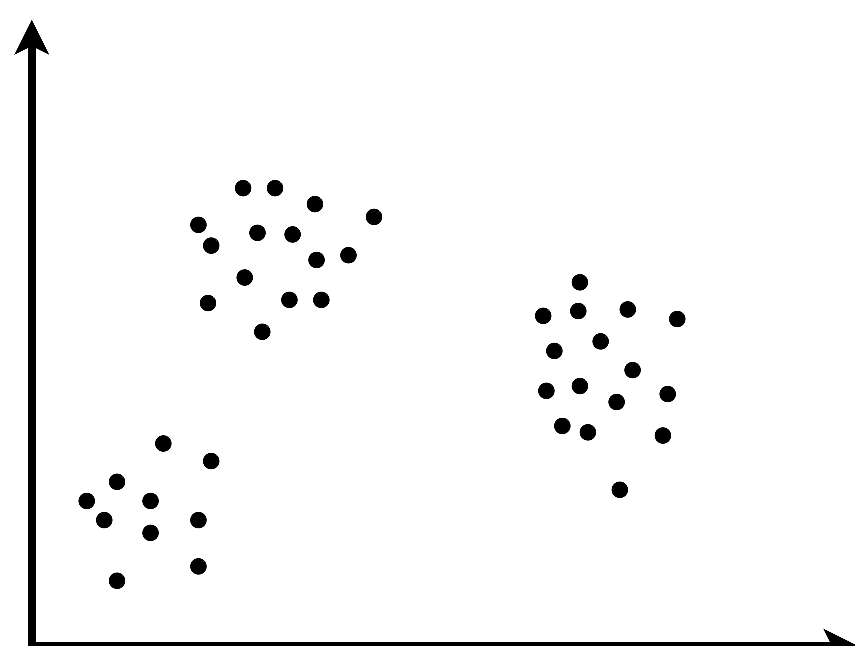

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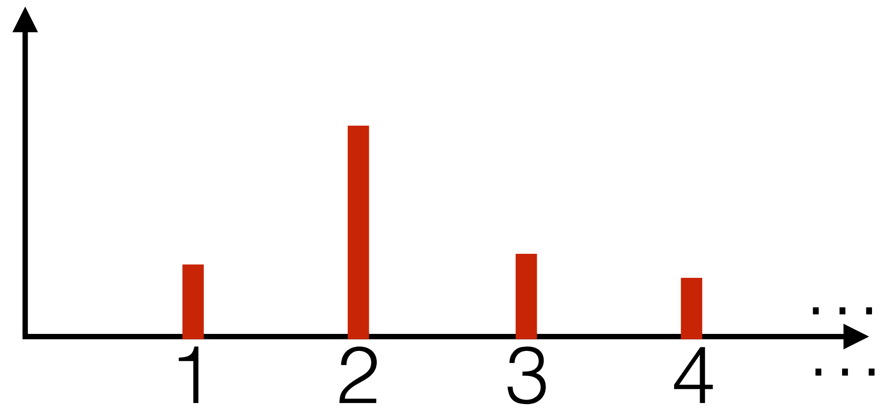

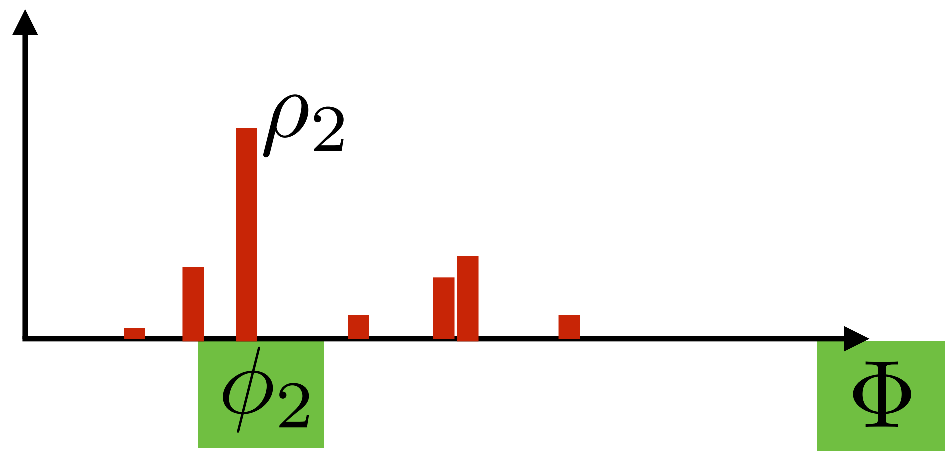

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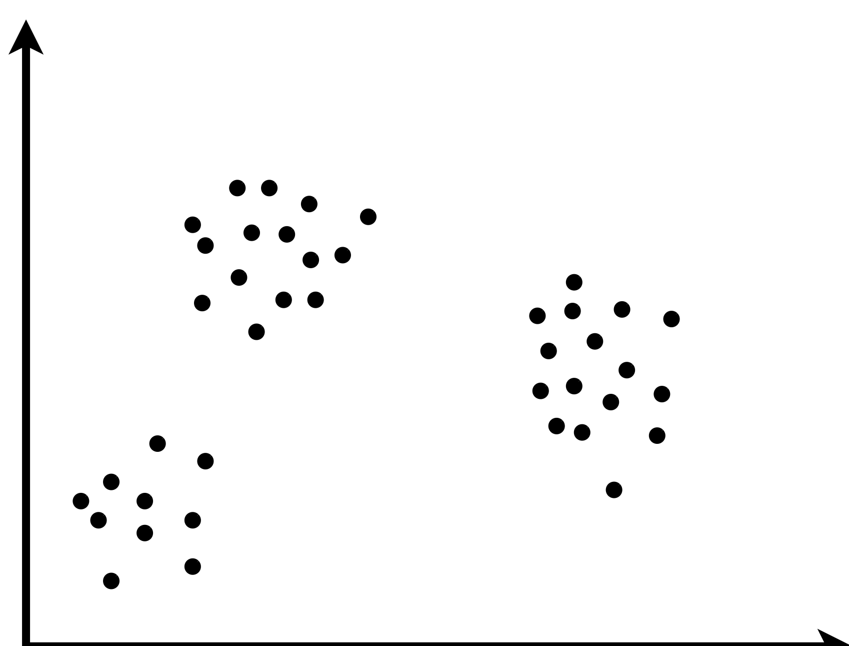

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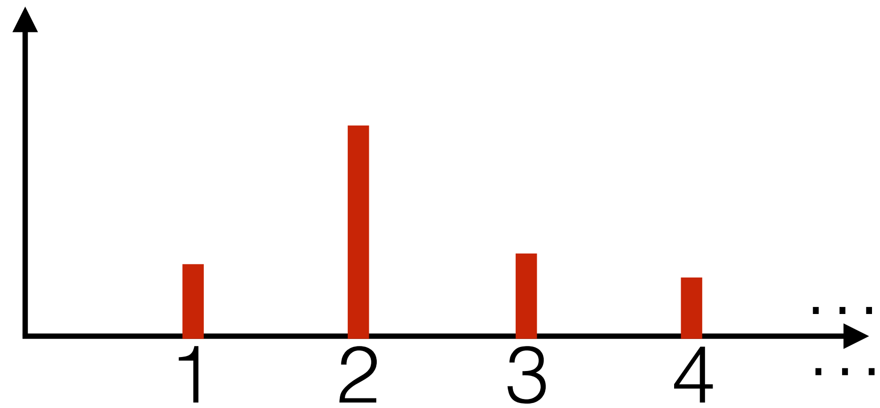

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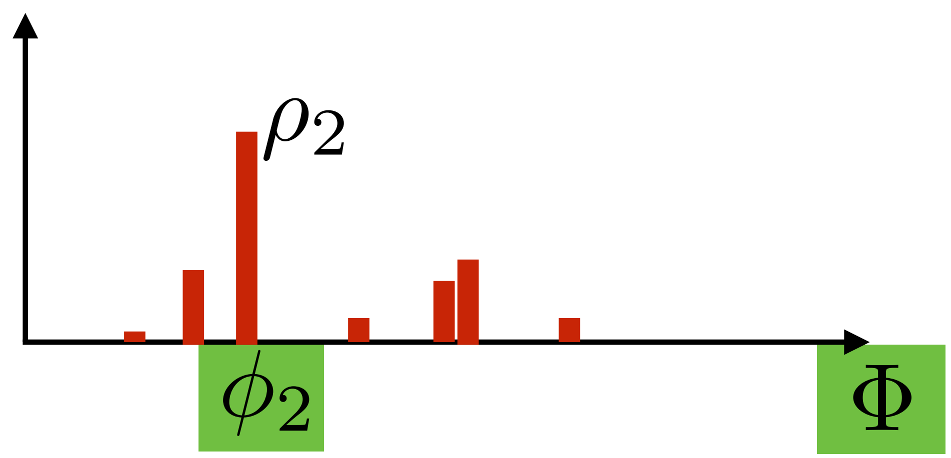

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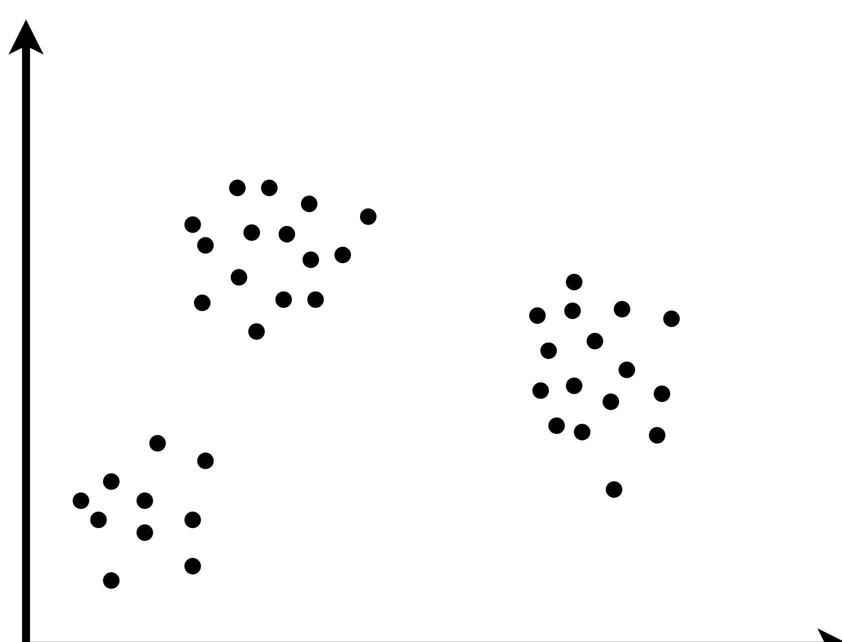
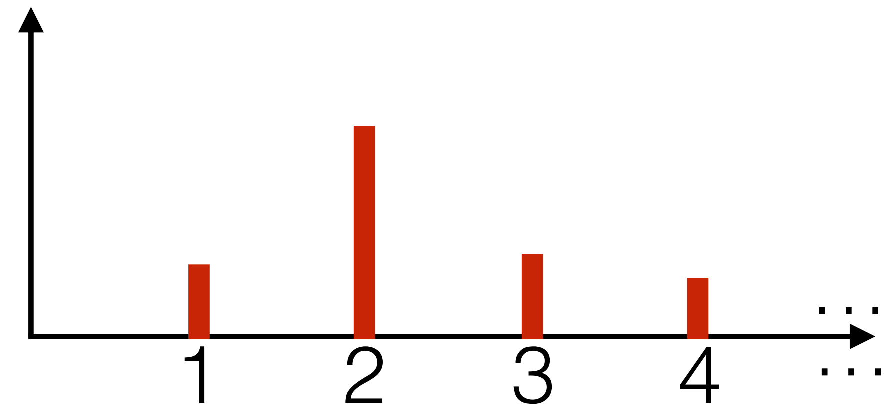


Dirichlet process mixture model

- More generally



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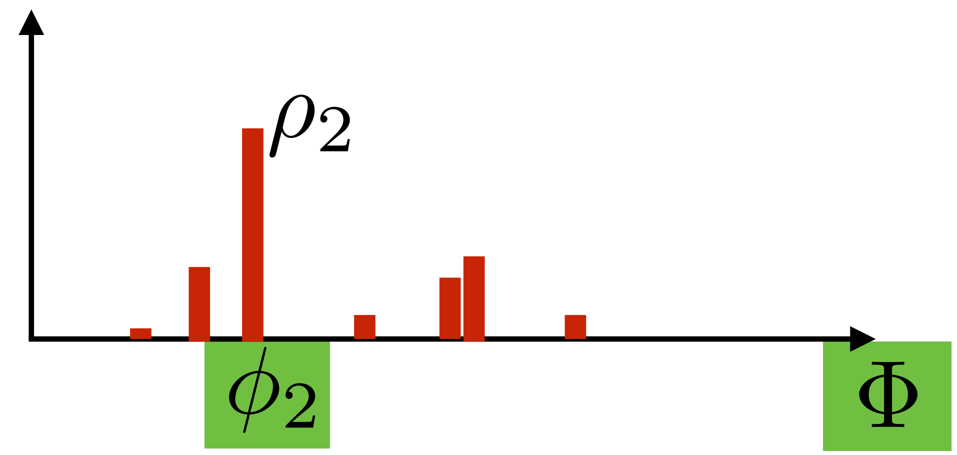

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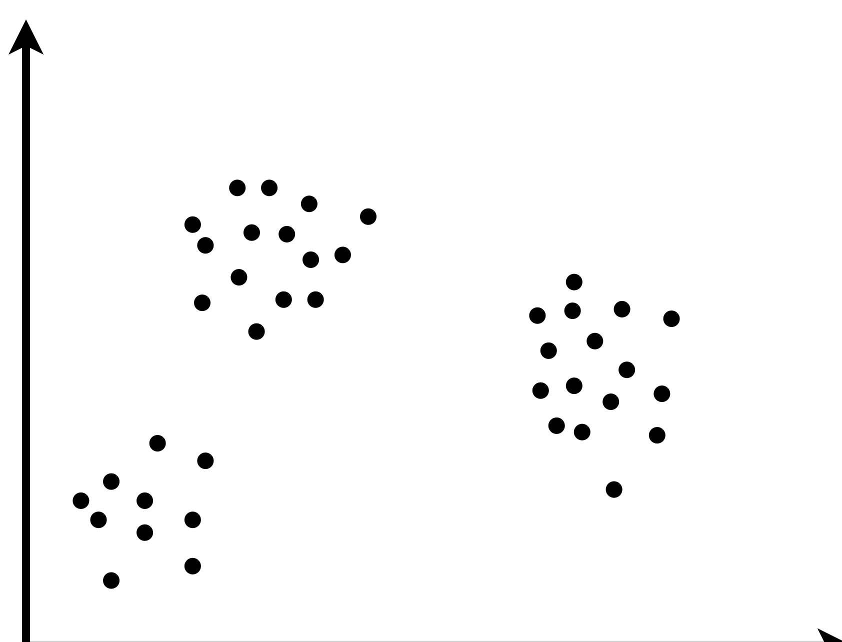

$$x_n \overset{indep}{\sim} F(\theta_n)$$

Dirichlet process mixture model

- More generally

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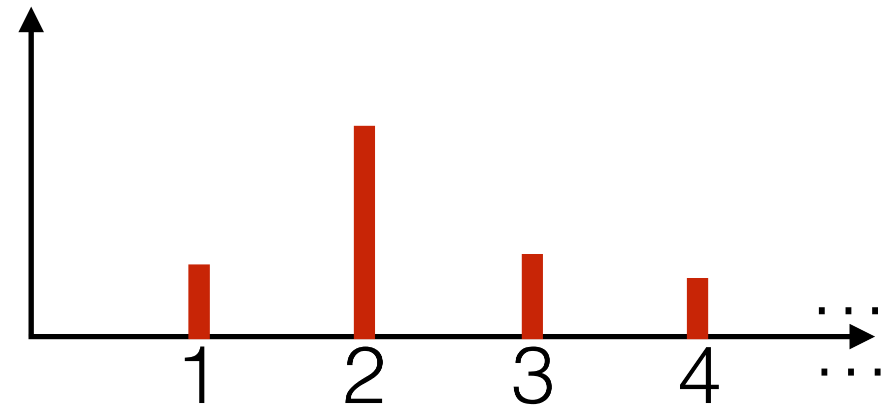

$$\phi_k \stackrel{iid}{\sim} G_0 \qquad k = 1, 2, \ldots$$

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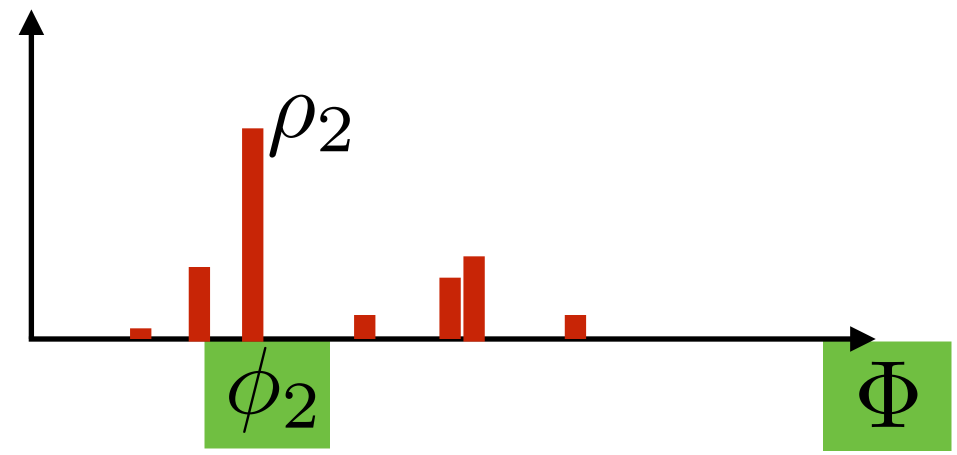

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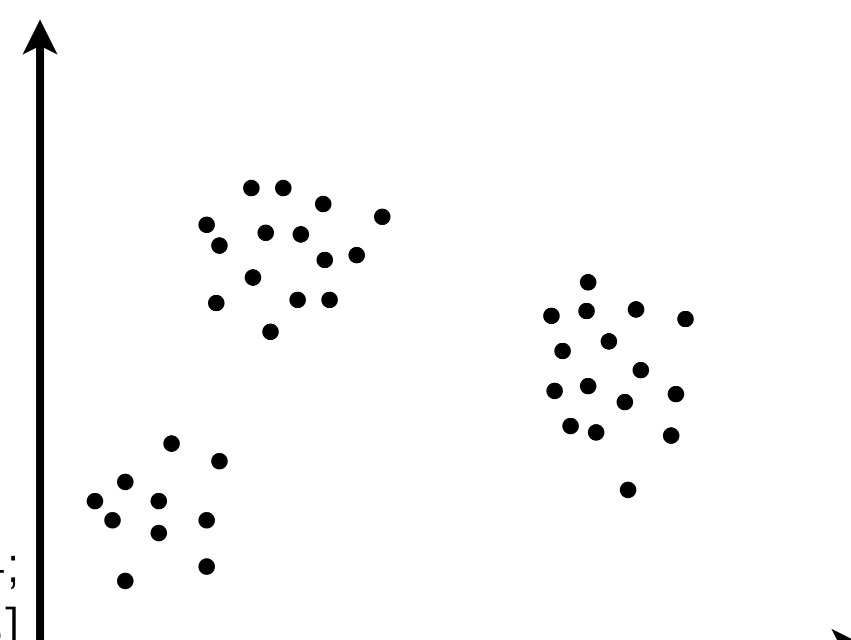
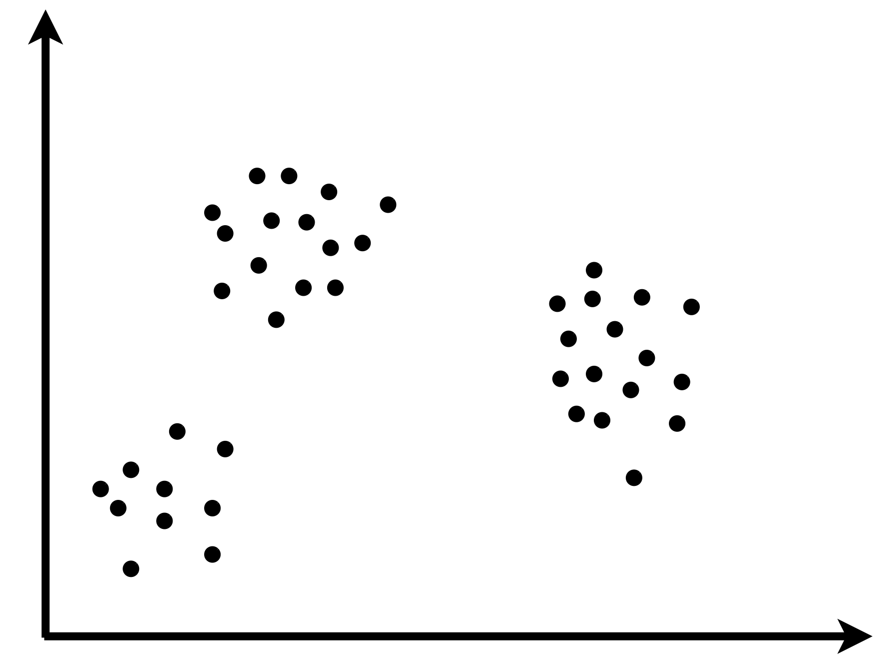


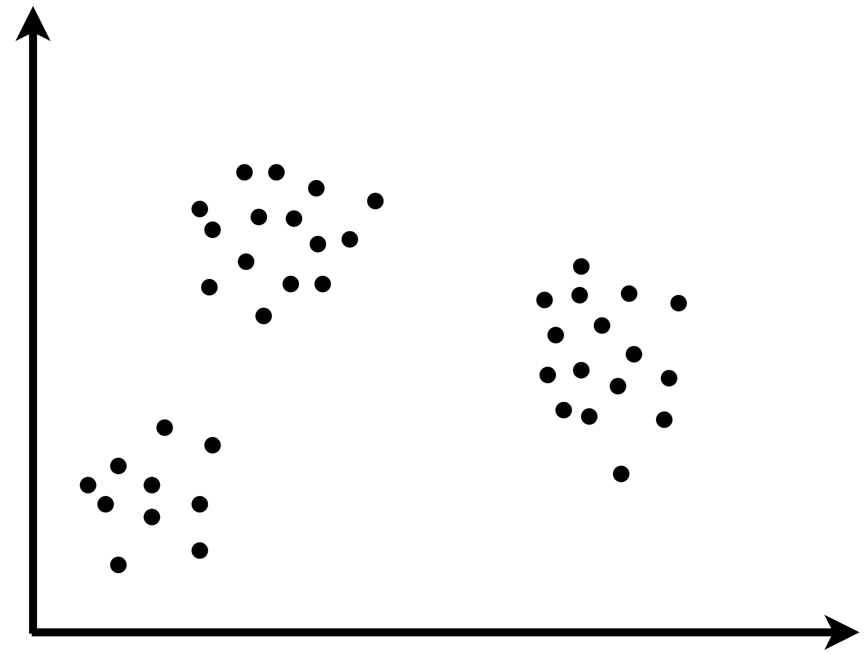
[Antoniak 1974; Ferguson 1983; West, Müller, Escobar 1994; Escobar, West 1995; MacEachern, Müller 1998]

DP or not DP, that is the question



DP or not DP, that is the question

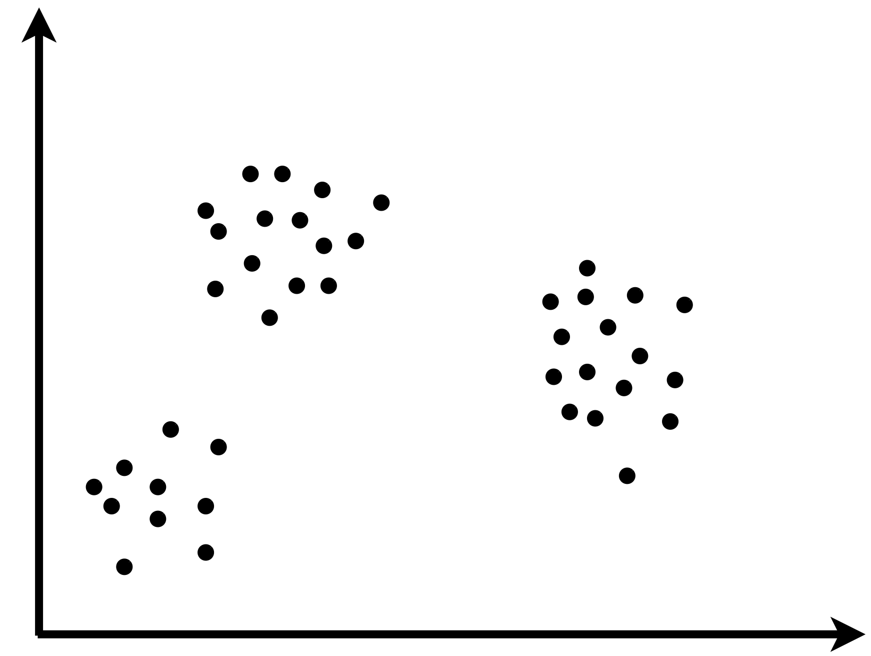
- GEM: 



DP or not DP, that is the question

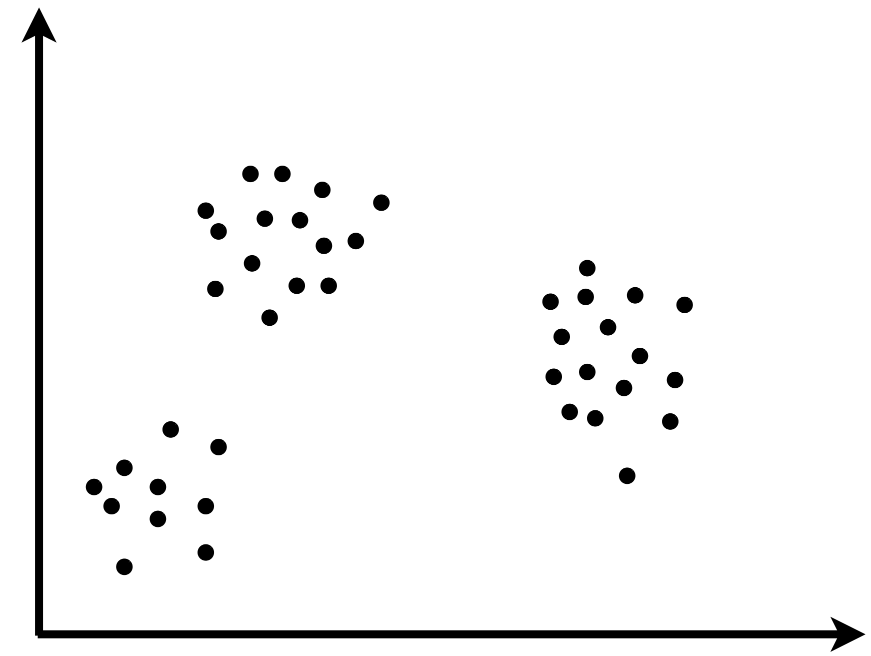
- GEM: 

- Compare to:



DP or not DP, that is the question

- GEM: 
- Compare to:
  - Finite (small *K*) mixture model



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- GEM: 
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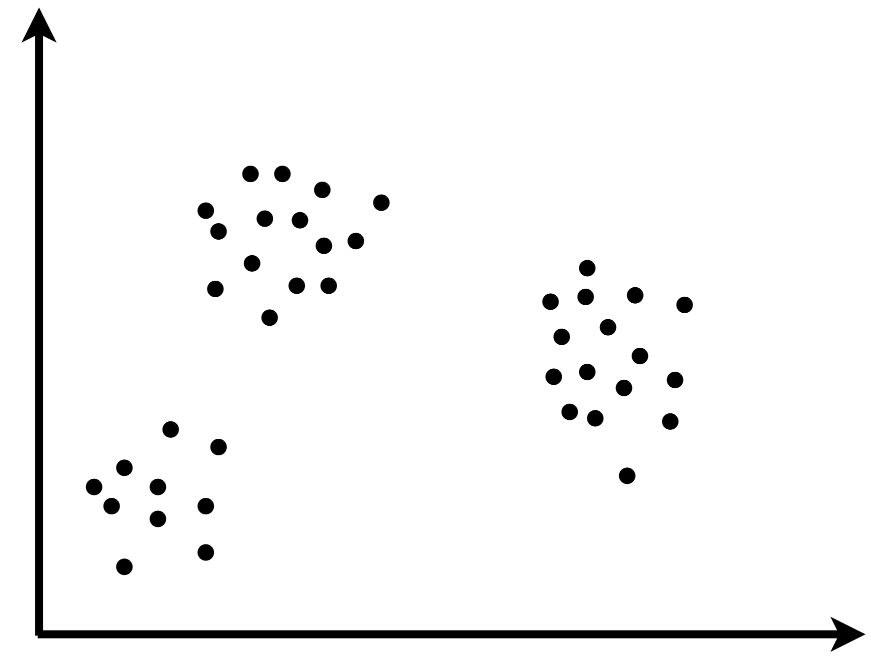

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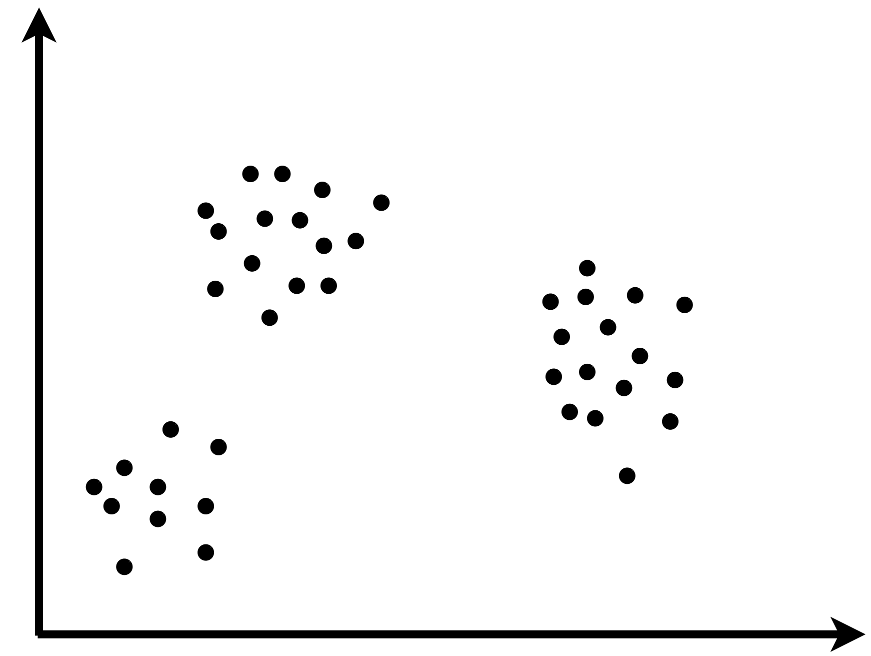
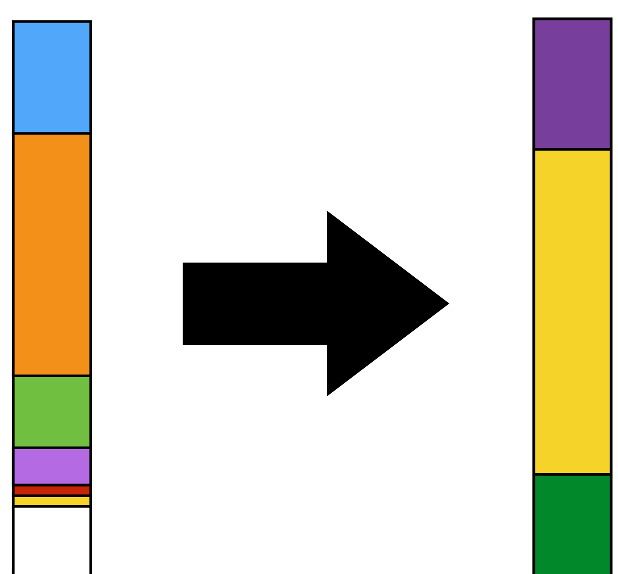


- Compare to:
  - Finite (small *K*) mixture model

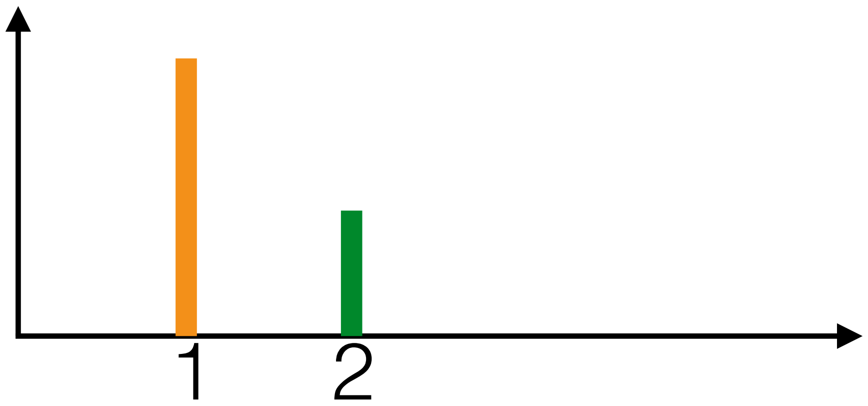
  - Finite (large *K*) mixture model

  - Time series



Marginal cluster assignments

$$\rho_1 \sim \mathrm{Beta}(a_1, a_2), z_n \overset{iid}{\sim} \mathrm{Cat}(\rho_1, \rho_2)$$



Marginal cluster assignments

- Integrate out the frequencies

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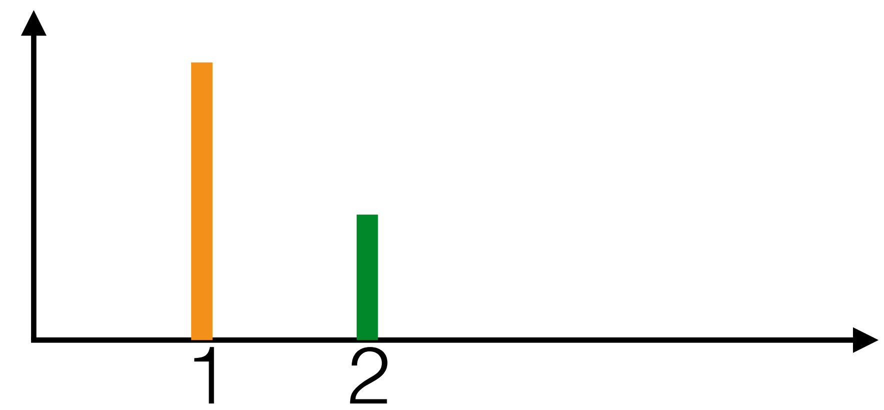
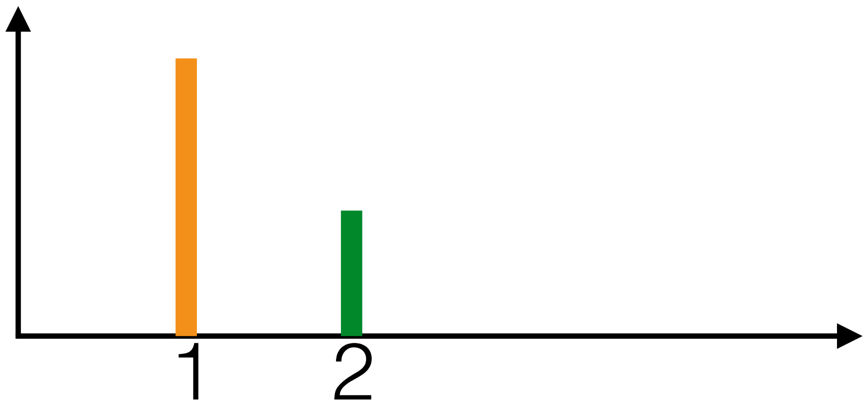


Marginal cluster assignments

- Integrate out the frequencies

$\rho_1 \sim \mathrm{Beta}(a_1, a_2), z_n \stackrel{iid}{\sim} \mathrm{Cat}(\rho_1, \rho_2)$

$p(z_n = 1 | z_1, \ldots, z_{n-1})$



Marginal cluster assignments

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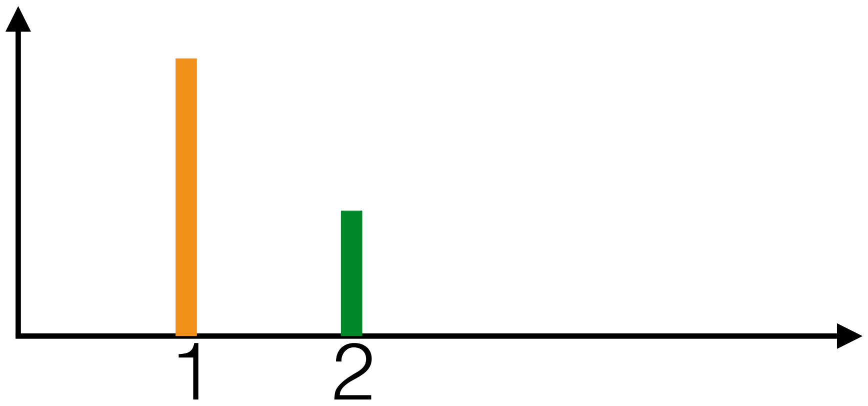
$$p(z_n = 1 | z_1, \ldots, z_{n-1})$$
$$= \int p(z_n = 1, \rho_1 | z_1, \ldots, z_{n-1}) d\rho_1$$



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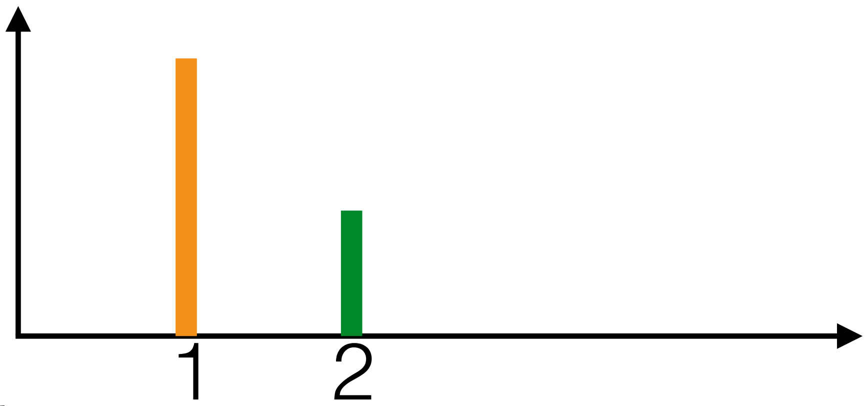

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Marginal cluster assignments

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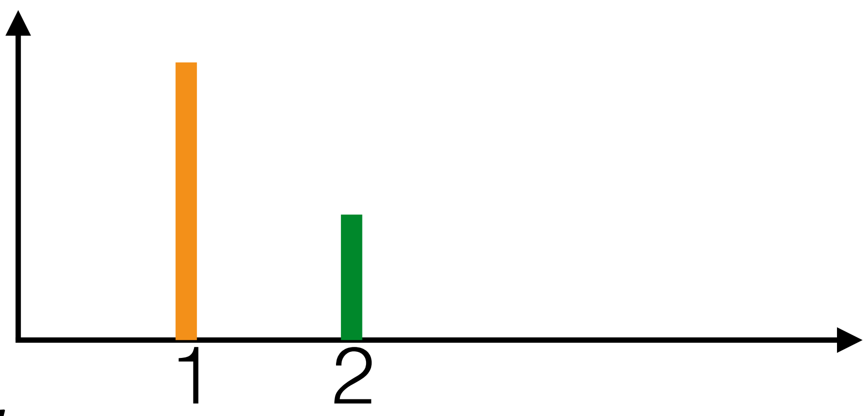

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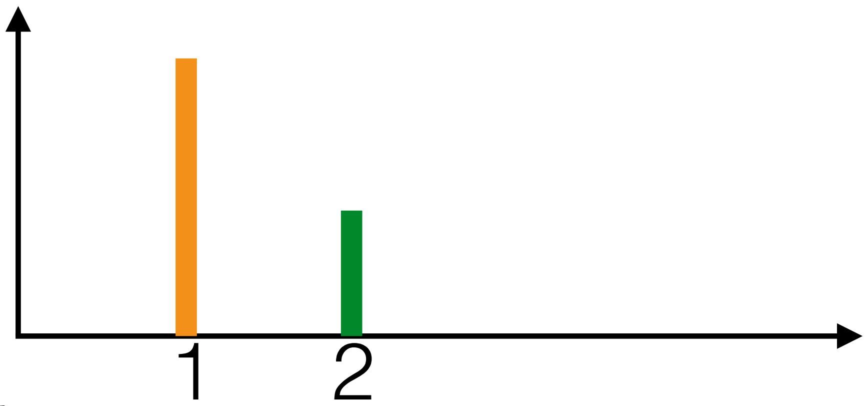

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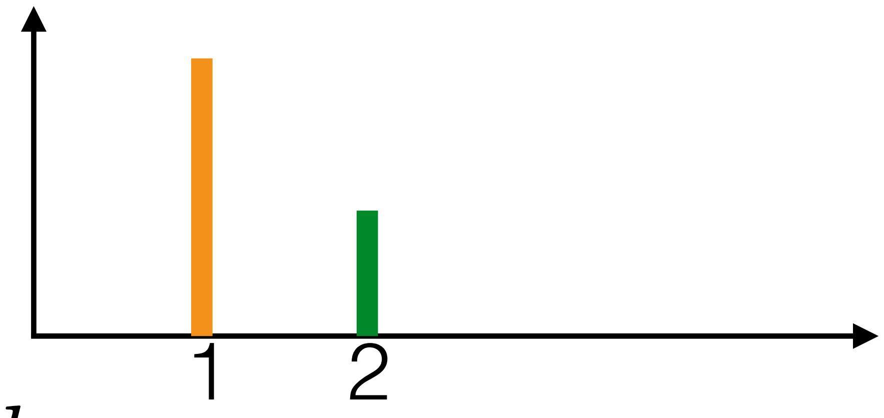

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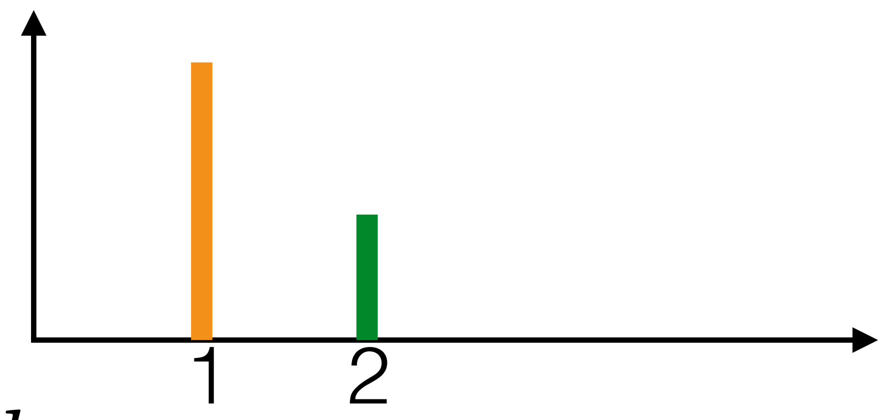

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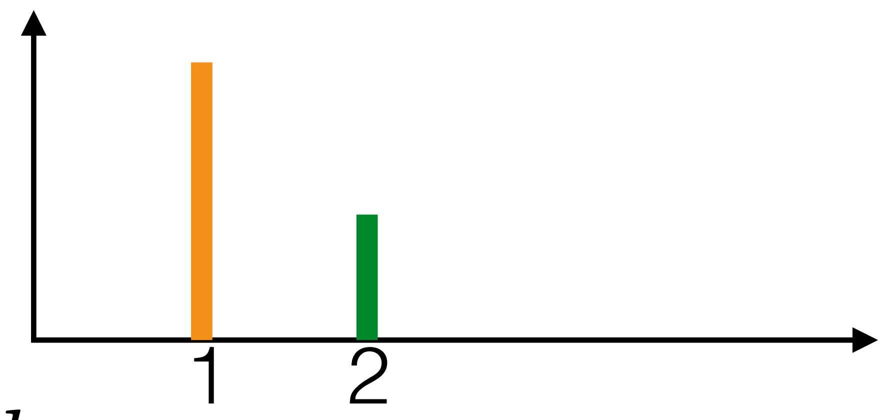

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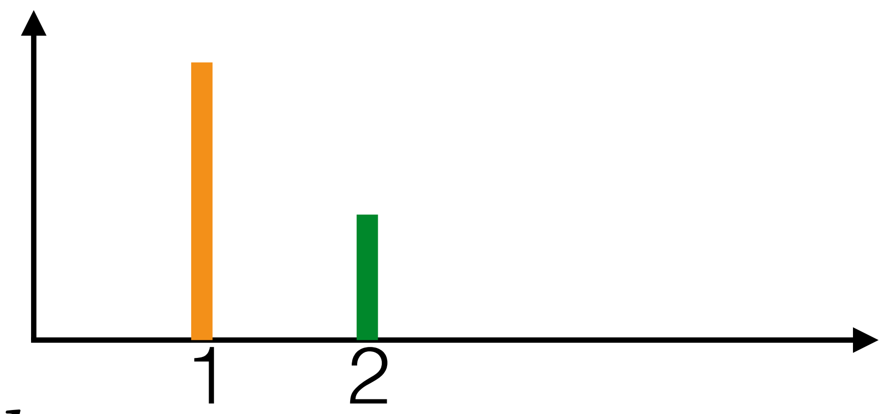

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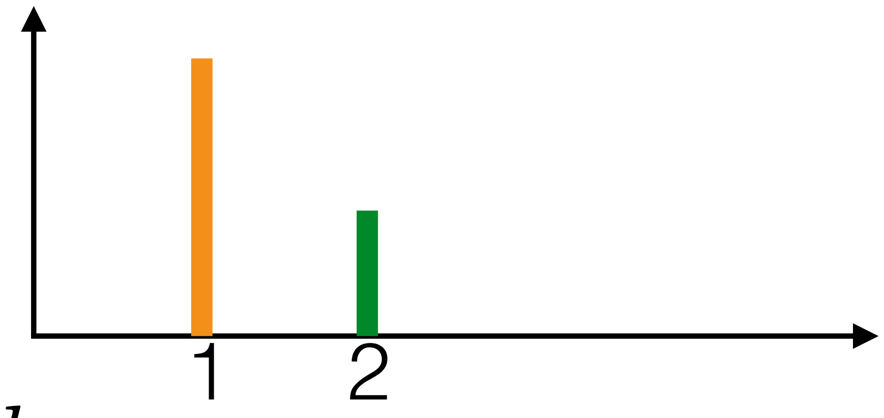
Marginal cluster assignments

- Integrate out the frequencies



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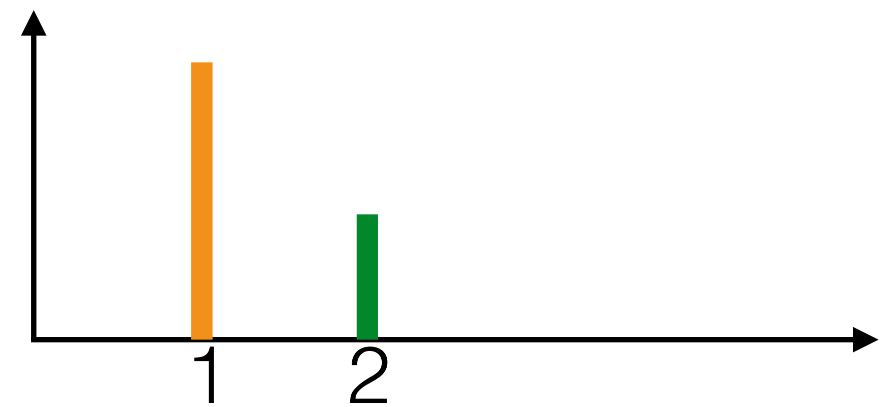
$$= \int \rho_1 \frac{\Gamma(a_{1,n} + a_{2,n})}{\Gamma(a_{1,n})\Gamma(a_{2,n})} \rho_1^{a_{1,n}-1} (1 - \rho_1)^{a_{2,n}-1} d\rho_1$$

Marginal cluster assignments

- Integrate out the frequencies



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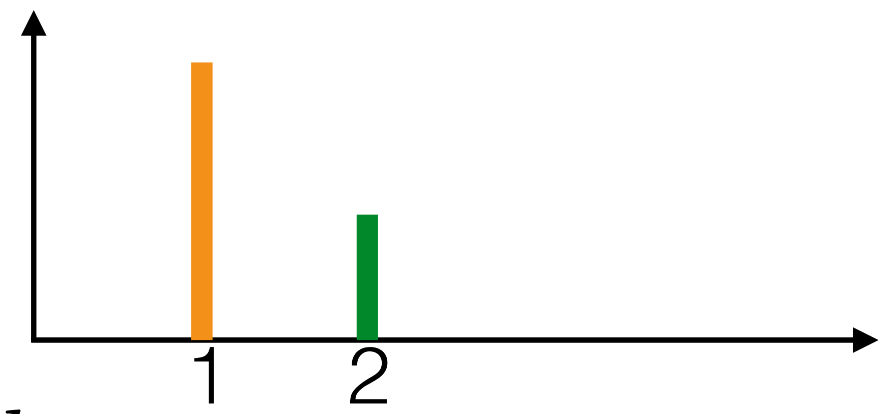
$$= \frac{\Gamma(a_{1,n} + a_{2,n})}{\Gamma(a_{1,n})\Gamma(a_{2,n})} \frac{\Gamma(a_{1,n}+1)\Gamma(a_{2,n})}{\Gamma(a_{1,n} + a_{2,n} + 1)}$$

Marginal cluster assignments

- Integrate out the frequencies



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$$p(z_n = 1 | z_1, \ldots, z_{n-1})$$

$$= \int p(z_n = 1|\rho_1) p(\rho_1 | z_1, \ldots, z_{n-1}) d\rho_1$$

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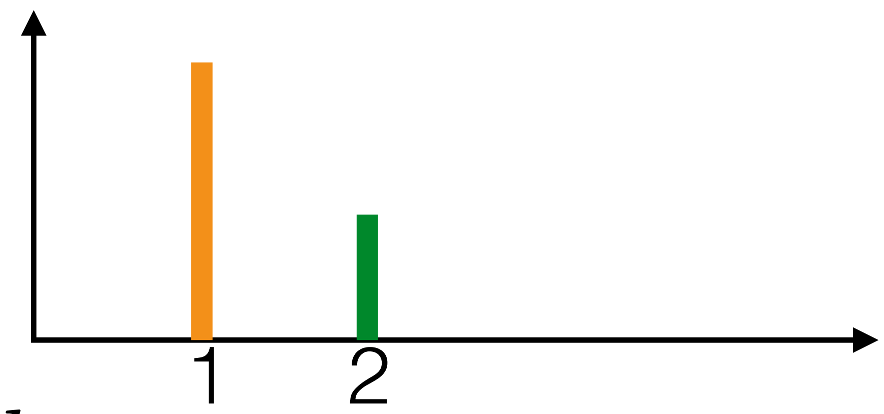
$$= \int \rho_1 \frac{\Gamma(a_{1,n} + a_{2,n})}{\Gamma(a_{1,n})\Gamma(a_{2,n})} \rho_1^{a_{1,n}-1} (1-\rho_1)^{a_{2,n}-1} d\rho_1$$

$$= \frac{\Gamma(a_{1,n} + a_{2,n})}{\Gamma(a_{1,n})\Gamma(a_{2,n})} \frac{\Gamma(a_{1,n} + 1)\Gamma(a_{2,n})}{\Gamma(a_{1,n} + a_{2,n} + 1)}$$

Recall
$\Gamma(x+1) = x\Gamma(x)$

Marginal cluster assignments

- Integrate out the frequencies



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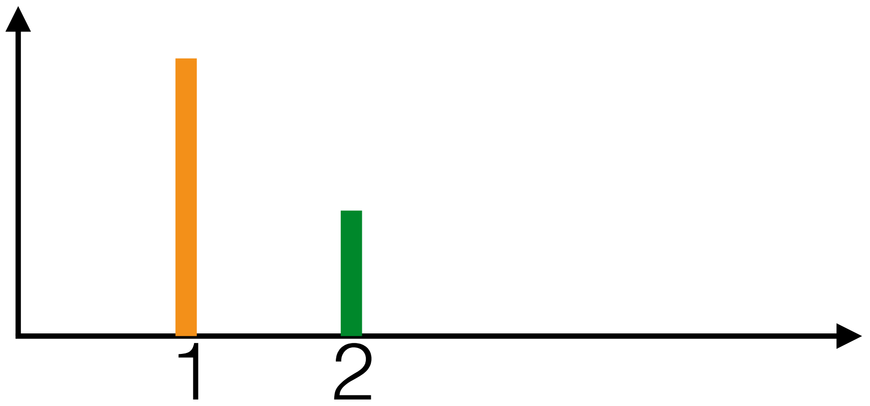

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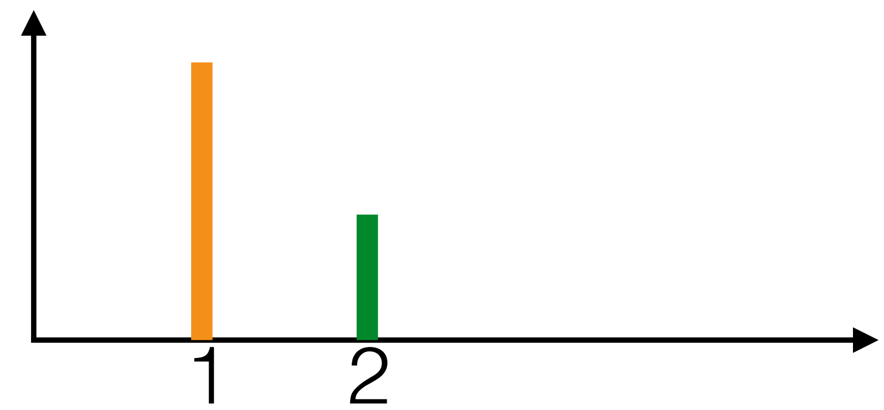

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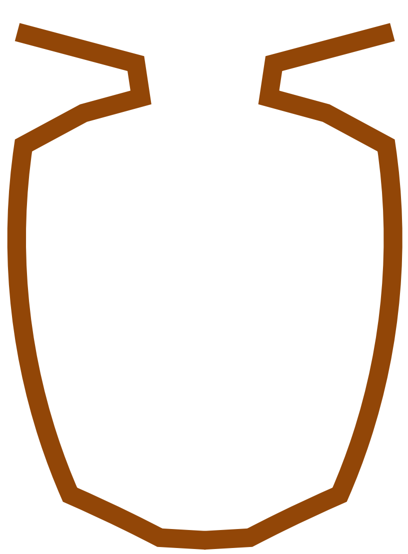

Marginal cluster assignments

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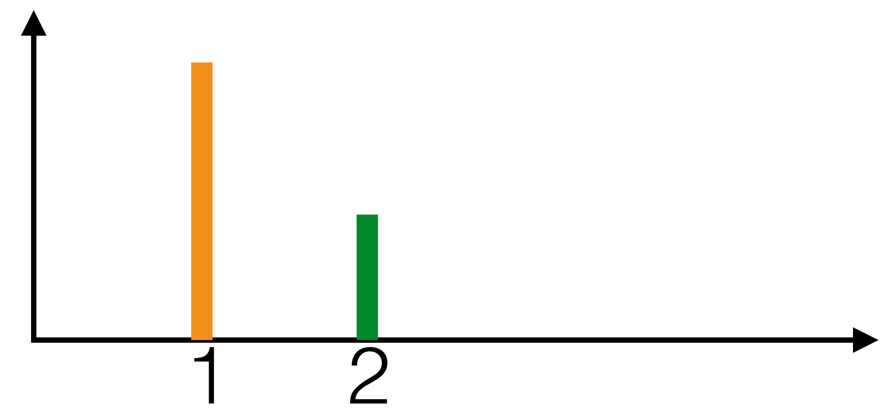

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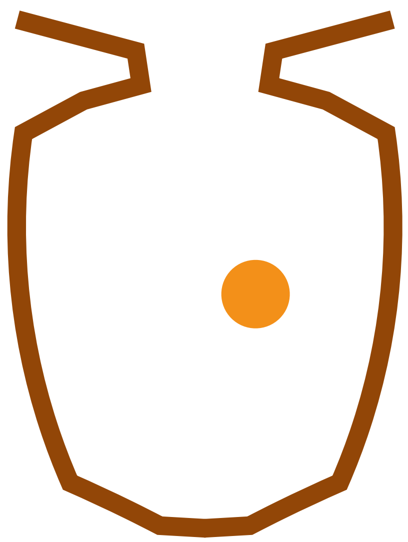

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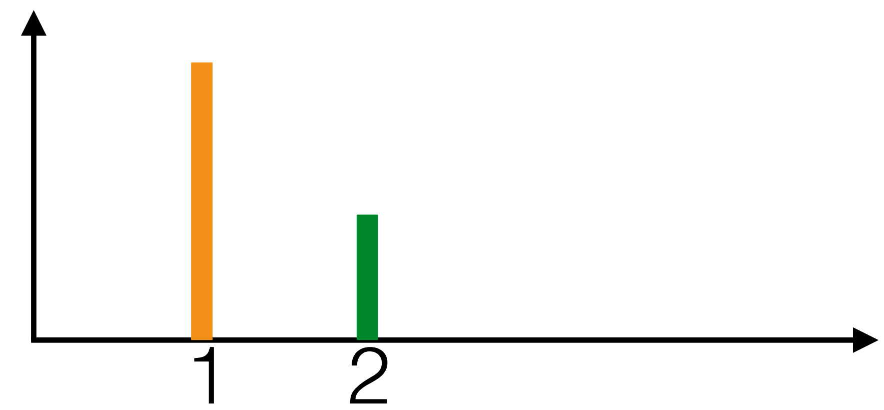

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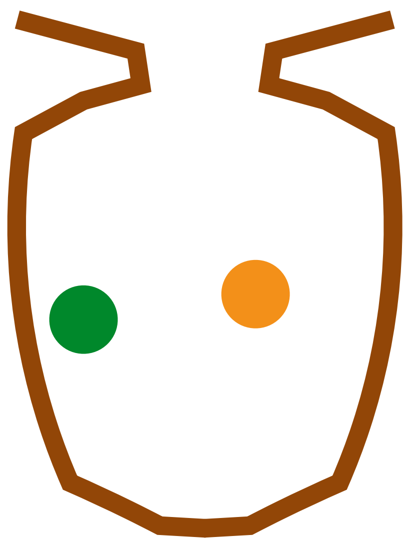

Marginal cluster assignments

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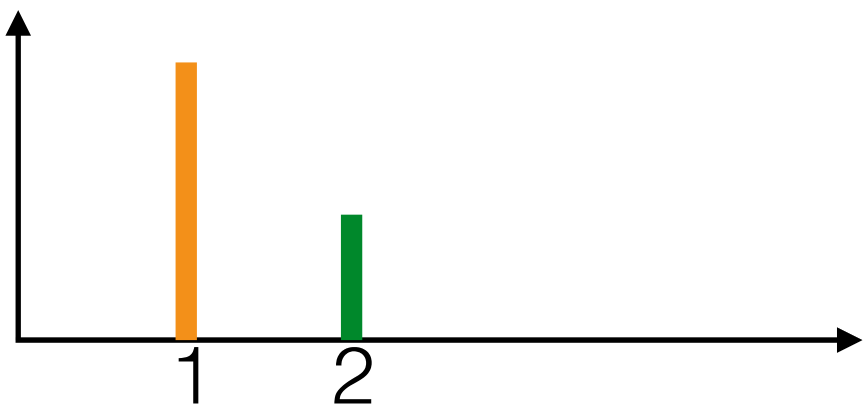

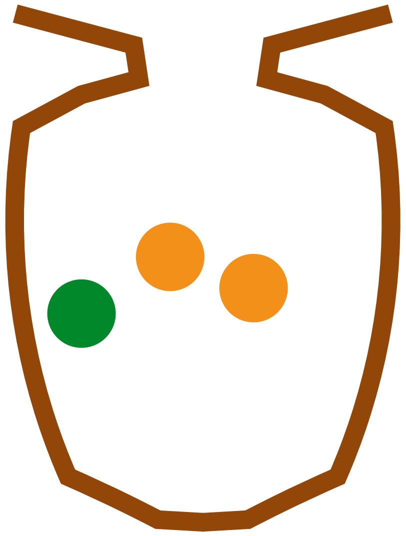

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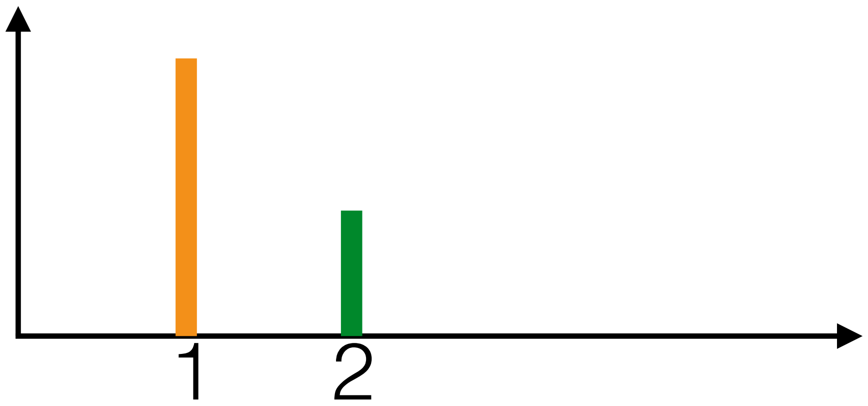

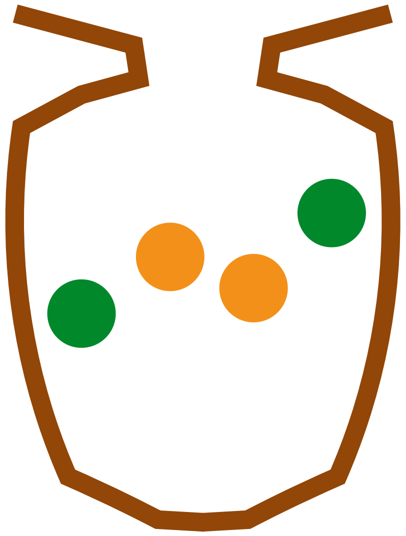

Marginal cluster assignments

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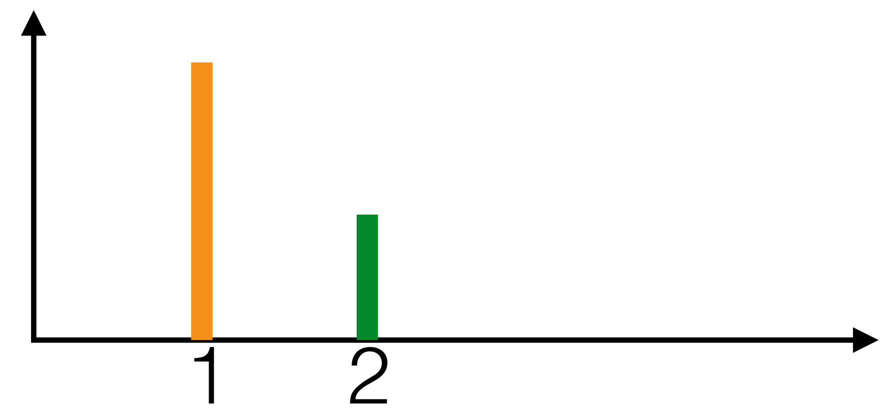

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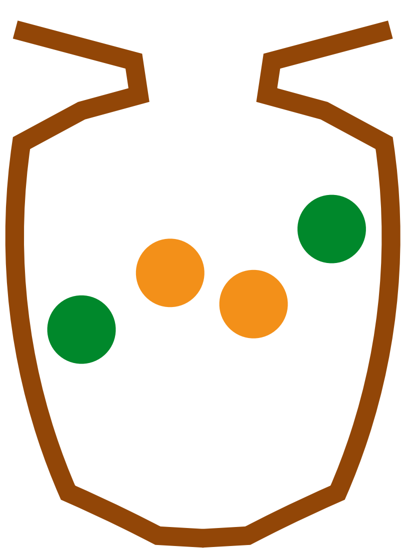

$$\lim_{n \to \infty} \frac{\#\ \text{orange}}{\#\ \text{total}}$$

Marginal cluster assignments

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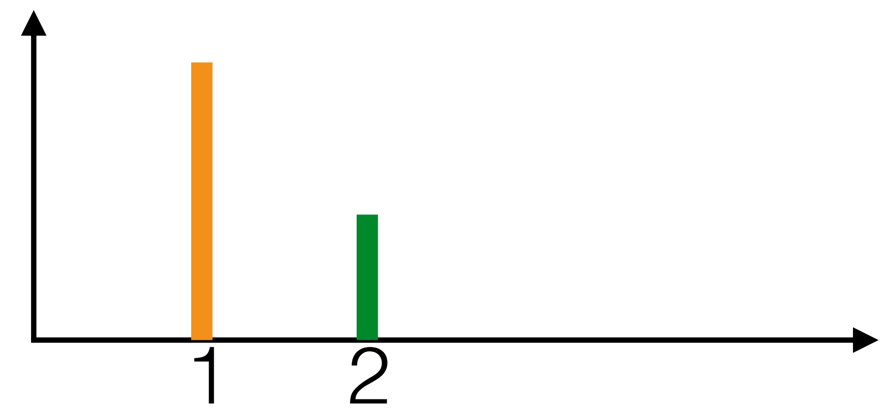

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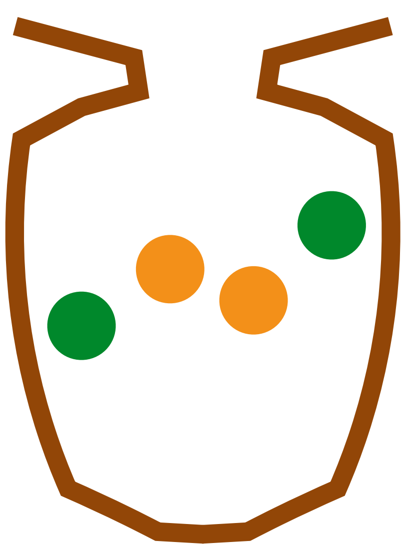

$$\lim_{n \to \infty} \frac{\#\ \text{orange}}{\#\ \text{total}} = \rho_{\text{orange}}$$

Marginal cluster assignments

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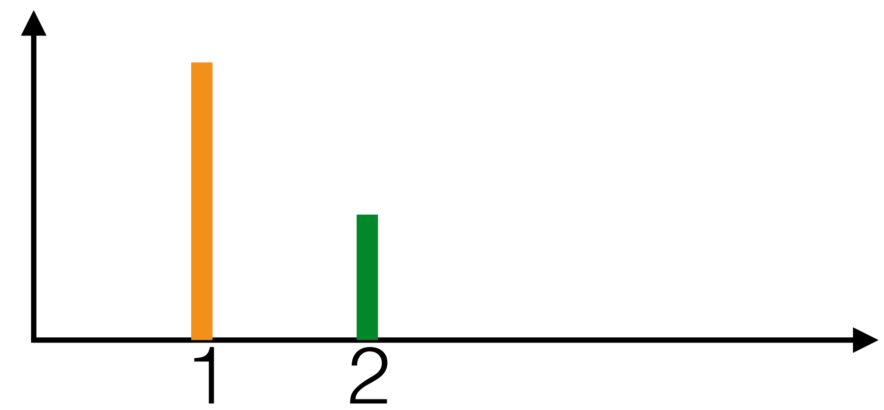

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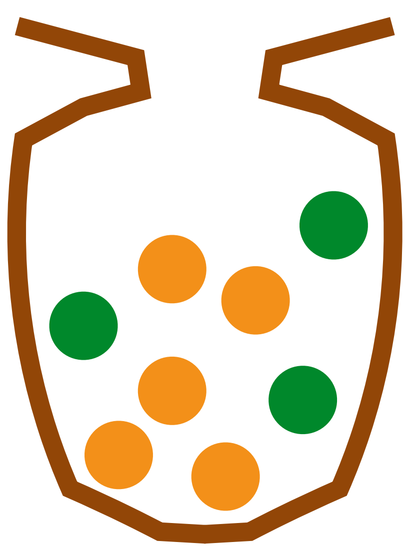

Marginal cluster assignments

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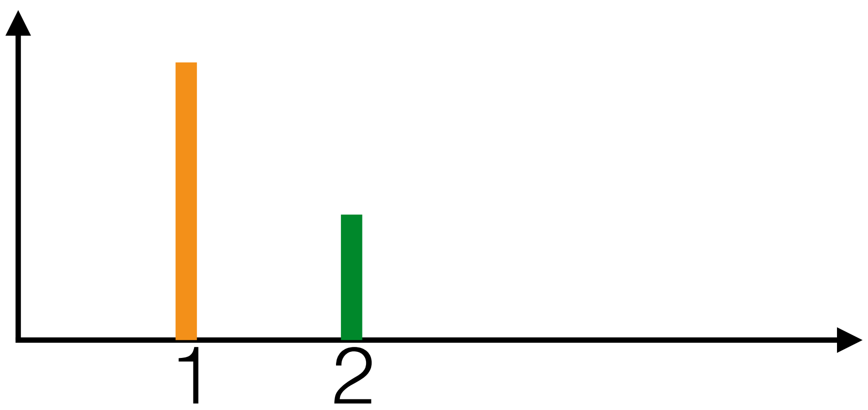

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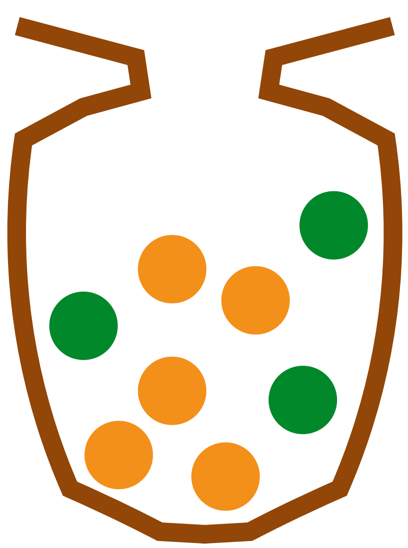

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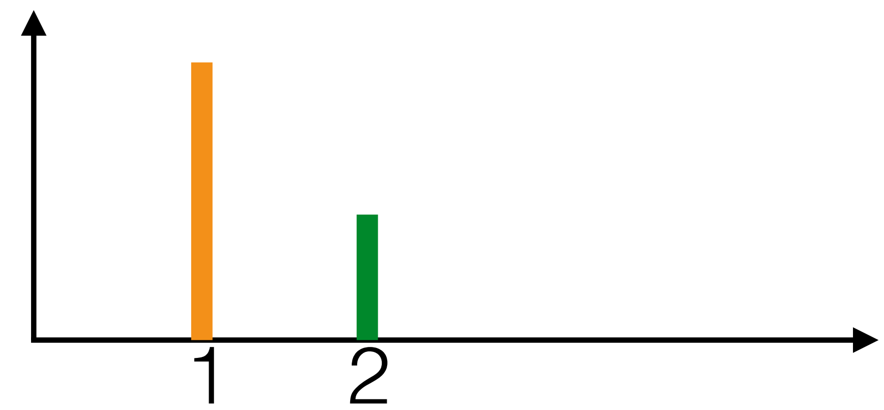

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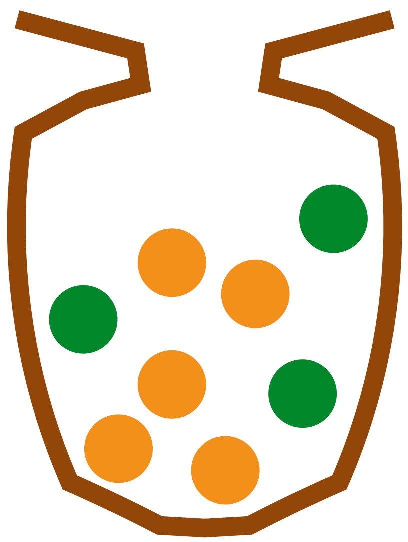

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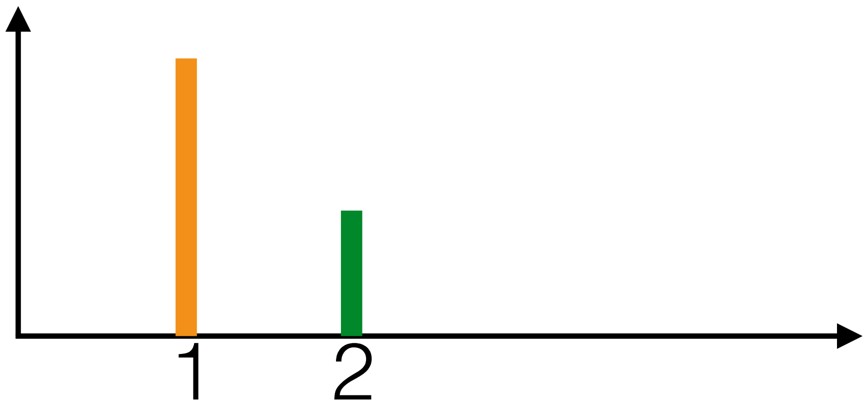

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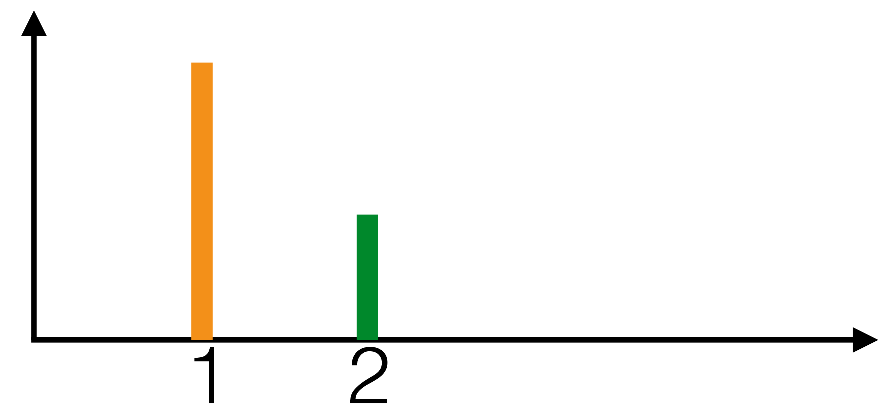

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- Pólya urn

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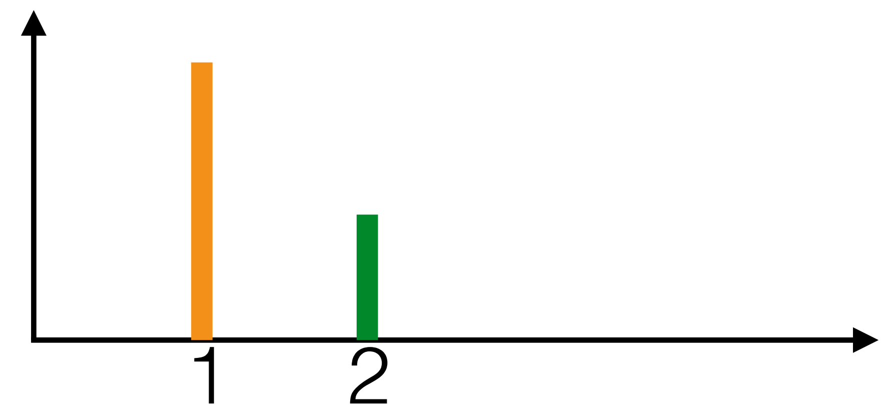

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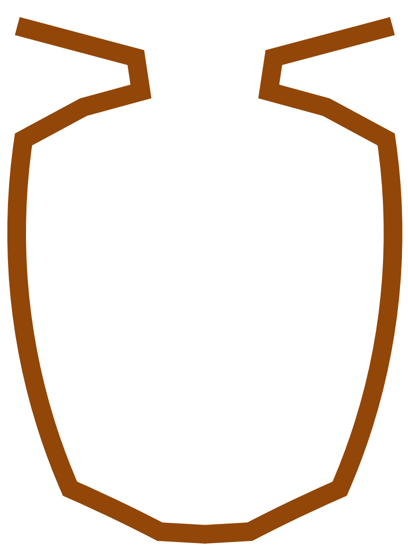

Marginal cluster assignments

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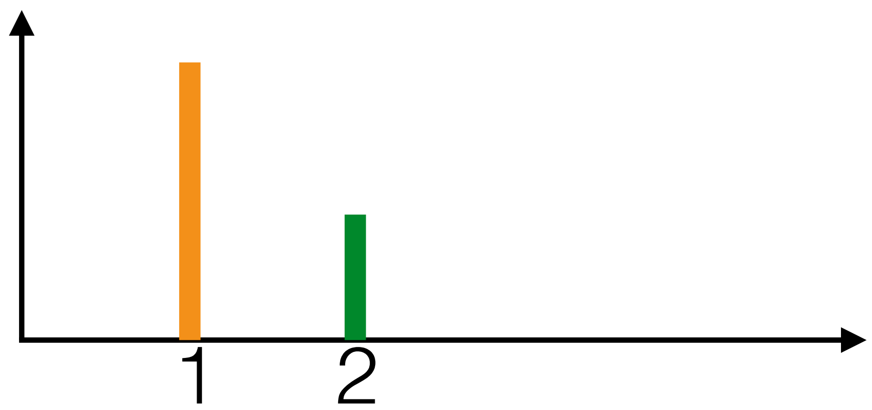

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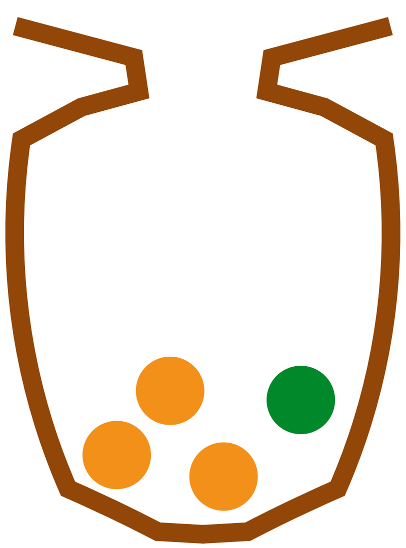

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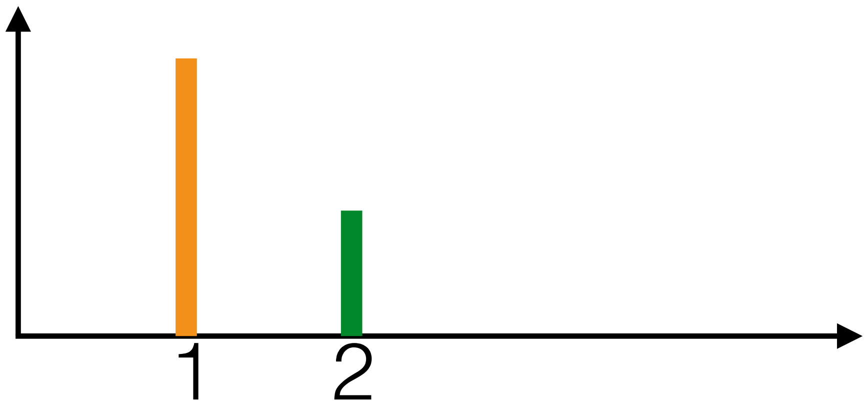

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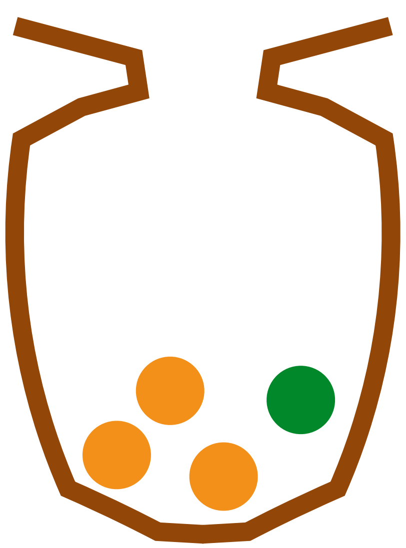
- Pólya urn
  - Choose any ball with equal probability



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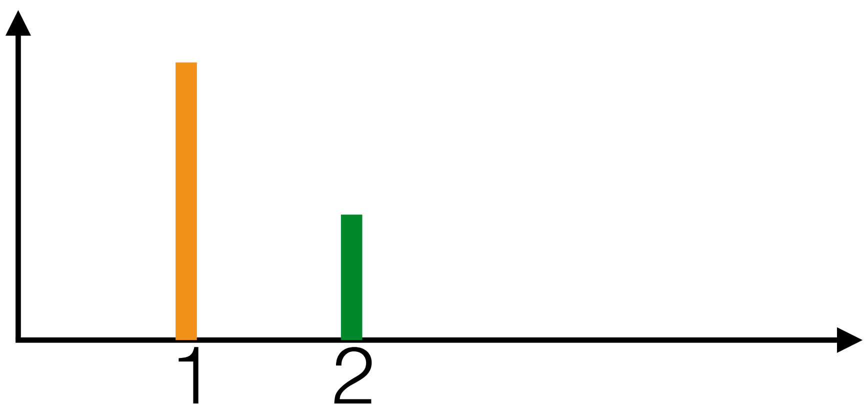

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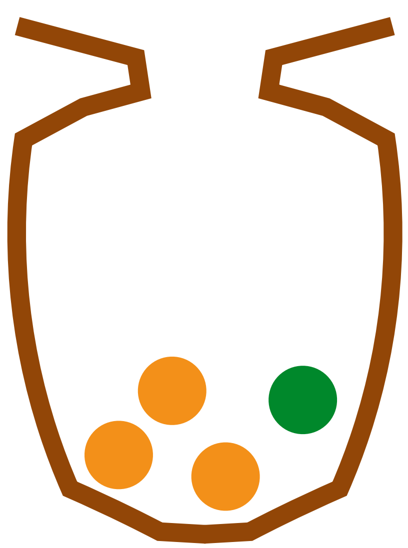
- Pólya urn
  - Choose any ball with equal probability
  - Replace and add ball of same color



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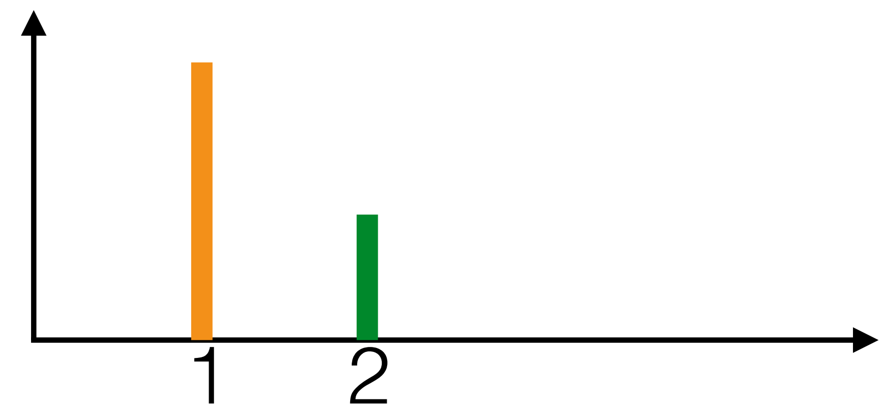

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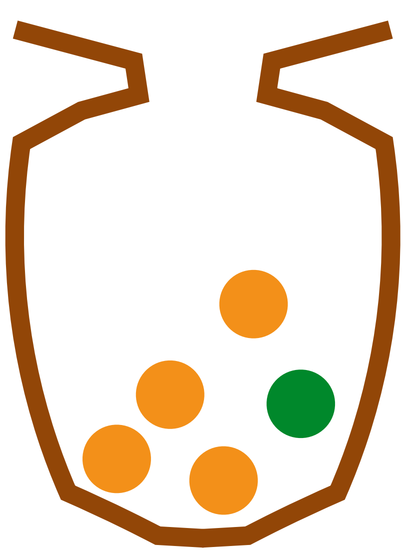
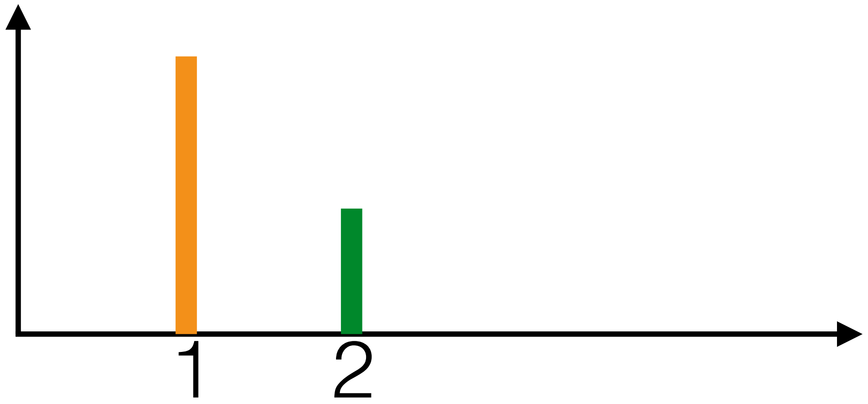


Marginal cluster assignments

- Integrate out the frequencies

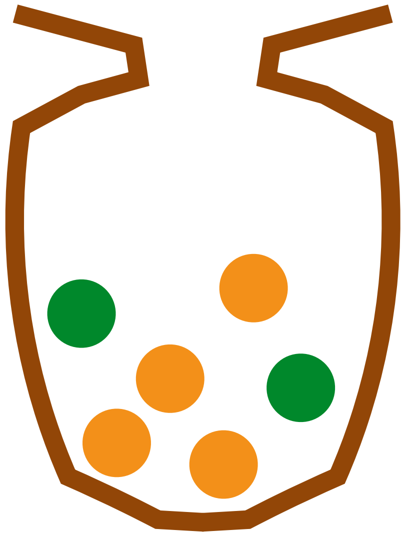
$$\rho_1 \sim \mathrm{Beta}(a_1, a_2),\, z_n \overset{iid}{\sim} \mathrm{Cat}(\rho_1, \rho_2)$$

$$p(z_n = 1 | z_1, \ldots, z_{n-1}) = \frac{a_{1,n}}{a_{1,n} + a_{2,n}}$$



$$a_{1,n} := a_1 + \sum_{m=1}^{n-1} \mathbf{1}\{z_m = 1\},\, a_{2,n} = a_2 + \sum_{m=1}^{n-1} \mathbf{1}\{z_m = 2\}$$

- Pólya urn
  - Choose any ball with equal probability
  - Replace and add ball of same color



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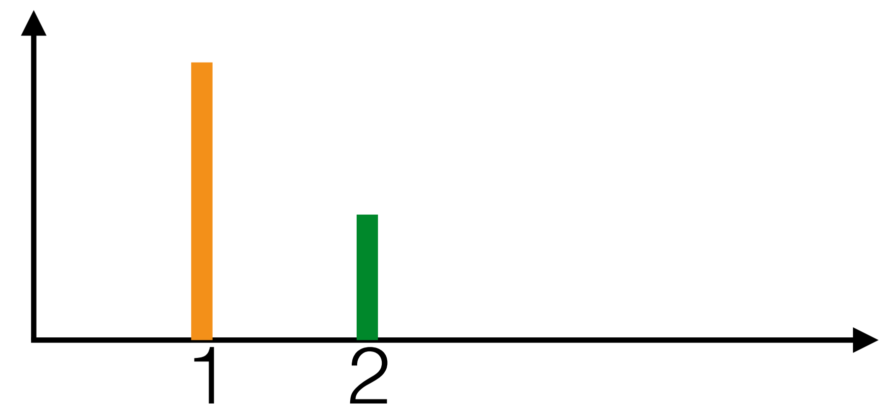

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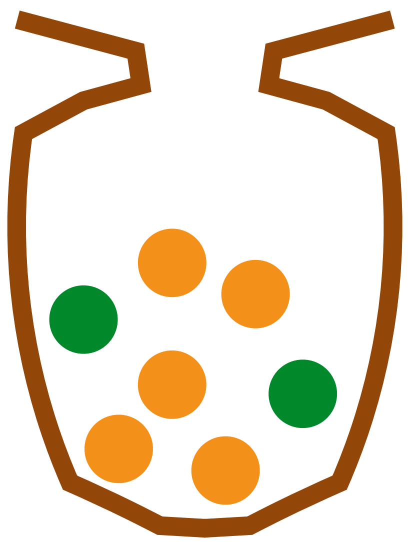

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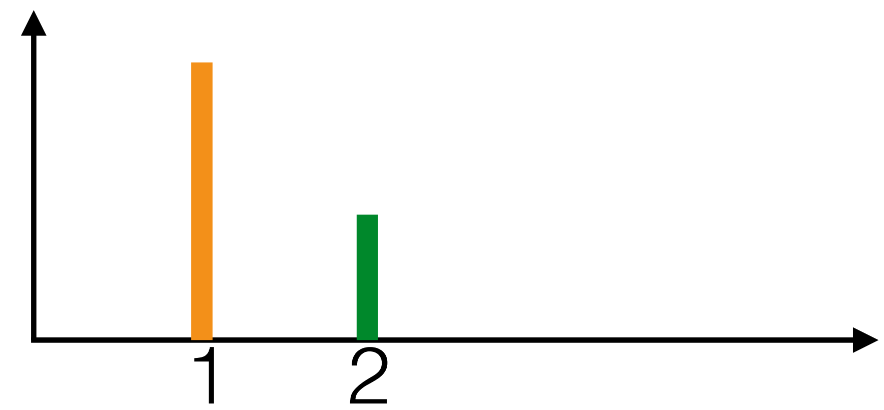

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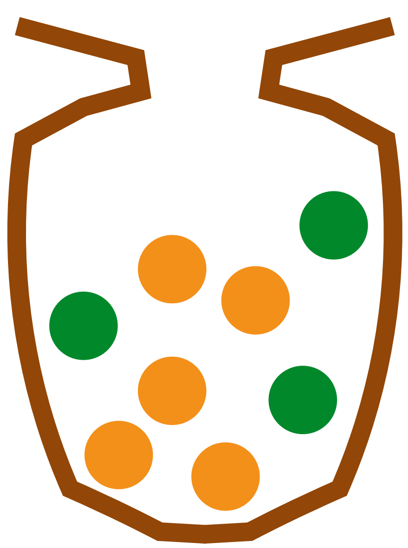

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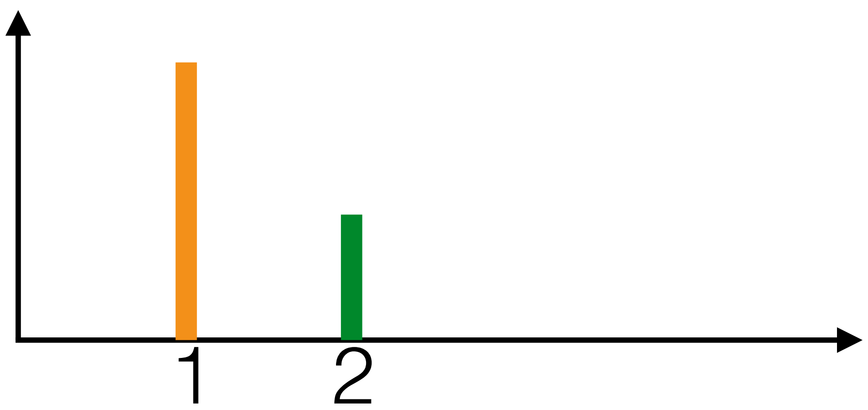

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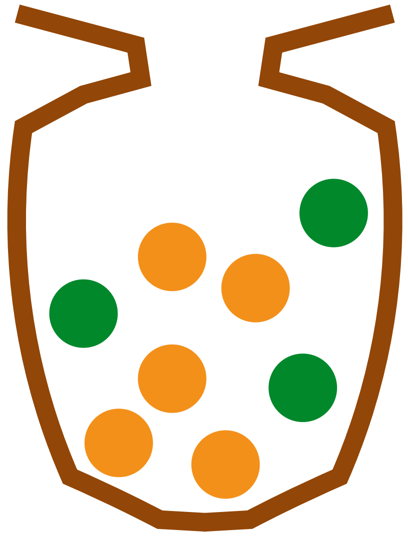

$$\lim_{n \to \infty} \frac{\#\ \text{orange}}{\#\ \text{total}}$$

Marginal cluster assignments

- Integrate out the frequencies

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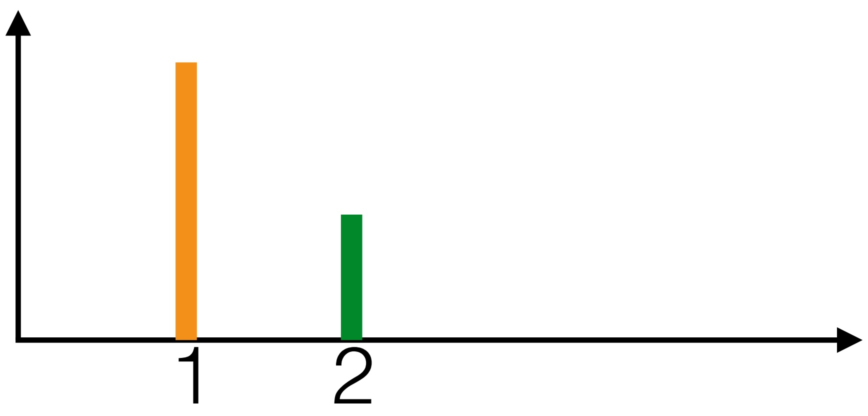

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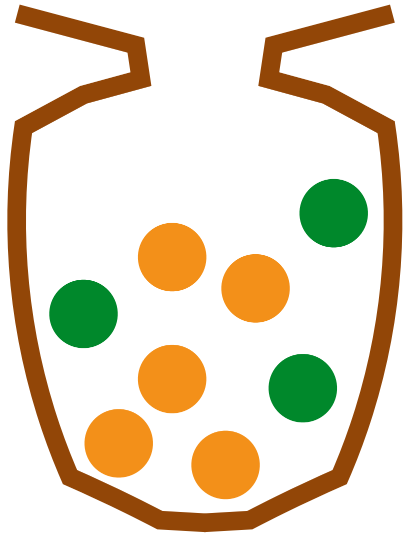

$$\lim_{n \to \infty} \frac{\#\ \text{orange}}{\#\ \text{total}} = \rho_{\text{orange}}$$

Marginal cluster assignments

- Integrate out the frequencies

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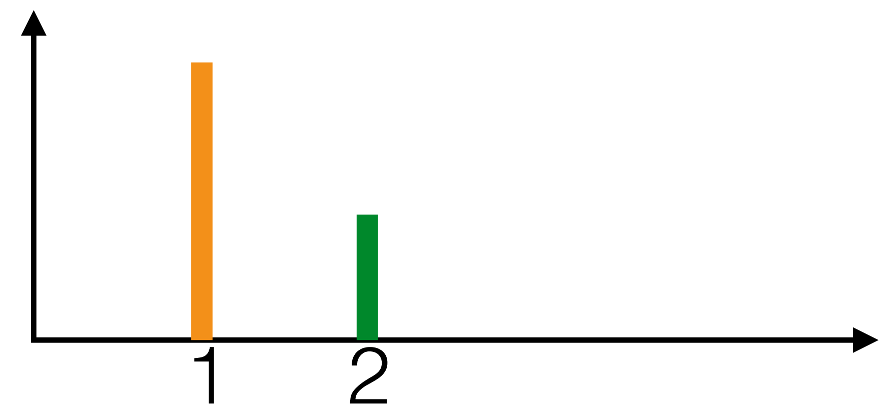

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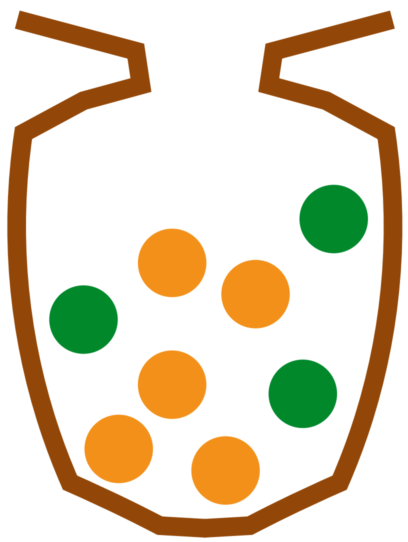

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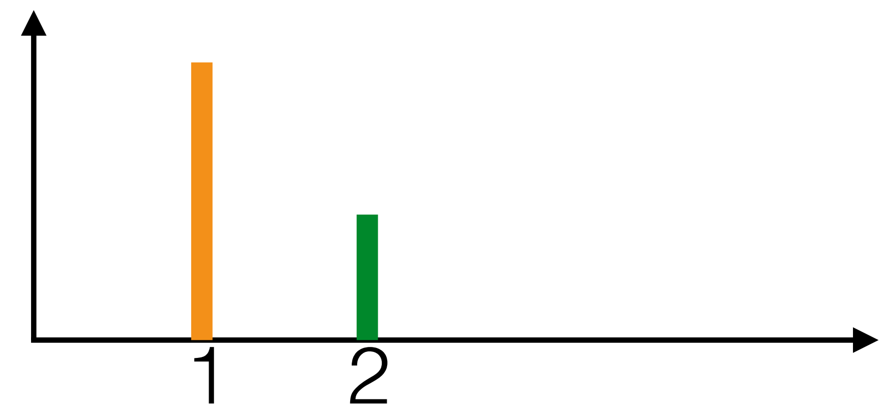

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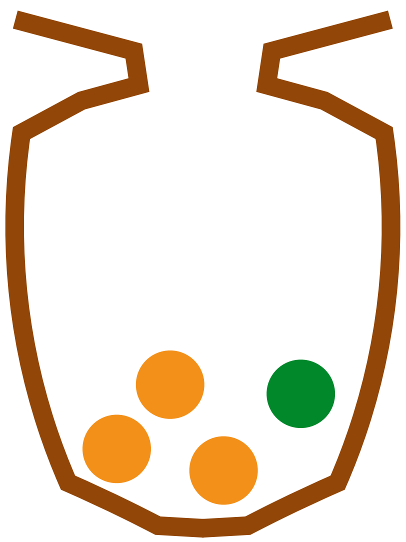

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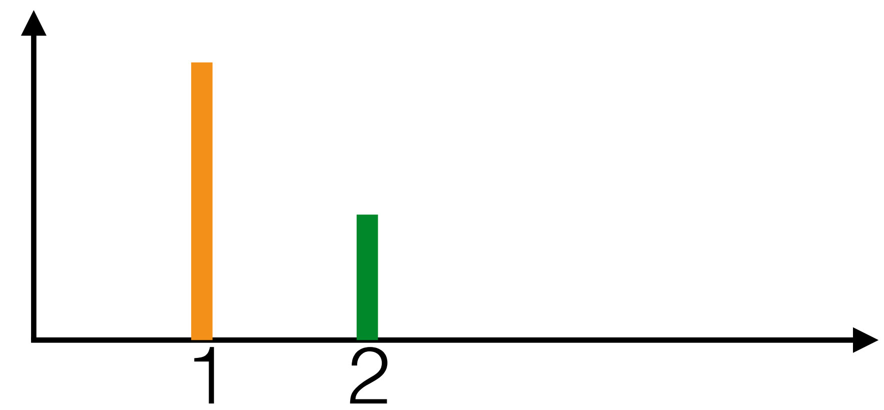

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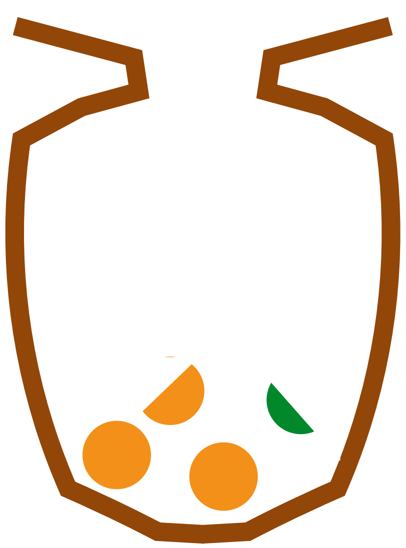

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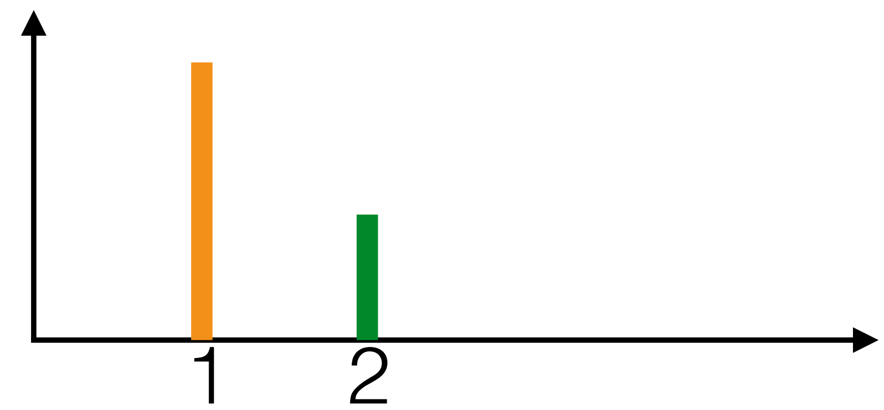

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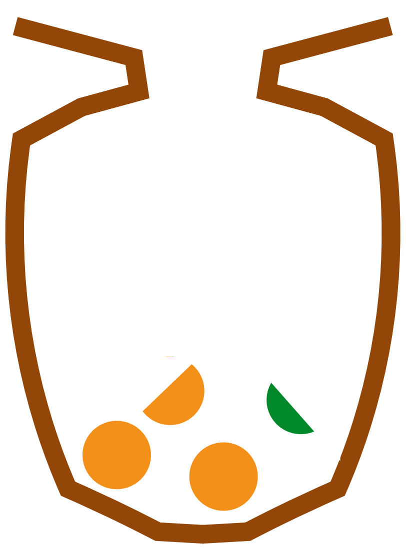

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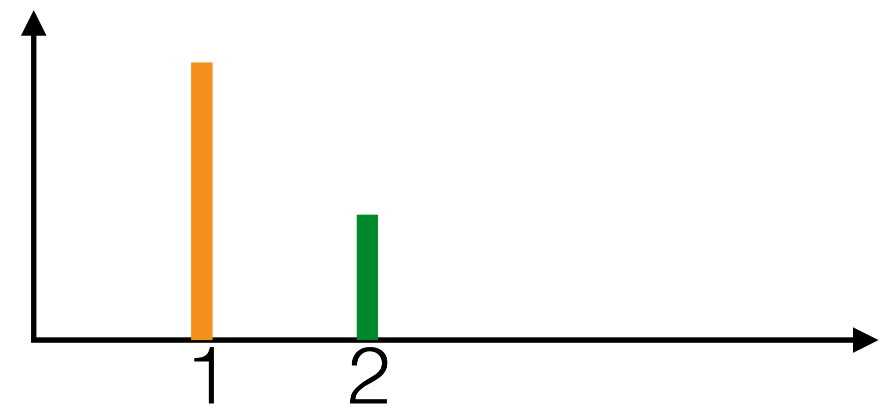
Marginal cluster assignments

- Integrate out the frequencies



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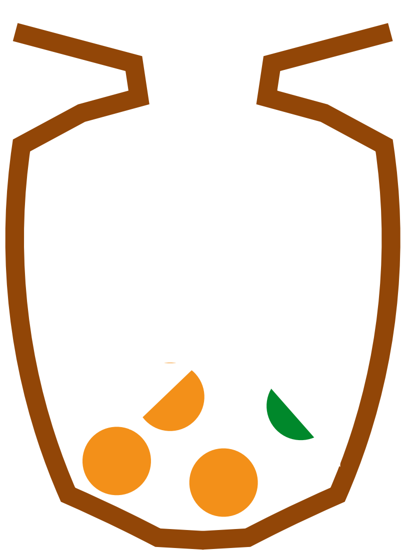

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$$\text{PolyaUrn}(a_{\text{orange}}, a_{\text{green}})$$

Marginal cluster assignments

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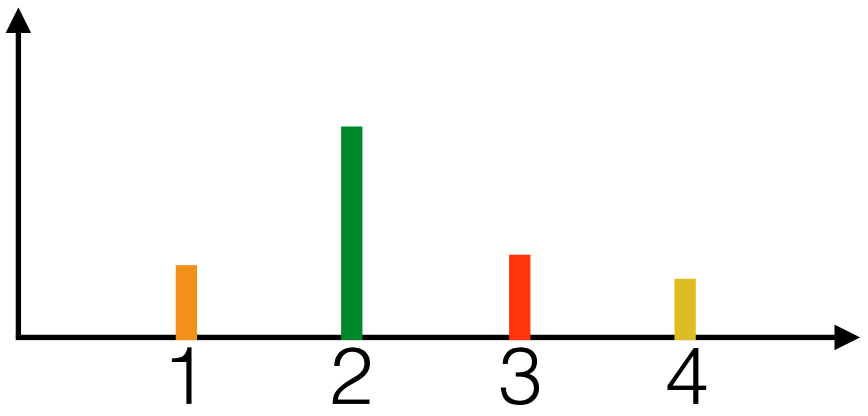
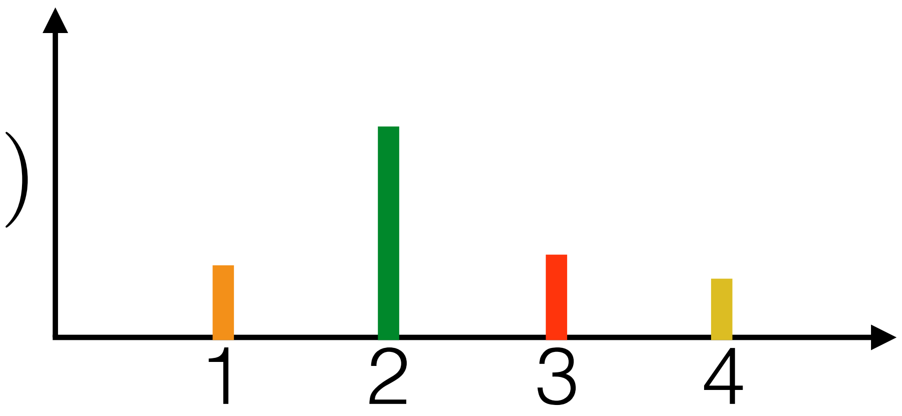


Marginal cluster assignments

- Integrate out the frequencies

$\rho_{1:K} \sim \mathrm{Dirichlet}(a_{1:K}), z_n \overset{iid}{\sim} \mathrm{Cat}(\rho_{1:K})$



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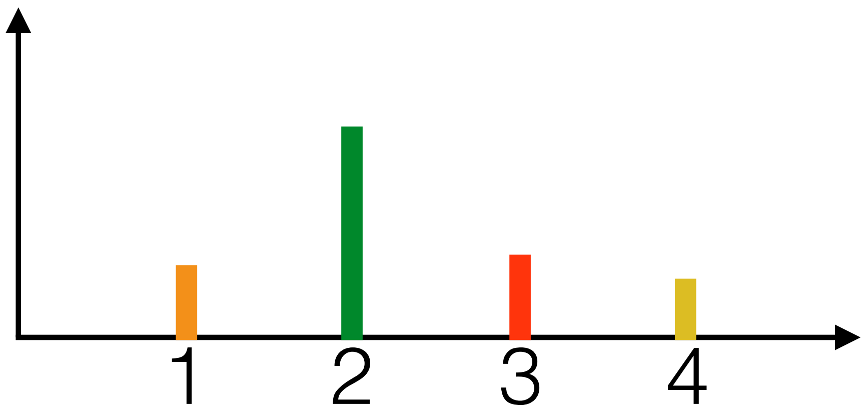

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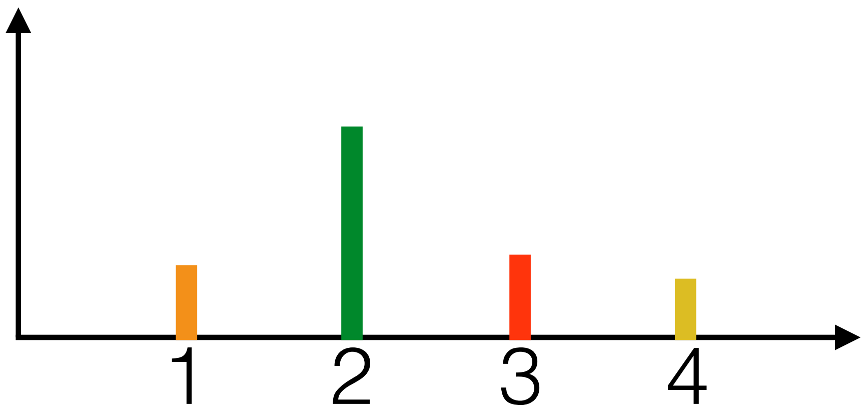

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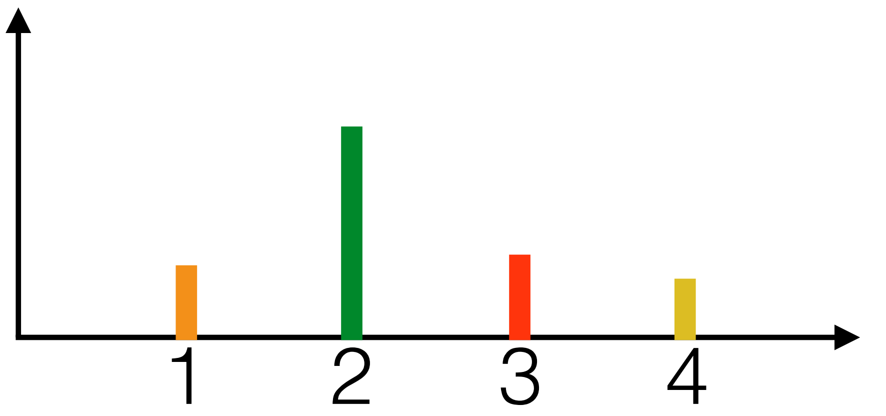

- multivariate Pólya urn

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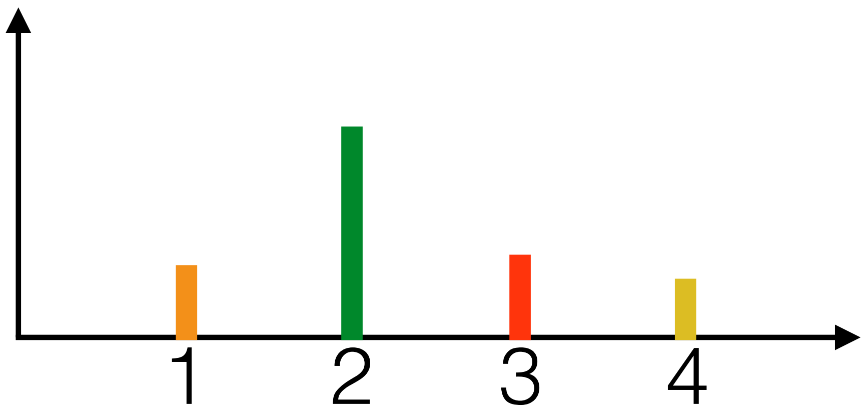

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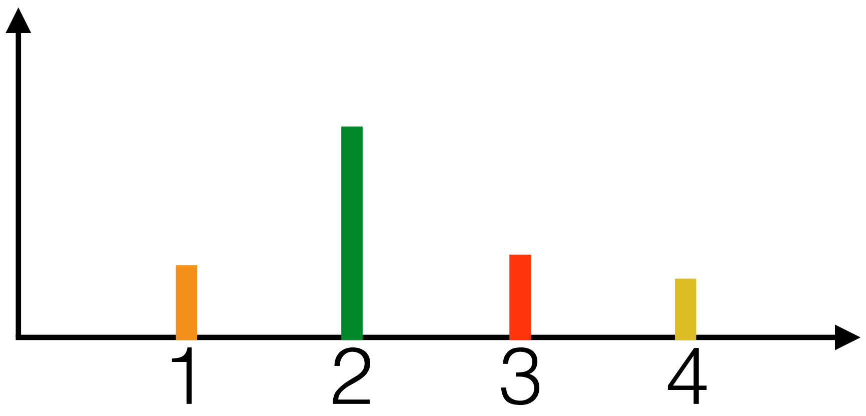

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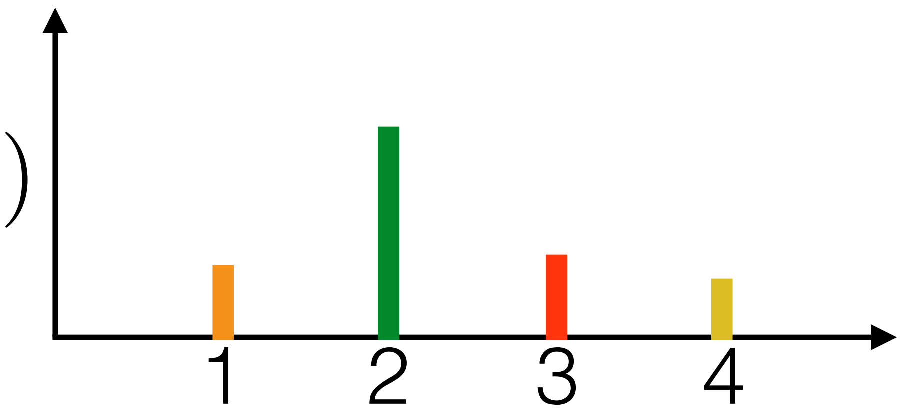

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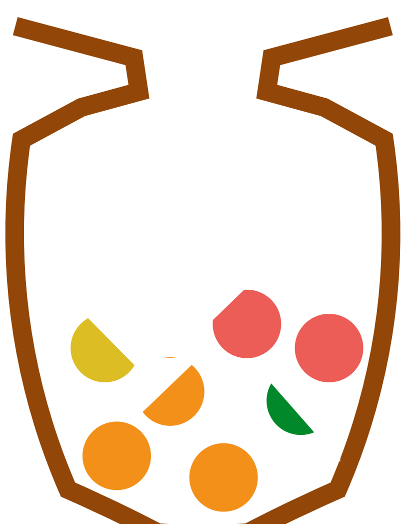

Marginal cluster assignments

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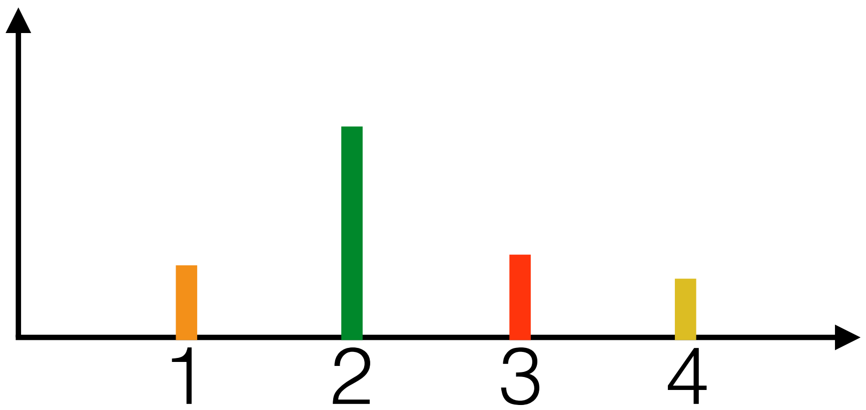

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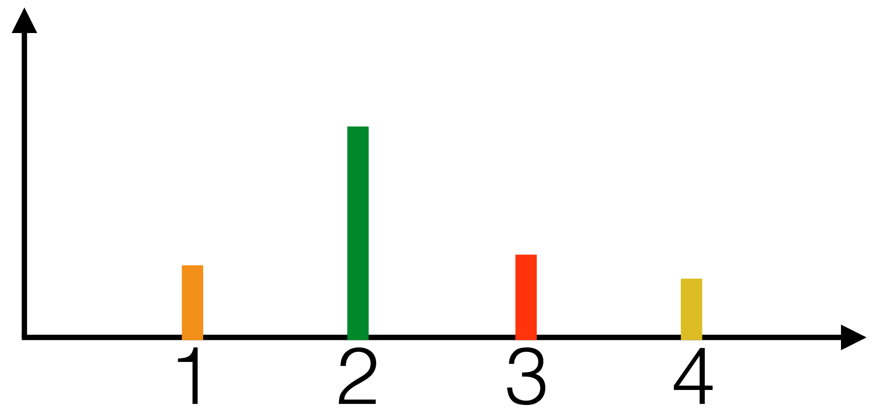
$$\lim_{n \to \infty} \frac{(\#\text{ orange}, \#\text{ green}, \#\text{ red}, \#\text{ yellow})}{\#\text{ total}}$$

Marginal cluster assignments

- Integrate out the frequencies



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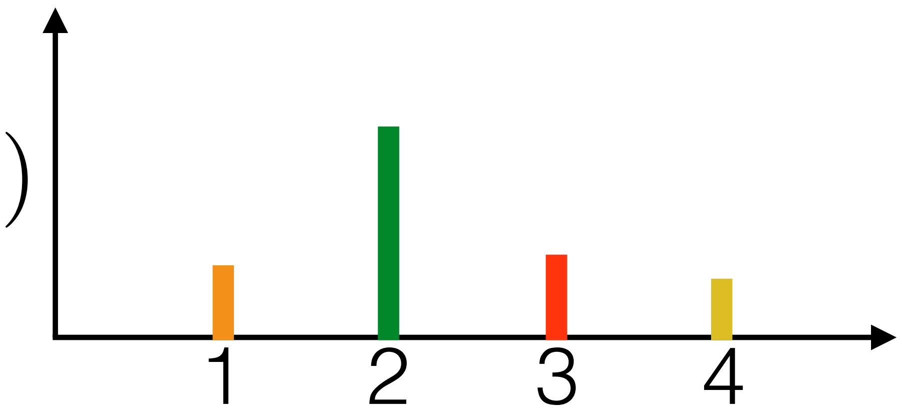

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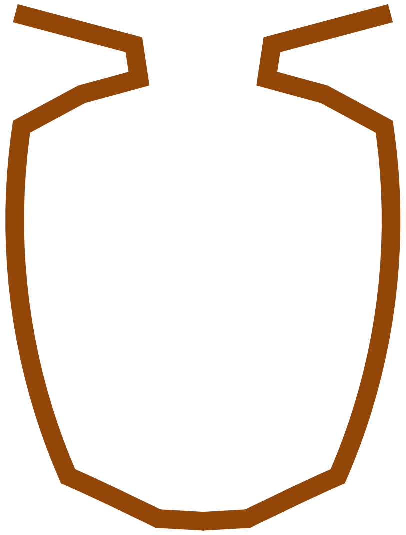
Marginal cluster assignments

- Hoppe urn / Blackwell-MacQueen urn

[Blackwell, MacQueen 1973; Hoppe 1984]

Marginal cluster assignments

- Hoppe urn / Blackwell-MacQueen urn



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Marginal cluster assignments

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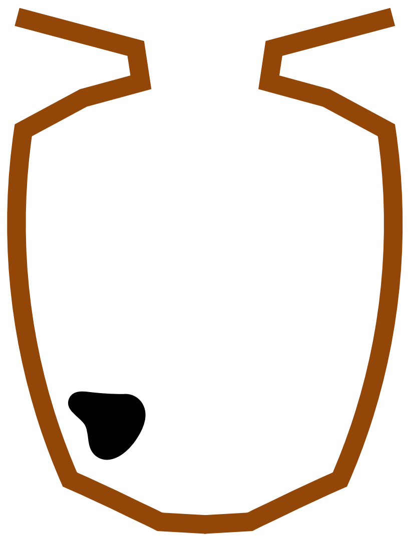

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Marginal cluster assignments

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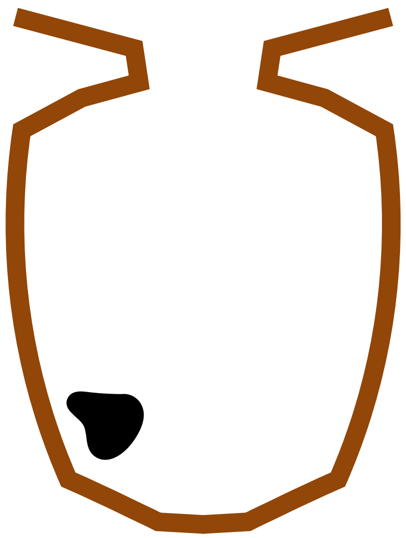

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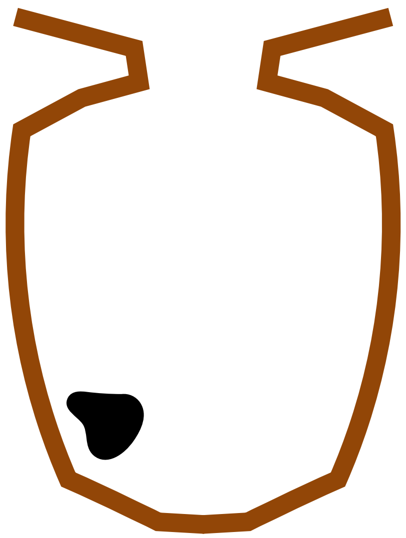
Marginal cluster assignments

- Hoppe urn / Blackwell-MacQueen urn



  - Choose ball with prob proportional to its mass
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[Blackwell, MacQueen 1973; Hoppe 1984]

Marginal cluster assignments

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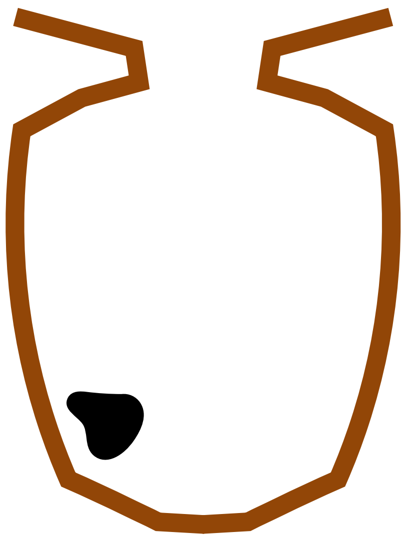

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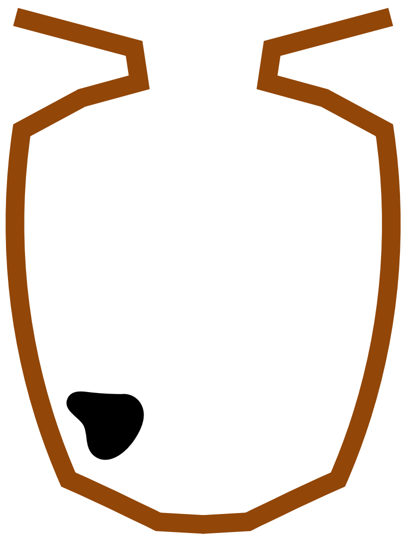

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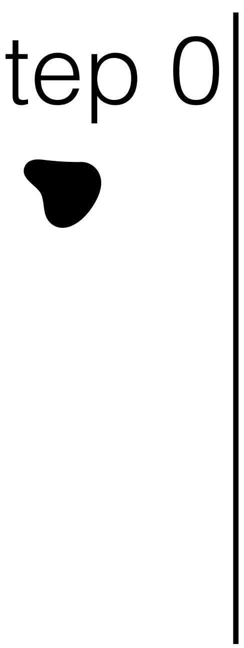
Step 0



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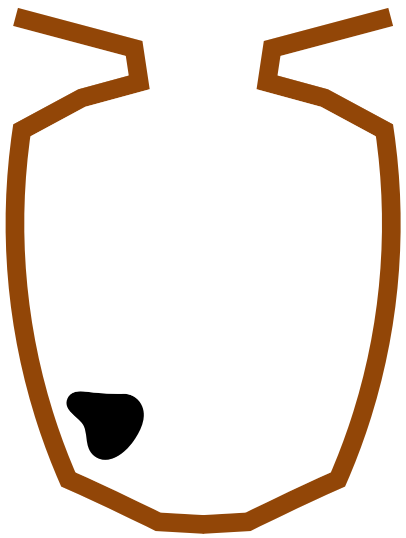
Marginal cluster assignments

- Hoppe urn / Blackwell-MacQueen urn



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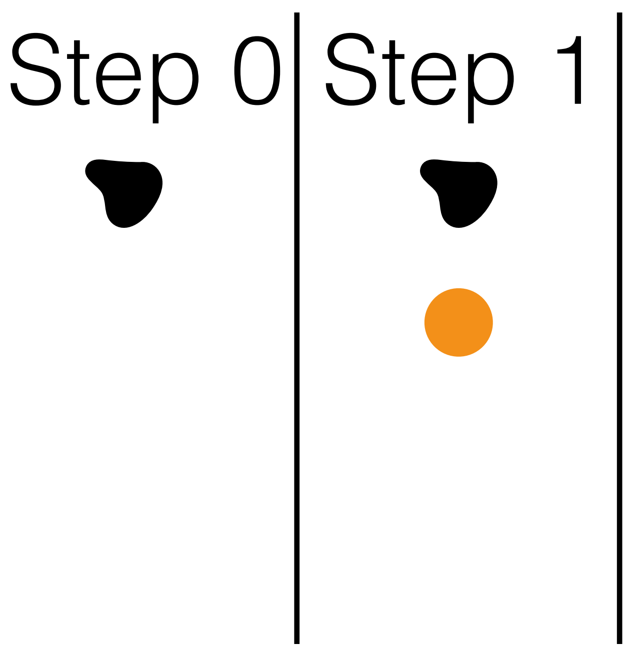

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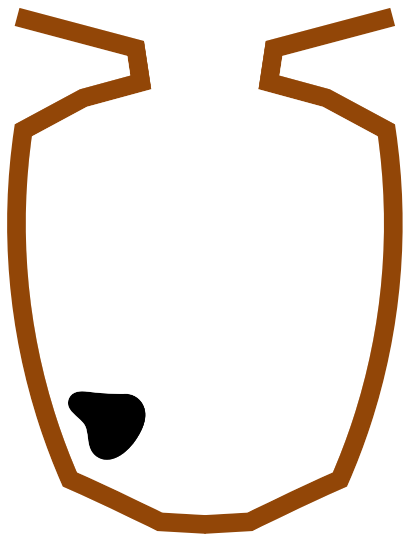
Marginal cluster assignments

- Hoppe urn / Blackwell-MacQueen urn



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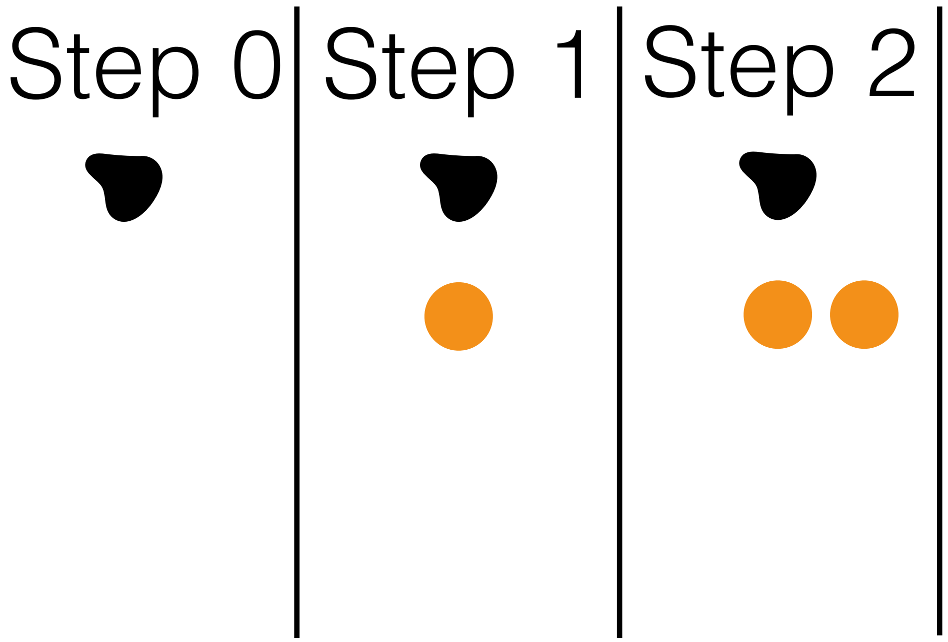

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Marginal cluster assignments

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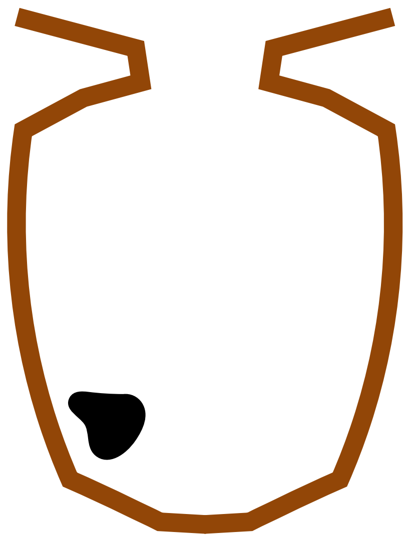

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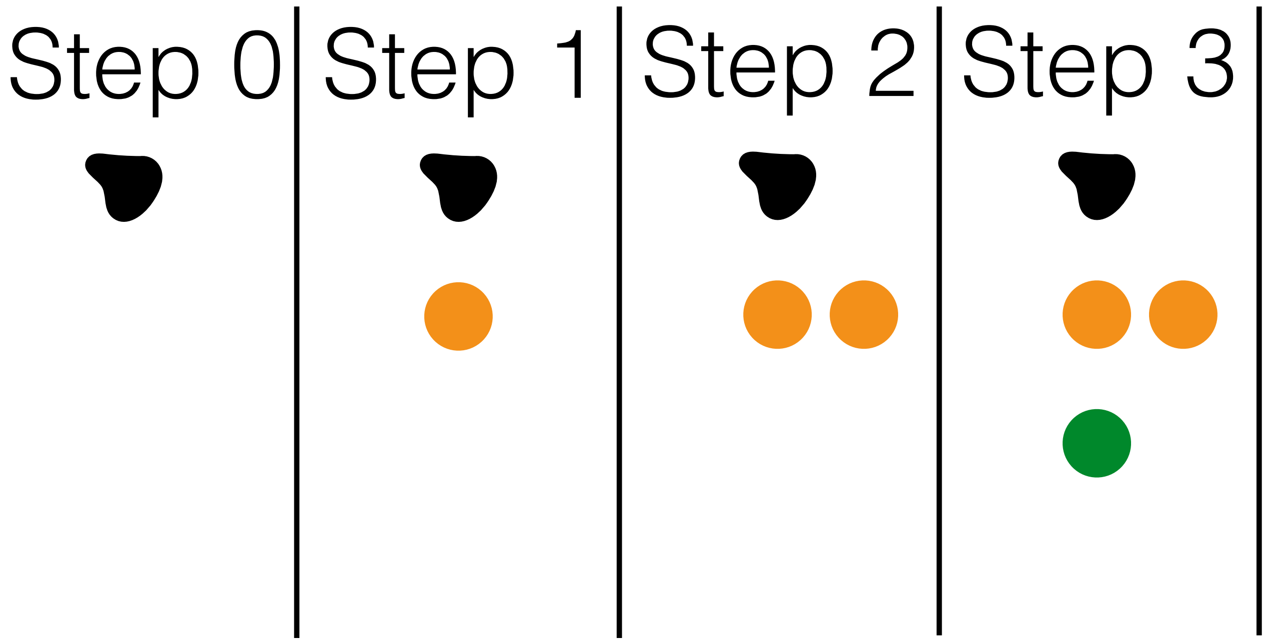
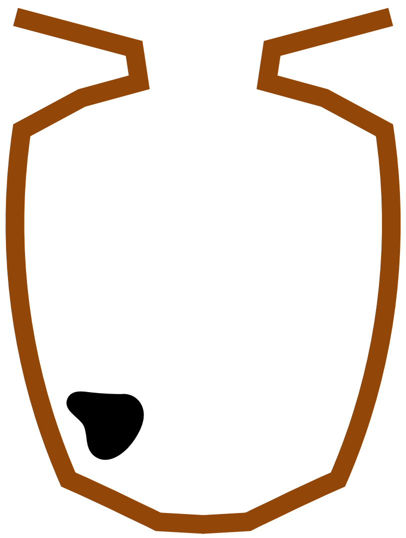


[Blackwell, MacQueen 1973; Hoppe 1984]

Marginal cluster assignments

- Hoppe urn / Blackwell-MacQueen urn



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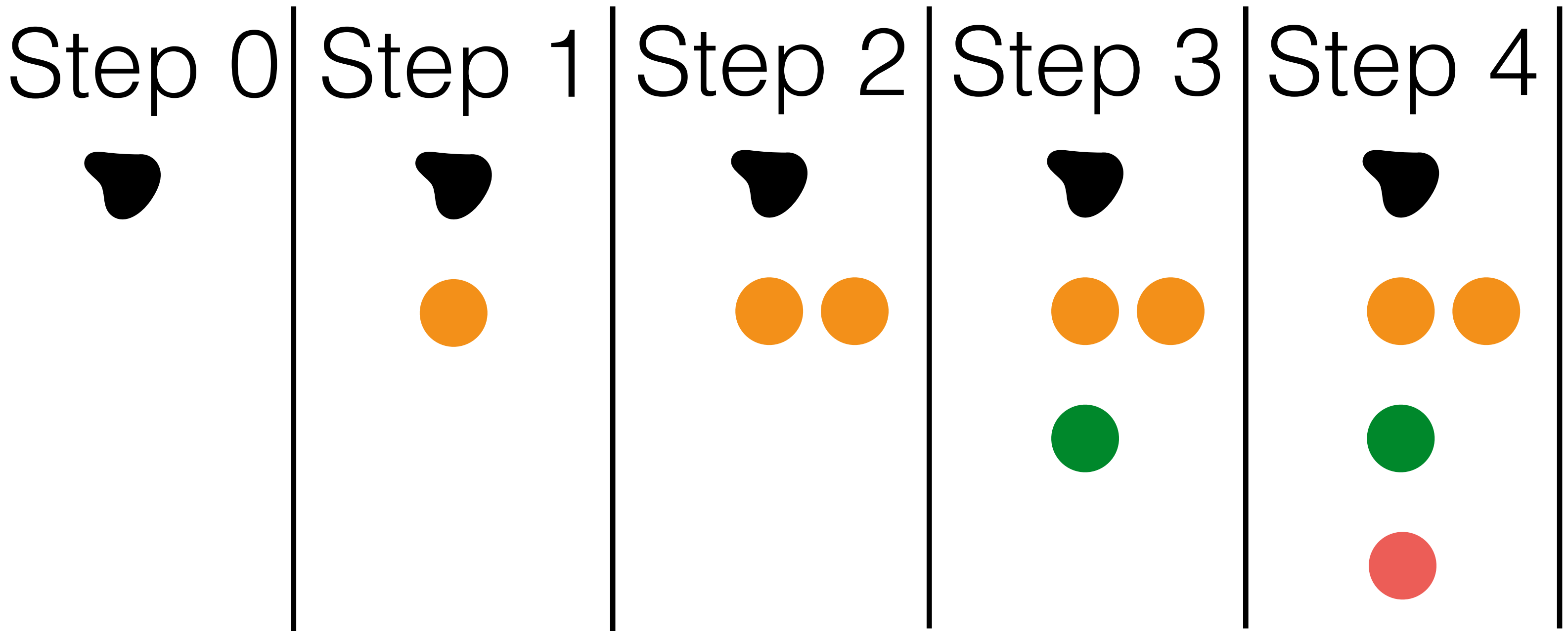

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Marginal cluster assignments

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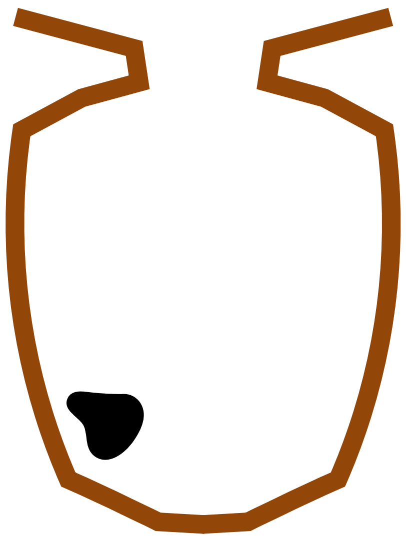

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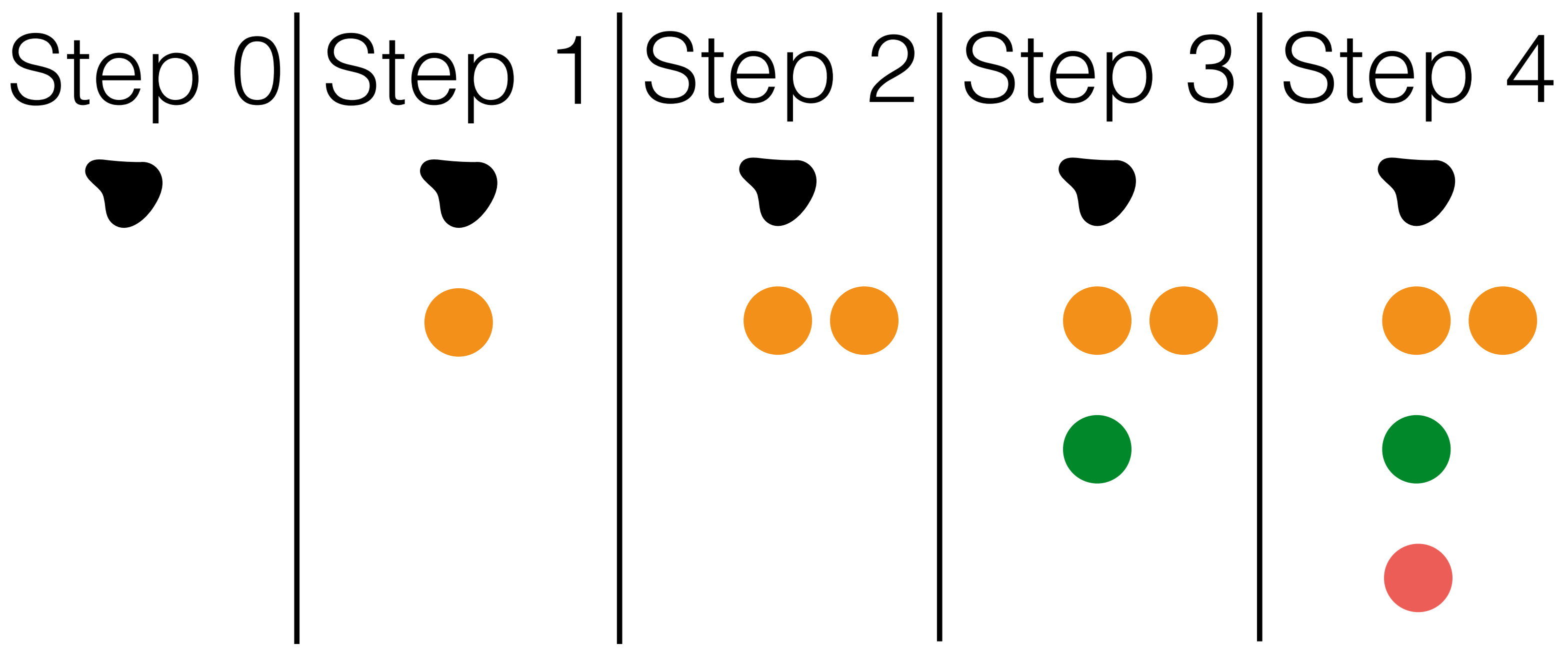
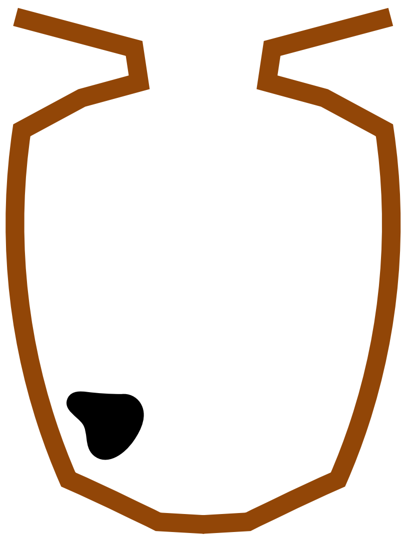


$$(\#\text{orange}, \#\text{other}) = \text{PolyaUrn}(1, \alpha)$$

Marginal cluster assignments

- Hoppe urn / Blackwell-MacQueen urn



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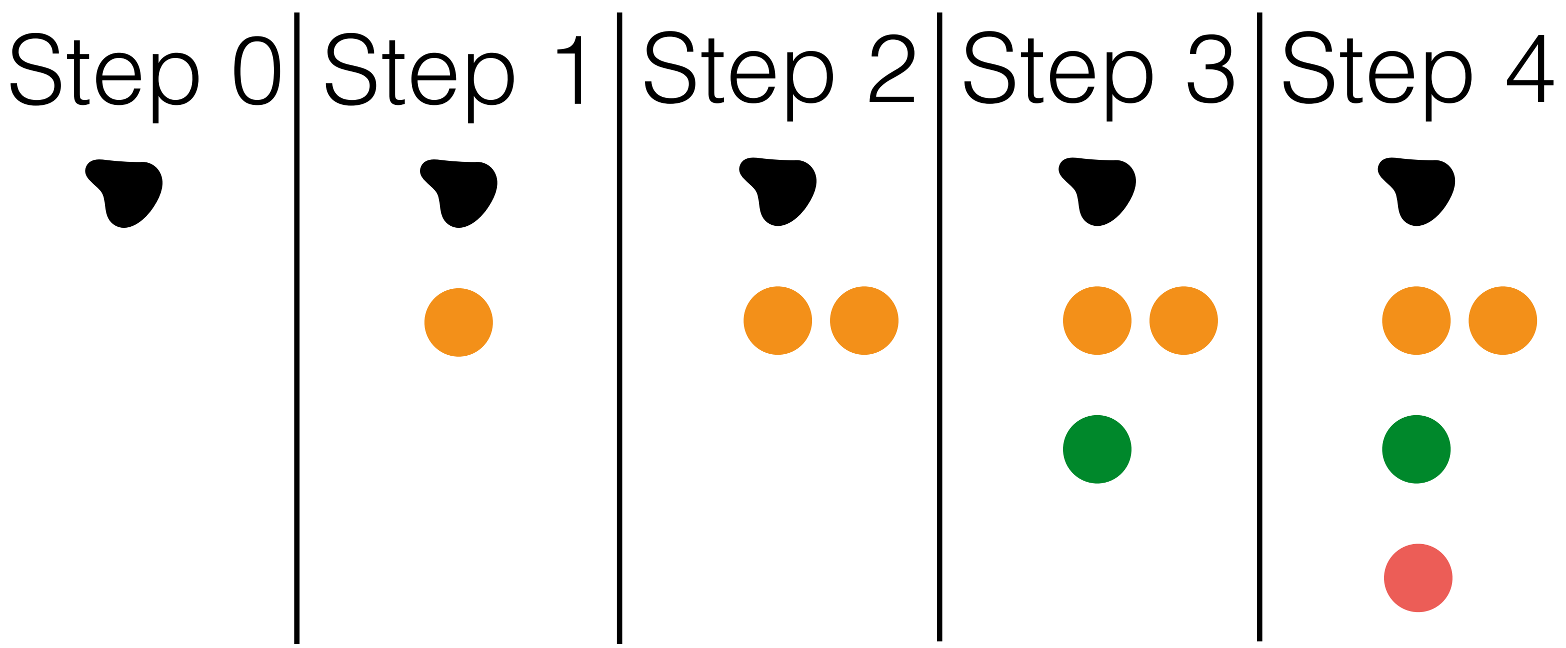

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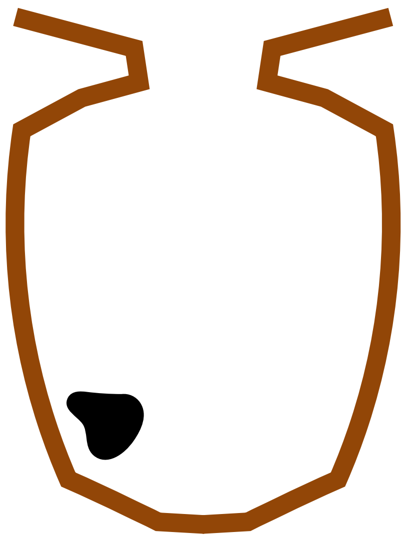
Marginal cluster assignments

- Hoppe urn / Blackwell-MacQueen urn



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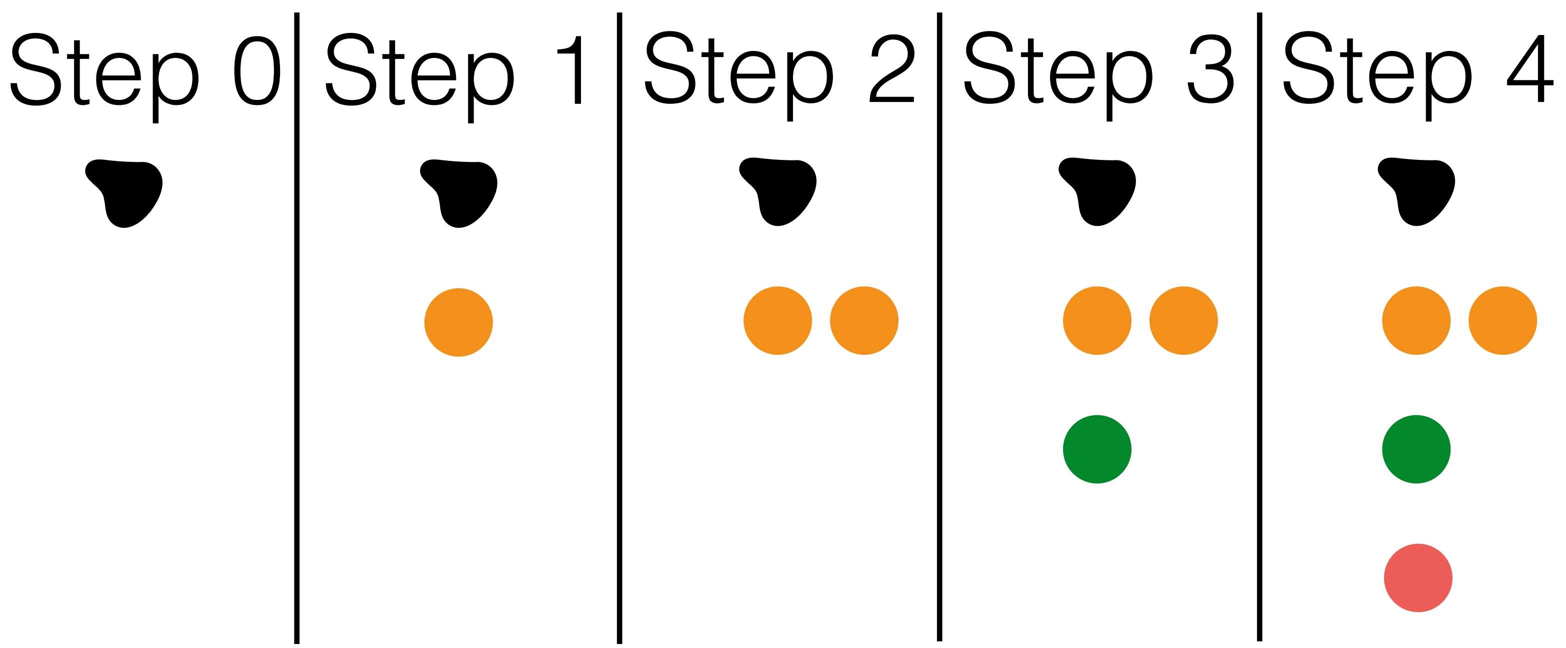

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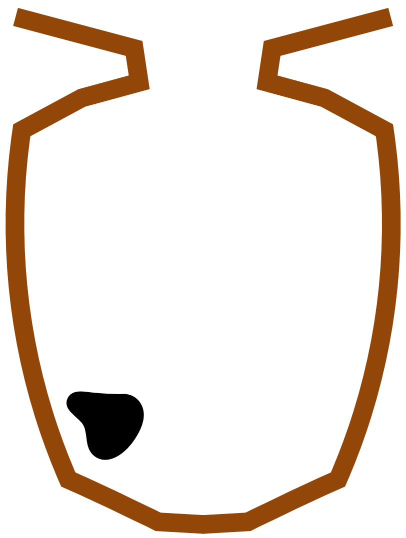
Marginal cluster assignments

- Hoppe urn / Blackwell-MacQueen urn



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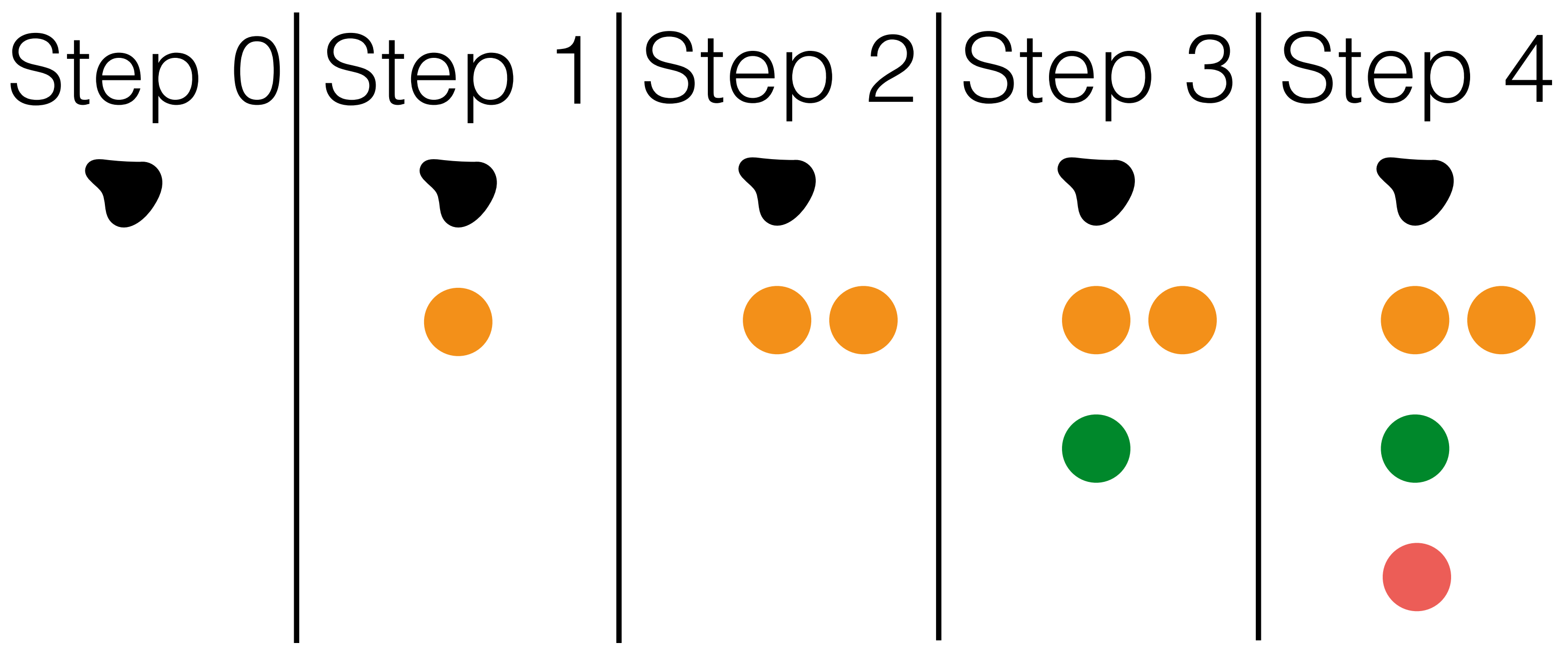

$V_k \overset{iid}{\sim} \text{Beta}(1, \alpha)$

$(\#\text{orange}, \#\text{other}) = \text{PolyaUrn}(1, \alpha)$

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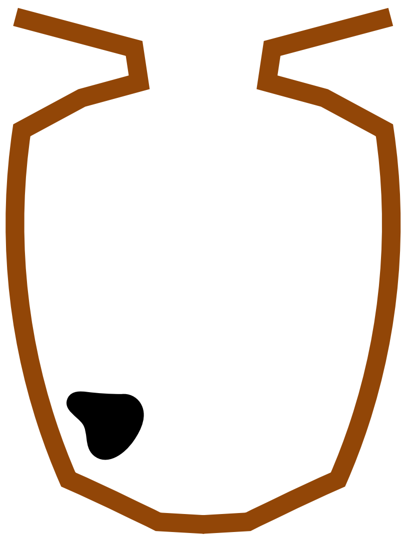
Marginal cluster assignments

- Hoppe urn / Blackwell-MacQueen urn



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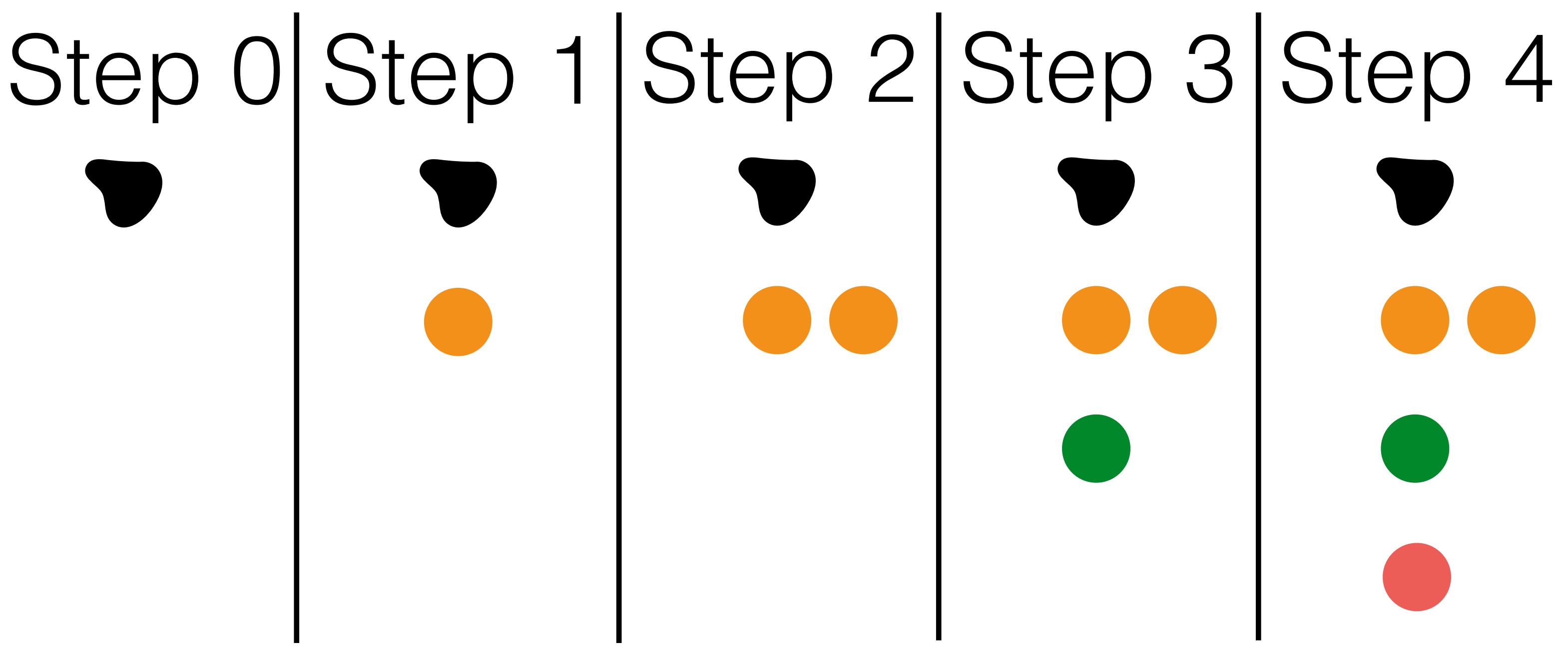

$$V_k \stackrel{iid}{\sim} \text{Beta}(1, \alpha)$$

$$\rho_1 = V_1$$

$$(\#\text{orange}, \#\text{other}) = \text{PolyaUrn}(1, \alpha)$$

- not orange: $(\#\text{green}, \#\text{other}) = \text{PolyaUrn}(1, \alpha)$
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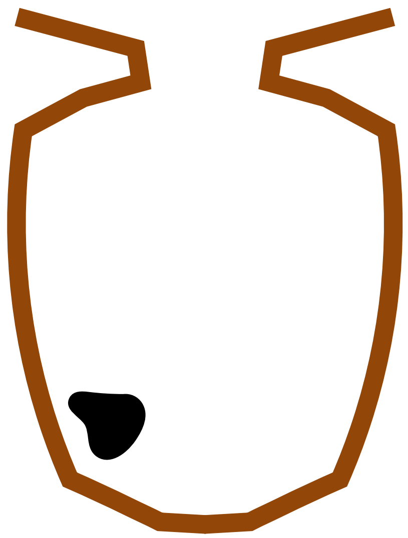
Marginal cluster assignments

- Hoppe urn / Blackwell-MacQueen urn



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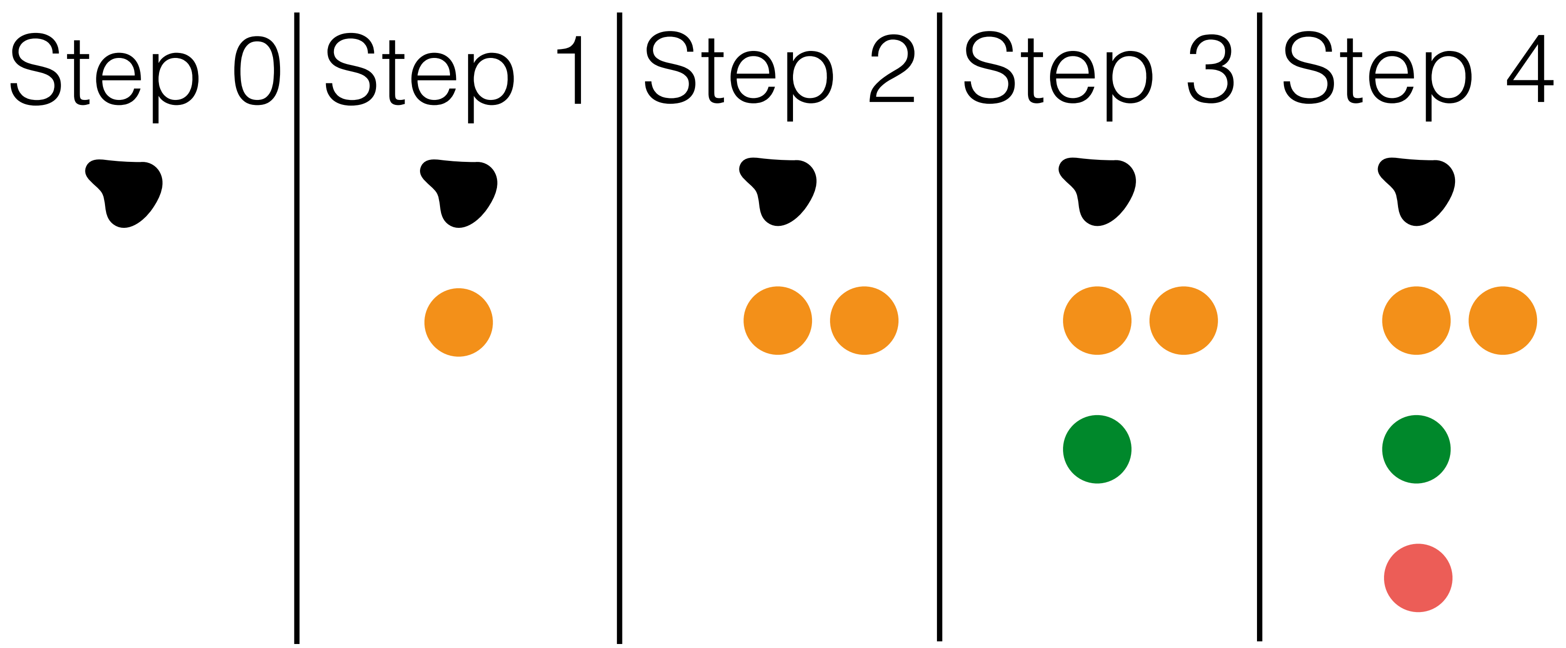

$$V_k \overset{iid}{\sim} \text{Beta}(1, \alpha)$$

$$\rho_1 = V_1$$

$$\rho_2 = (1 - V_1)V_2$$

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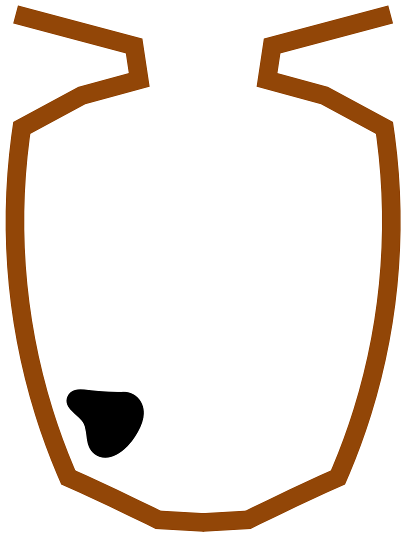
Marginal cluster assignments

- Hoppe urn / Blackwell-MacQueen urn



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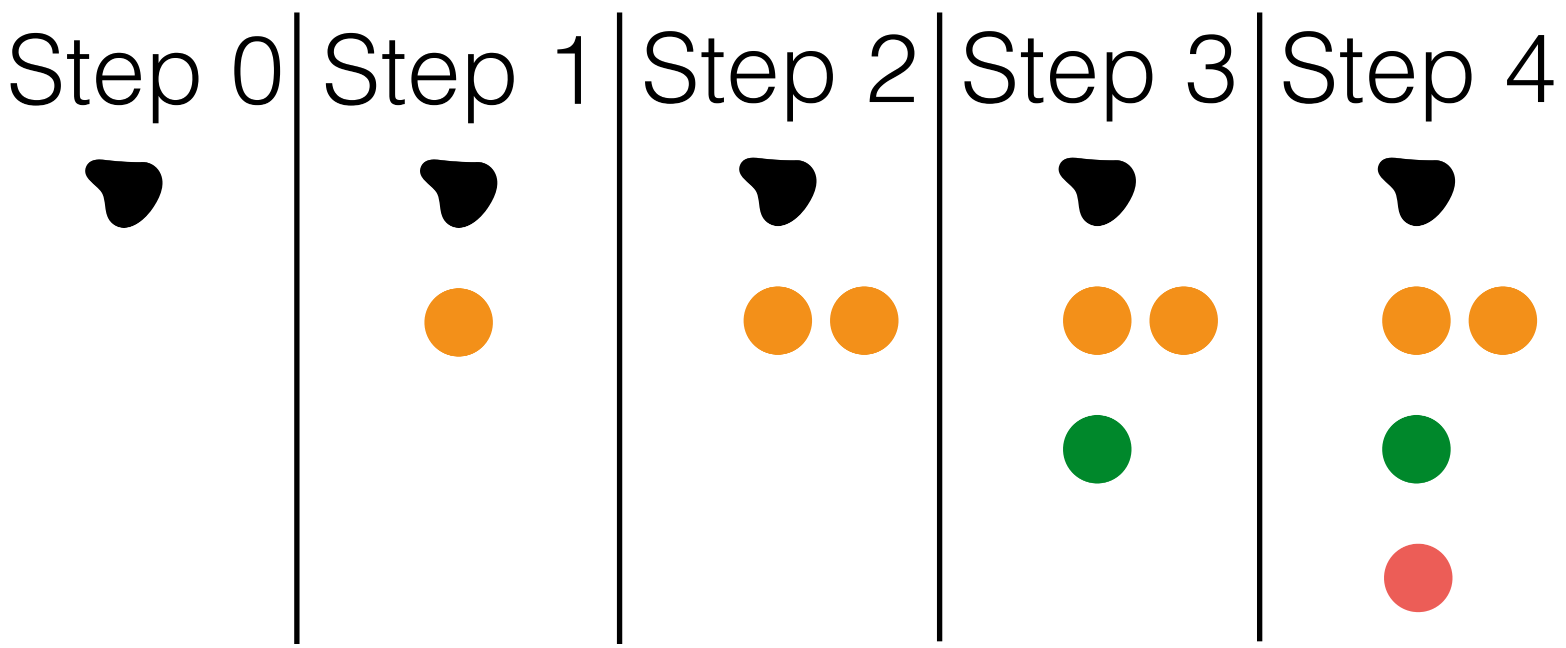

$$V_k \overset{iid}{\sim} \text{Beta}(1, \alpha)$$

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$$\rho_2 = (1 - V_1)V_2$$

$$\rho_3 = [\prod_{k=1}^{2}(1 - V_k)]V_3$$

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Chinese restaurant process

Chinese restaurant process

- Same thing we just did

Chinese restaurant process

- Same thing we just did
- Each customer walks into the restaurant

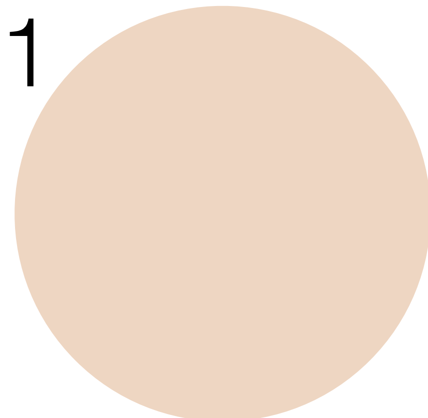
Chinese restaurant process

- Same thing we just did
- Each customer walks into the restaurant
  - Sits at existing table with prob proportional to # people there

Chinese restaurant process

- Same thing we just did
- Each customer walks into the restaurant
  - Sits at existing table with prob proportional to # people there
  - Forms new table with prob proportional to α

Chinese restaurant process



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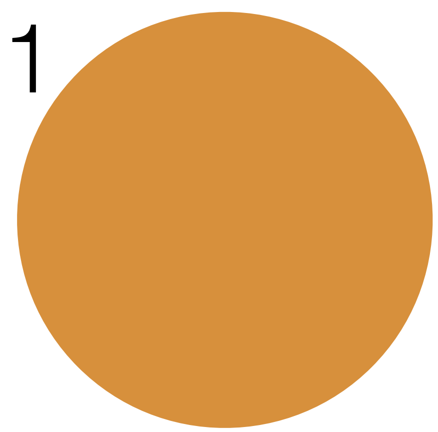
Chinese restaurant process



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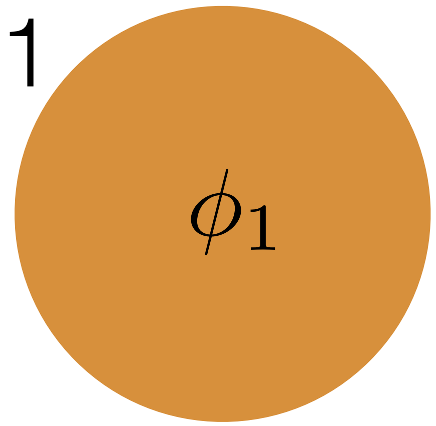

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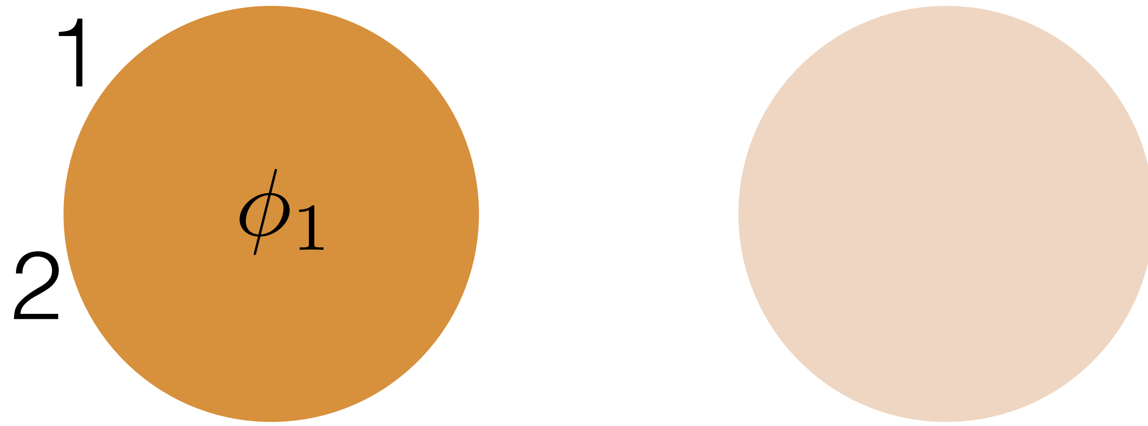

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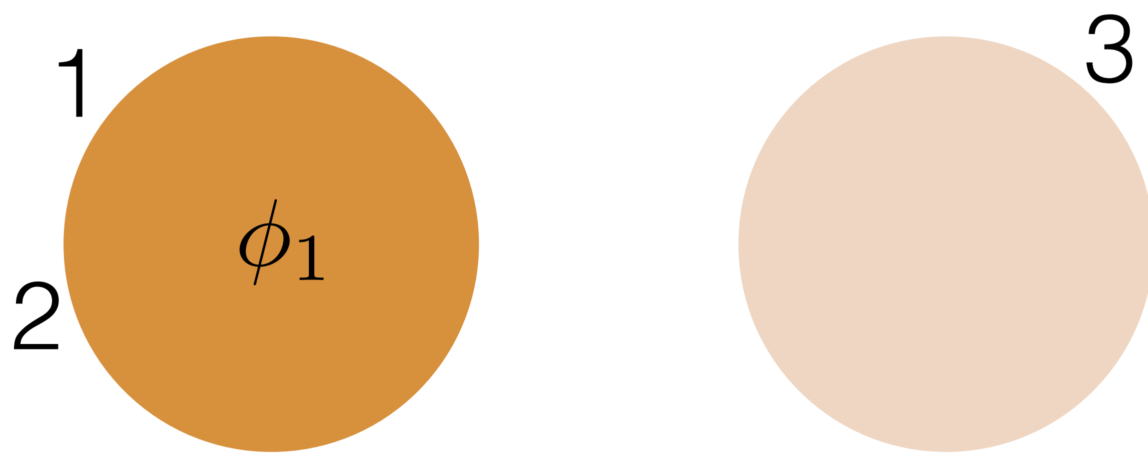

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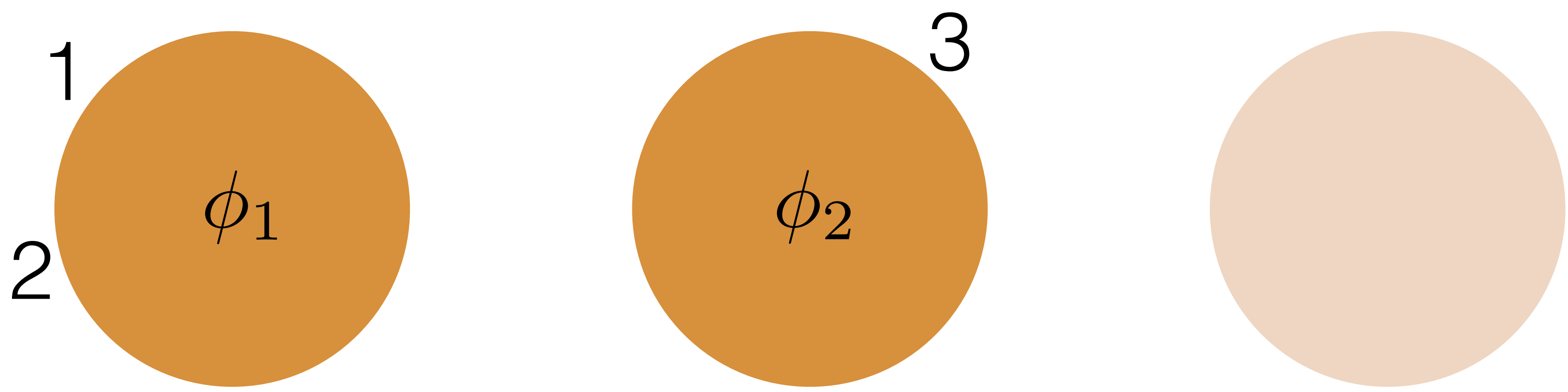

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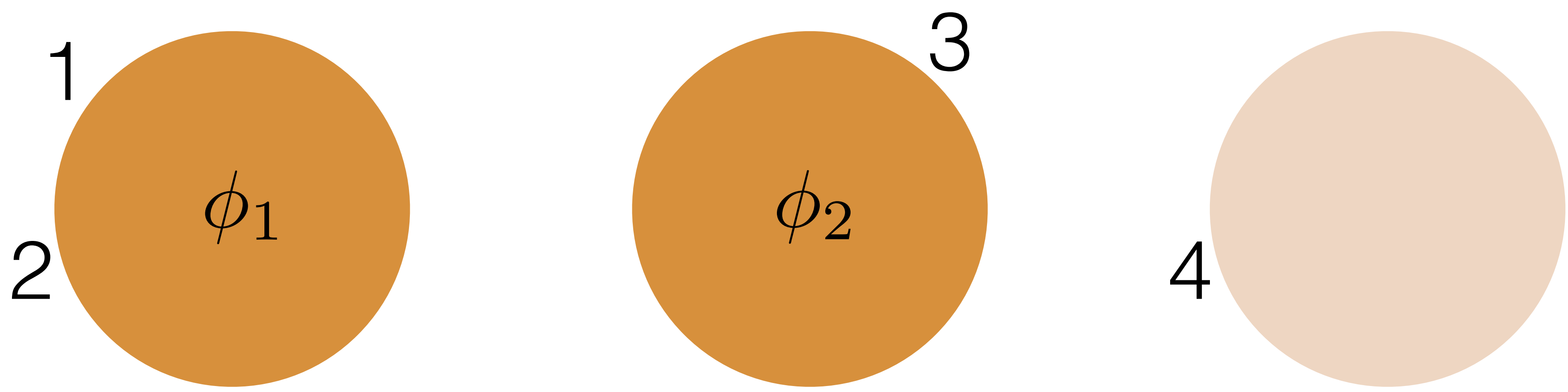

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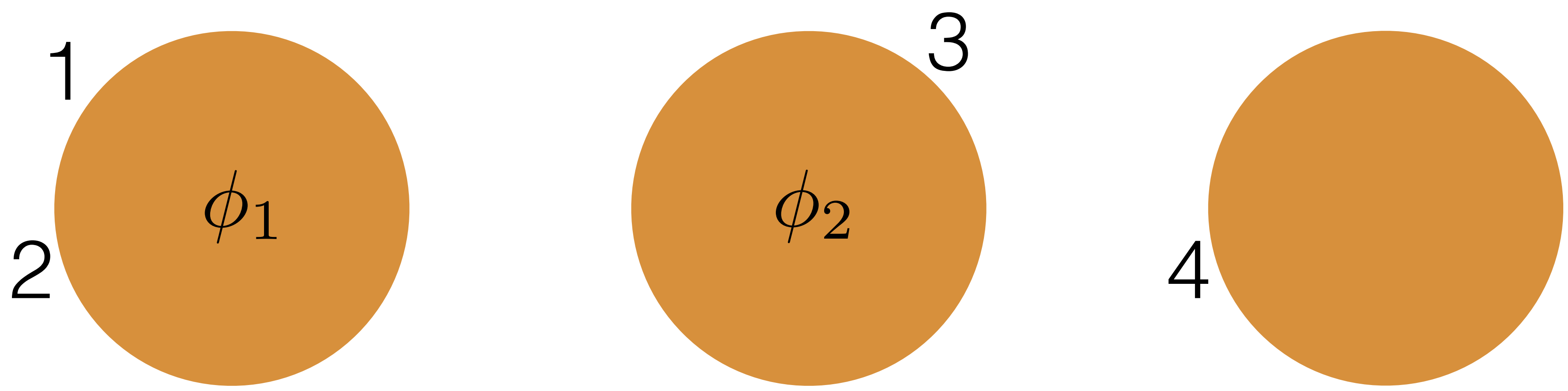
Chinese restaurant process



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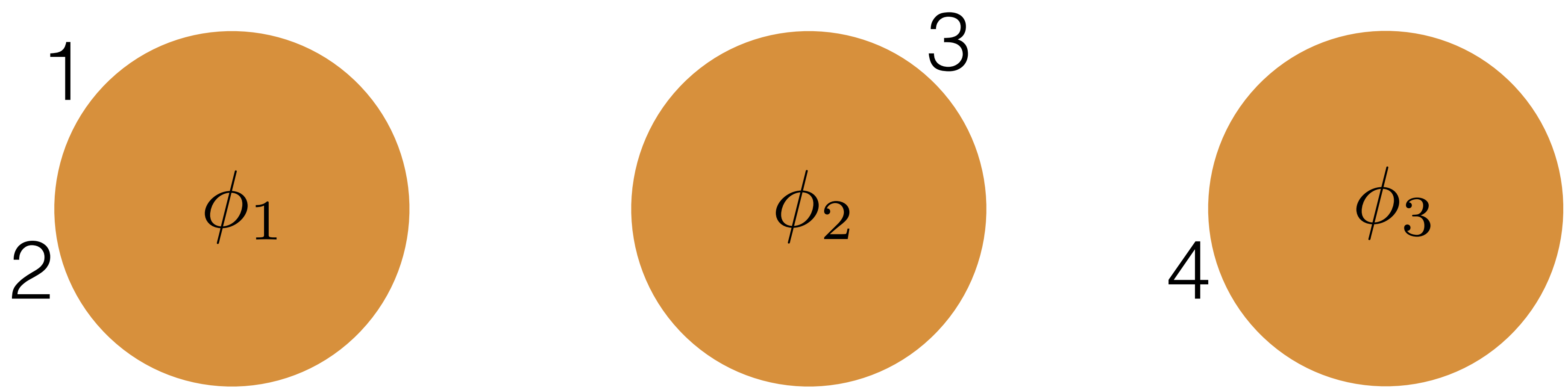

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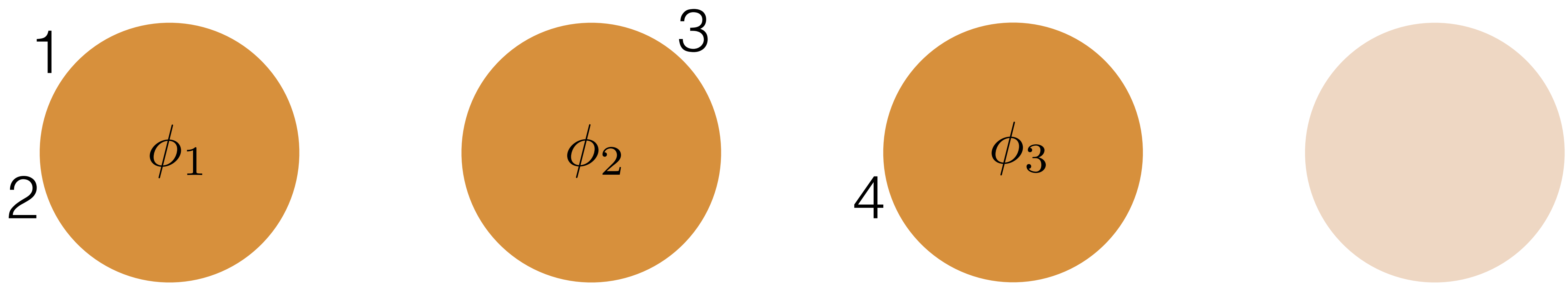

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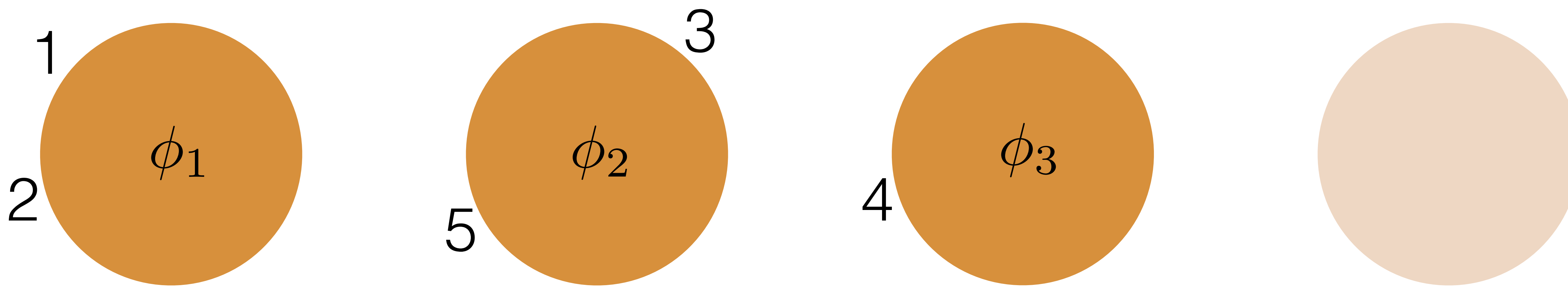

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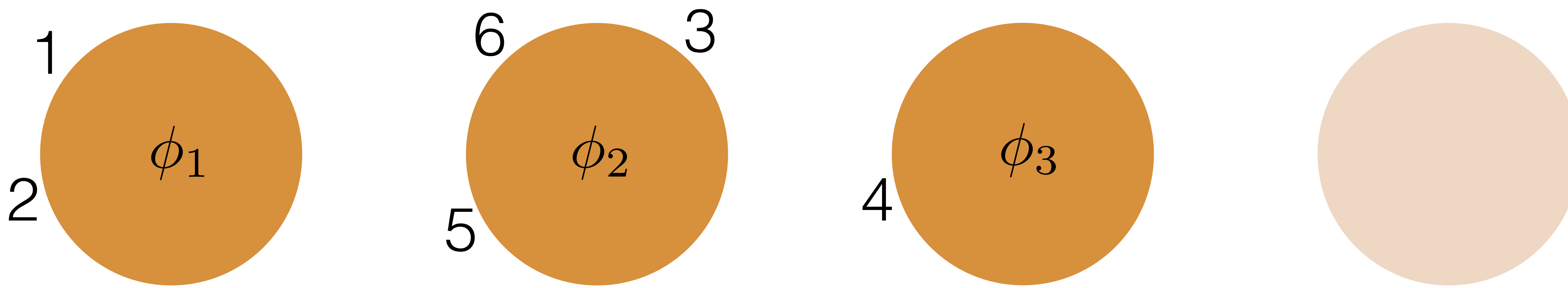

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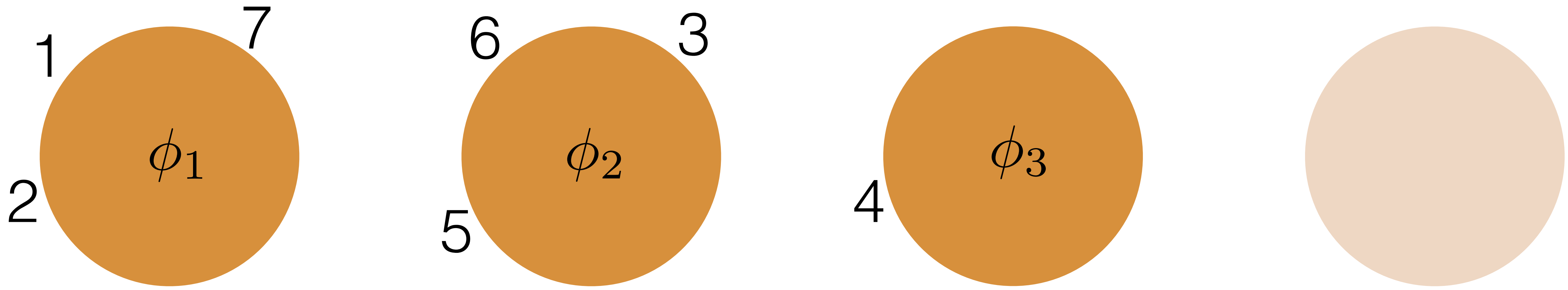
Chinese restaurant process



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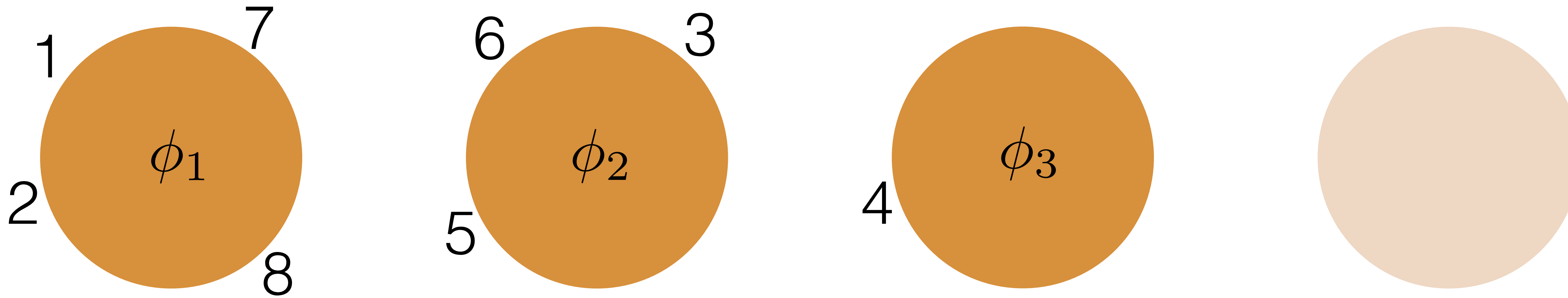

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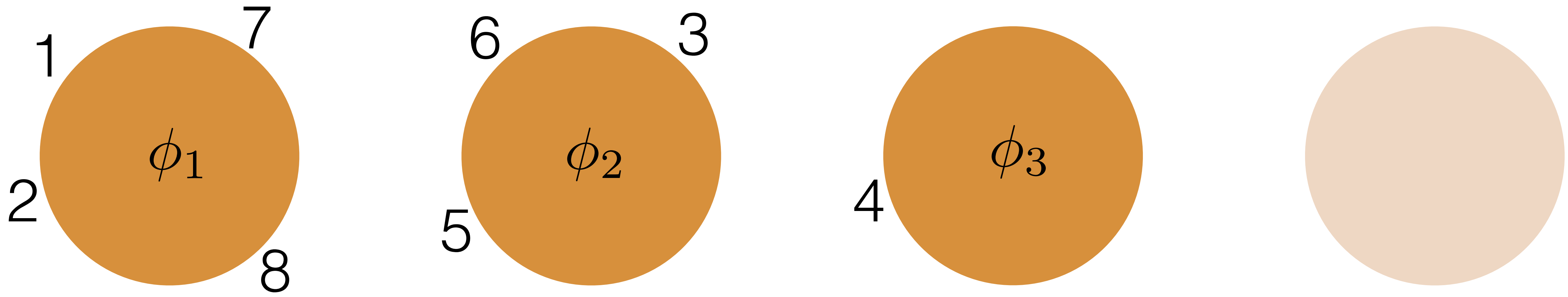
Chinese restaurant process



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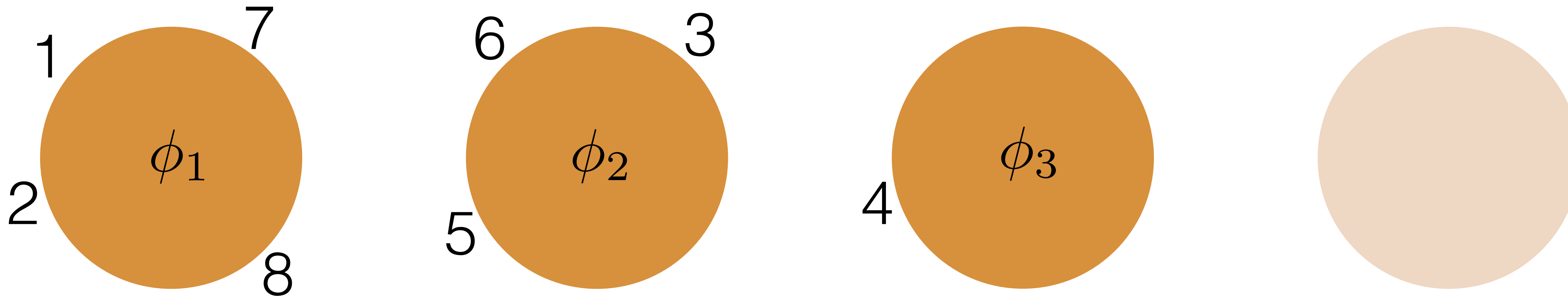
[Aldous 1983]

Chinese restaurant process



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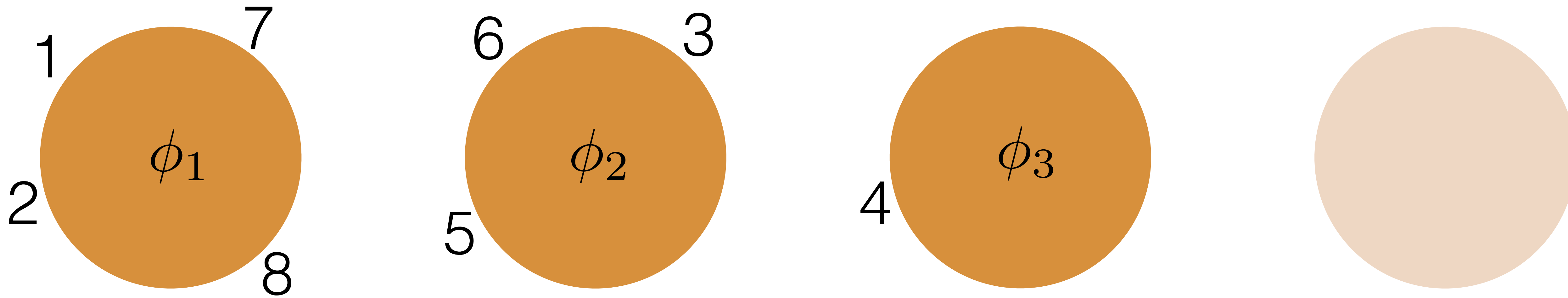
- Marginal for the Categorical likelihood with GEM prior

Chinese restaurant process



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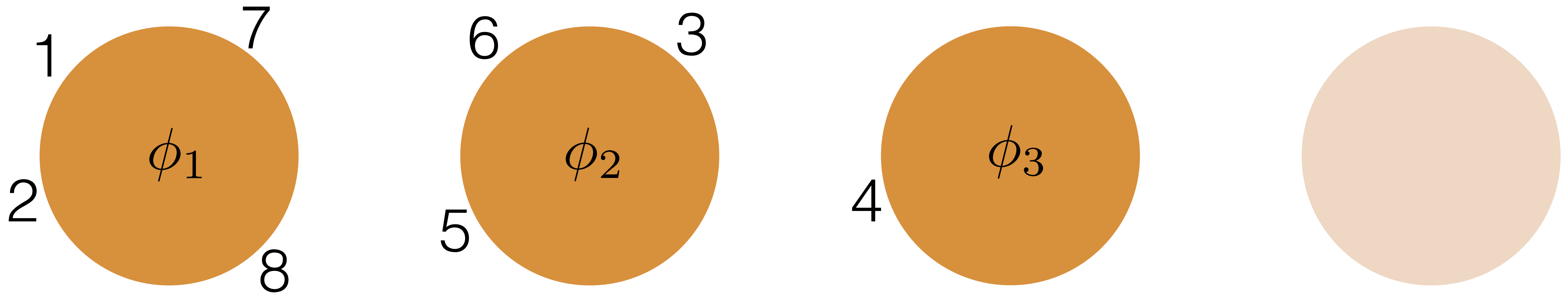
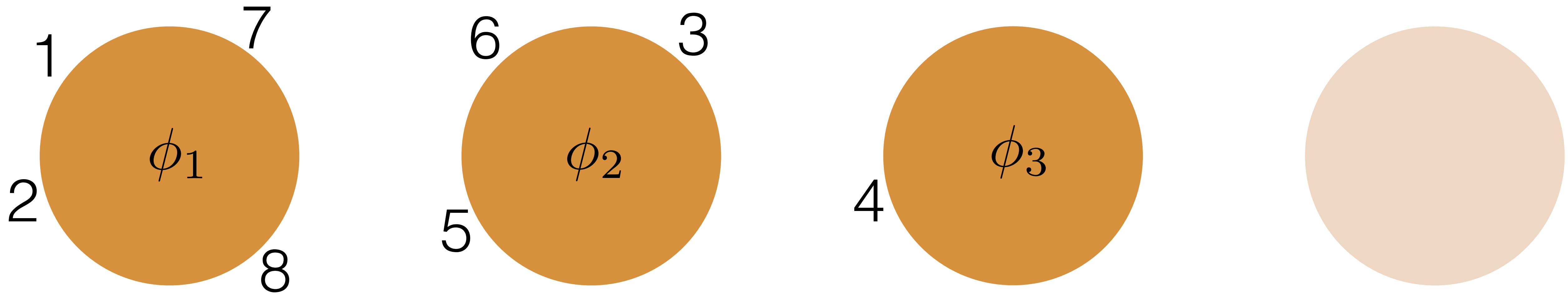
So far: Dirichlet process, Chinese restaurant process

- Infinity of parameters, growing number of parameters

Exercises

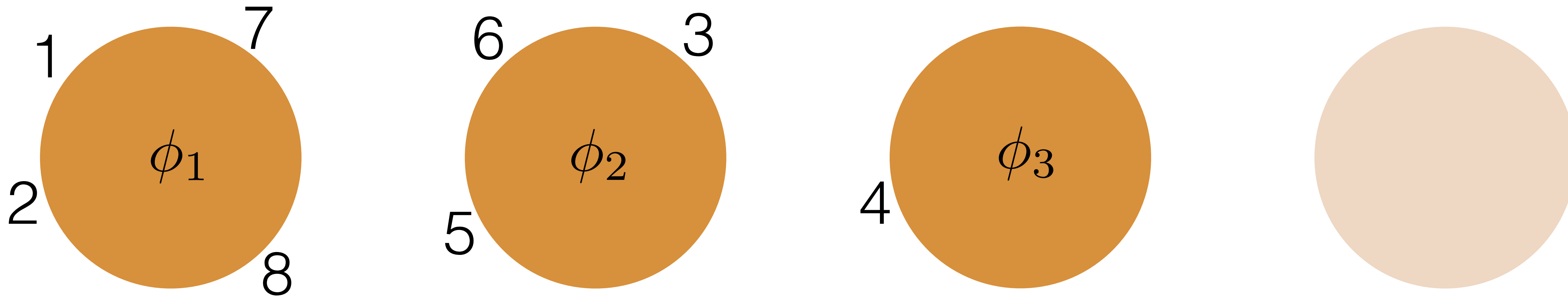


Exercises



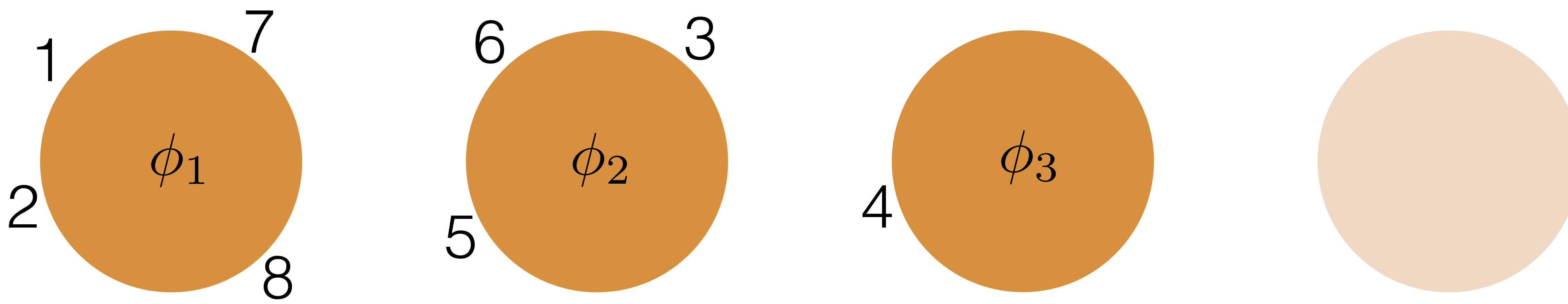
- Review Gibbs sampling

Exercises



- Review Gibbs sampling
- What are the advantages and disadvantages of the DP and CRP representations?

Exercises



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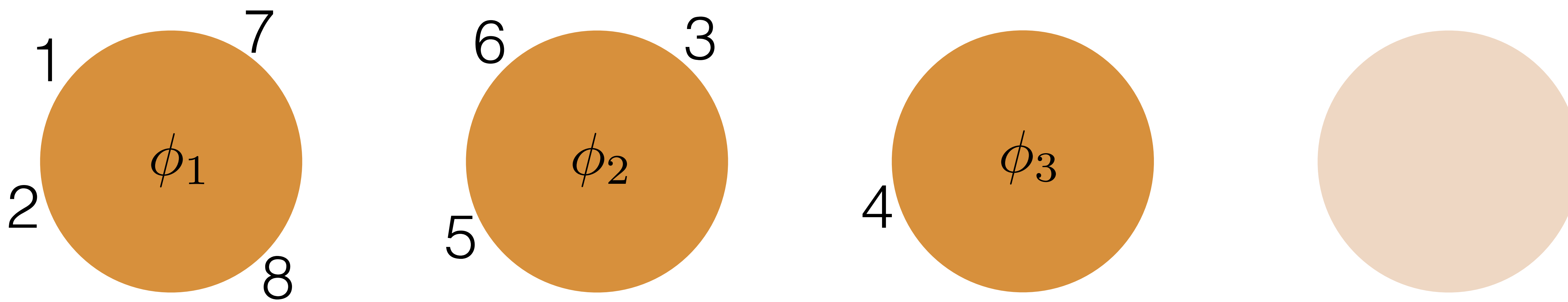
- What are the advantages and disadvantages of the DP and CRP representations?
- What is the expected number of clusters generated by a CRP(α) after *N* data points?

Exercises



- Review Gibbs sampling
- What are the advantages and disadvantages of the DP and CRP representations?
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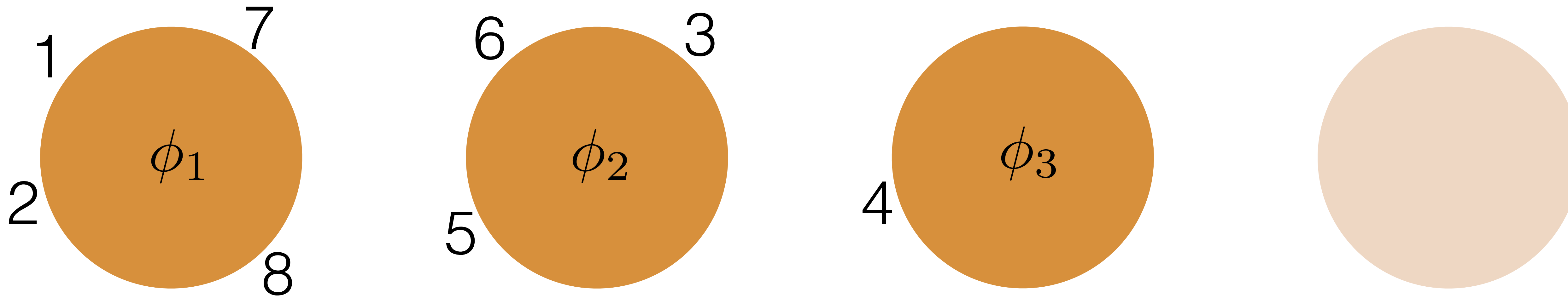
- What do you think about the answer to the previous question when it comes to real-life data modeling?

Exercises



- Review Gibbs sampling
- What are the advantages and disadvantages of the DP and CRP representations?
- What is the expected number of clusters generated by a CRP(α) after *N* data points?
- What do you think about the answer to the previous question when it comes to real-life data modeling?
- Code a CRP sampler. Examine the empirical distribution of the number of clusters after *N* customers.

References

A full reference list is provided at the end of the “Part III” slides.