Nonparametric Bayesian Methods: Part IV

Tamara Broderick
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Electrical Engineering & Computer Science
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[slides, code: http://www.tamarabroderick.com/tutorials.html]

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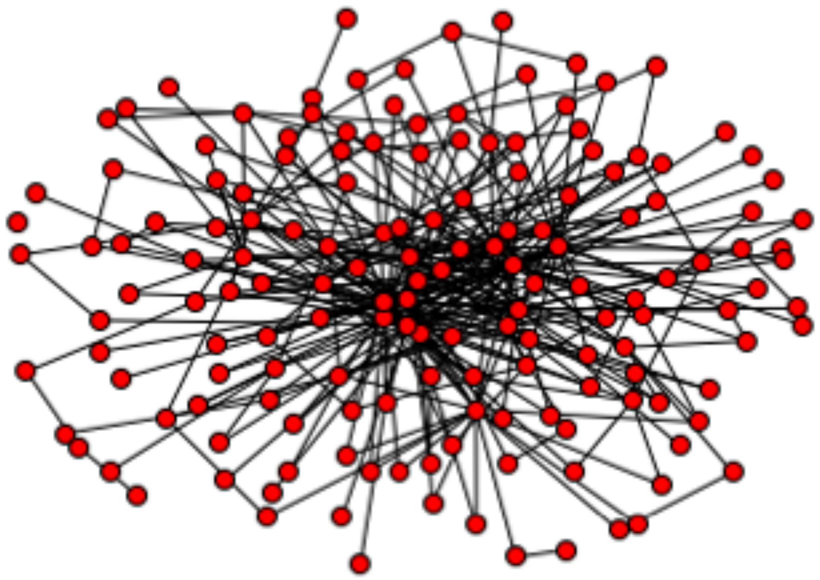
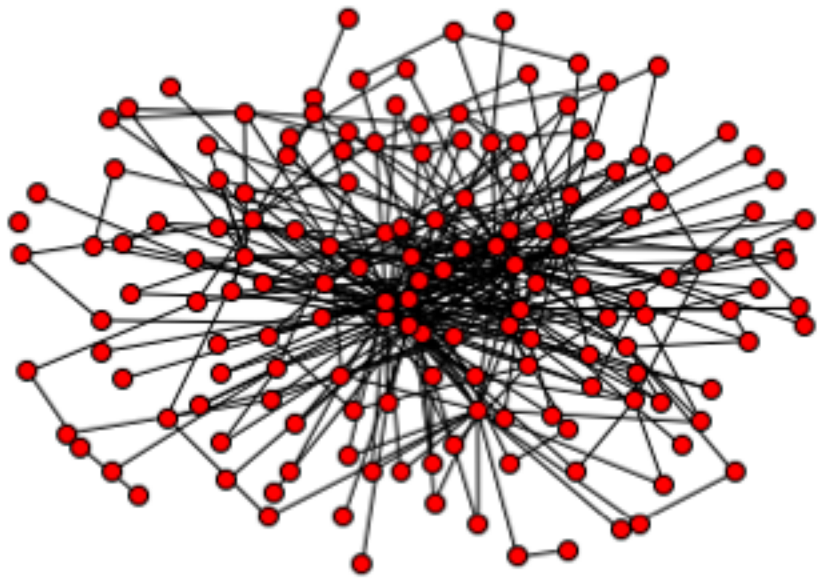


[Holland et al 1983; Kemp et al 2006; Xu et al 2007; Airoldi et al 2008; Lloyd et al 2012]



social: *Facebook, Twitter, email*
biological: *ecological, protein, gene*
transportation: *roads, railways*

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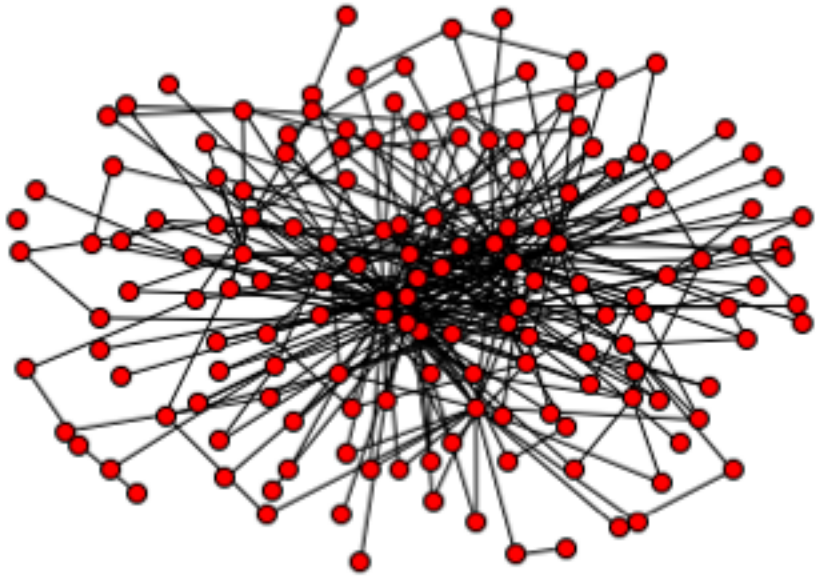
Probabilistic models for graphs

$p($  $)$

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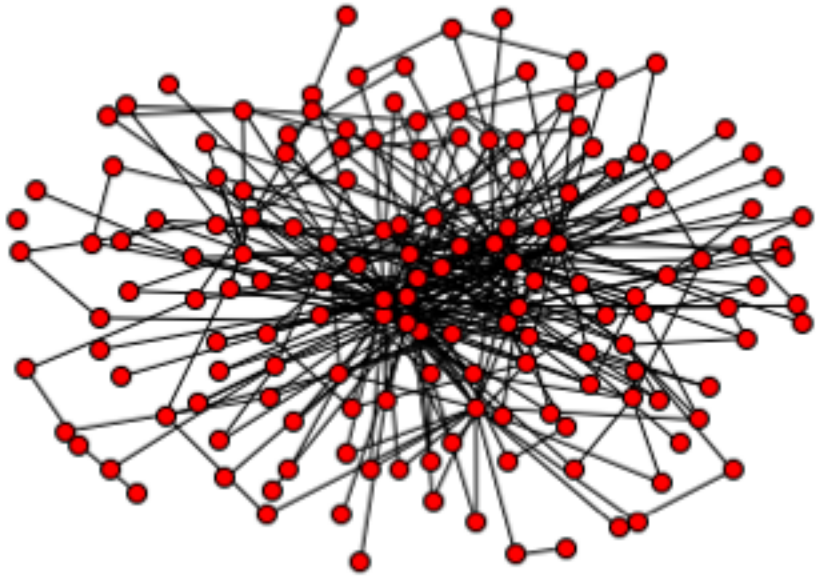
$p($  $)$

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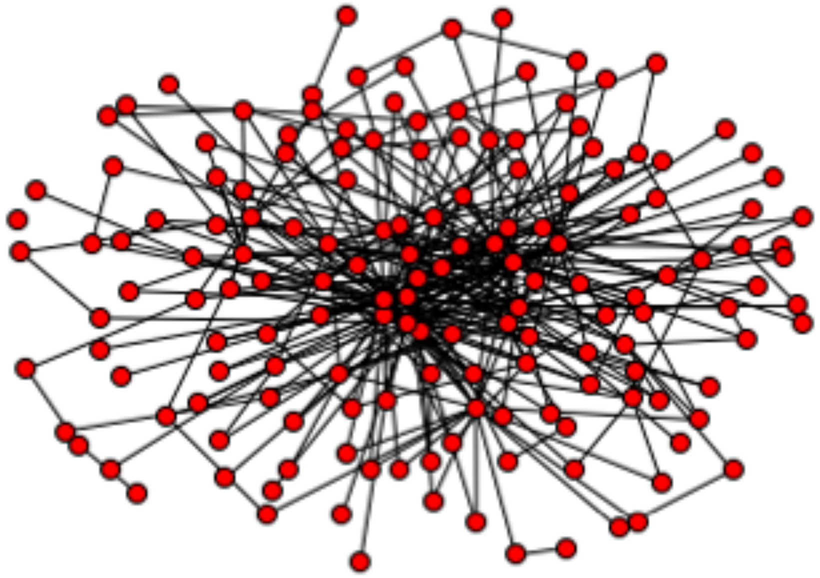
$p($  $)$

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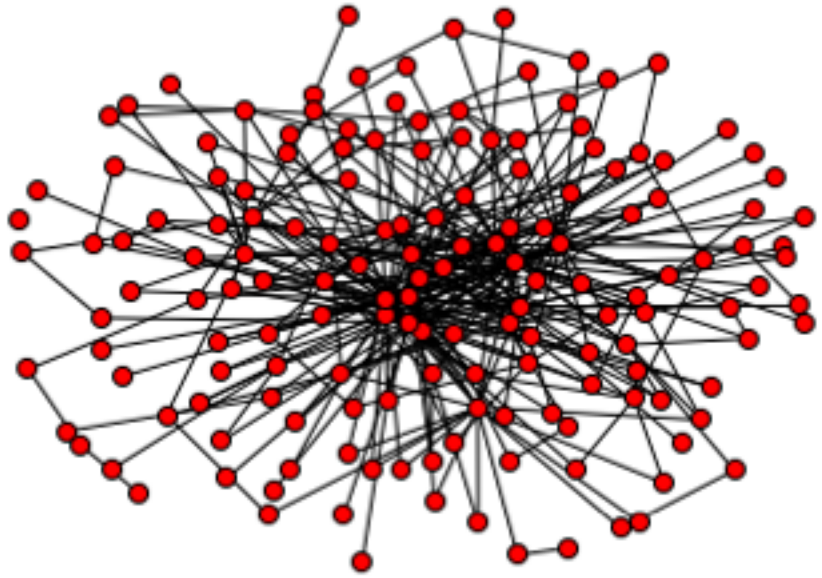
$p($  $)$

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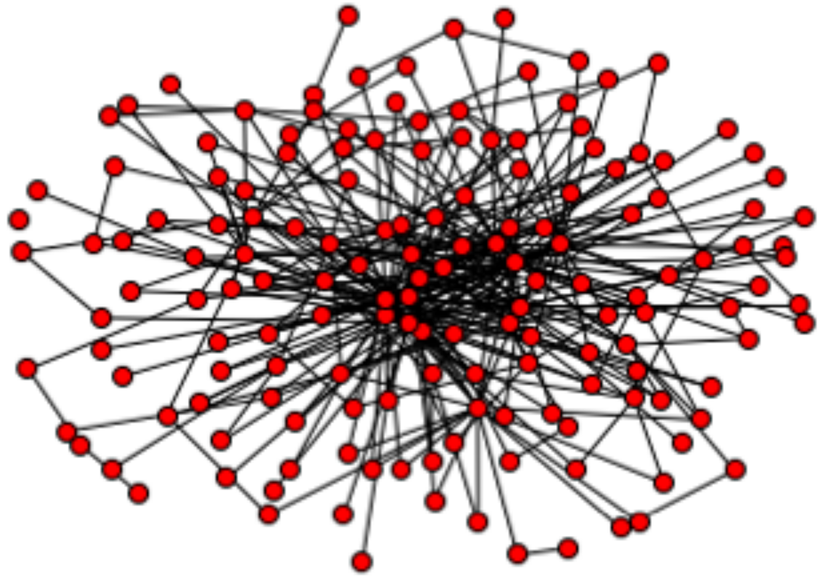
$p($  $)$

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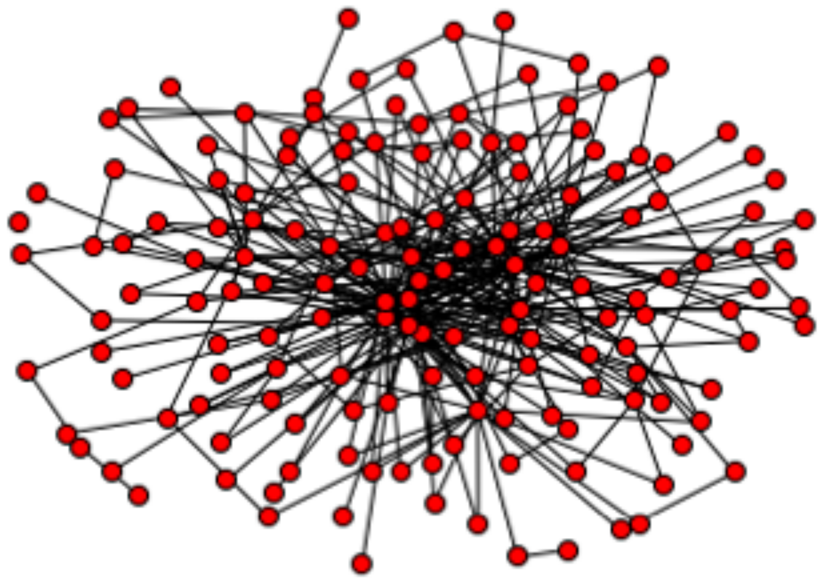
$p($  $)$

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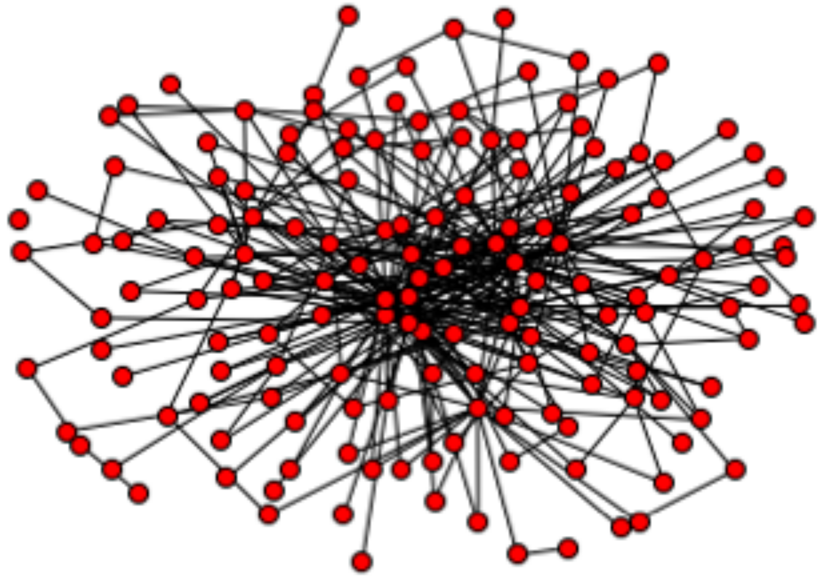
$p($  $)$

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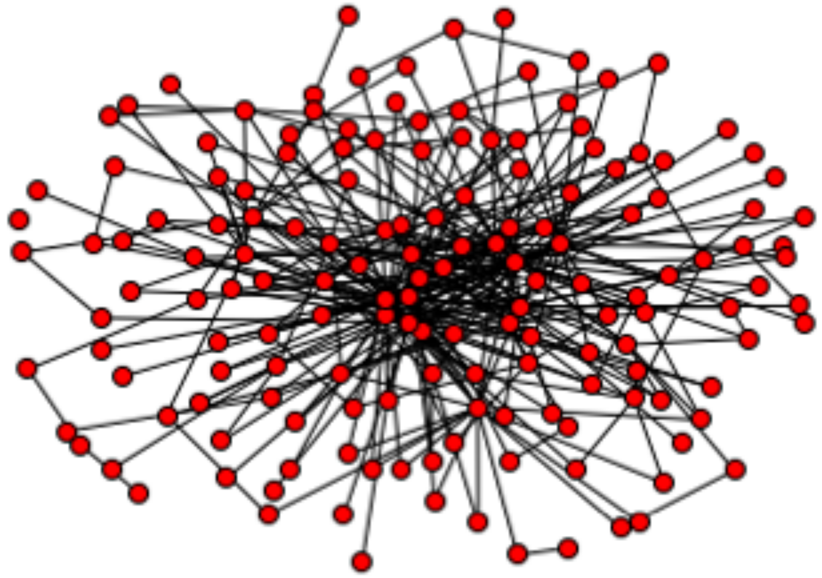
$p($  $)$

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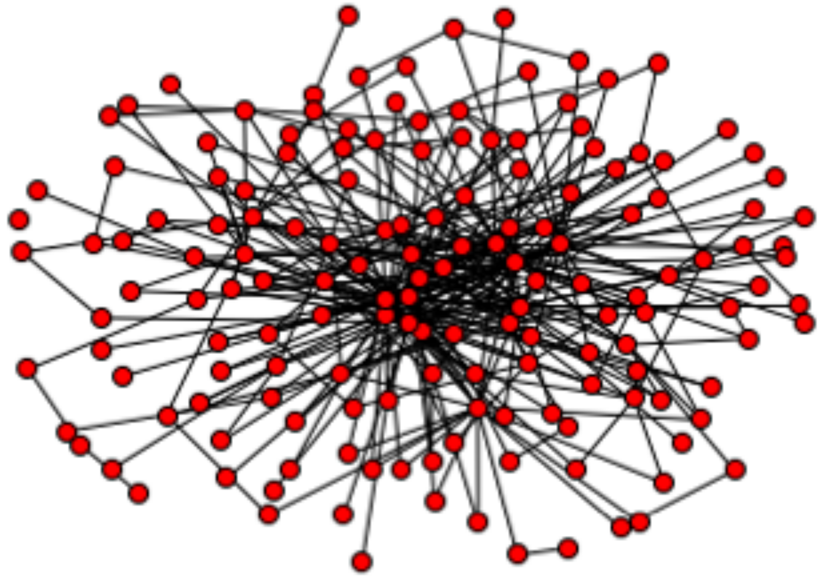
$p($  $)$

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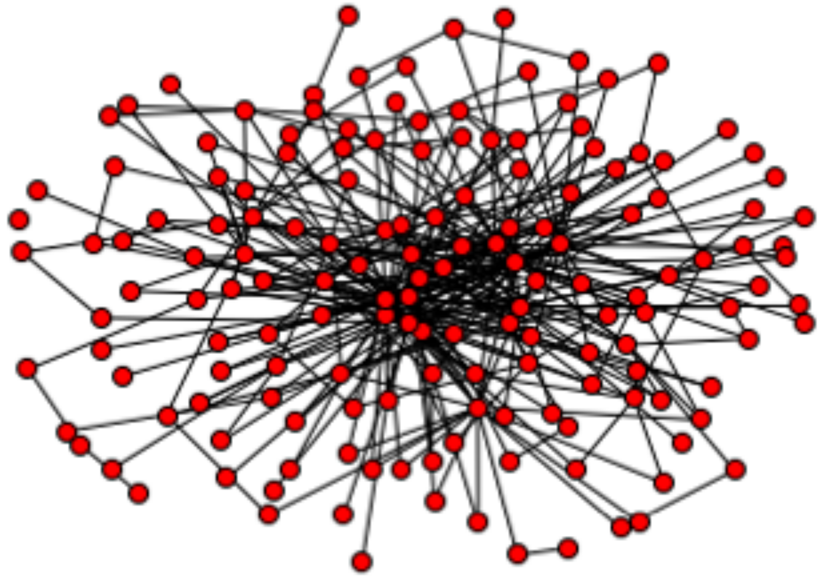
$p($  $)$

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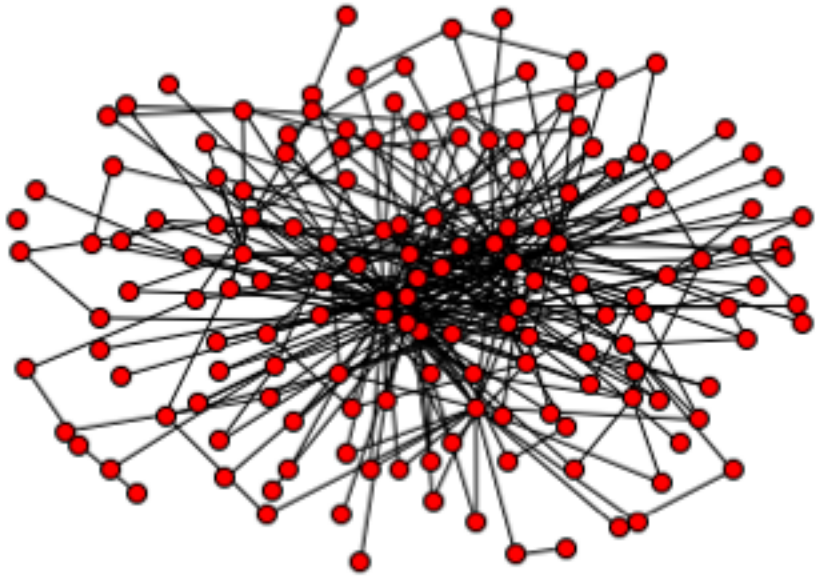
- **Solution**: a new framework for **sparse graphs**

[Holland et al 1983; Kemp et al 2006; Xu et al 2007; Airoldi et al 2008; Lloyd et al 2012]

Probabilistic models for graphs

$p($  $)$

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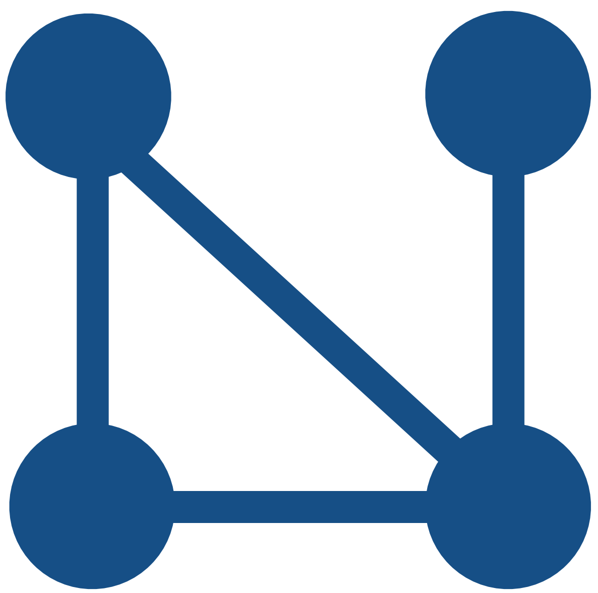
- Assume: Adding more data doesn't change distribution of earlier data (*projectivity*)
- **Problem**: model misspecification, dense graphs
- **Solution**: a new framework for sparse graphs
  - Concurrent & independent graphs work by Crane & Dempsey

1 [B, Cai 2015; Cai, B 2015a,b; Crane, Dempsey 2015a,b,16a,b; Cai, Campbell, B 2016; Campbell, Cai, B 2016]

Sequence of graphs



G

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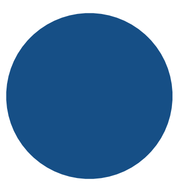
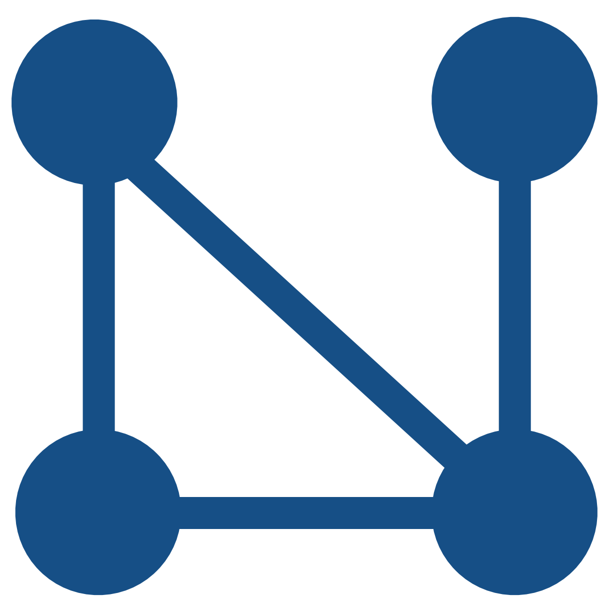


G_1



G

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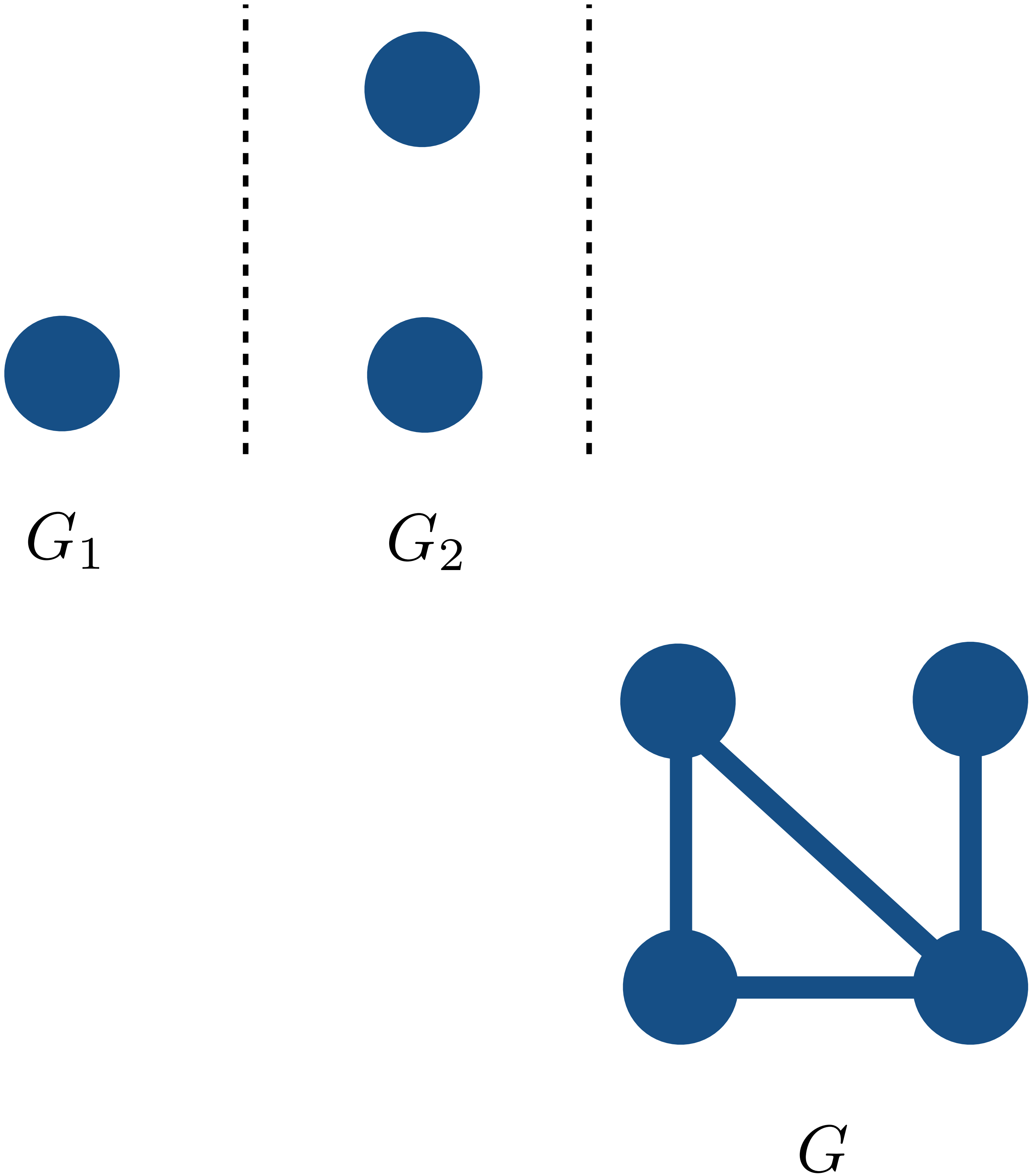

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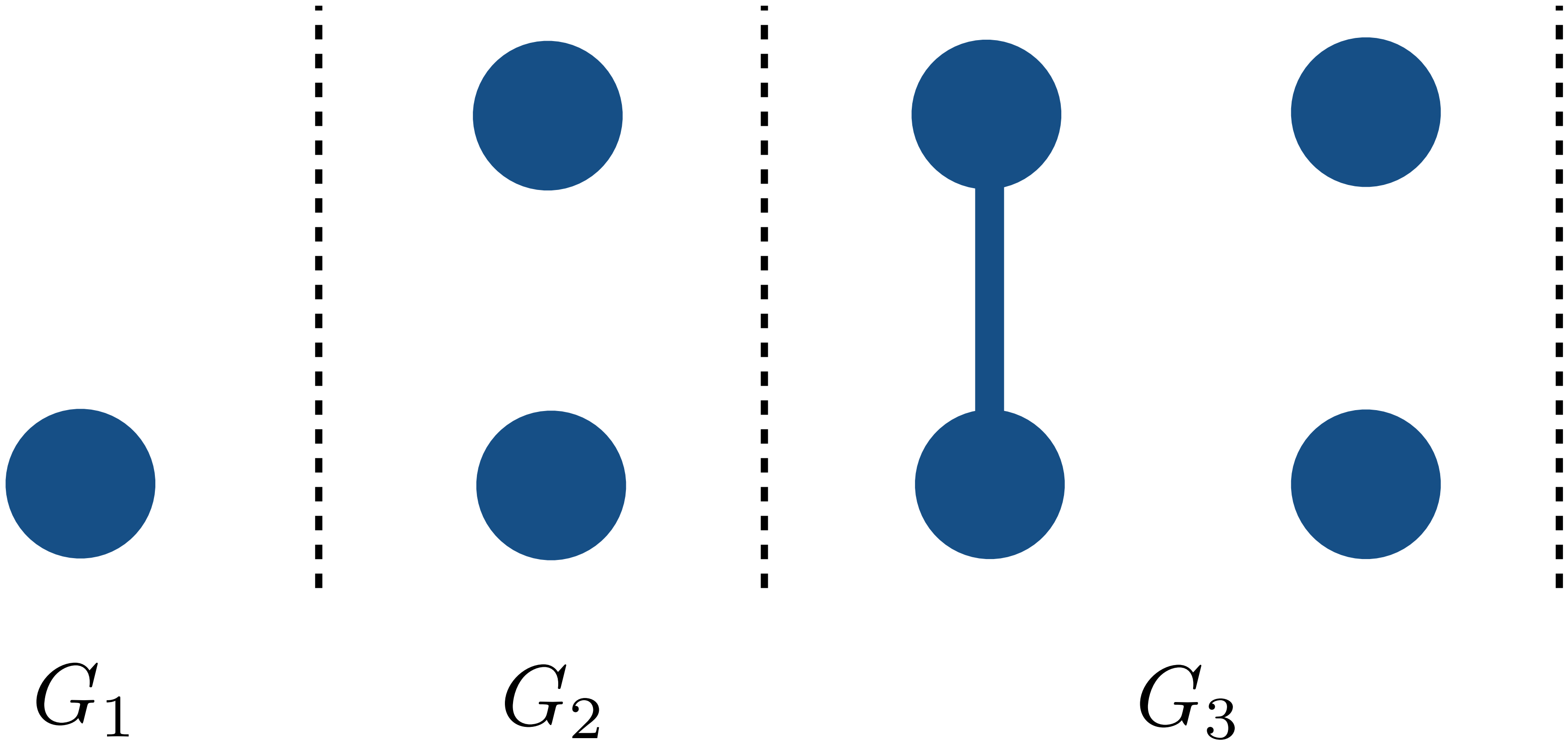

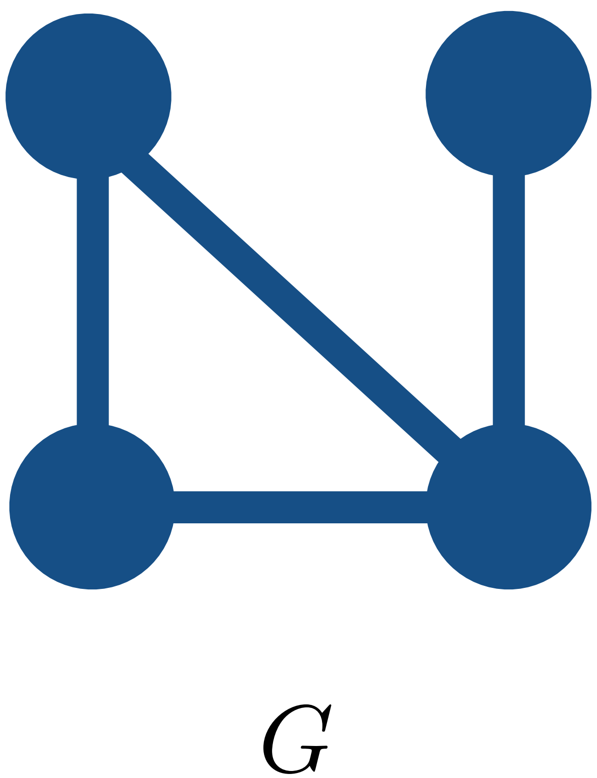

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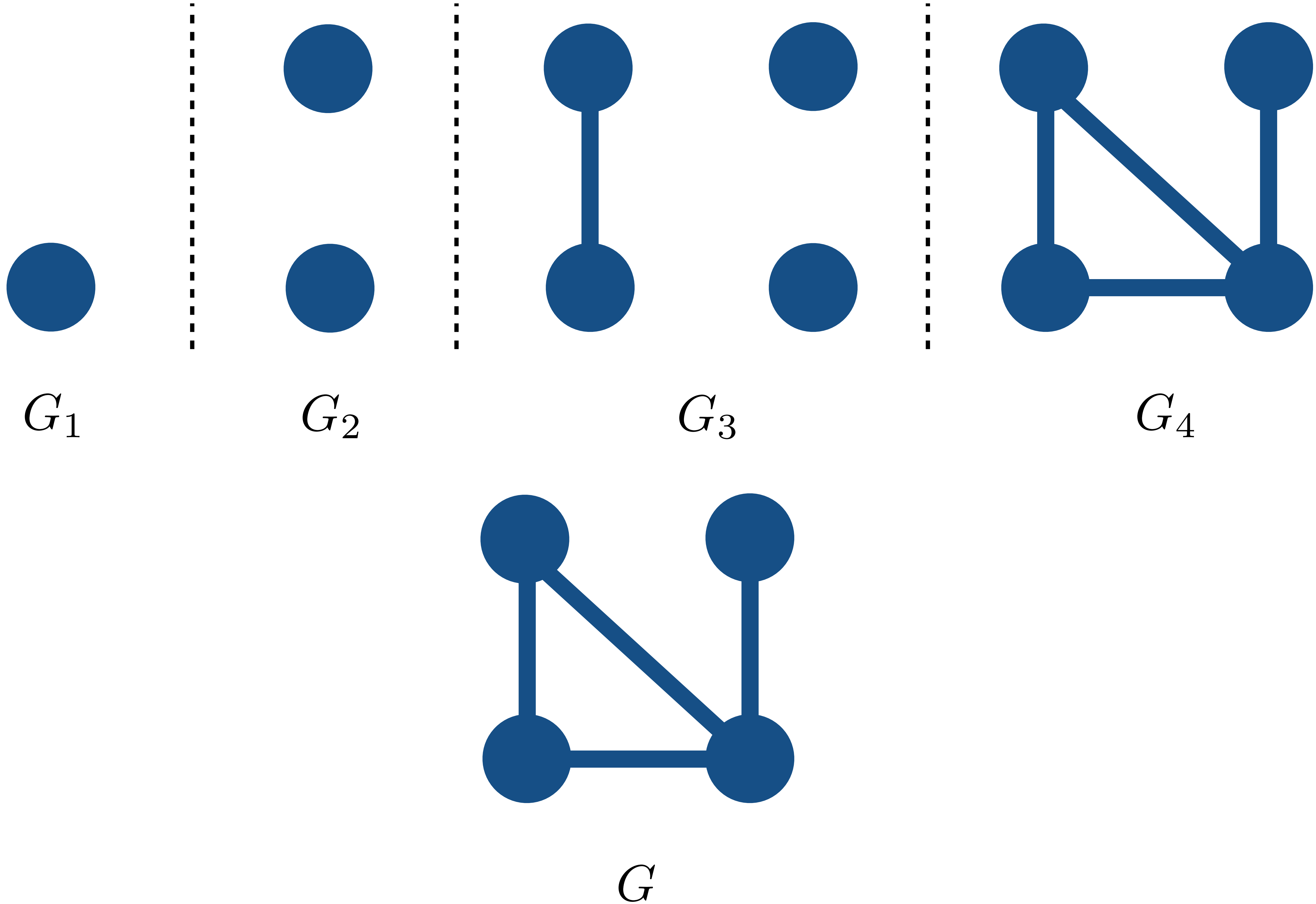

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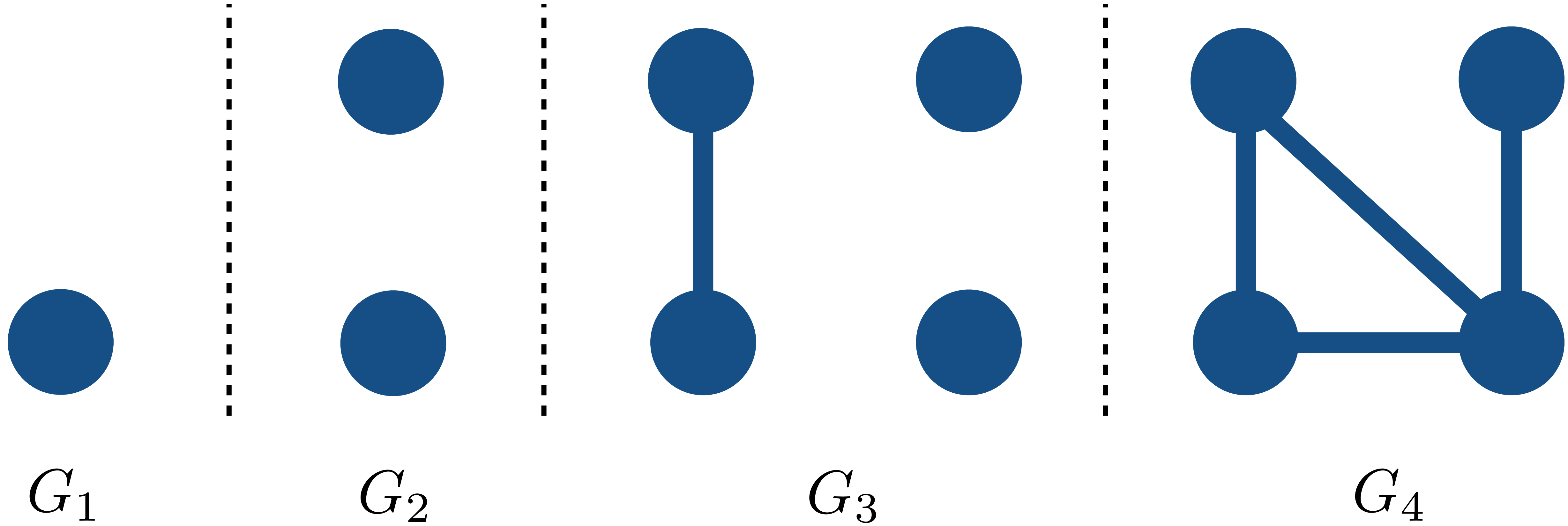

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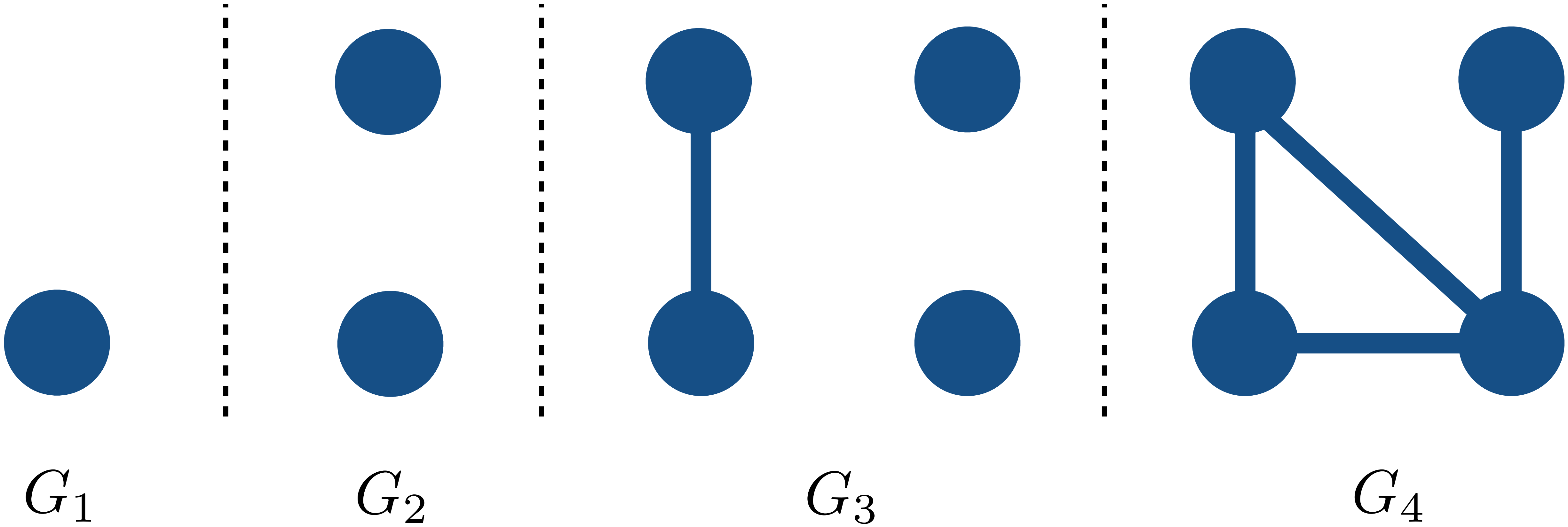
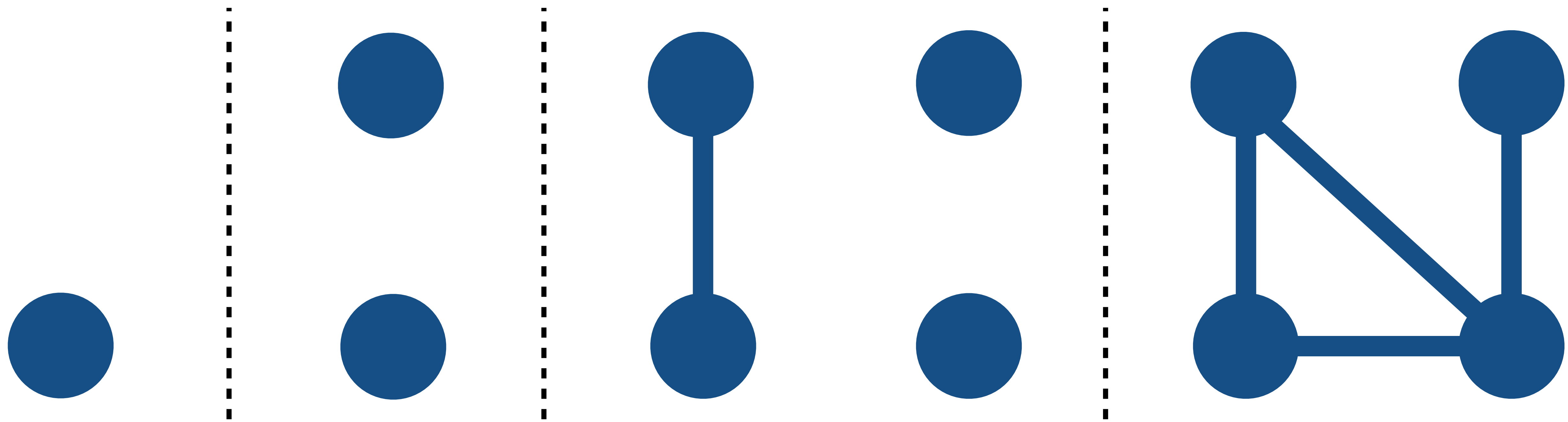


If $\#\text{nodes}(G_n) \to \infty$,

Sequence of graphs

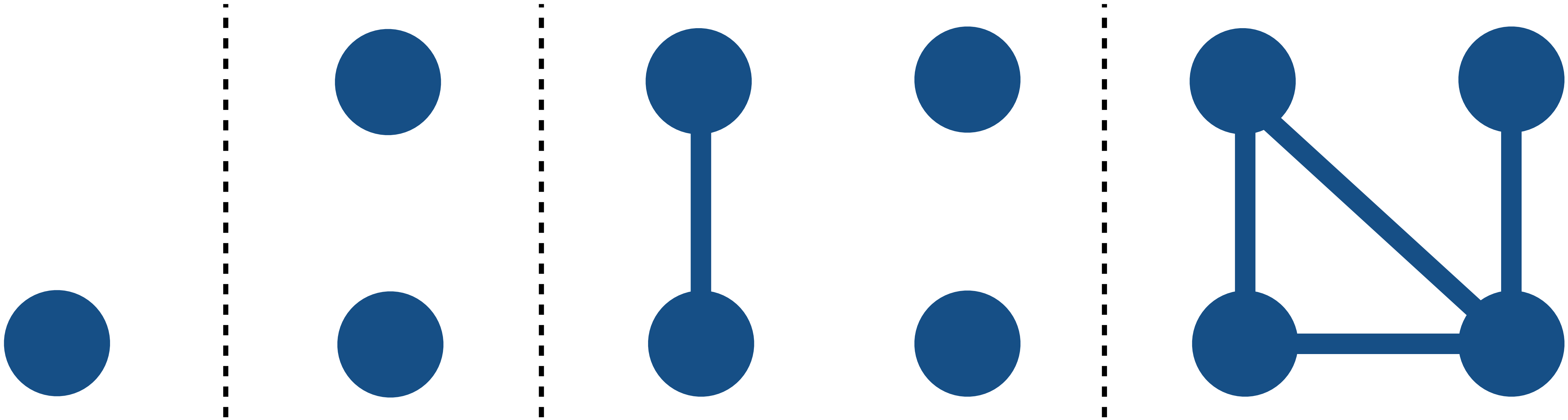


G_1 G_2 G_3 G_4

If $\#\text{nodes}(G_n) \to \infty$,

- *Dense* graph sequence $\#\text{edges}(G_n) \geq c \cdot [\#\text{nodes}(G_n)]^2$

Sequence of graphs



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If $\#\text{nodes}(G_n) \to \infty$,

- *Dense* graph sequence $\#\text{edges}(G_n) \geq c \cdot [\#\text{nodes}(G_n)]^2$
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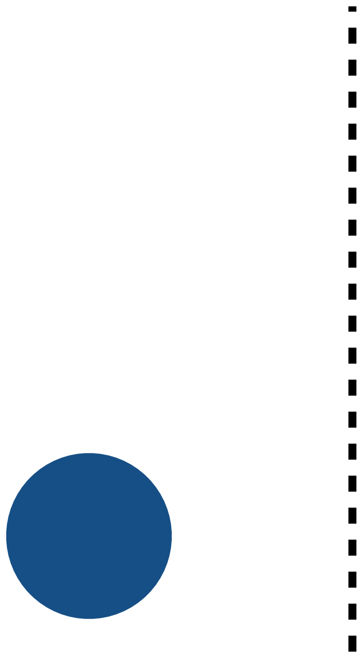
The Old Way: Nodes



G_1

[Hoover 1979, Aldous 1981]

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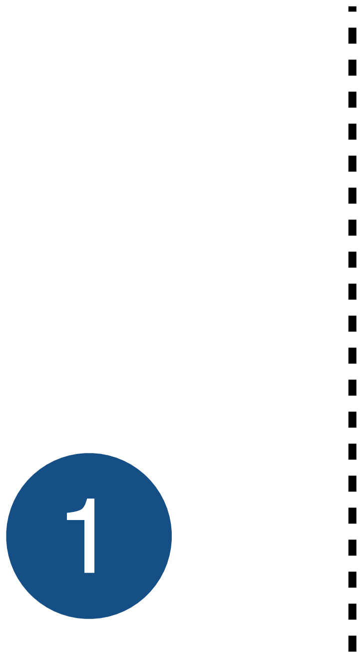

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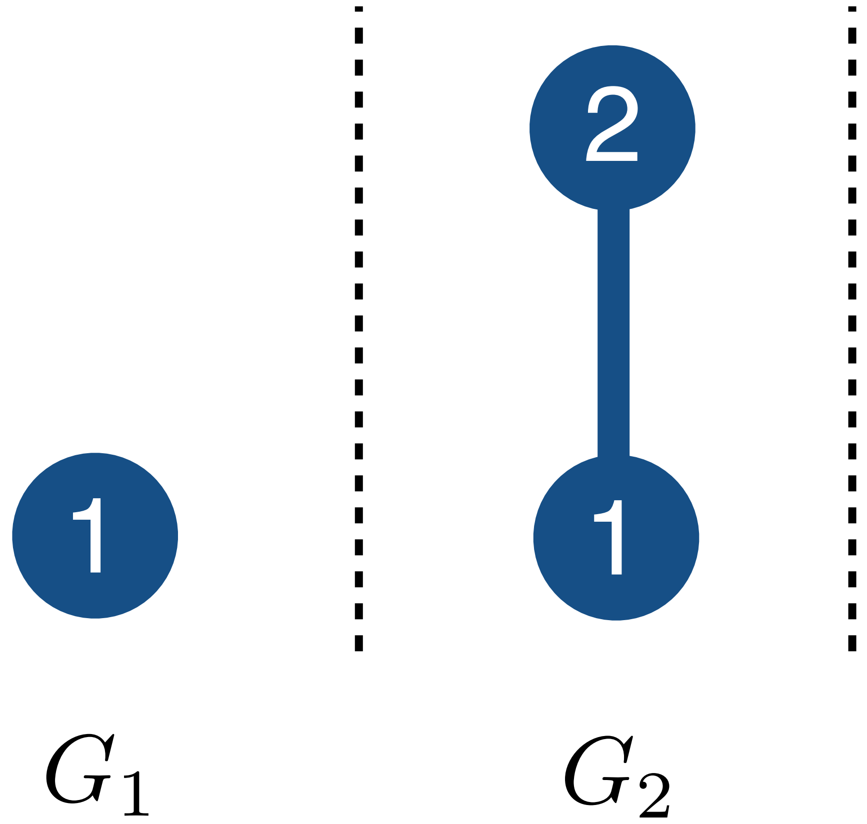

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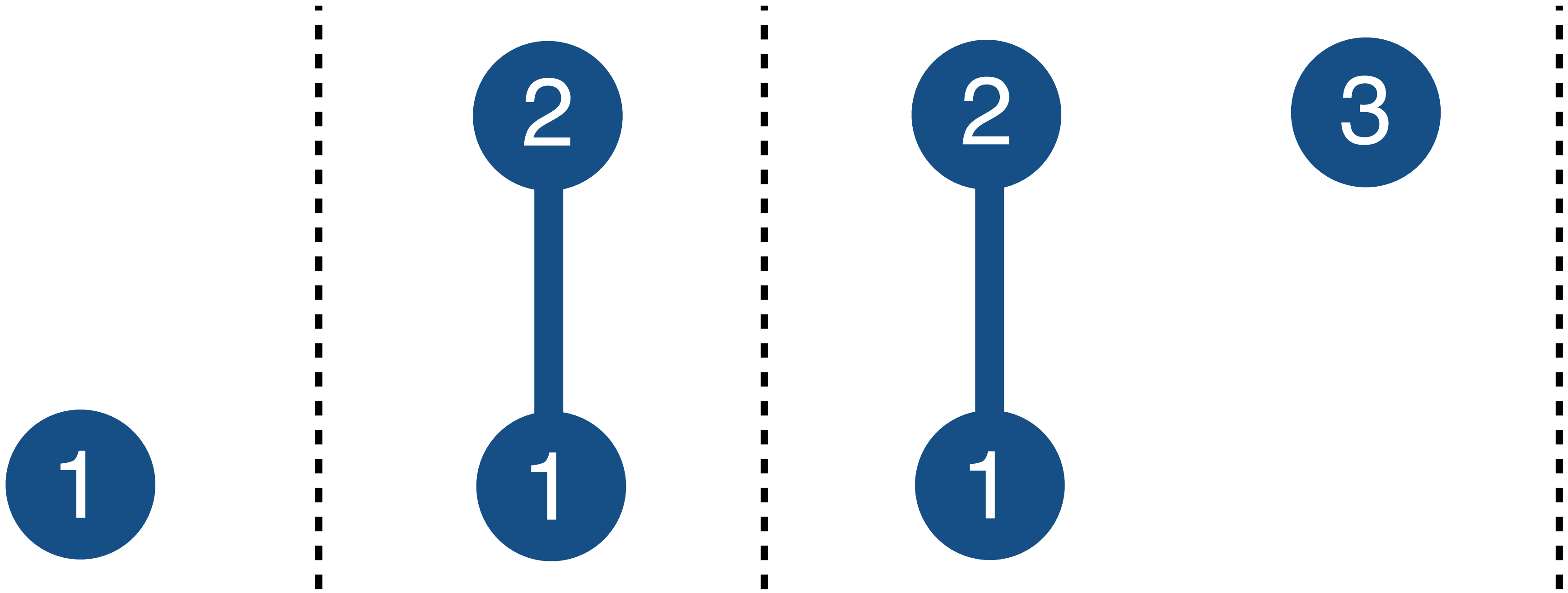

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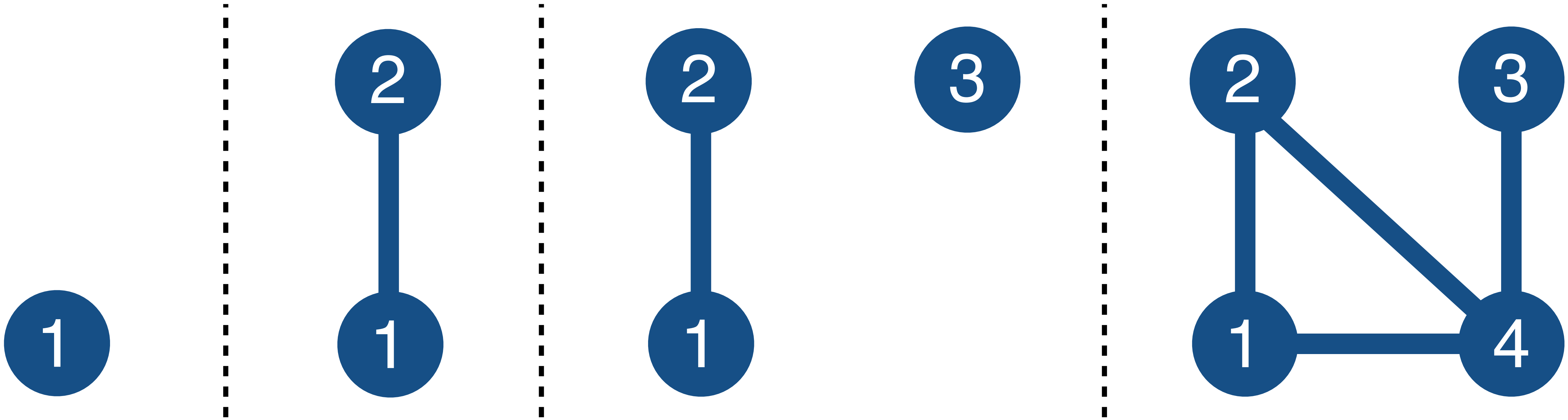

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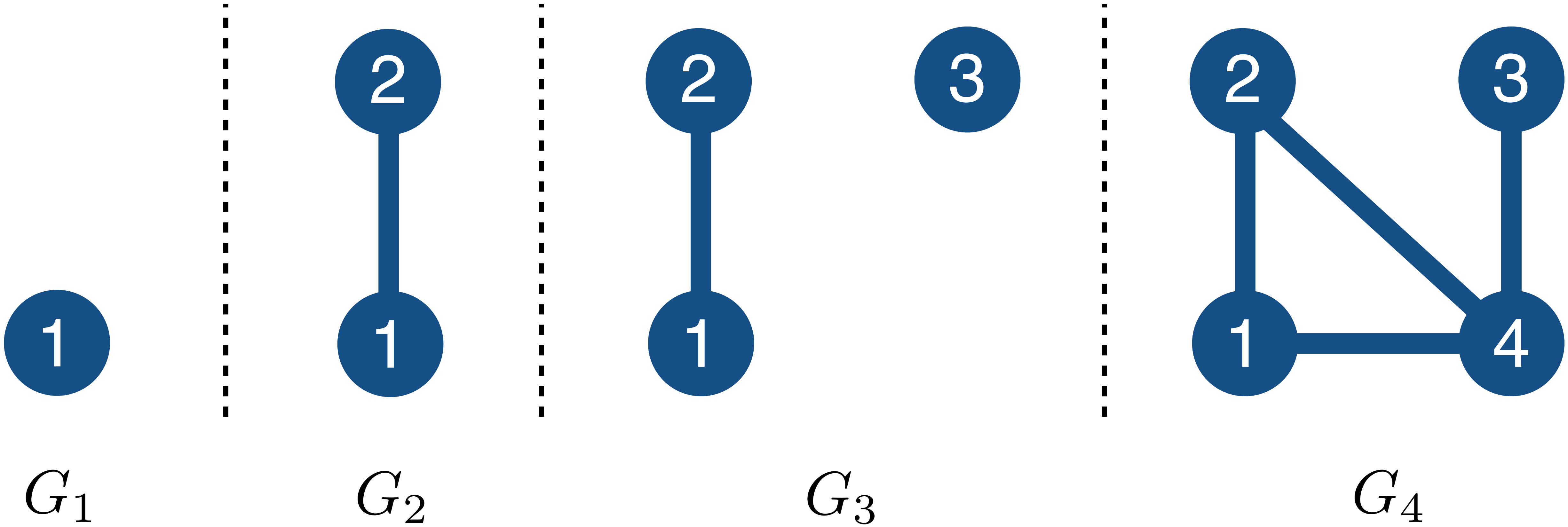
The Old Way: Exchangeability



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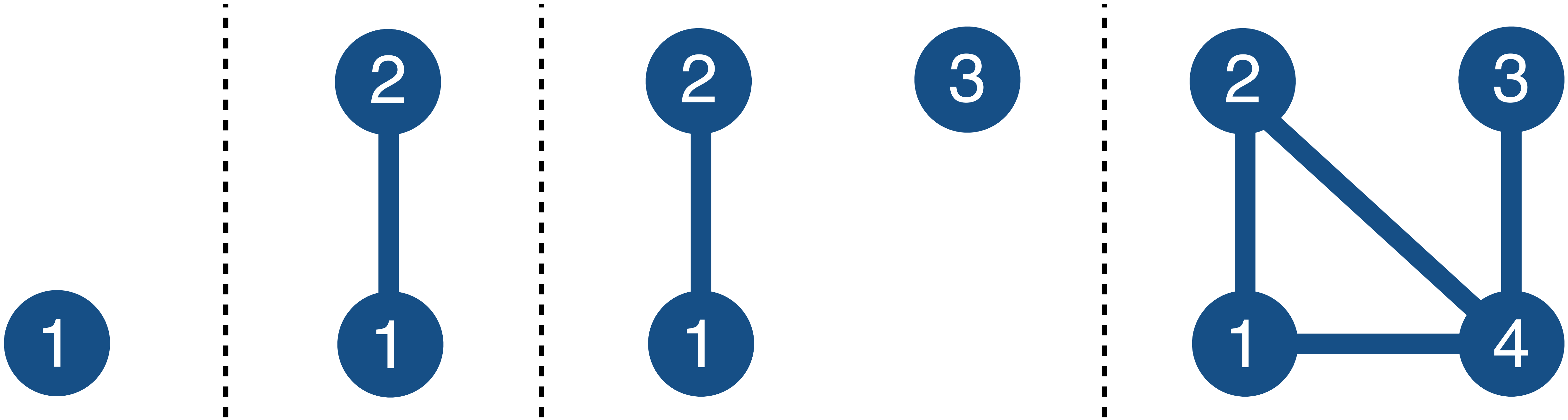

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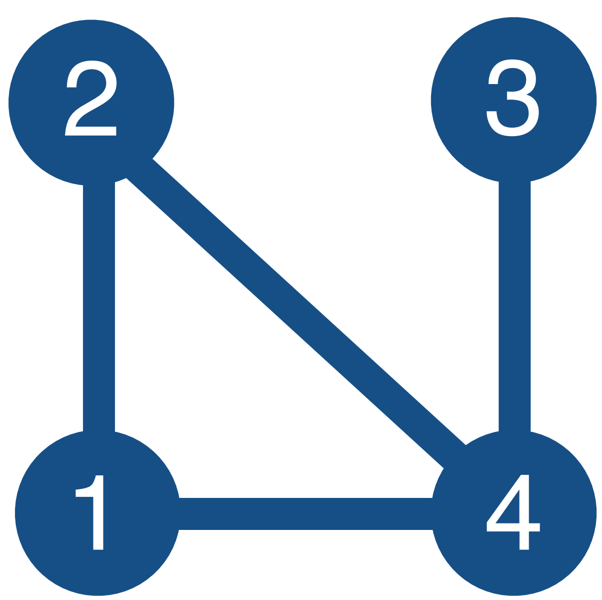

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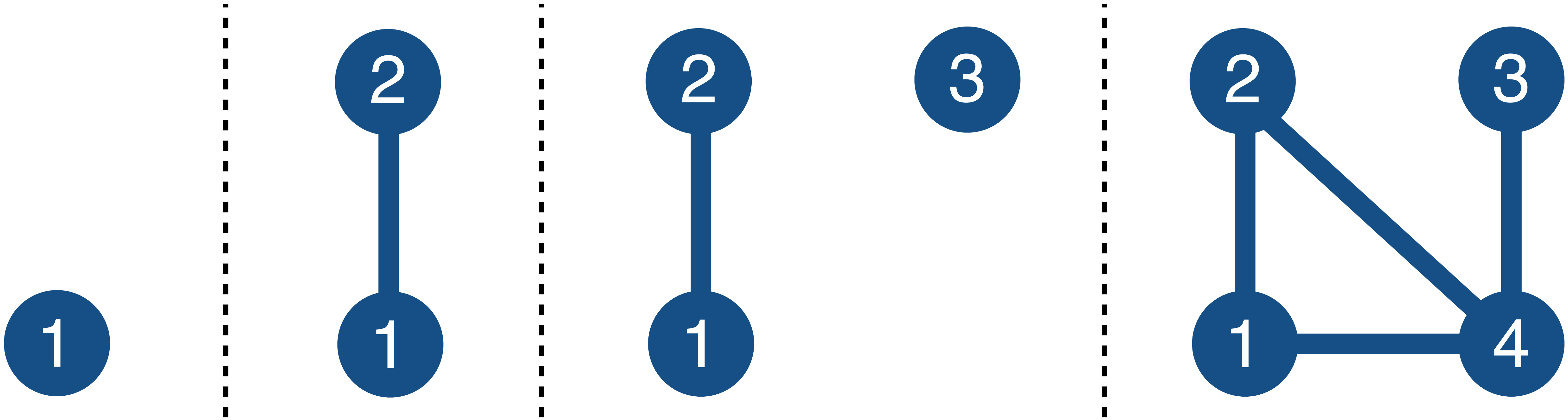

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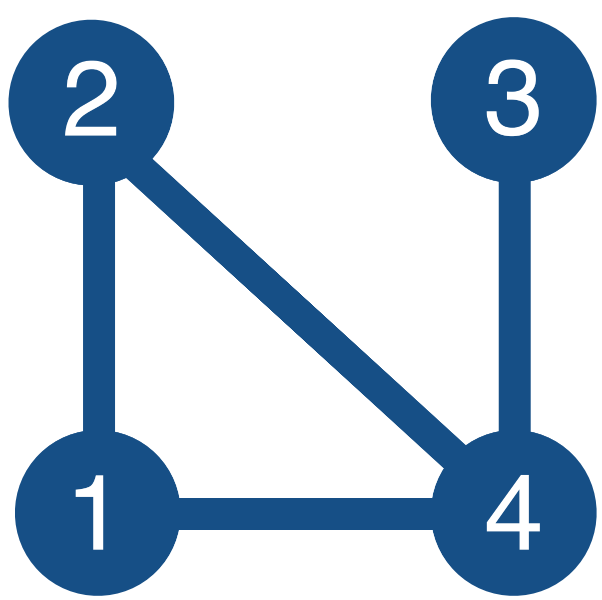
$p($  $)$

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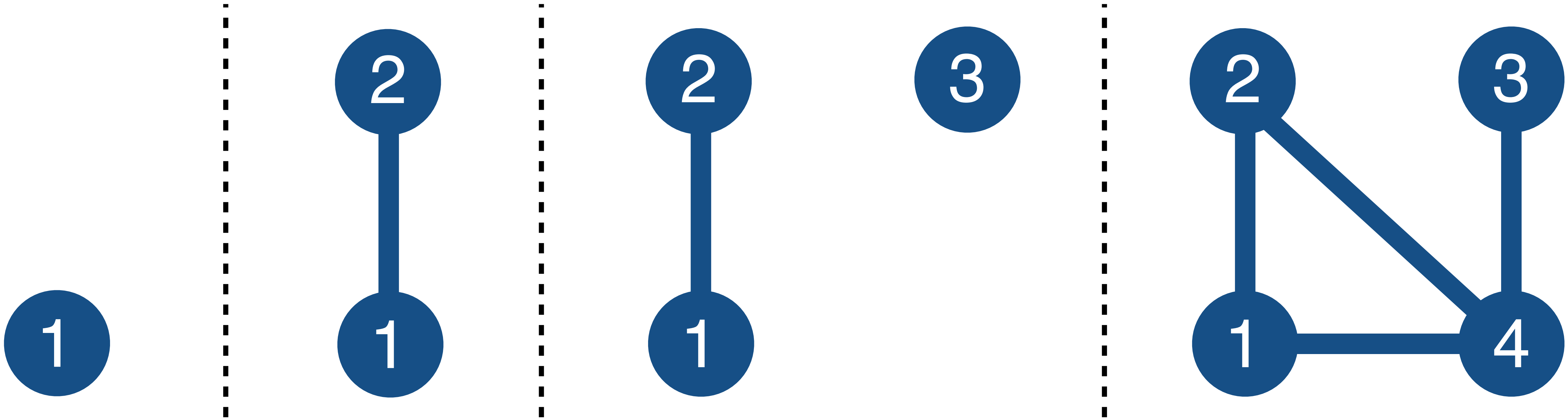

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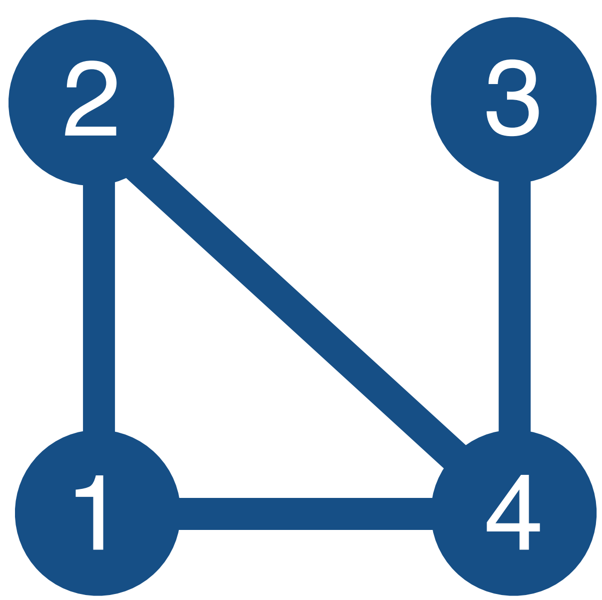
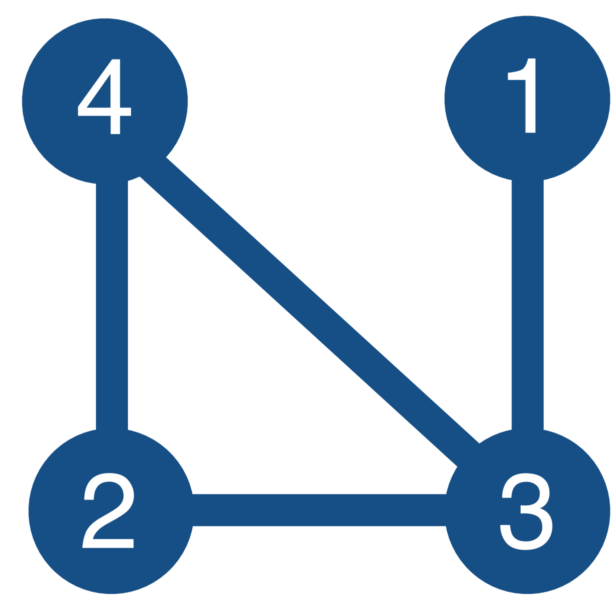
$p($ 

$) = p($ 

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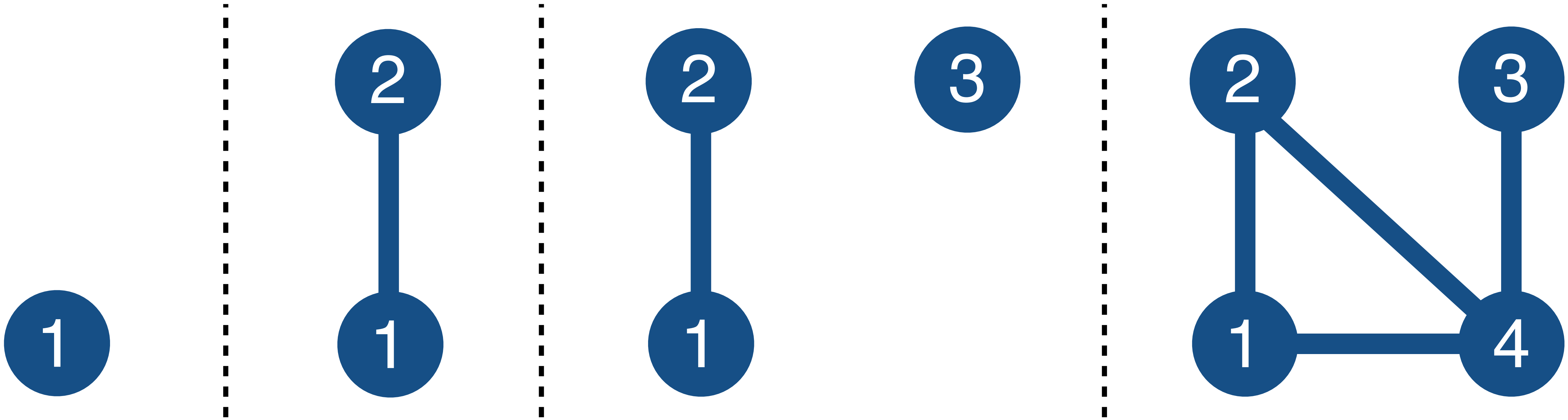

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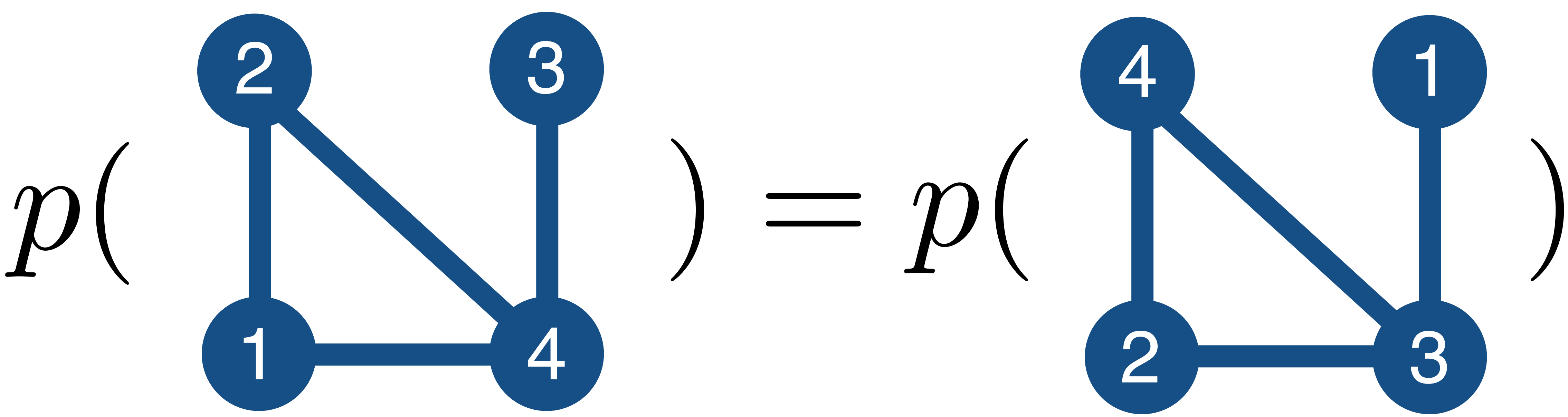

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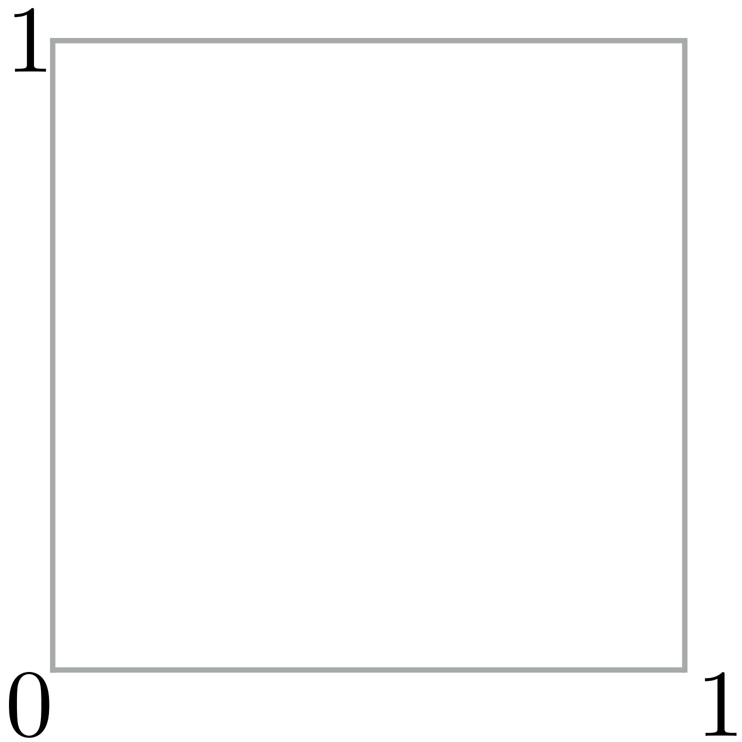

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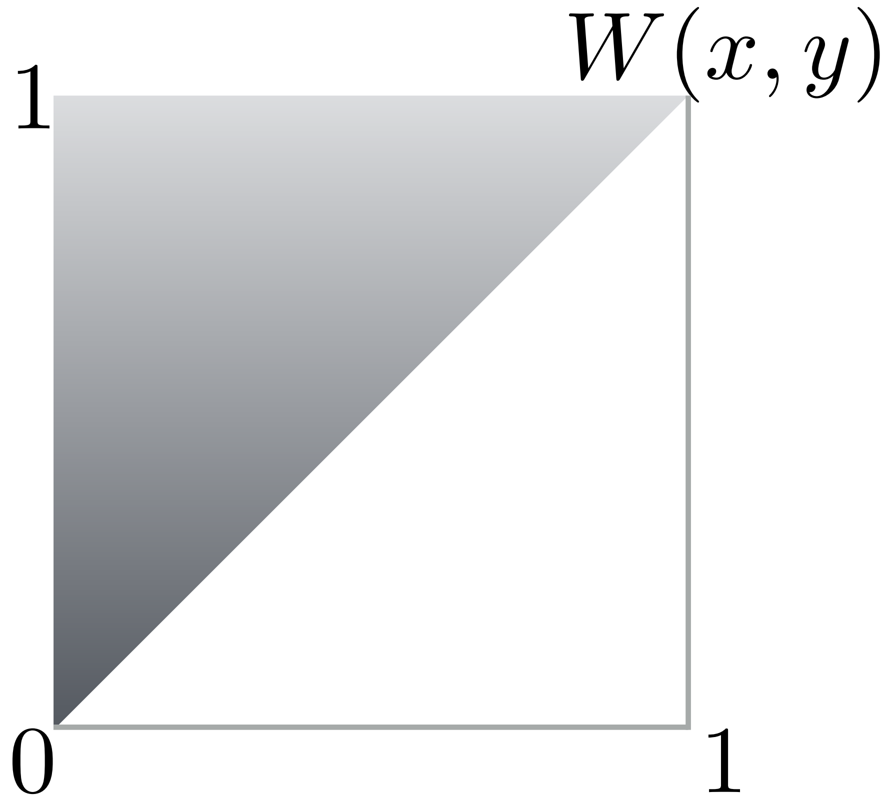

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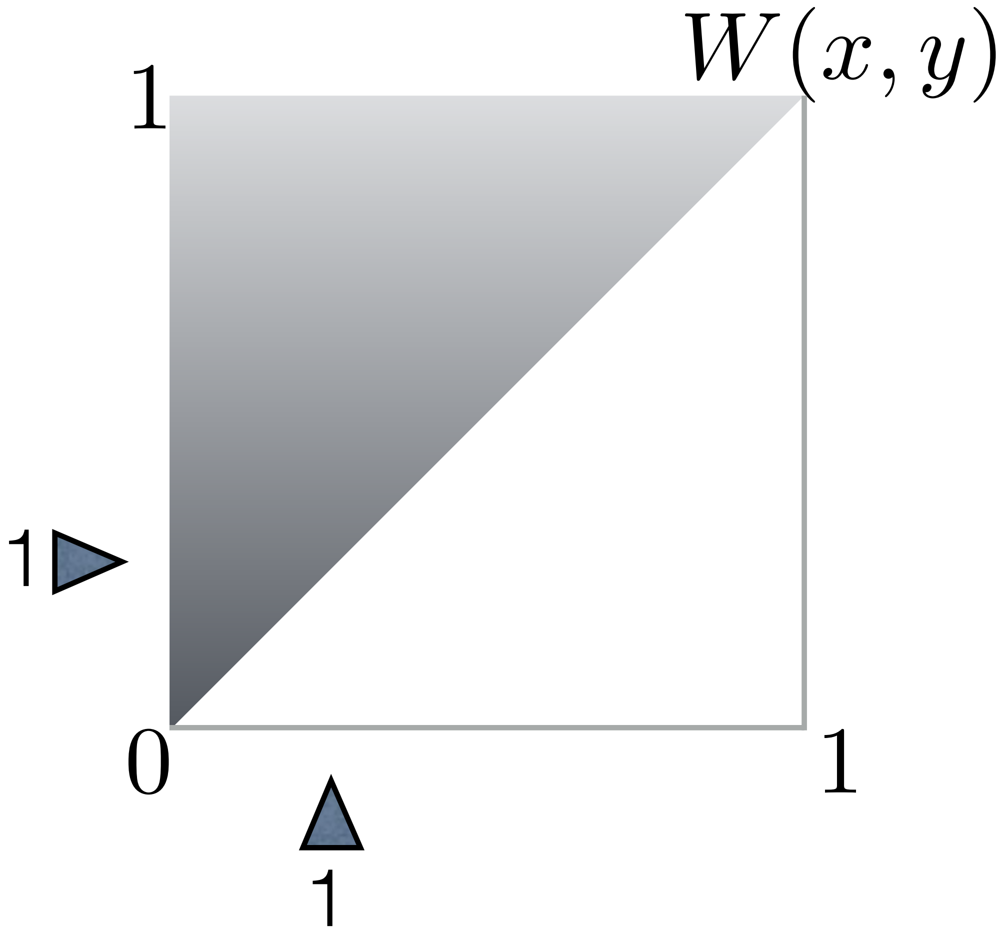

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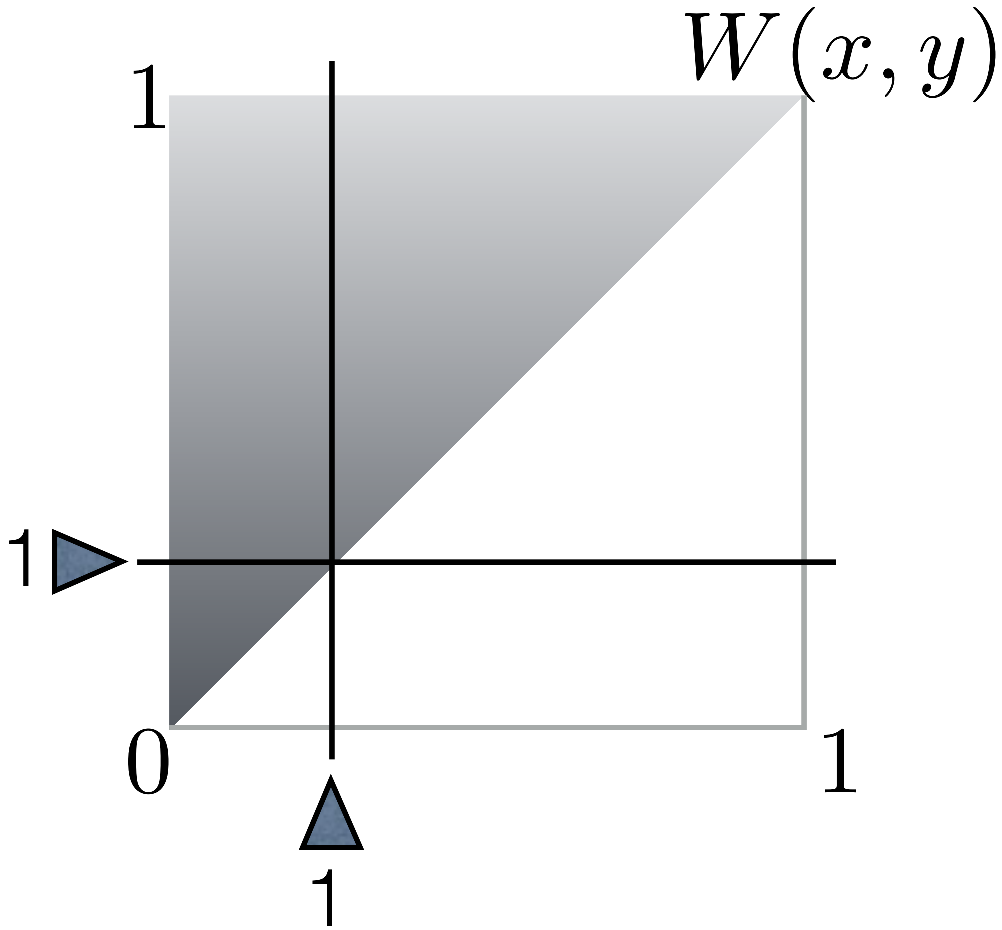

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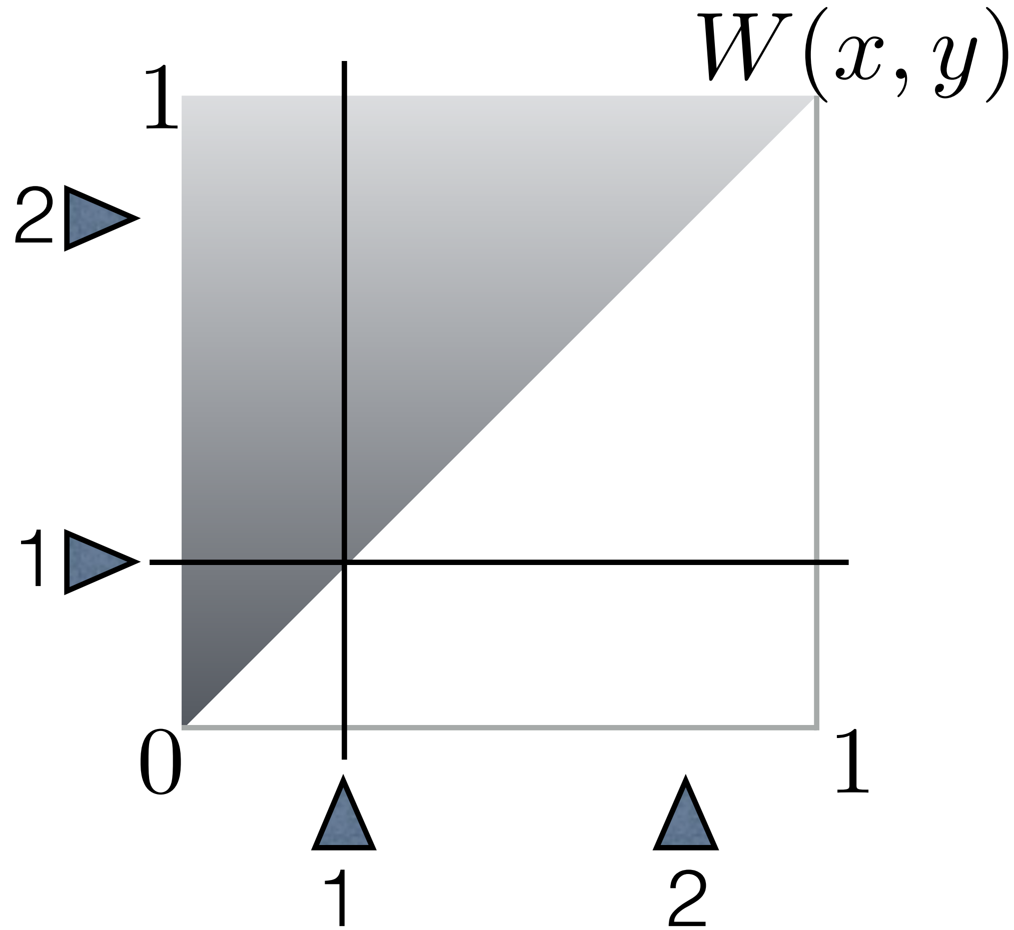

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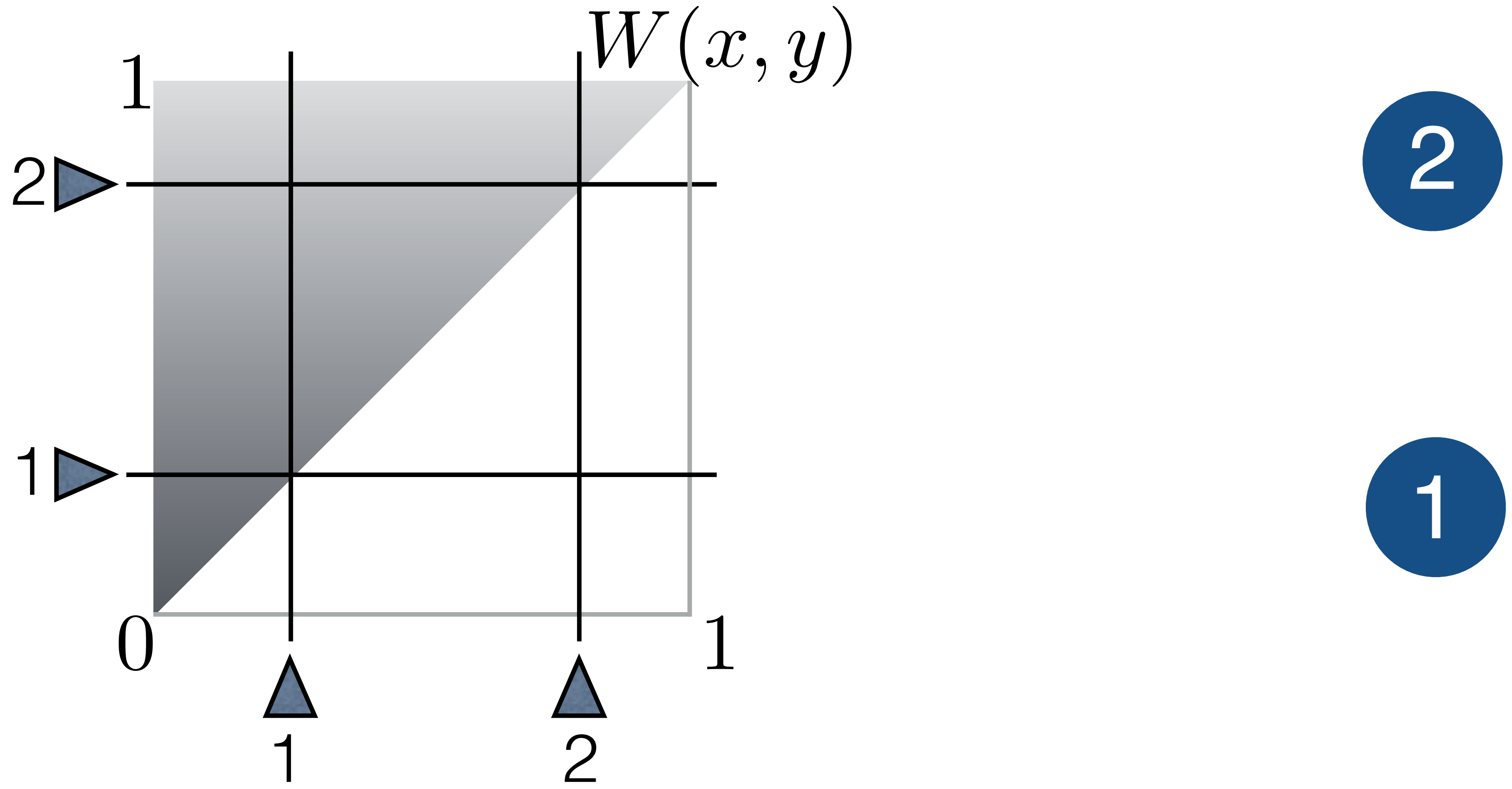

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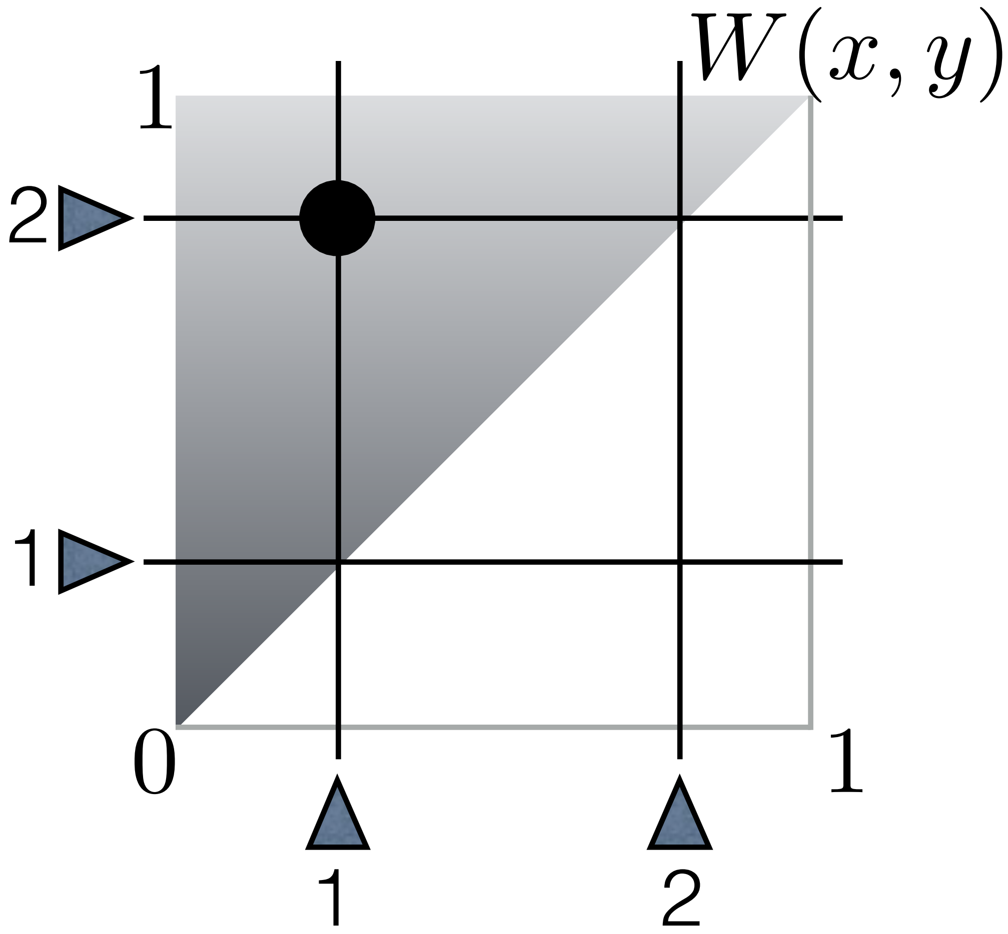

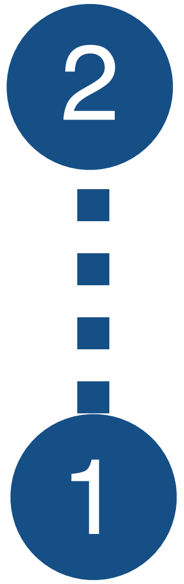

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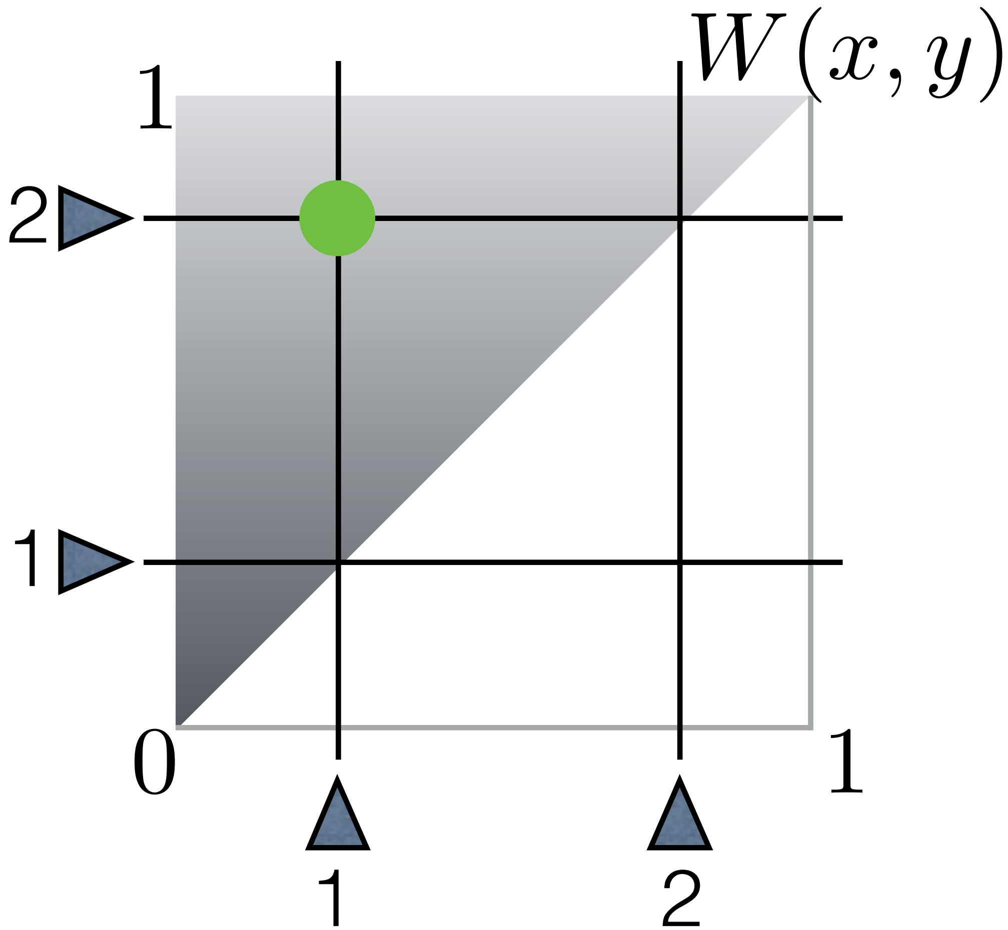

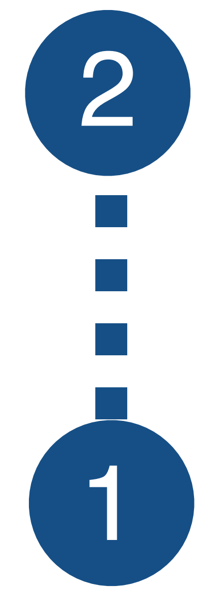

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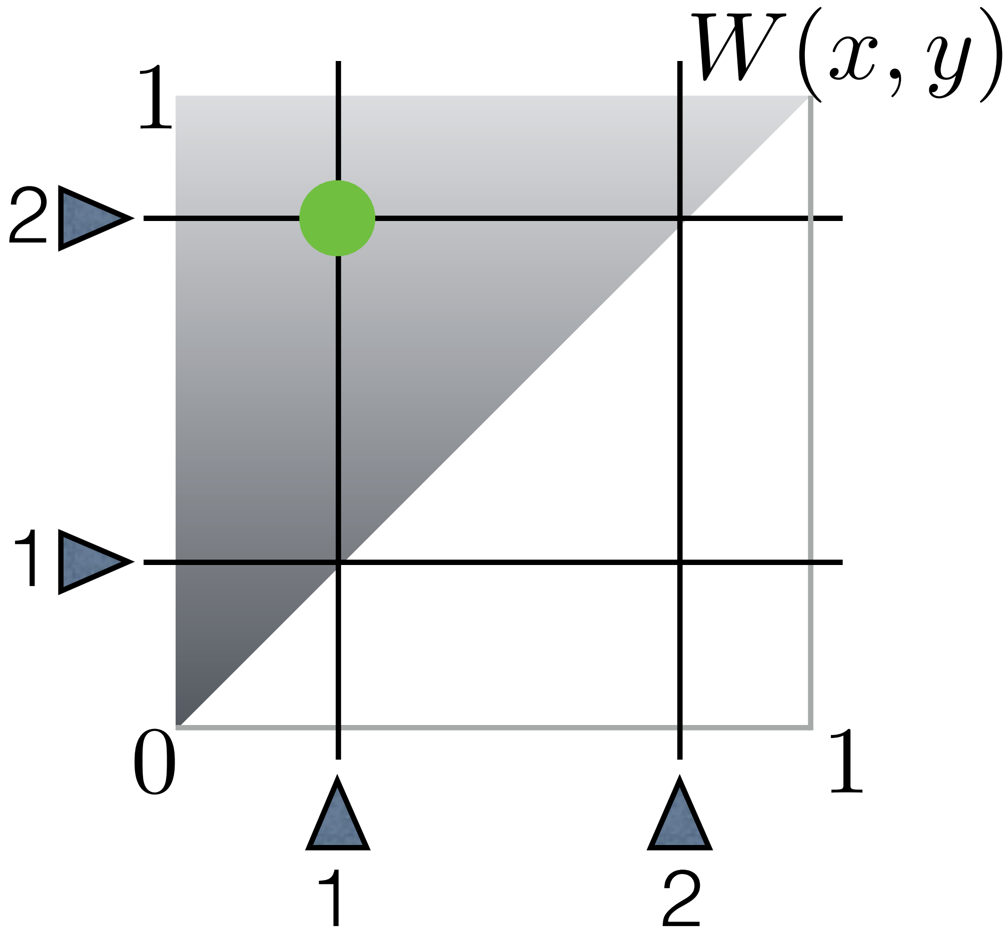

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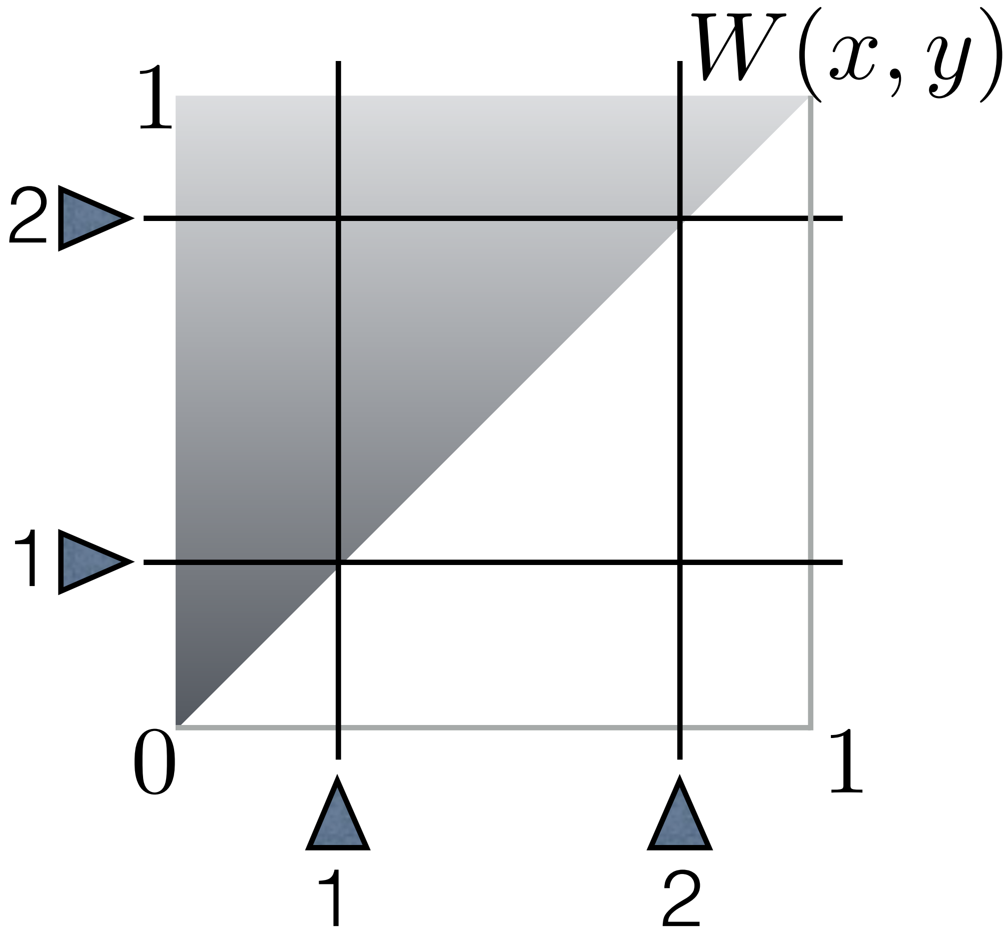

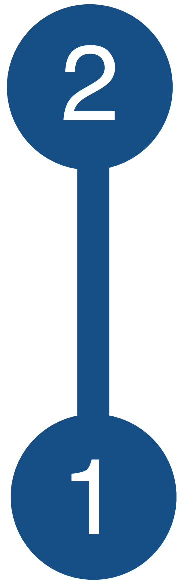

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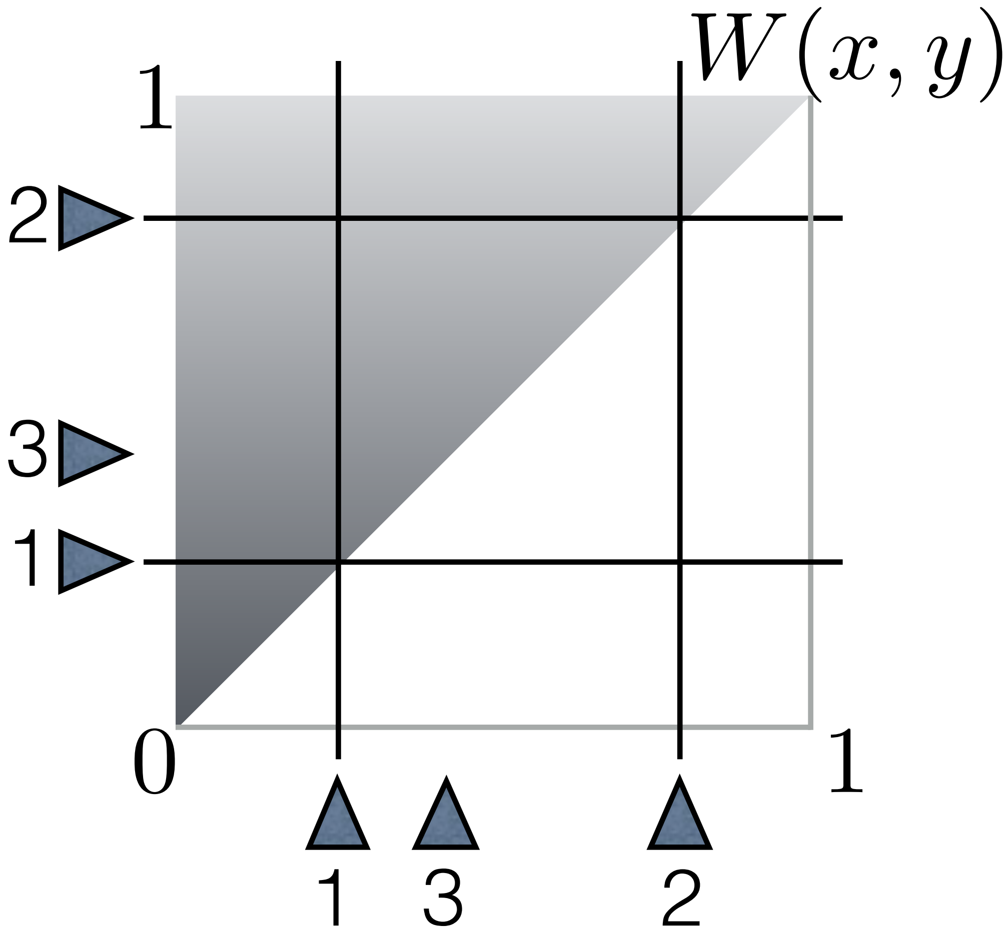

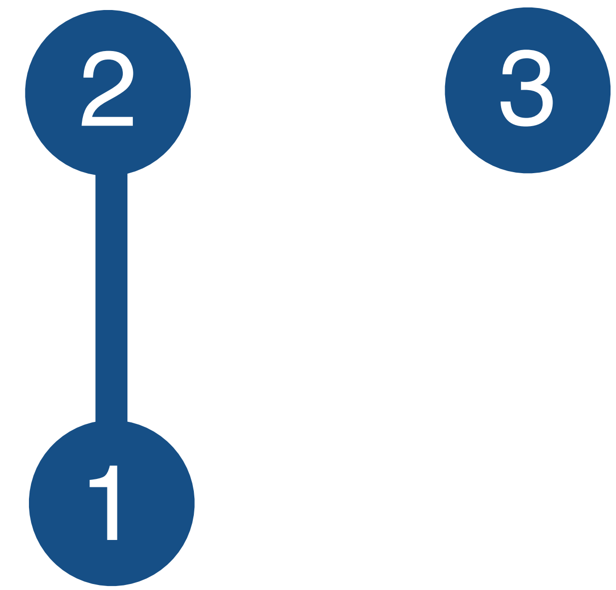

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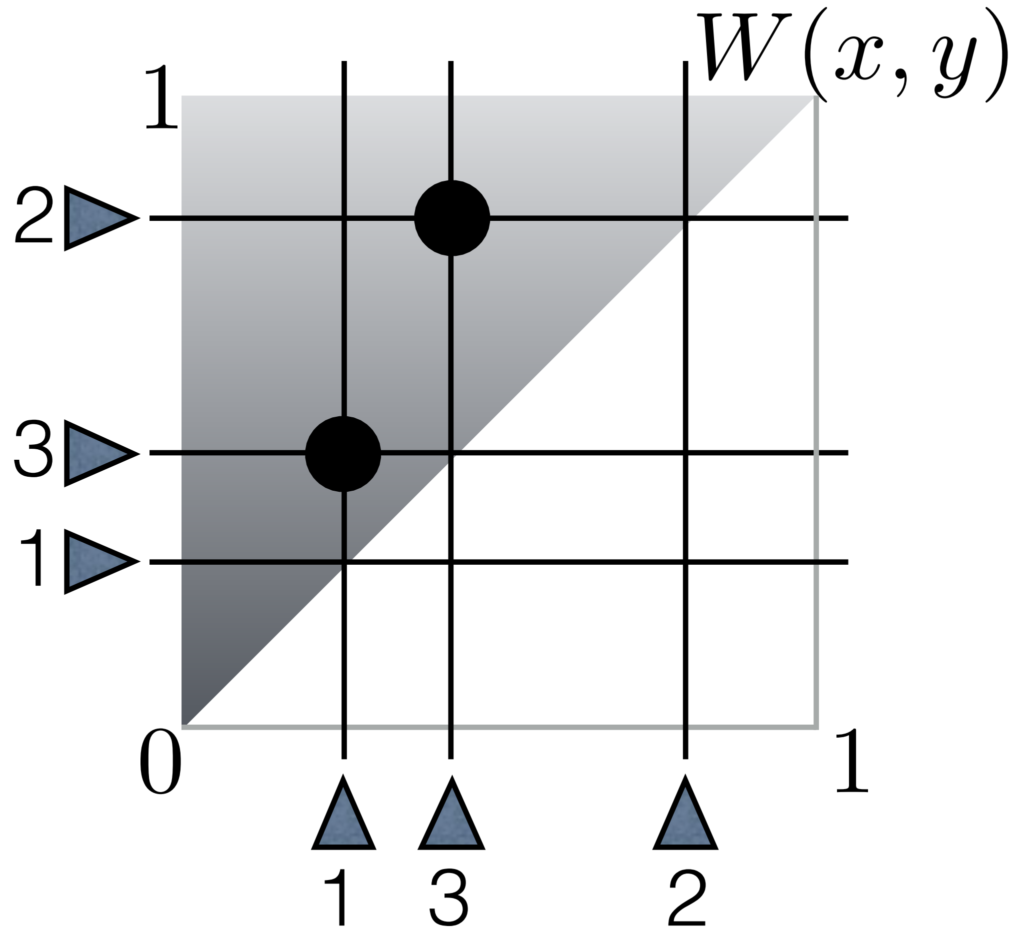
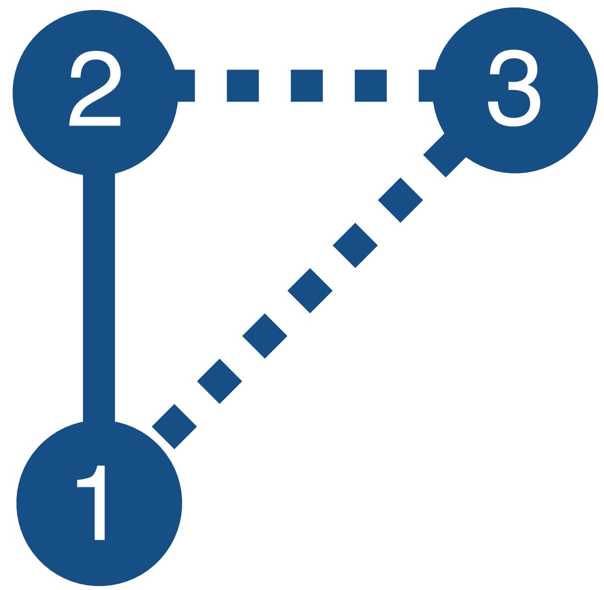
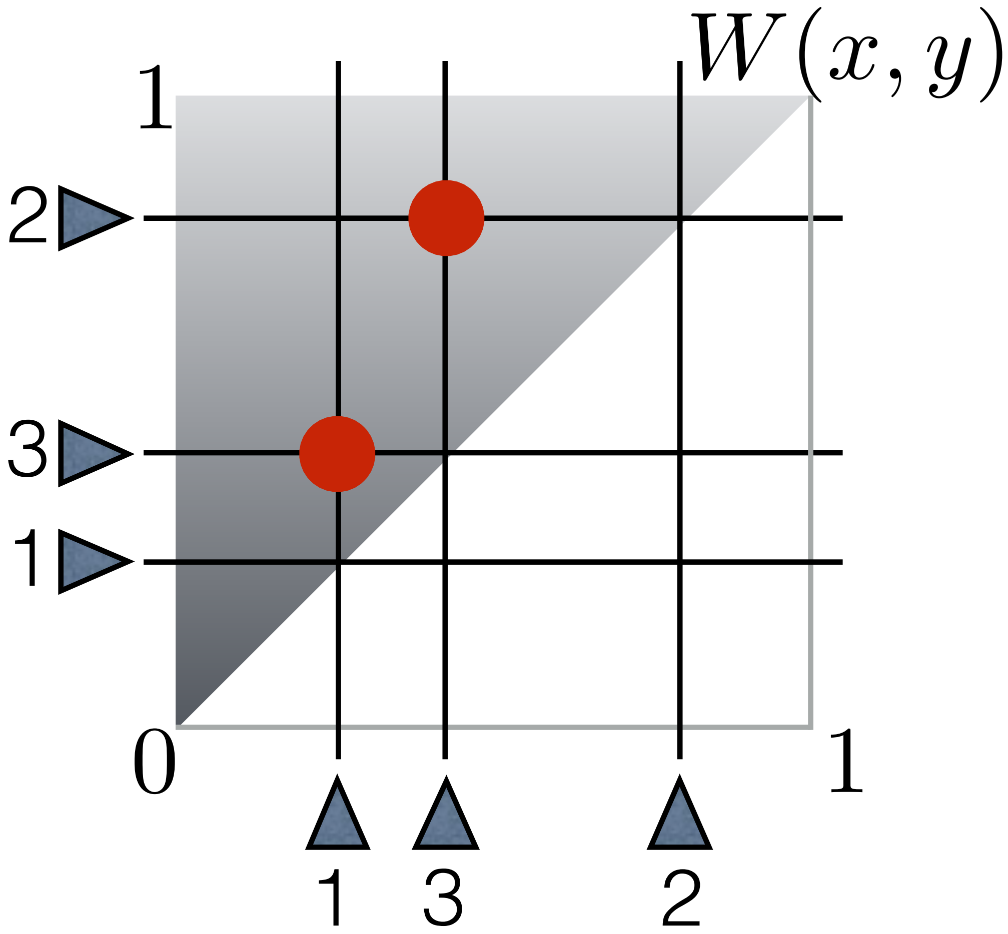
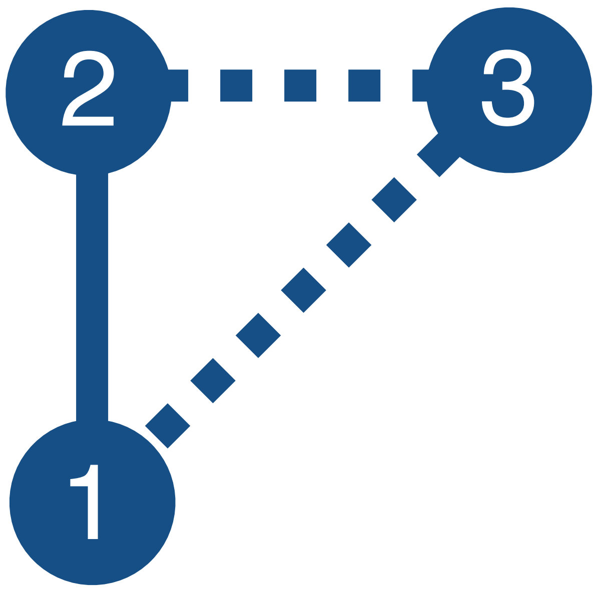




[Hoover 1979, Aldous 1981]

Aldous-Hoover





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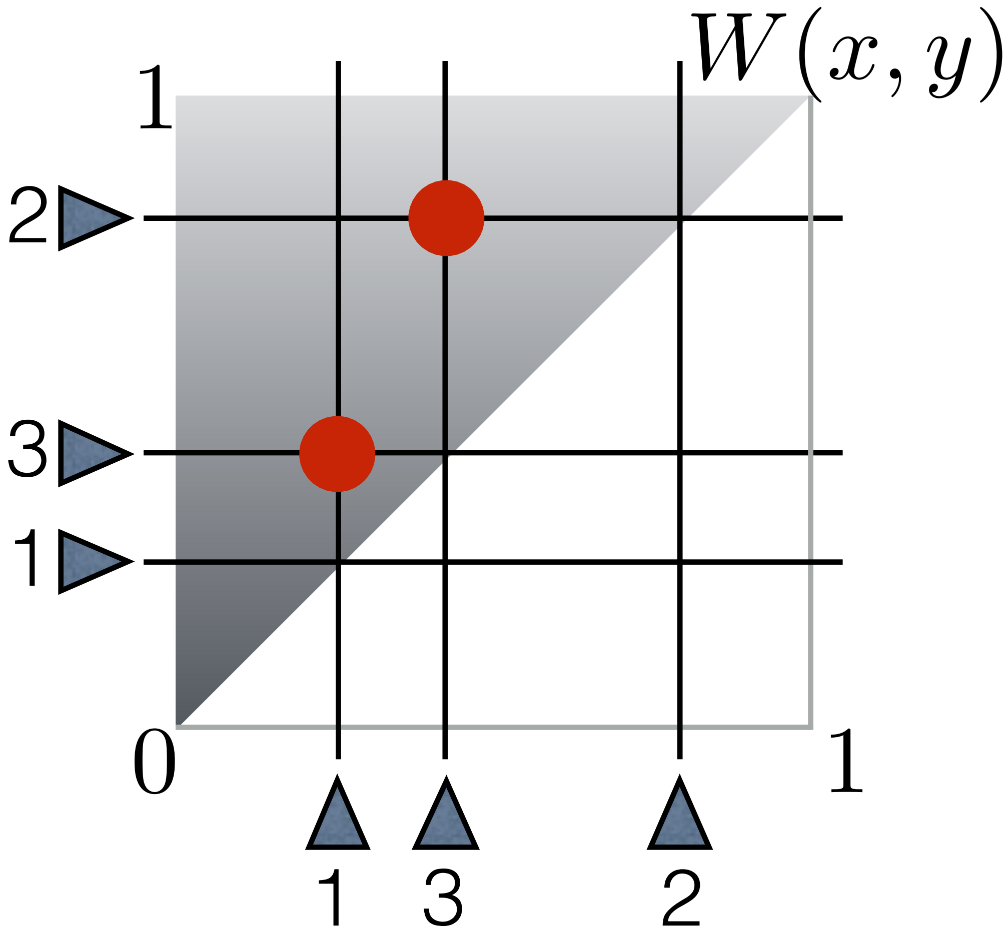

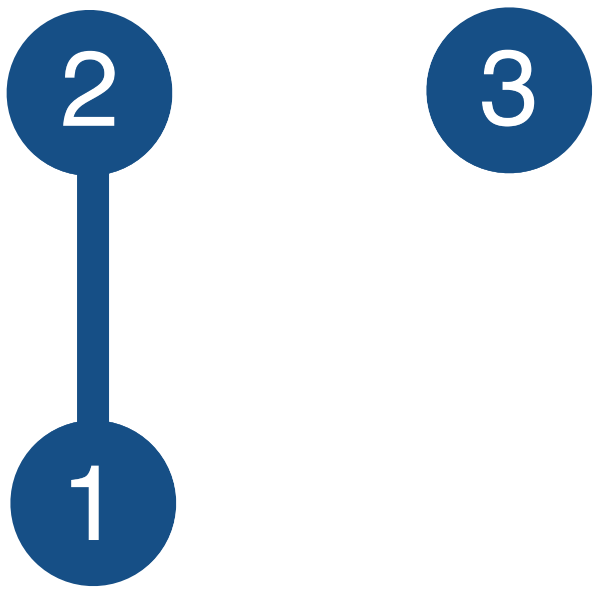

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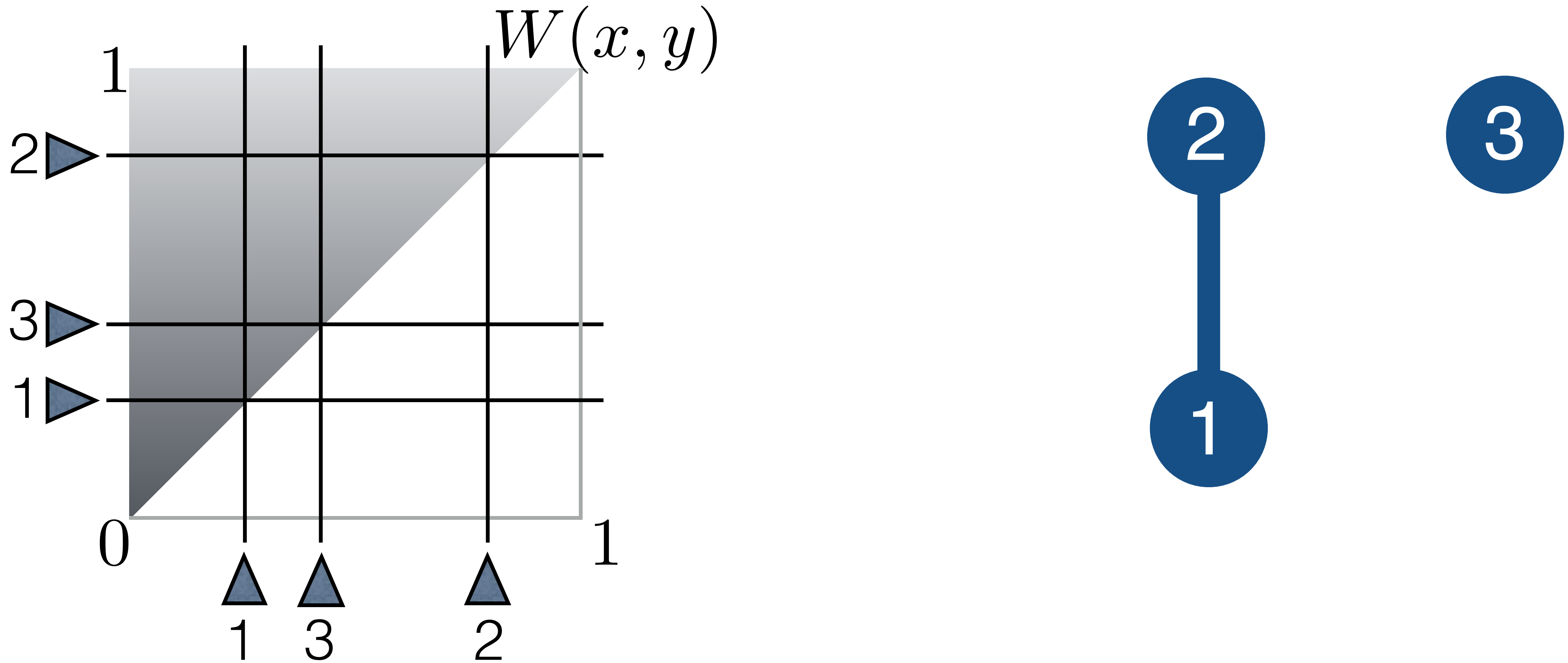

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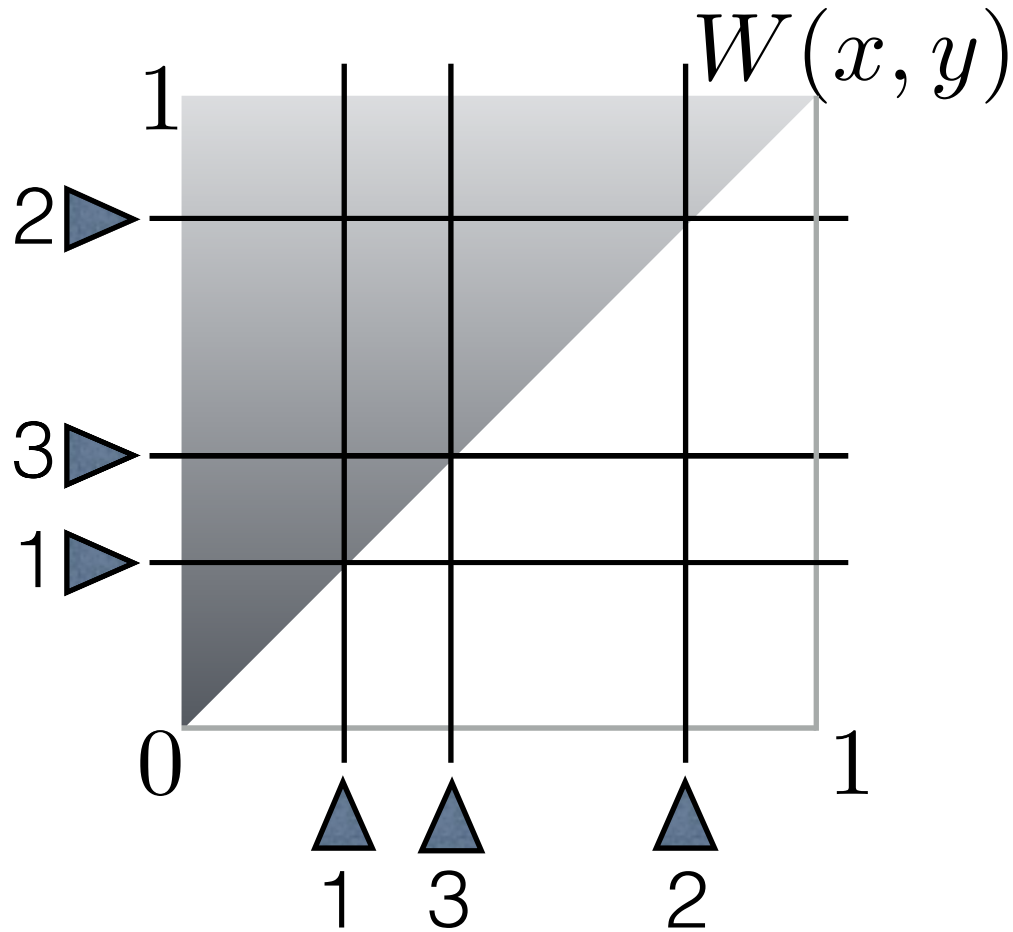
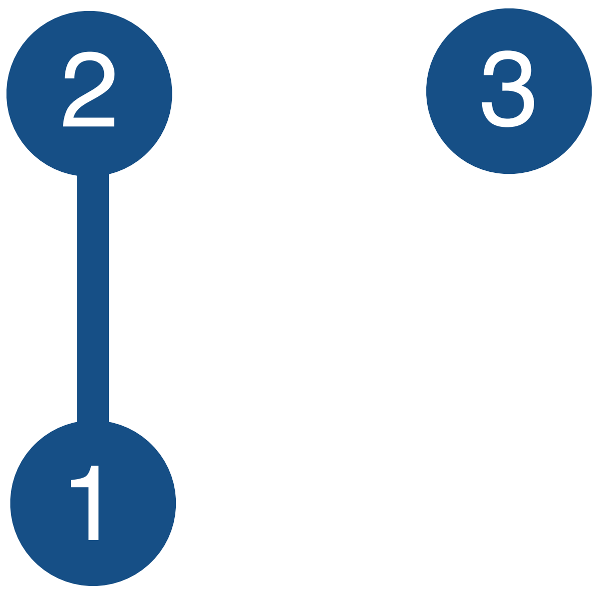
Aldous-Hoover

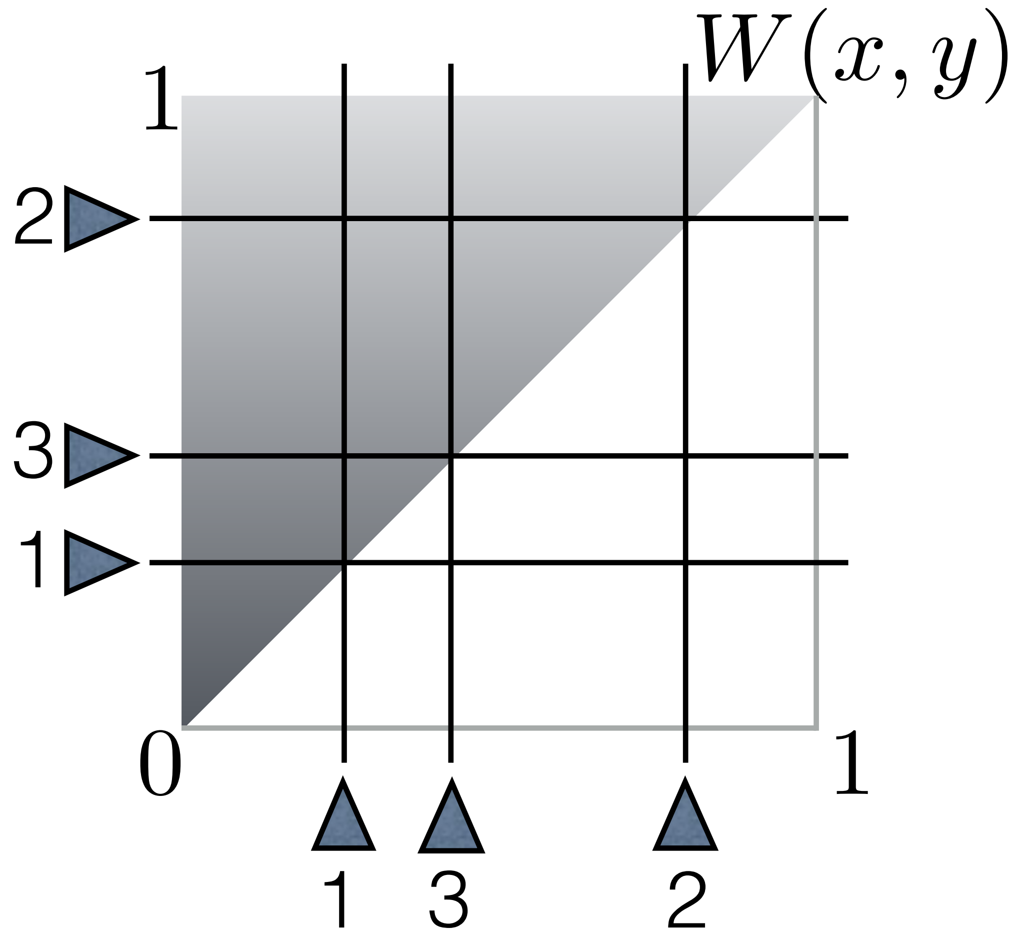


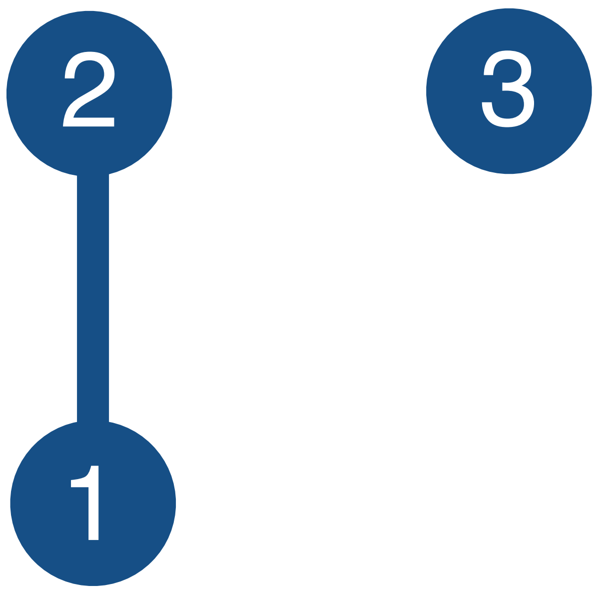


Thm (AH). Every node-exchangeable graph has a *graphon* rep

[Hoover 1979, Aldous 1981]

Aldous-Hoover



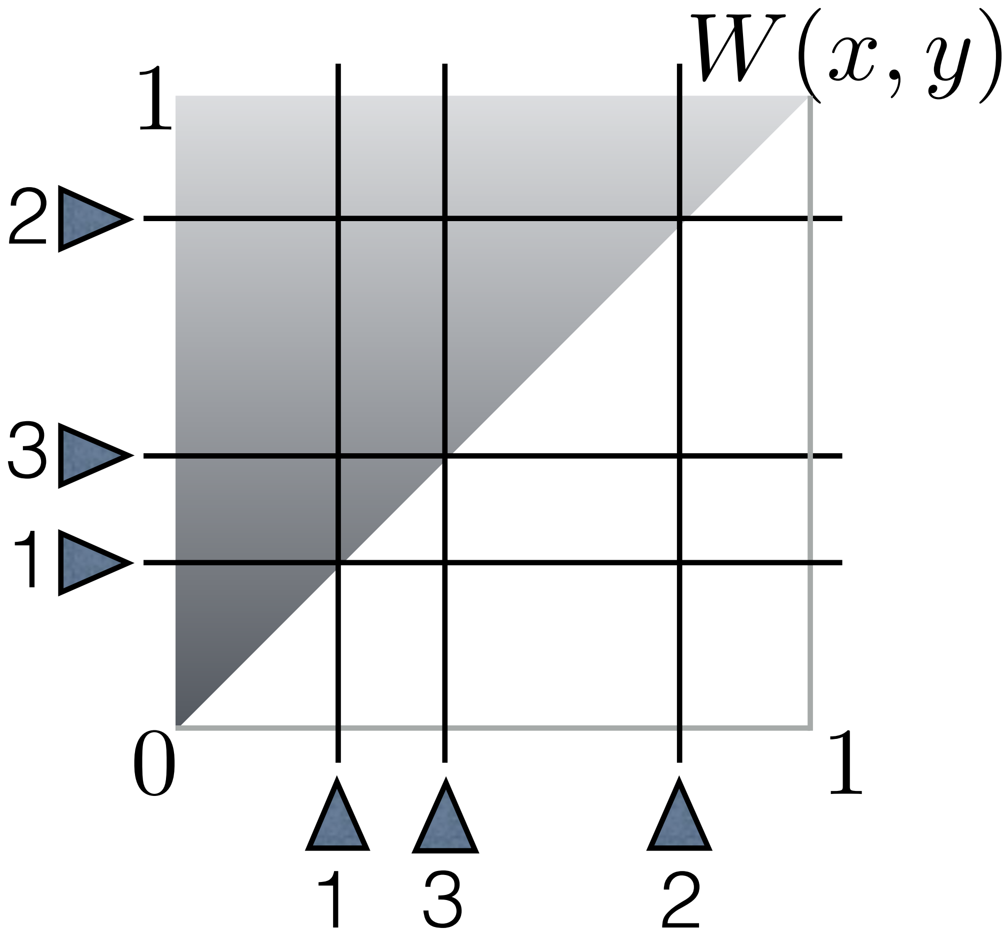


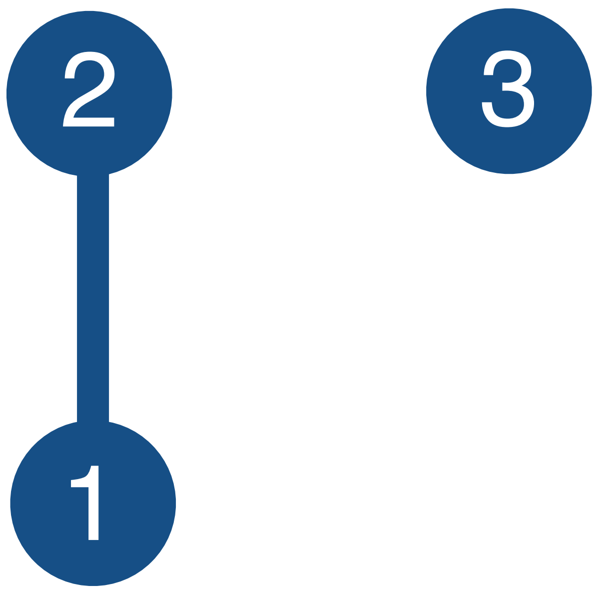
Thm (AH). Every node-exchangeable graph has a *graphon* rep

$$\mathbb{E}[\#\text{edges}(G_n)]$$

[Hoover 1979, Aldous 1981, Orbanz, Roy 2015]

Aldous-Hoover





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$$\mathbb{E}[\#\text{edges}(G_n)] = \mathbb{E}\left[\binom{n}{2}\frac{1}{2}\int_0^1\int_0^1 W(x,y)\,dx\,dy\right]$$

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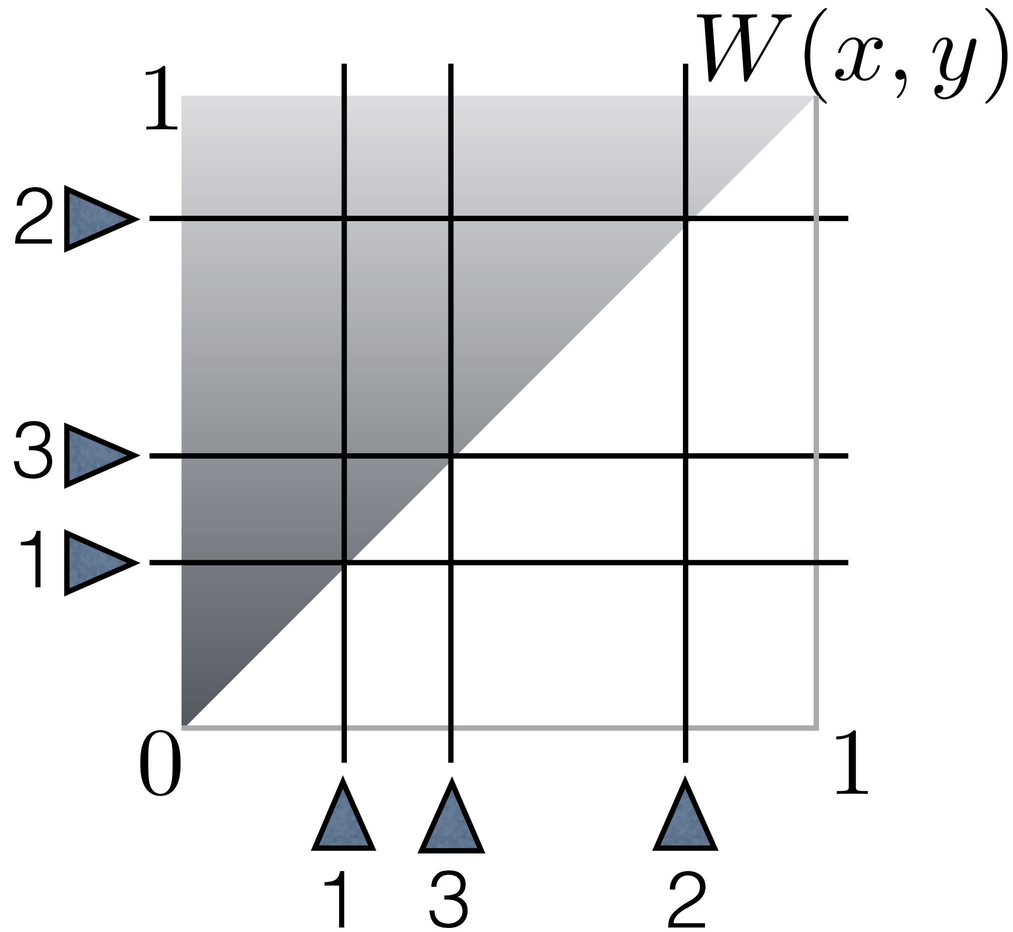
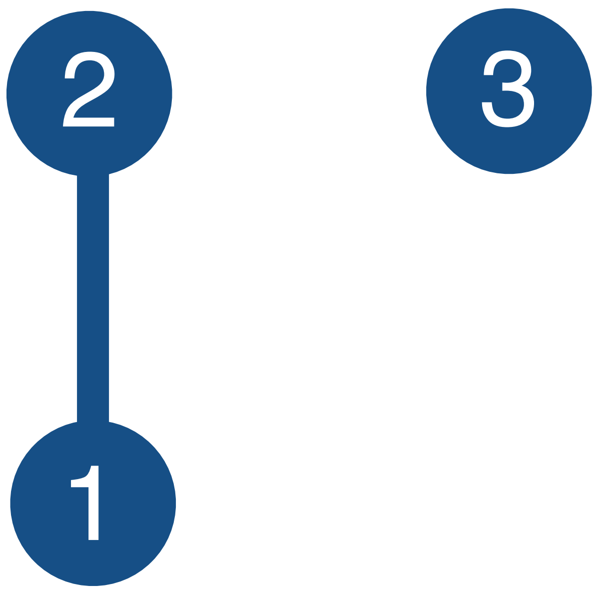
Aldous-Hoover





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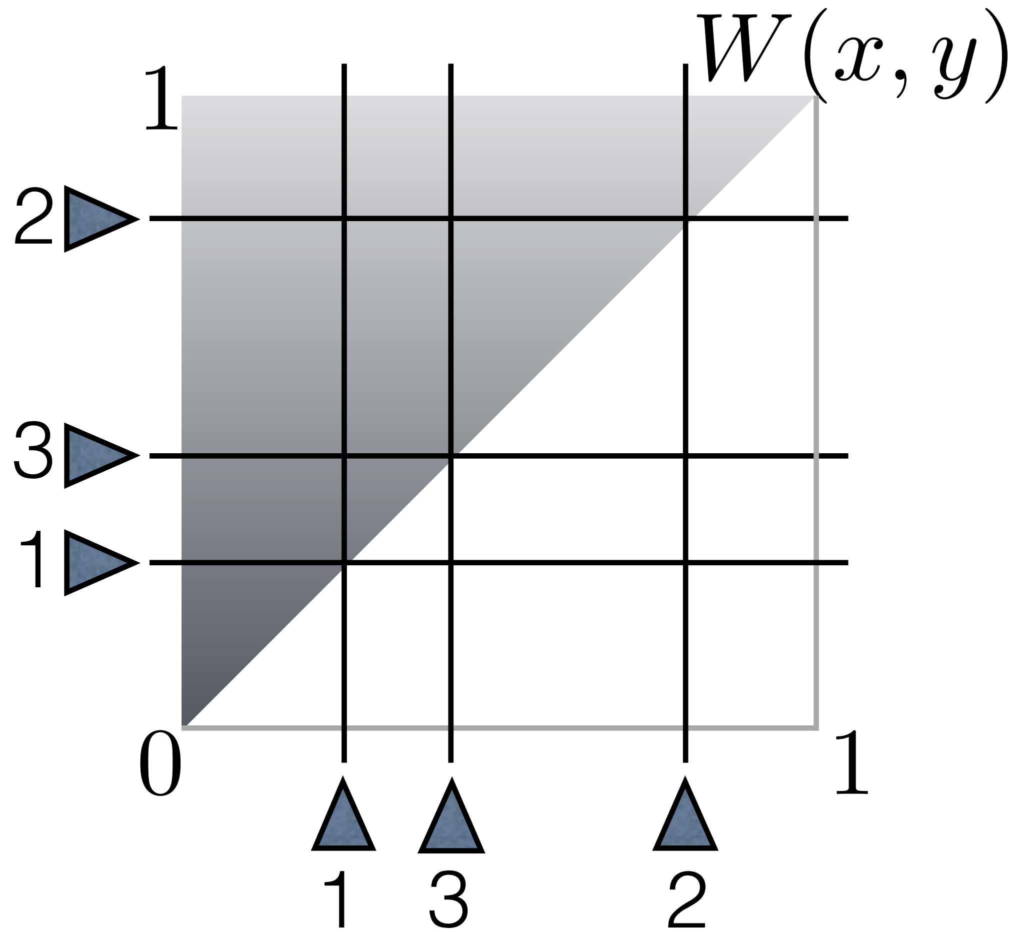

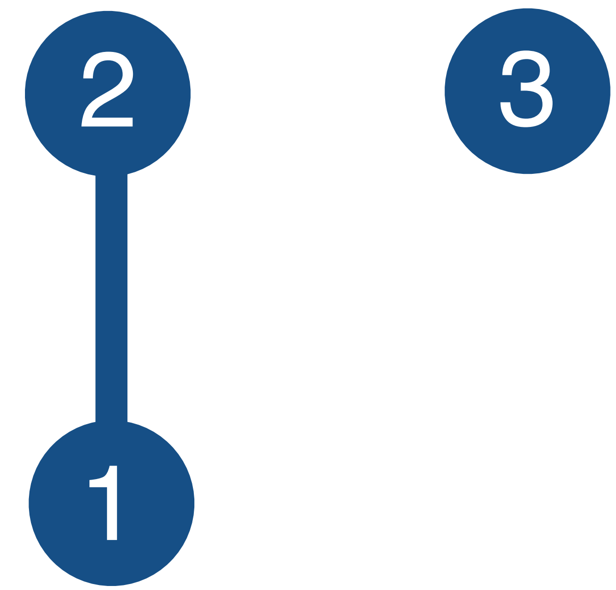

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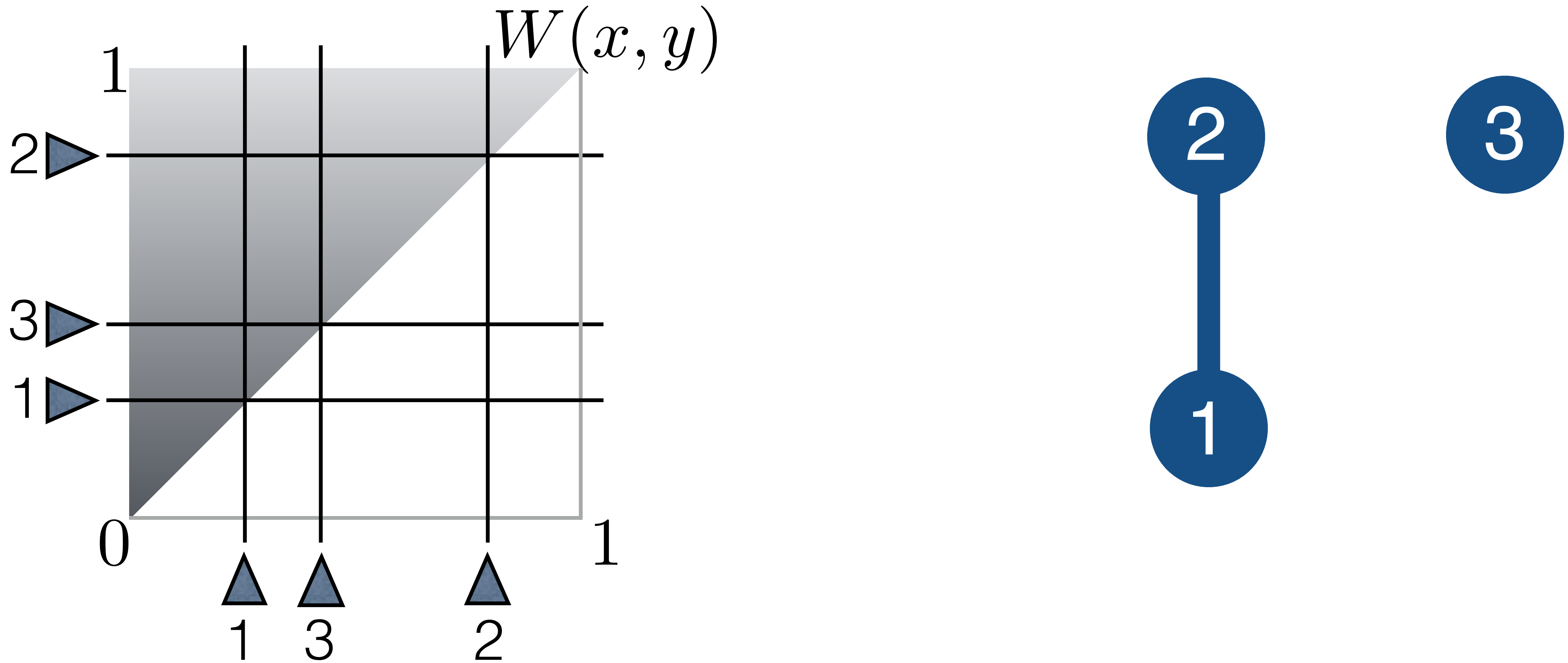

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Cor. Every node-exch graph sequence is dense (or empty) a.s.

[Hoover 1979, Aldous 1981, Orbanz, Roy 2015]

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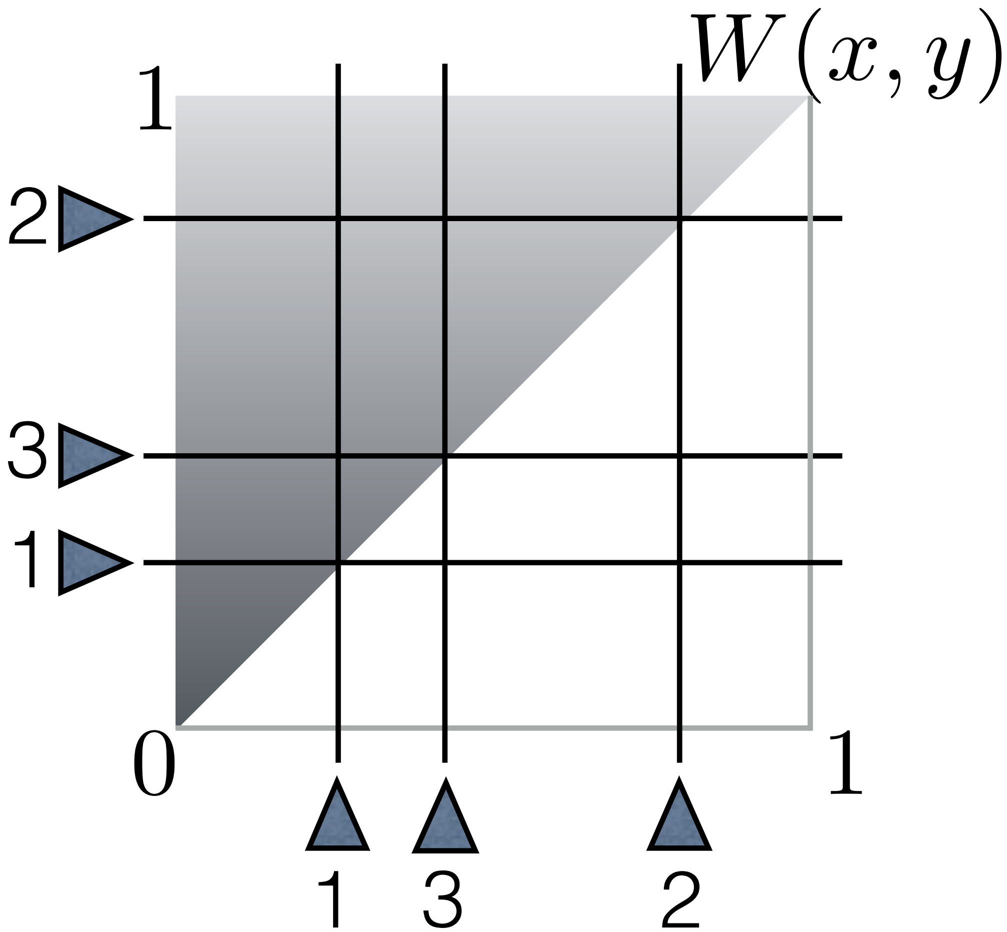

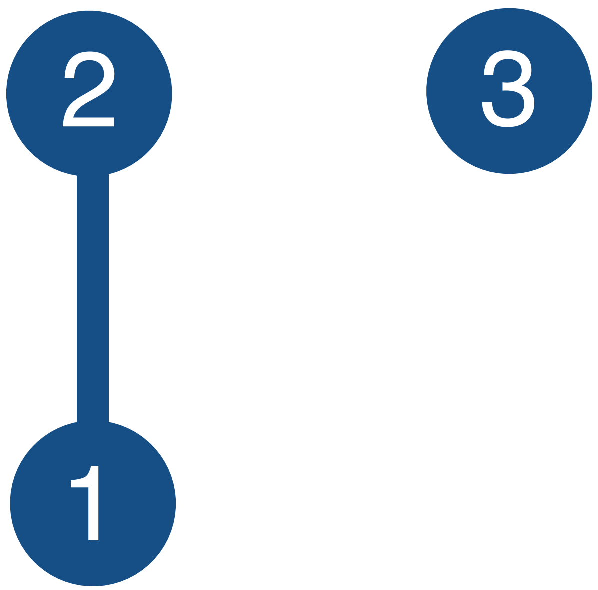

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Intuition: To a given node, all other nodes look the same.

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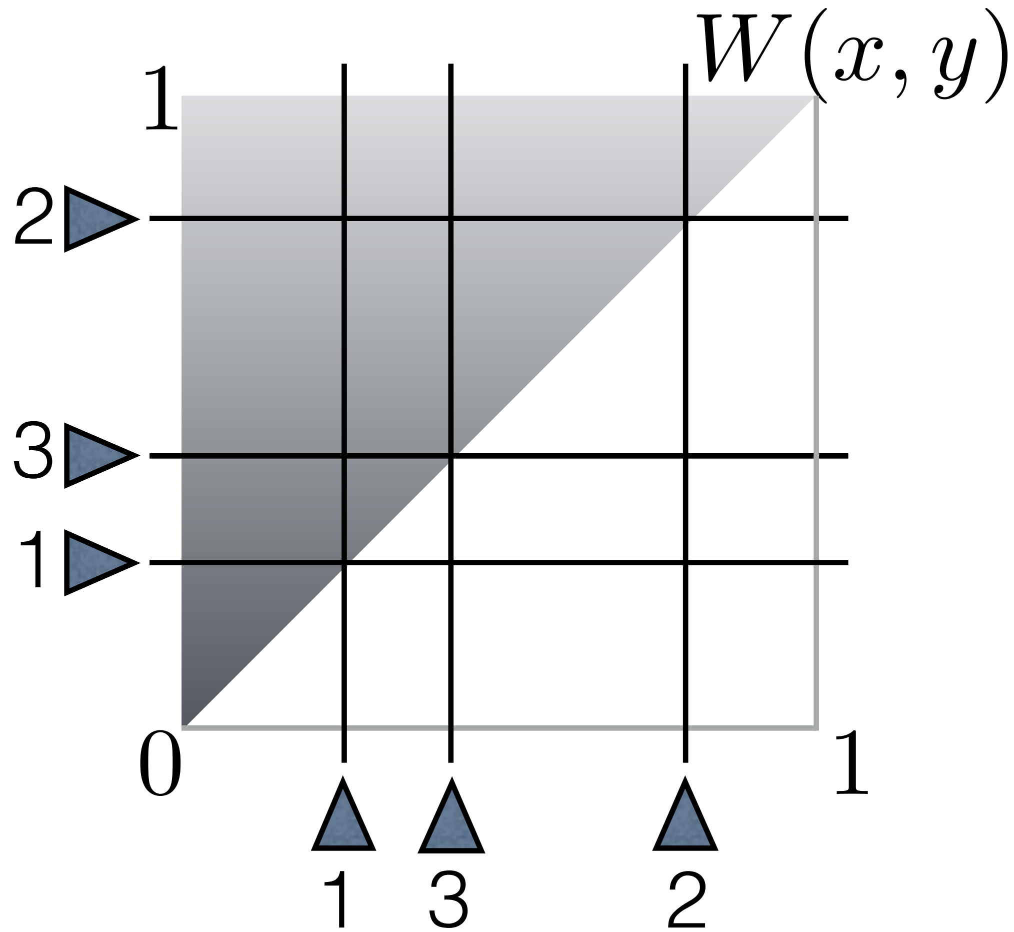

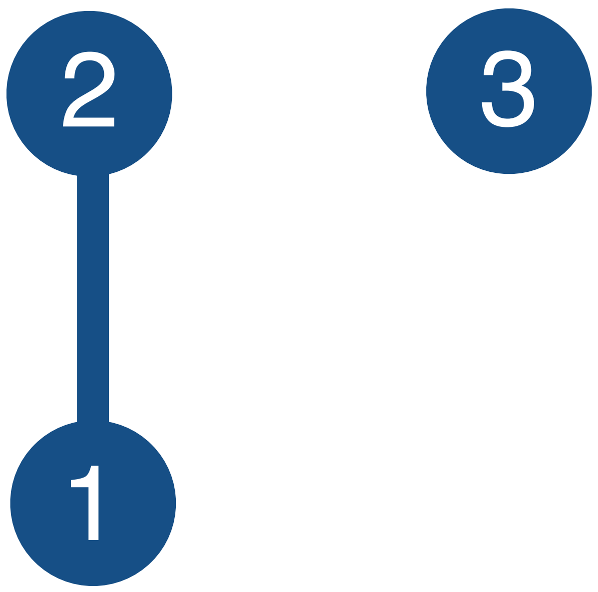

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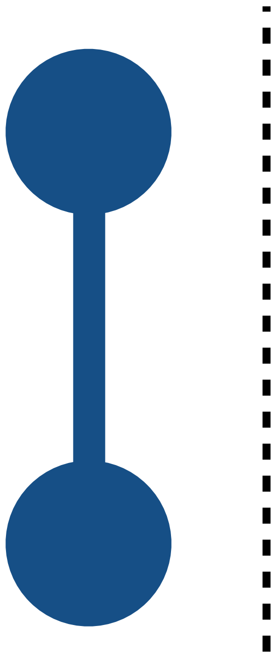
[Caron, Fox 2014; Veitch, Roy 2015; Borgs, Chayes, Cohn, Holden 2016;
Broderick, Cai 2015; Cai, Broderick 2015a,b; Crane, Dempsey 2015a,b, 2016a; Cai, Campbell, Broderick 2016]

A New Way: Edges



G_1

 [Broderick, Cai 2015; Cai, Broderick 2015a,b; Crane, Dempsey 2015a,b,16a; Cai, Campbell, Broderick 2016]

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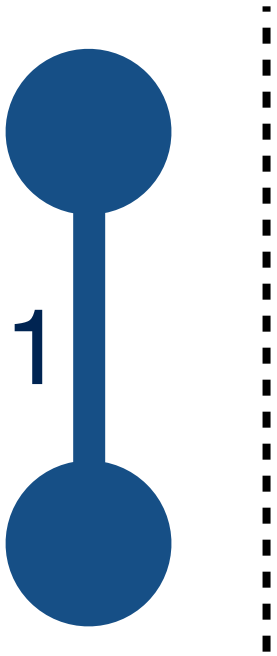

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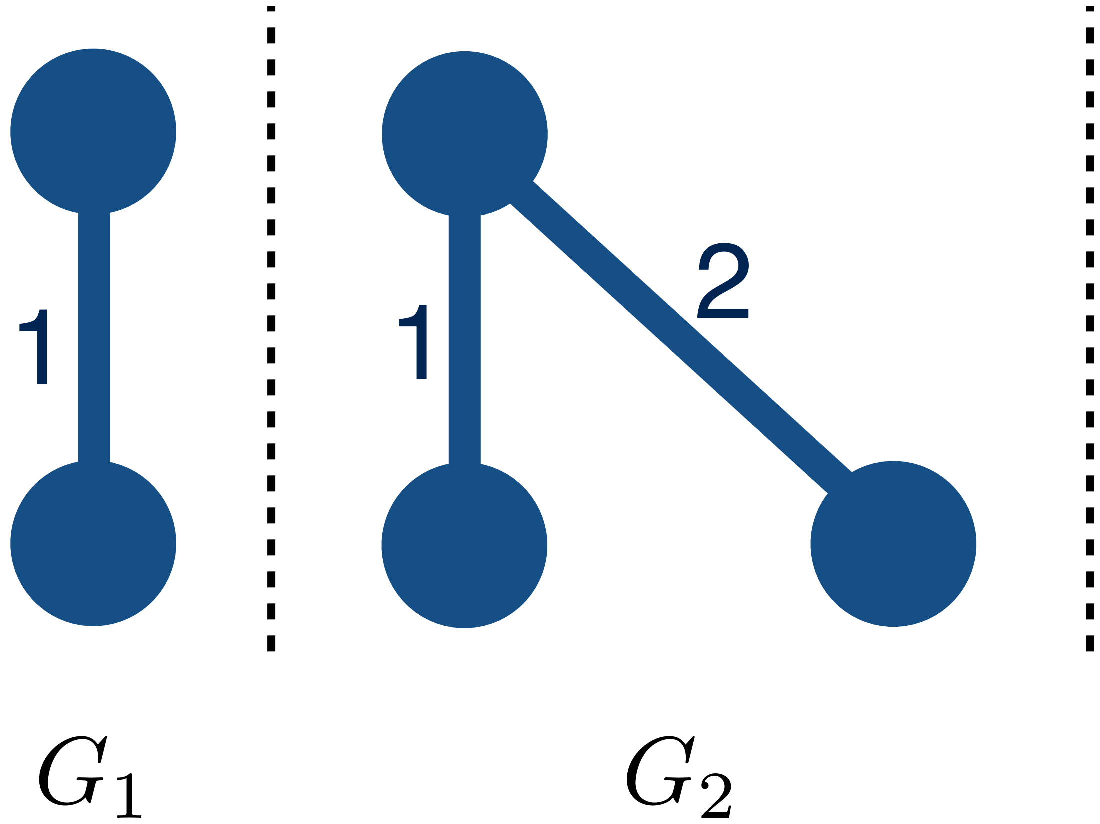

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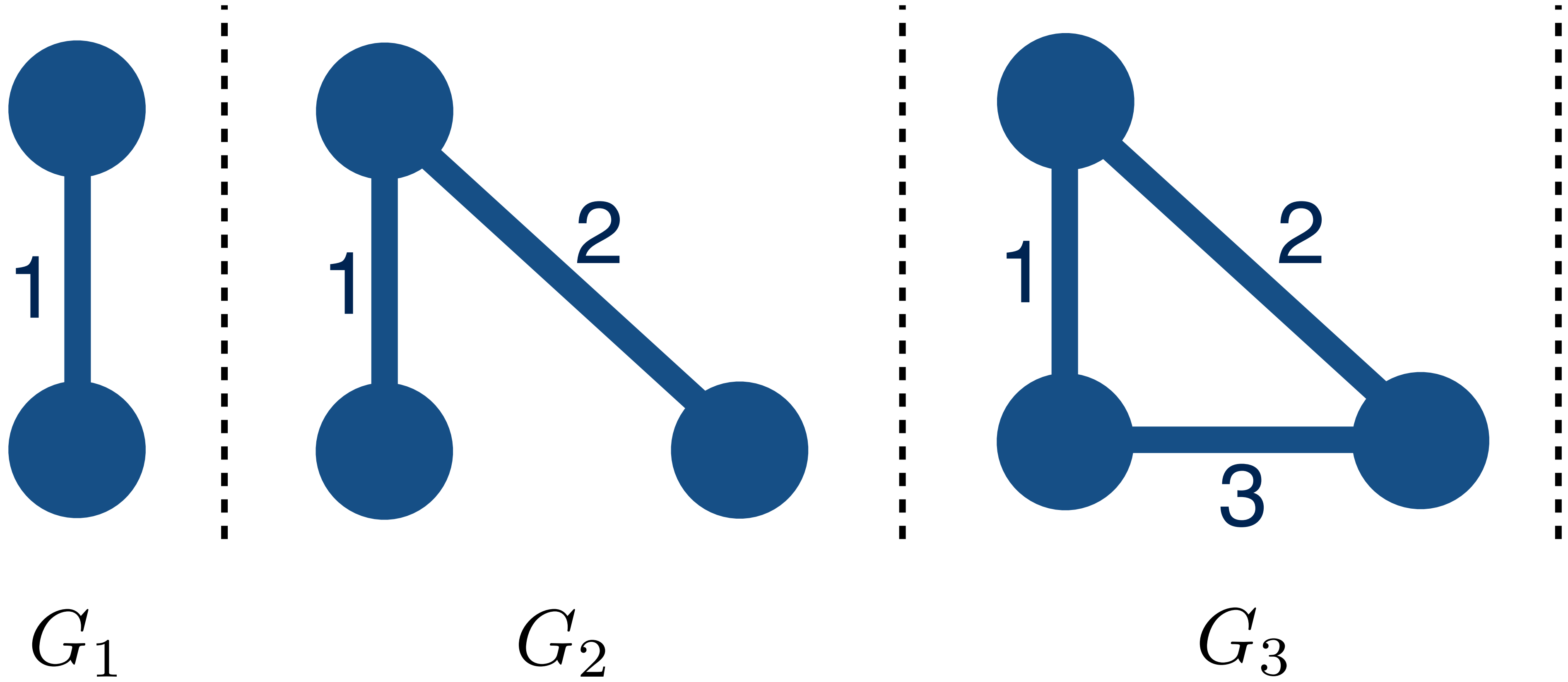

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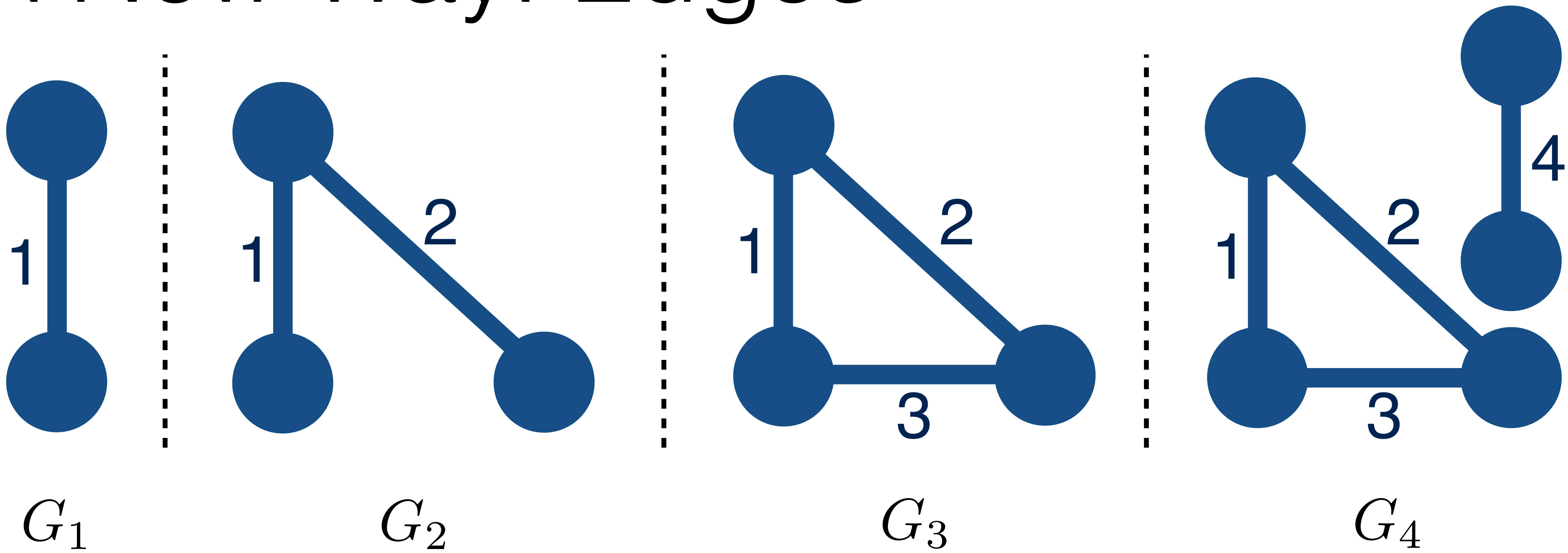

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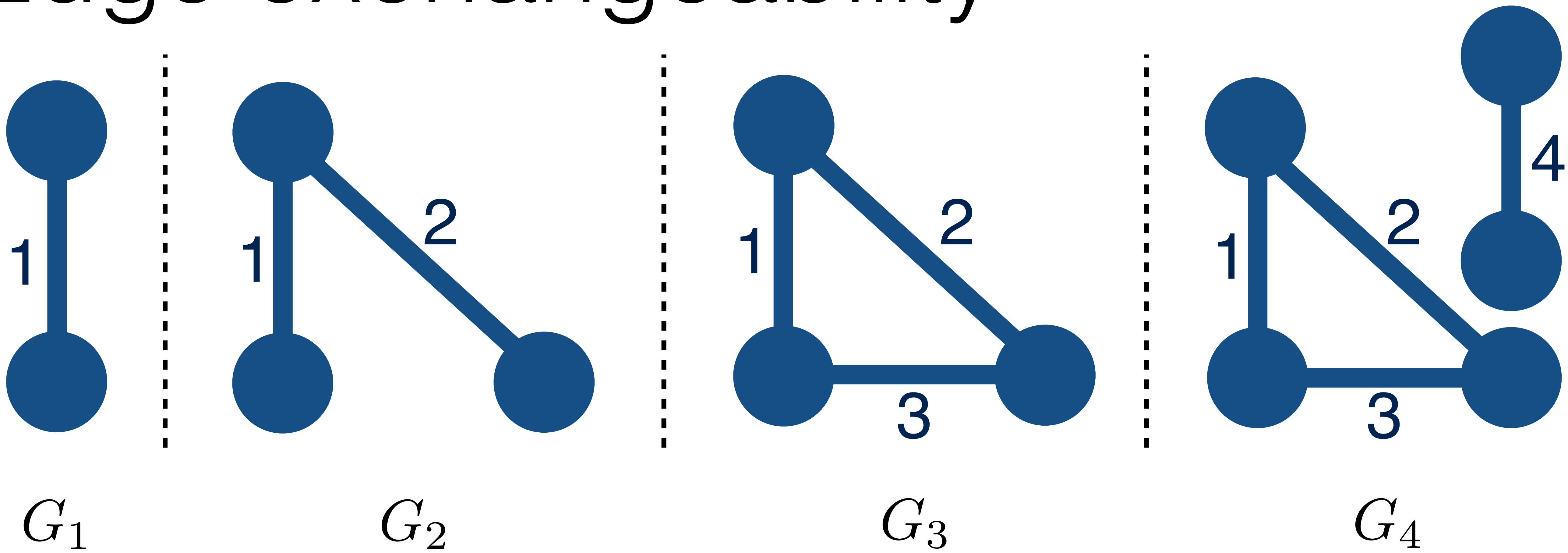
Edge exchangeability



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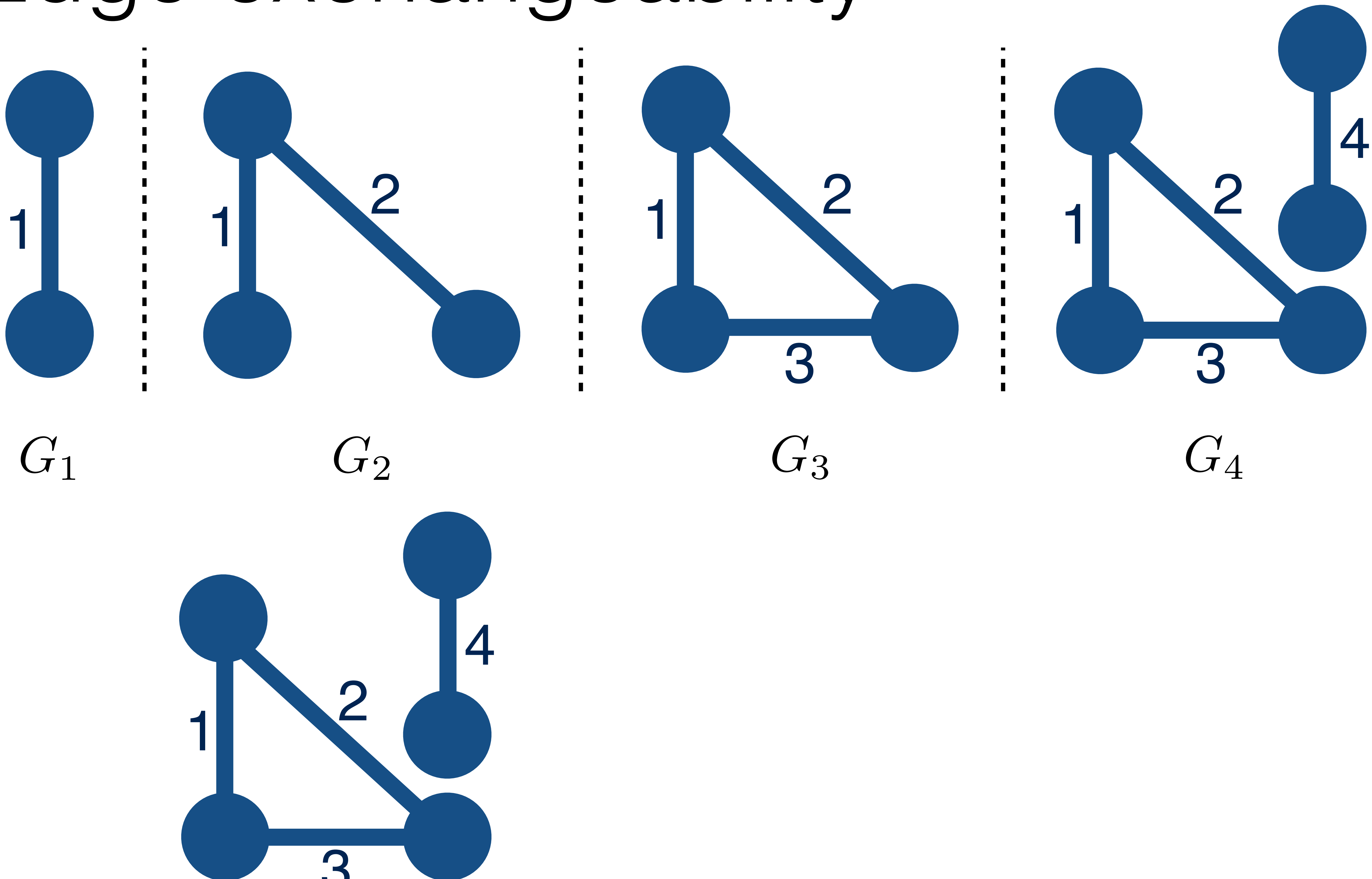

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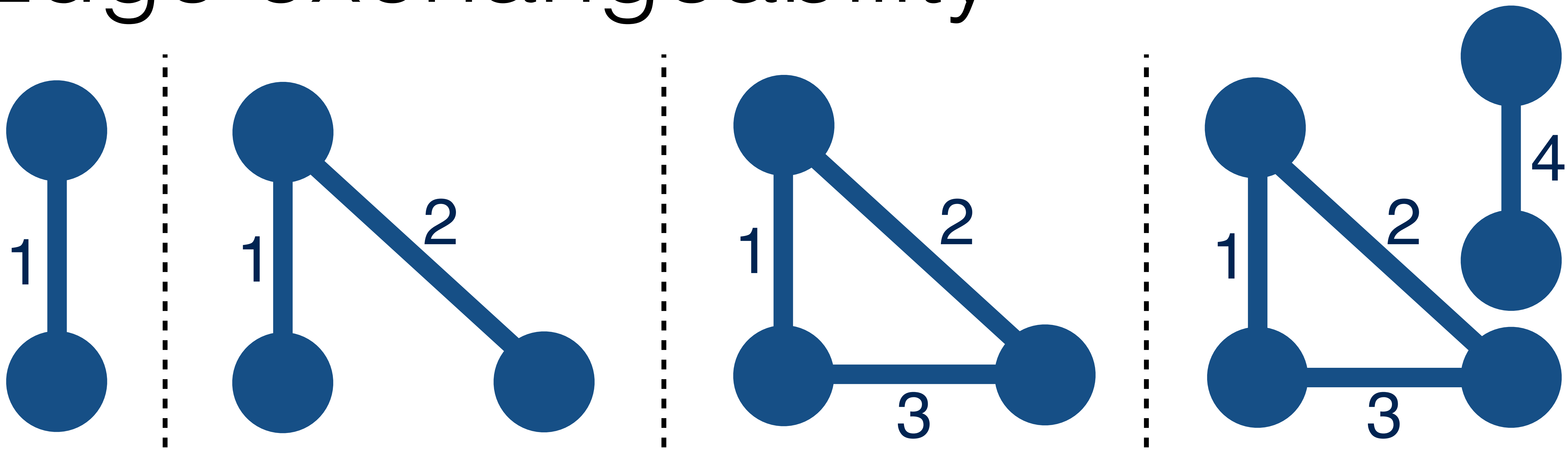
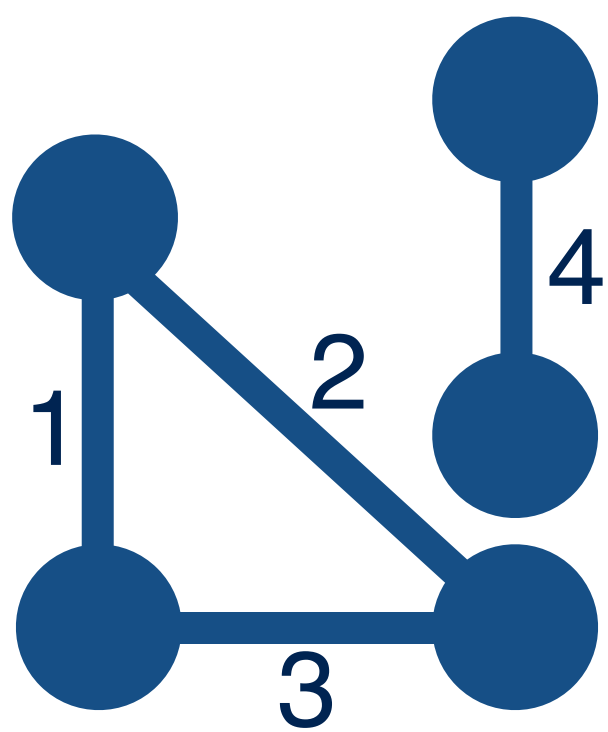


G_1 G_2 G_3 G_4

$p($



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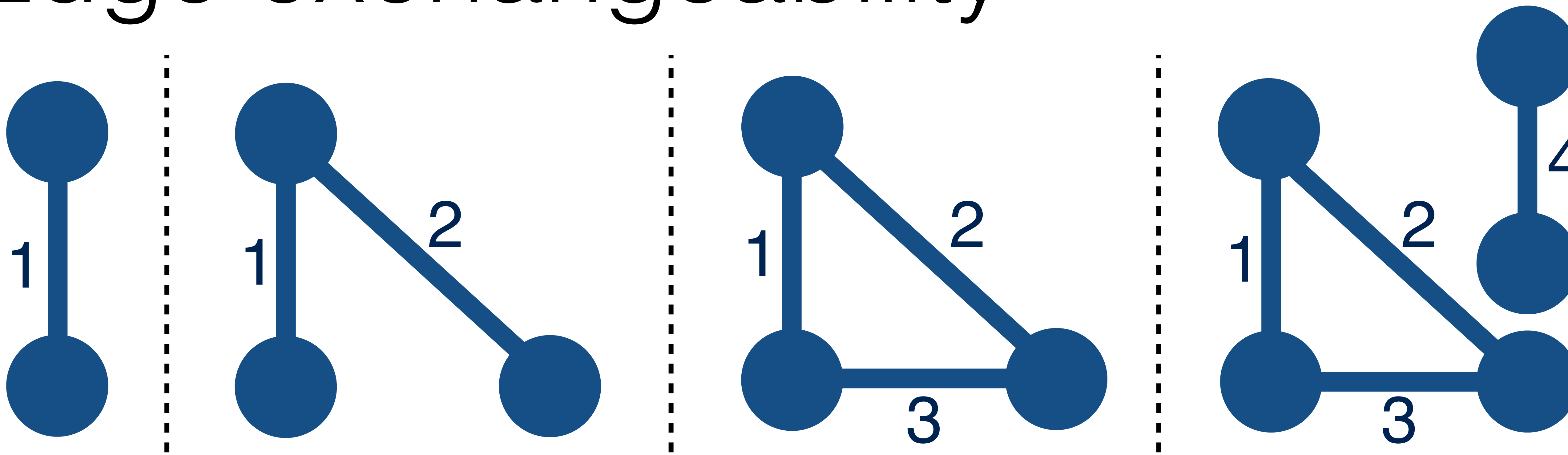
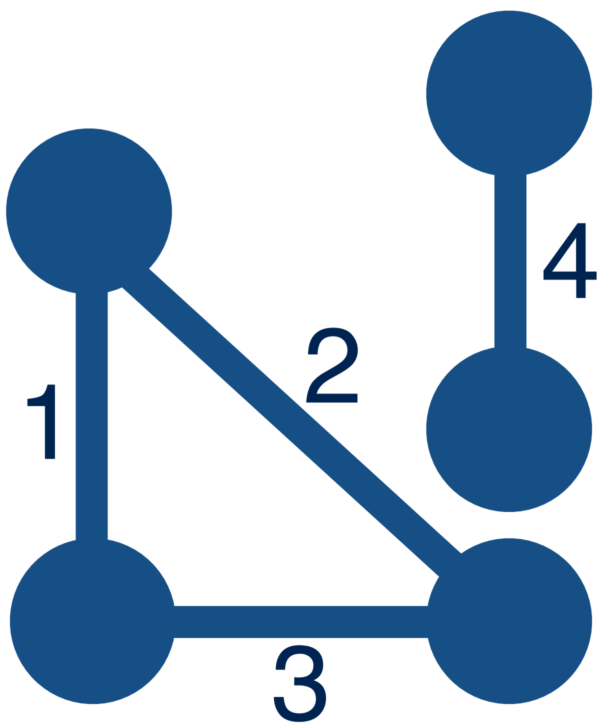
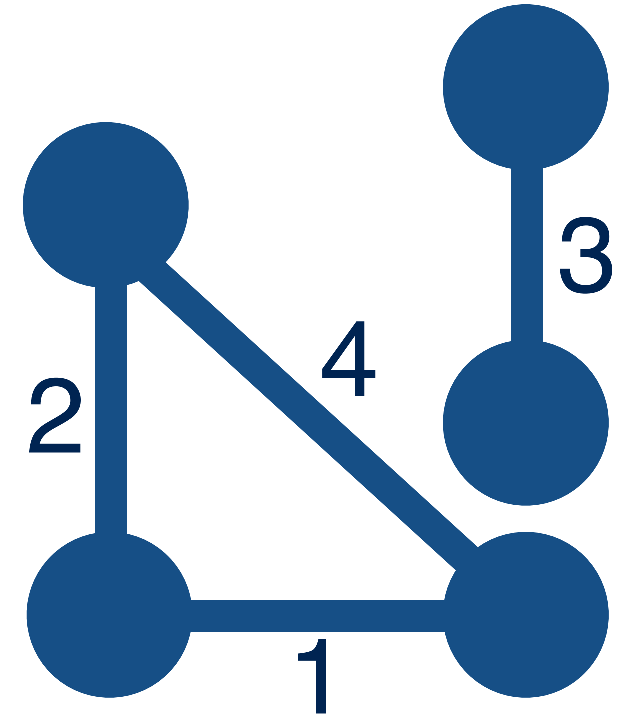


G_1 G_2 G_3 G_4

$p($ 

$) = p($ 

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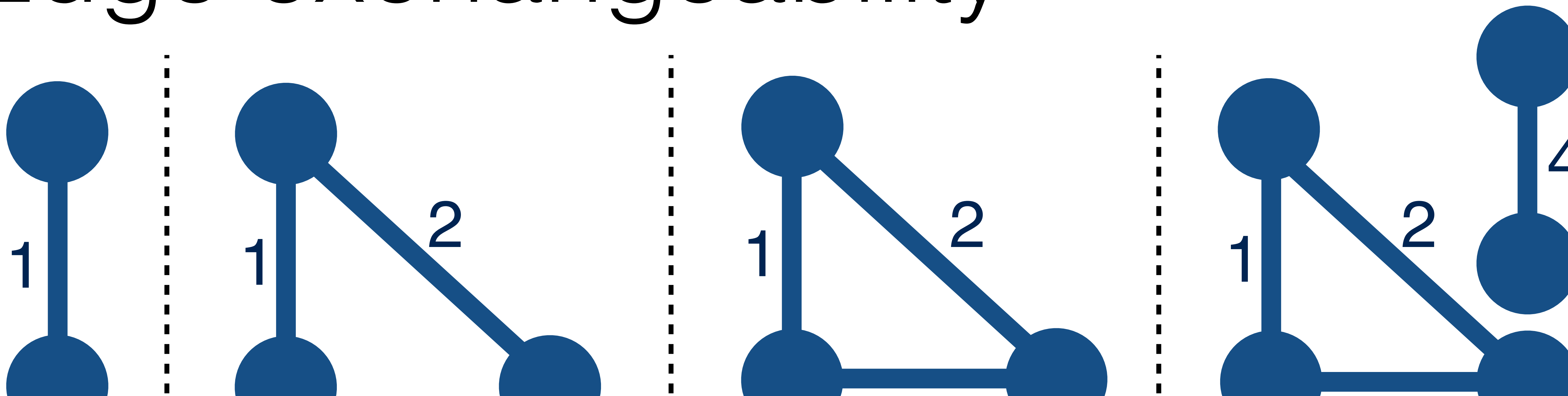
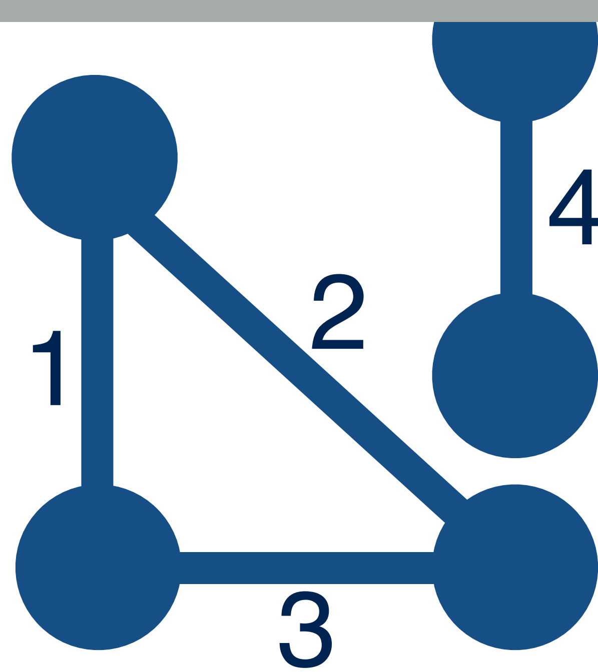
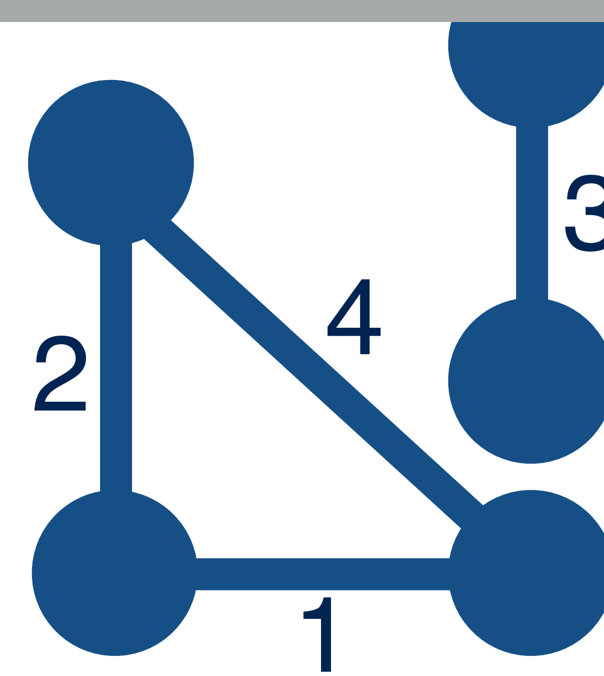


Thm (CCB). A wide class of edge-exchangeable graph models yields sparse graph sequences.

$p($ 

$) = p($ 

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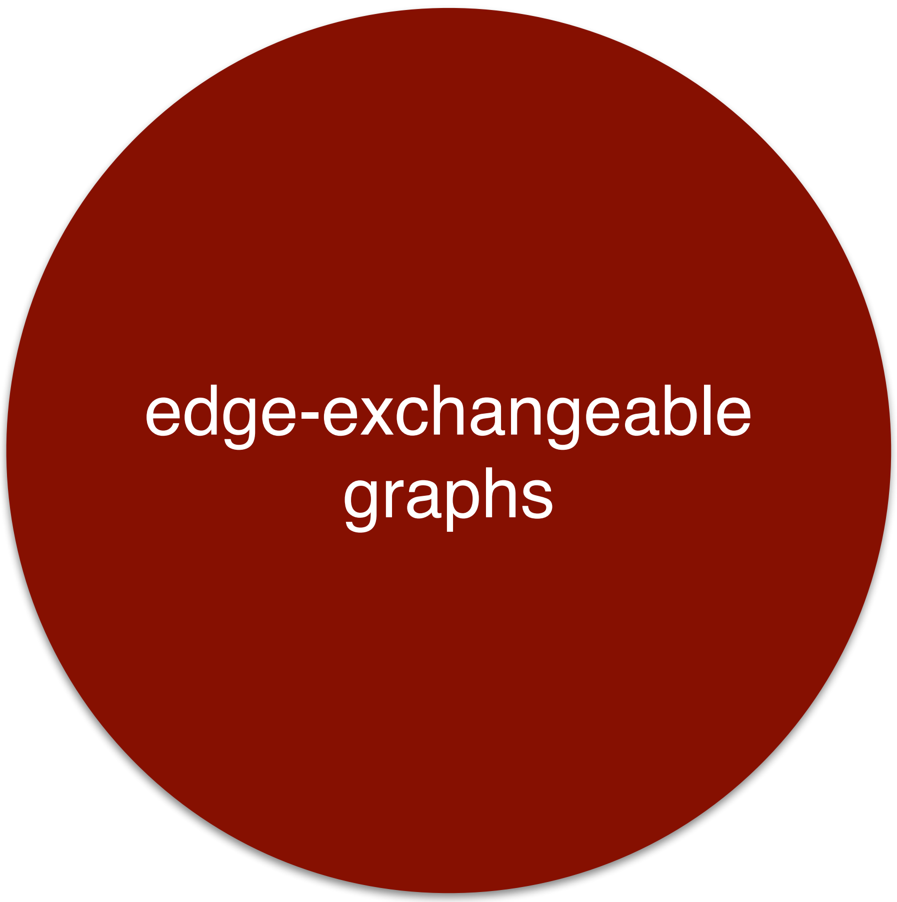
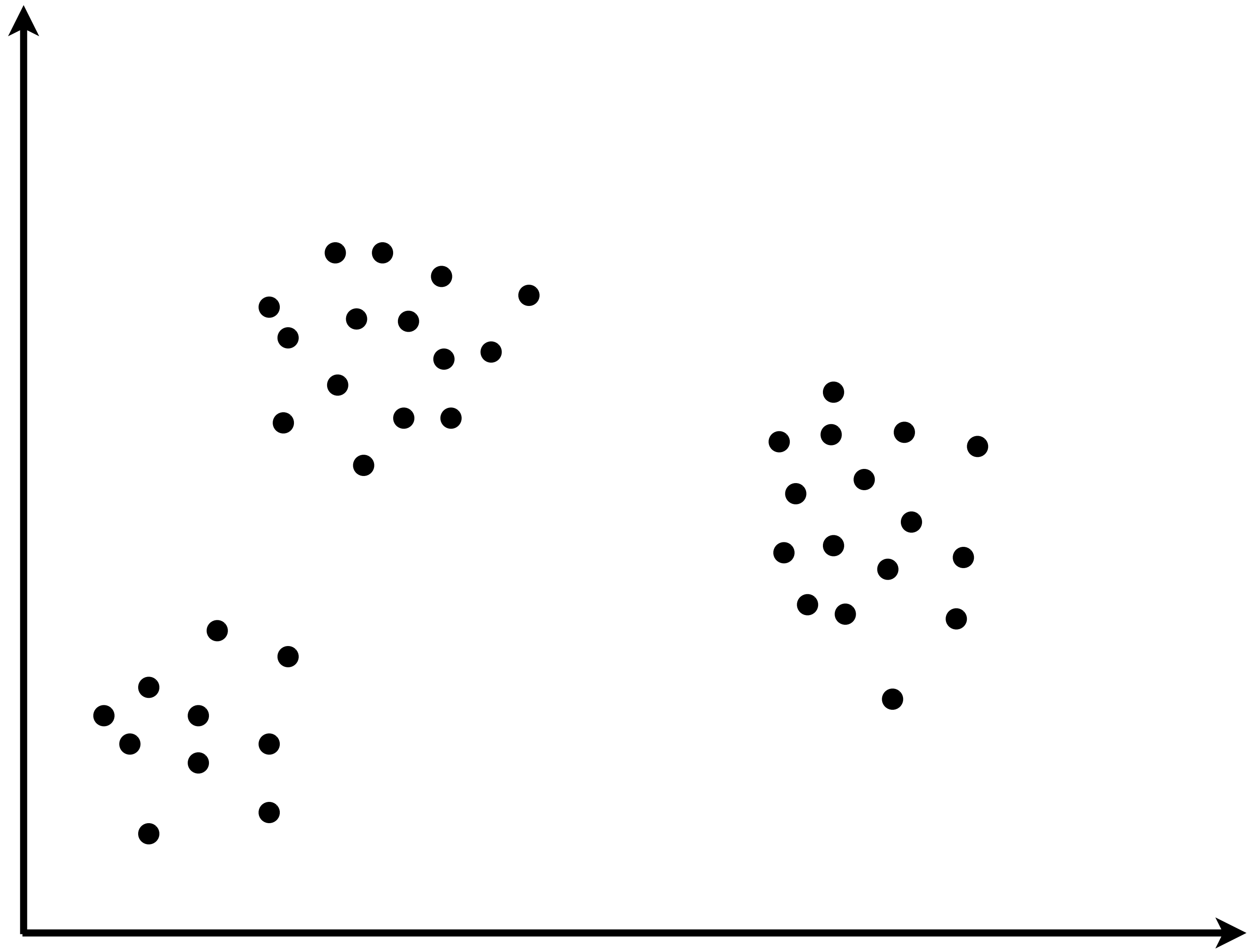
What we know so far



- Goal 1: characterization theorem for edge-exchangeable graphs
- Goal 2: sparsity theorem for edge-exchangeable graphs

Clustering



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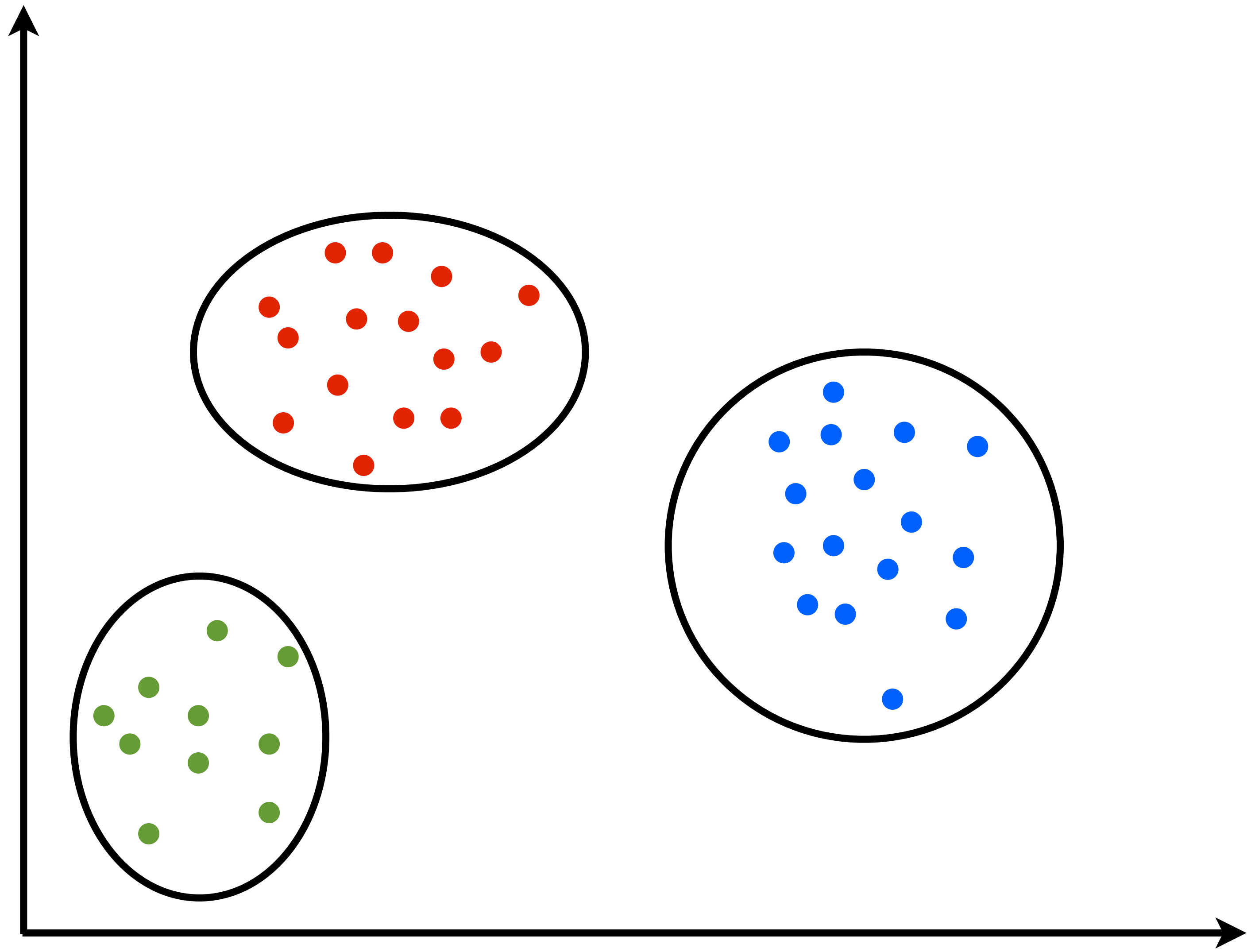
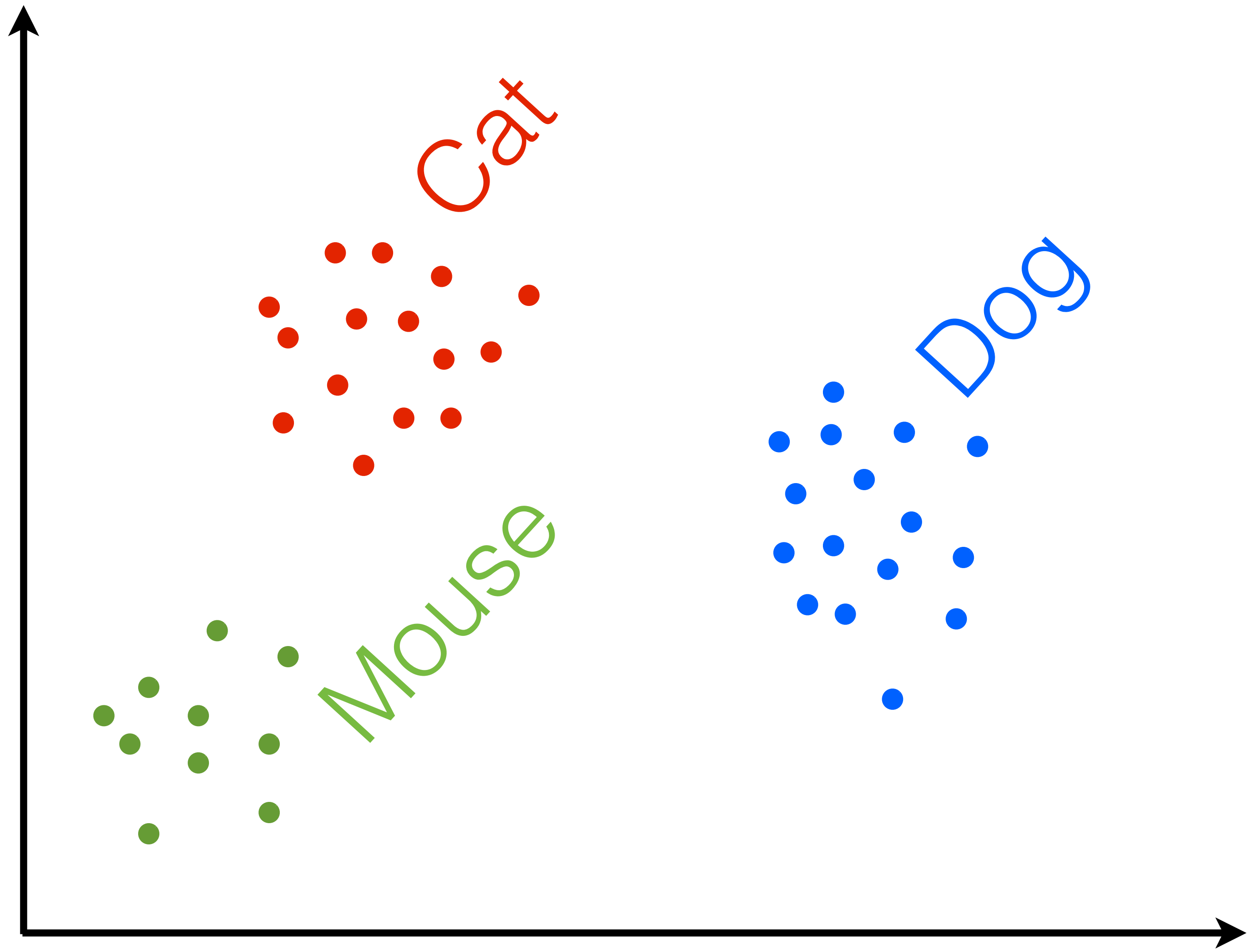


“Clusters”

Clustering



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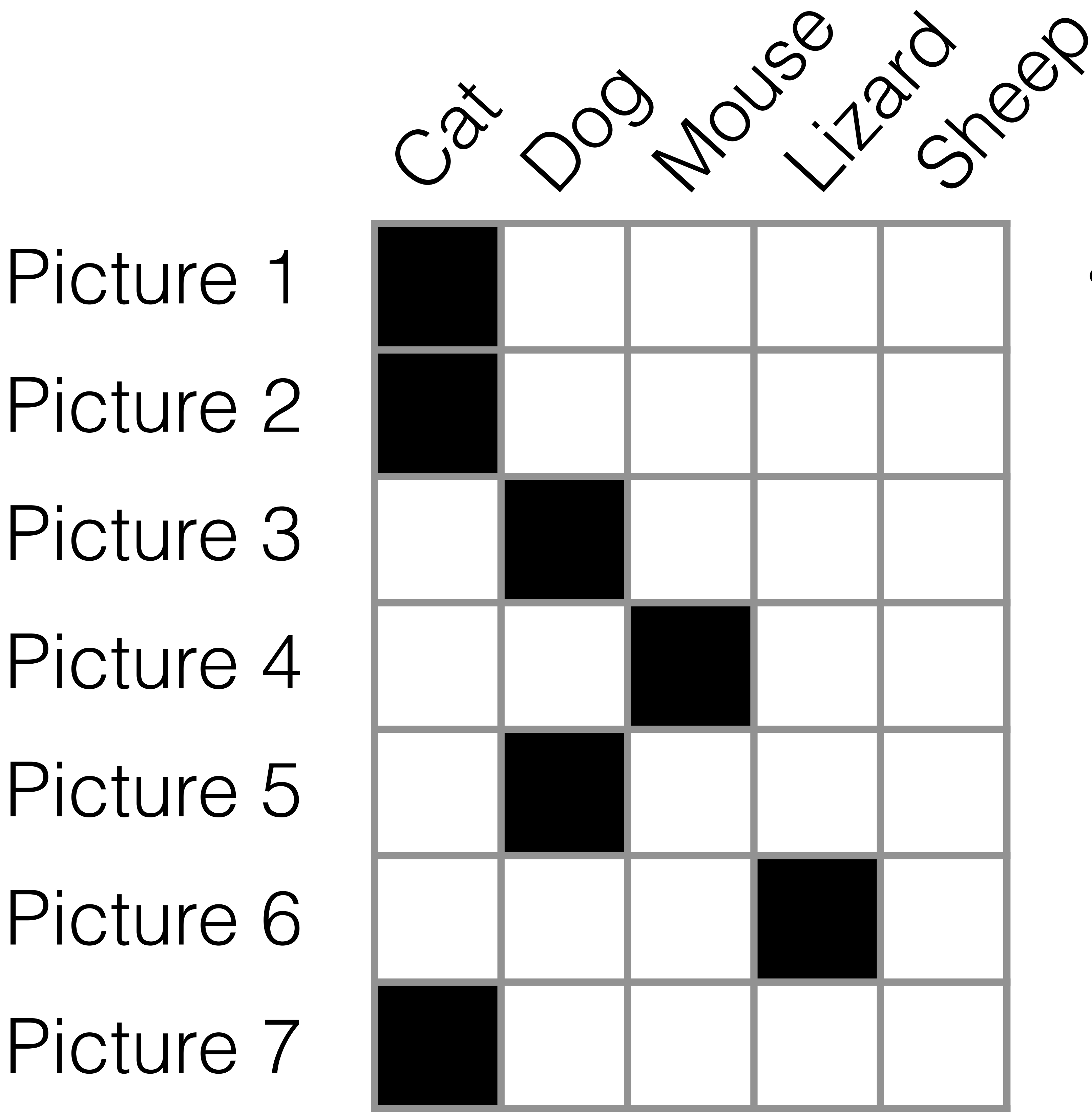
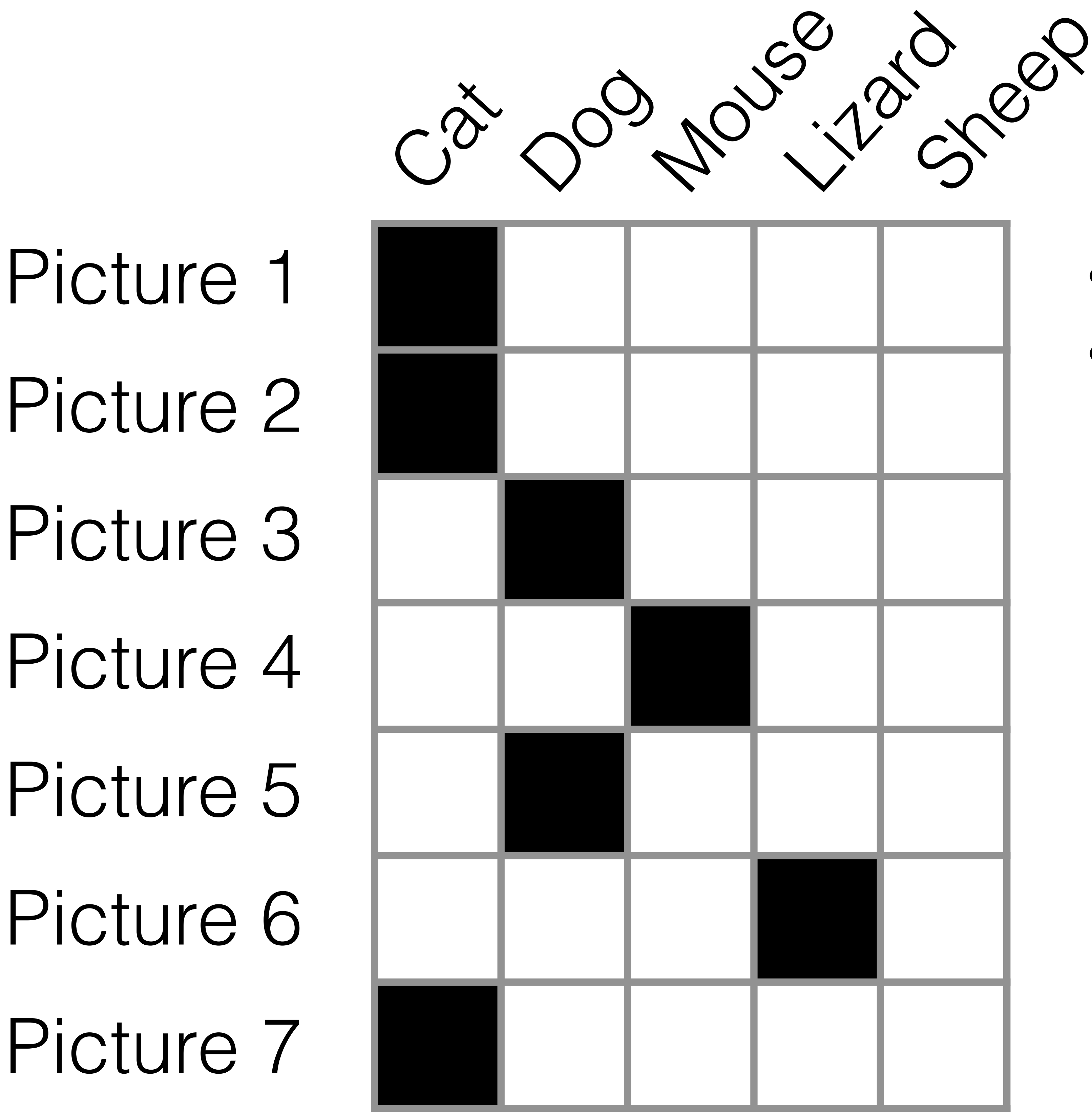


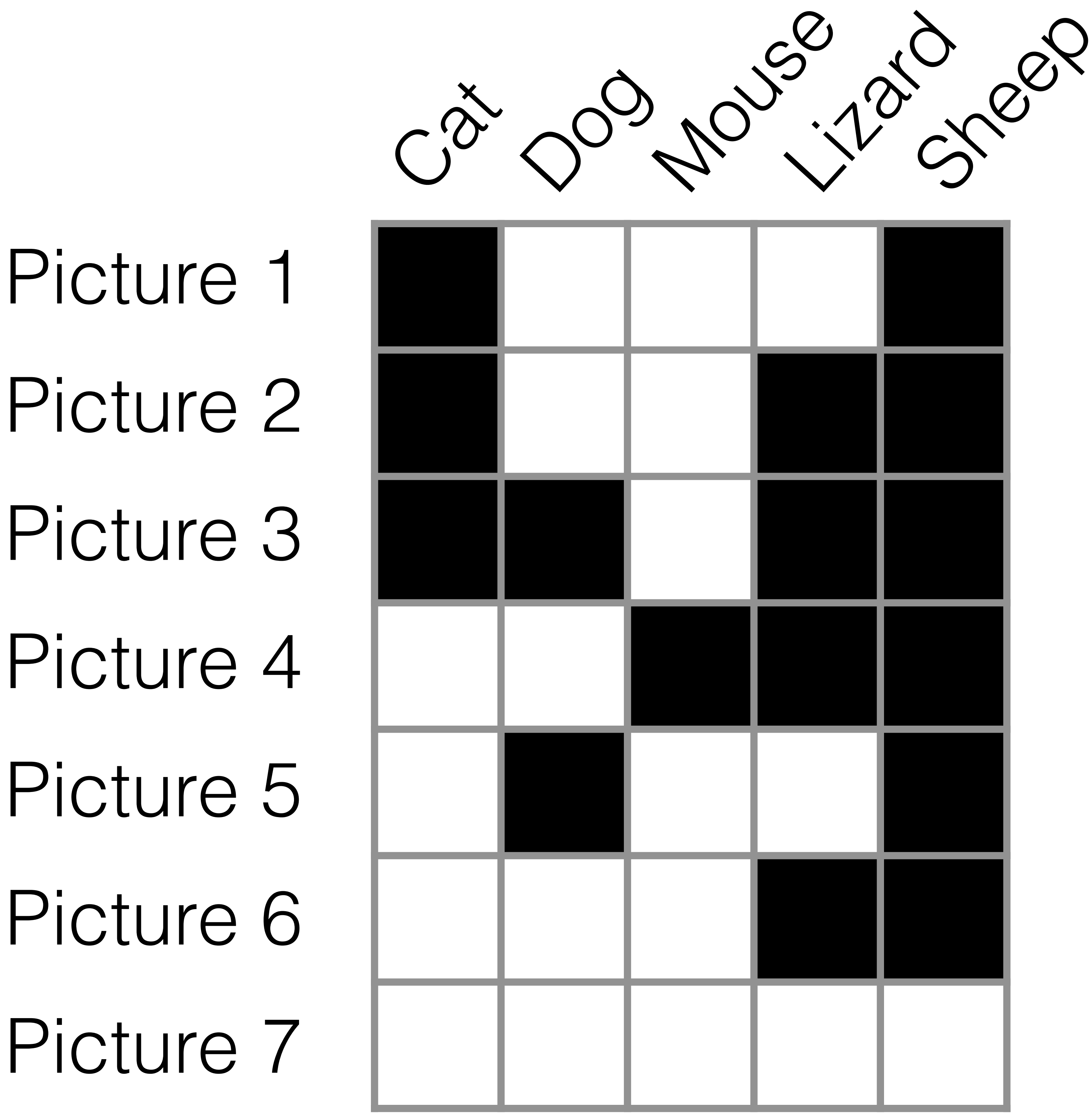
- Groups: clusters

Clustering



- Groups: clusters
- Exchangeable

Feature allocation



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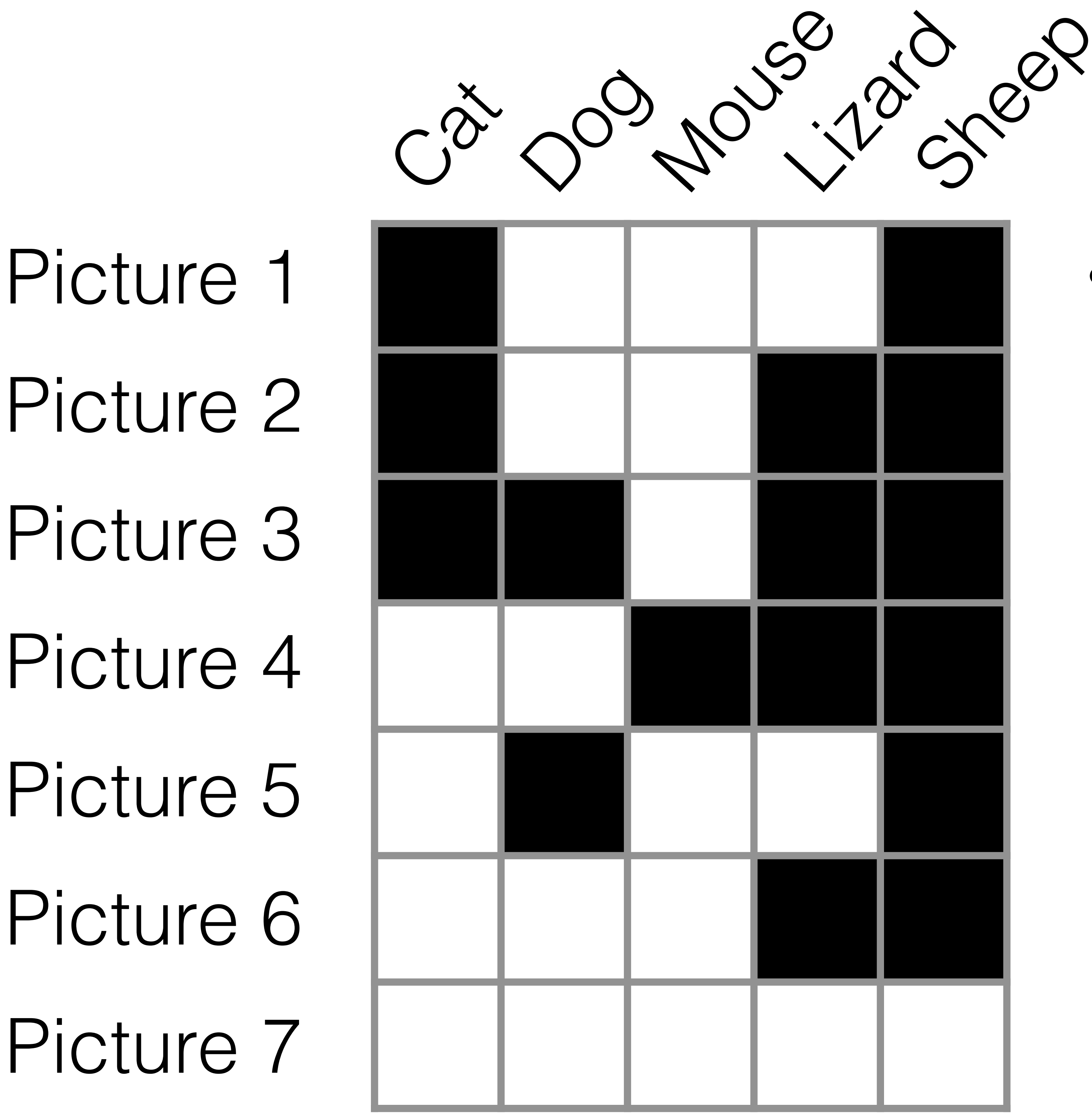
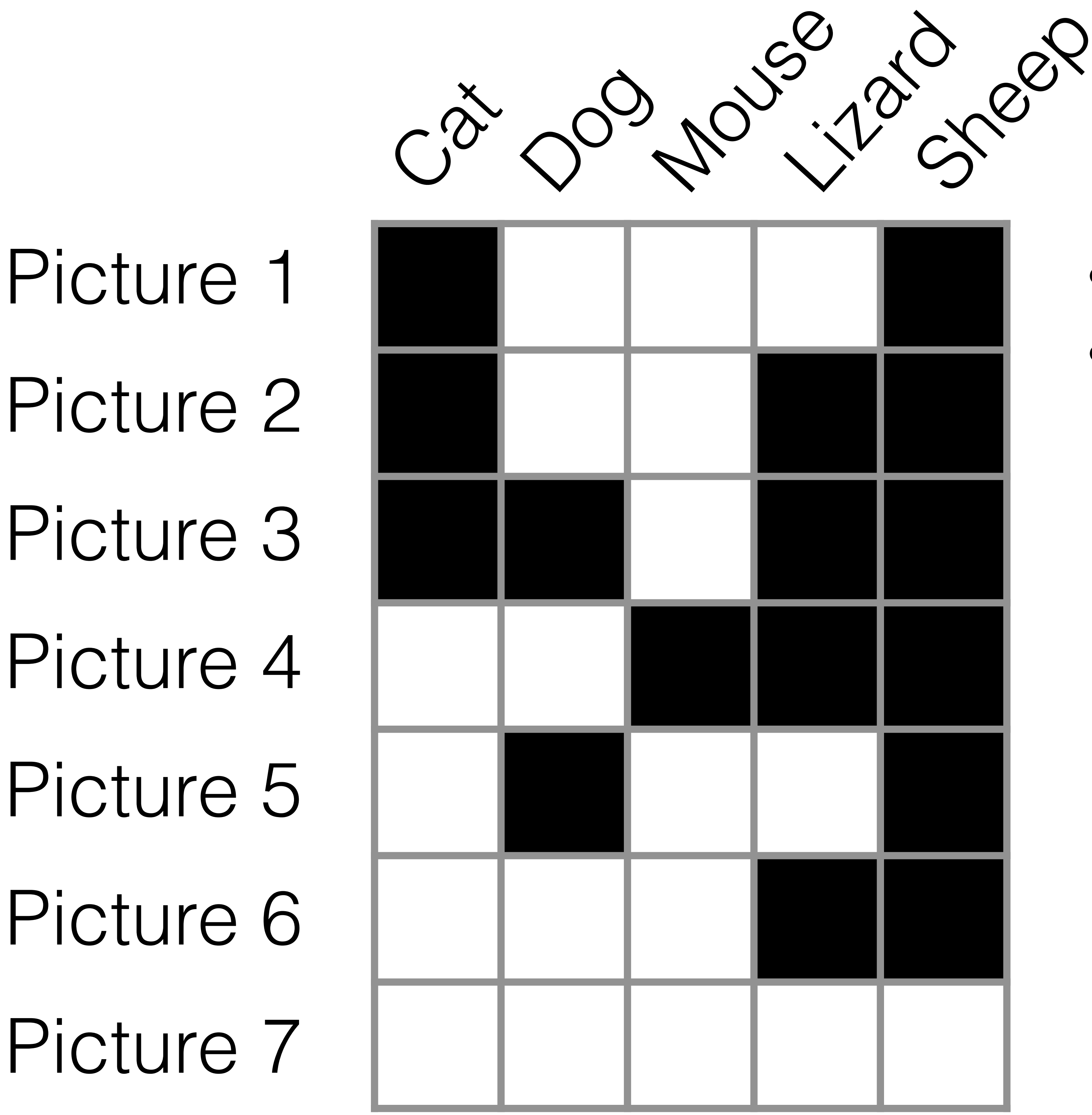


- Groups: features

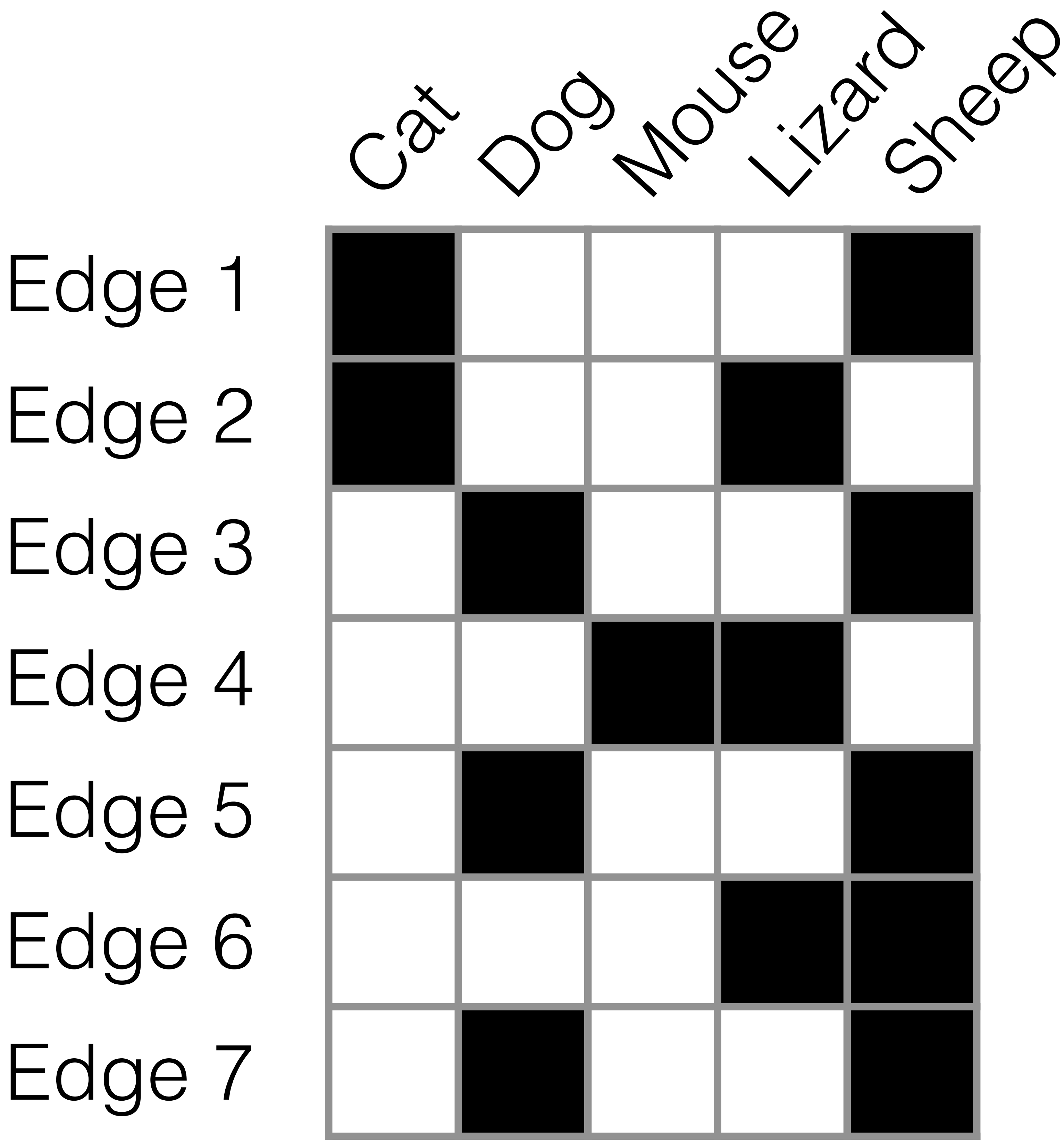
Feature allocation



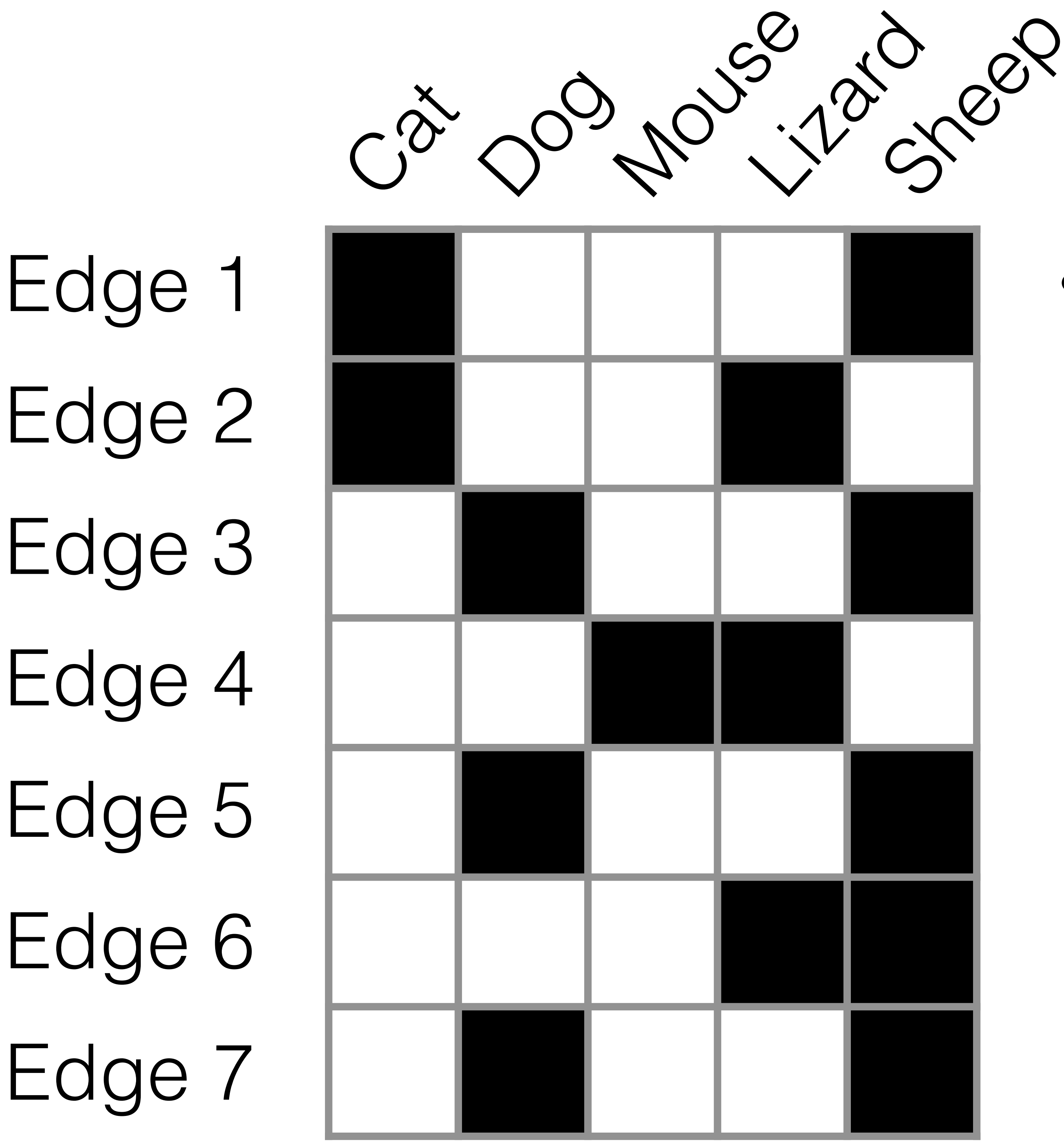
- Groups: features
- Exchangeable

[Broderick, Jordan, Pitman 2013; Broderick, Pitman, Jordan 2013]

Graph

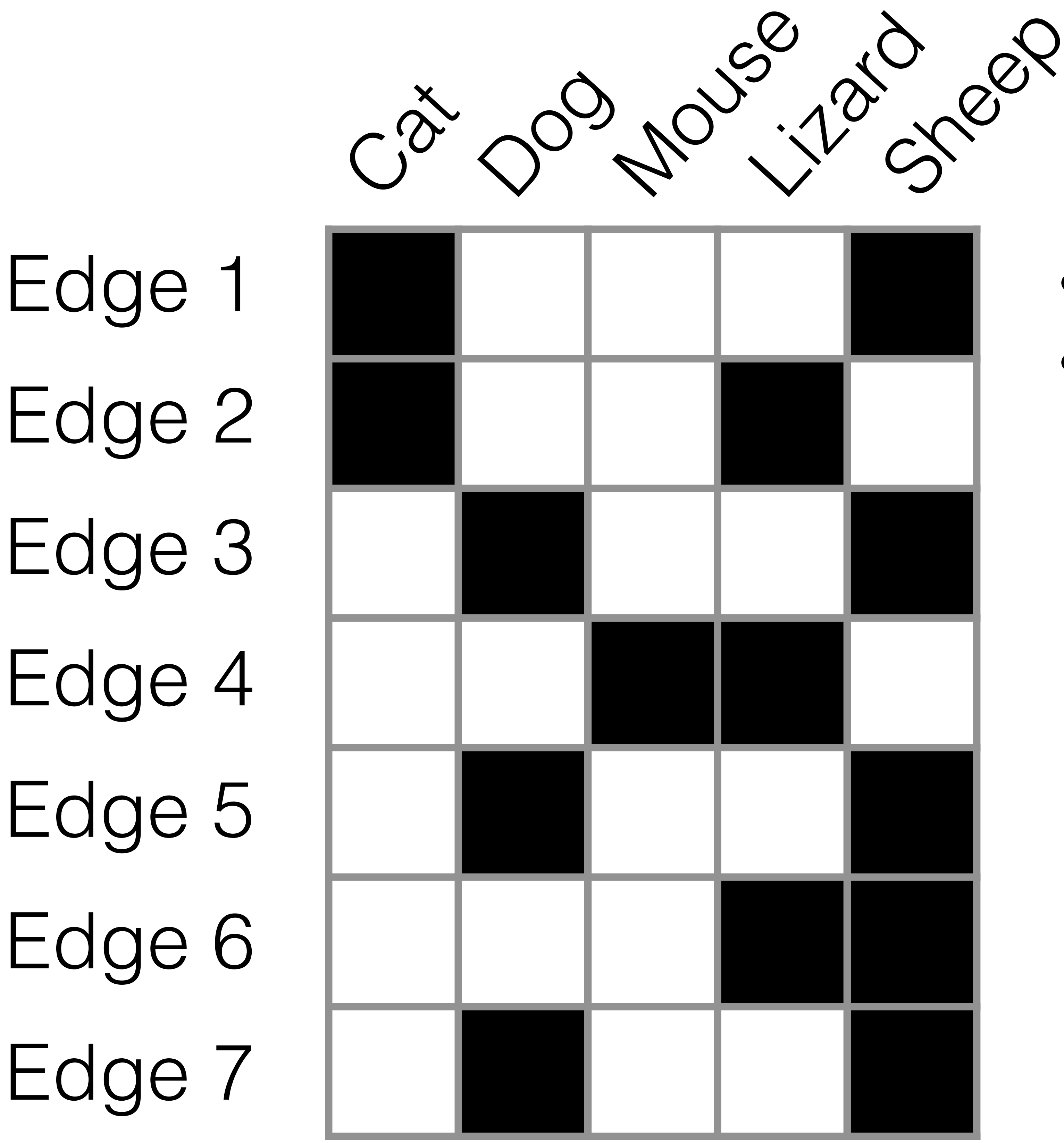

Cat
Dog
Mouse
Lizard
Sheep
Edge 1
Edge 2
Edge 3
Edge 4
Edge 5
Edge 6
Edge 7

Graph



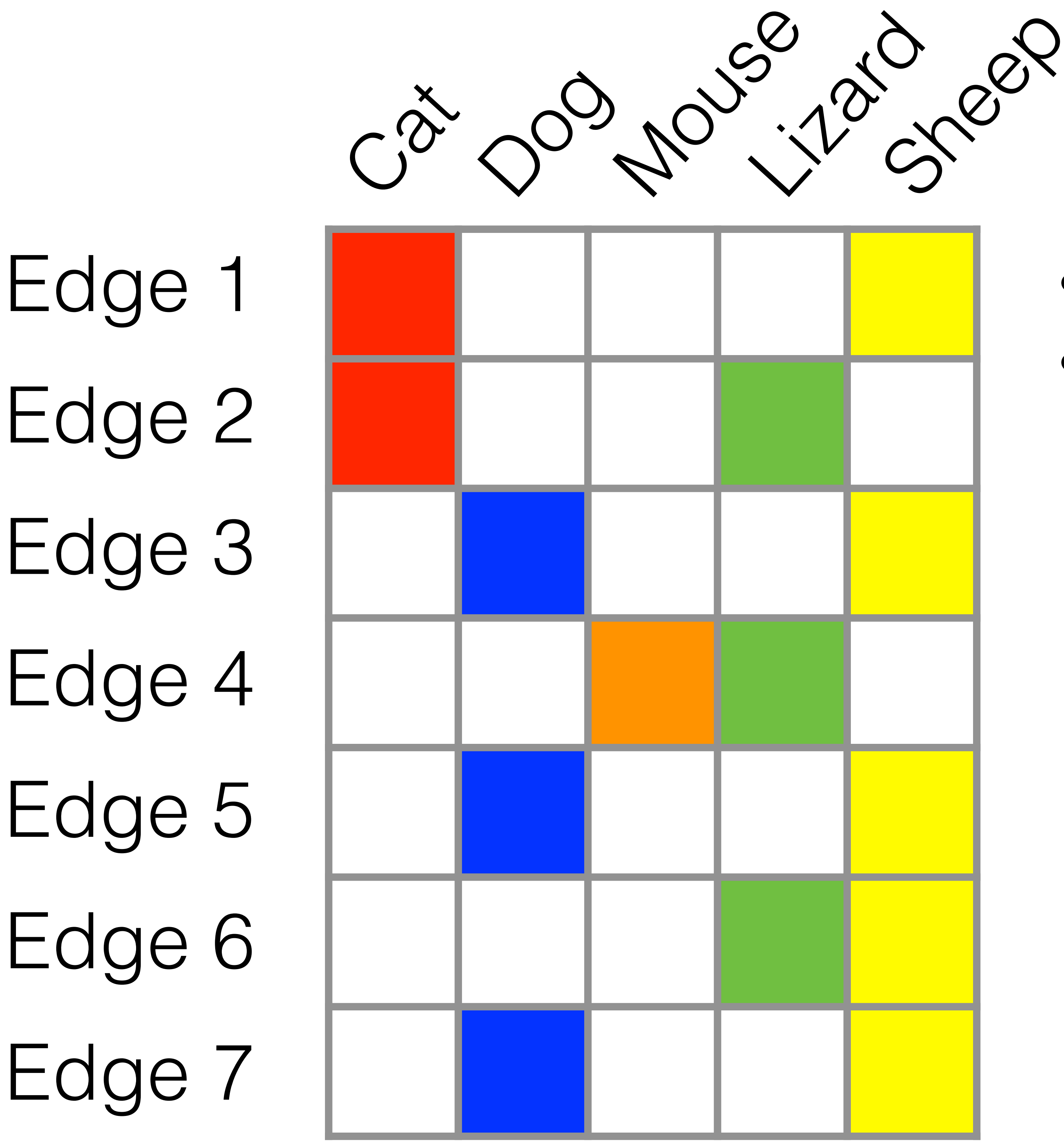
- Groups: vertices

Graph



- Groups: vertices
- Edge-exchangeable

Graph



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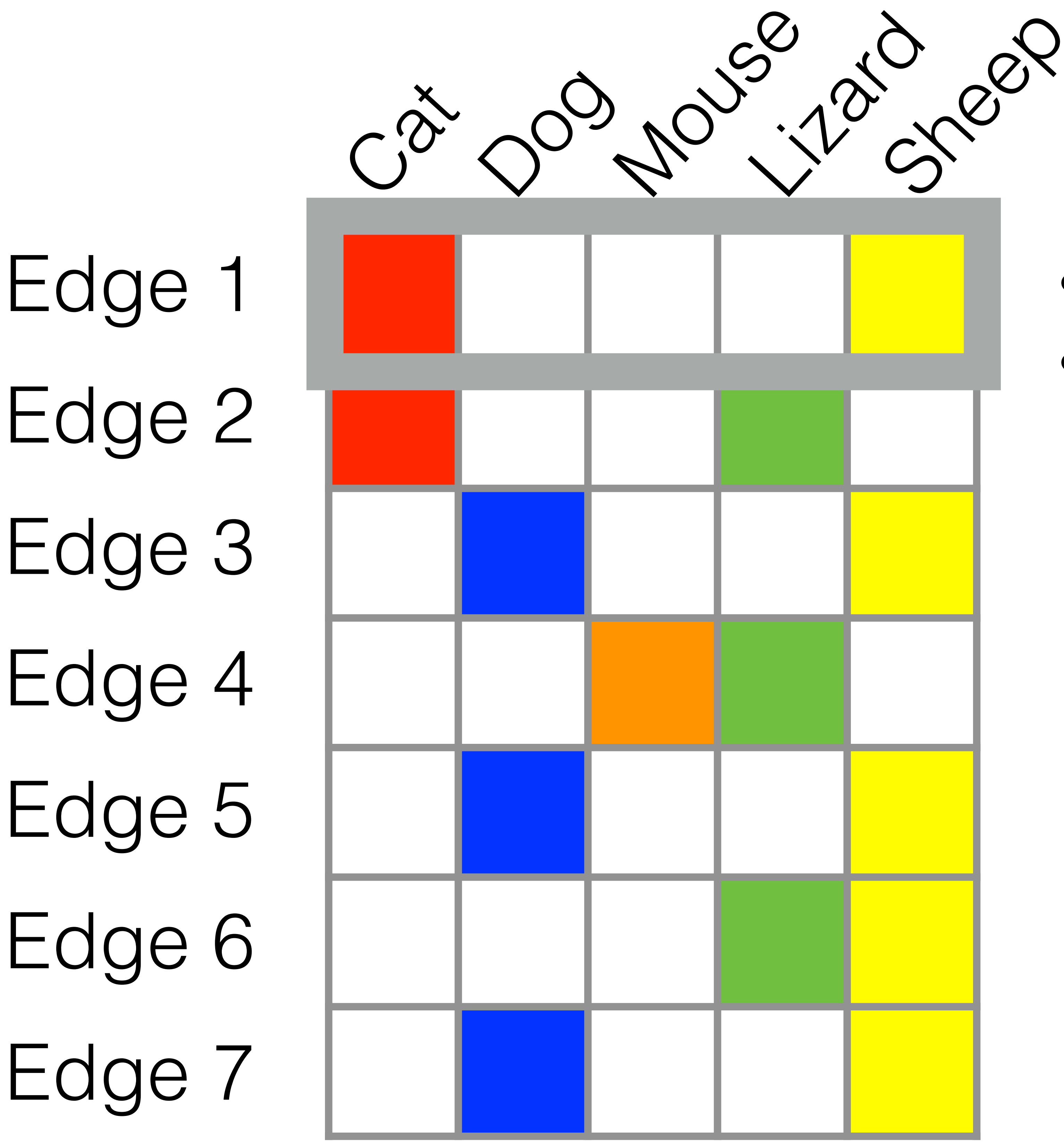

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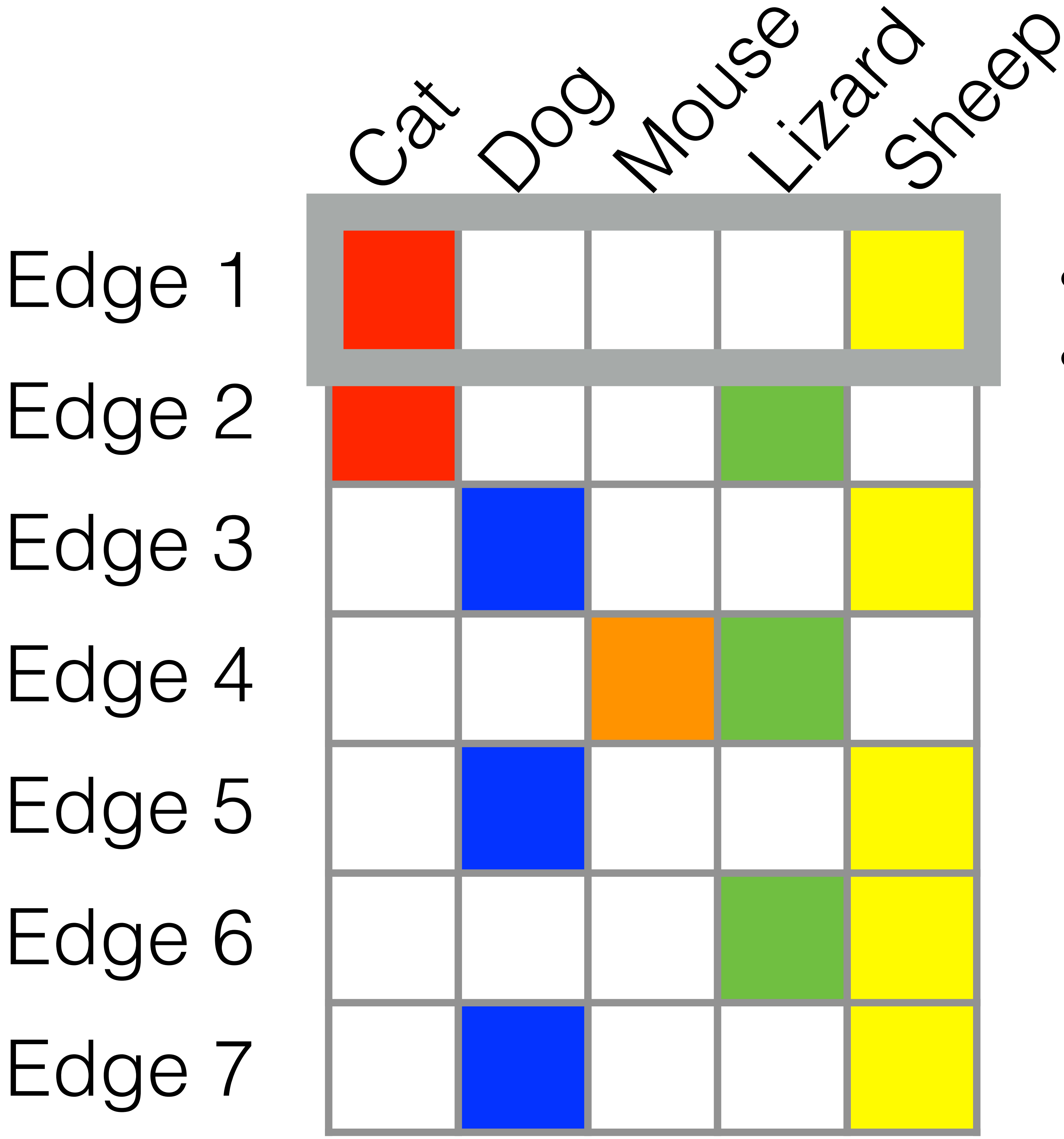

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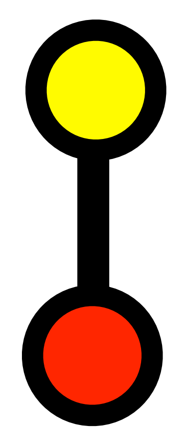

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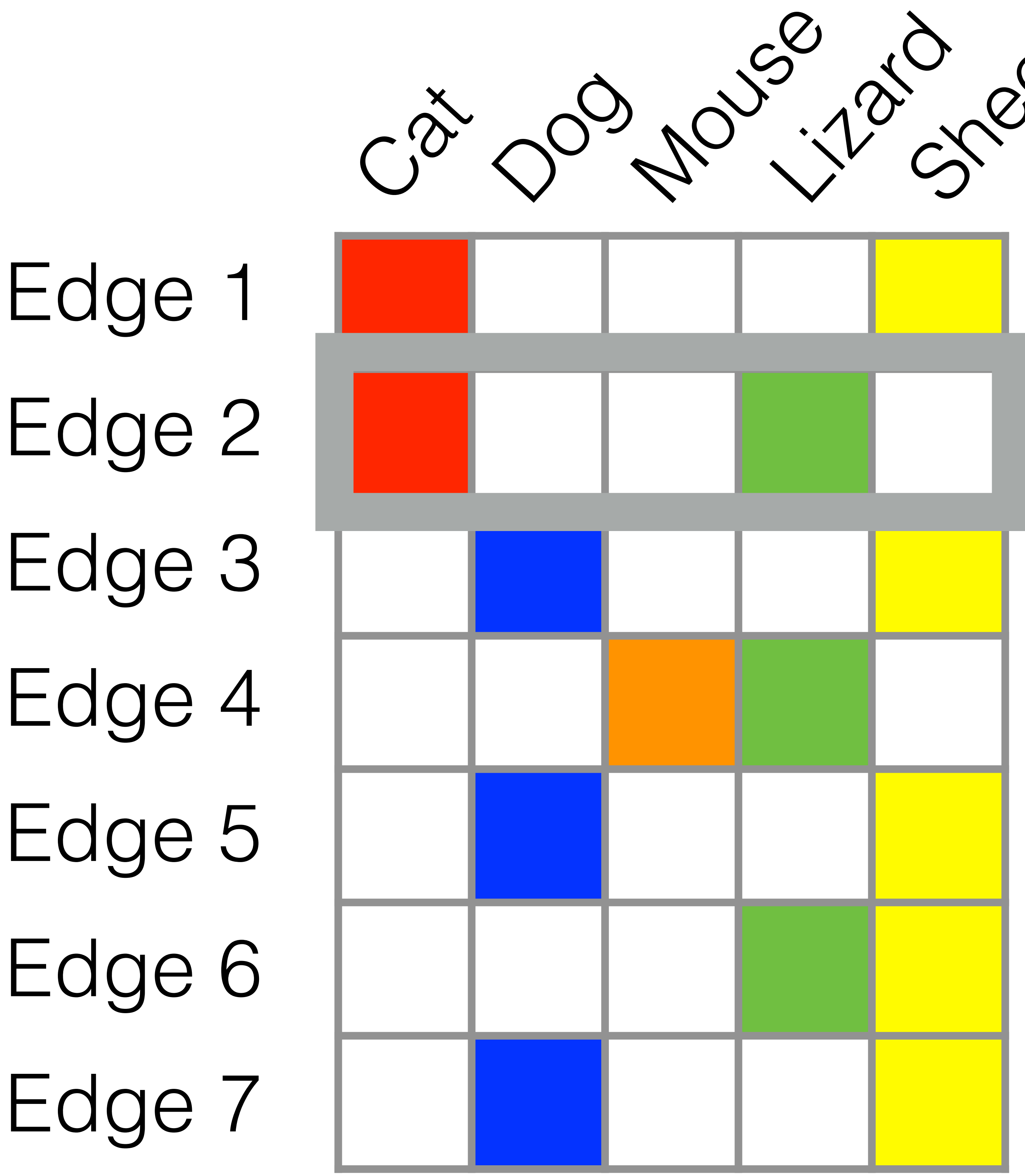

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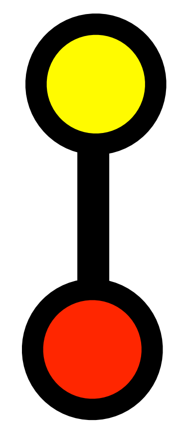
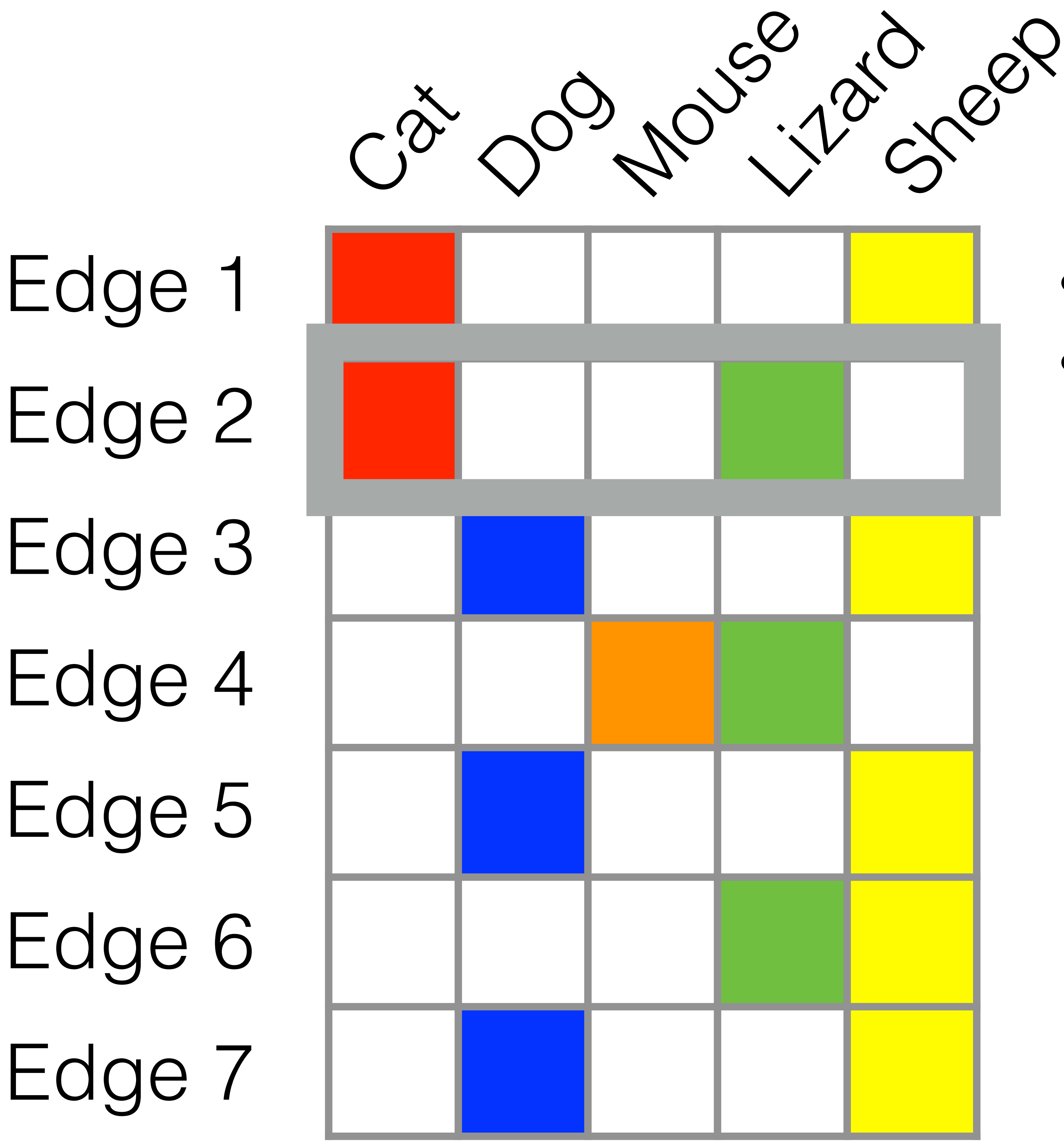


Graph



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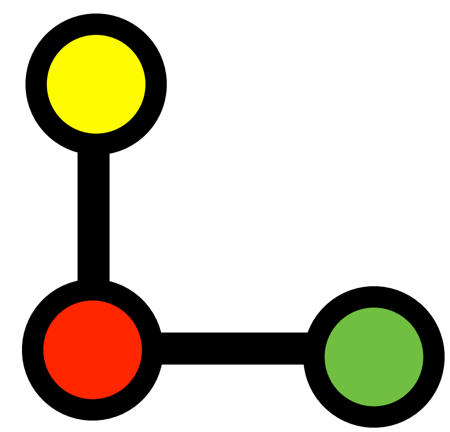

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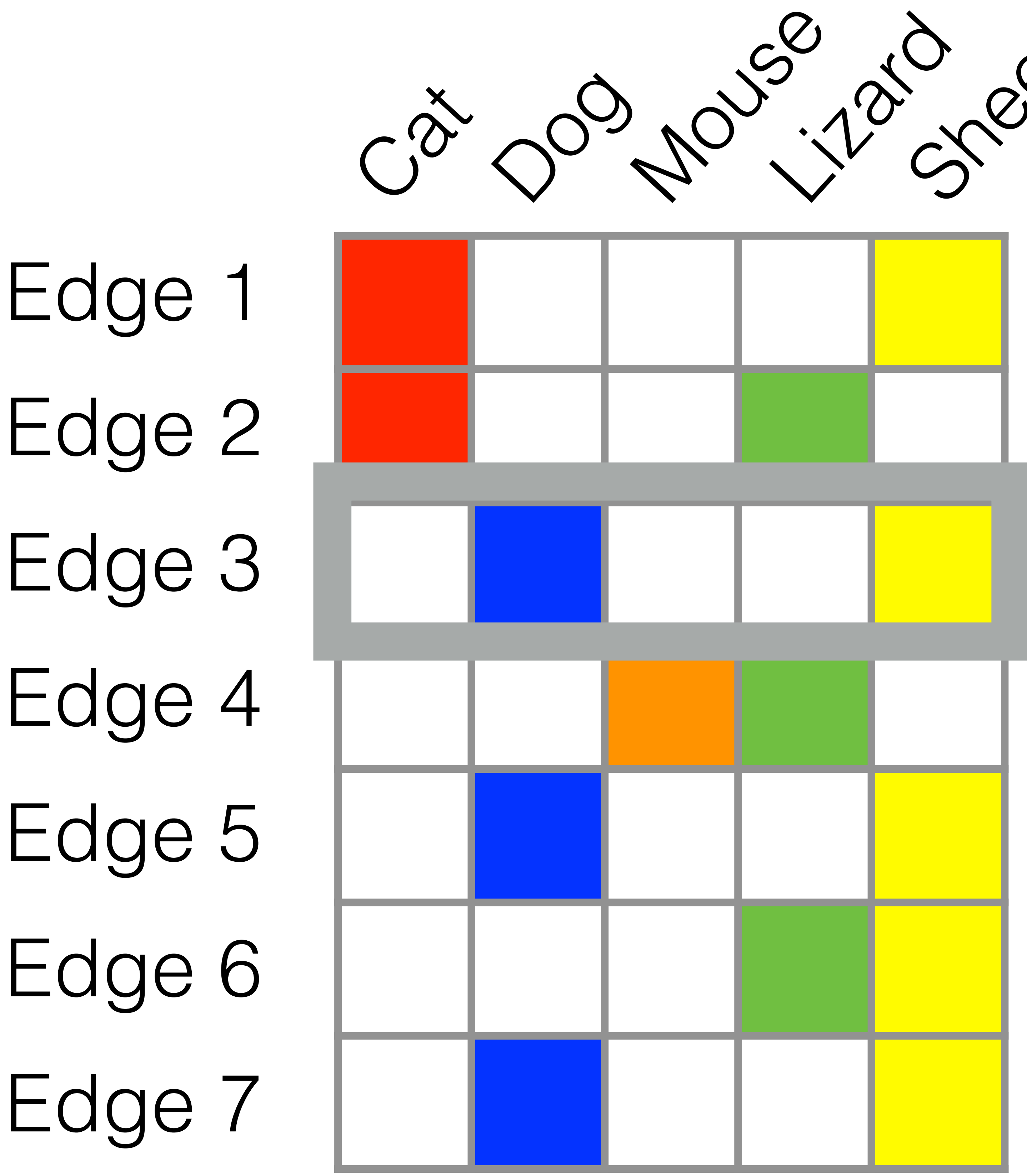

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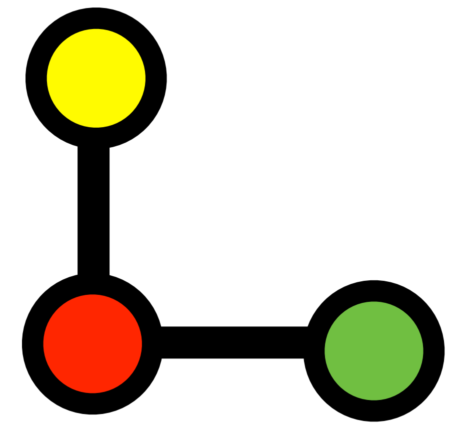
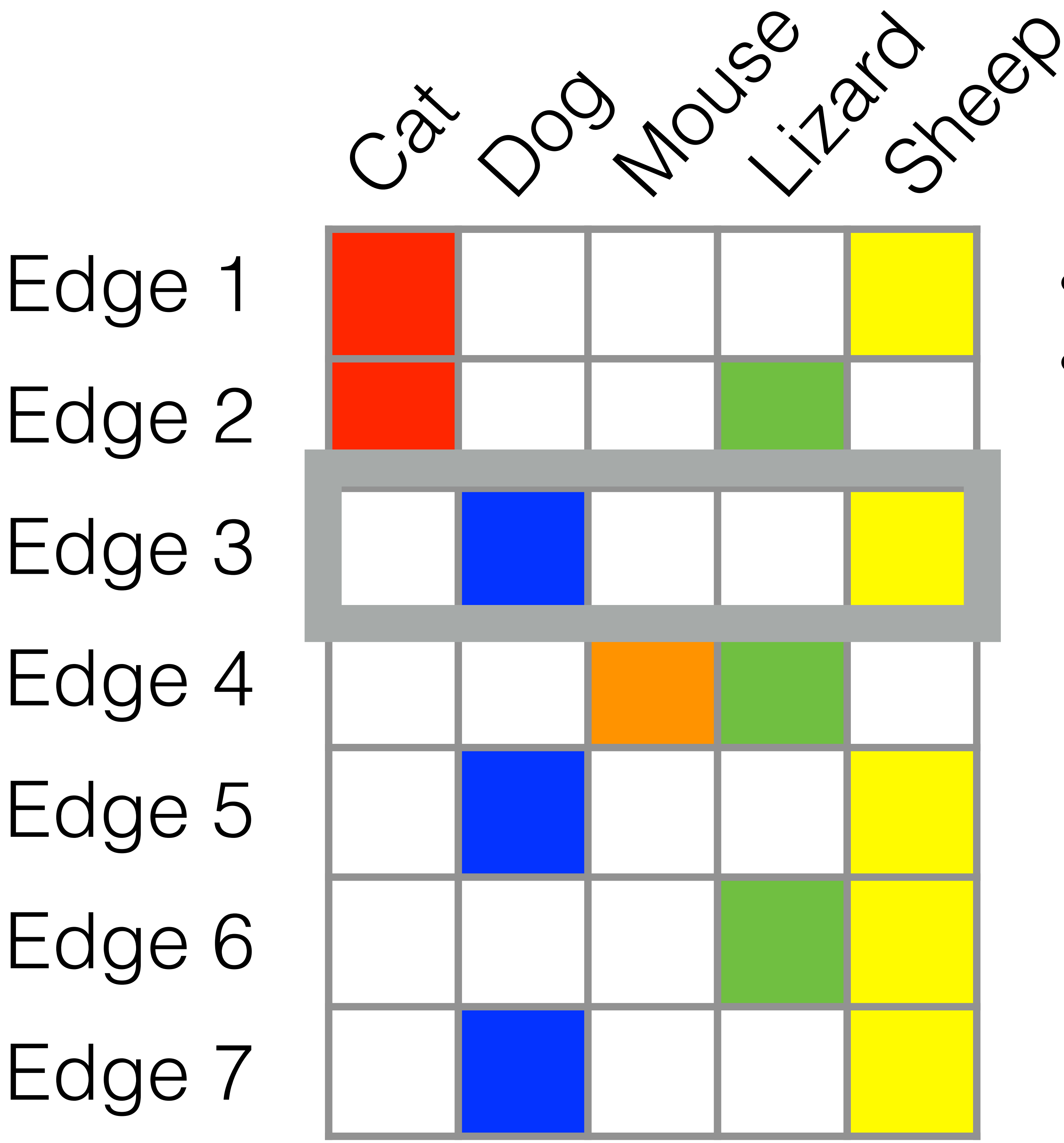
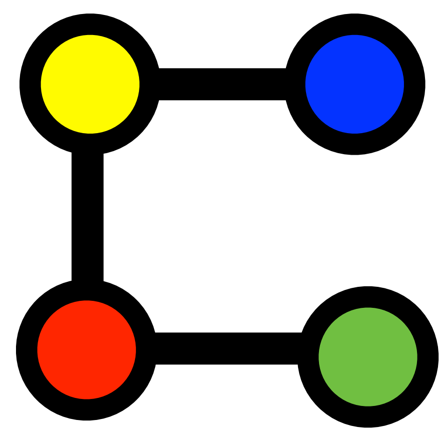


Graph



- Groups: vertices
- Edge-exchangeable



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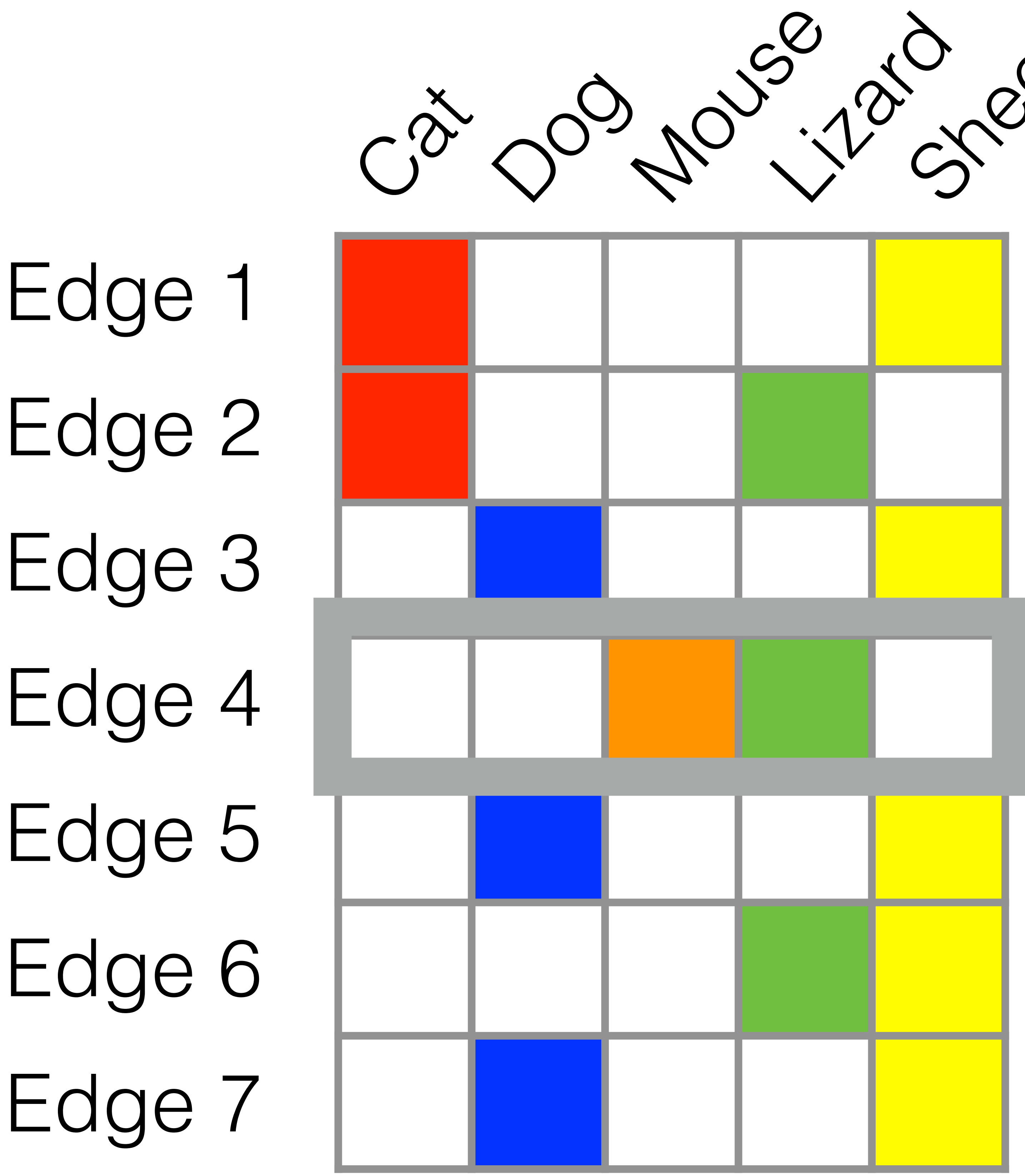

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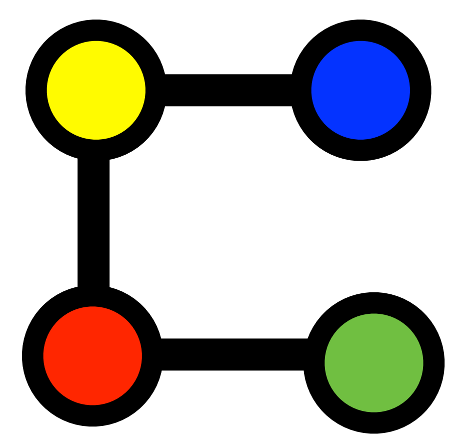

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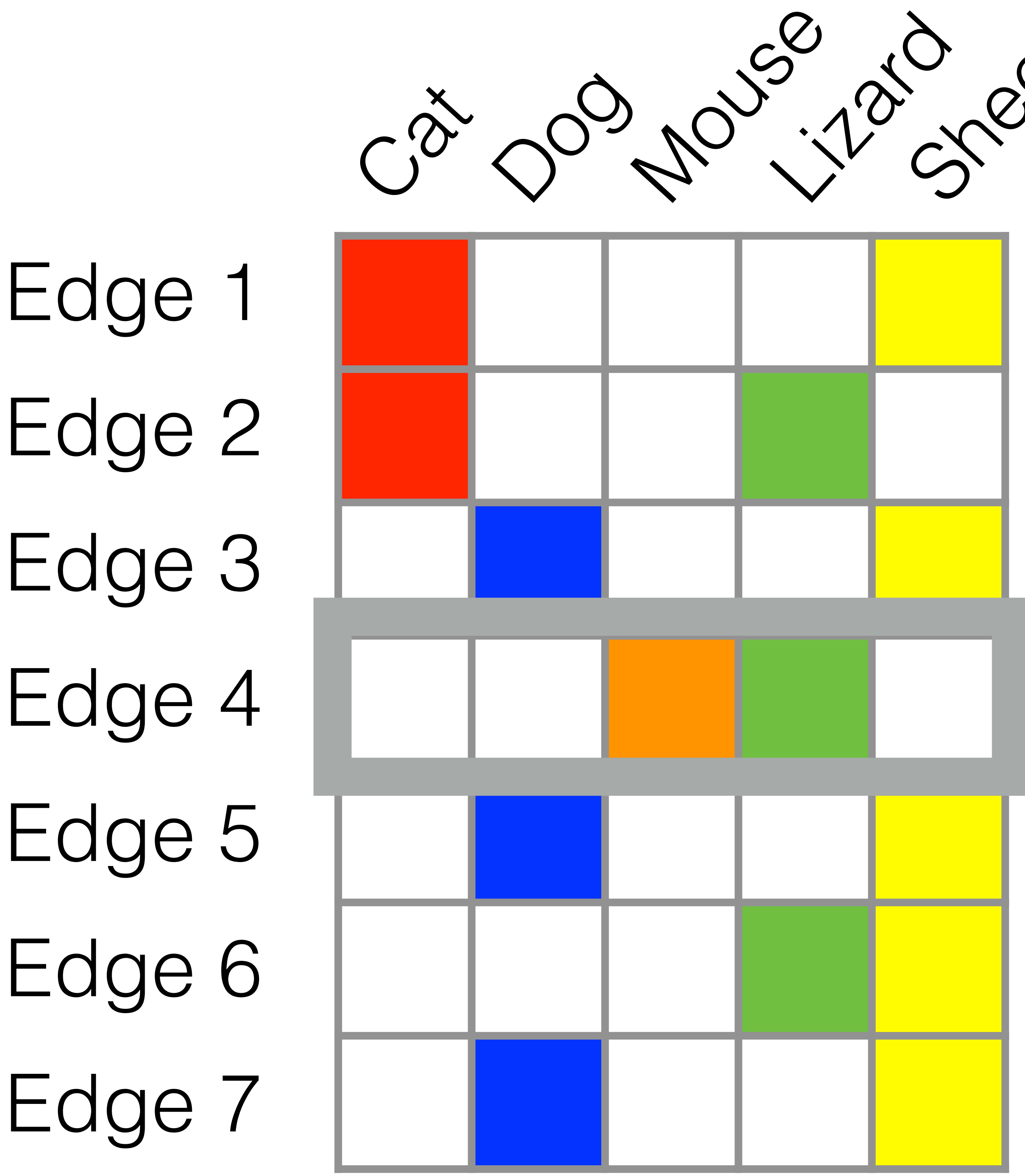

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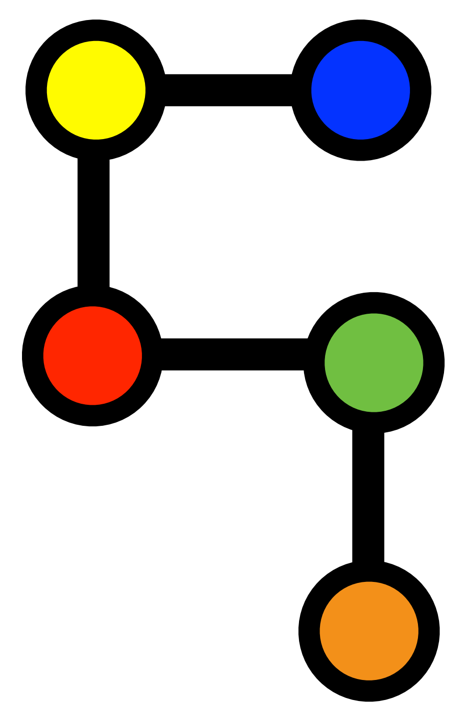

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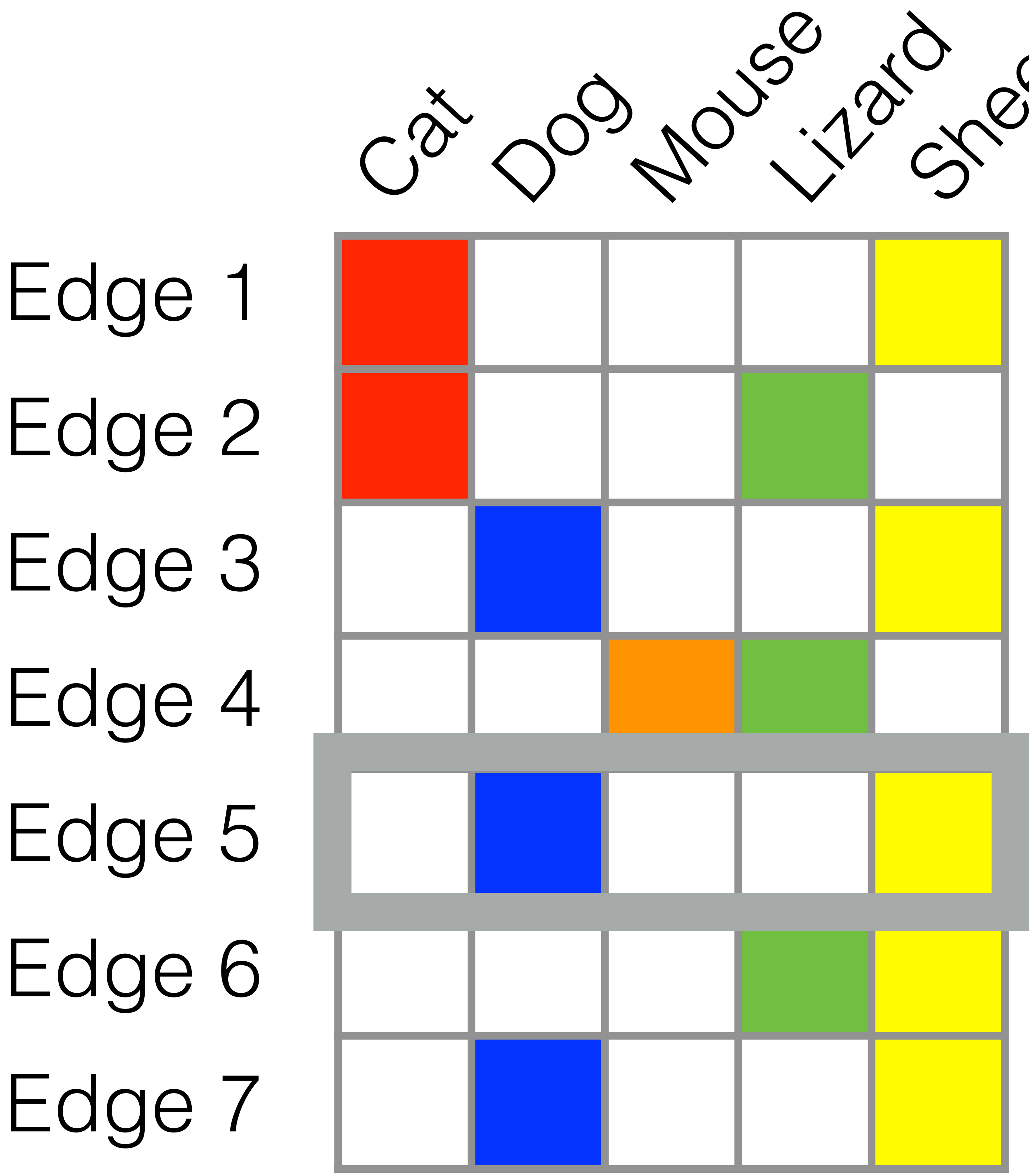

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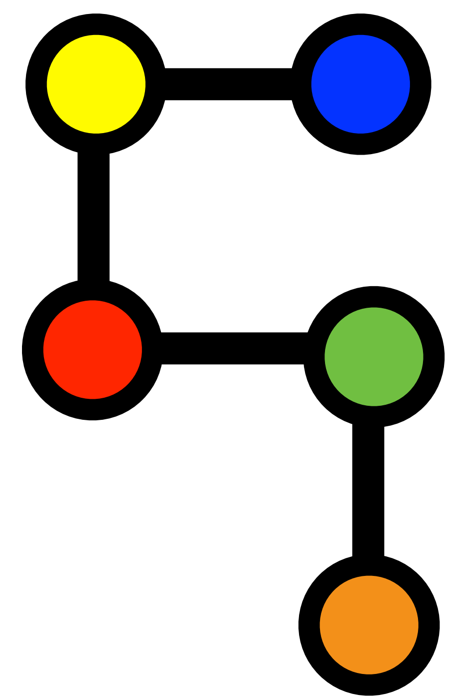

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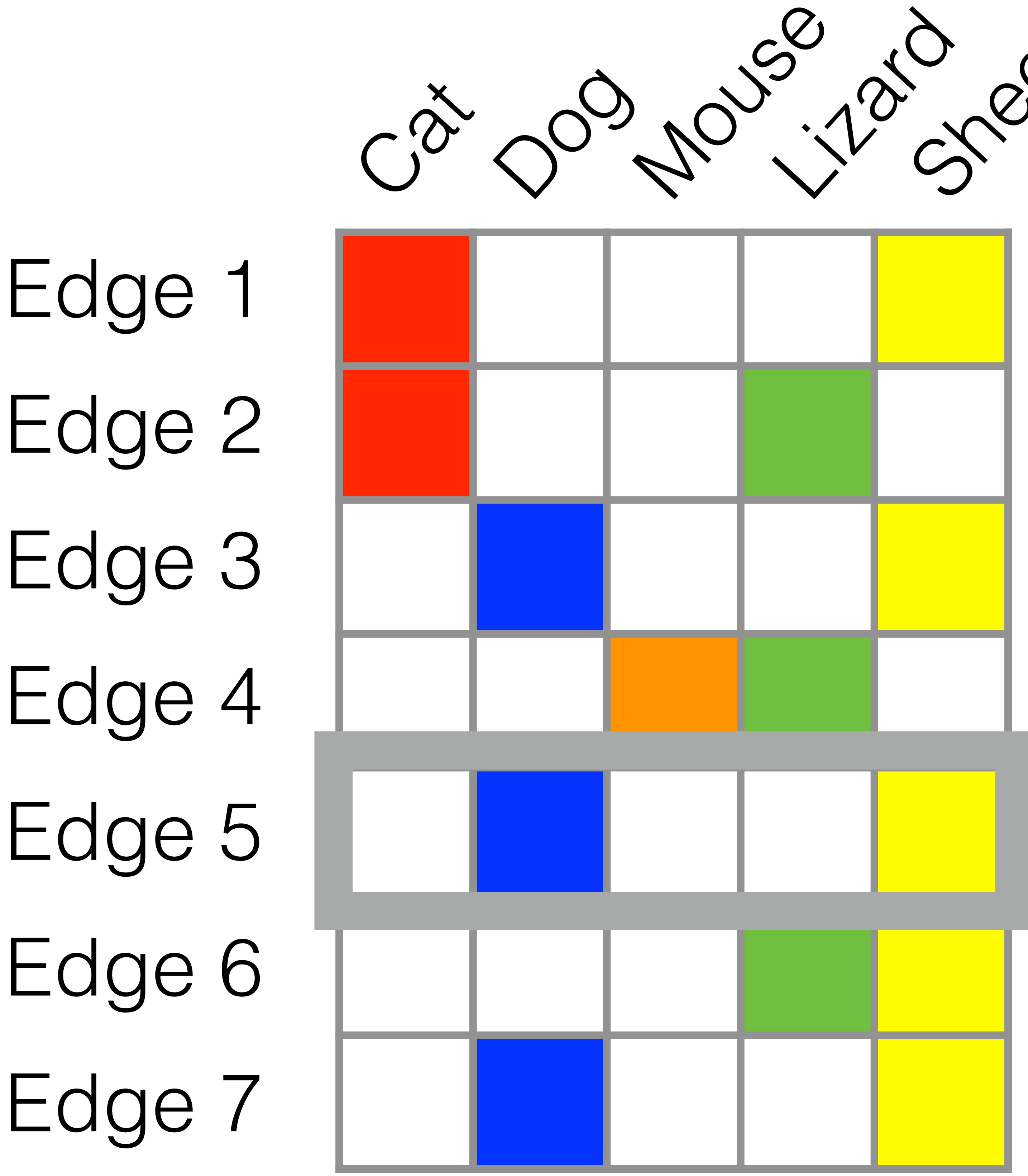

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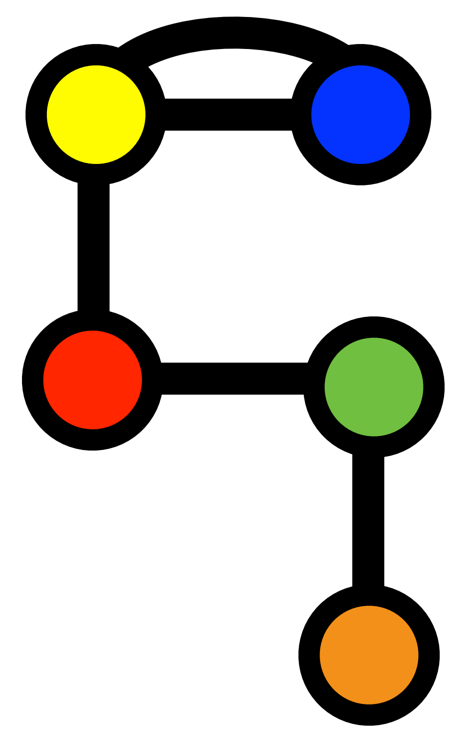

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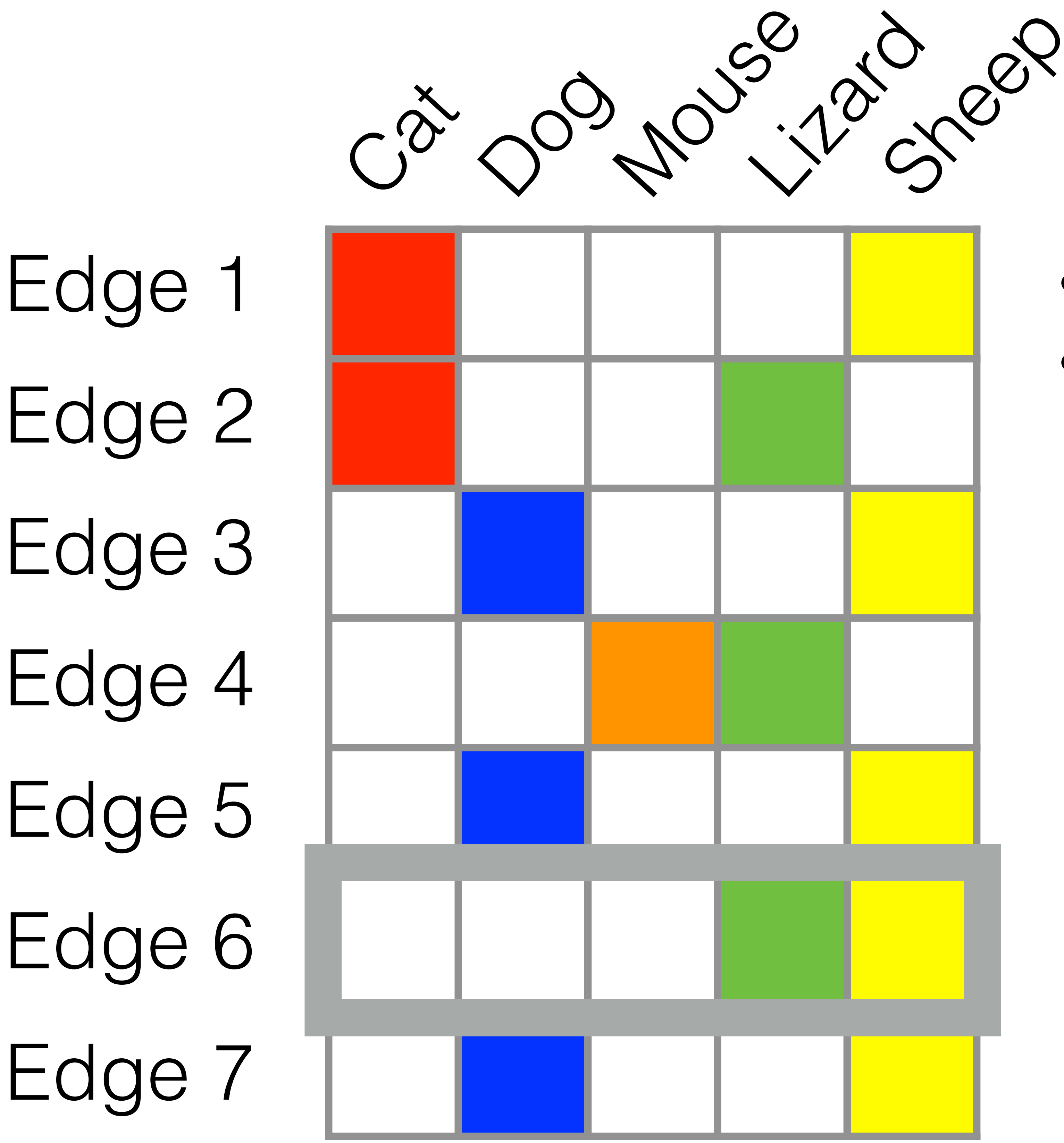

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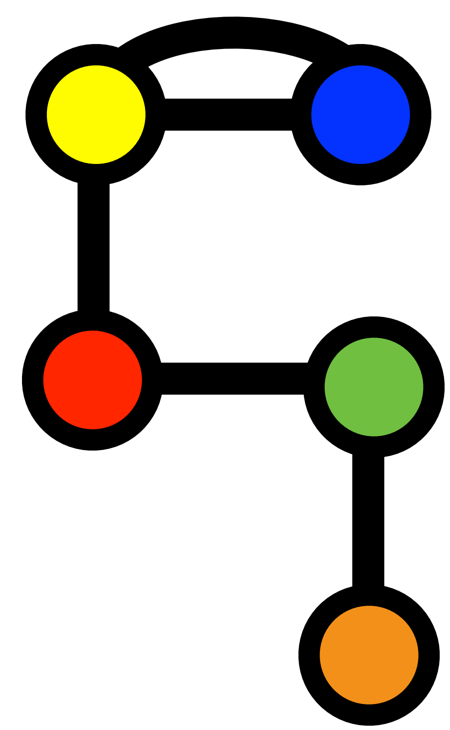

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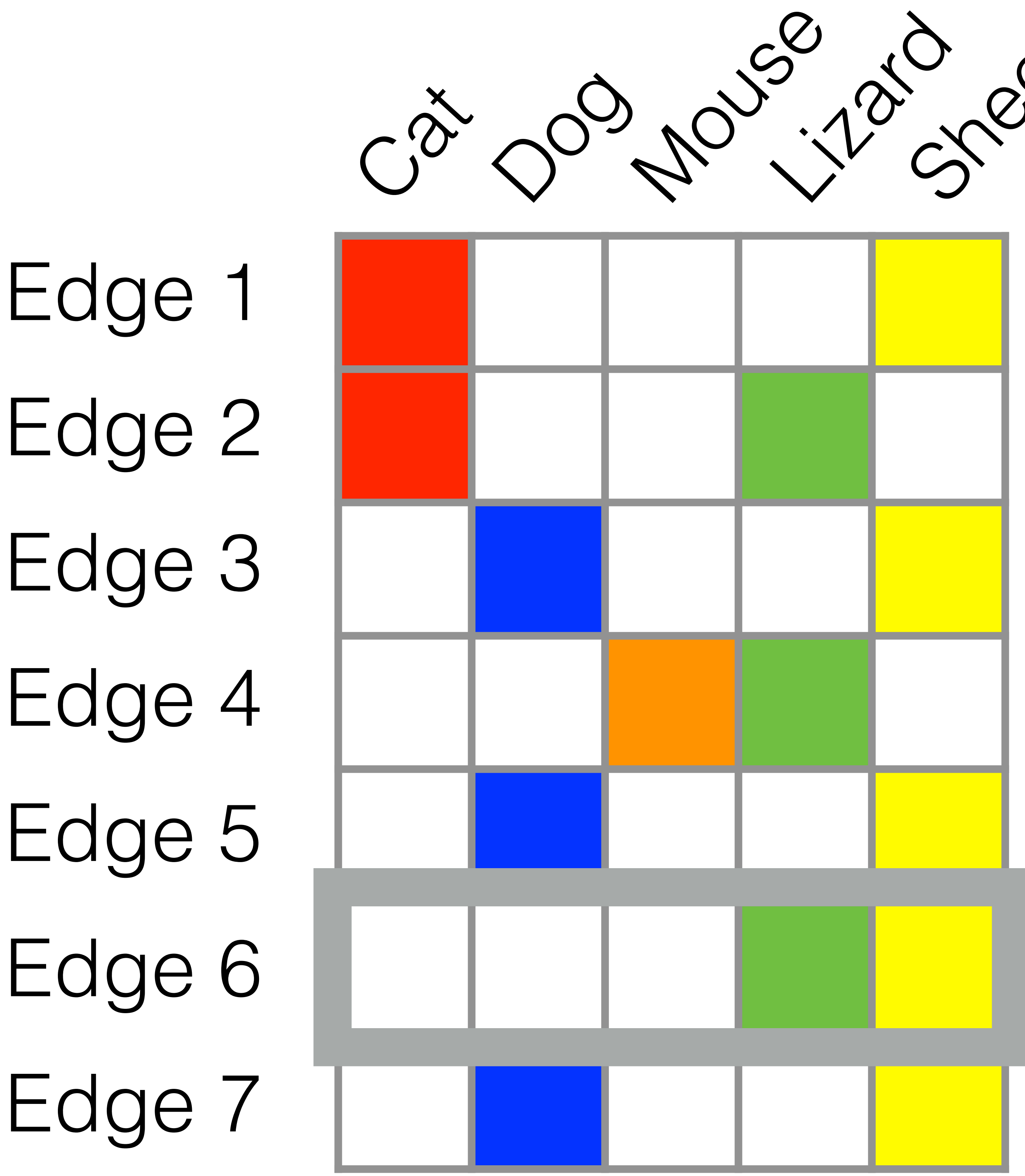

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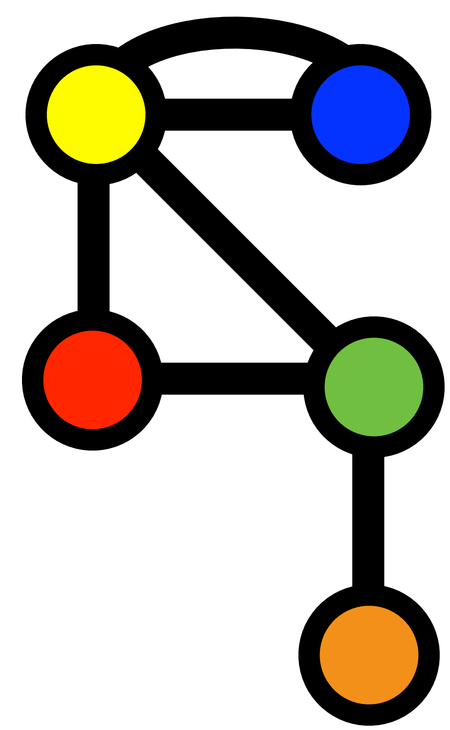

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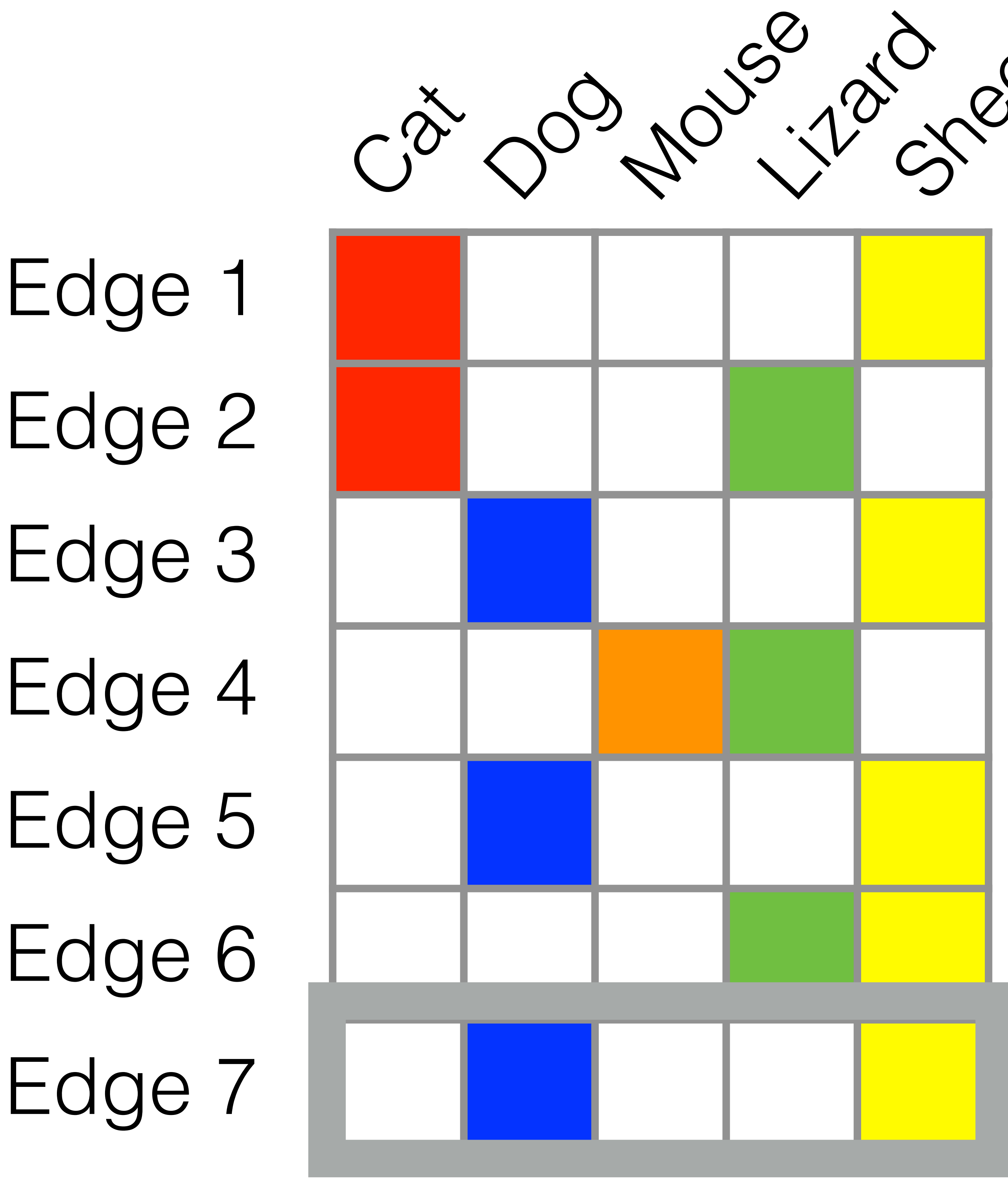

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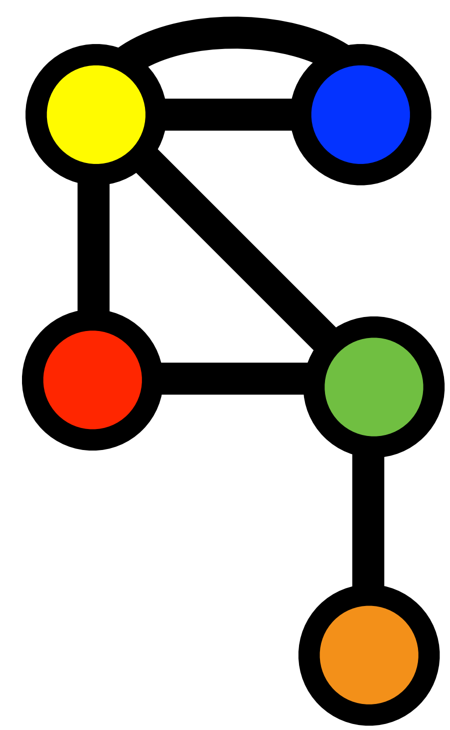

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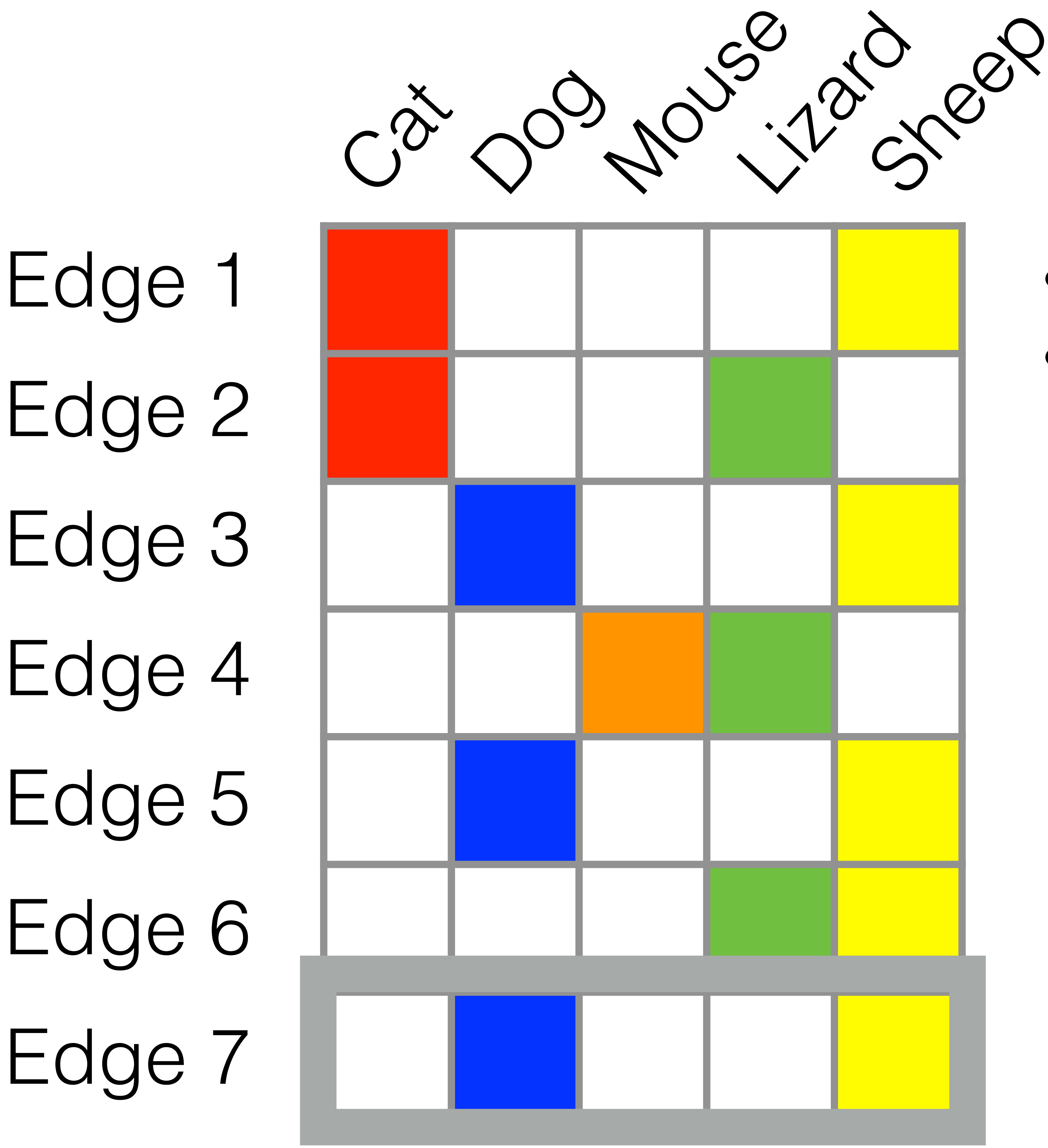

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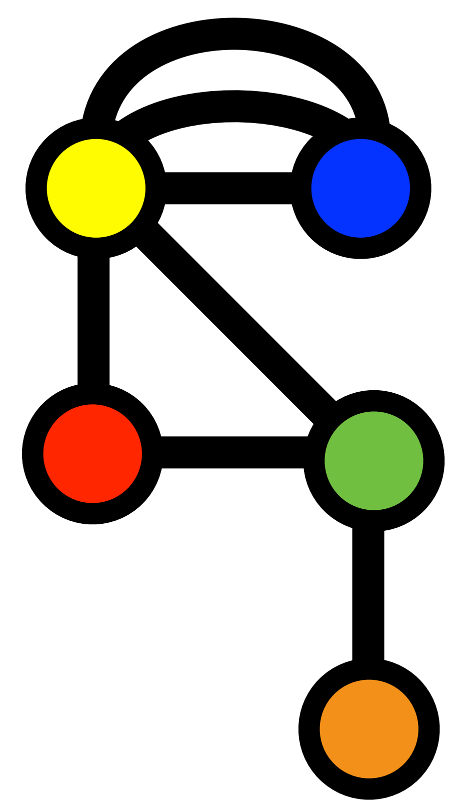
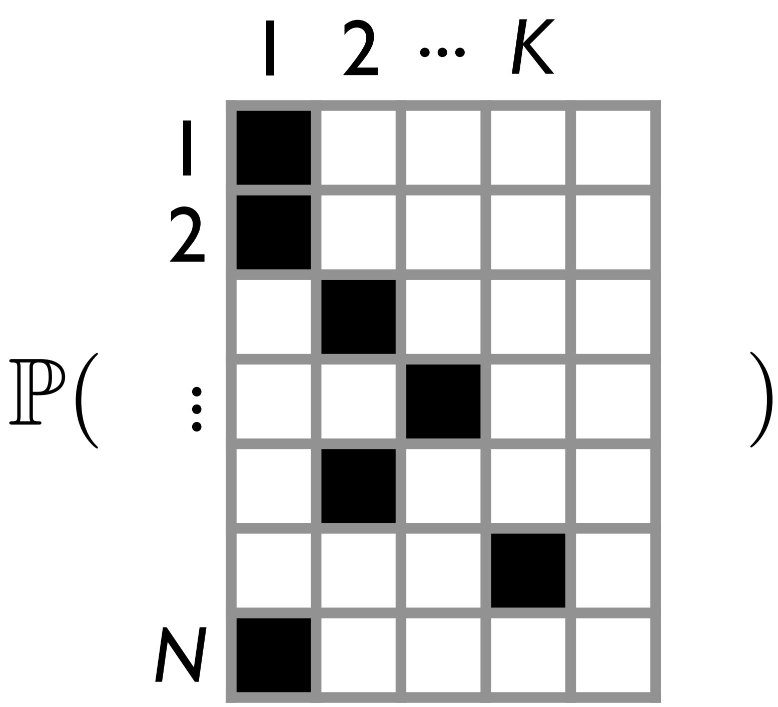


Exchangeable clustering distributions
are characterized

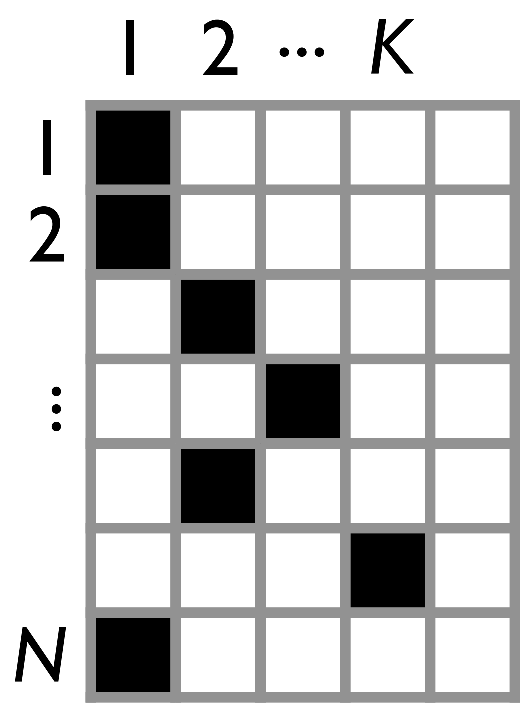
What about:
Exchangeable feature allocations?
Edge-exchangeable graphs?

Exchangeable probability functions



[Pitman 1995]

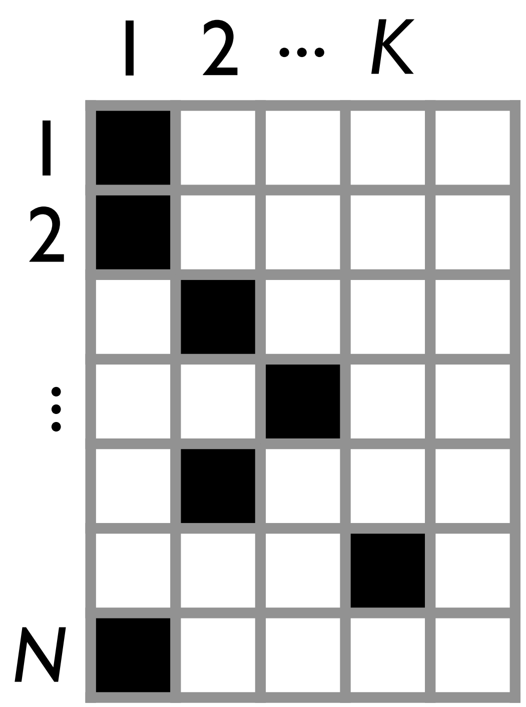
Exchangeable probability functions

$\mathbb{P}($  $) = p\left(S_{N,1}, \ldots, S_{N,K}\right)$

[Pitman 1995]

Exchangeable probability functions

Size of *K*th cluster



$$\mathbb{P}(\quad) = p\,(S_{N,1}, \ldots, S_{N,K})$$

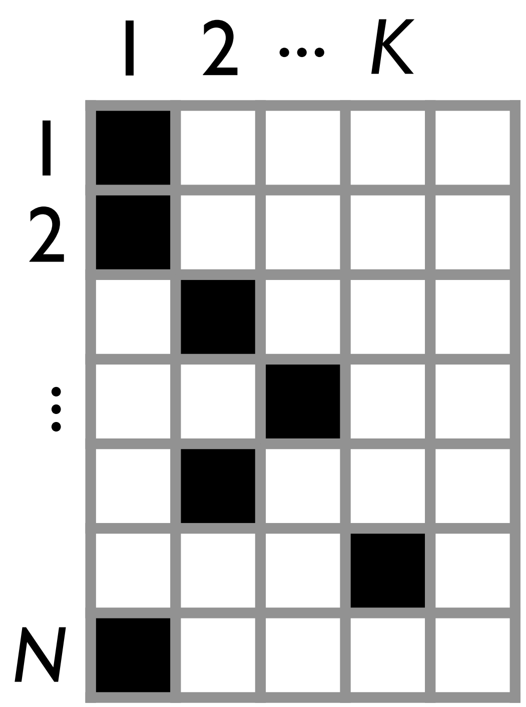
[Pitman 1995]

Exchangeable probability functions

exchangeable **partition** probability function (E**P**PF)

Size of Kth cluster

$\mathbb{P}($



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Exchangeable probability functions

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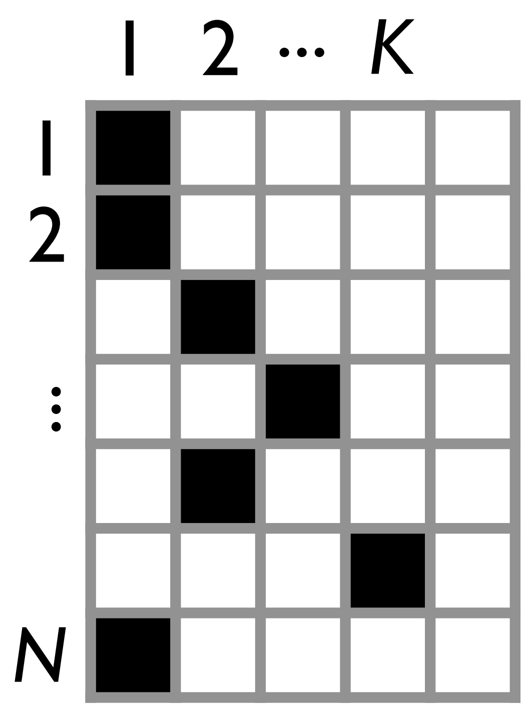
$\mathbb{P}($ 

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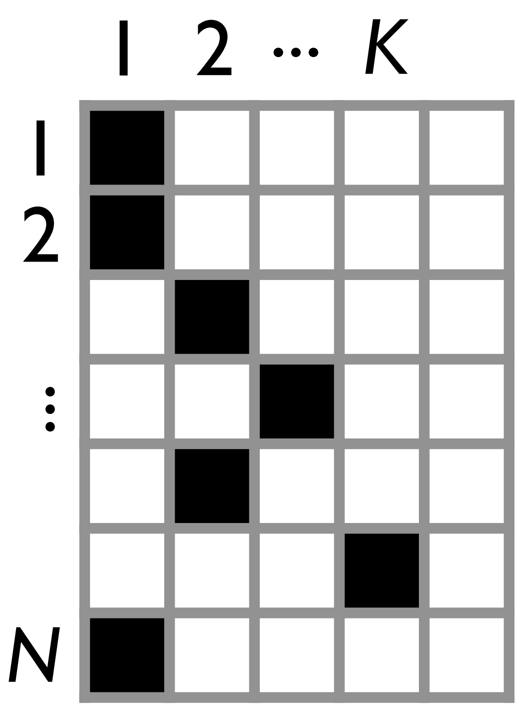

$$\mathbb{P}(\quad) = p(N; S_{N,1}, \ldots, S_{N,K})$$

Size of *K*th cluster

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Exchangeable probability functions

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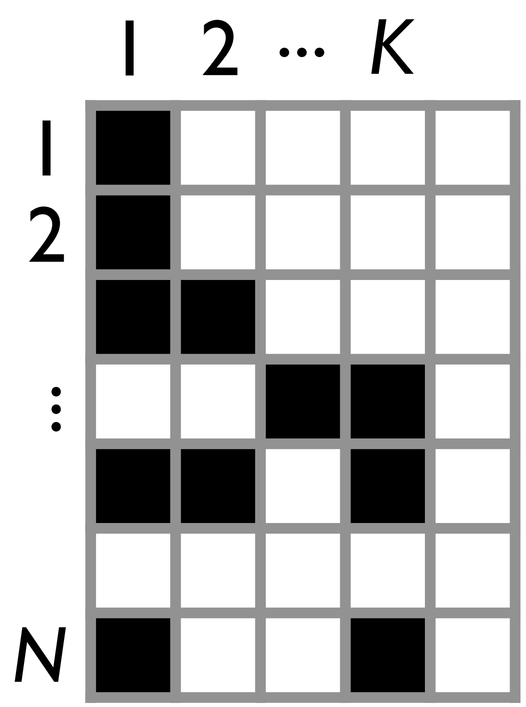

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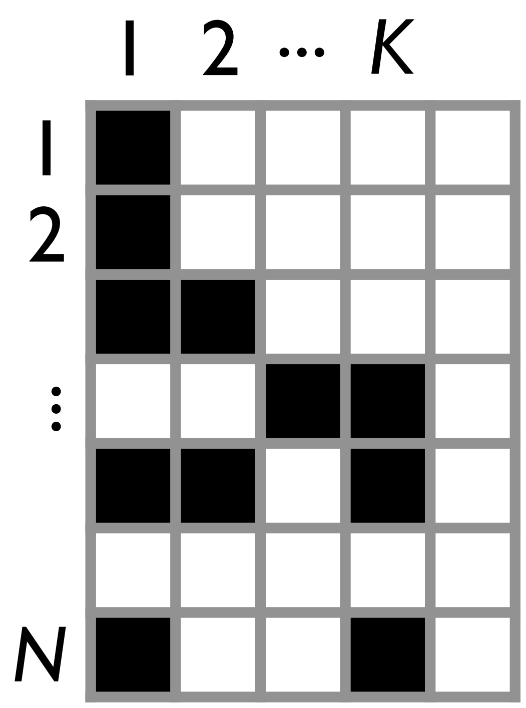

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Size of *K*th feature

[Pitman 1995]

Exchangeable probability functions

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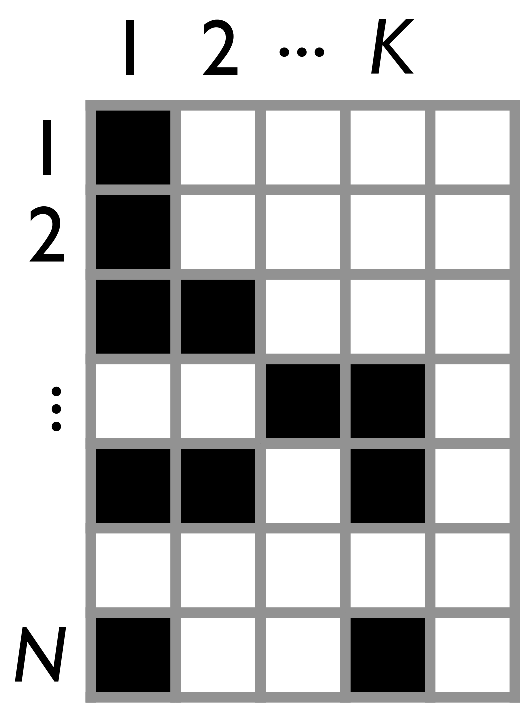
exchangeable **feature** probability function (E**F**PF)



Size of *K*th feature

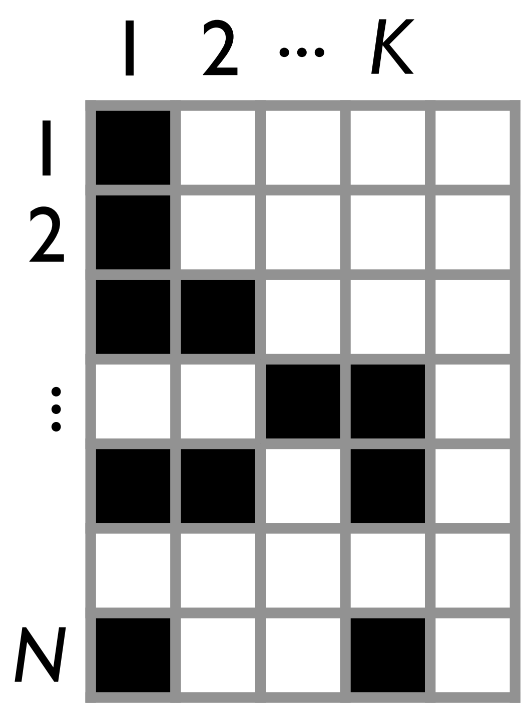
$$\mathbb{P}(\quad) = p(N; S_{N,1}, \ldots, S_{N,K})$$

[Pitman 1995; Broderick, Pitman, Jordan 2013]

Exchangeable probability functions

Thm (Pitman). Every exchangeable clustering has an exchangeable **partition** probability function (E**P**PF)

Prop (BPJ). Many, but not all, exchangeable feature allocations have an exchangeable **feature** probability function (E**F**PF)



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[Pitman 1995; Broderick, Pitman, Jordan 2013]

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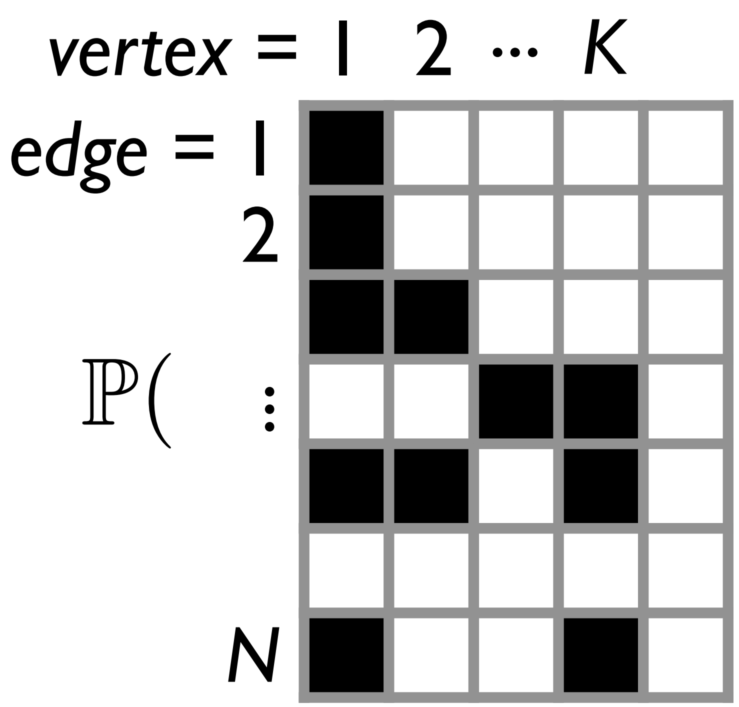
Size of *K*th feature



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[Pitman 1995; Broderick, Pitman, Jordan 2013; Campbell, Cai, Broderick 2016]

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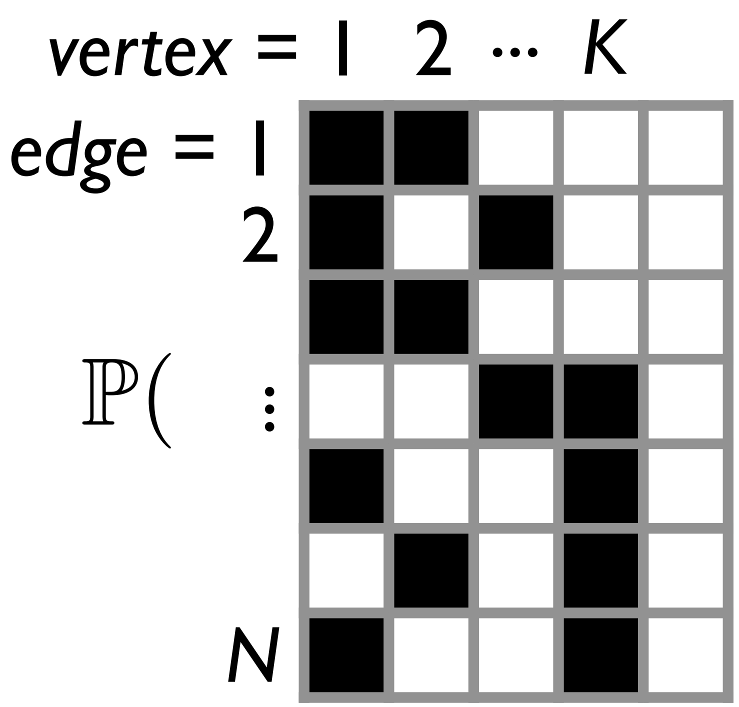

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[Pitman 1995; Broderick, Pitman, Jordan 2013; Campbell, Cai, Broderick 2016]

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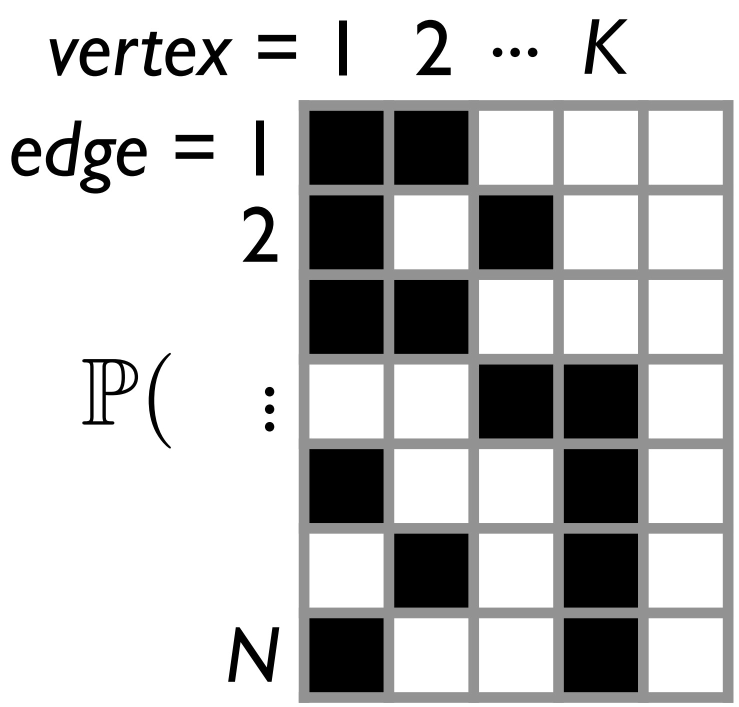

$$\mathbb{P}(\quad \text{[matrix]} \quad) = p(N; S_{N,1}, \ldots, S_{N,K})$$

Size of *K*th vertex

13

[Pitman 1995; Broderick, Pitman, Jordan 2013; Campbell, Cai, Broderick 2016]

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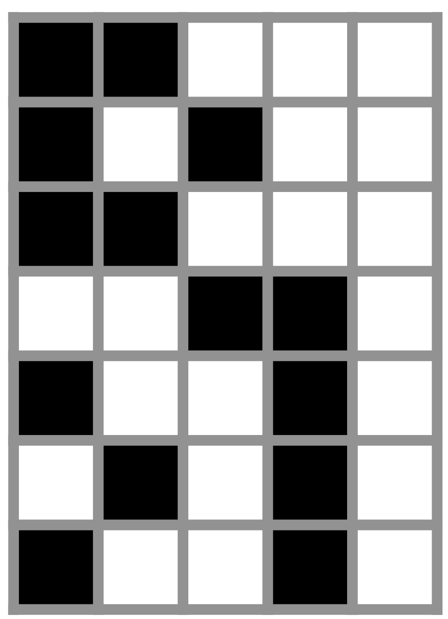

$$\mathbb{P}(\quad \cdot \quad) = p(N; S_{N,1}, \ldots, S_{N,K})$$

Degree of *K*th vertex

[Pitman 1995; Broderick, Pitman, Jordan 2013; Campbell, Cai, Broderick 2016]

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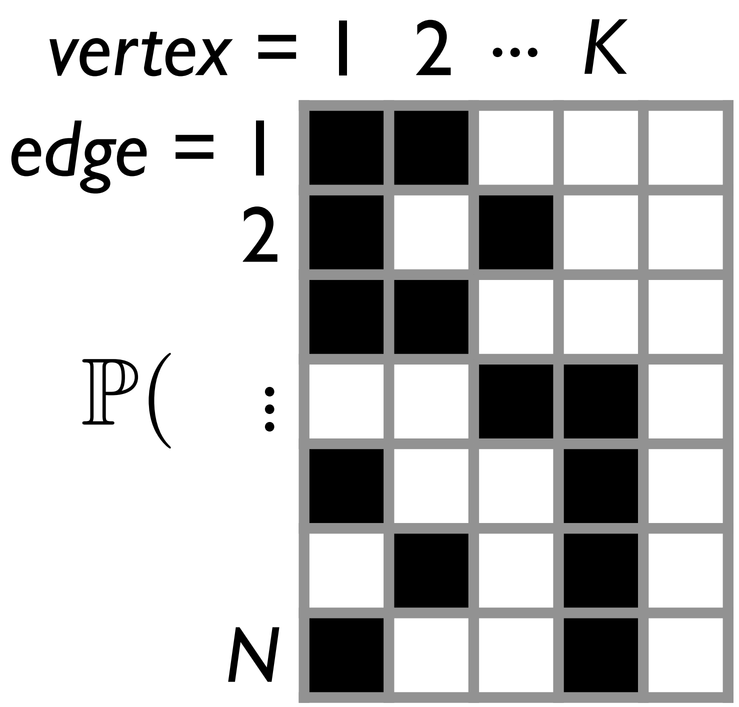

$$\mathbb{P}(\quad\cdot\quad) = p\left(S_{N,1}, \ldots, S_{N,K}\right)$$

Degree of *K*th vertex

[Pitman 1995; Broderick, Pitman, Jordan 2013; Campbell, Cai, Broderick 2016]

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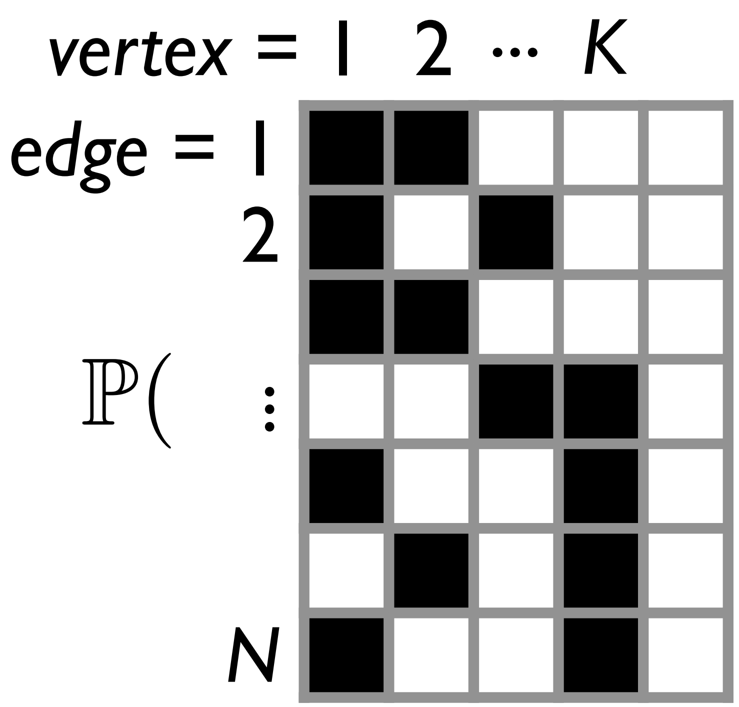
Degree of *K*th vertex



$$\mathbb{P}(\quad) = p\left(S_{N,1}, \ldots, S_{N,K}\right)$$

Definition (CCB). Exchangeable **vertex** probability function (E**V**PF)

[Pitman 1995; Broderick, Pitman, Jordan 2013; Campbell, Cai, Broderick 2016]

Exchangeable probability functions

Q: Does every edge-exchangeable graph have an EVPF?

[Broderick, Jordan, Pitman 2013]

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A: No. Counterexample:

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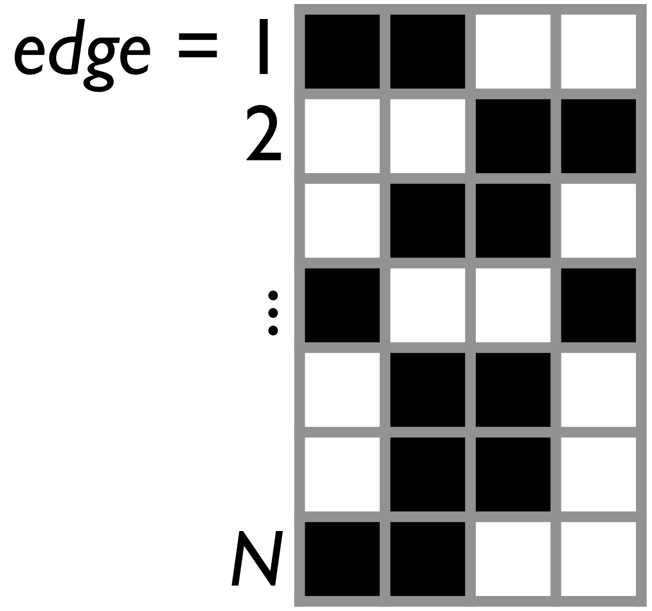

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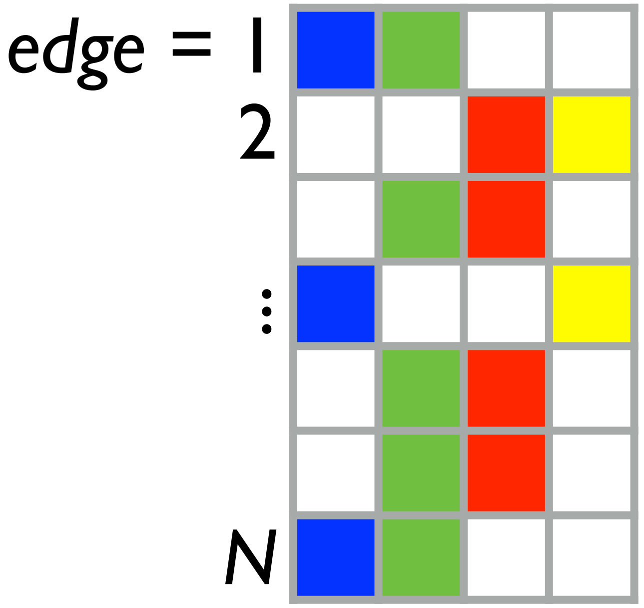

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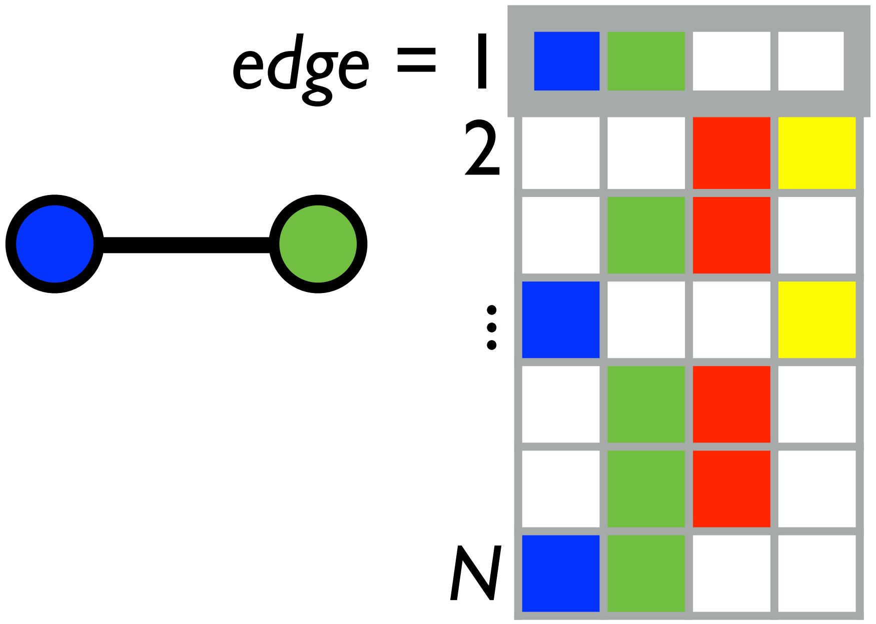

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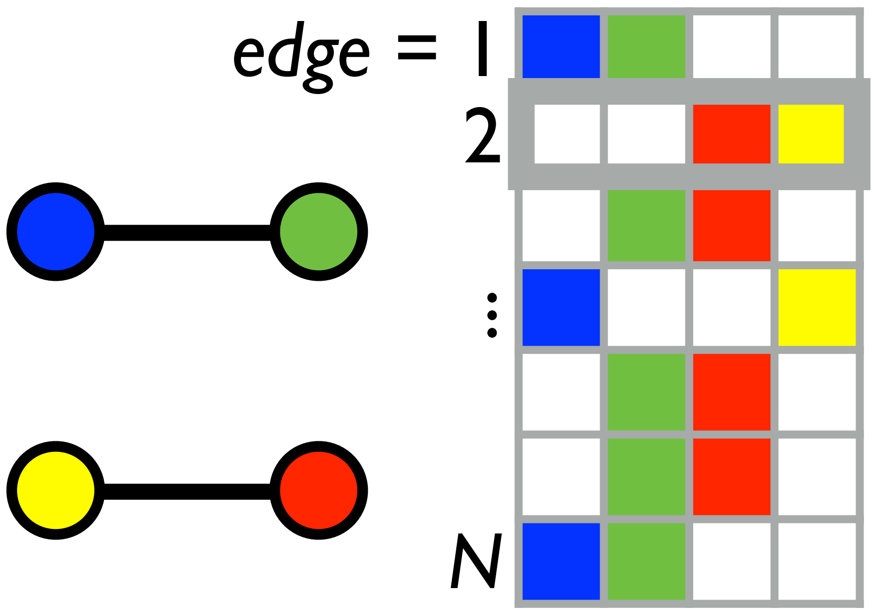

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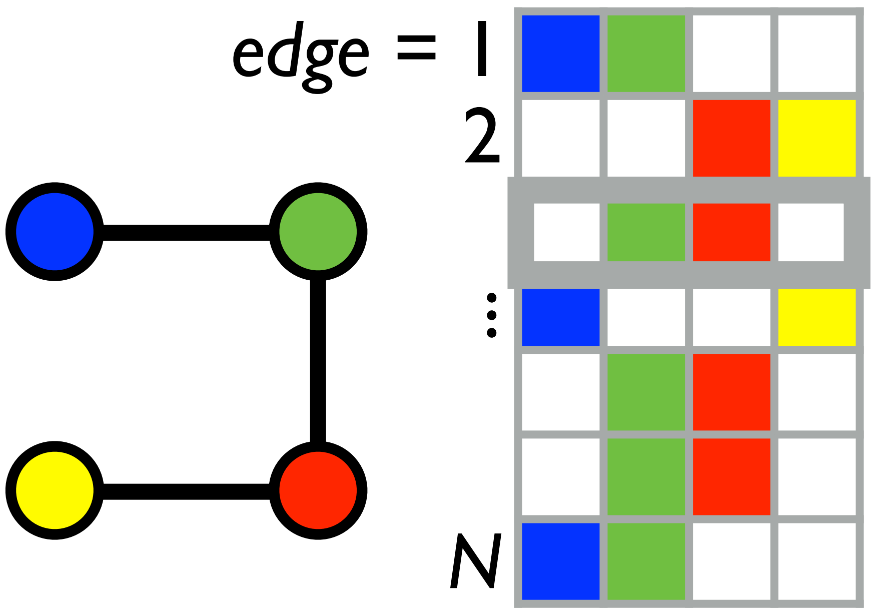

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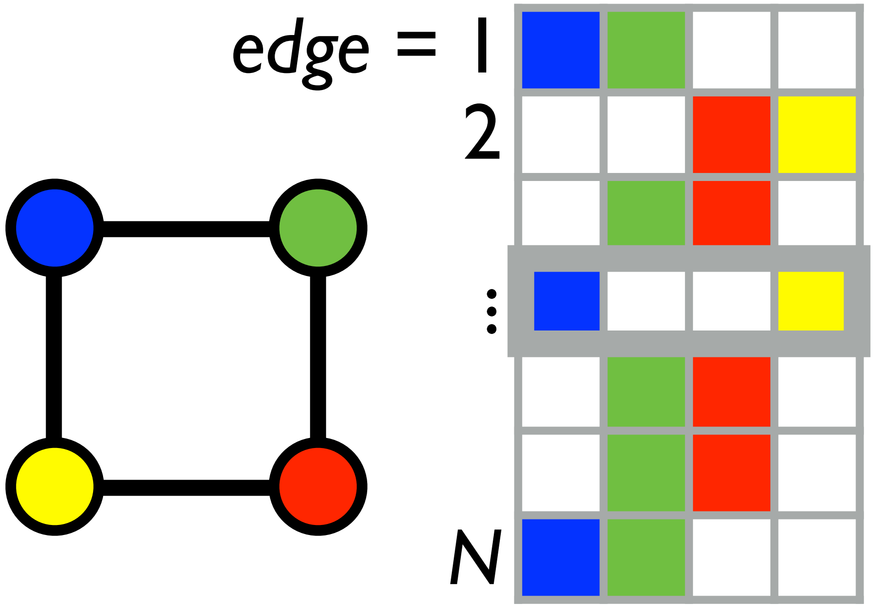

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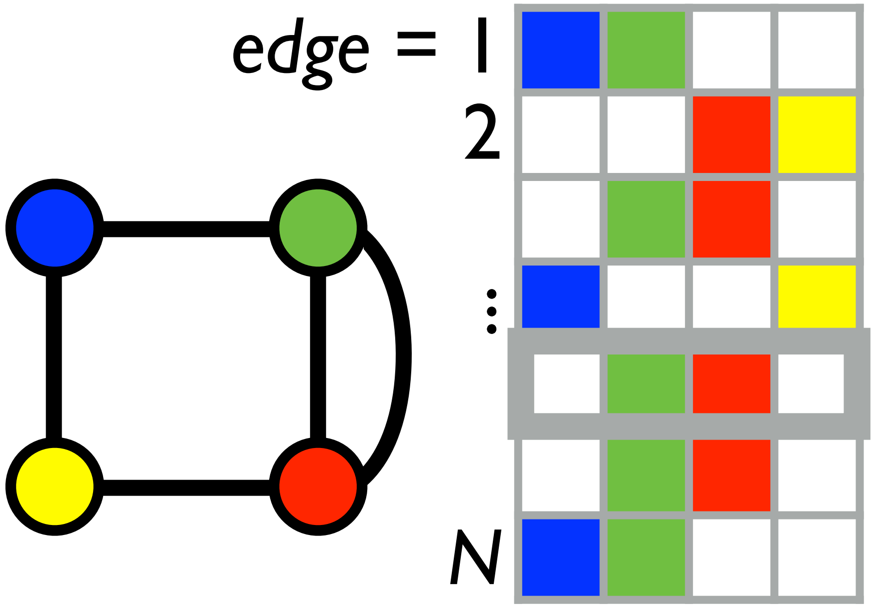

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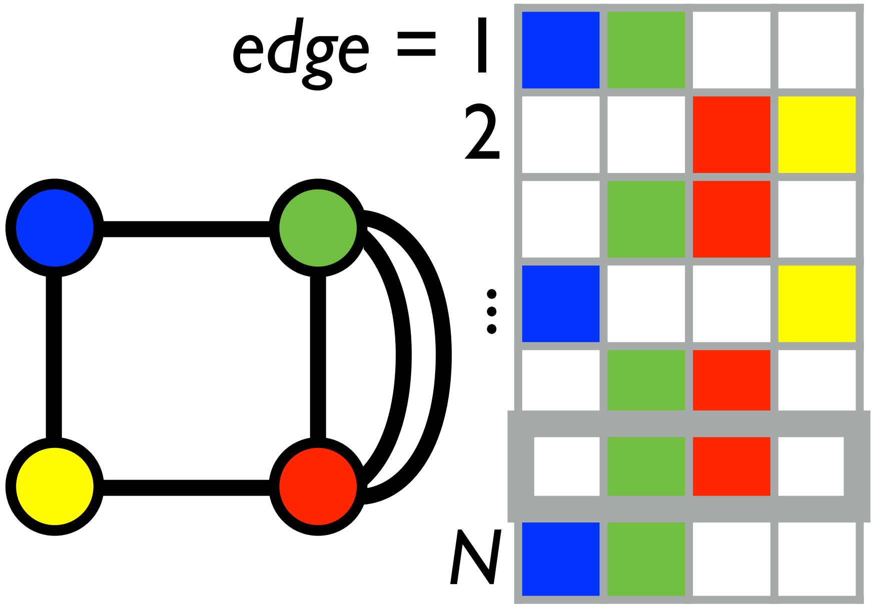

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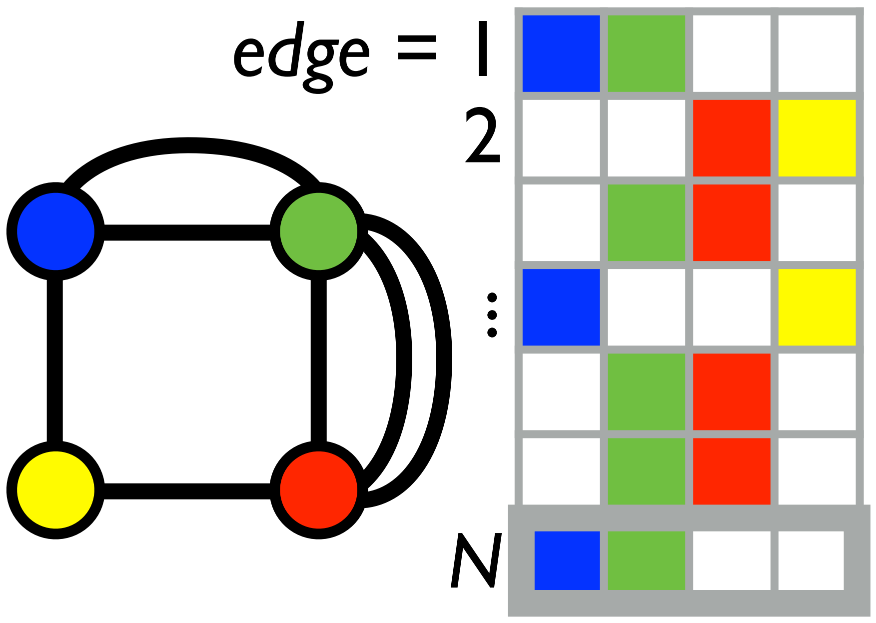

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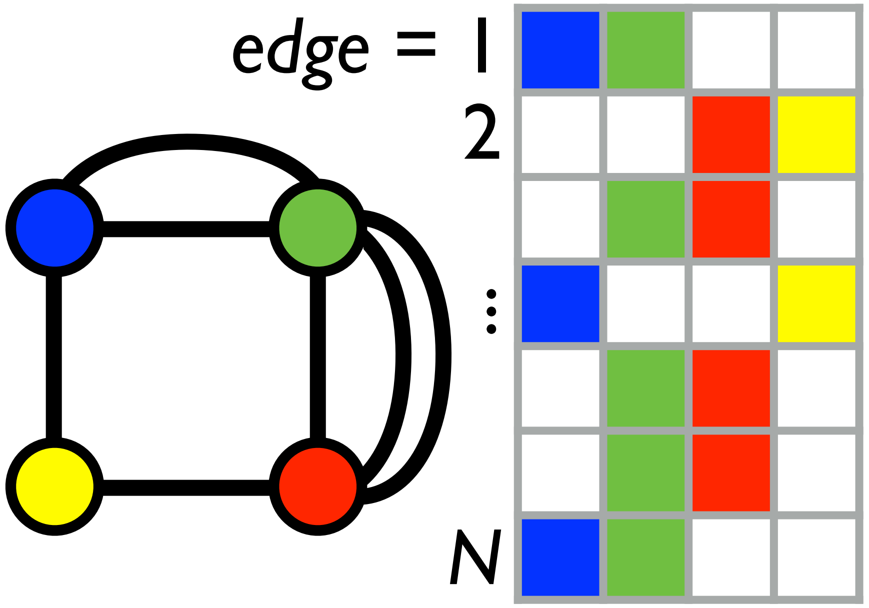

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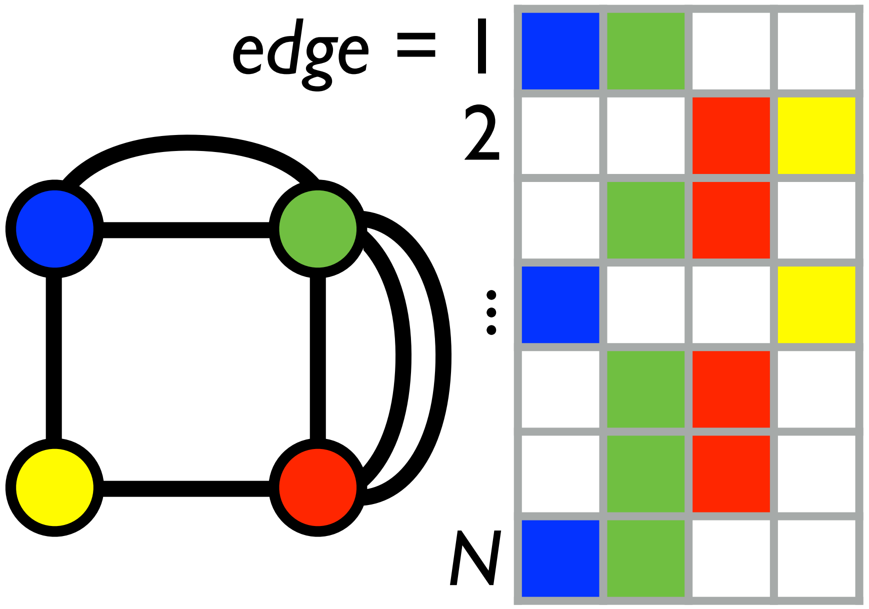
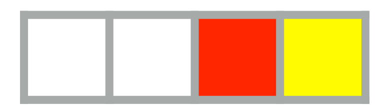
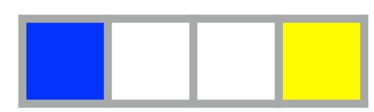


$\mathbb{P}(\text{row} =$  $) = p_1$

$\mathbb{P}(\text{row} =$  $) = p_2$

$\mathbb{P}(\text{row} =$  $) = p_3$

$\mathbb{P}(\text{row} =$  $) = p_4$

[Broderick, Jordan, Pitman 2013]

Exchangeable probability functions

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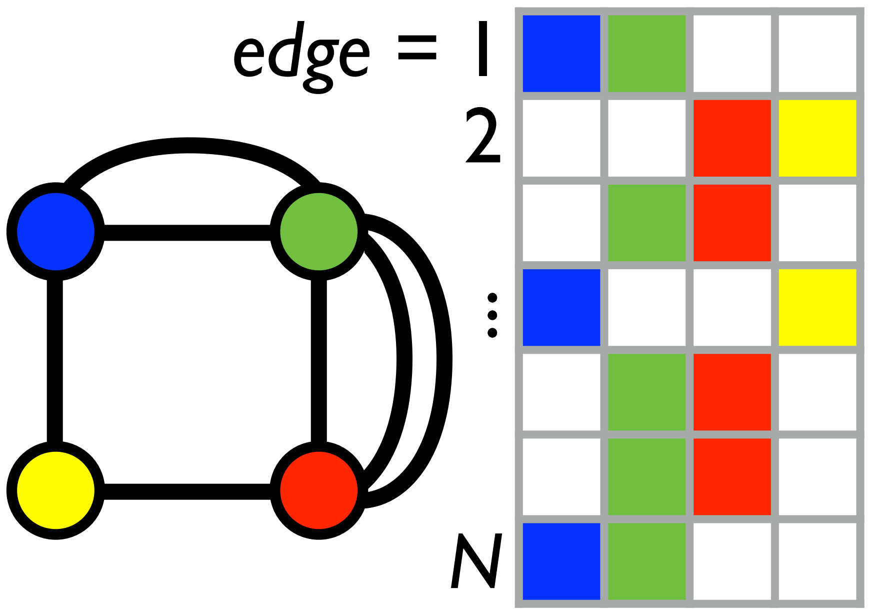
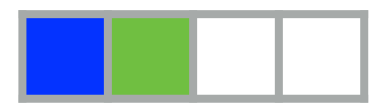
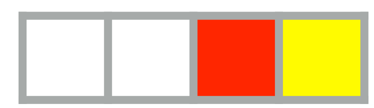
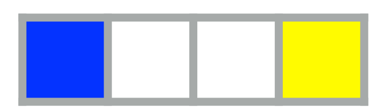
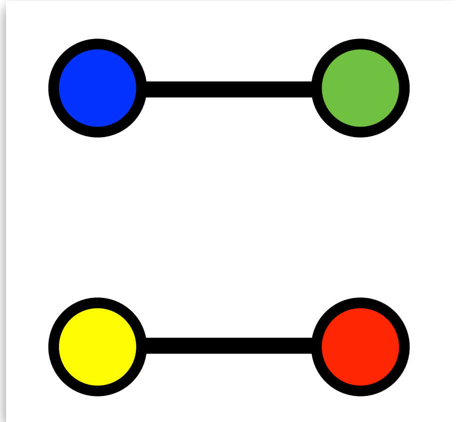


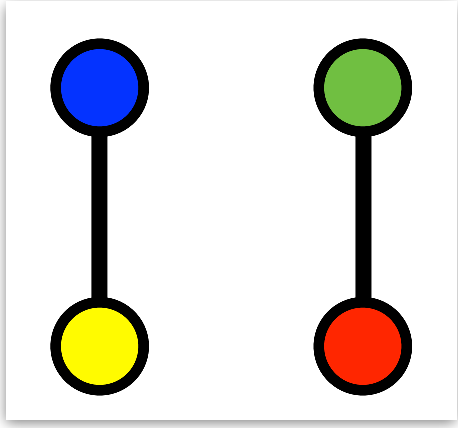
$\mathbb{P}(\text{row} =$  $) = p_1$

$\mathbb{P}(\text{row} =$  $) = p_2$

$\mathbb{P}(\text{row} =$  $) = p_3$

$\mathbb{P}(\text{row} =$  $) = p_4$





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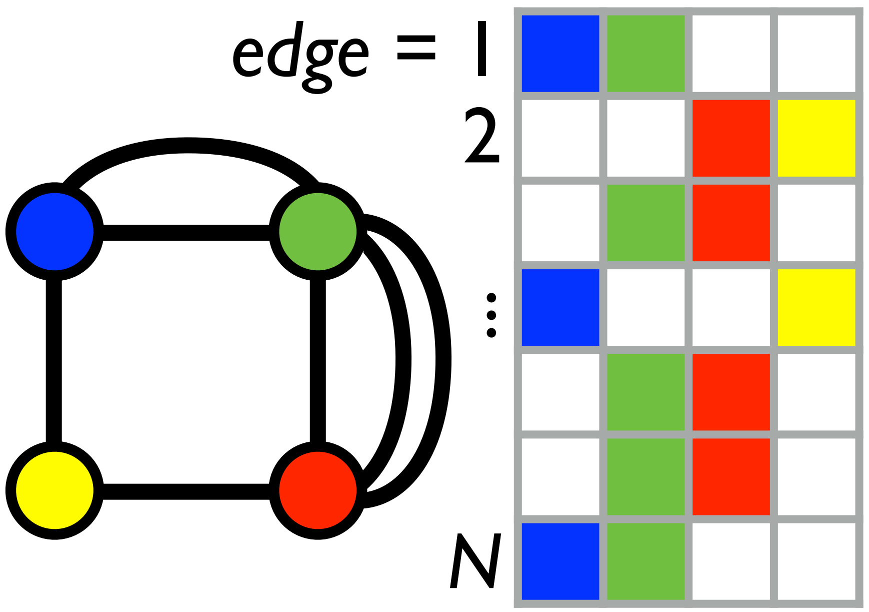
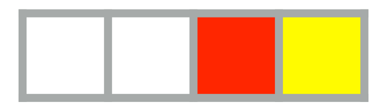


$\mathbb{P}(\text{row} =$  $) = p_1$

$\mathbb{P}(\text{row} =$  $) = p_2$

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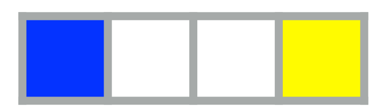
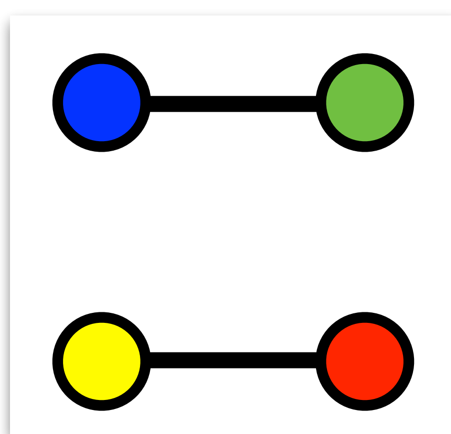
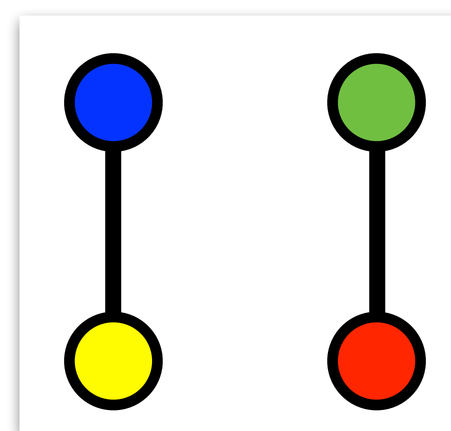
$\mathbb{P}(\text{row} =$  $) = p_4$

$\mathbb{P}($  $)$ $\mathbb{P}($  $)$

[Broderick, Jordan, Pitman 2013]

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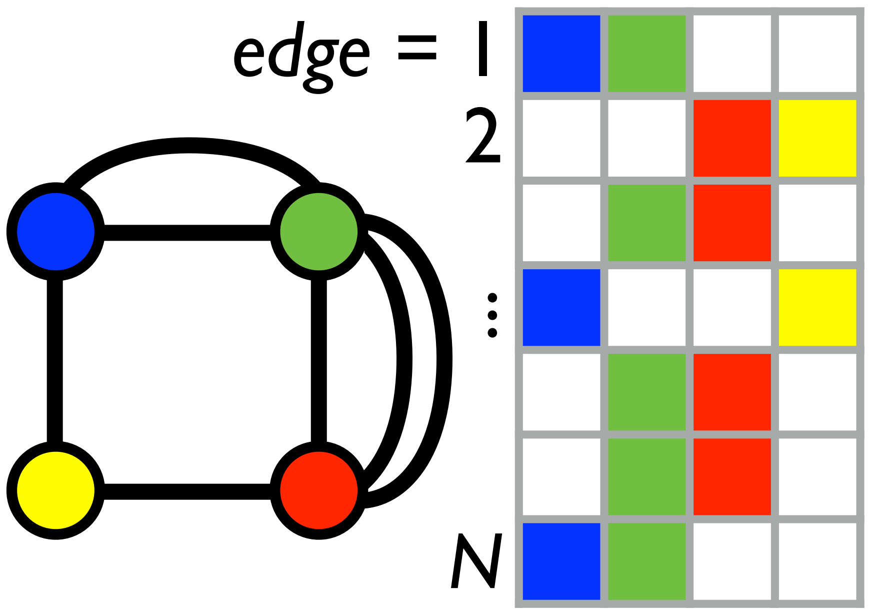

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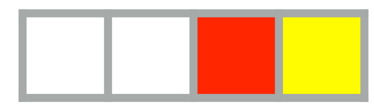
$\mathbb{P}(\text{row} =$  $) = p_2$

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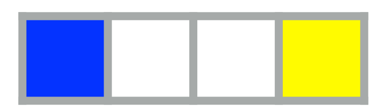
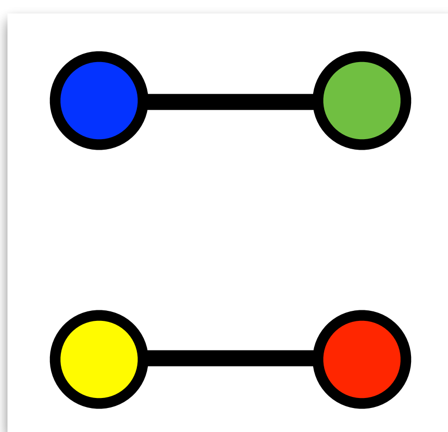
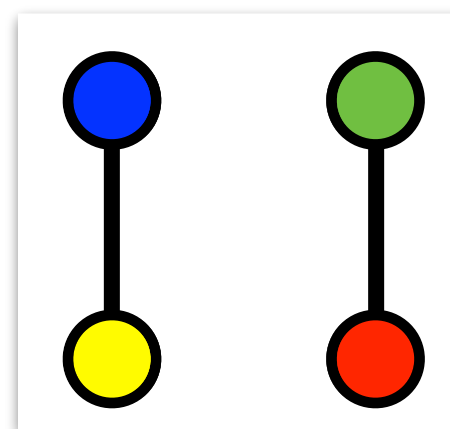
$\mathbb{P}(\text{row} =$  $) = p_4$

$\mathbb{P}($  $)$ $\mathbb{P}($  $)$

$p_1 p_2$ $p_3 p_4$

[Broderick, Jordan, Pitman 2013]

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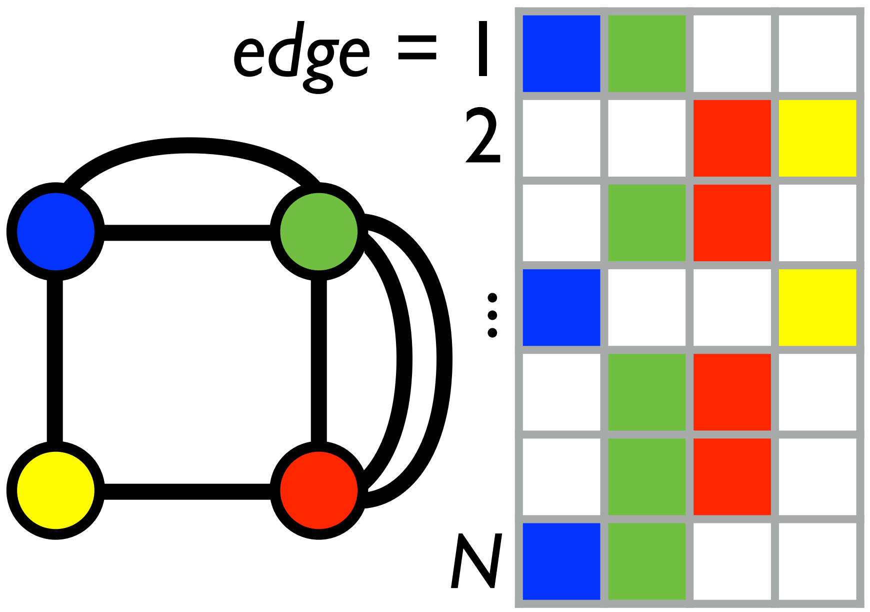

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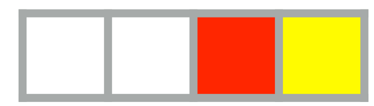
$\mathbb{P}(\text{row} =$  $) = p_2$

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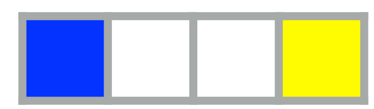
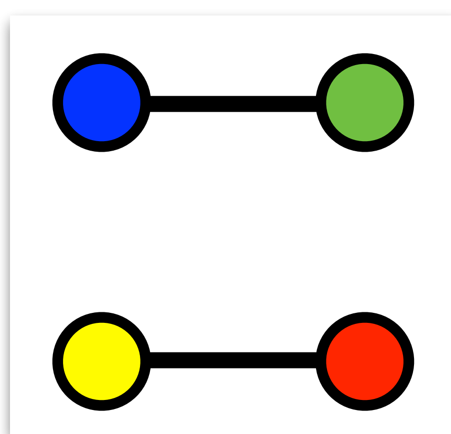
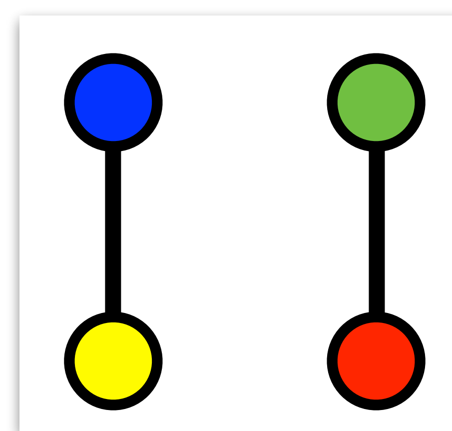
$\mathbb{P}(\text{row} =$  $) = p_4$

$\mathbb{P}($  $) \neq \mathbb{P}($  $)$

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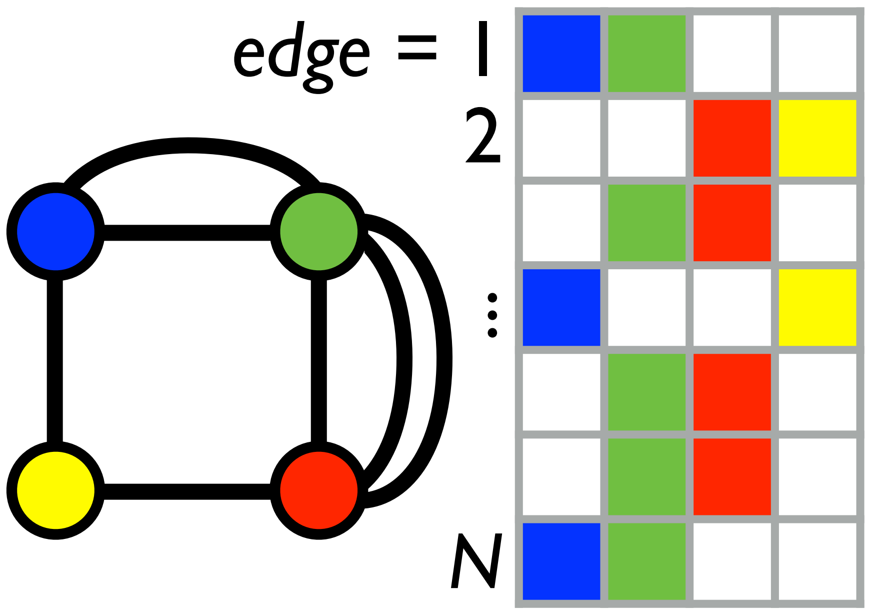
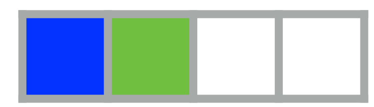
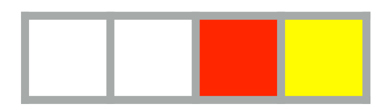


$\mathbb{P}($row $=$  $) = p_1$

$\mathbb{P}($row $=$  $) = p_2$

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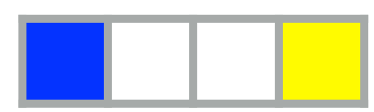
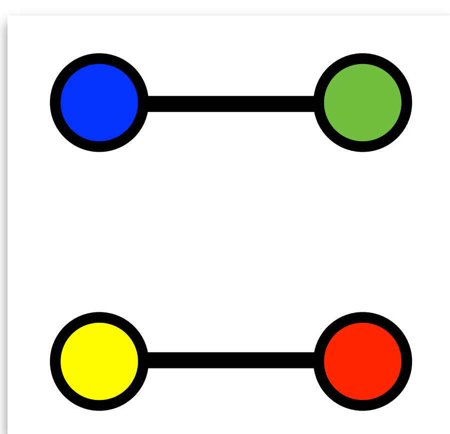
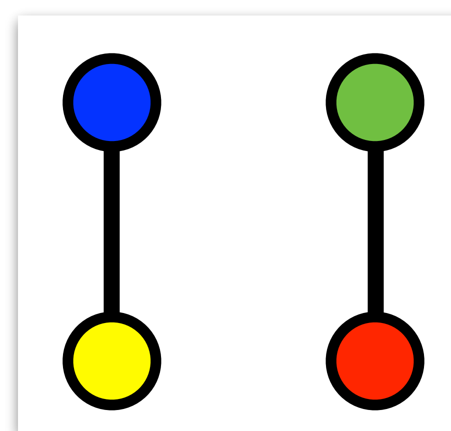
$\mathbb{P}($row $=$  $) = p_4$

$\mathbb{P}($  $) \neq \mathbb{P}($  $)$

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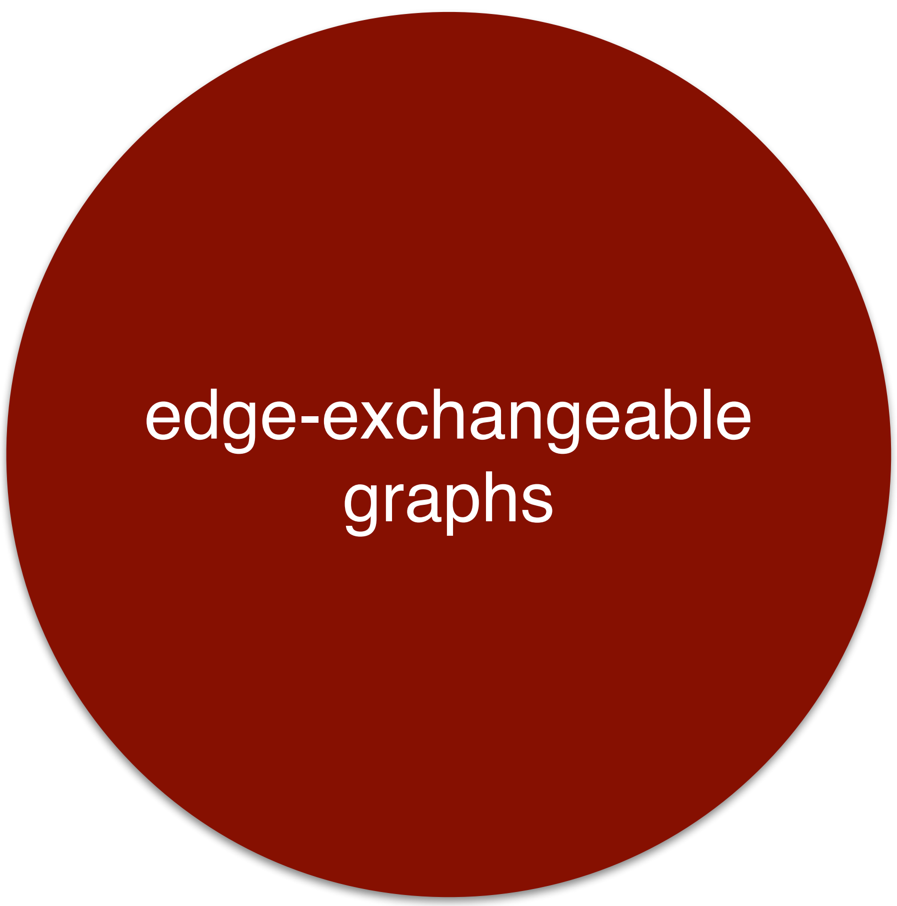
Cor. Not every exchangeable feature allocation has an EFPF.

[Broderick, Jordan, Pitman 2013]

What we know so far



- Goal 1: characterization theorem for edge-exchangeable graphs
- Goal 2: sparsity theorem for edge-exchangeable graphs

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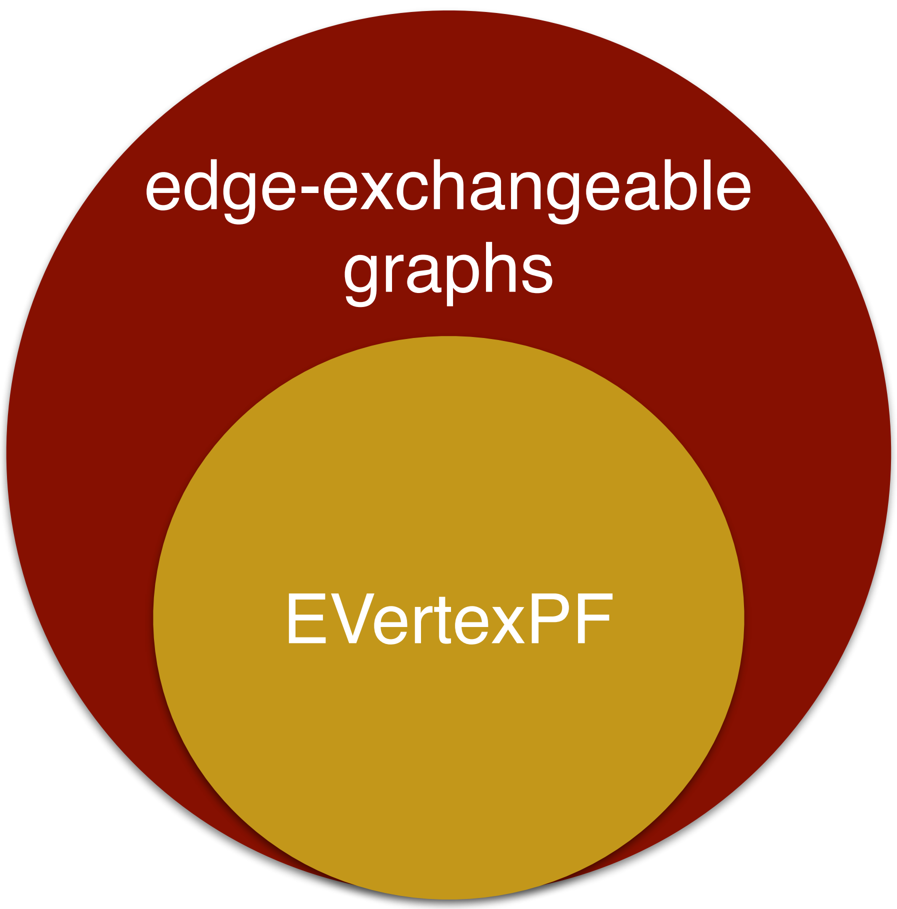

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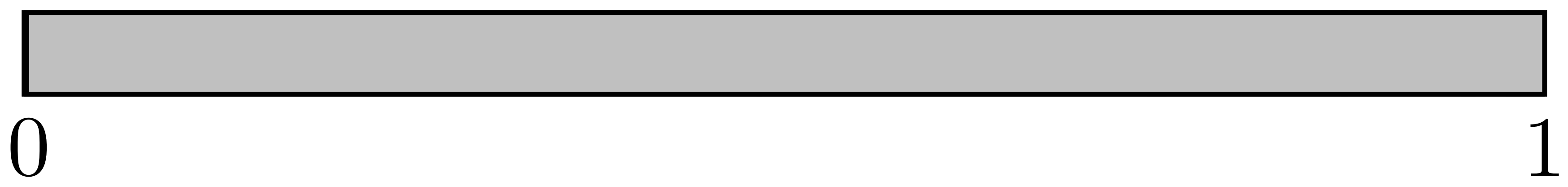
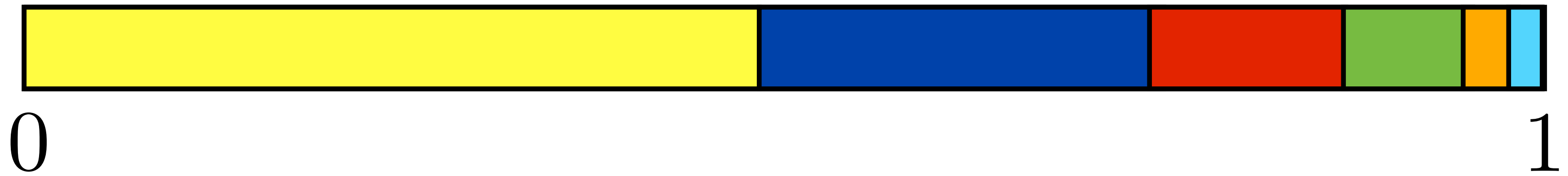
Clustering



[Kingman 1978]

Clustering



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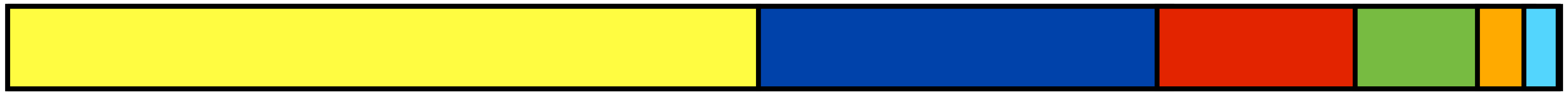

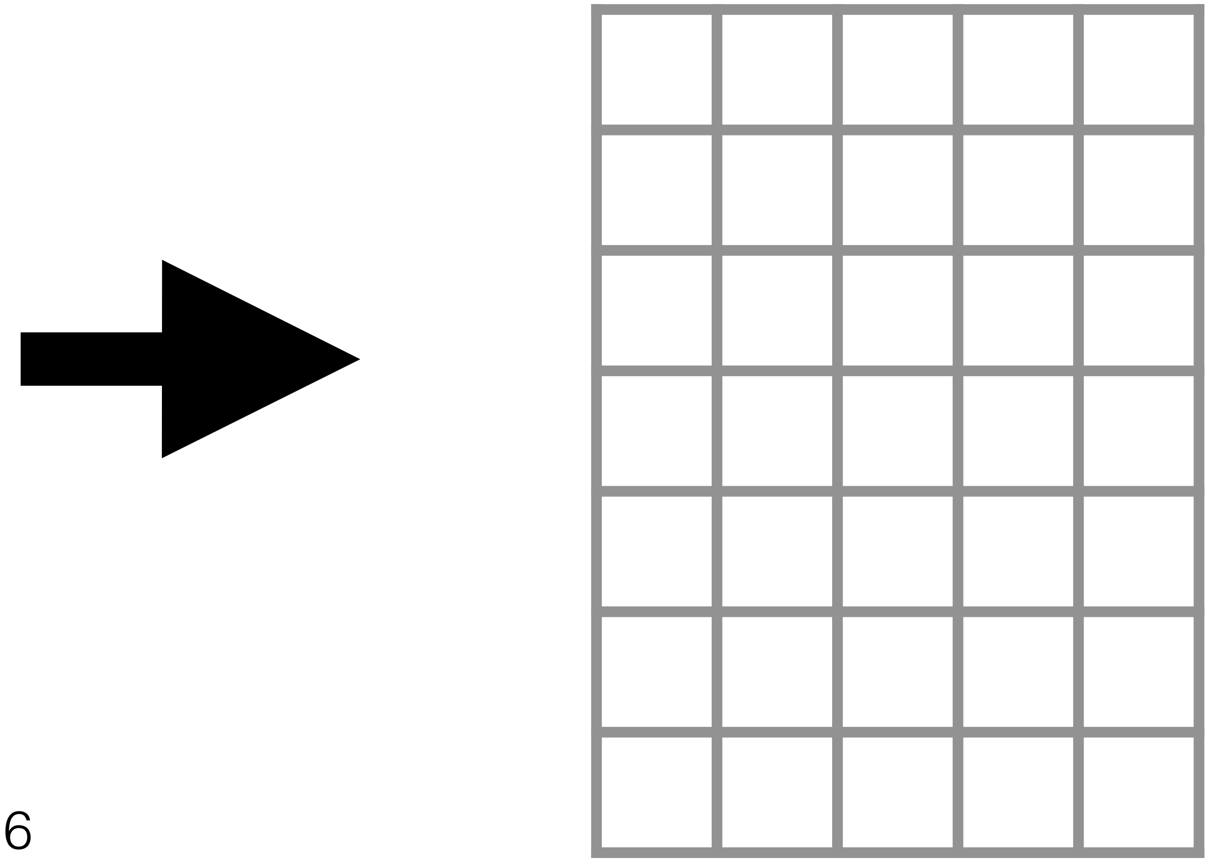

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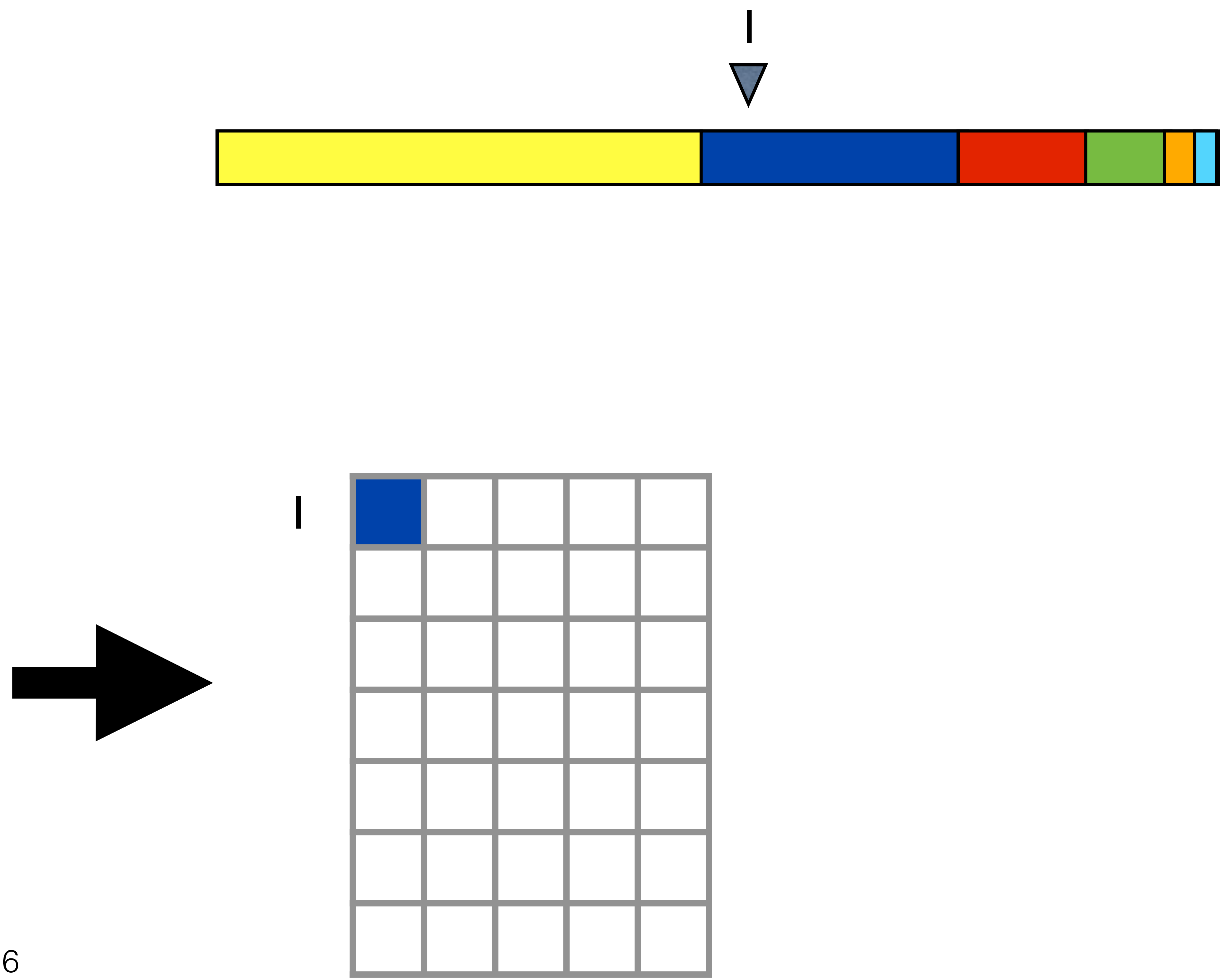

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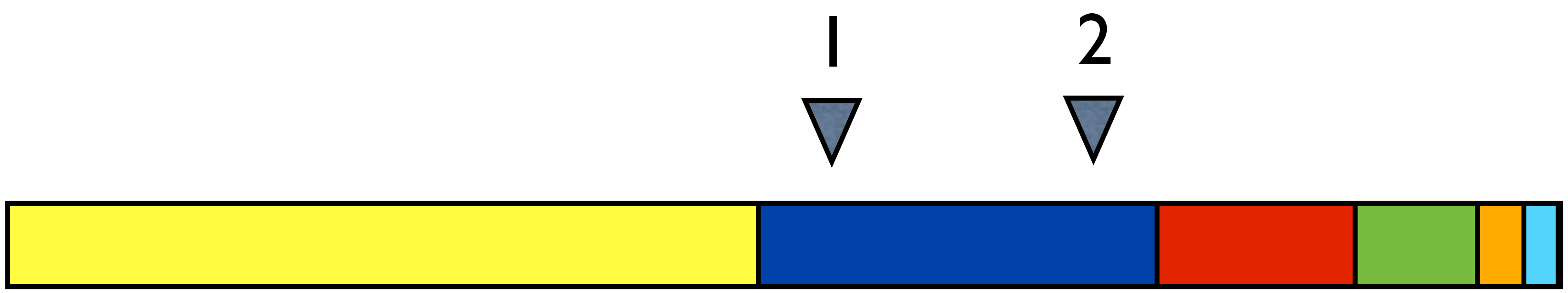

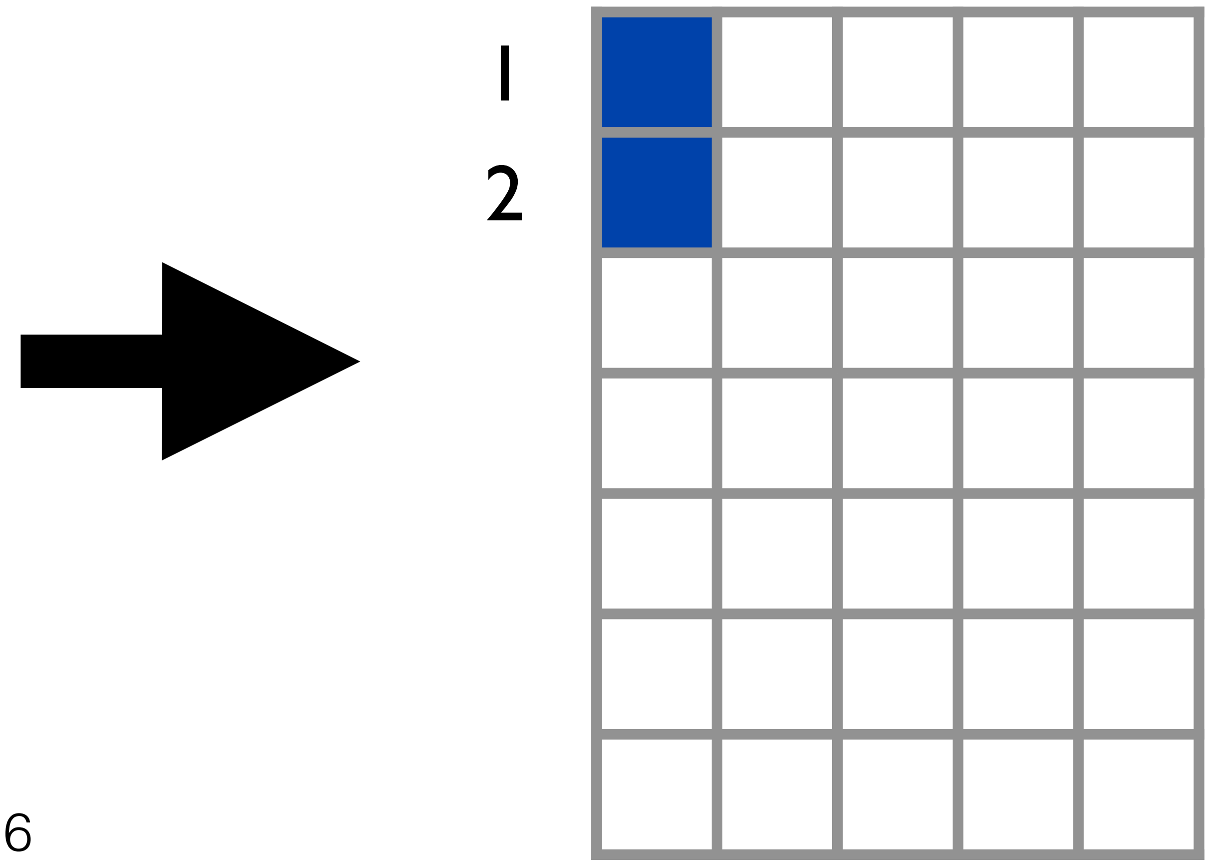

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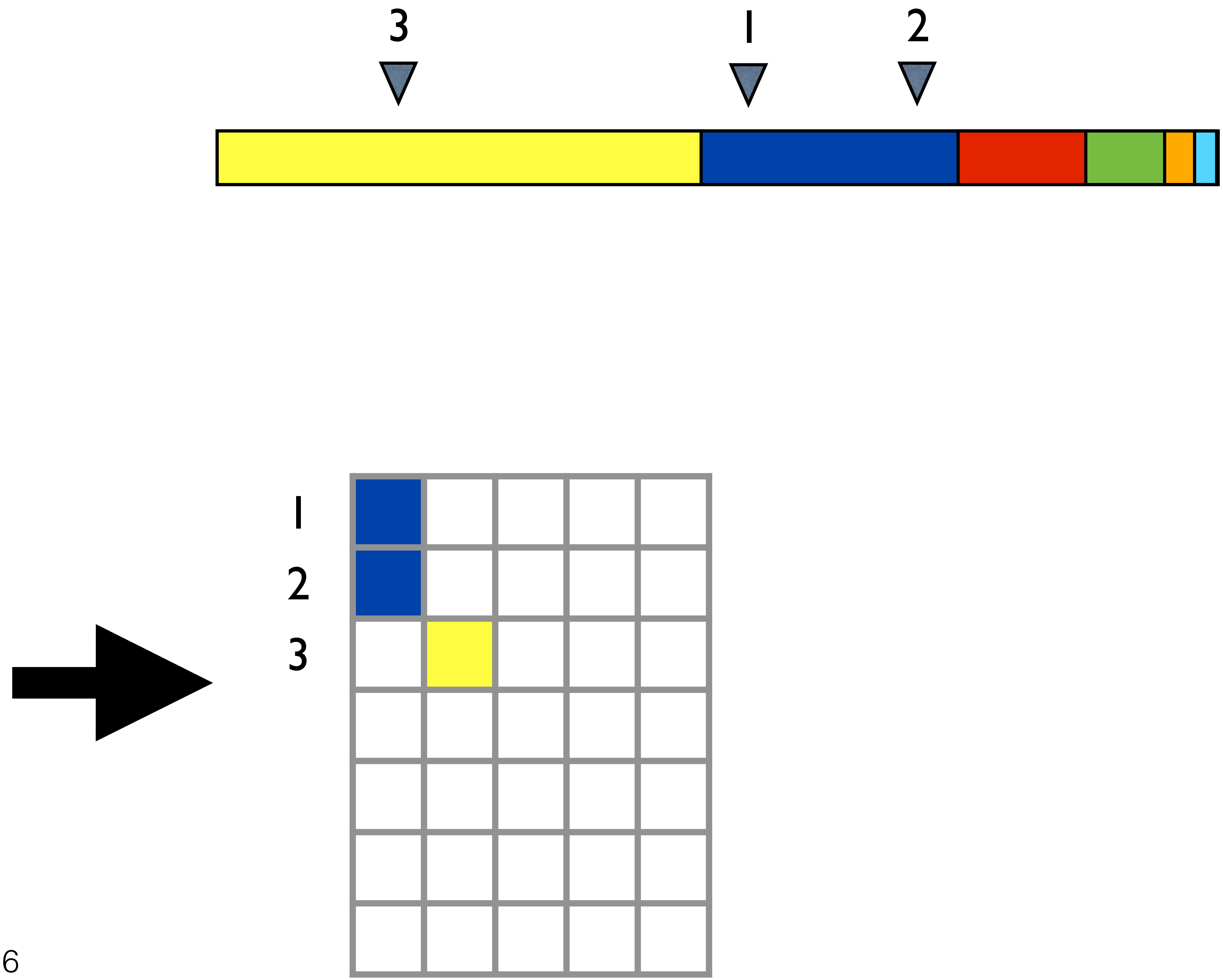

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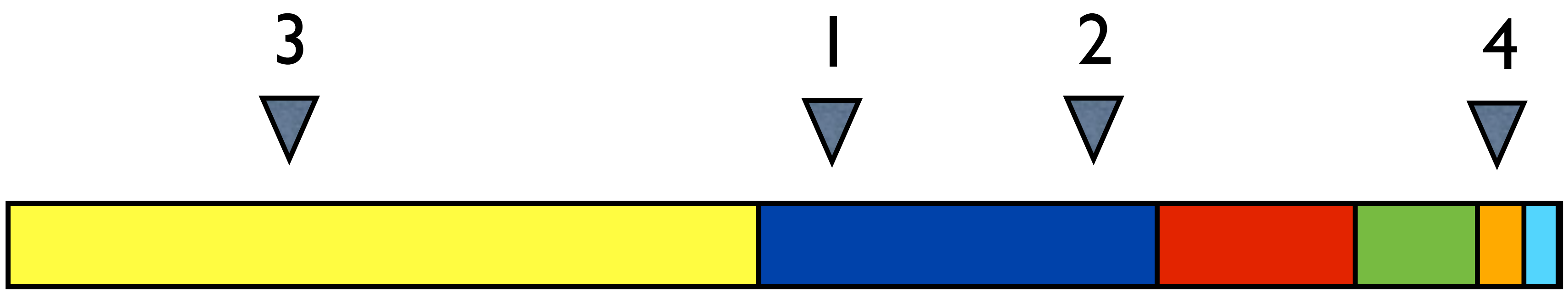

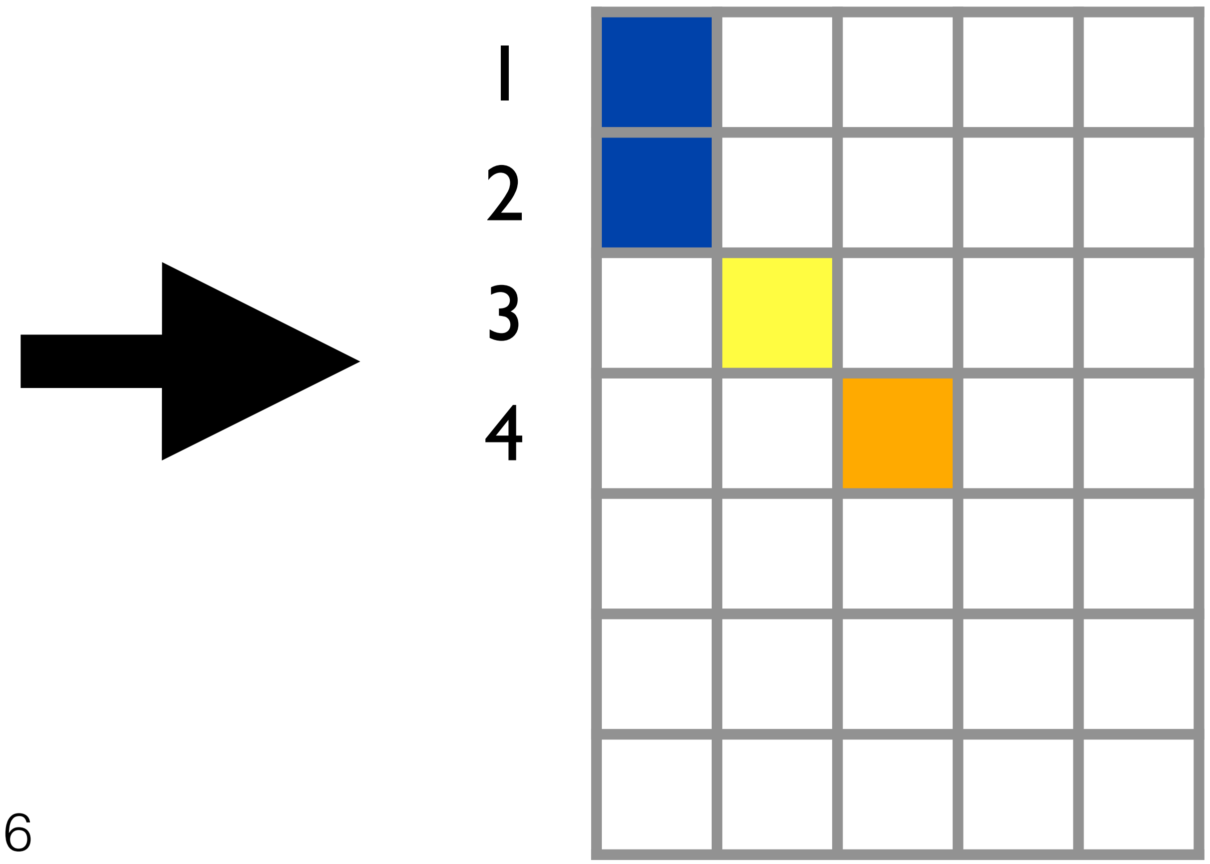

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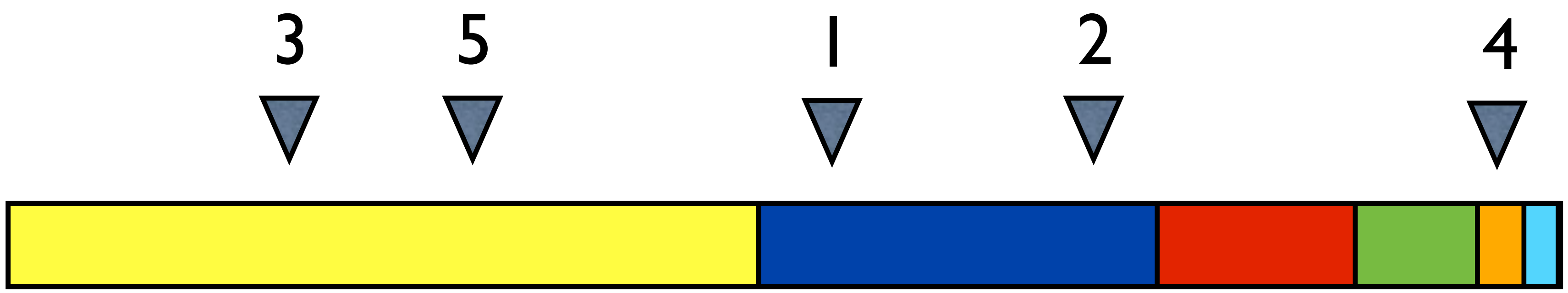

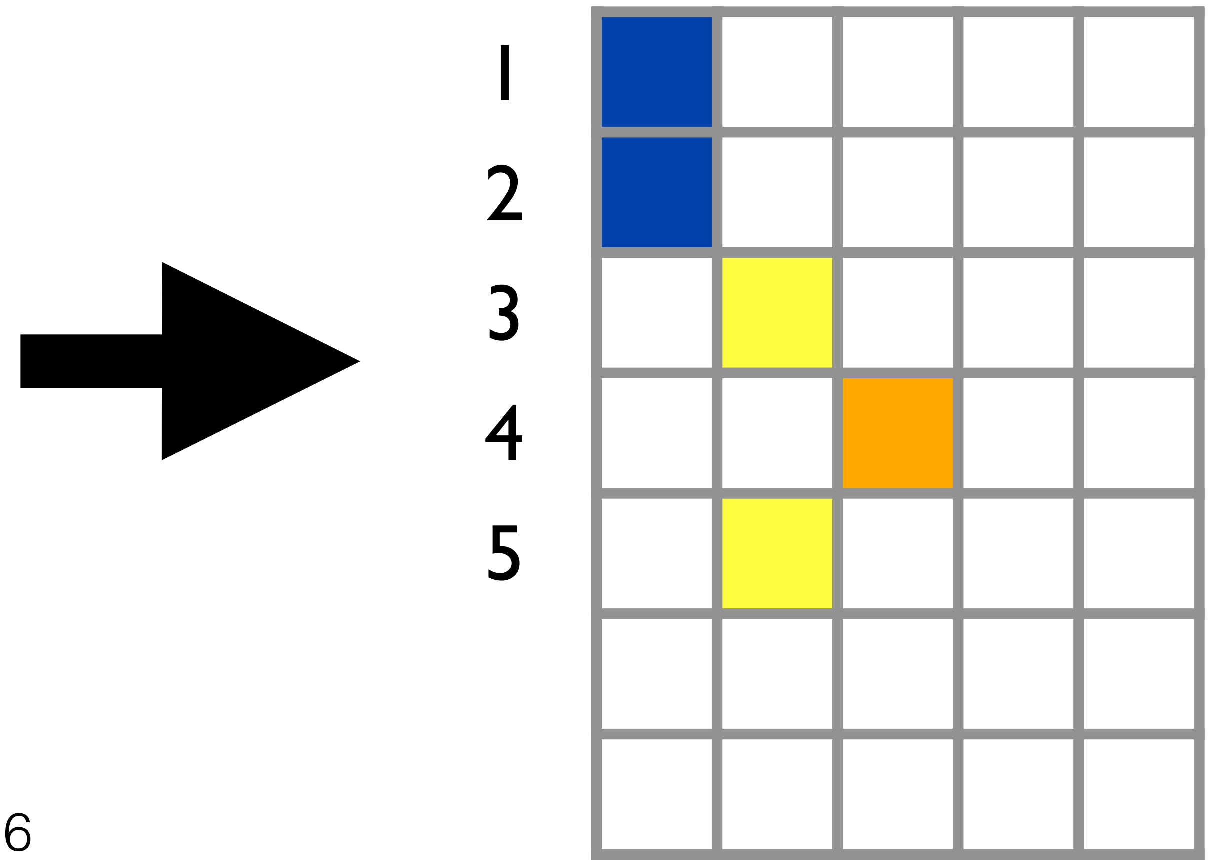

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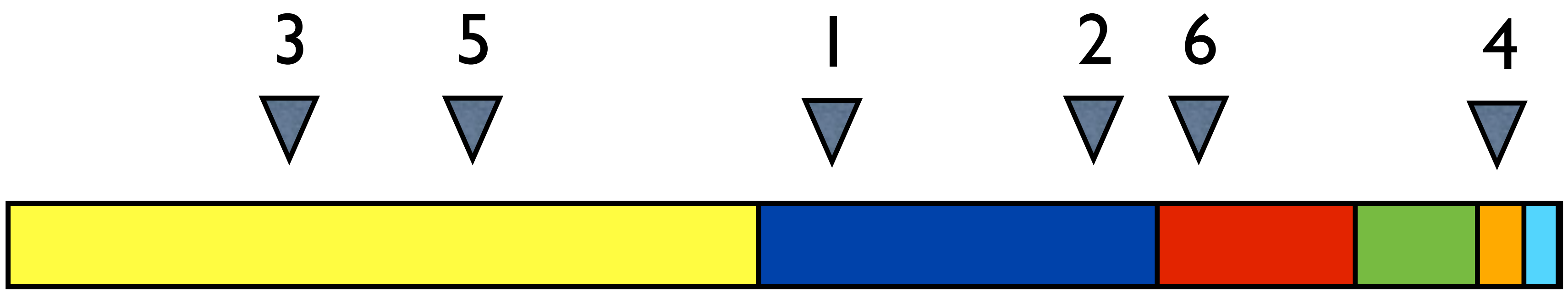

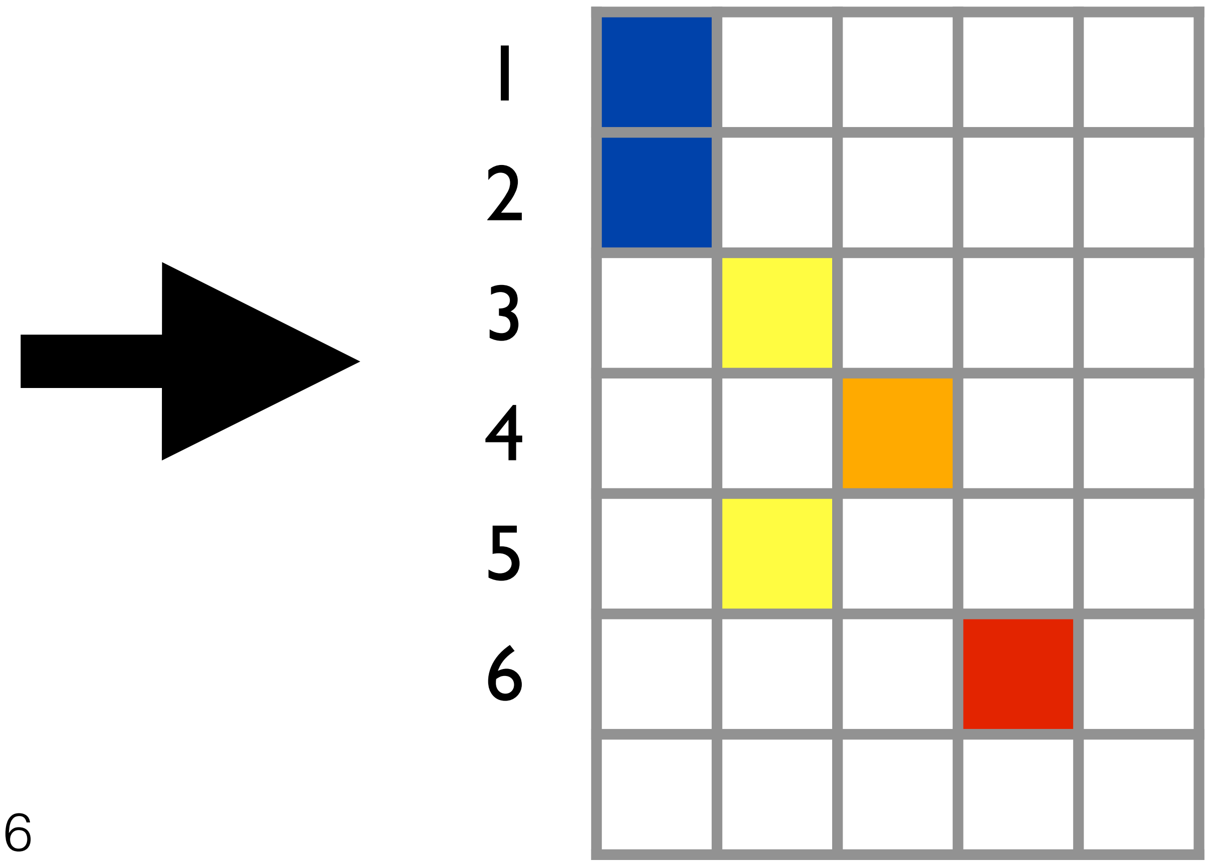

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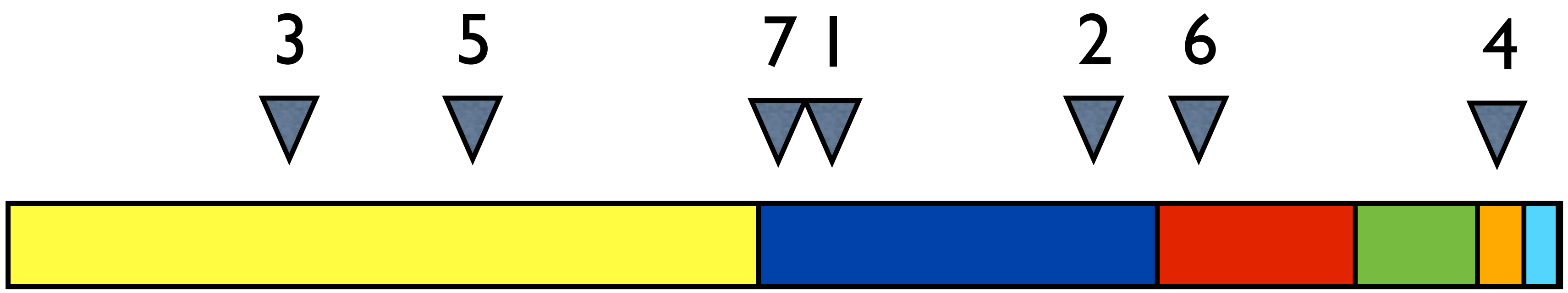

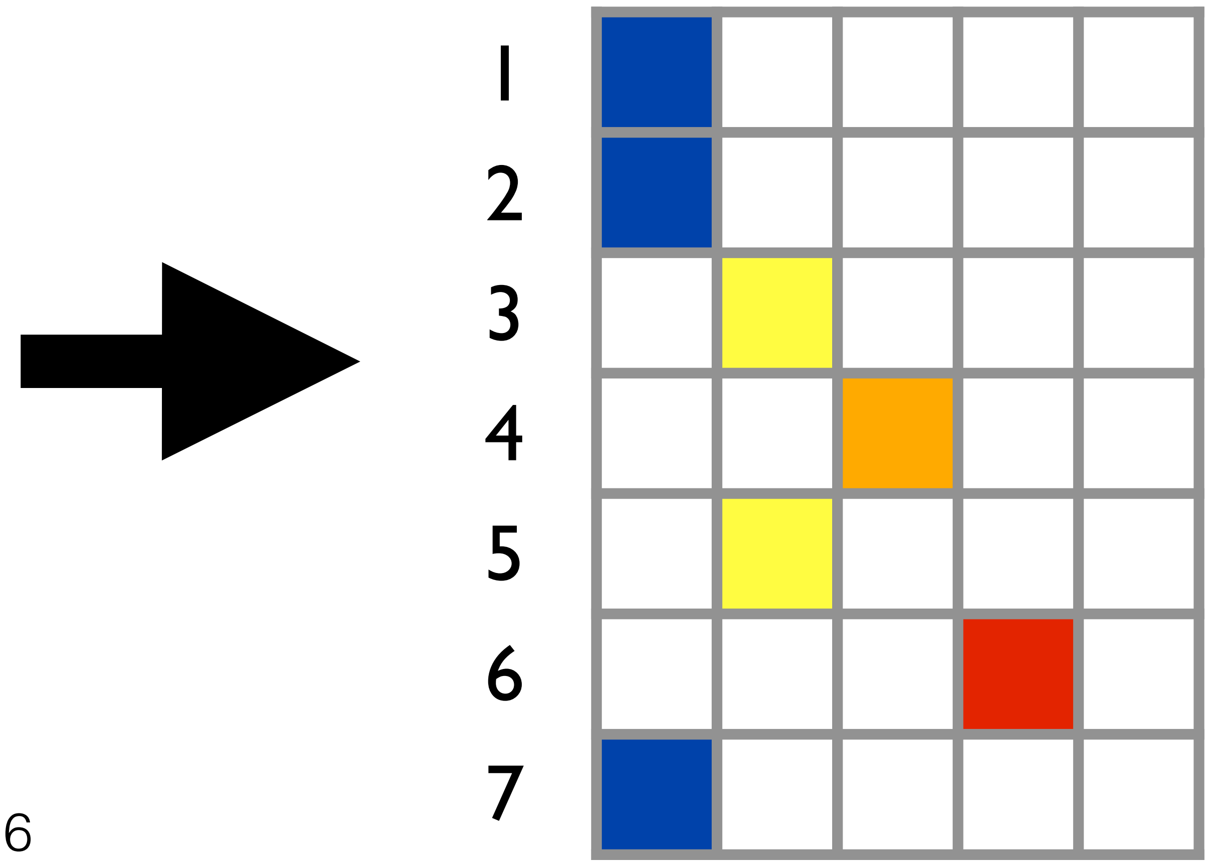

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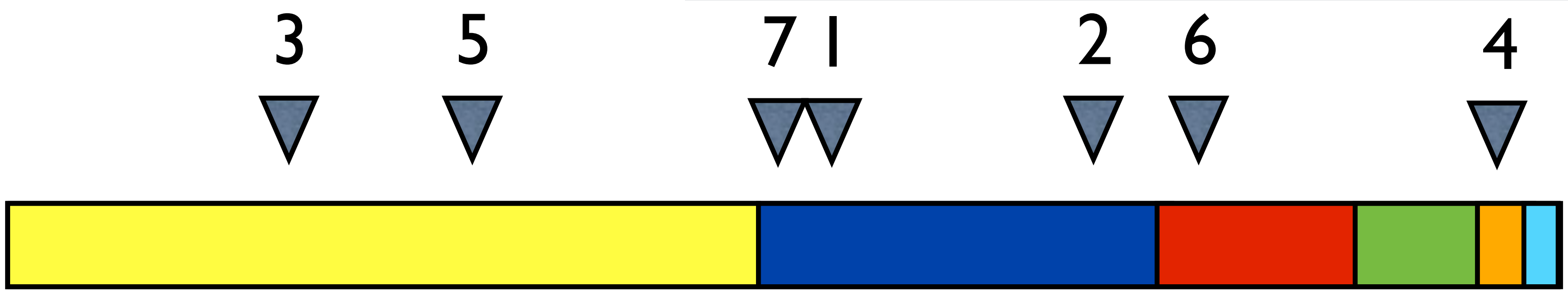
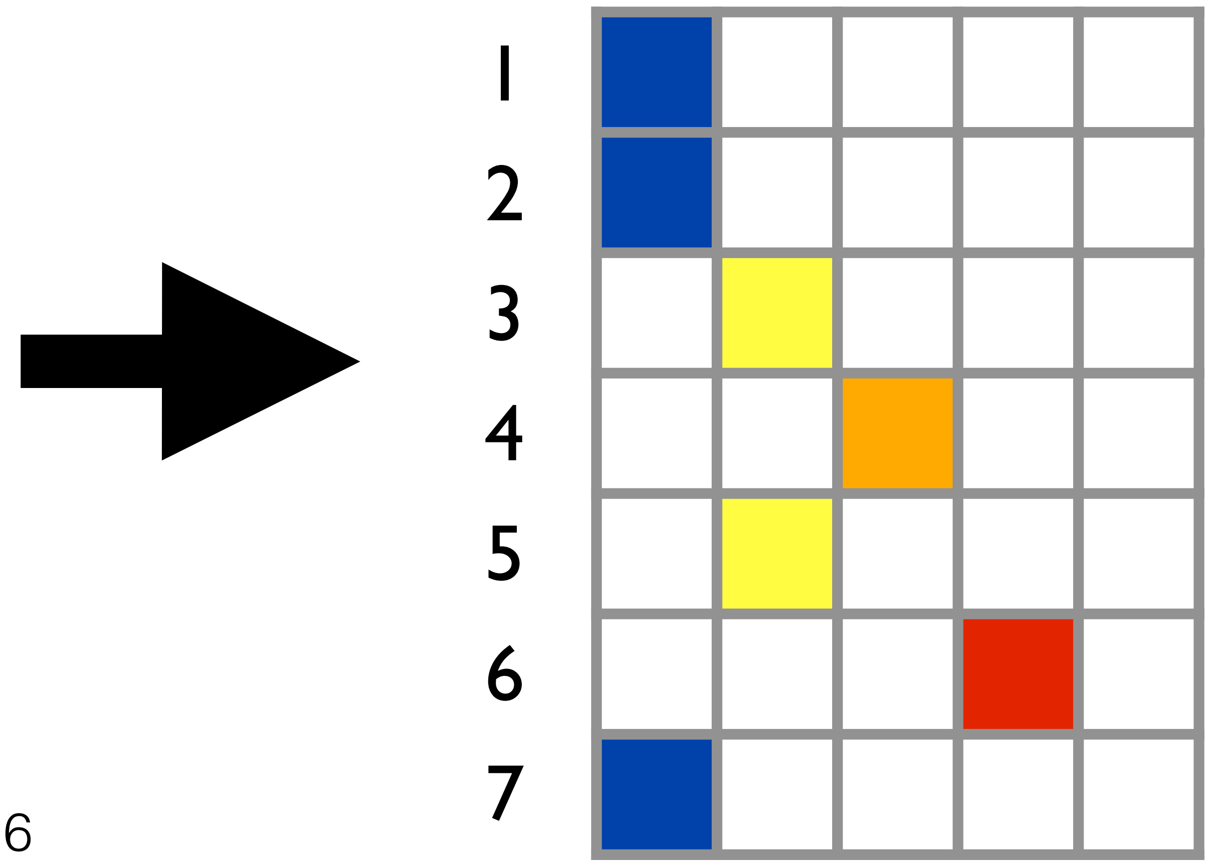
Clustering

Thm (Kingman). A clustering is exchangeable iff it has a Kingman paintbox representation





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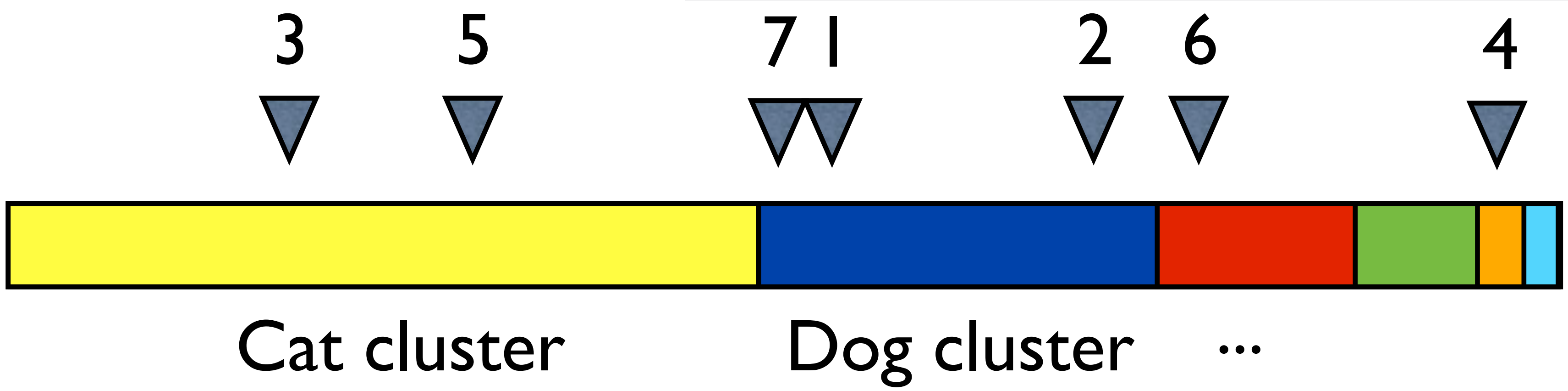

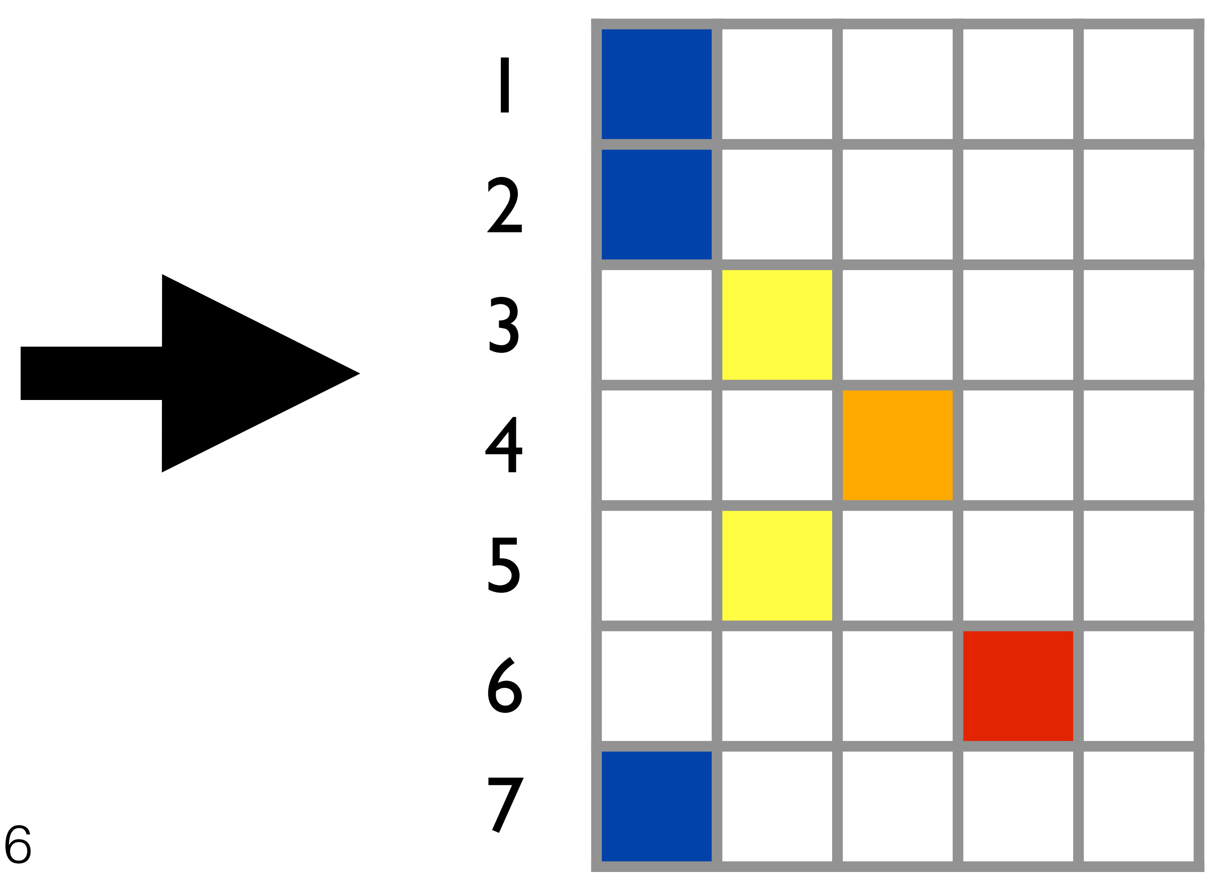

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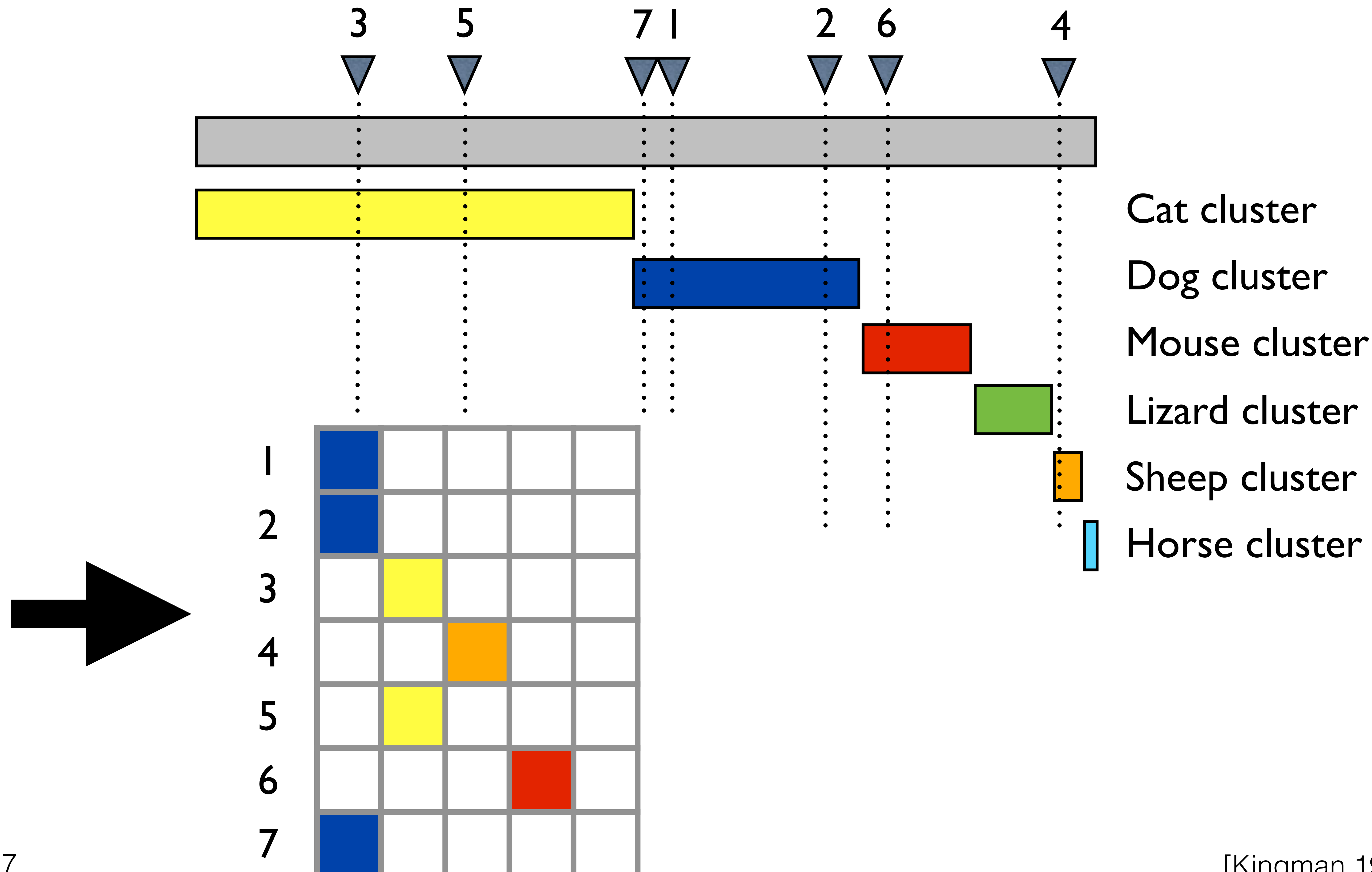

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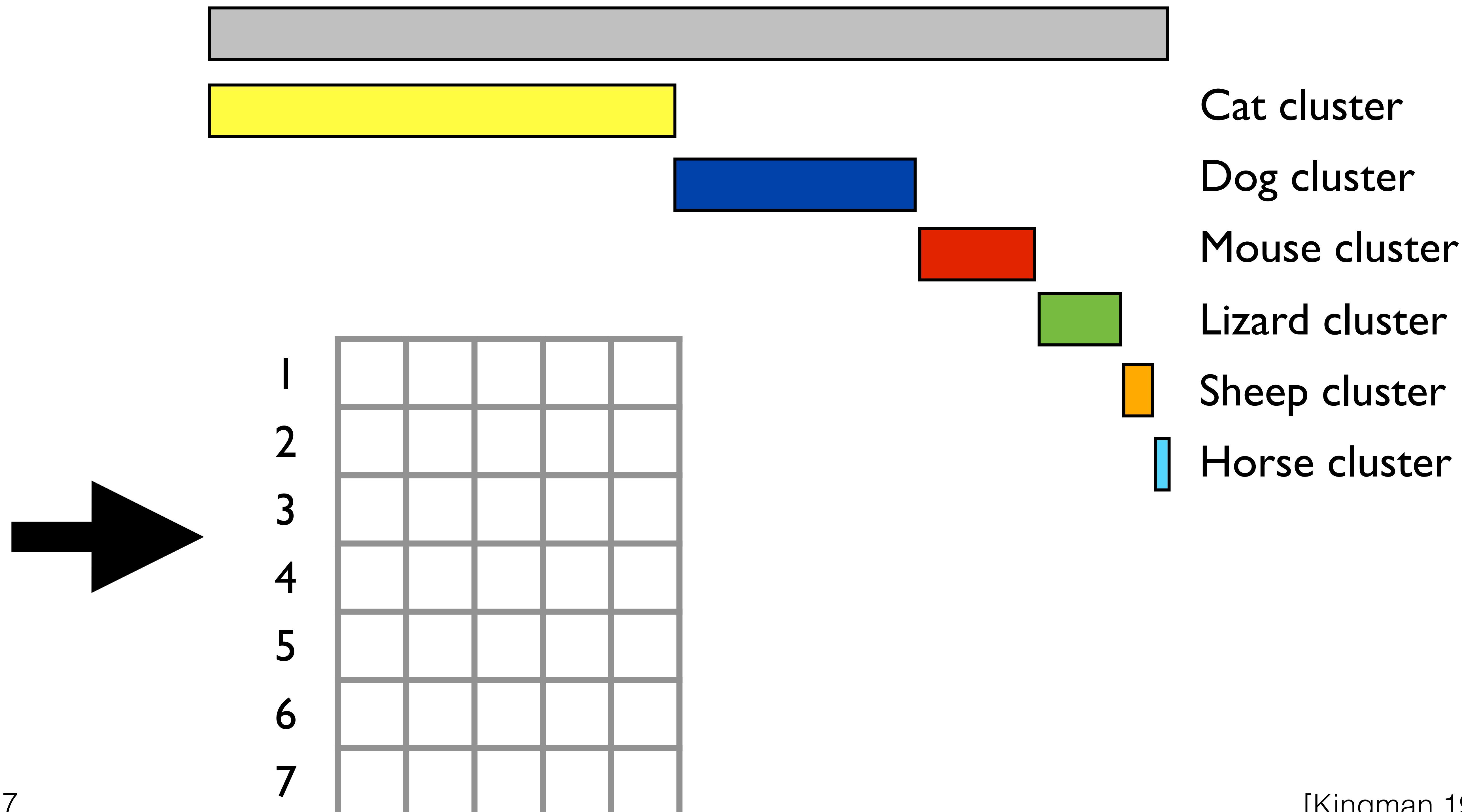

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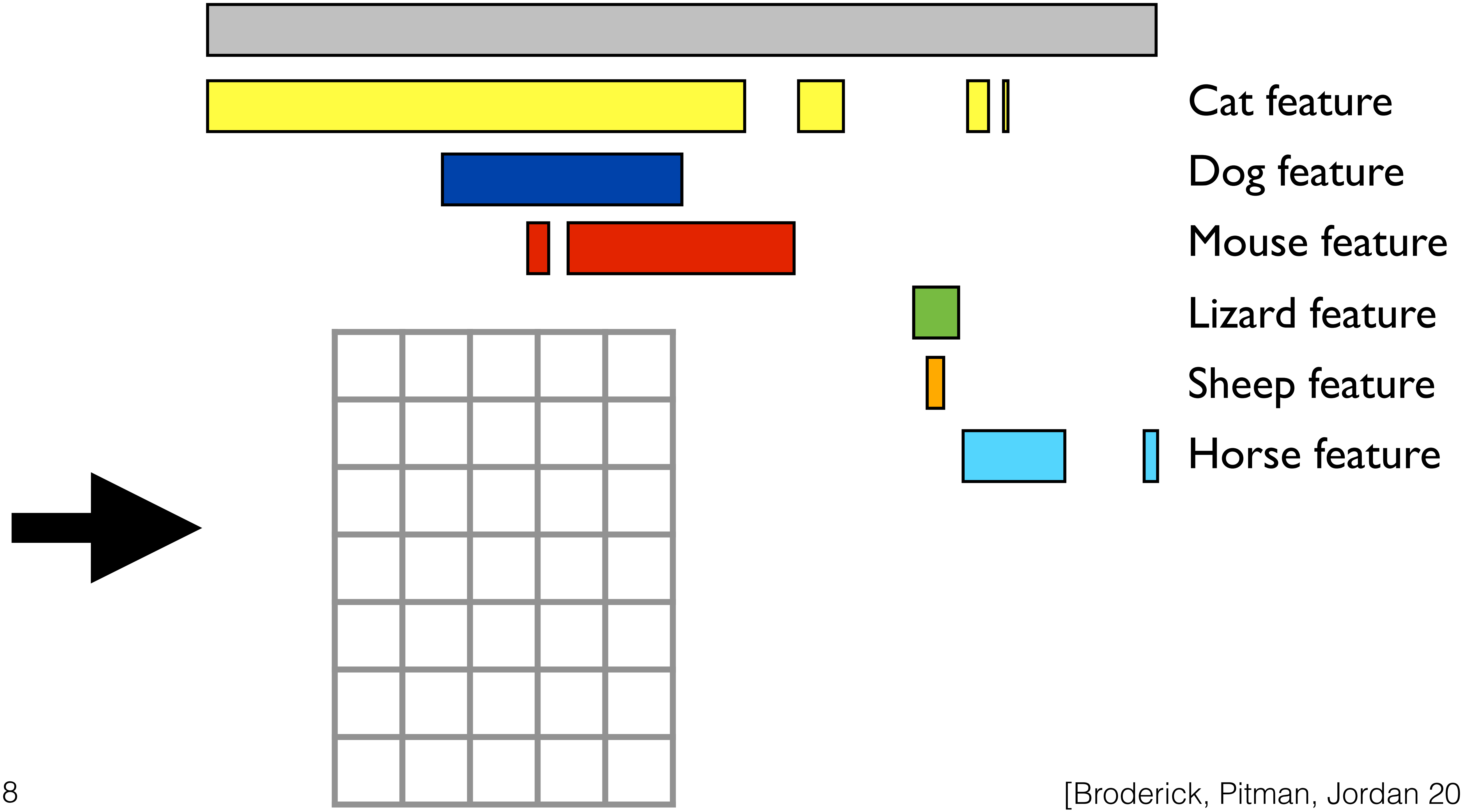
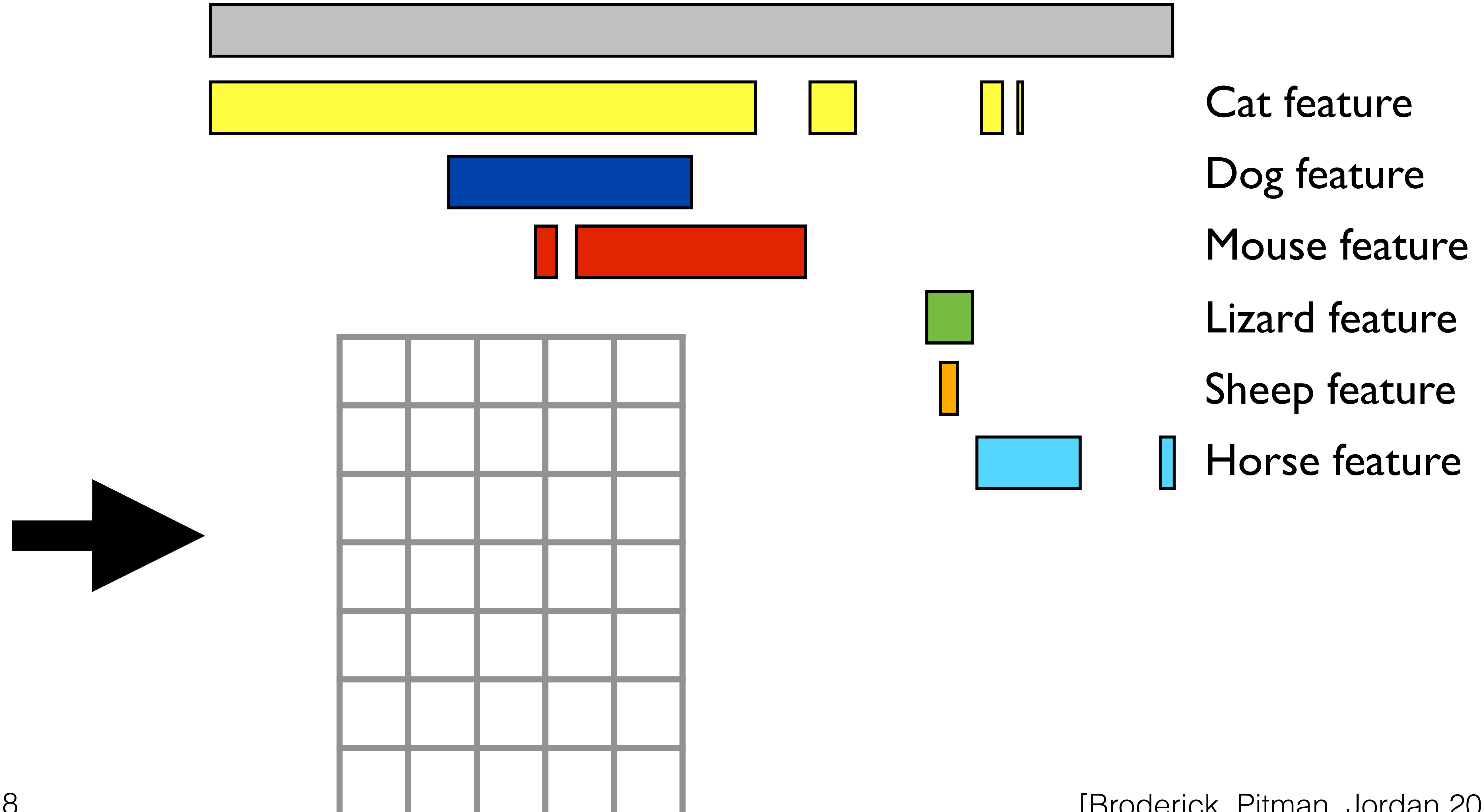


[Broderick, Pitman, Jordan 2013]

Feature allocation



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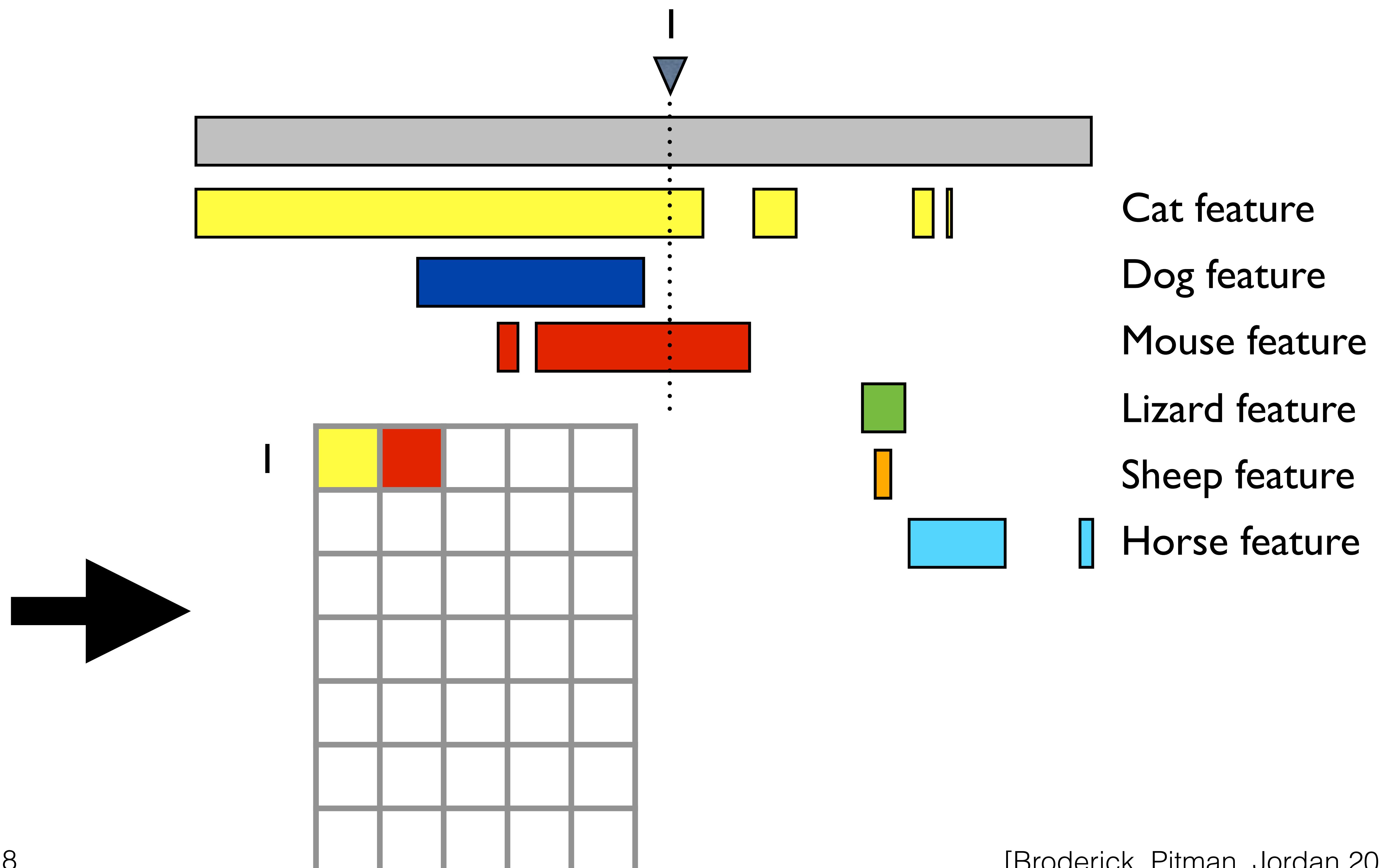

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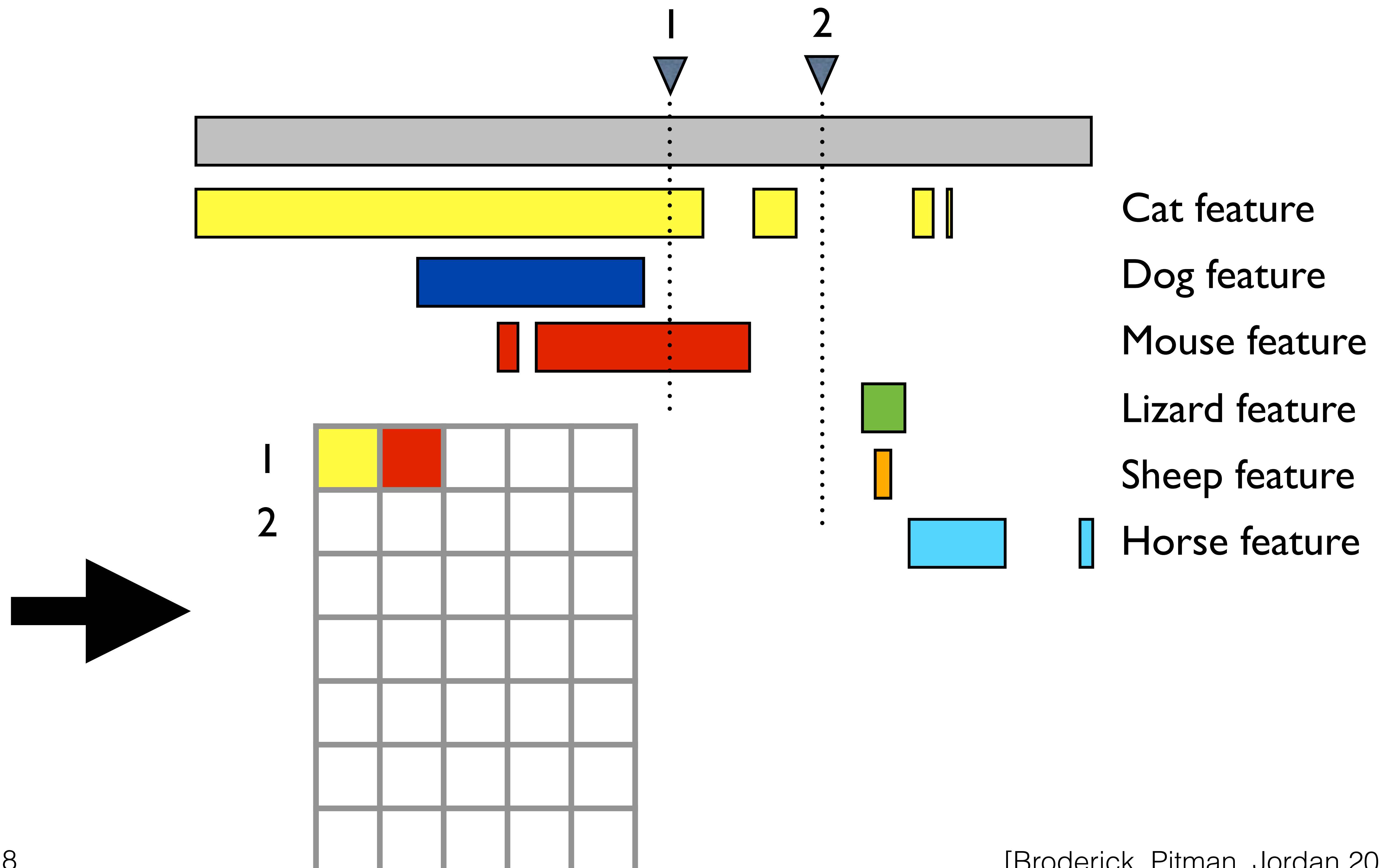

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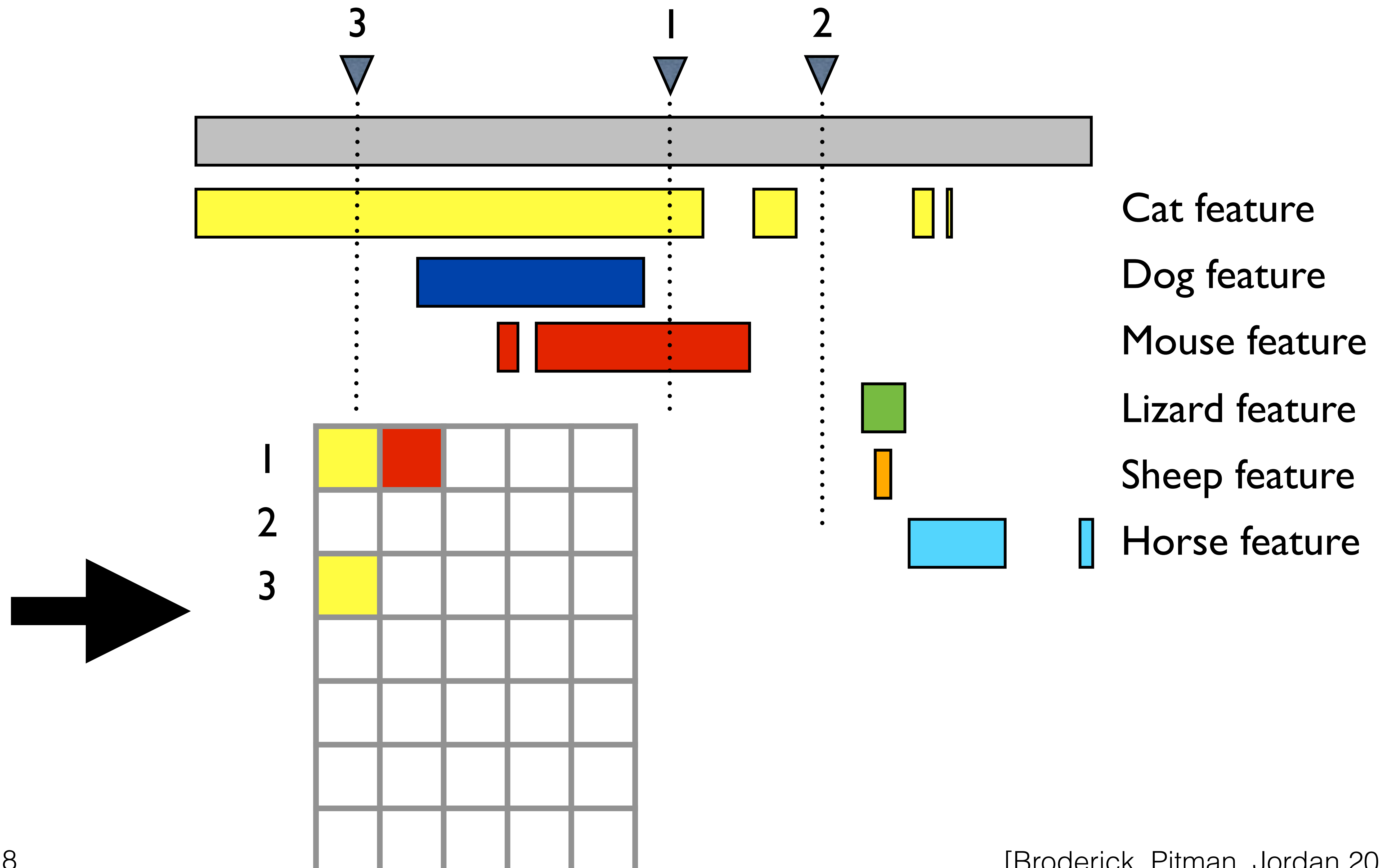

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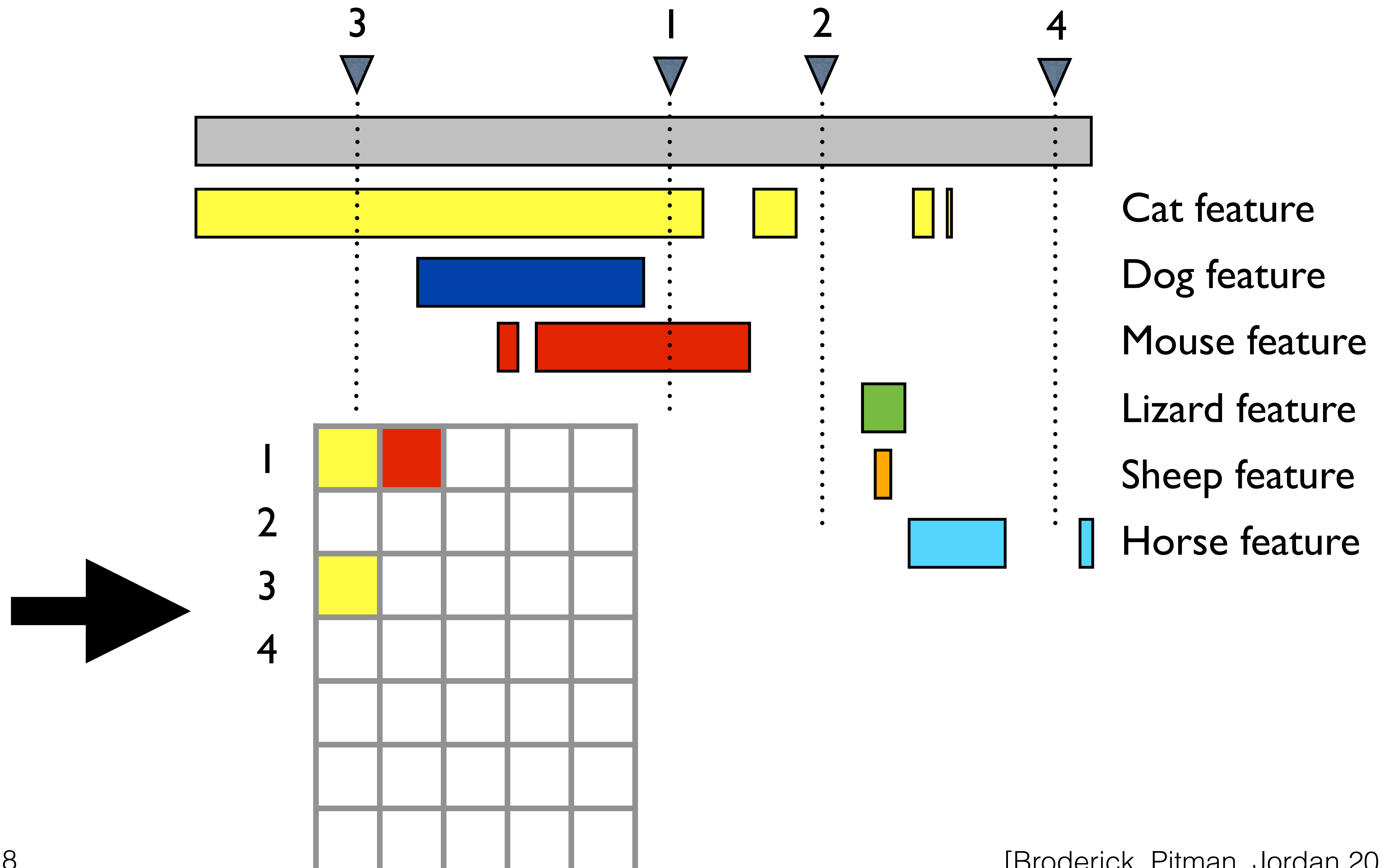

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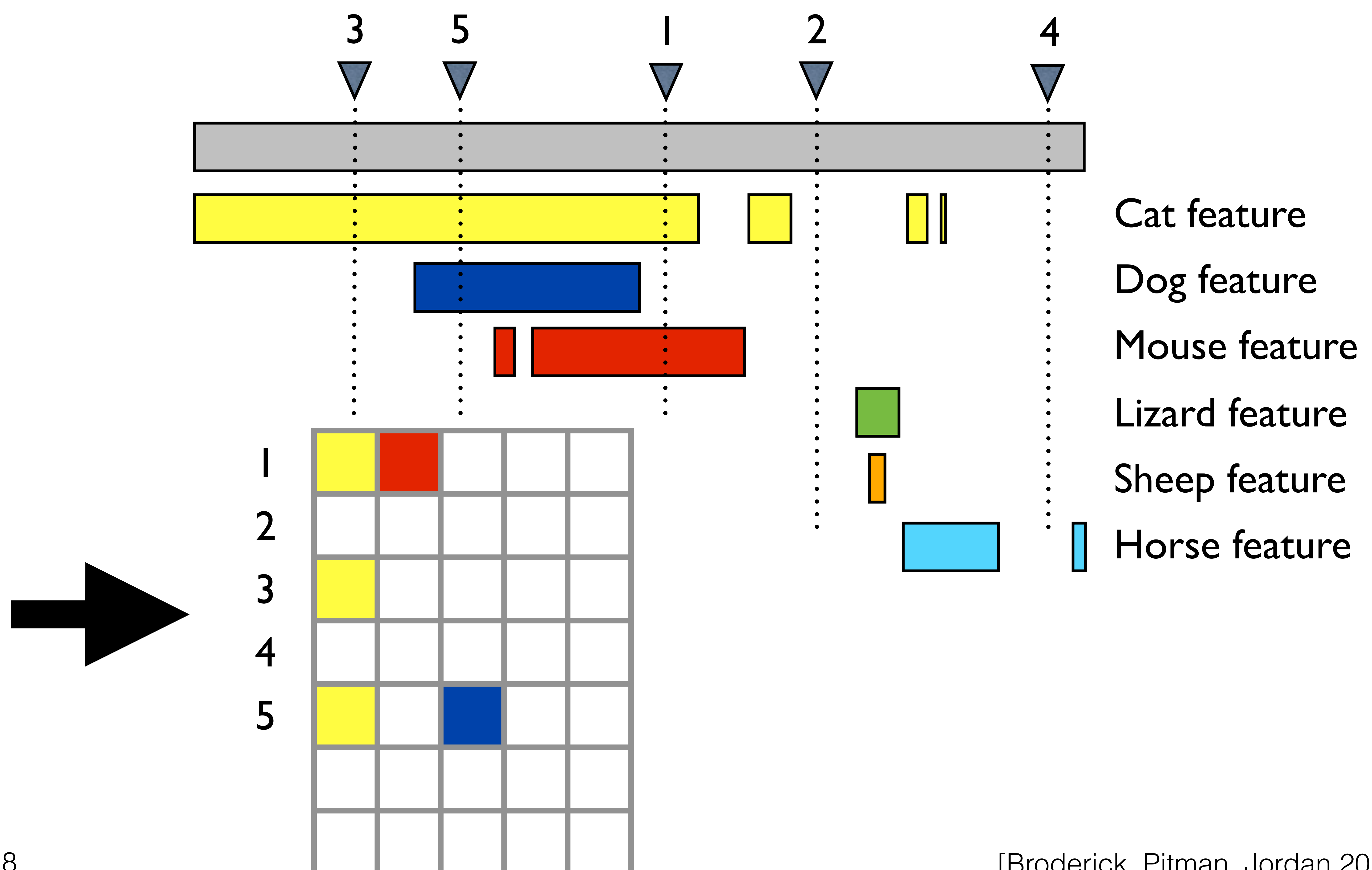

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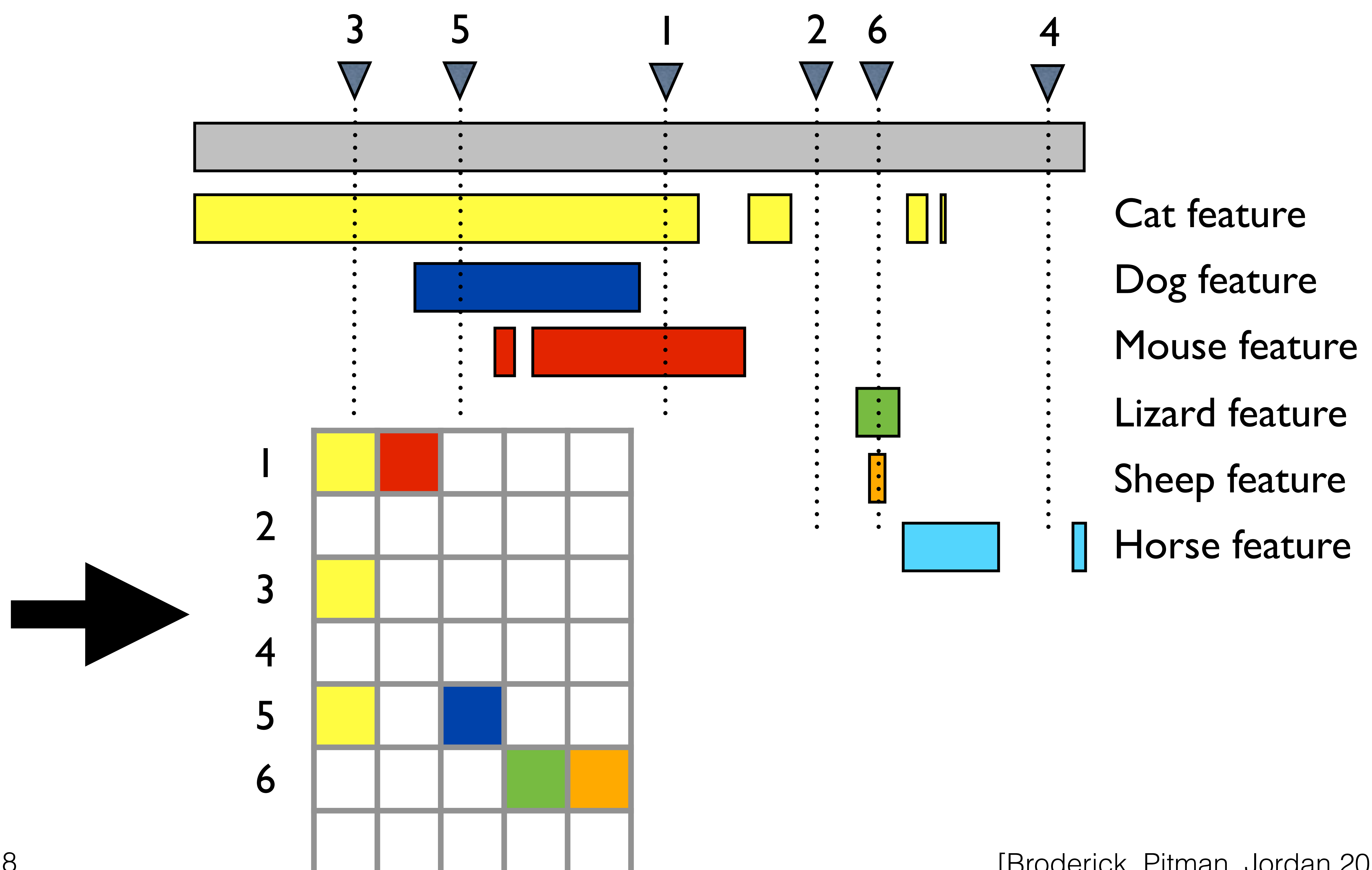

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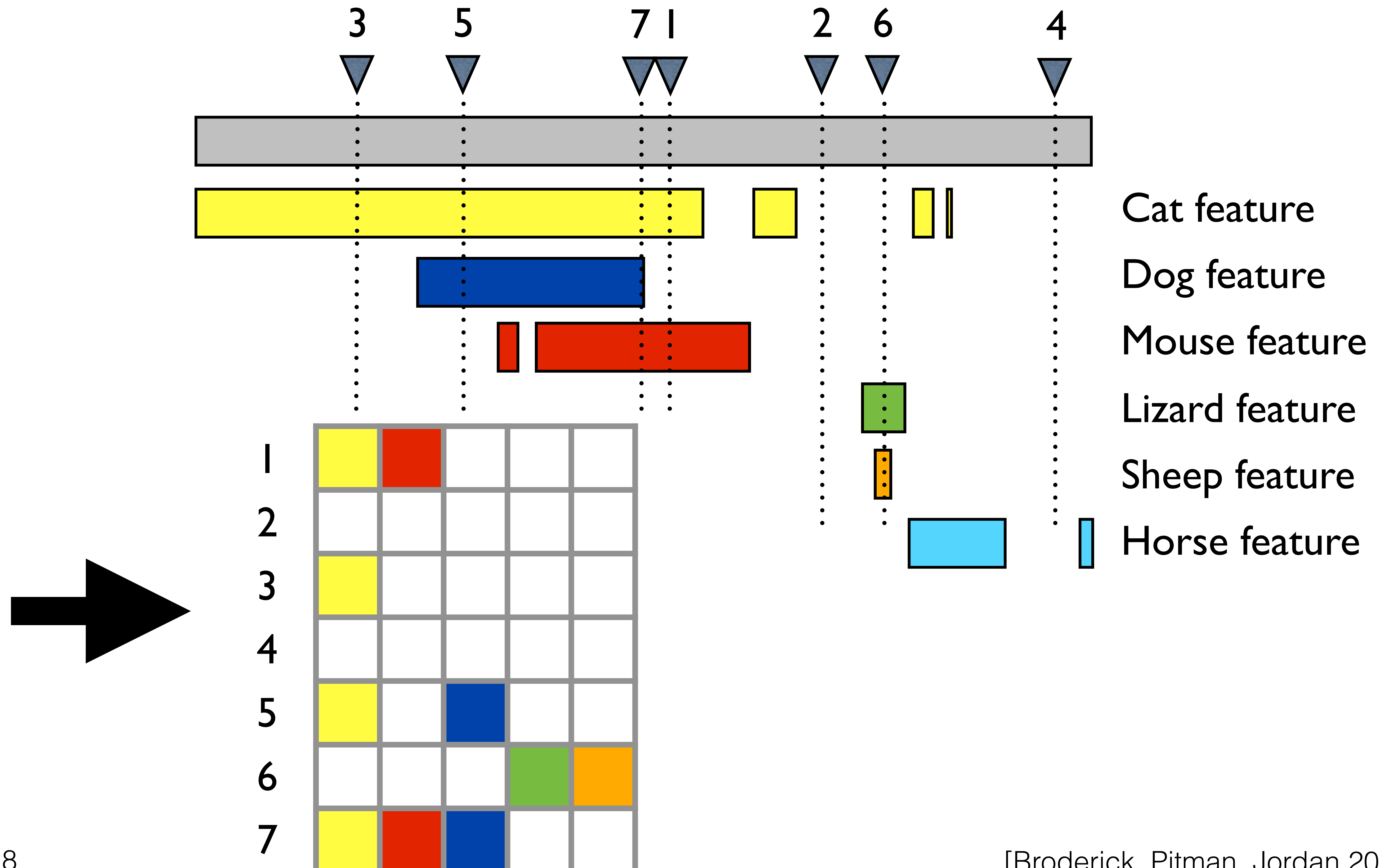

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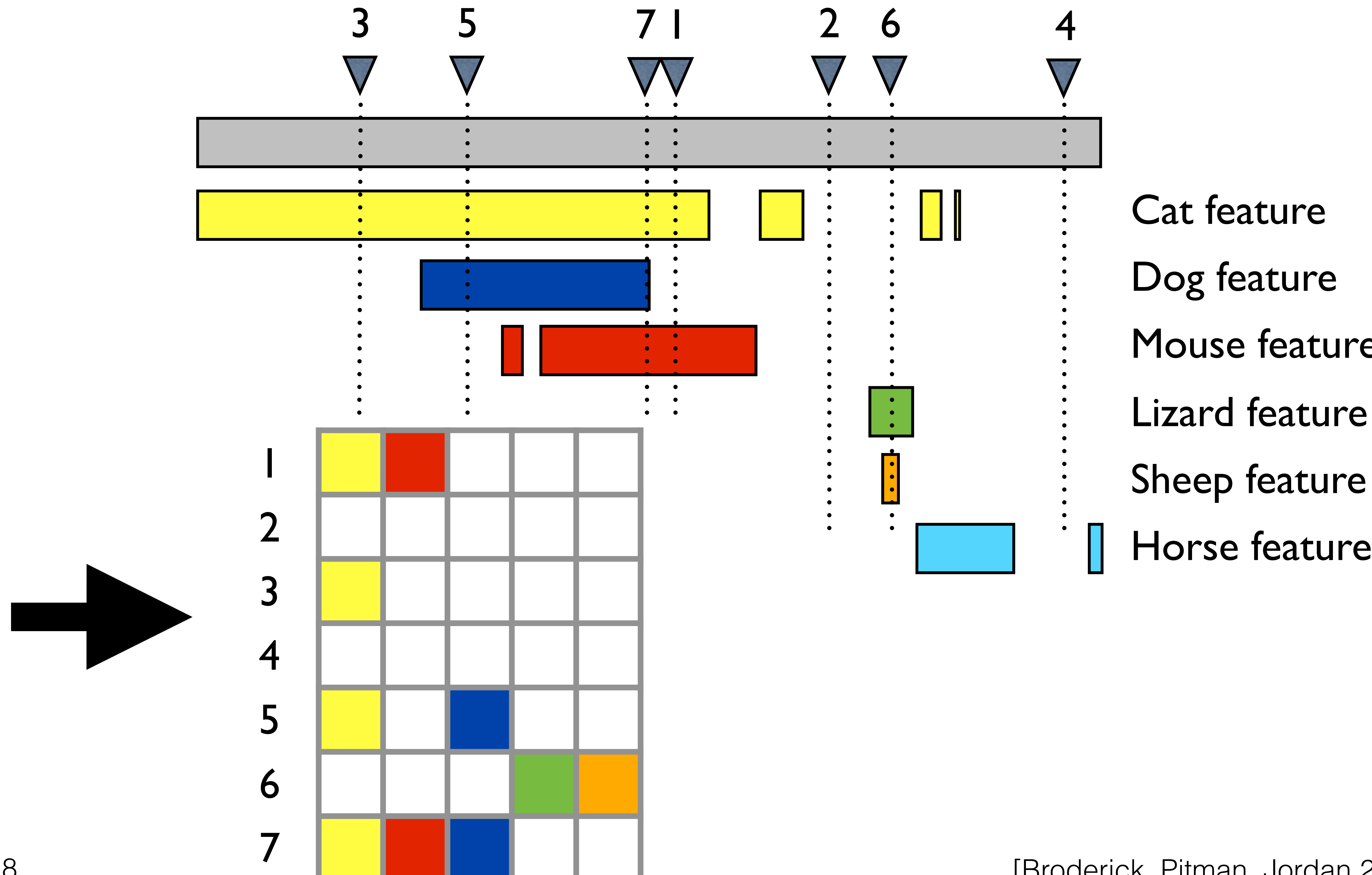
Feature allocation

Thm (BPJ). A feature allocation is exchangeable iff it has a feature paintbox representation



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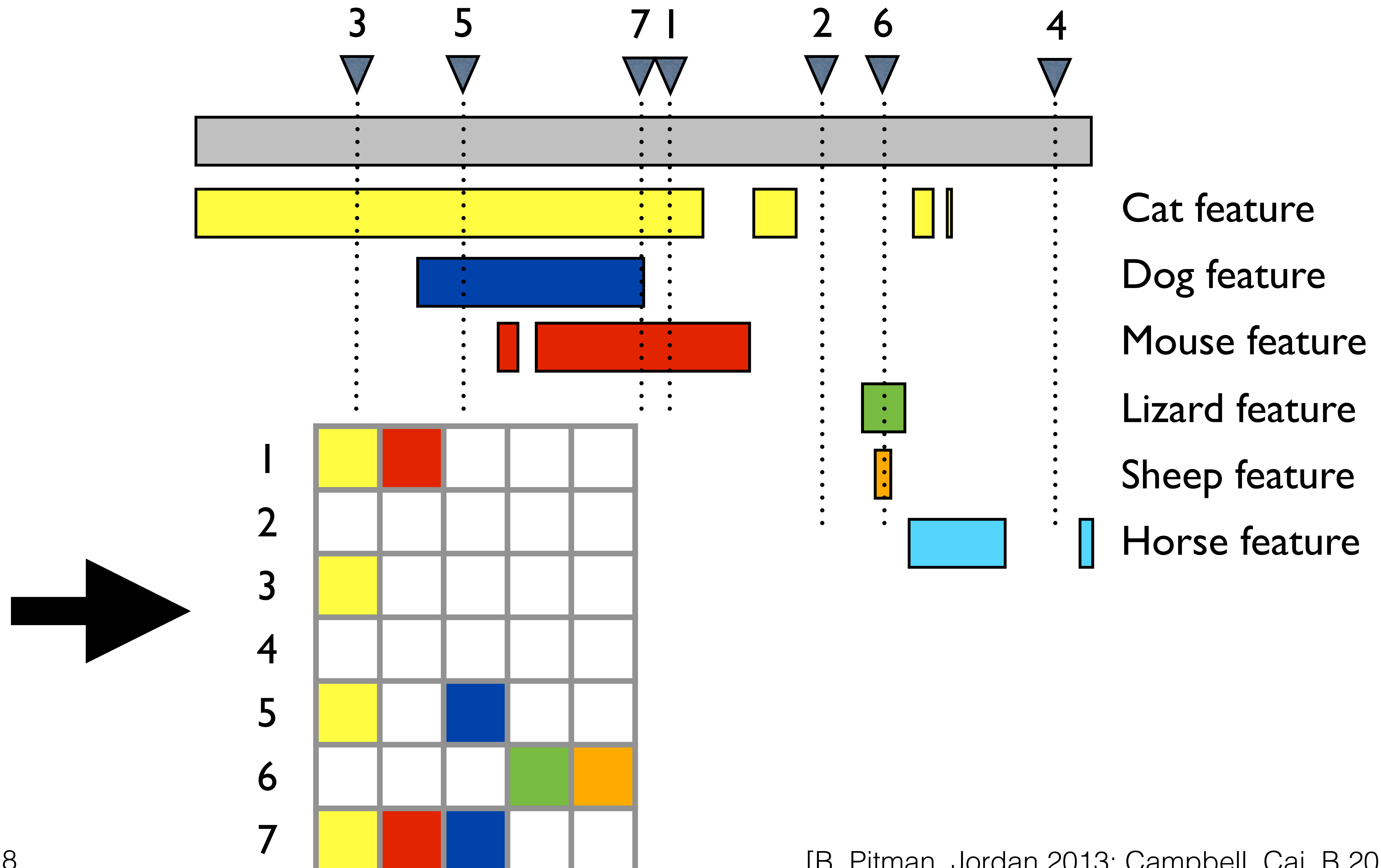
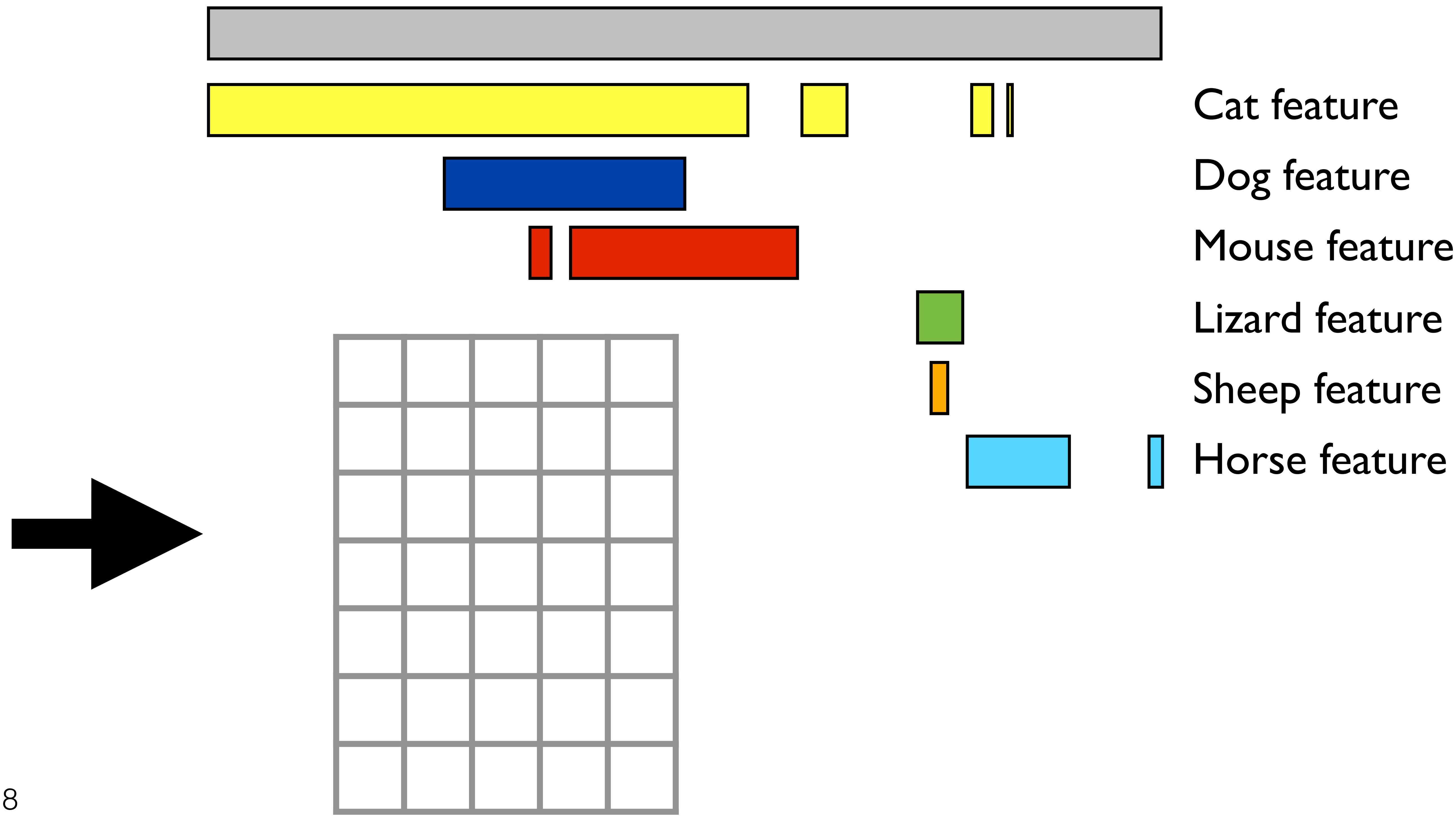
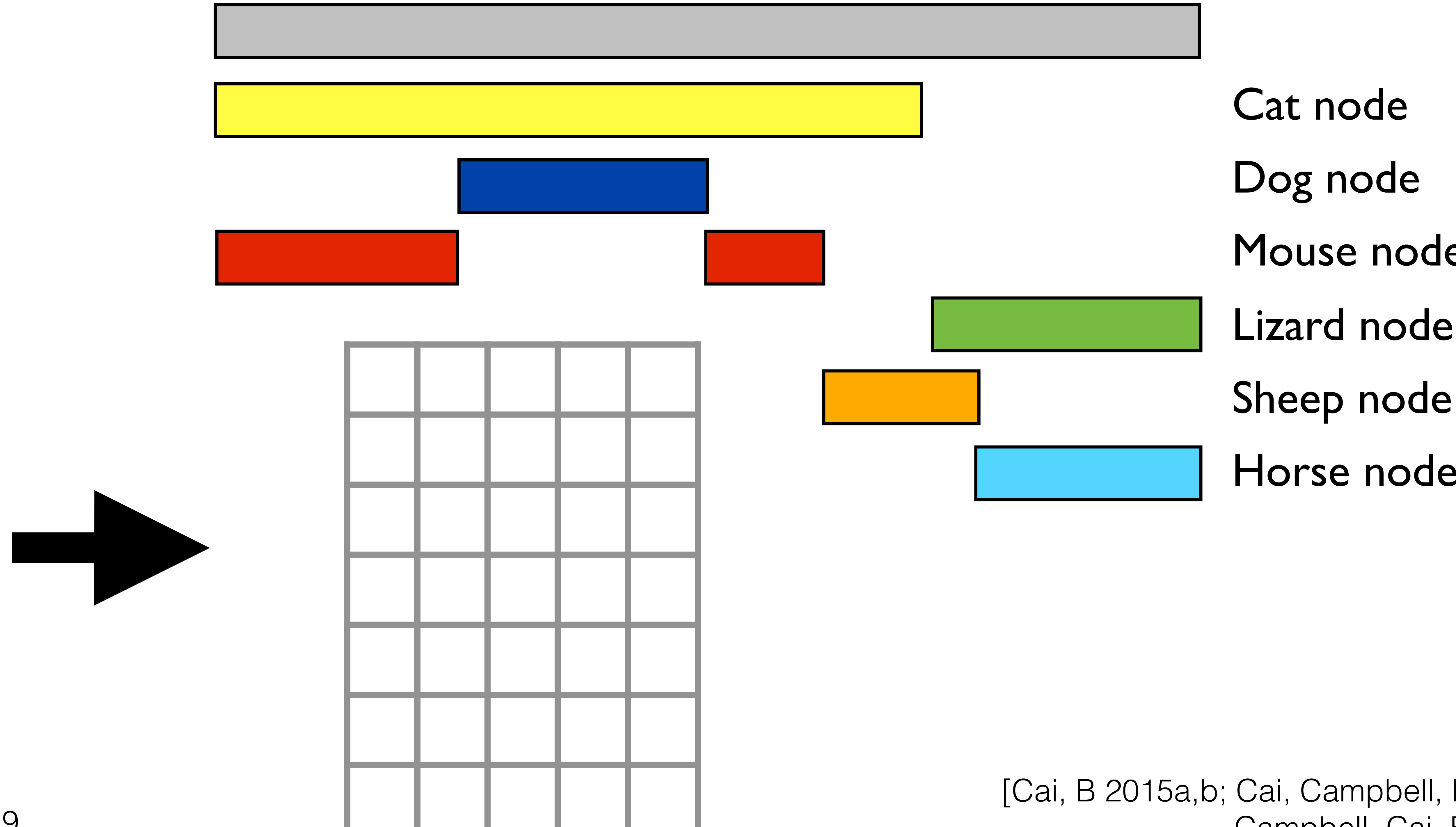


[B, Pitman, Jordan 2013; Campbell, Cai, B 2016]


Cat feature
Dog feature
Mouse feature
Lizard feature
Sheep feature
Horse feature



[Cai, B 2015a,b; Cai, Campbell, B, 2016;
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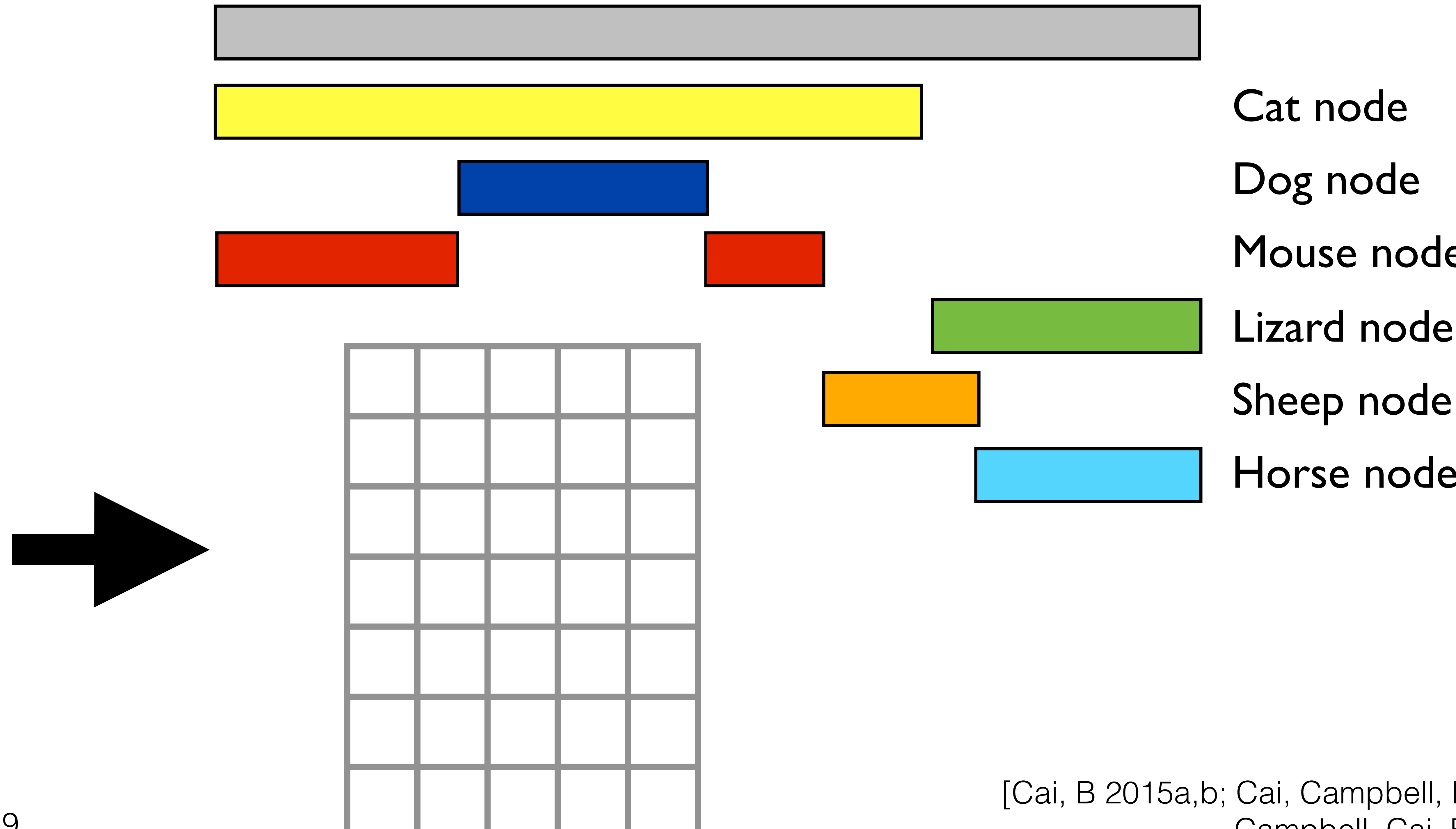
Edge-exchangeable graph



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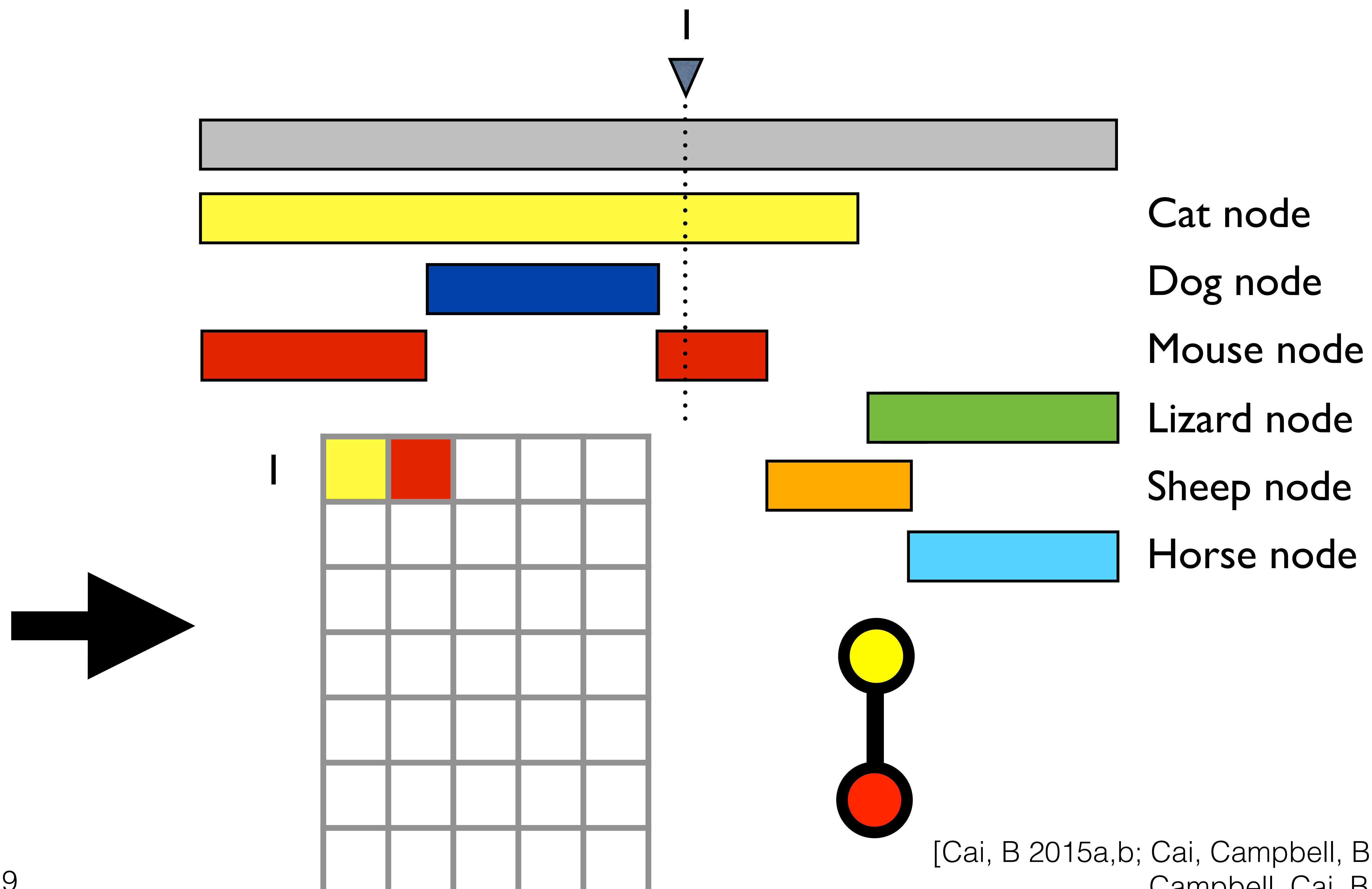

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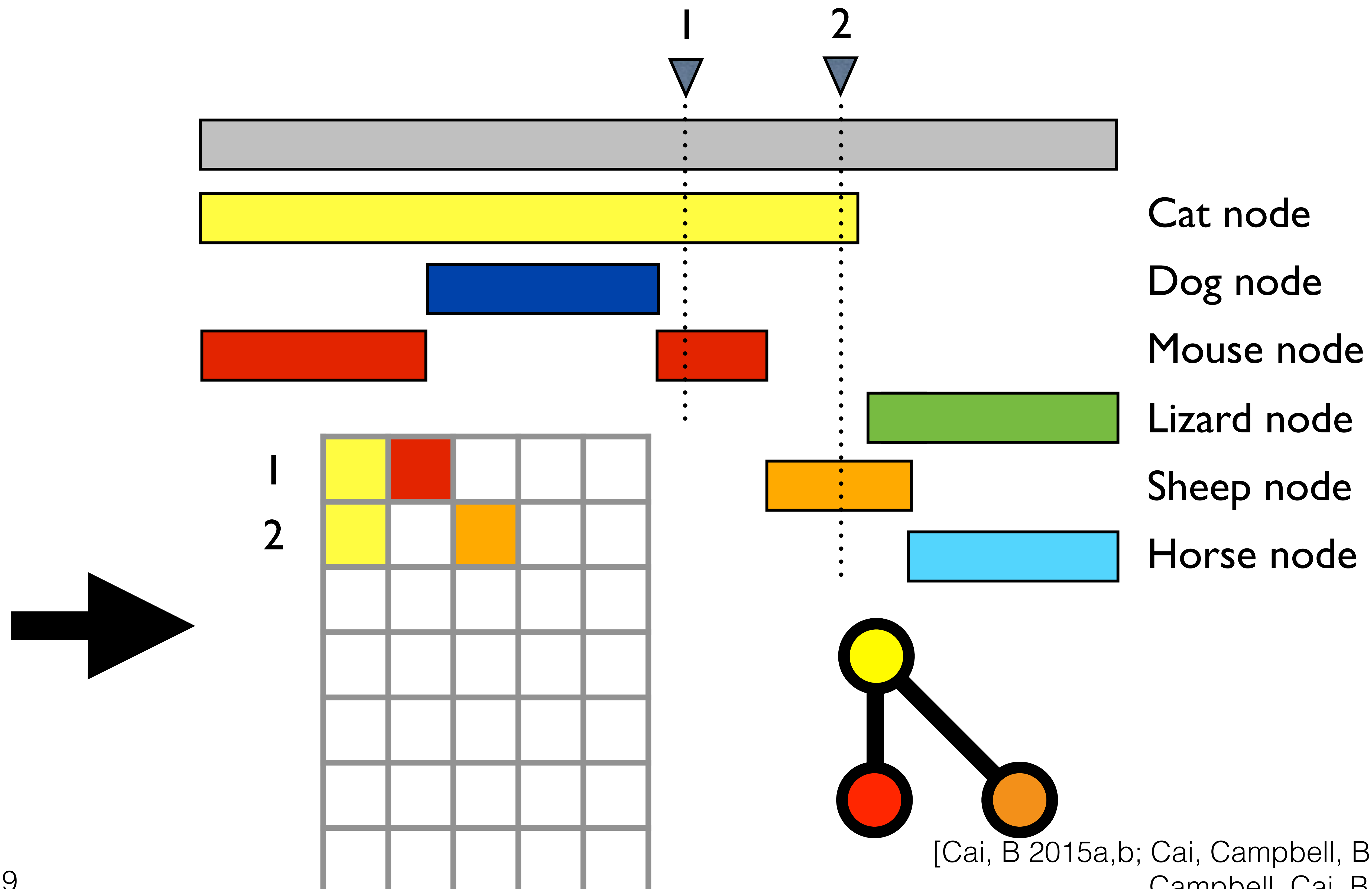

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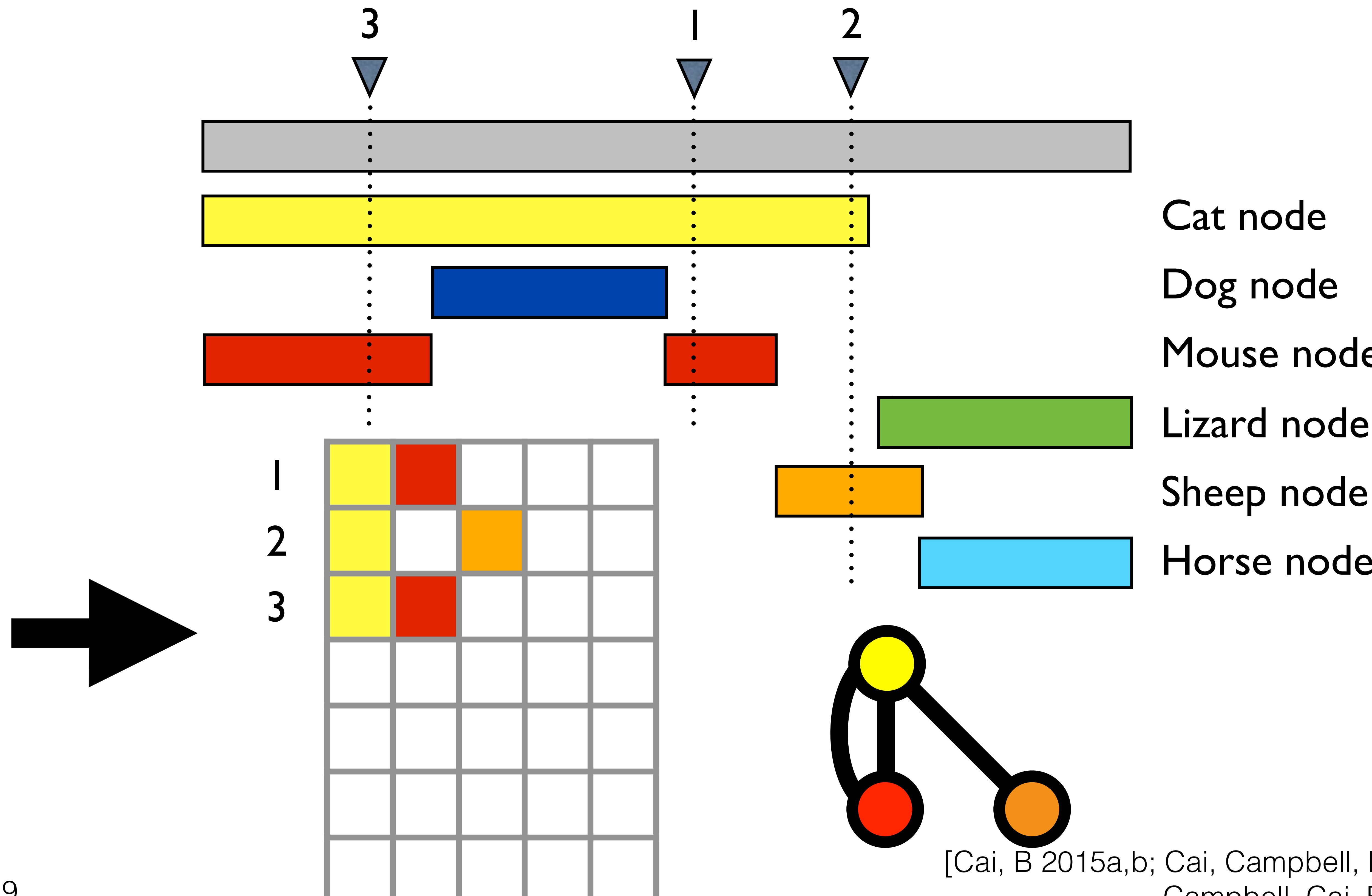

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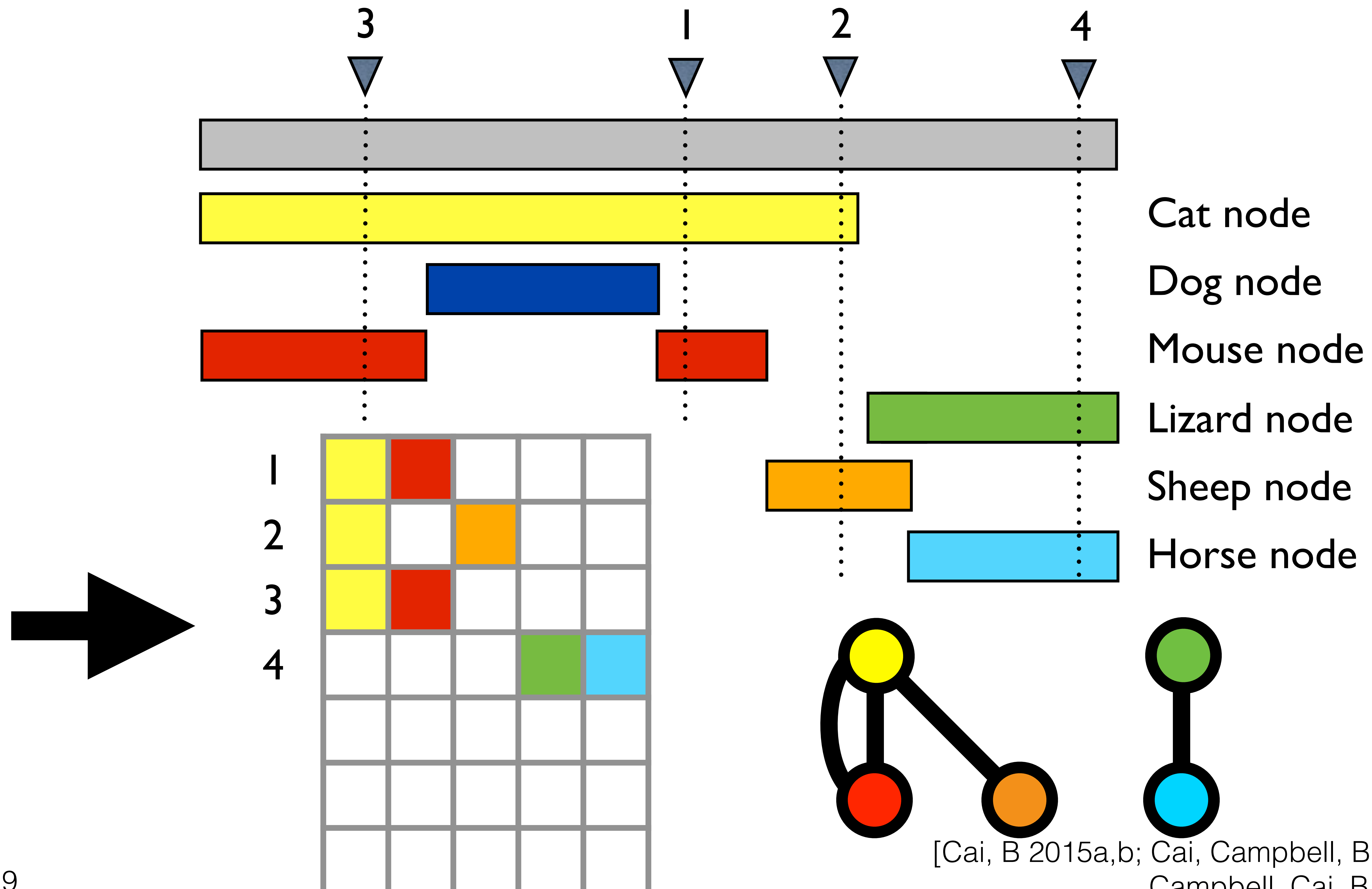

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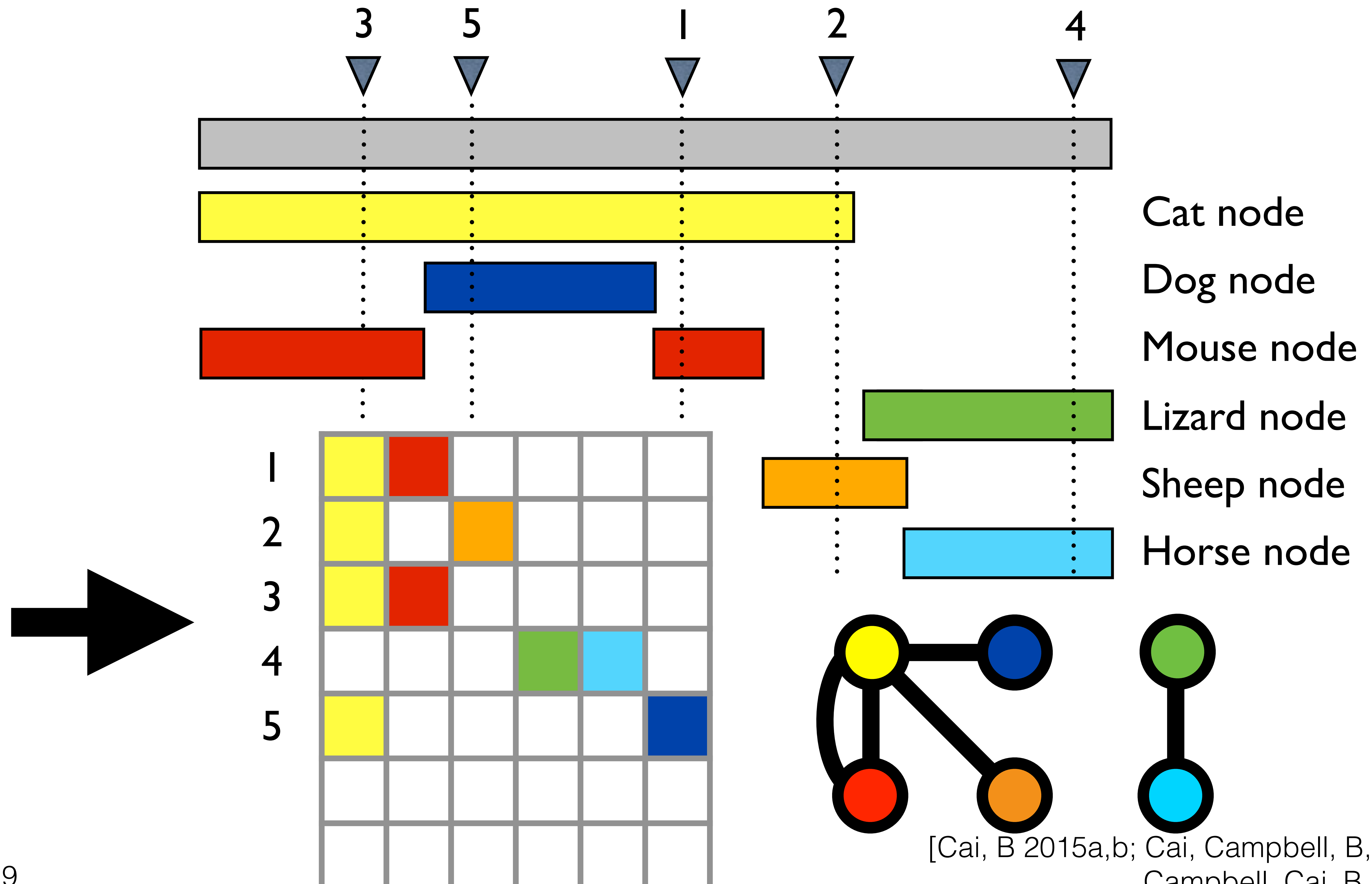

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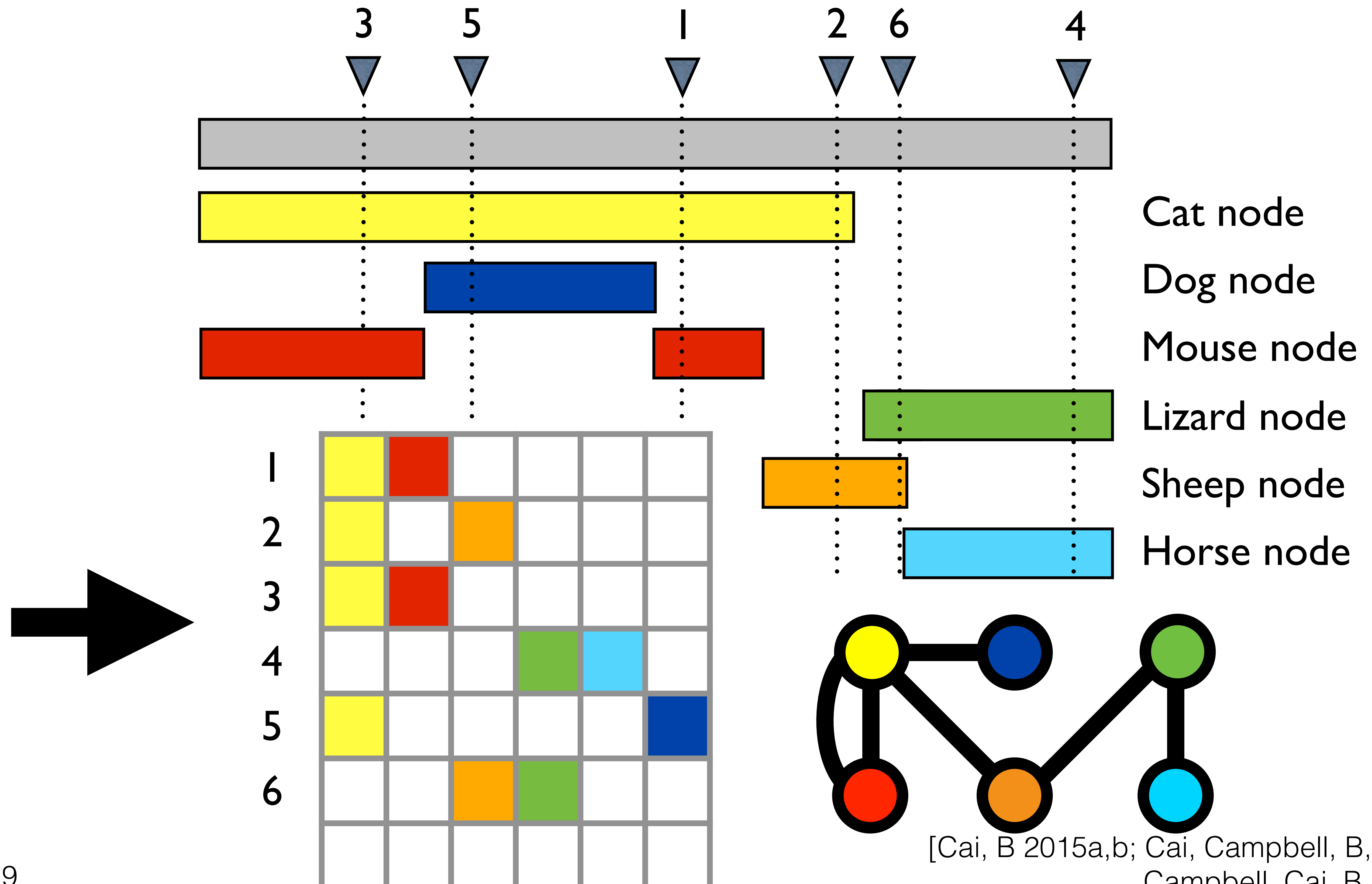
Edge-exchangeable graph



[Cai, B 2015a,b; Cai, Campbell, B, 2016;
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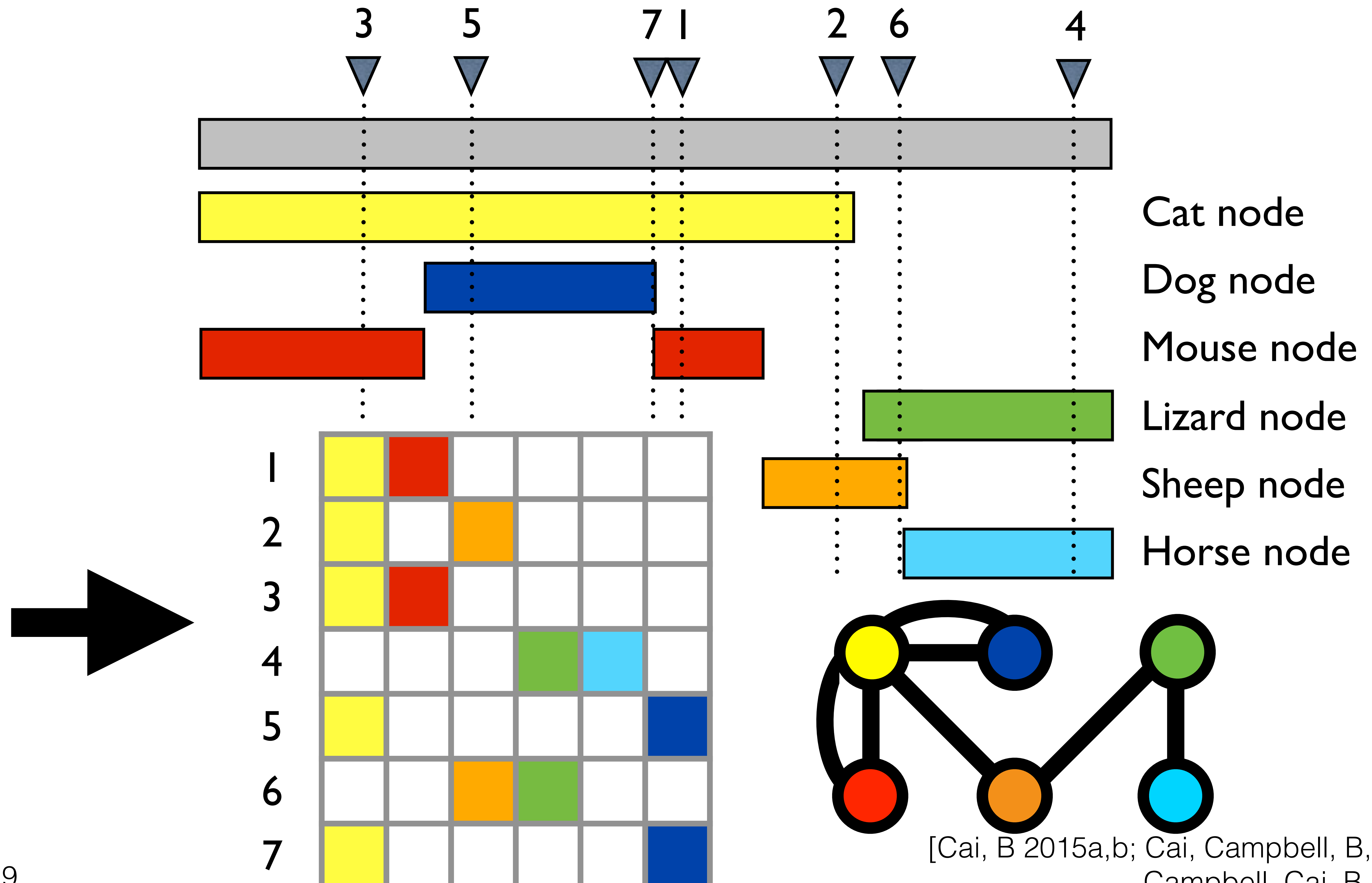

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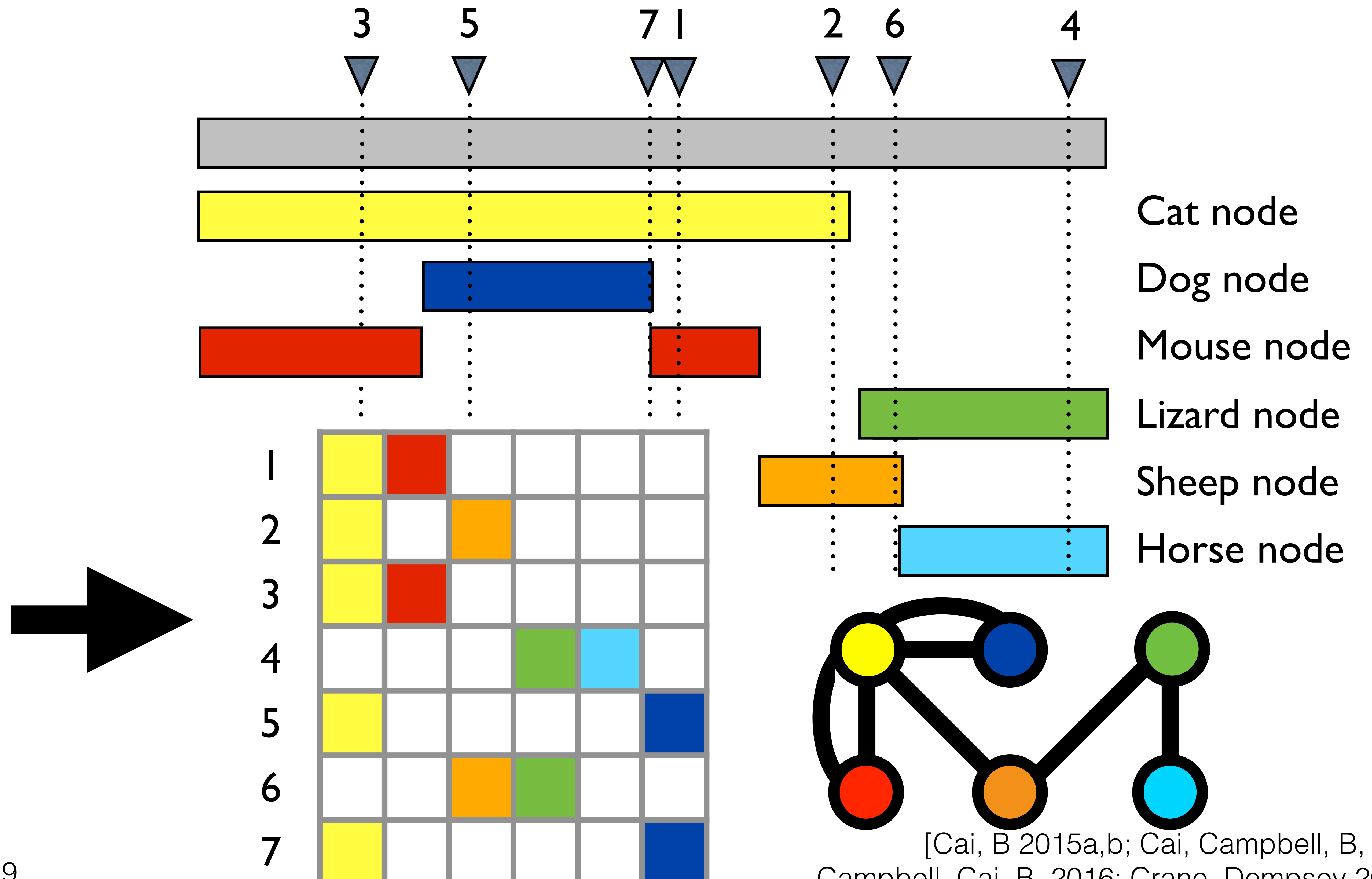
Cor (CCB). A graph sequence is edge-exchangeable iff it has a graph paintbox



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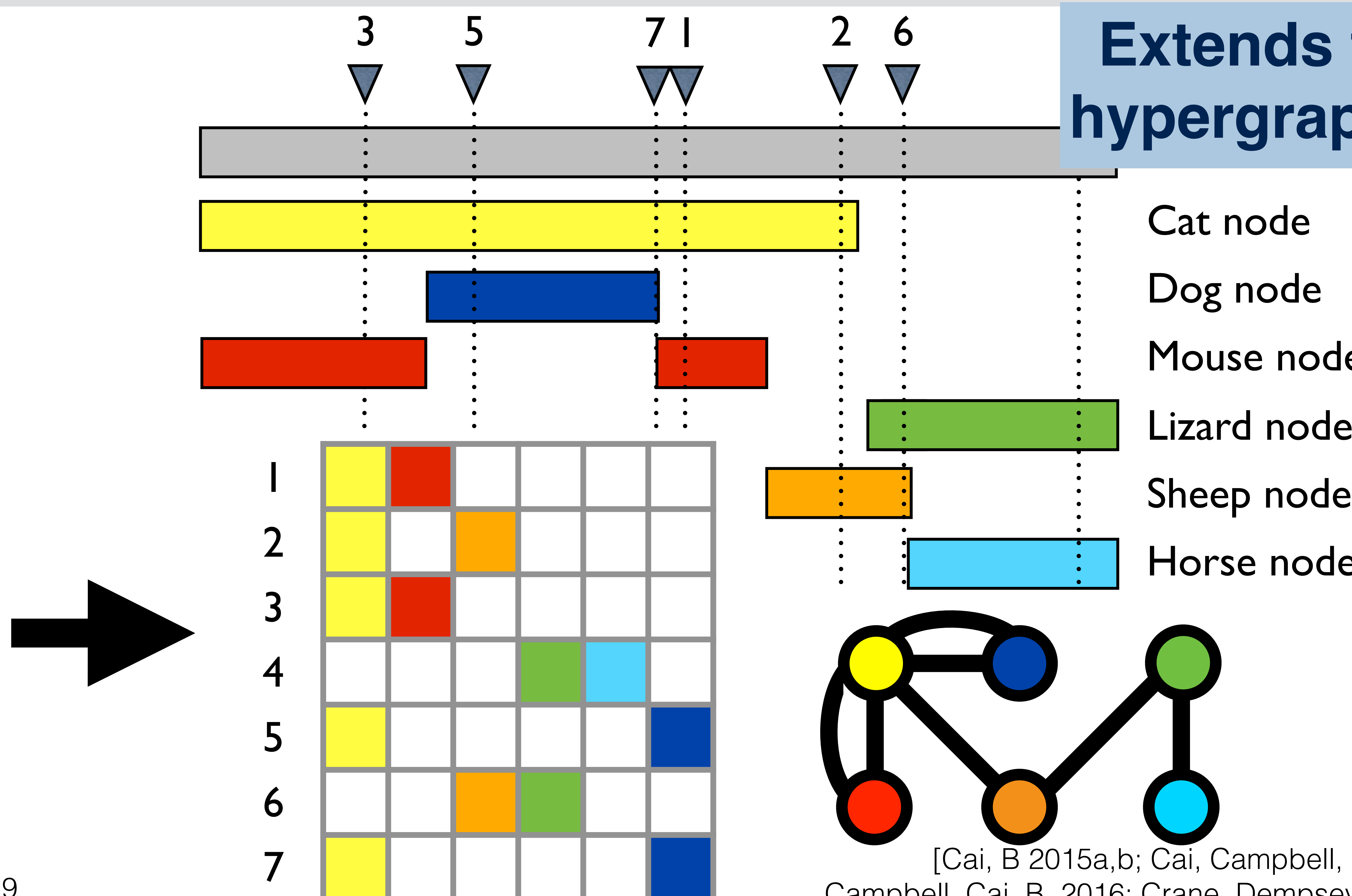

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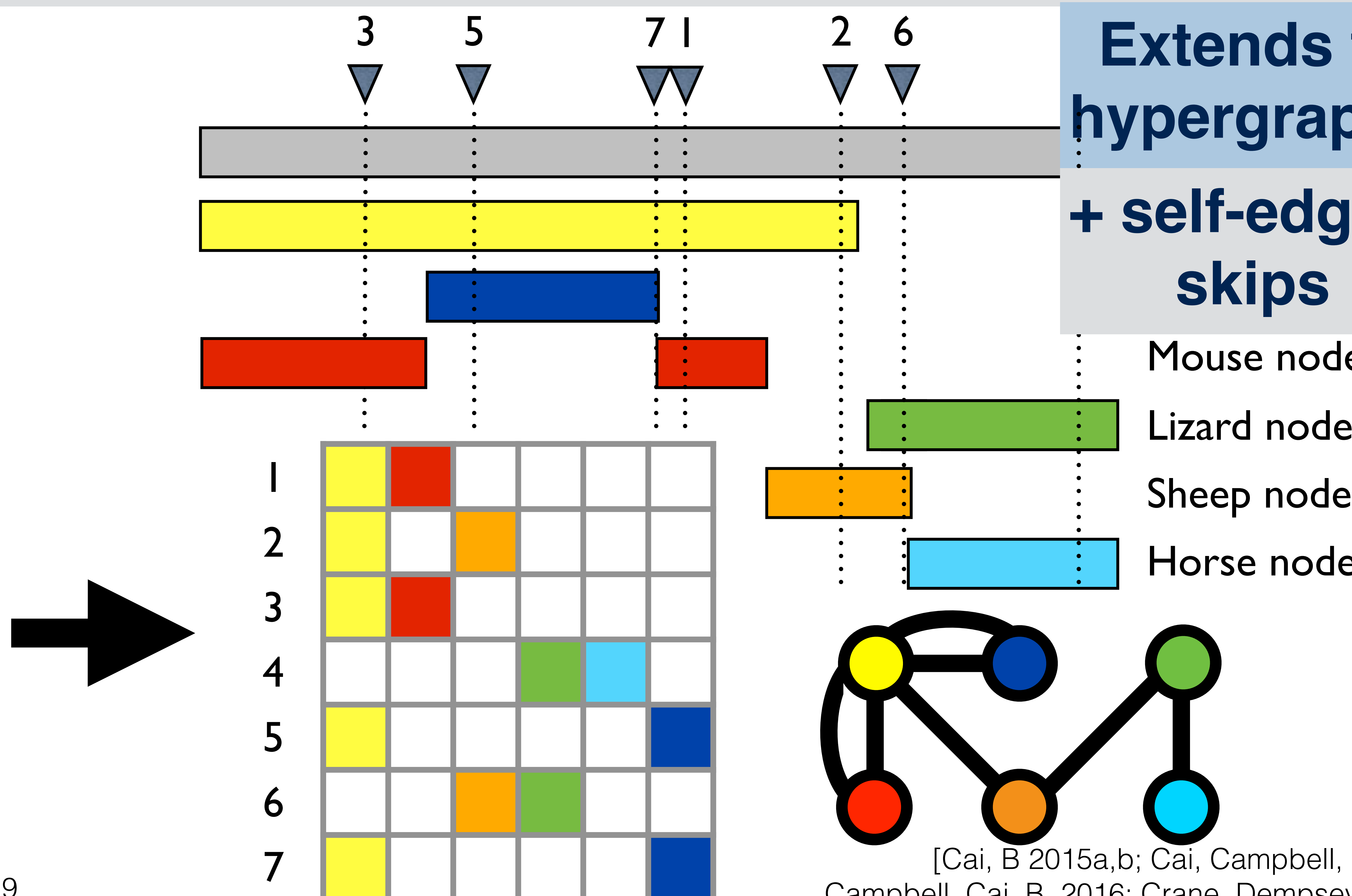
Extends to hypergraphs

+ self-edges, skips



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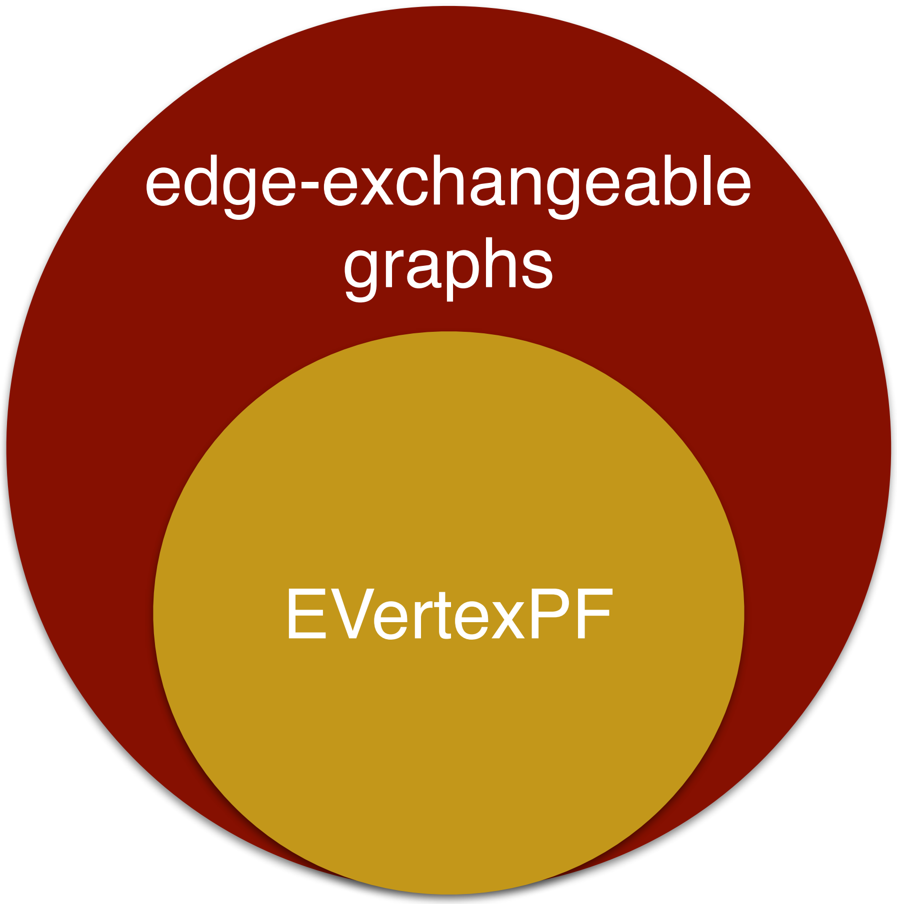

- Goal 1: characterization theorem for edge-exchangeable graphs
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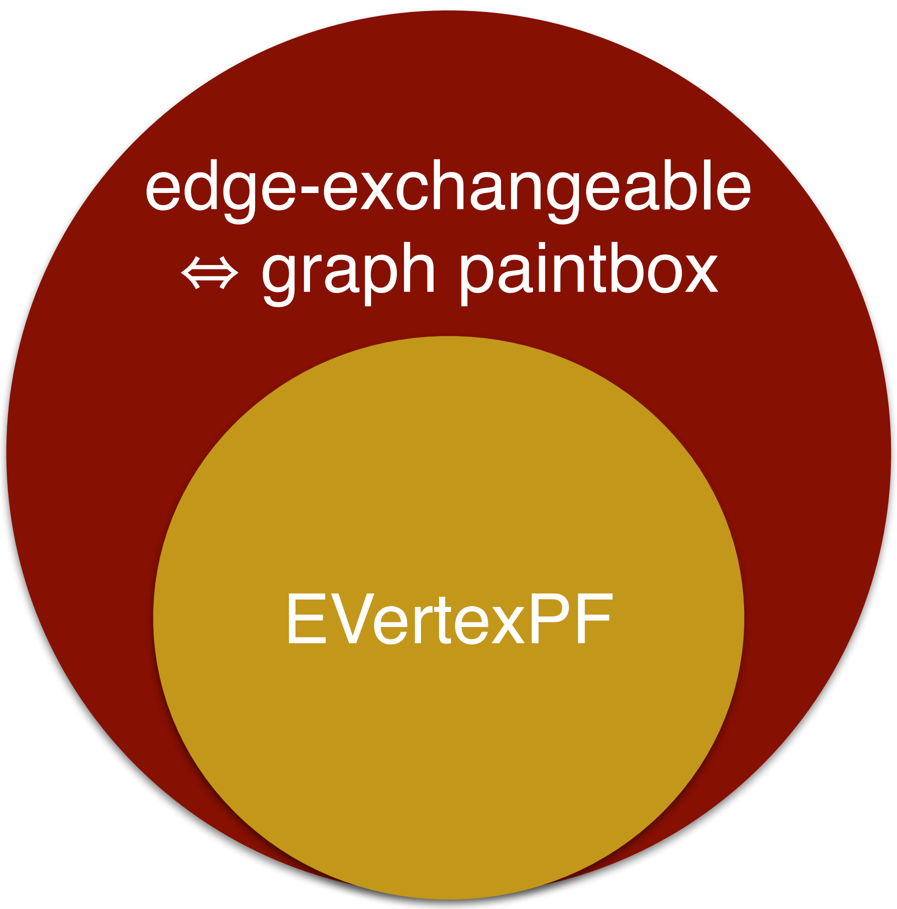

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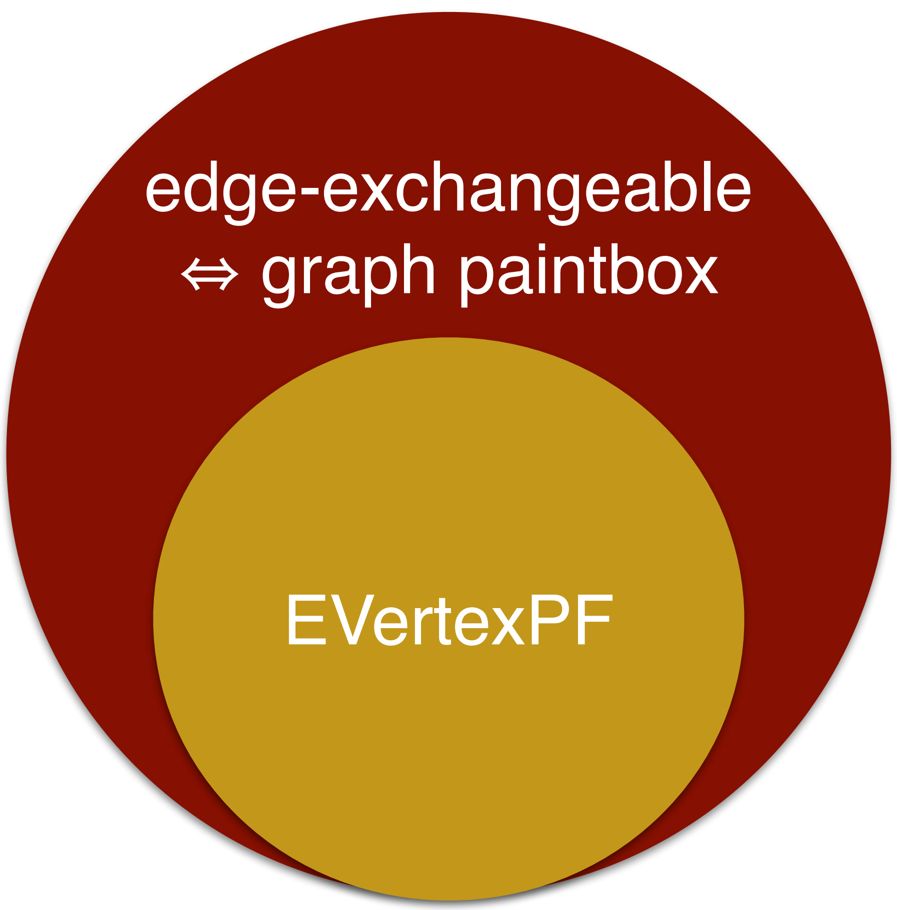

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How to prove sparsity?

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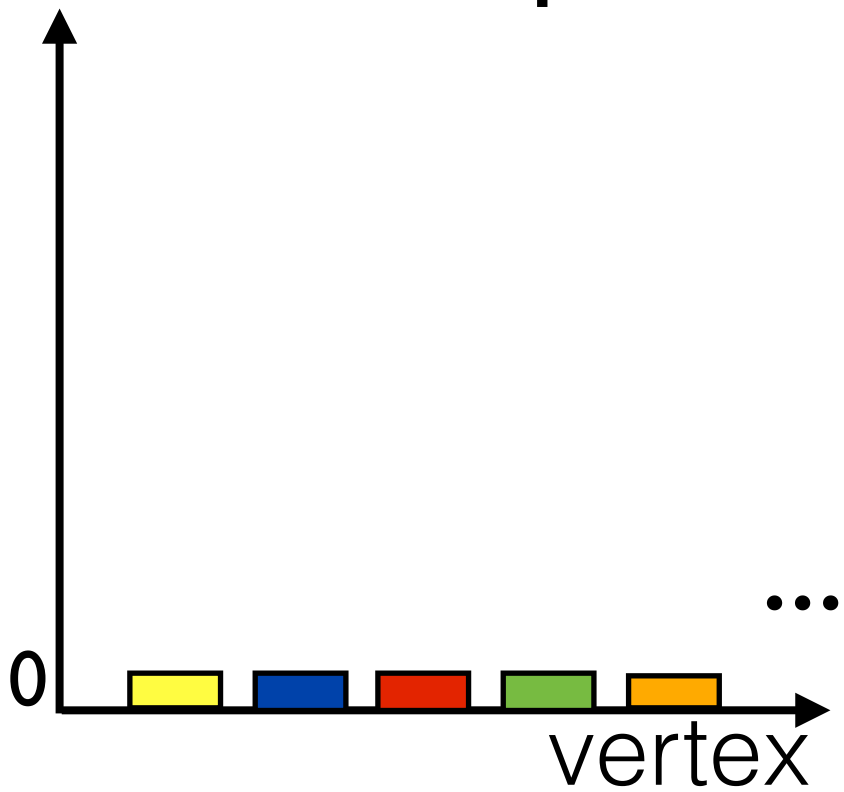

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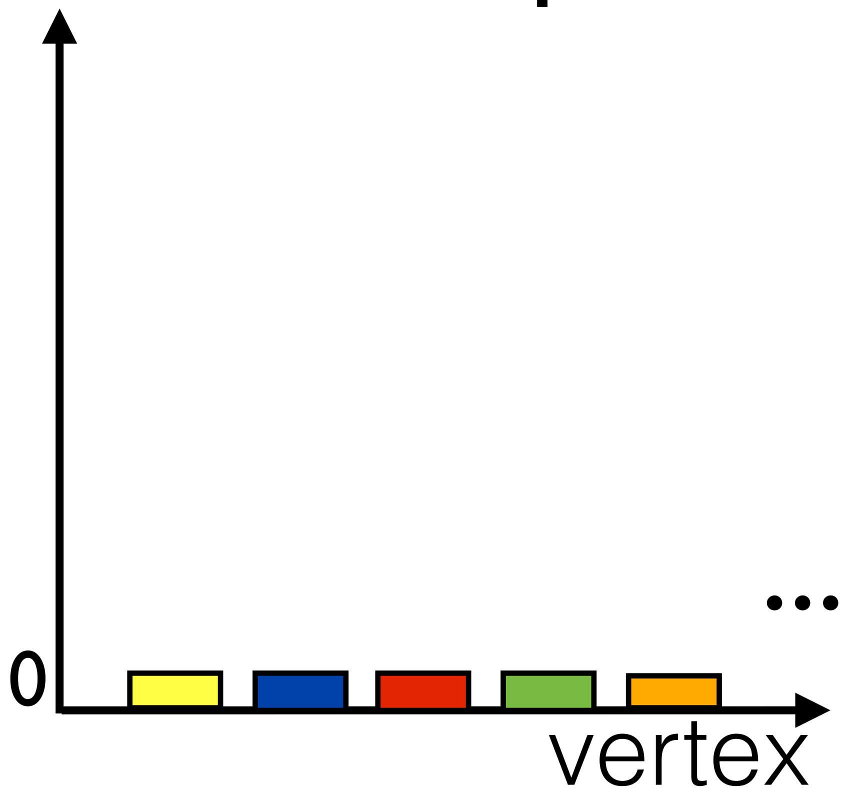

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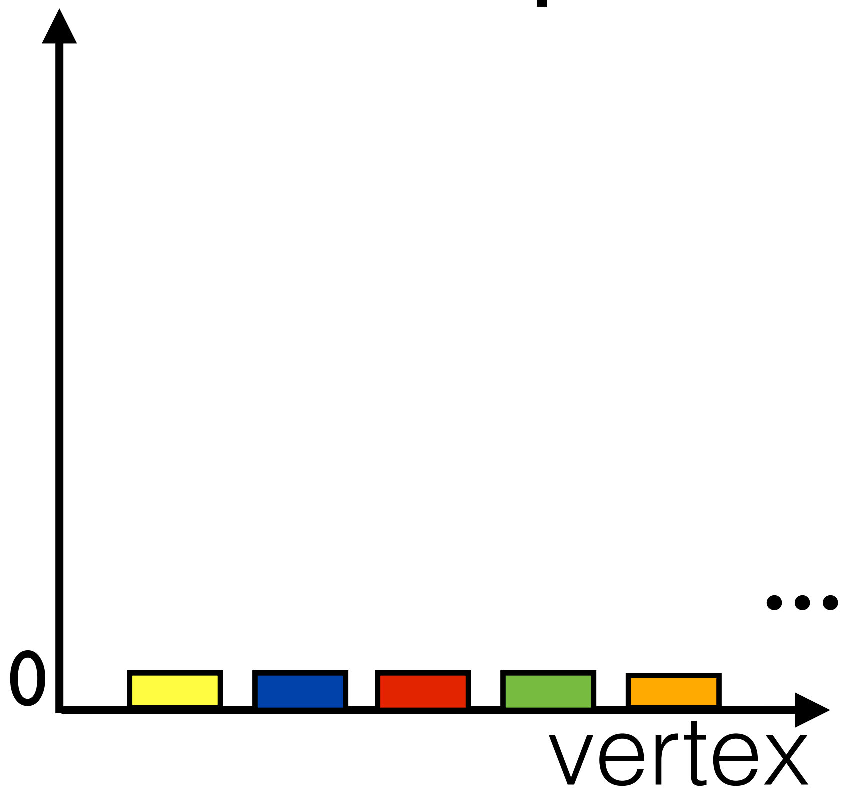

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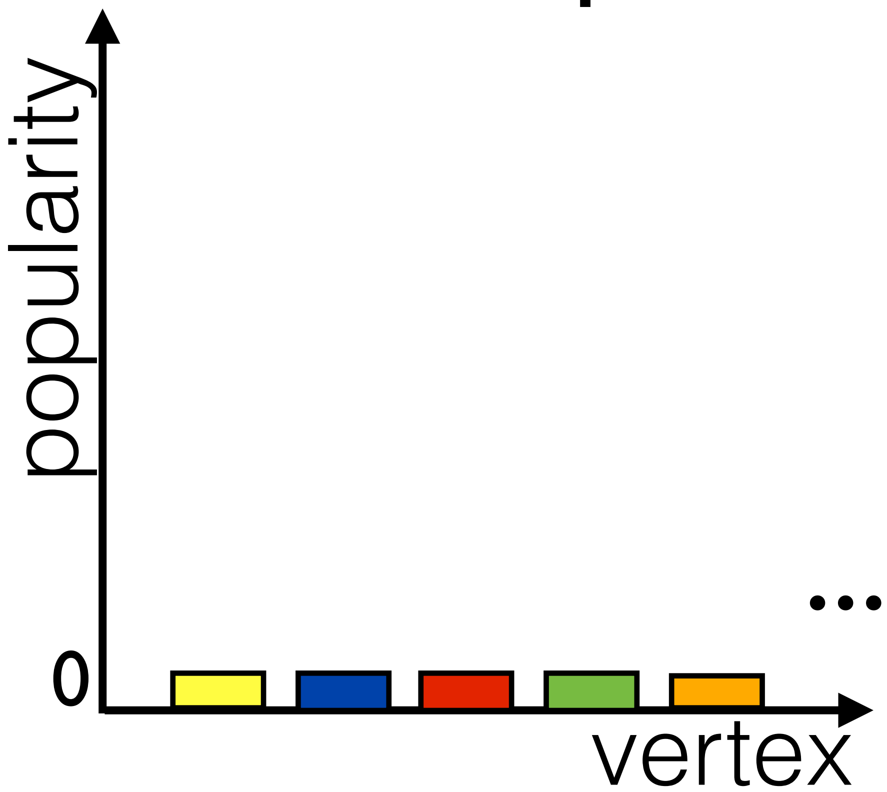

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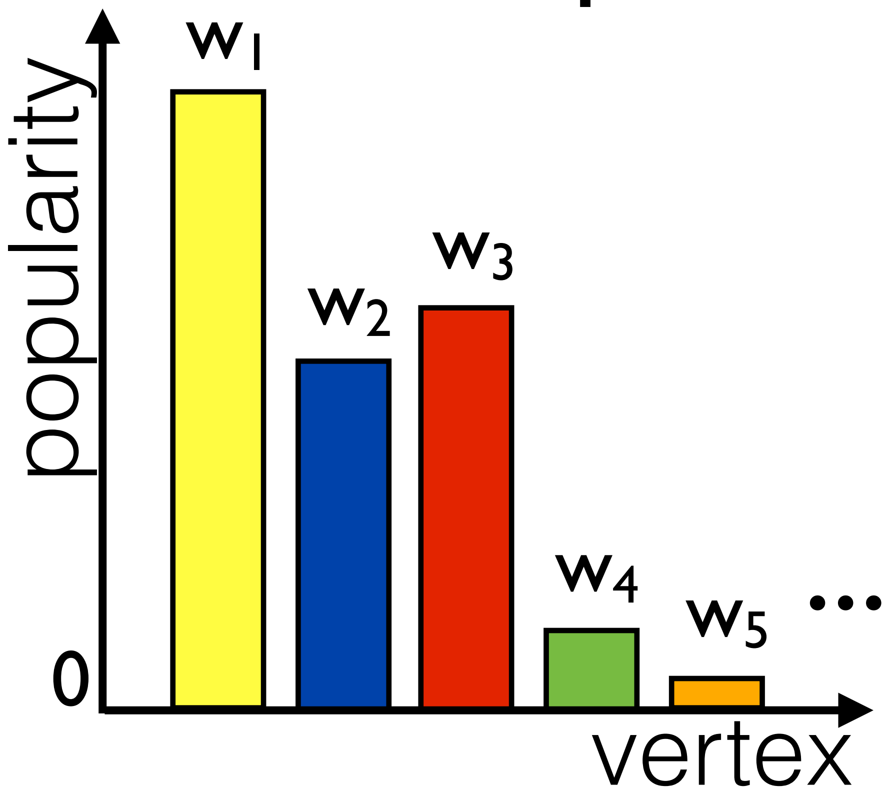

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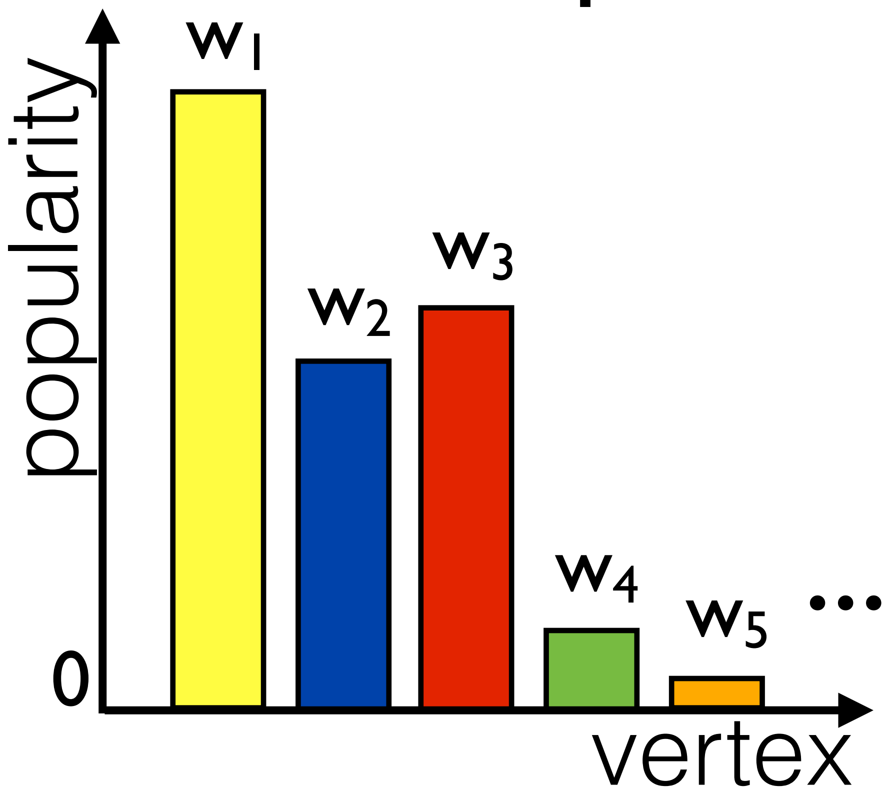

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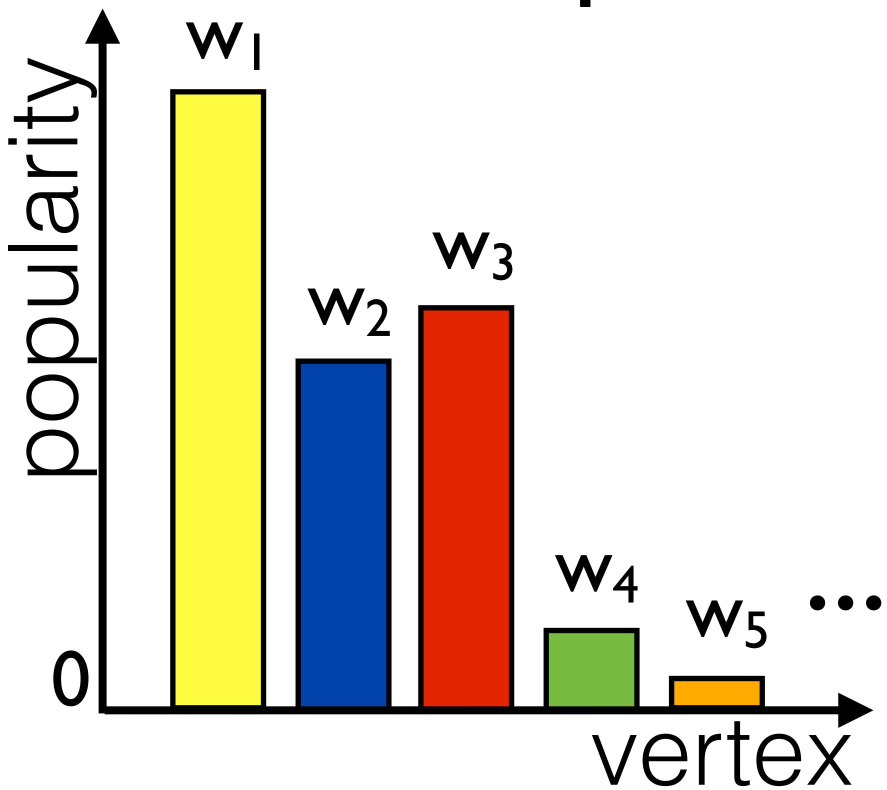

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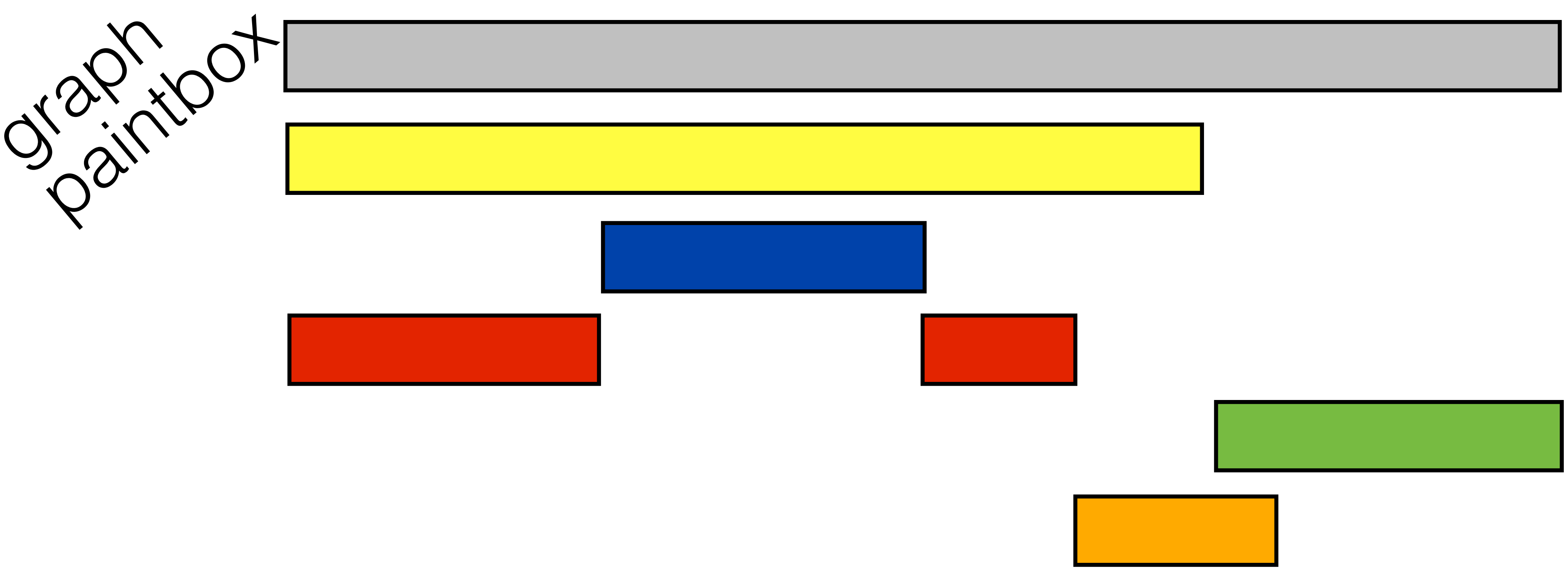

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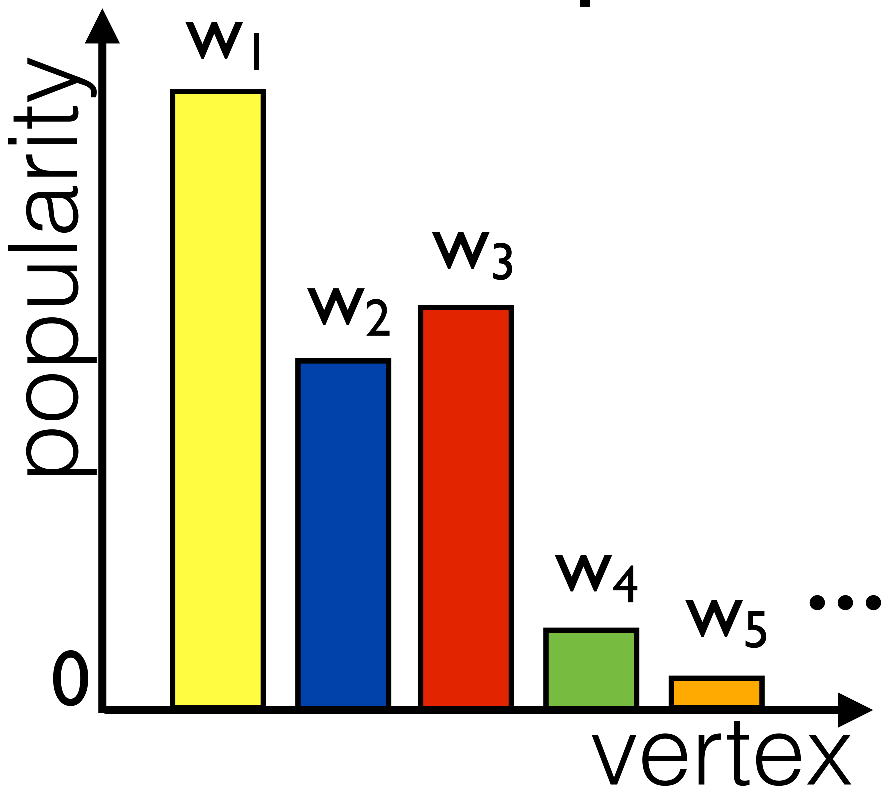

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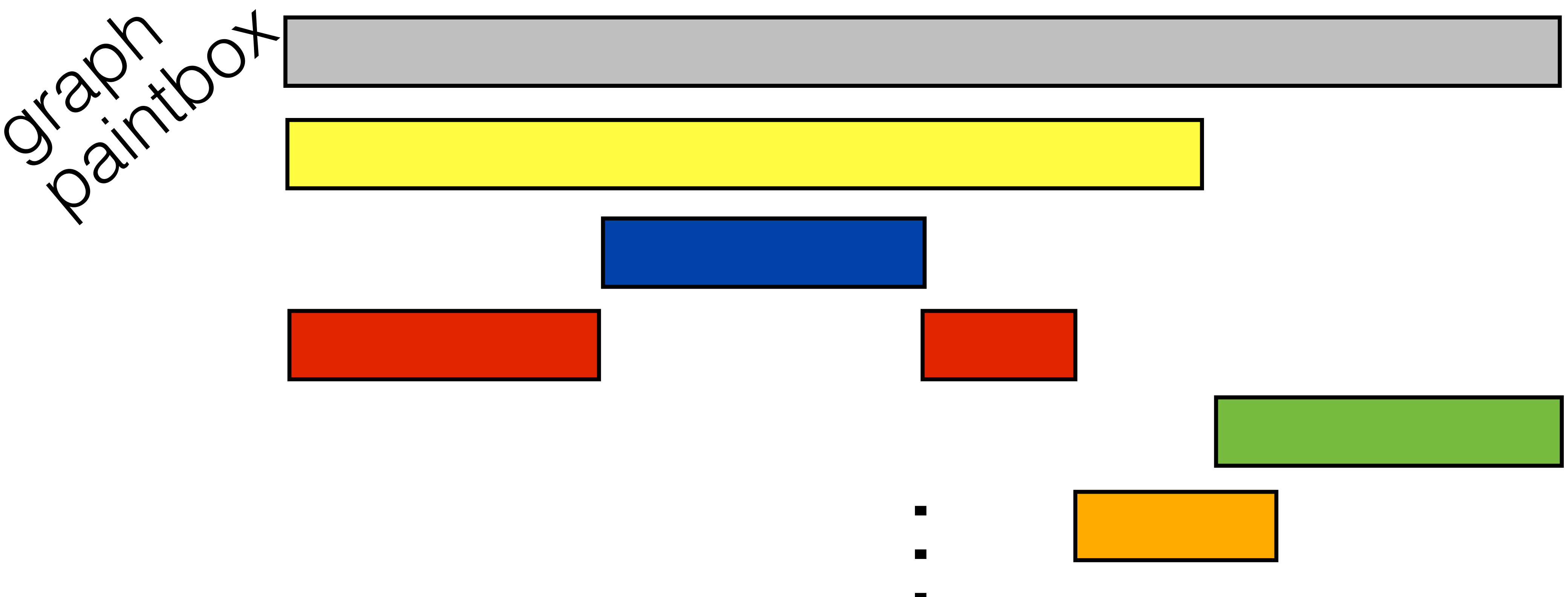

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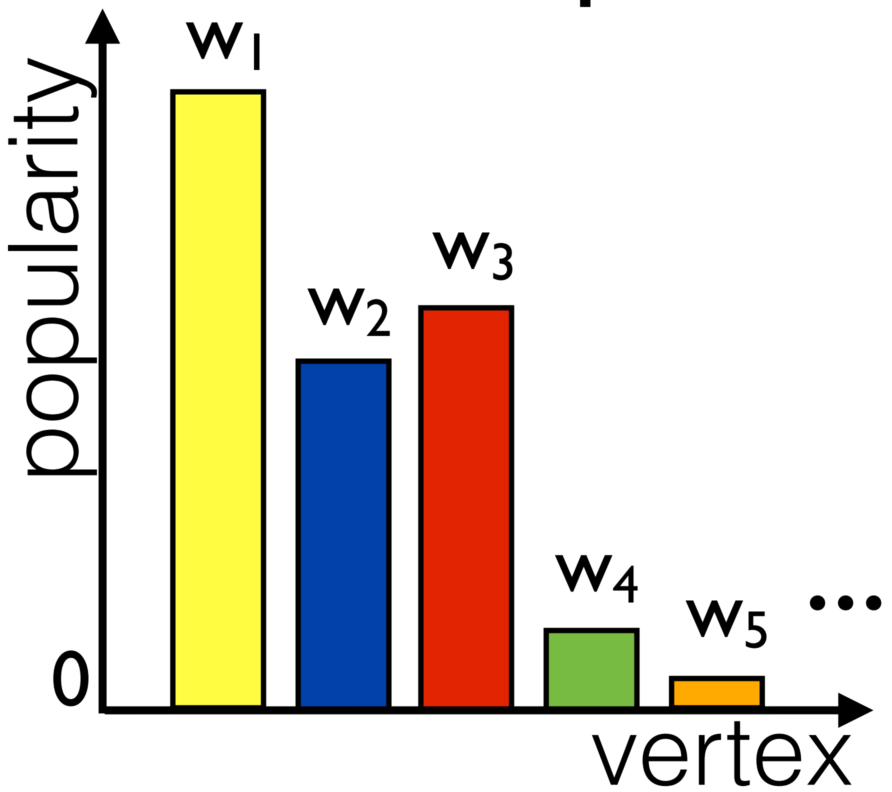

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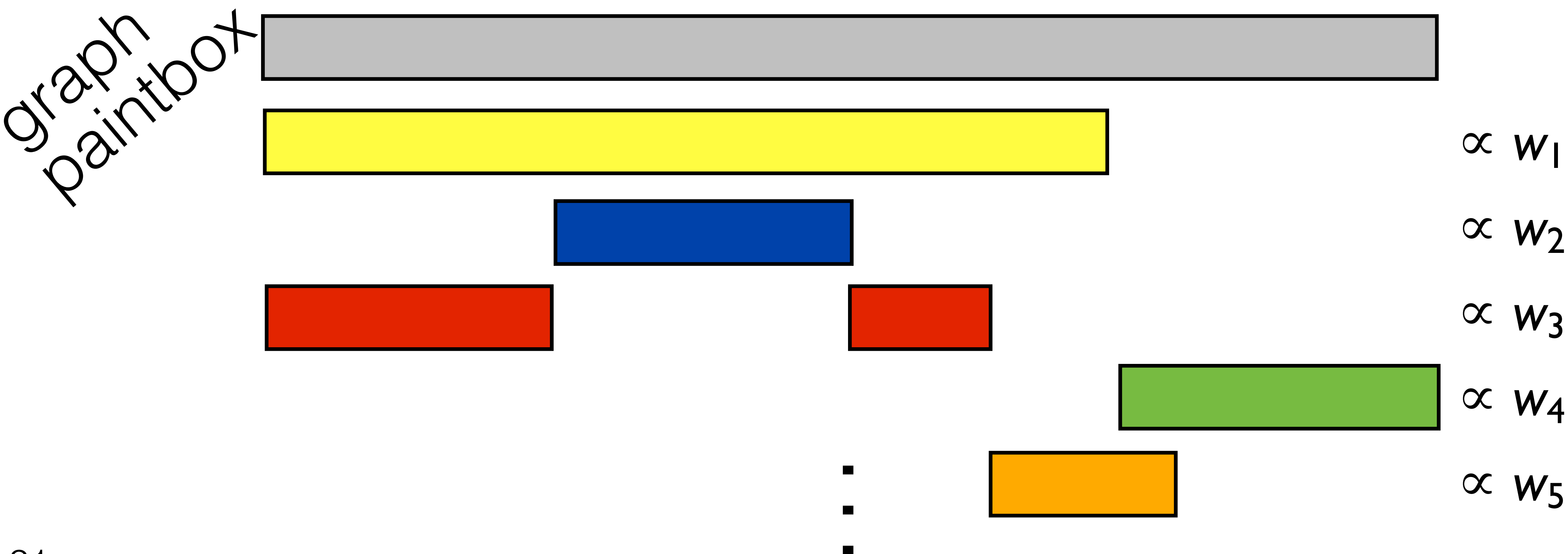

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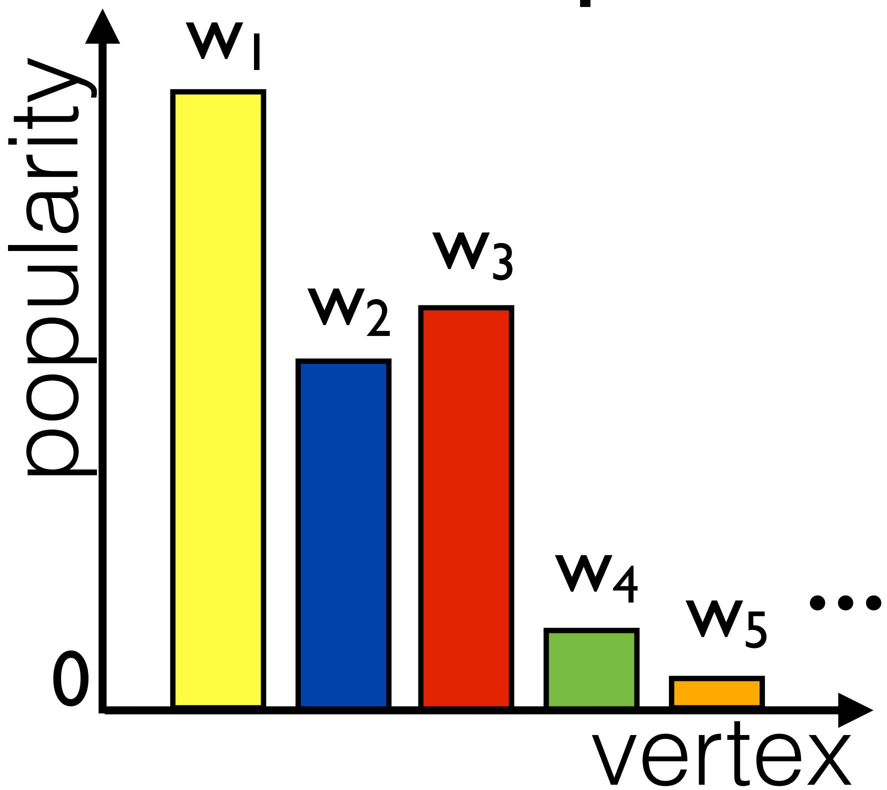

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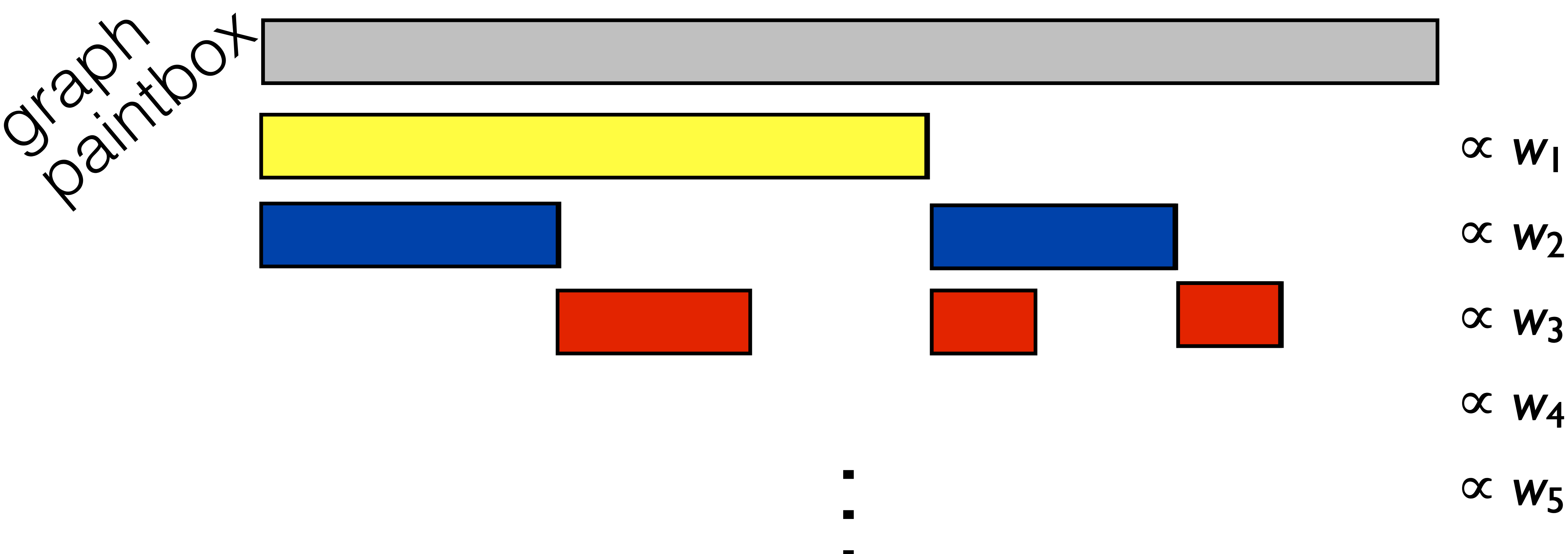

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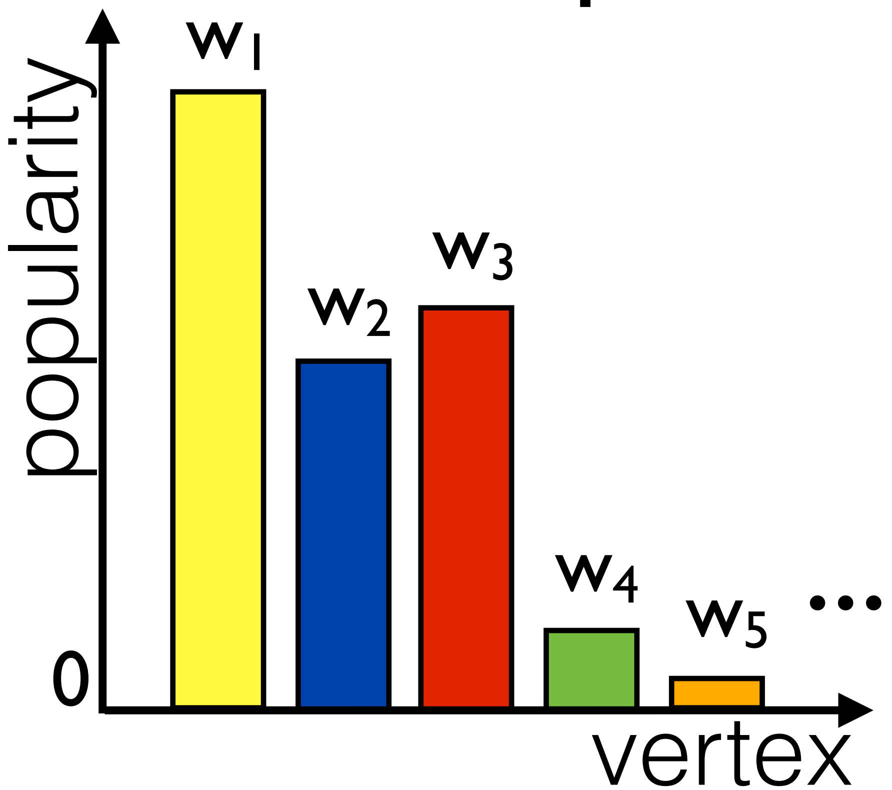

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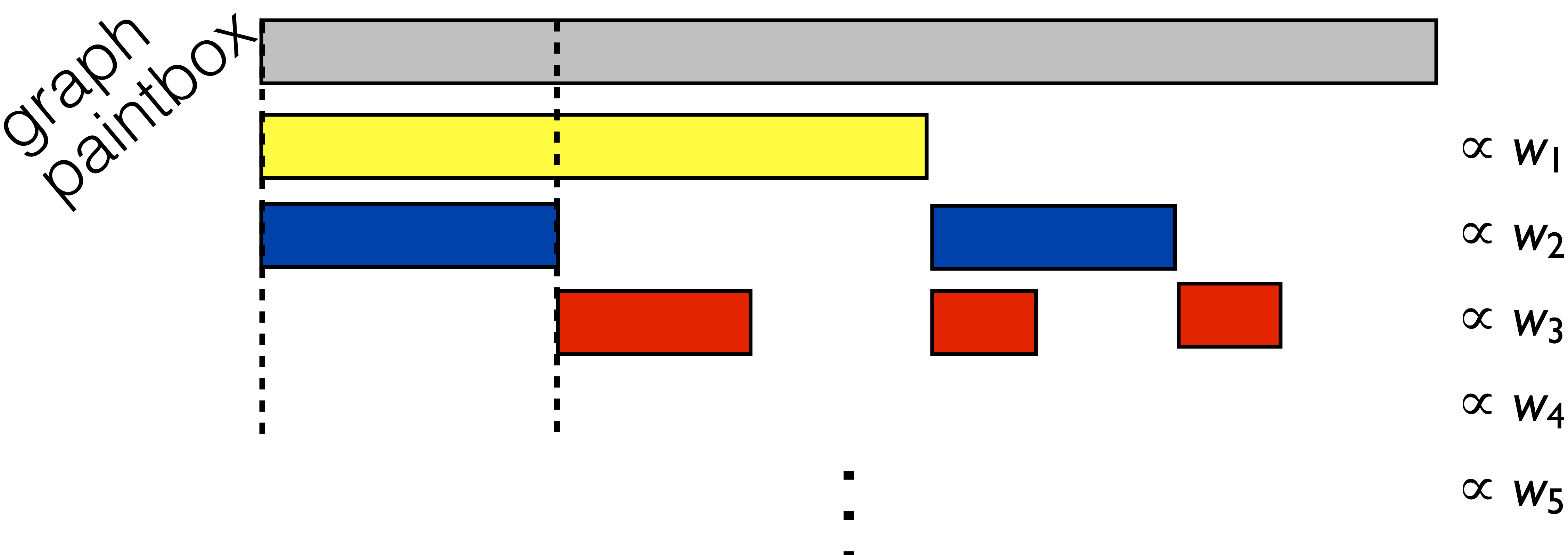

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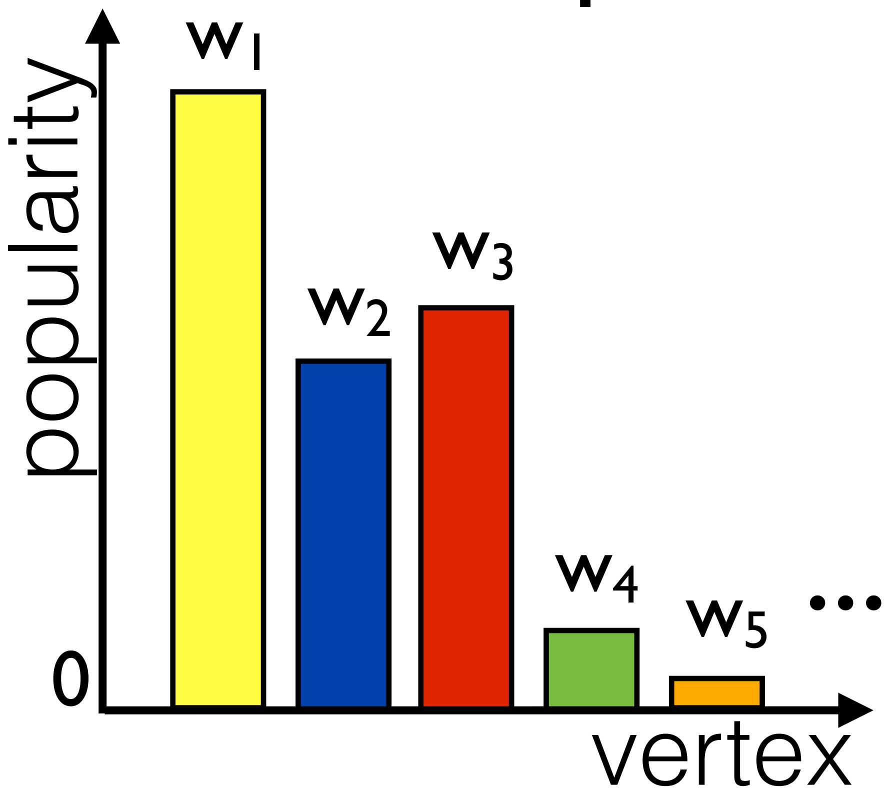

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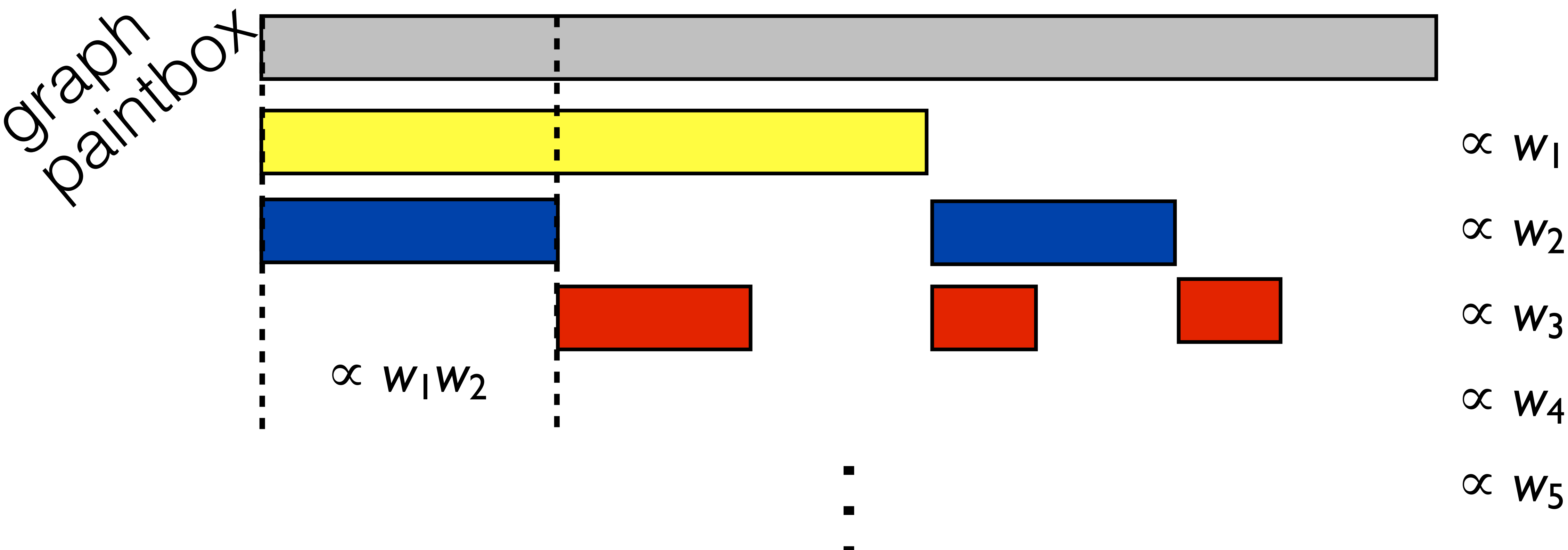

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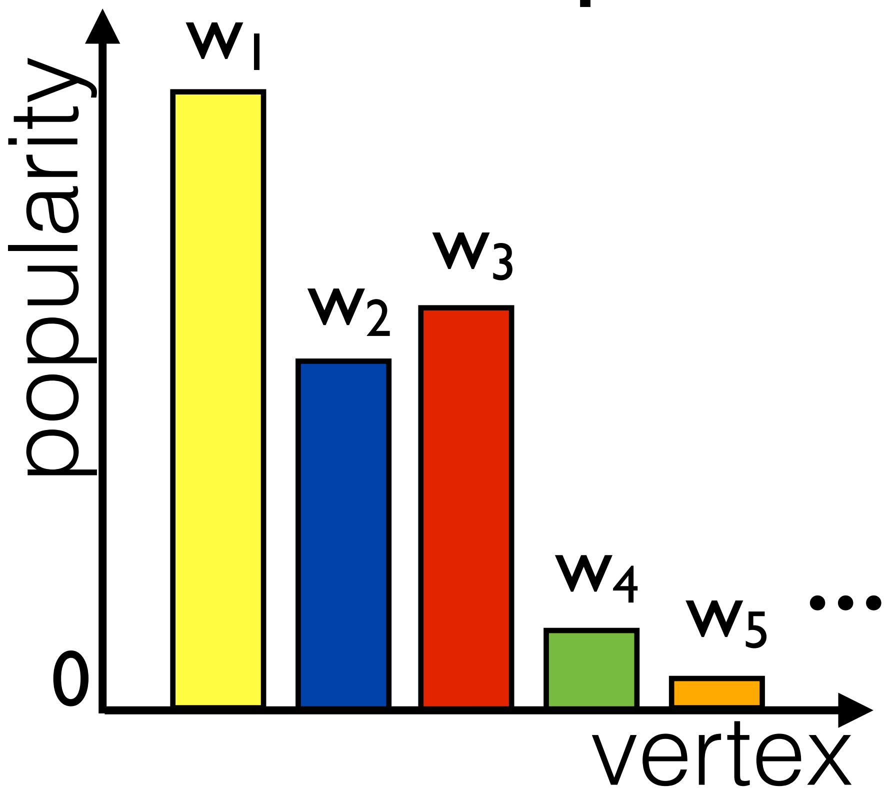

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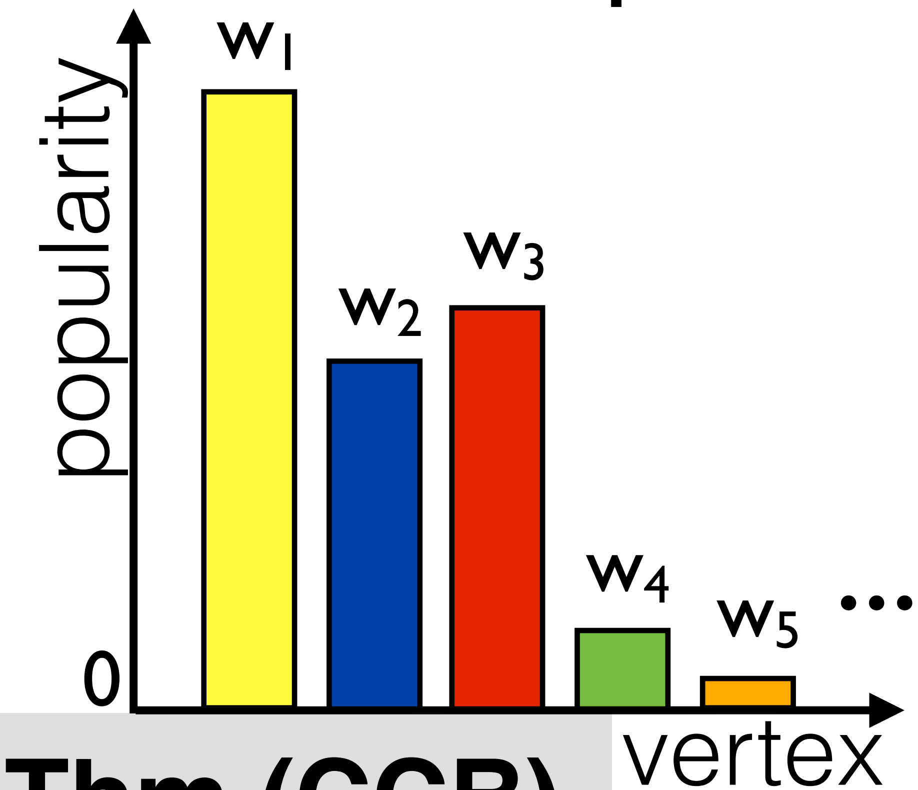

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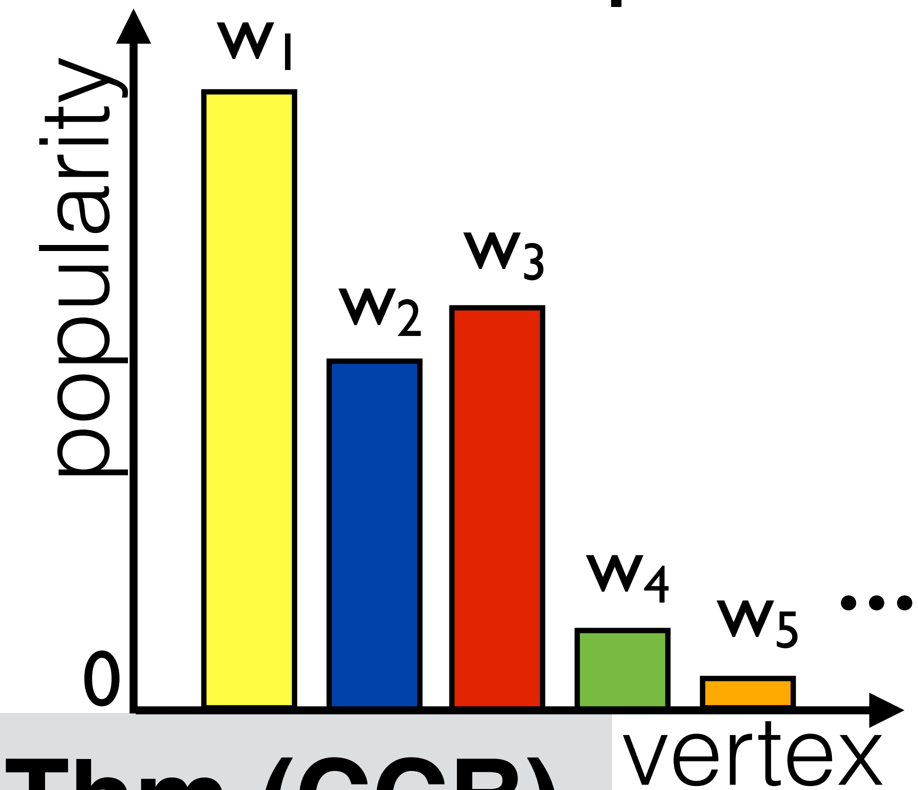

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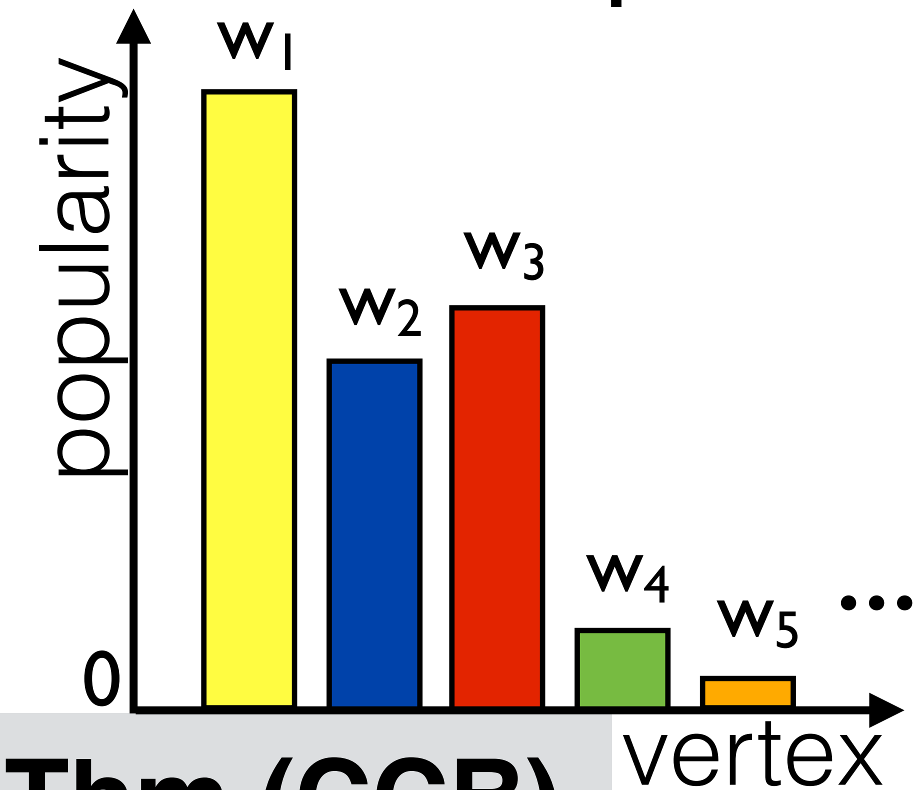

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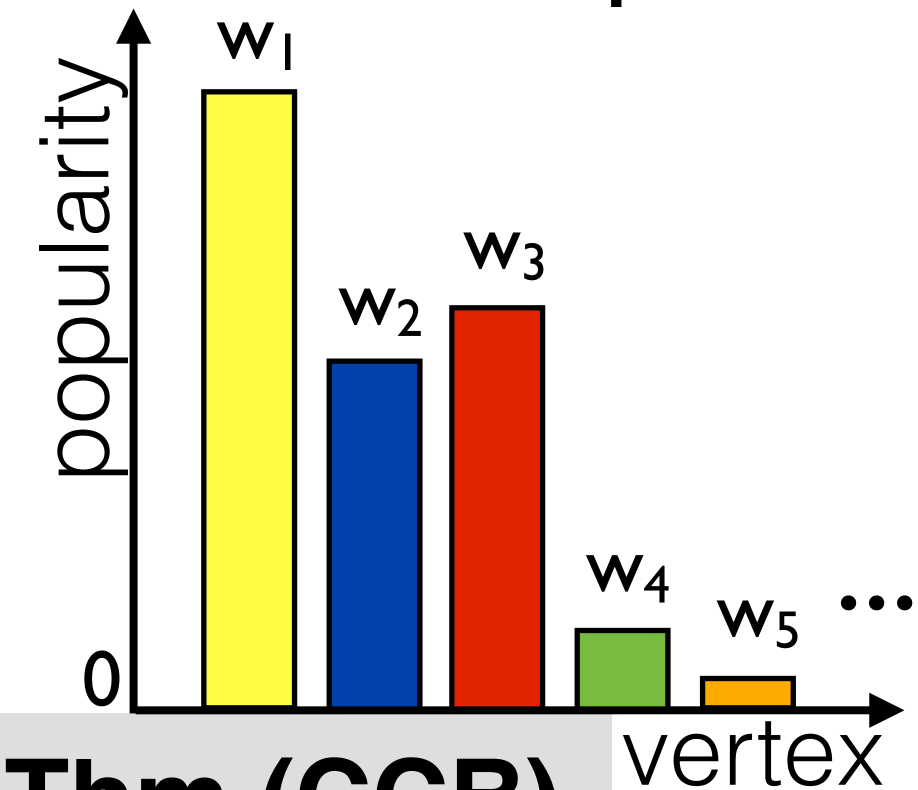

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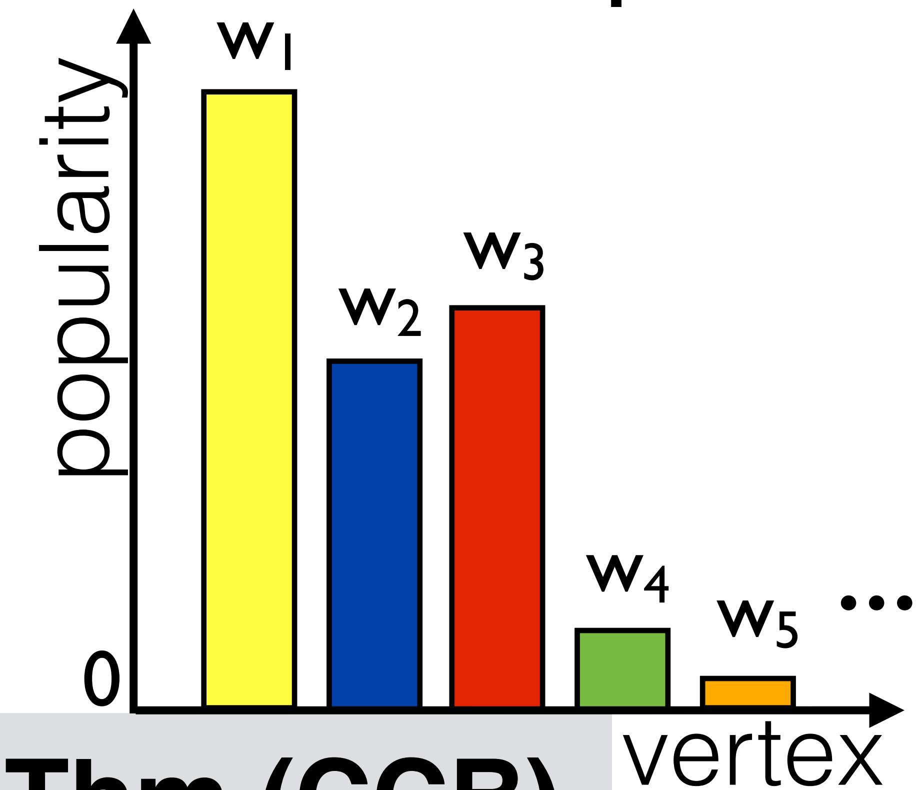

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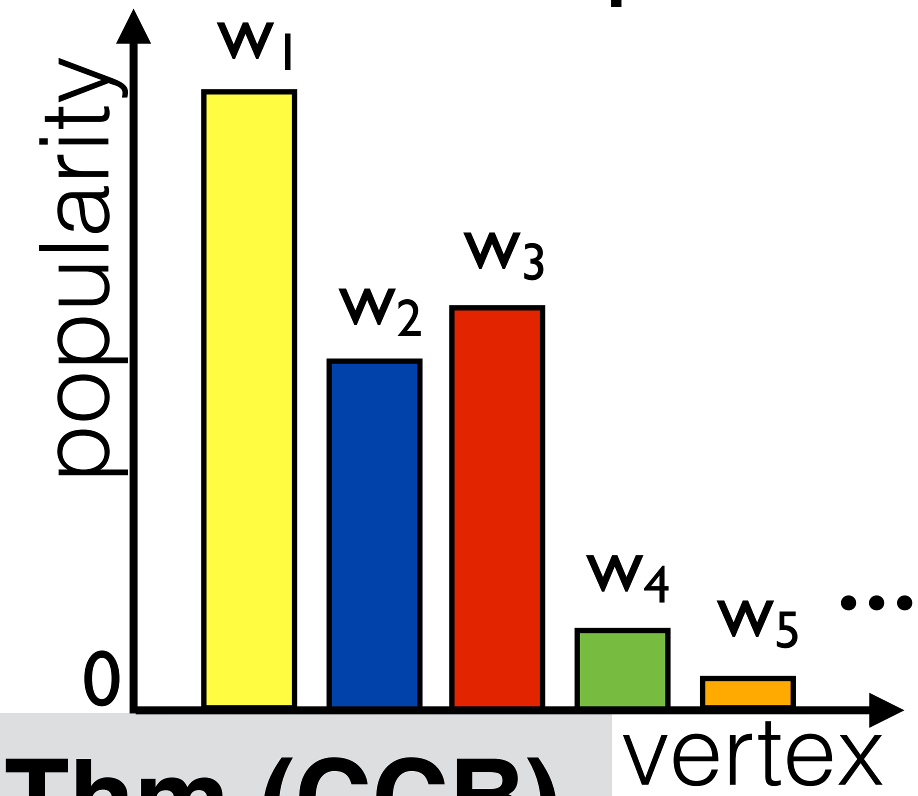

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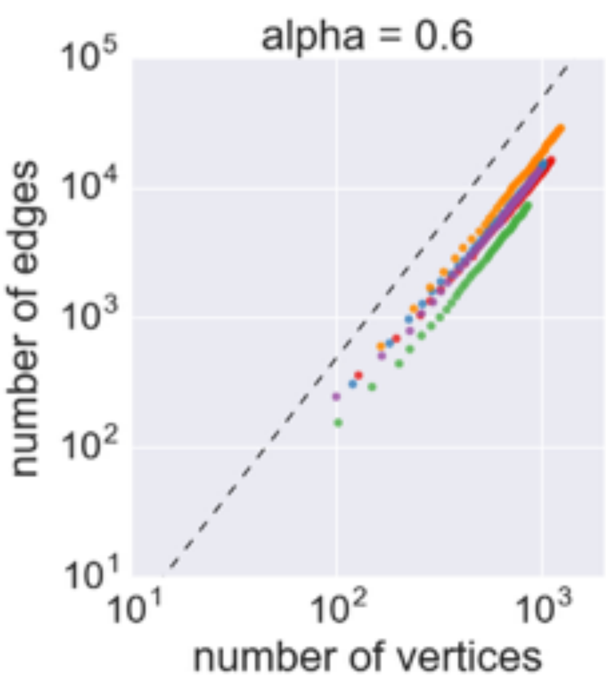
Cor (CCB). A wide class of edge-exchangeable graph models yields sparse graph sequences.

[Cai, Campbell, Broderick 2016]

Example graph frequency model

Empirical: can achieve range of (sparse & dense) power laws



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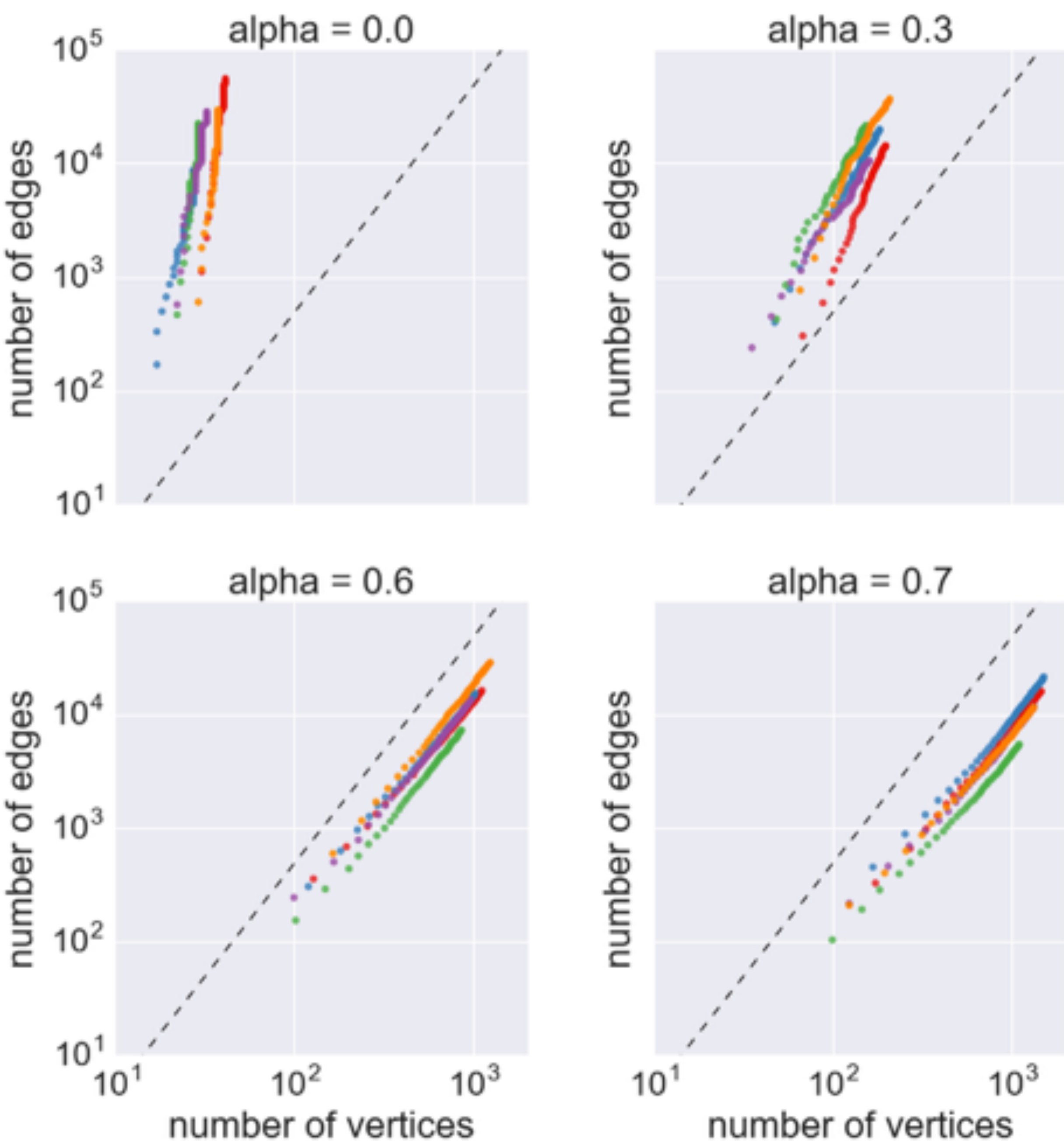

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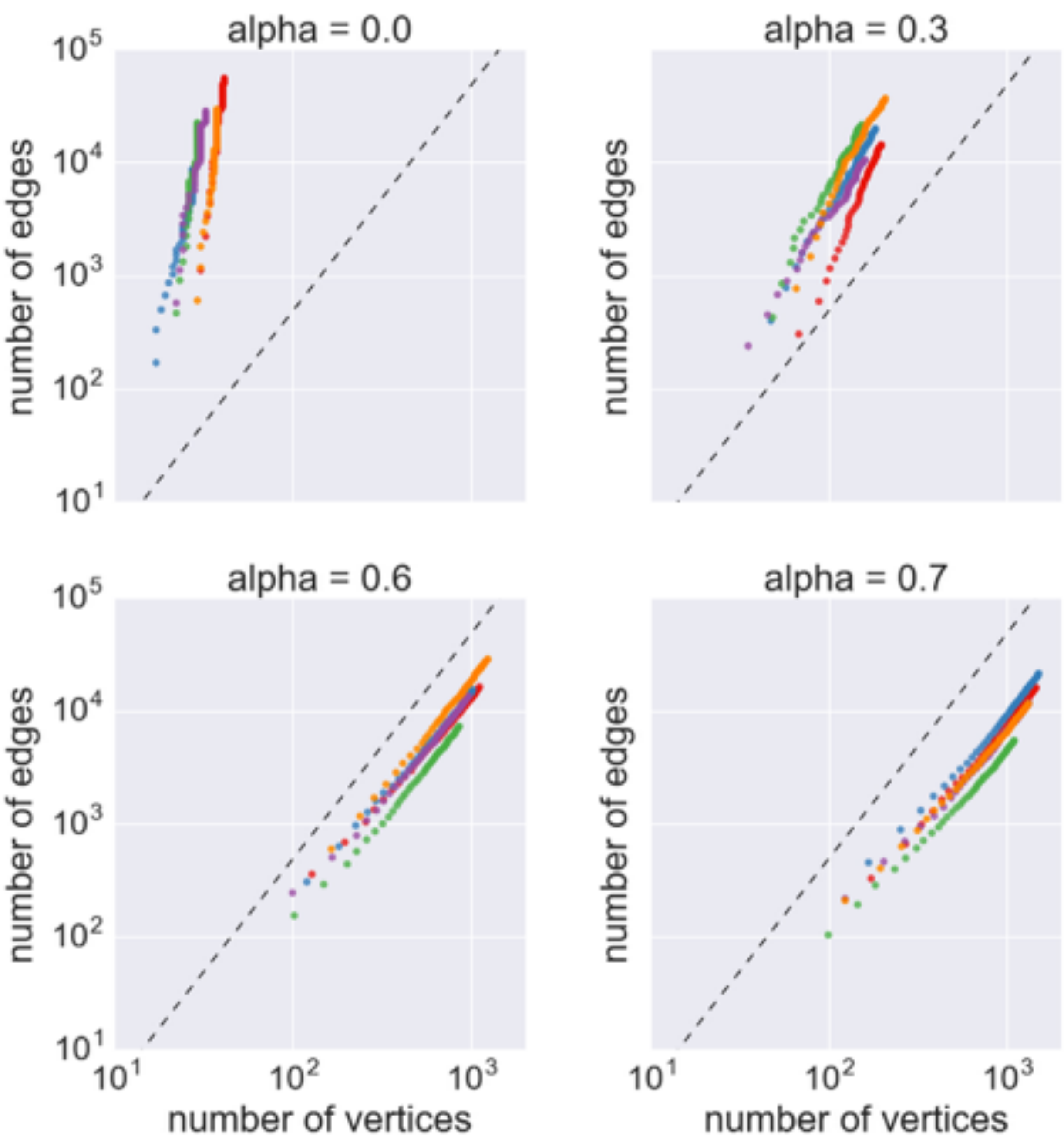

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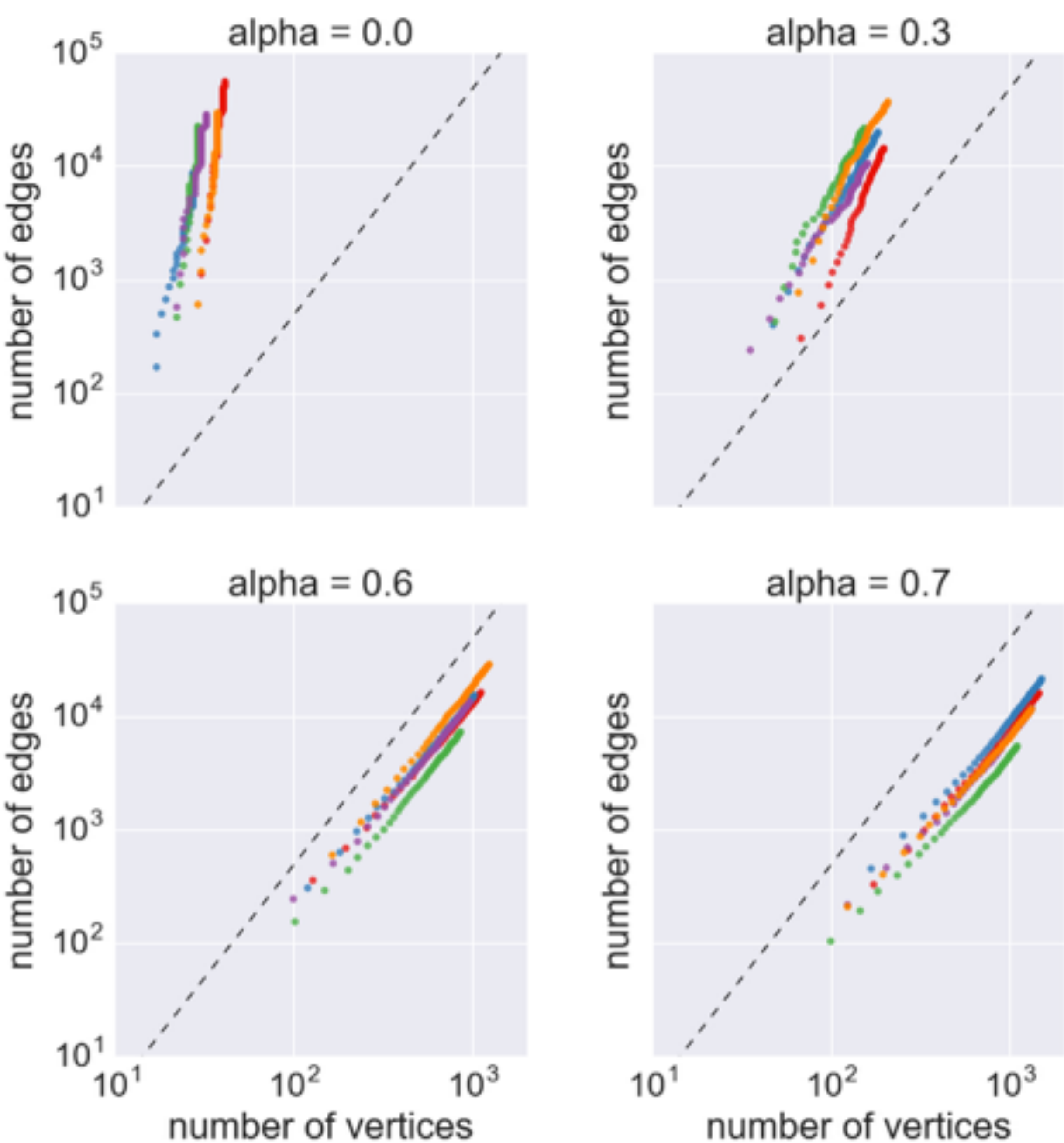
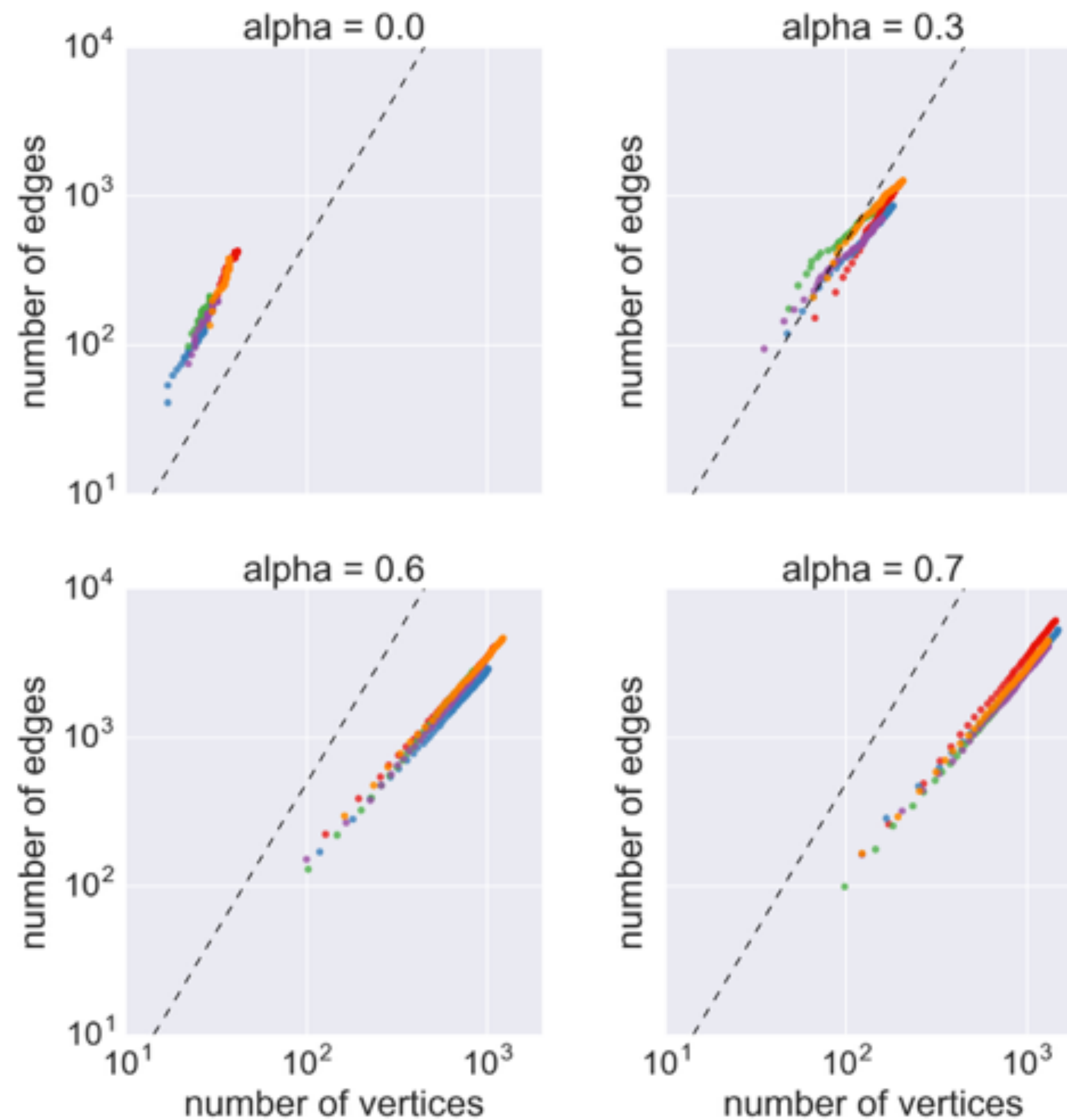


- Binary graph



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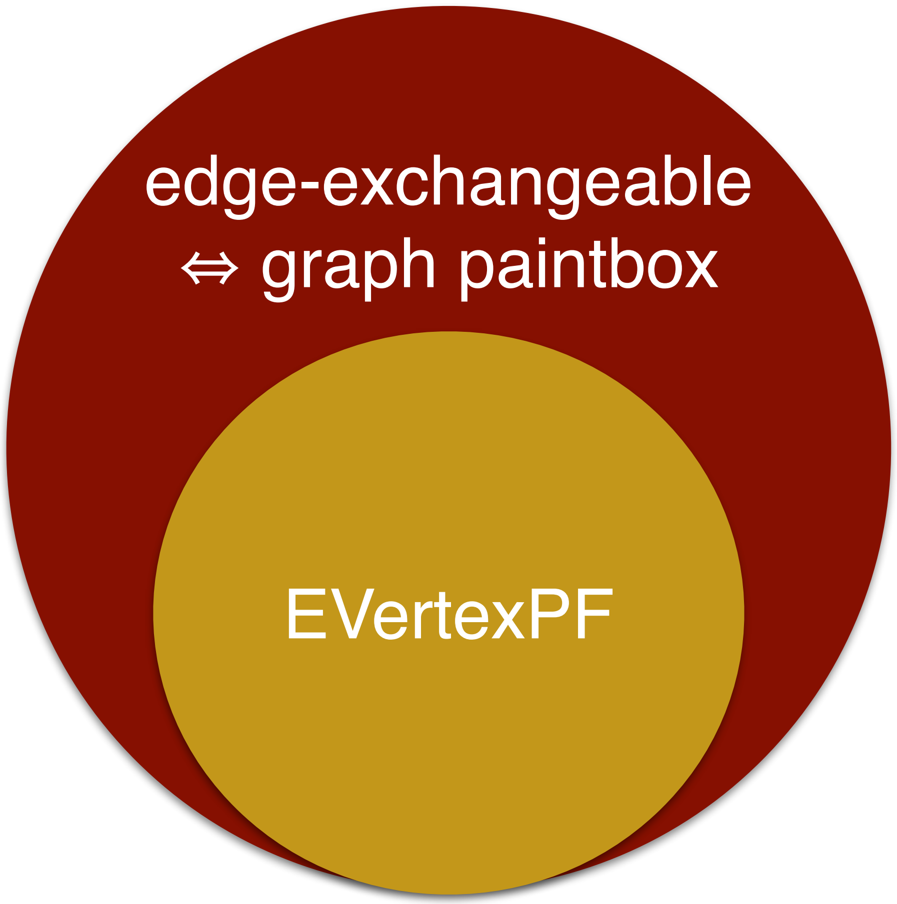

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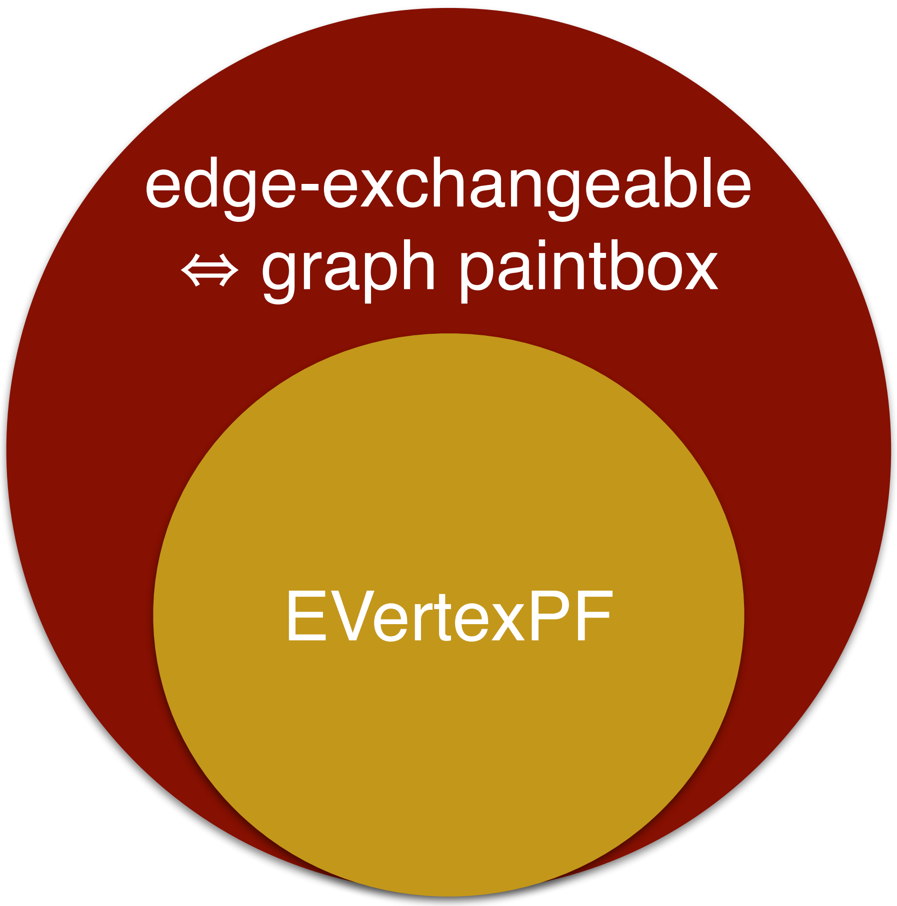

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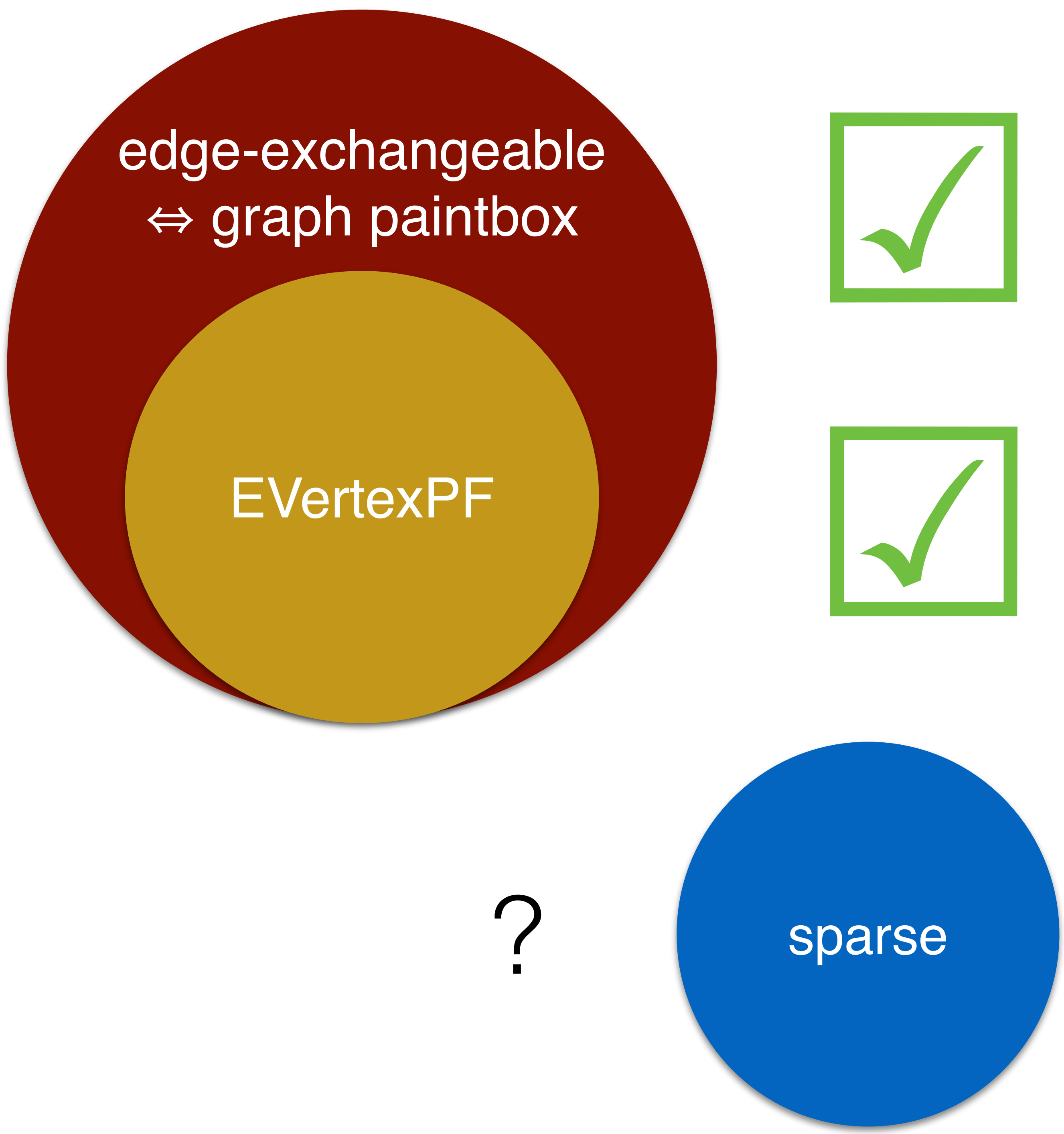

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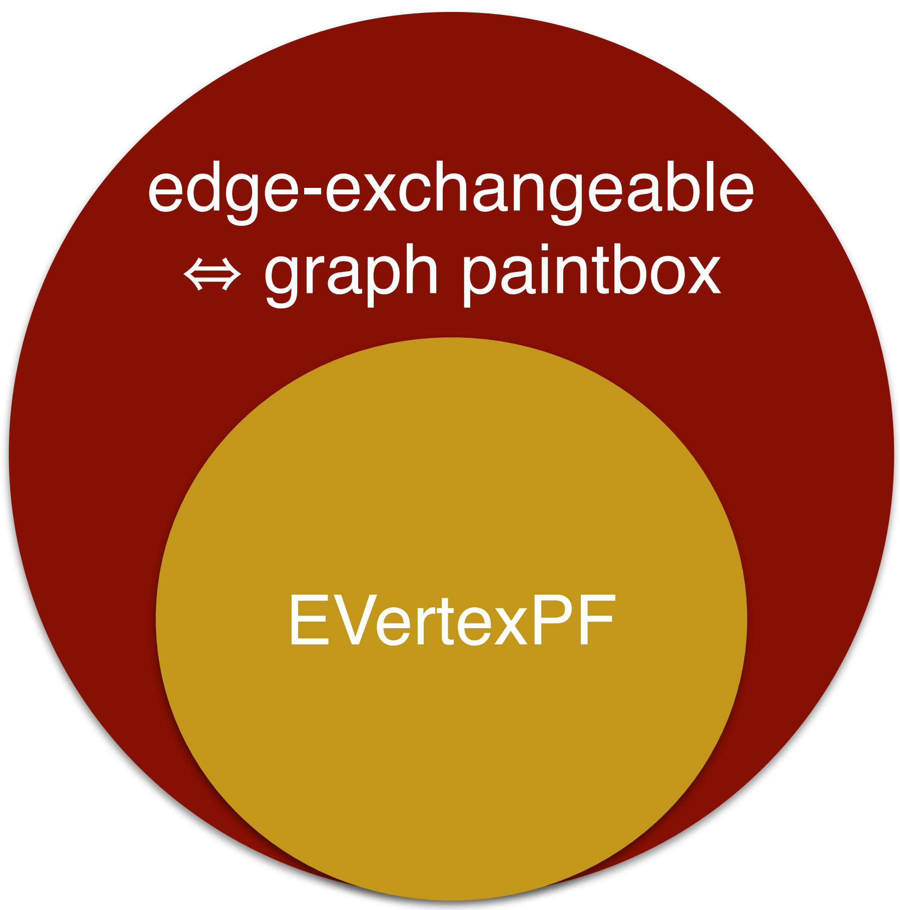

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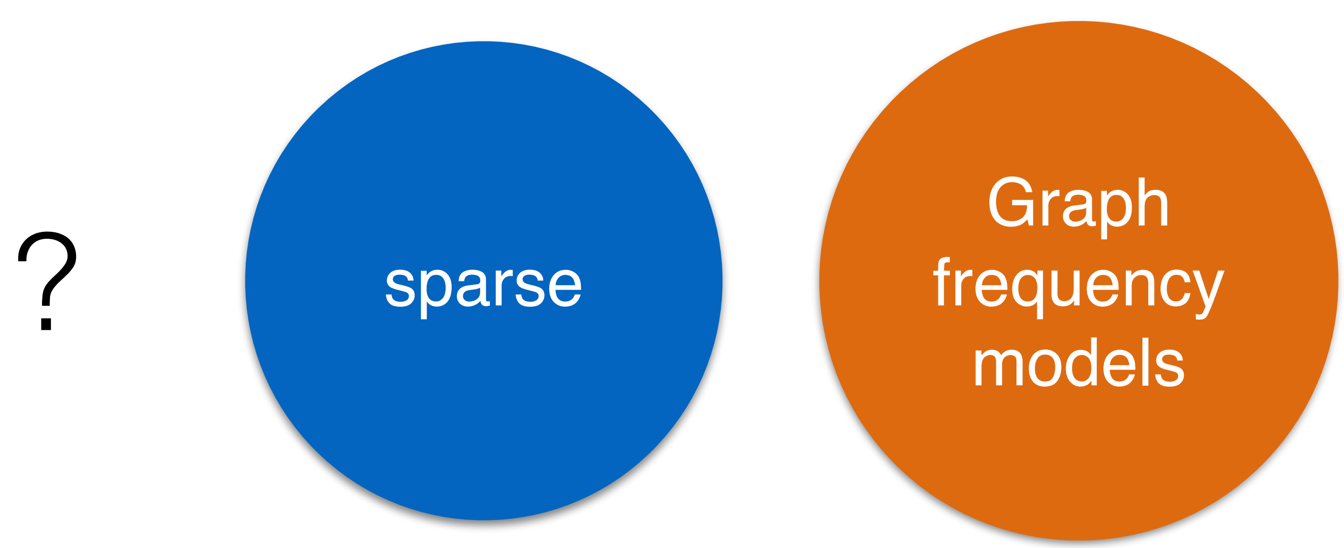

What we know so far

edge-exchangeable
⇔ graph paintbox

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?

sparse

Graph
frequency
models

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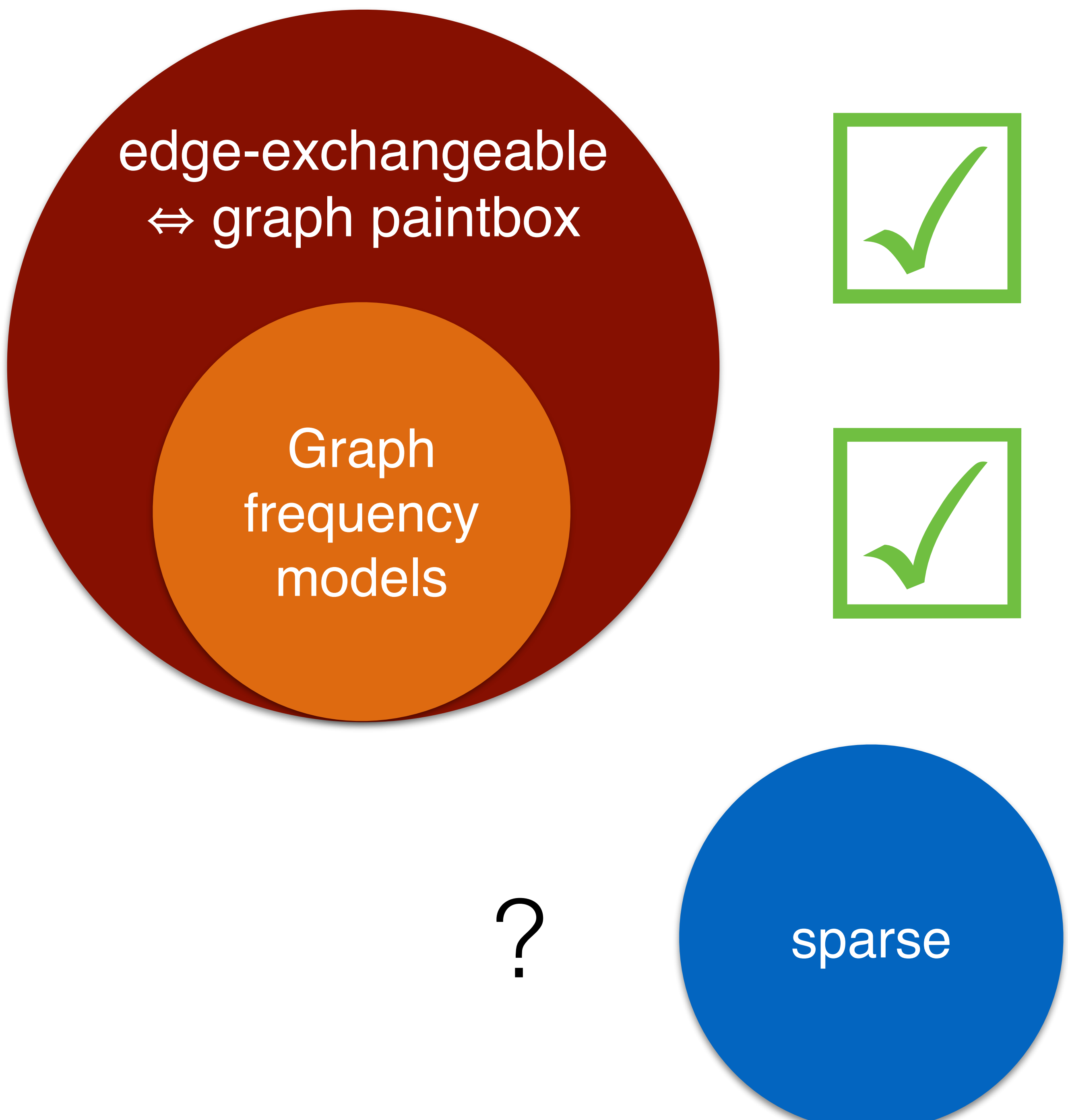

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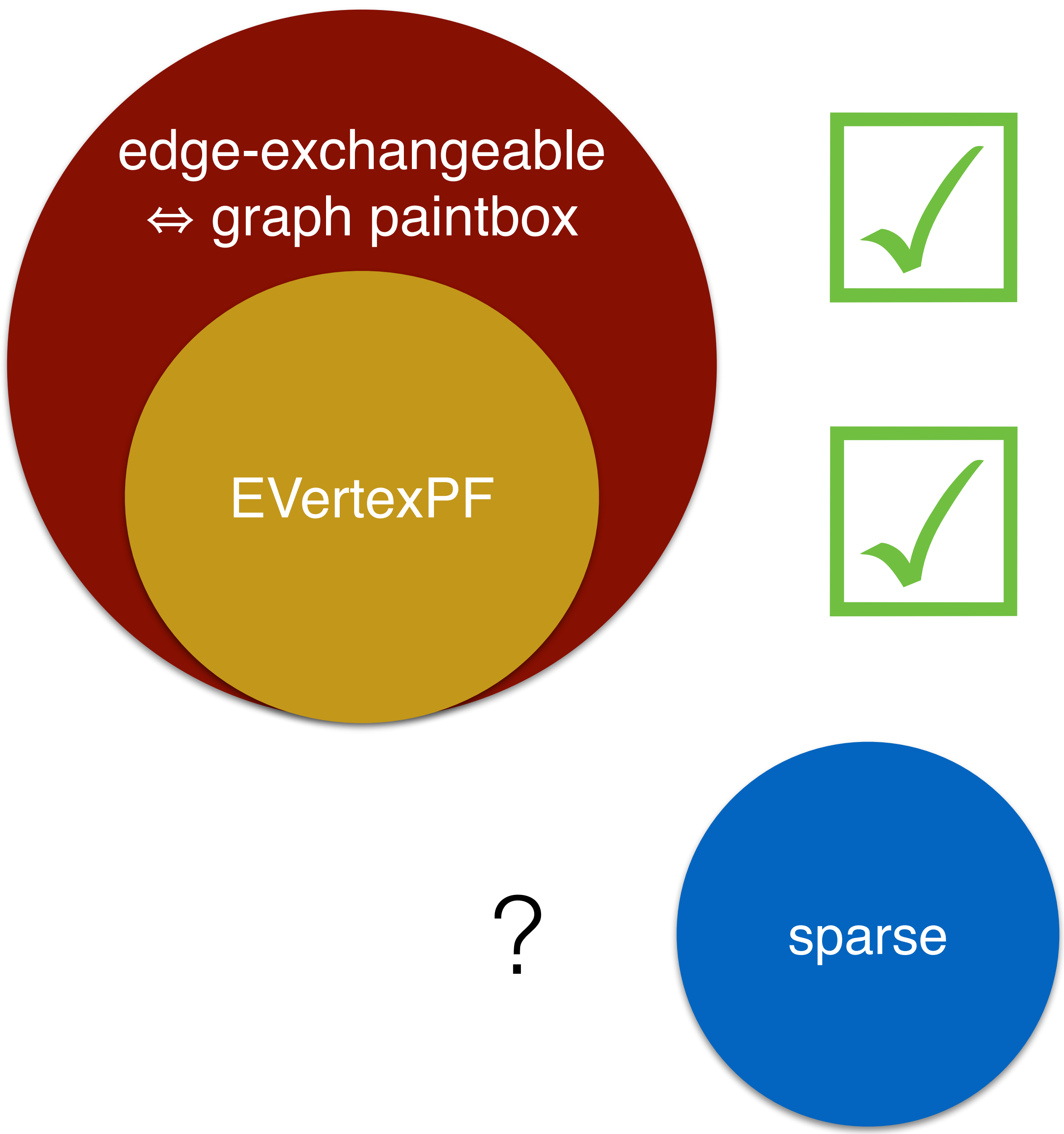

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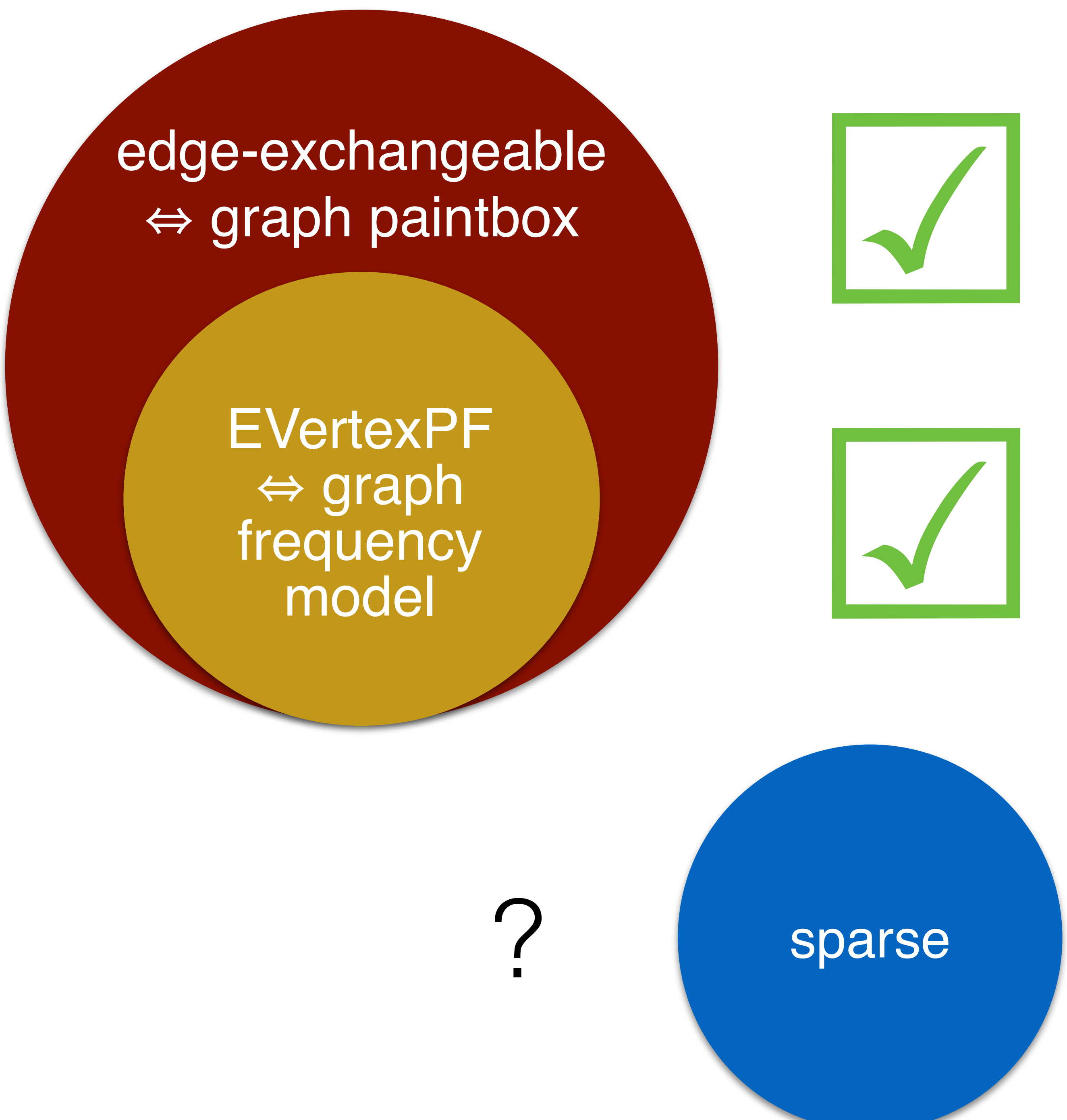

- Thm 1: characterization theorem for edge-exchangeable graphs
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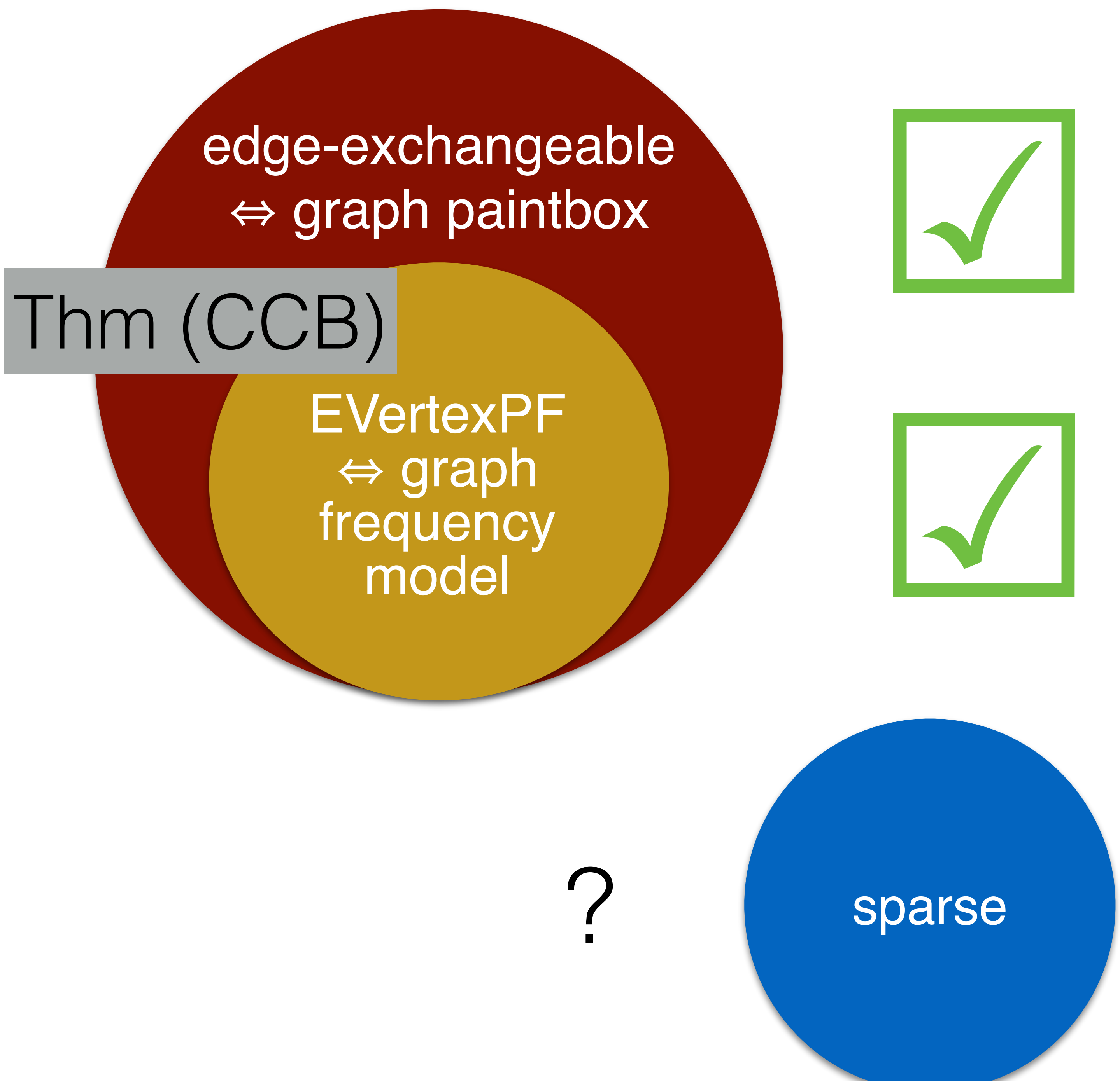

- Thm 1: characterization theorem for edge-exchangeable graphs
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[Campbell, Cai, Broderick 2016]

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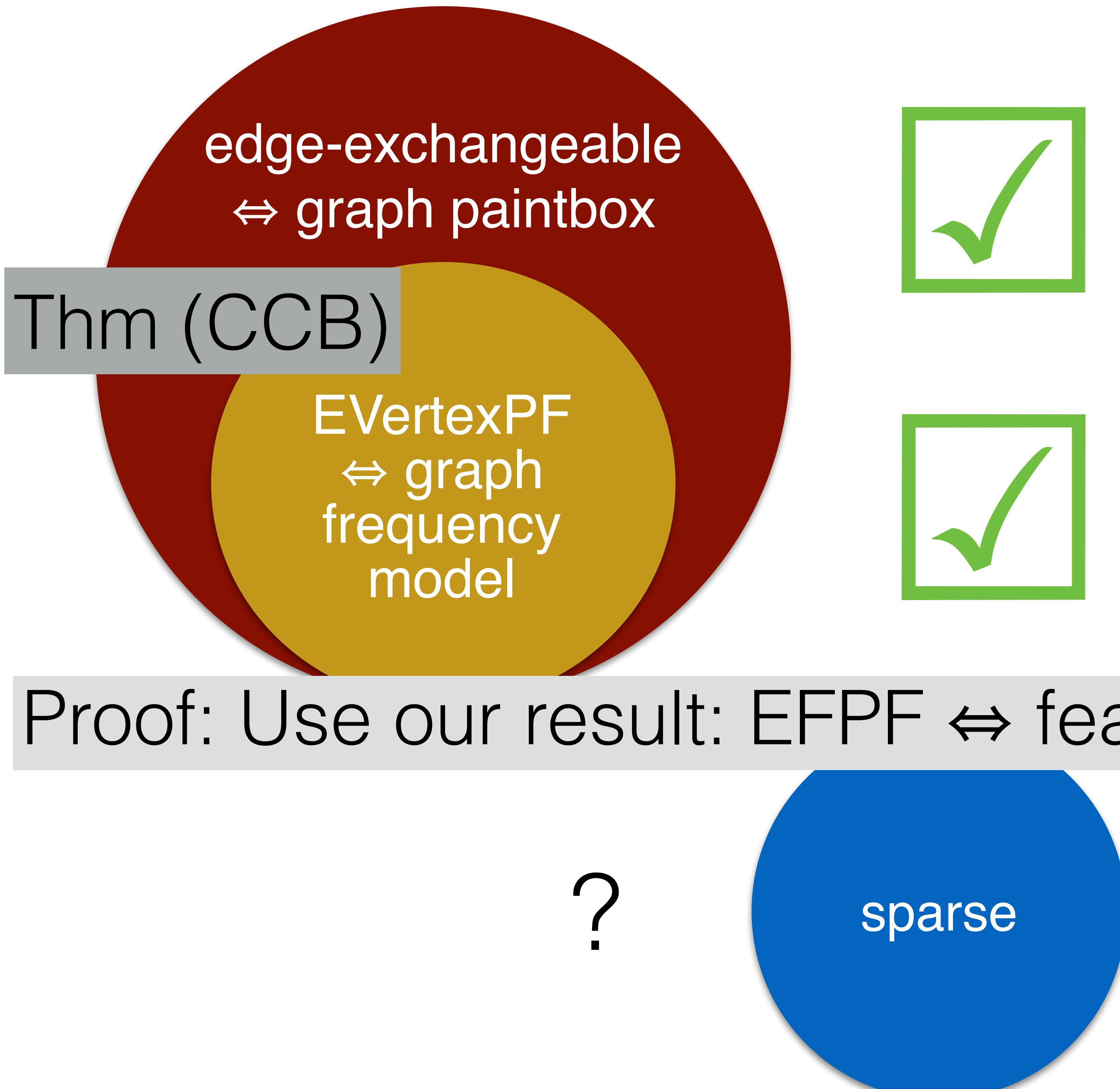

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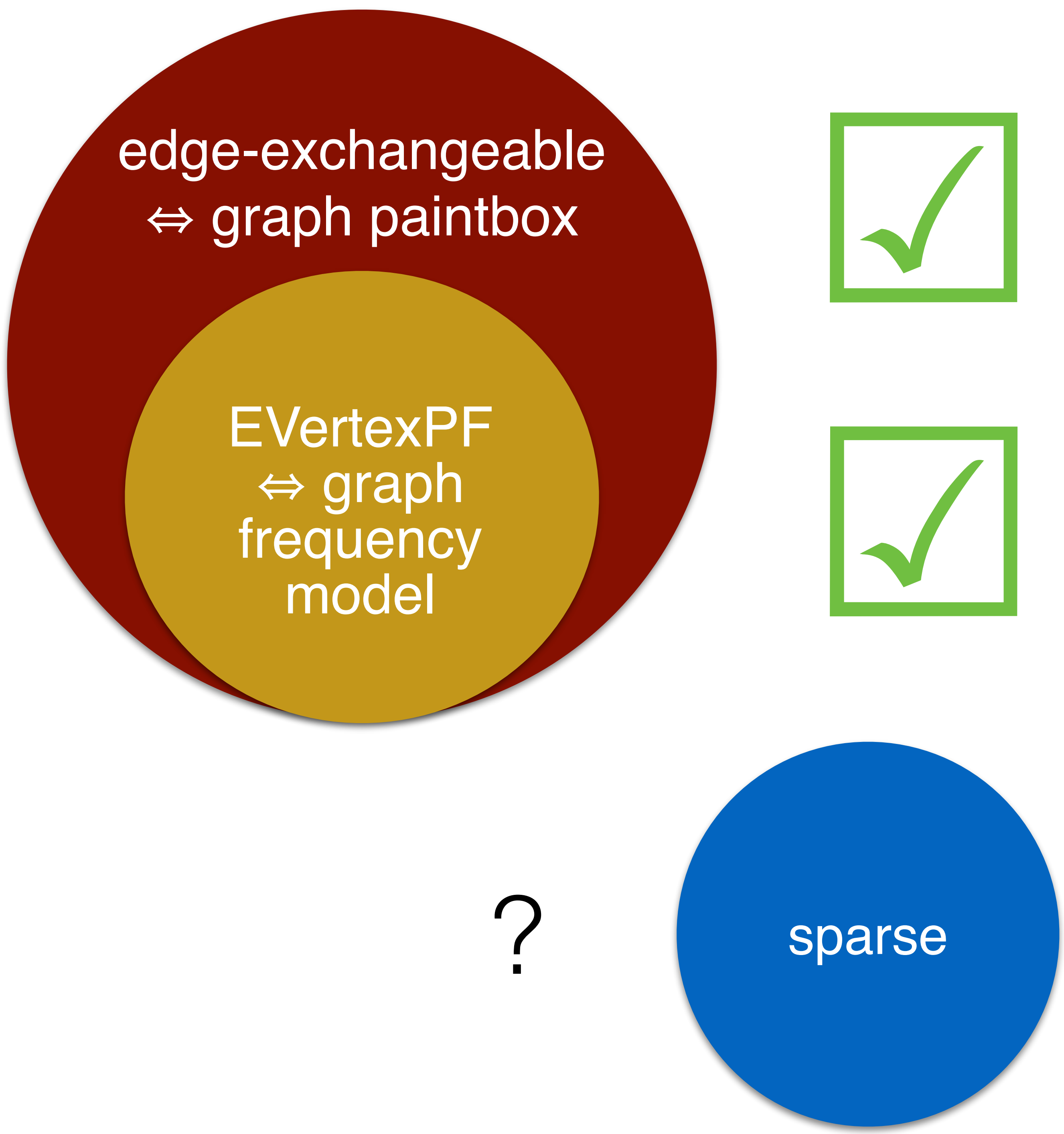
Proof: Use our result: EFPF ⇔ feature frequency model

[Campbell, Cai, Broderick 2016; Broderick, Jordan, Pitman 2013]

What we know so far



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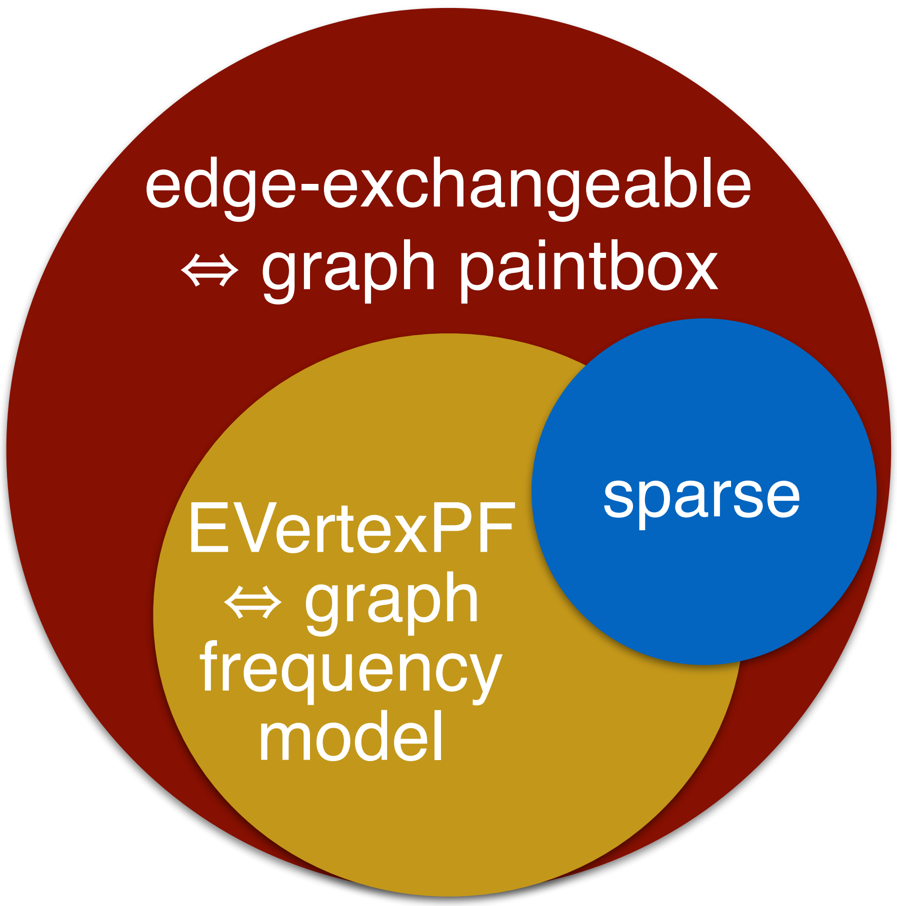

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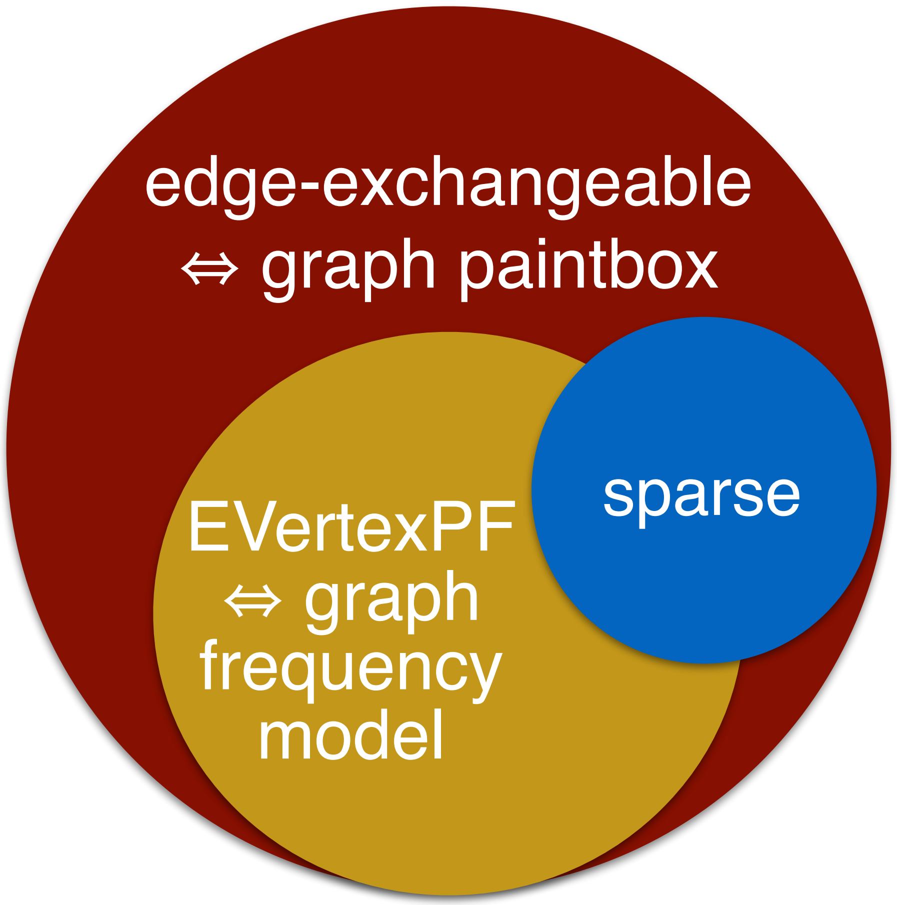

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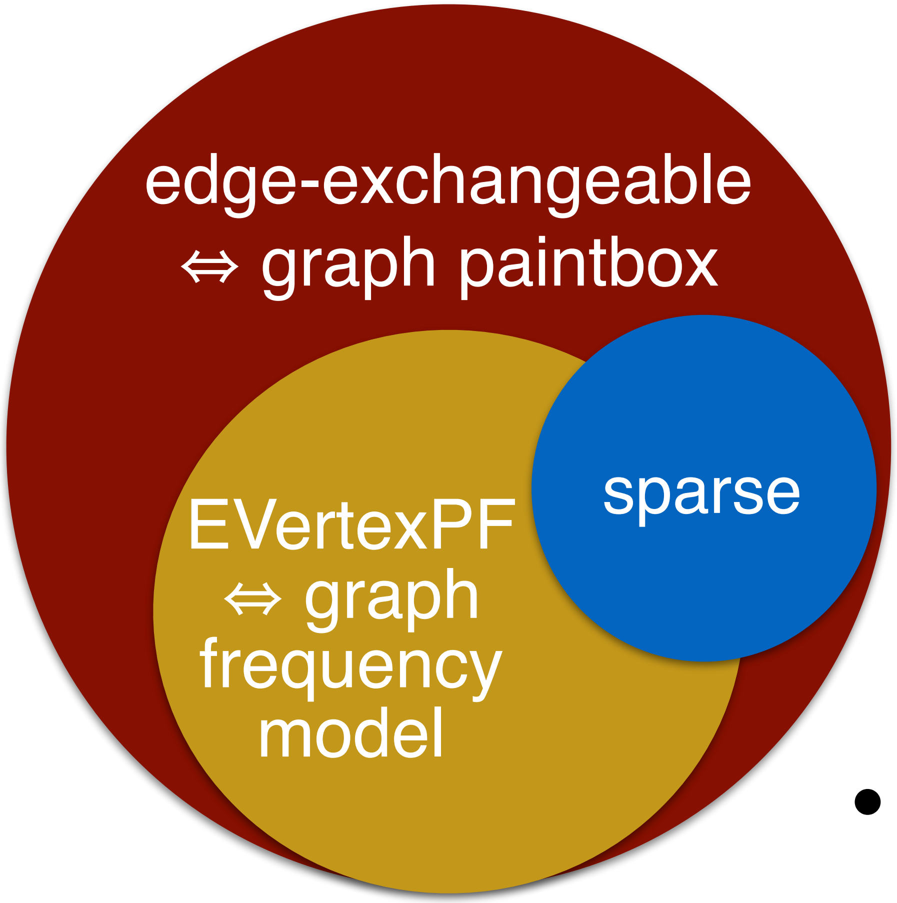

- ✓ Thm 1: characterization theorem for edge-exchangeable graphs
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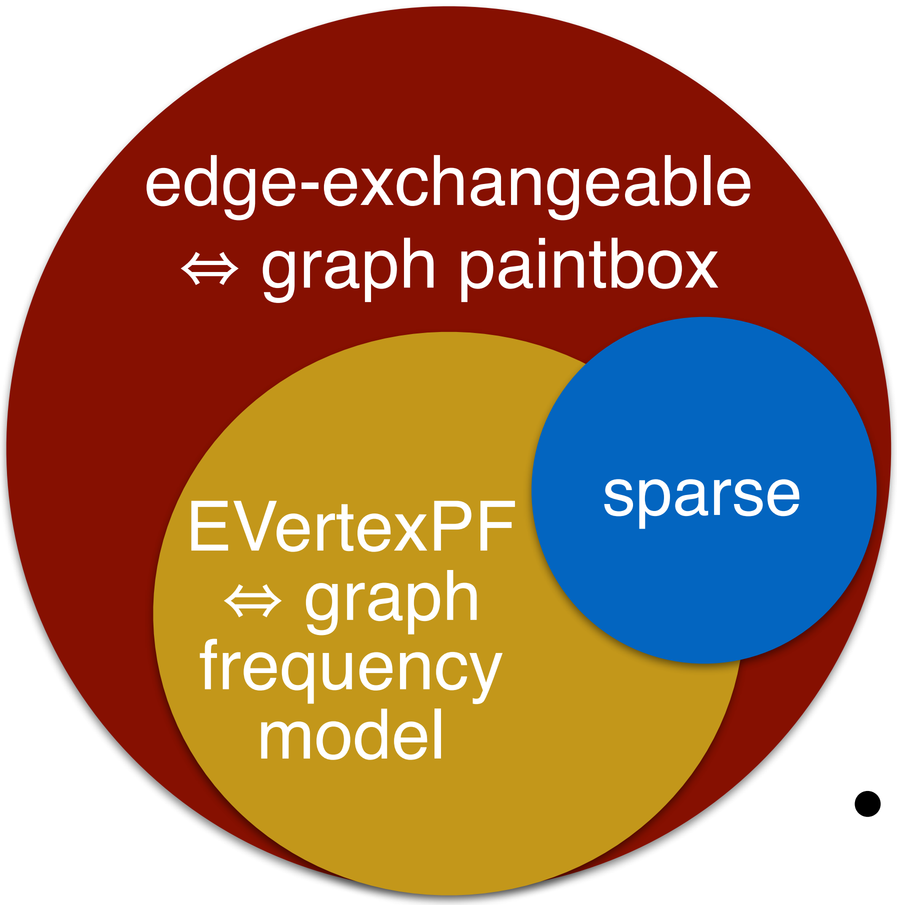
- Also: Characterization of “dust” (new for features, traits, and graphs)

[Campbell, Cai, Broderick 2016]

What we know so far



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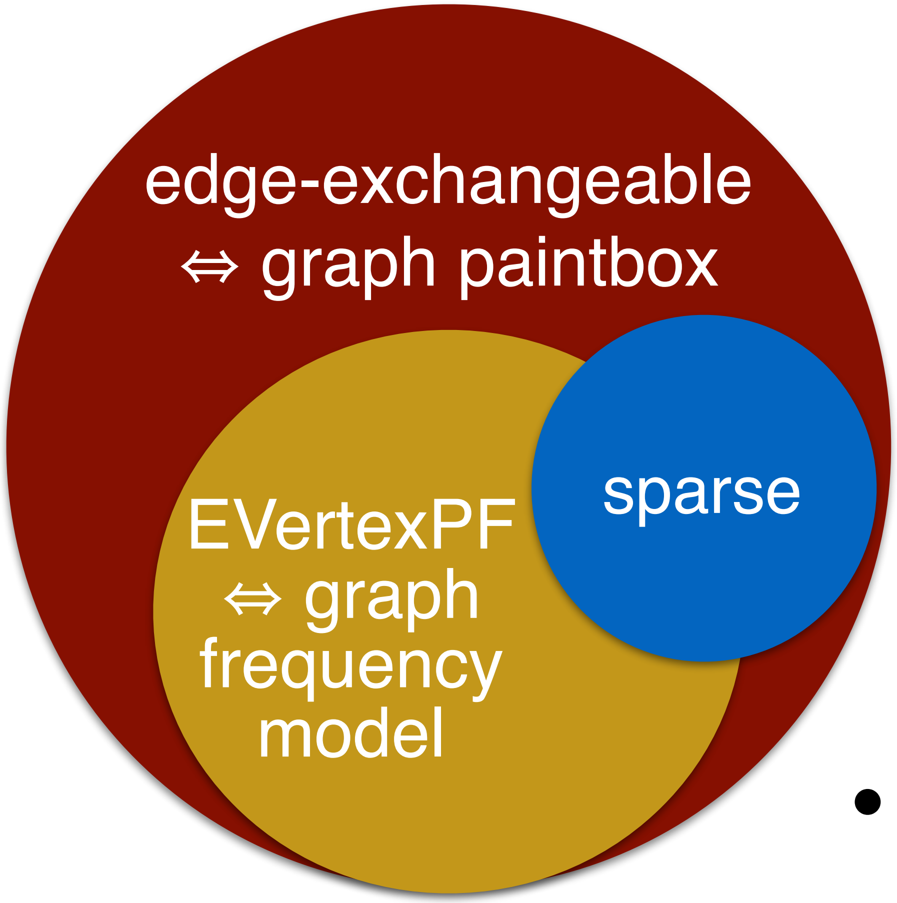
What’s next

[Campbell, Cai, Broderick 2016]

What we know so far



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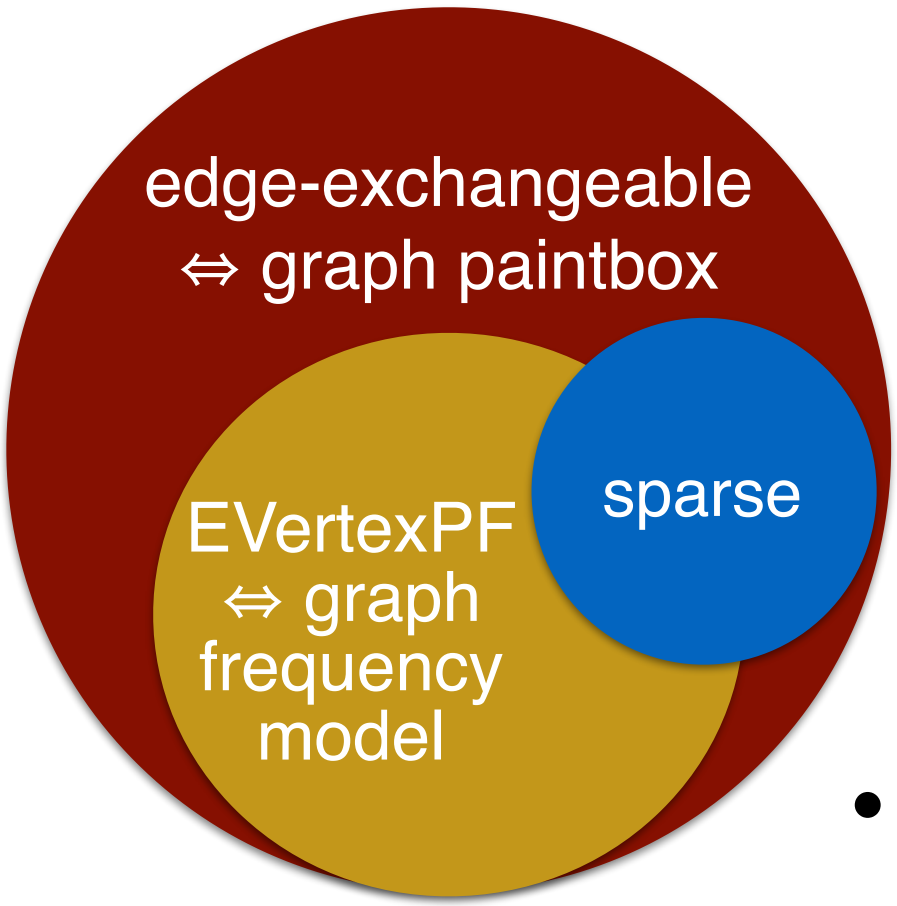
What’s next

- Characterize all sparse, edge-exchangeable graphs

[Campbell, Cai, Broderick 2016]

What we know so far



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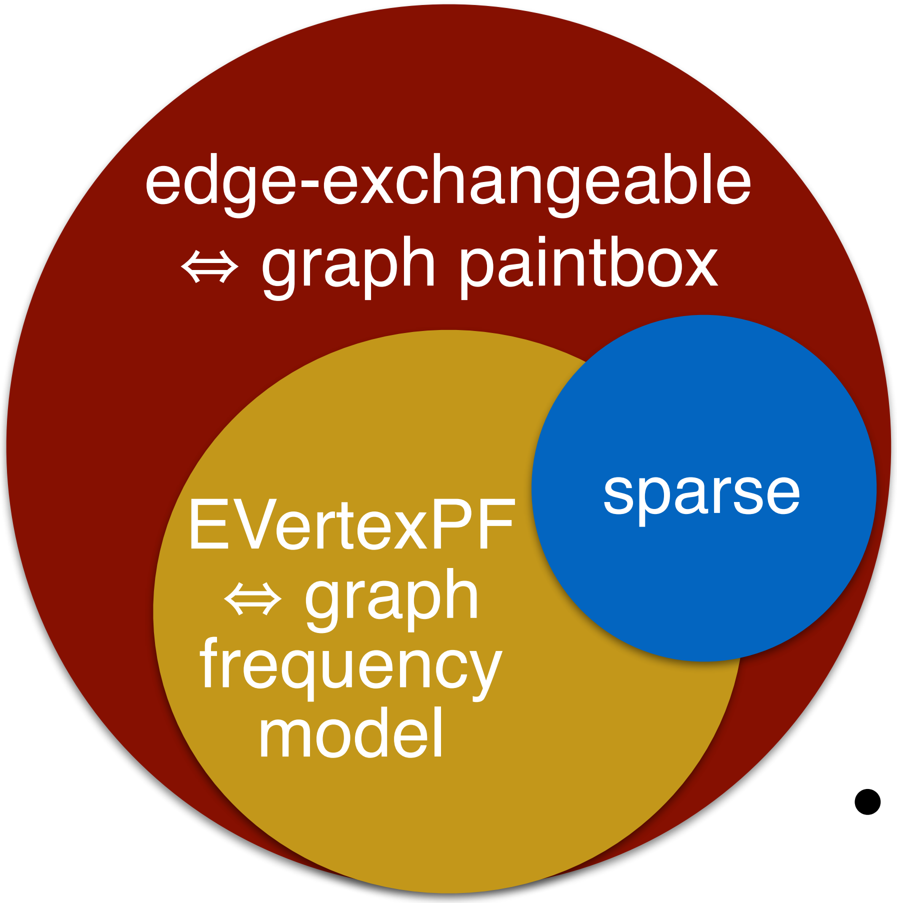
What’s next

- Characterize all sparse, edge-exchangeable graphs
- Characterize the different types of power laws (edges, triangles, degree distributions, etc.)

[Campbell, Cai, Broderick 2016; Broderick, Jordan, Pitman 2012]

What we know so far



- Thm 1: characterization theorem for edge-exchangeable graphs
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What’s next

- Characterize all sparse, edge-exchangeable graphs
- Characterize the different types of power laws (edges, triangles, degree distributions, etc.)
- Models and inference; truncation approximations

 [Campbell, Cai, Broderick 2016; Broderick, Jordan, Pitman 2012; Campbell*, Huggins*, How, Broderick 2016]

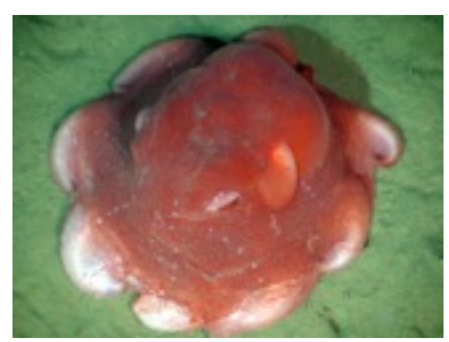
Nonparametric Bayes

- Bayesian

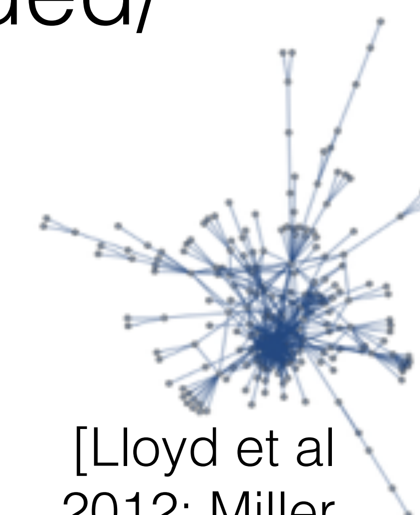
$$\mathbb{P}(\text{parameters}|\text{data}) \propto \mathbb{P}(\text{data}|\text{parameters})\mathbb{P}(\text{parameters})$$

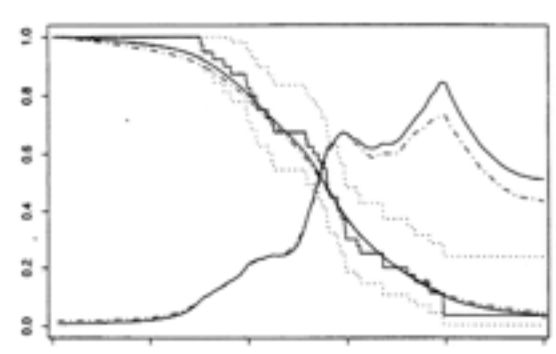
- Not parametric (i.e. not finite parameter, unbounded/ growing/infinite number of parameters)

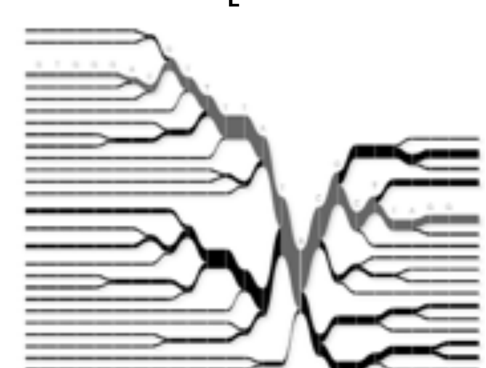
[wikipedia.org]


[Ed Bowlby, NOAA]

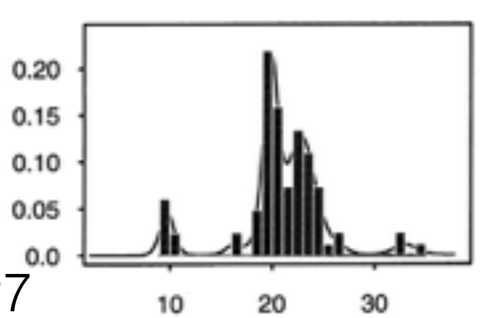

[Fox et al 2014]


[Lloyd et al 2012; Miller et al 2010]

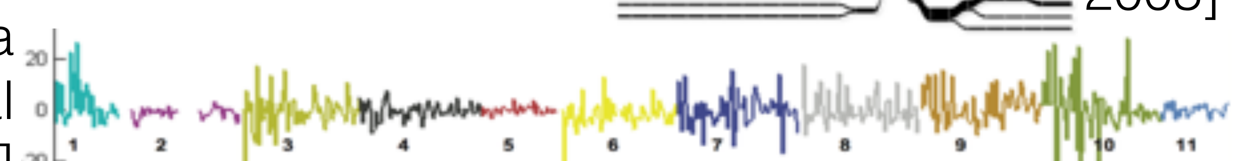

[Arjas, Gasbarra 1994]


[Ewens 1972; Hartl, Clark 2003]


[Sudderth, Jordan 2009]



[Escobar, West 1995; Ghosal et al 1999]

[Saria et al 2010]


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