Nonparametric Bayesian Methods: Models, Algorithms, and Applications (Part II)

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Electrical Engineering & Computer Science
MIT

Roadmap

- Example problem: clustering
- Example NPBayes model: Dirichlet process
- Chinese restaurant process
- Inference
- Venture further into the wild world of Nonparametric Bayes

- Big questions
    - Why NPBayes?
    - What does an infinite/growing number of parameters really mean (in NPBayes)?
    - Why is NPBayes challenging but practical?

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  - Why is NPBayes challenging but practical? Infinite dimensional parameter; more on this now!

Distributions

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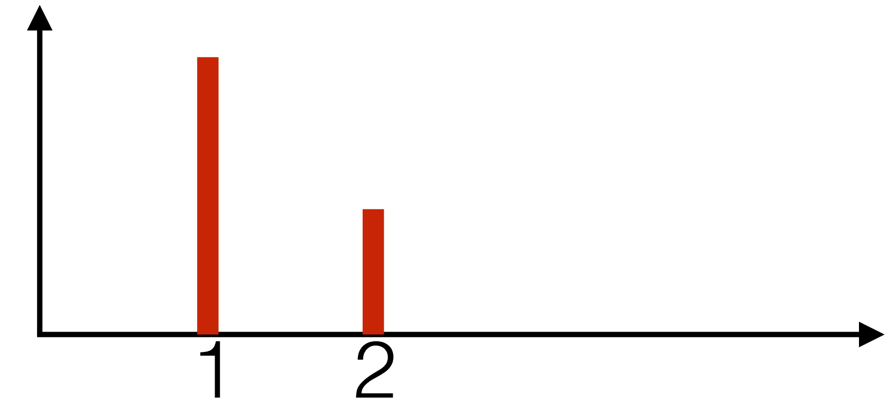
- Beta → random distribution over $1, 2$



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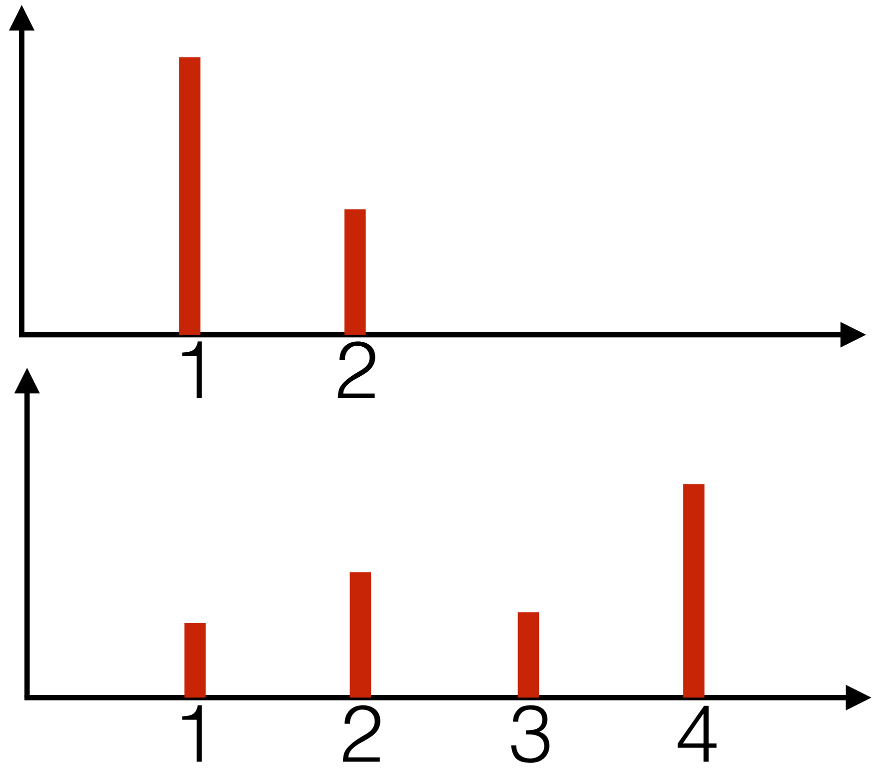

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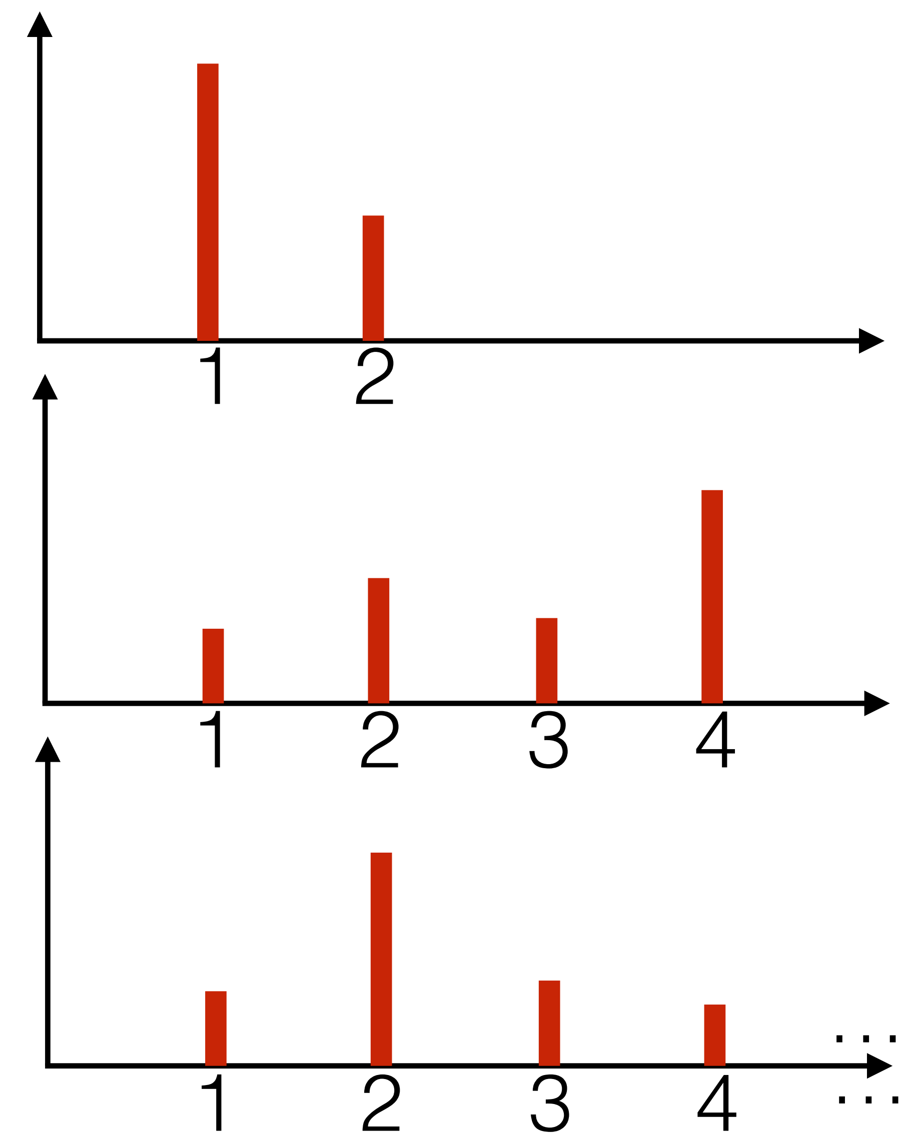

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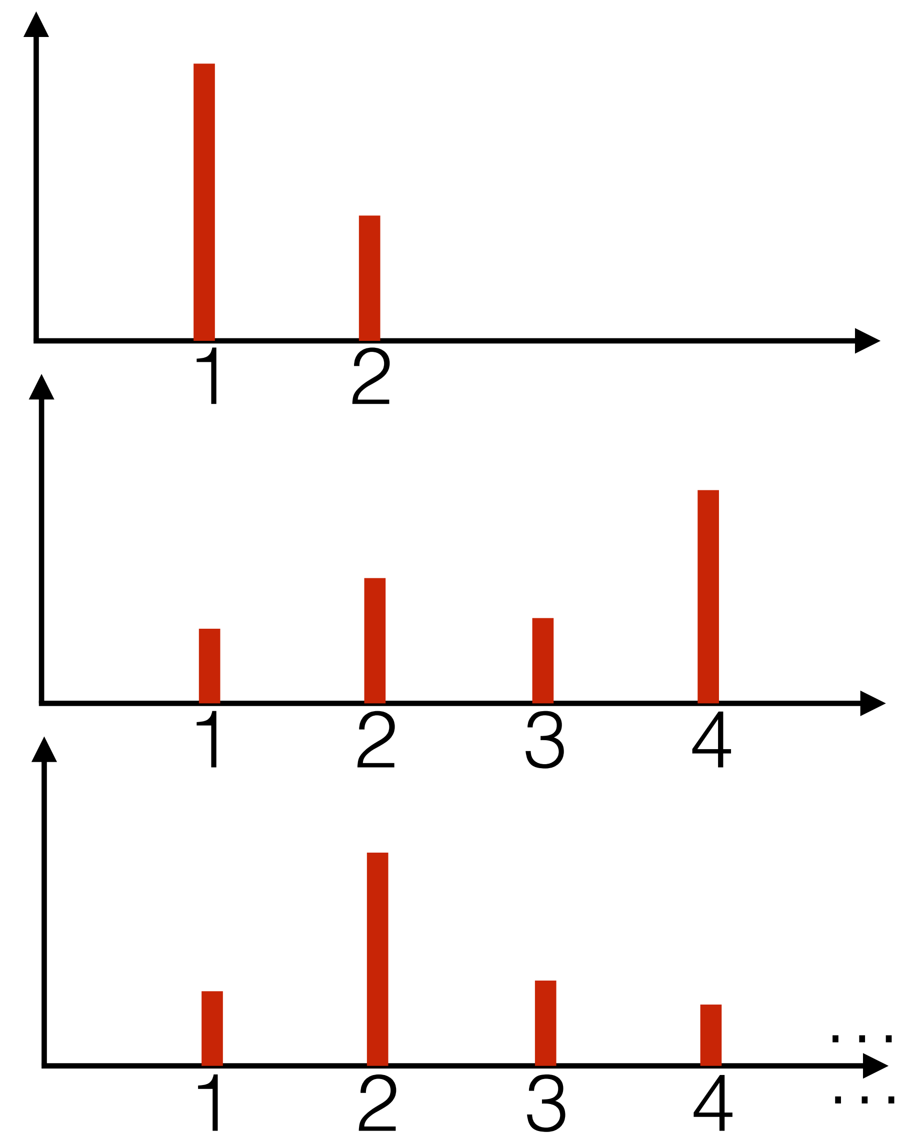

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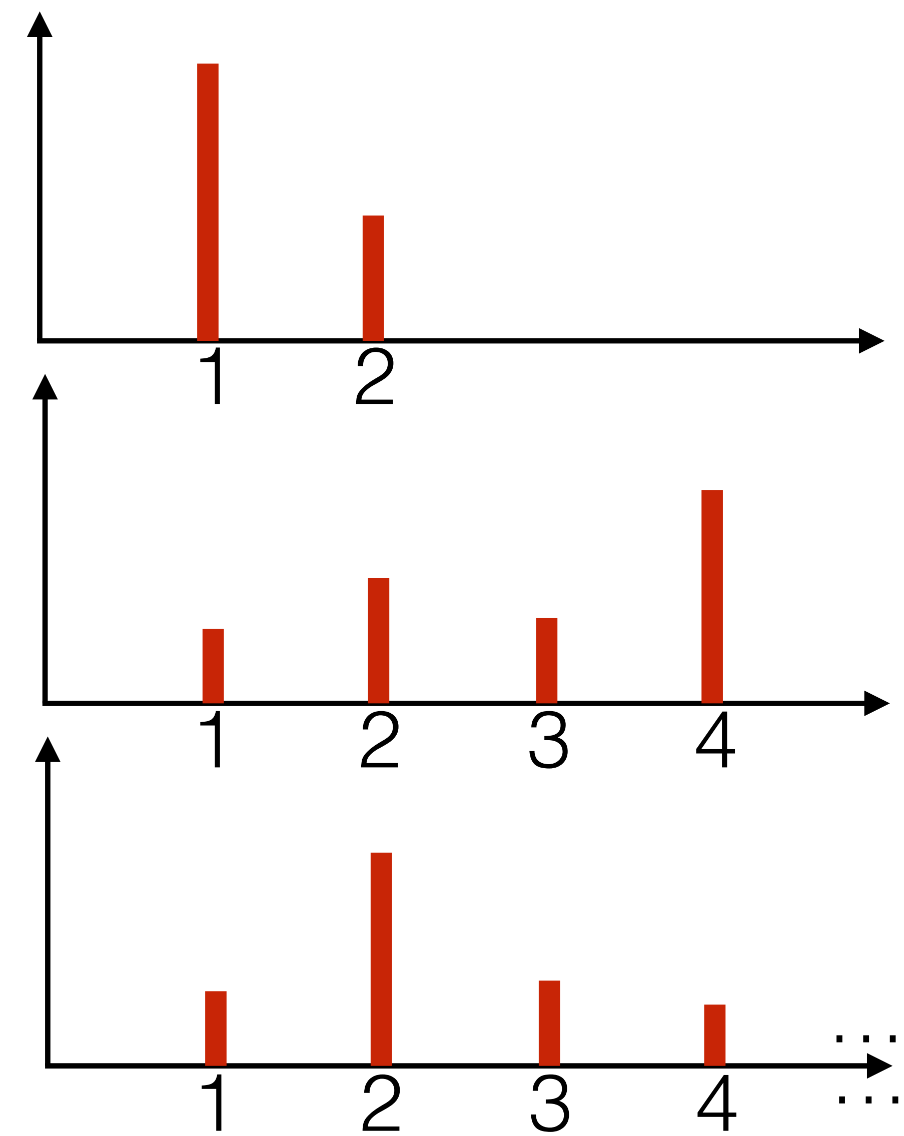

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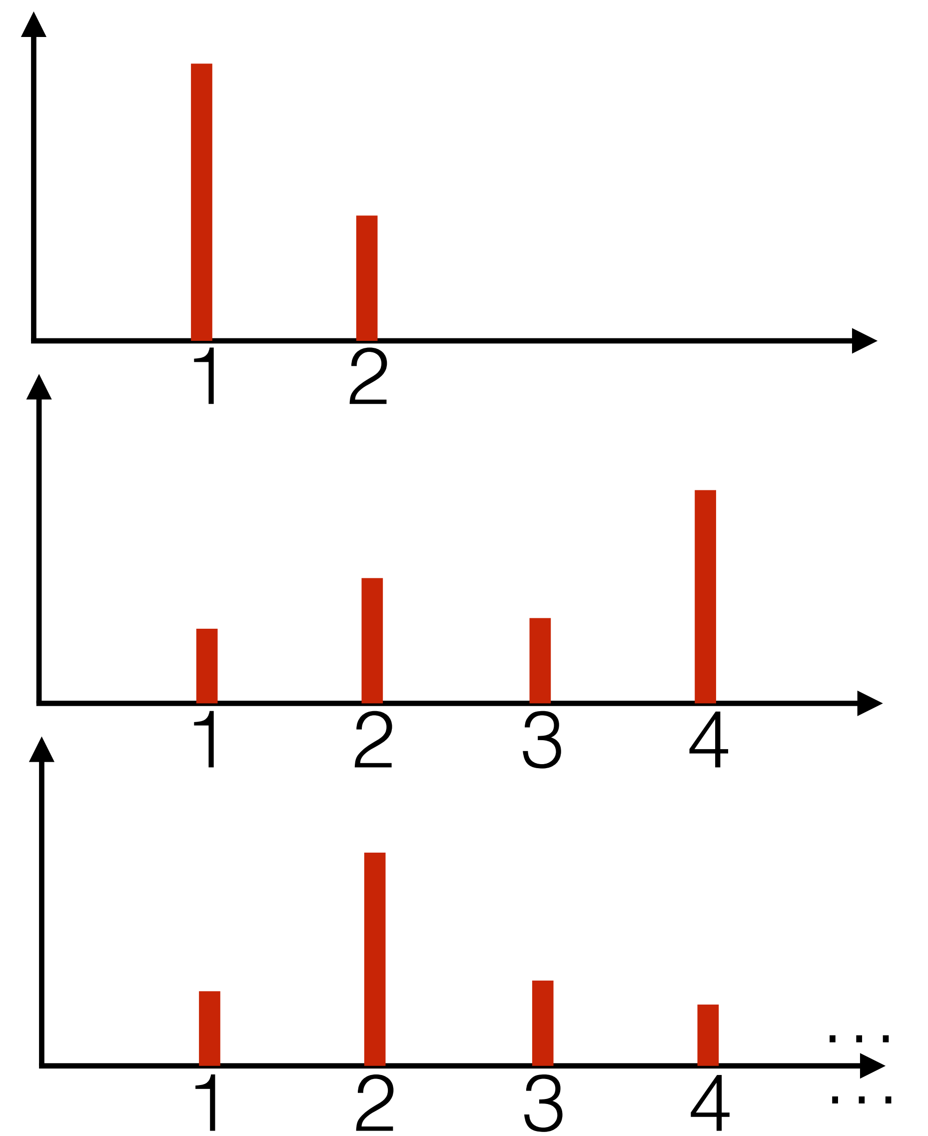

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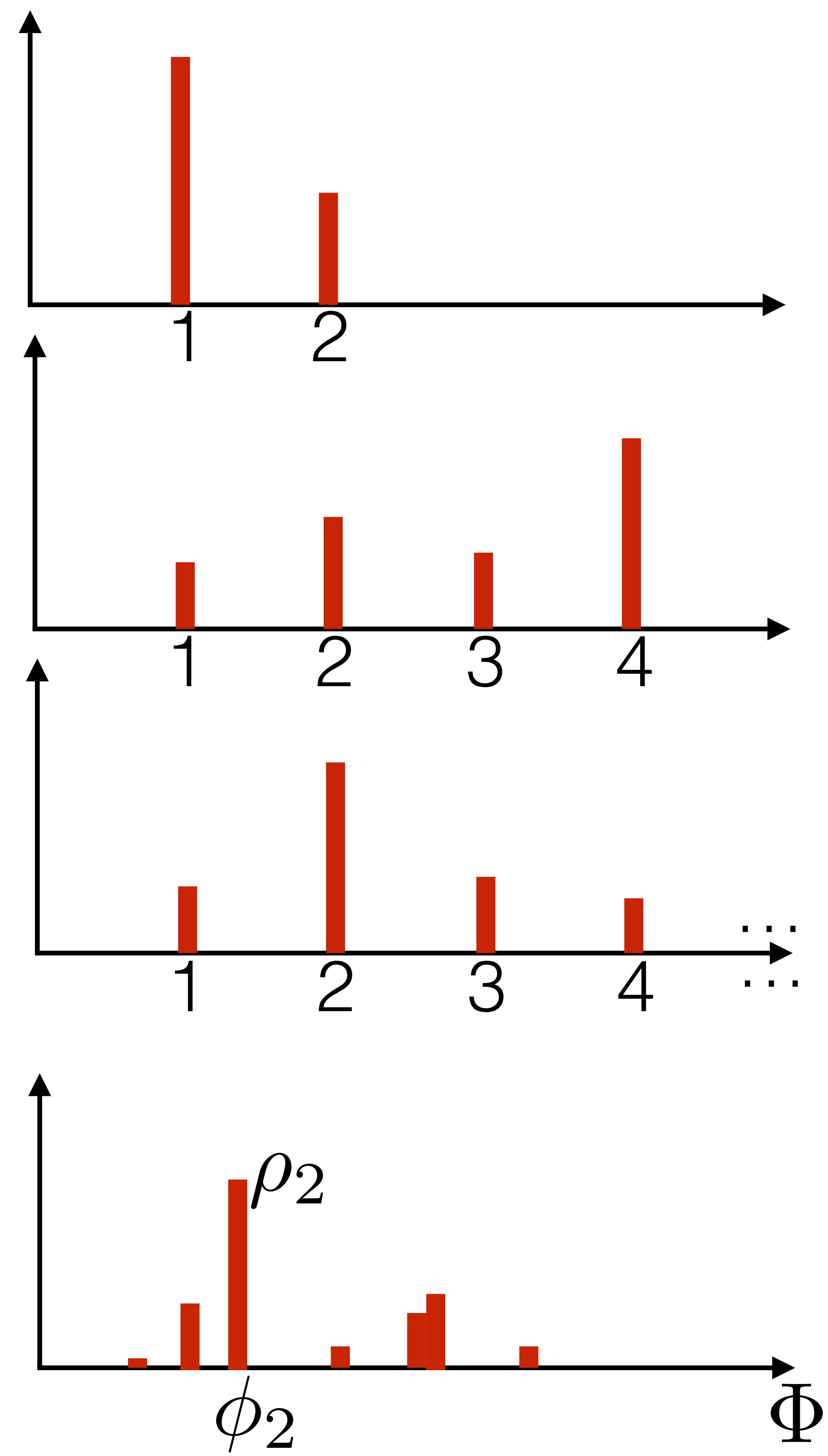

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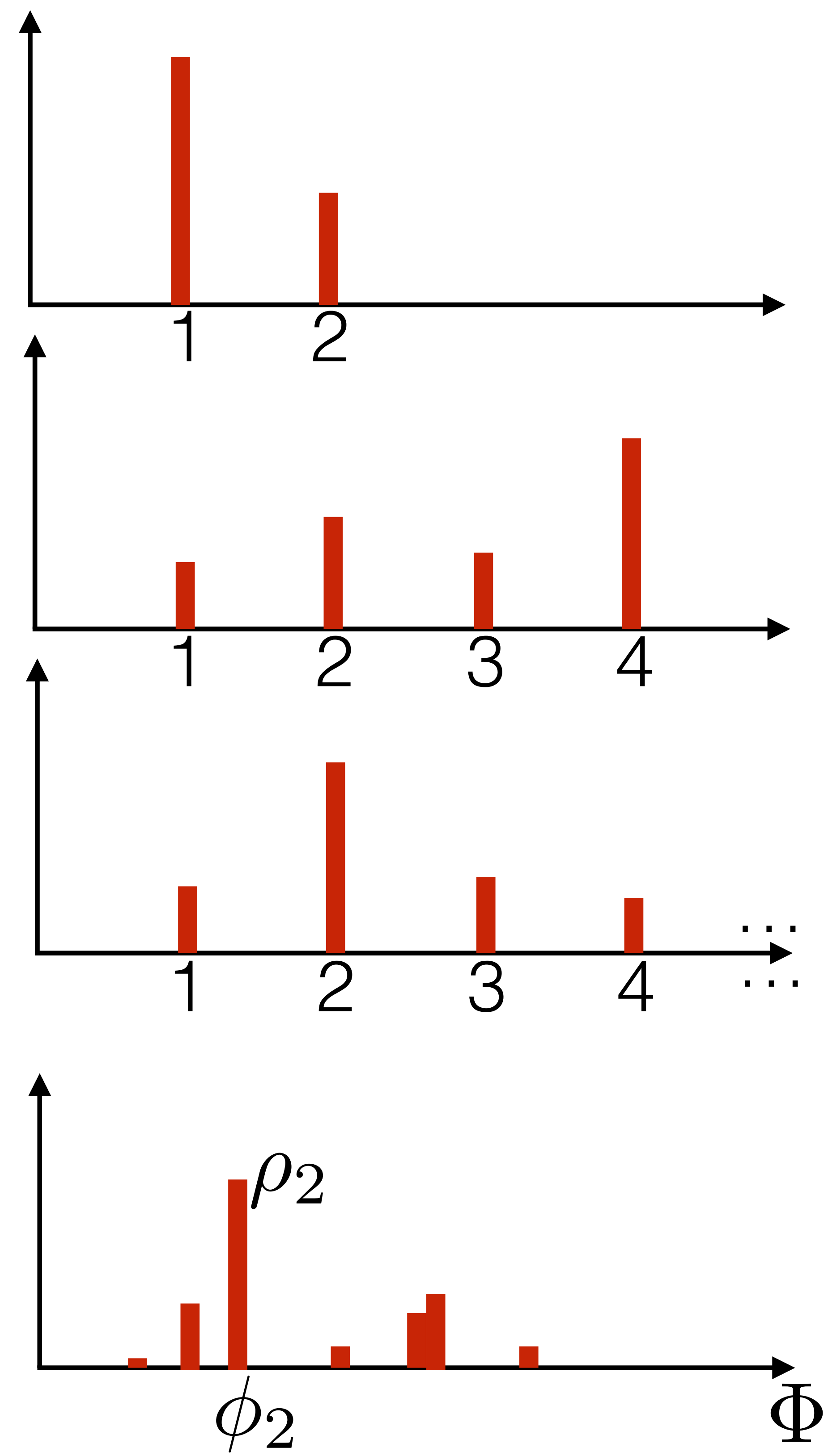

[Ferguson 1973]

Dirichlet process mixture model

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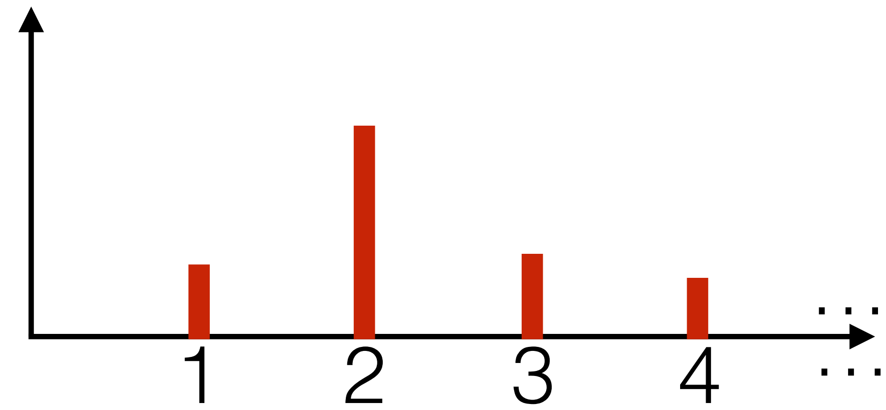

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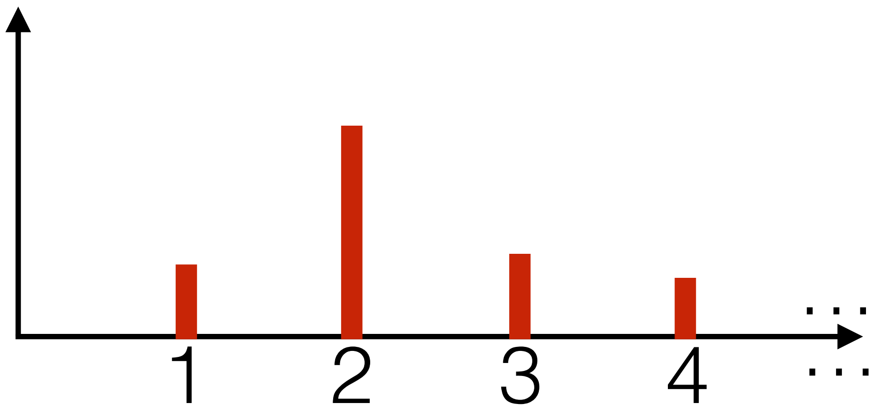

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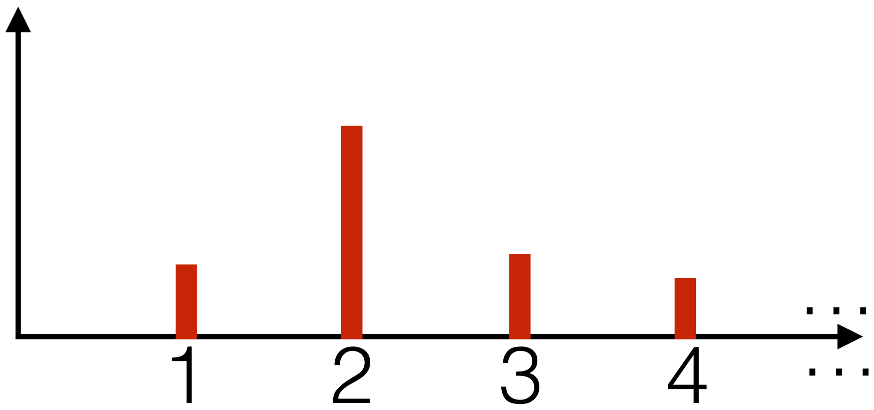

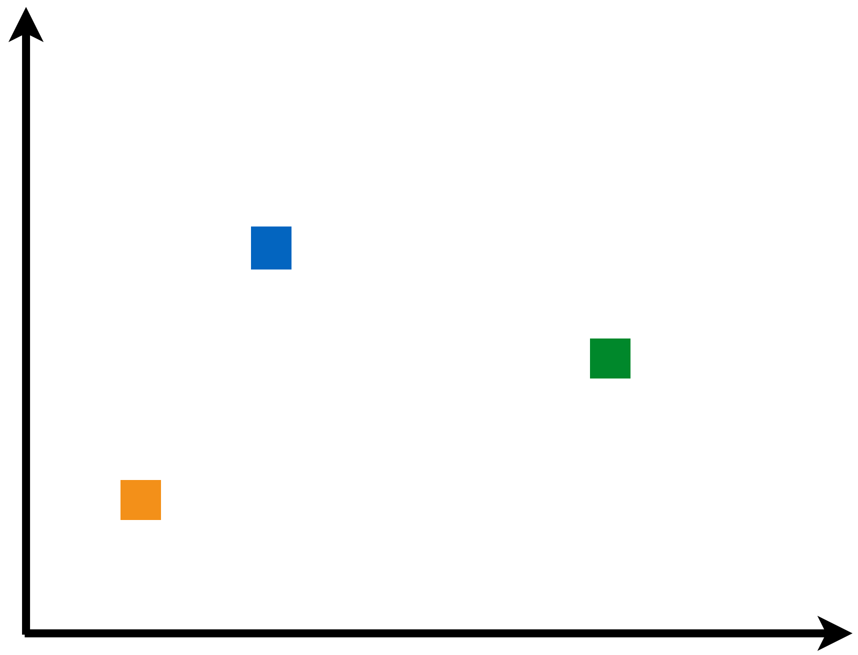

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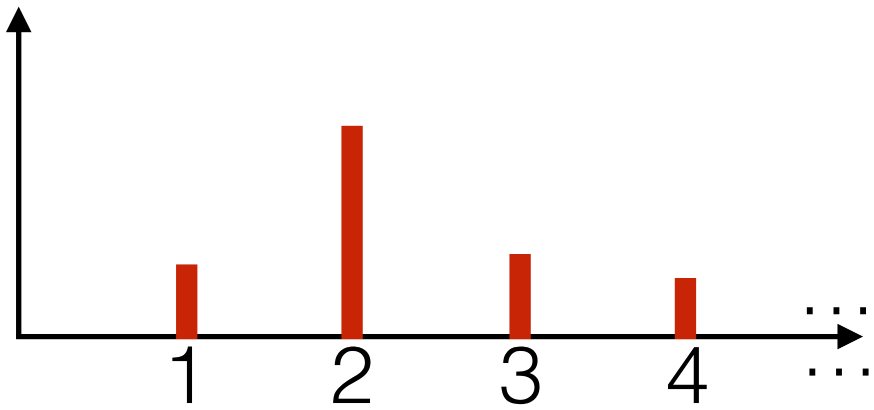

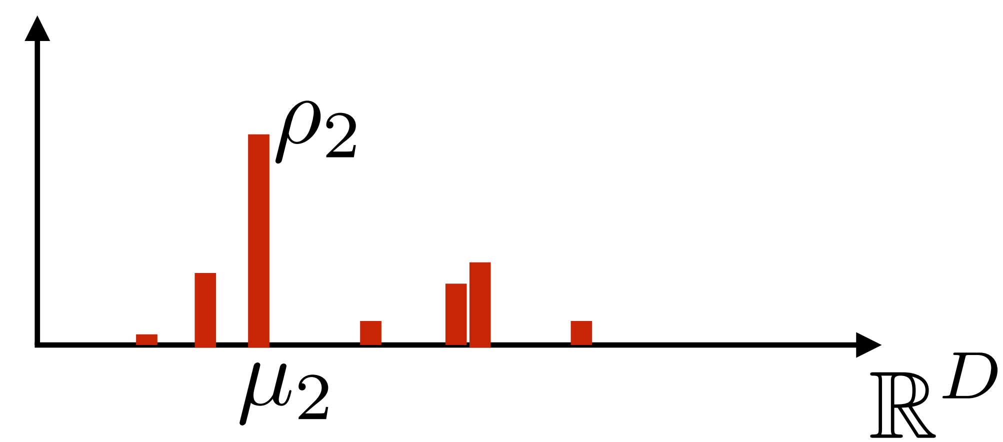

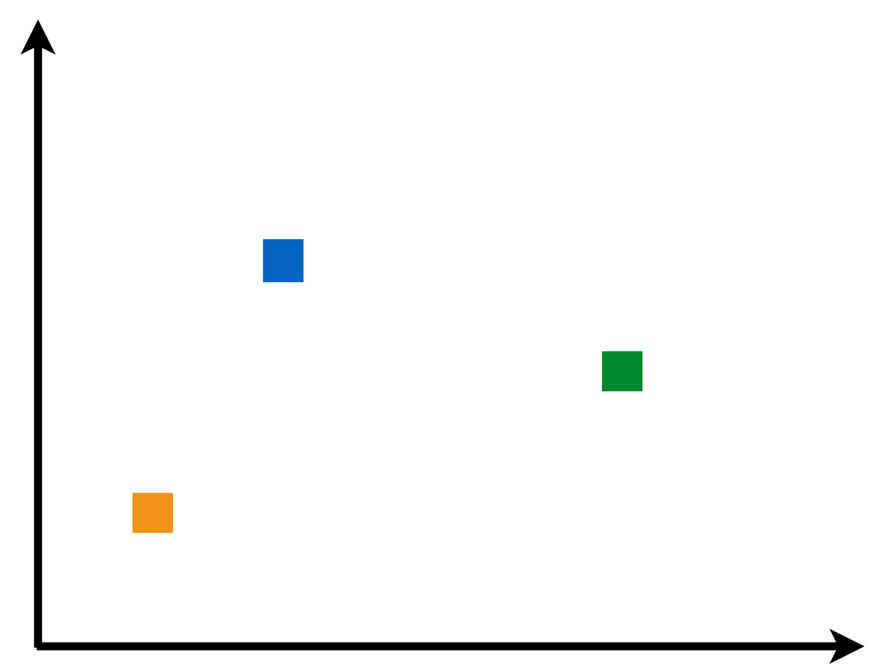

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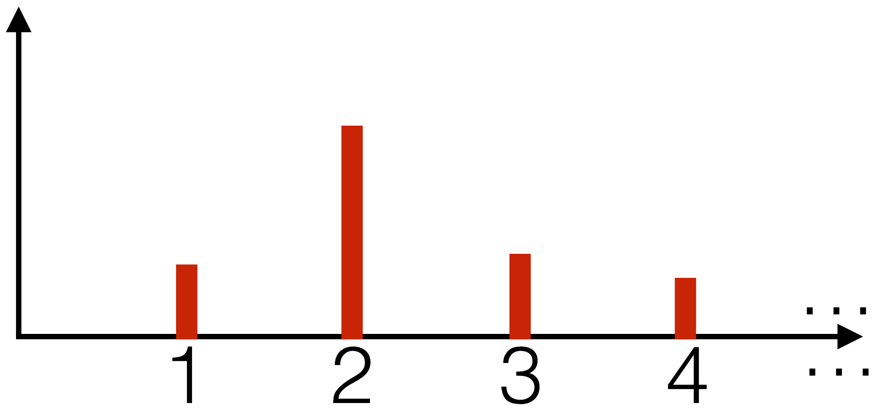

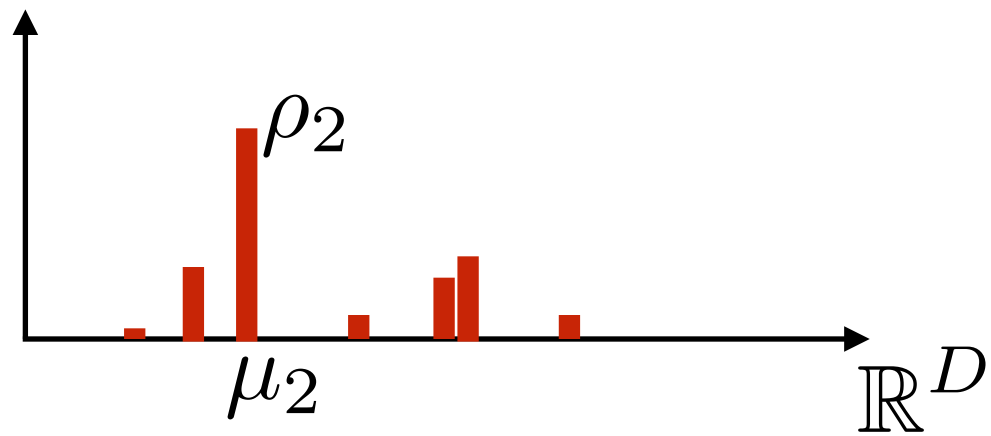

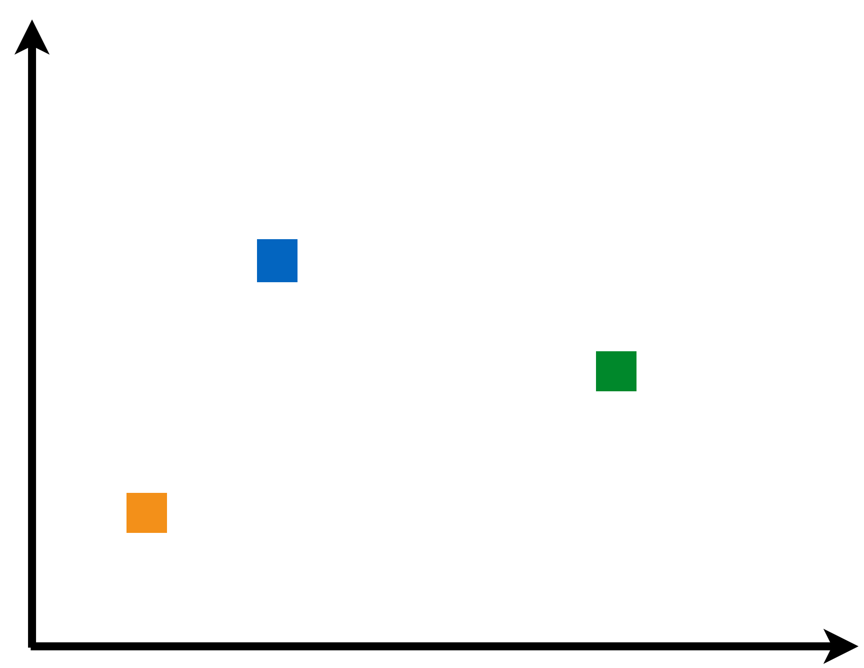

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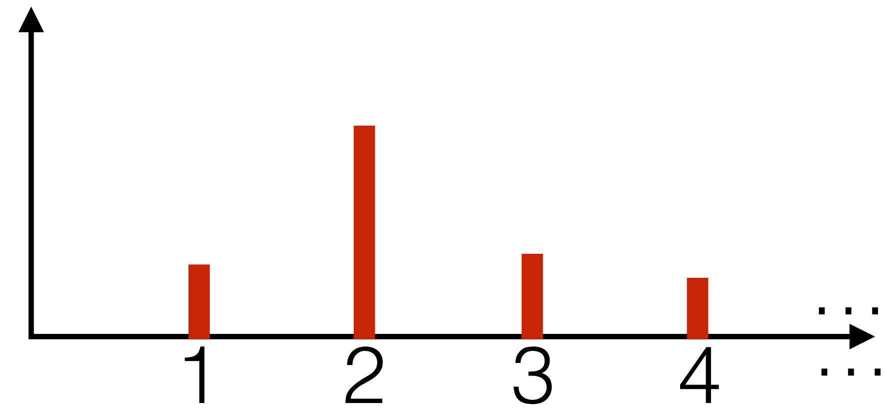

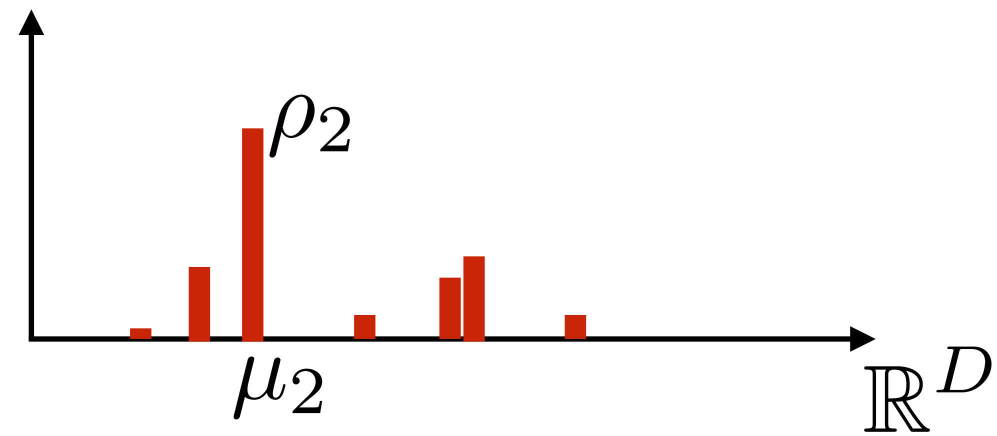

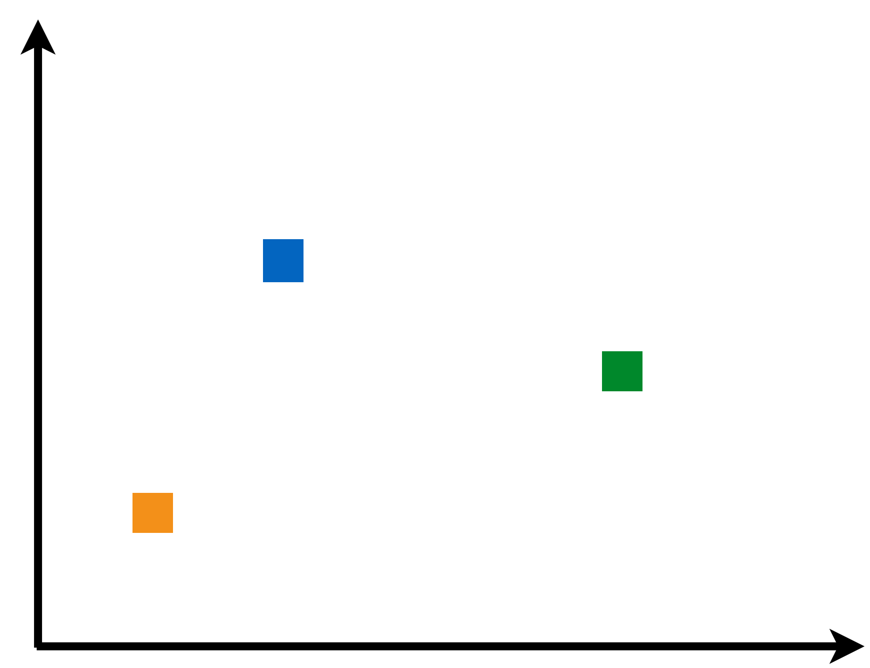

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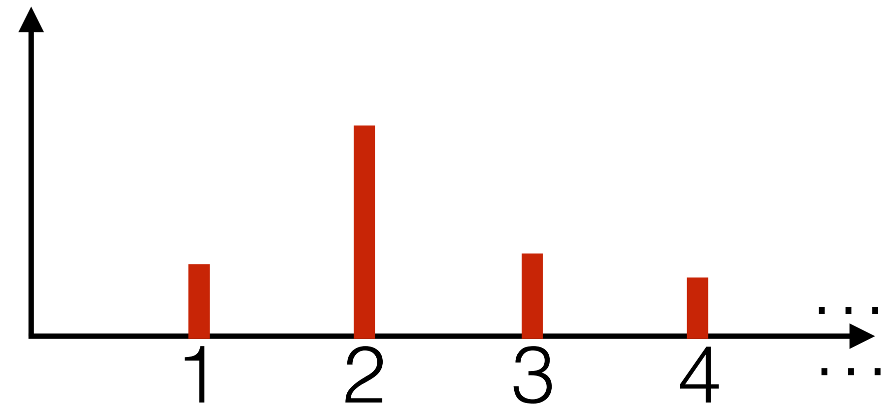

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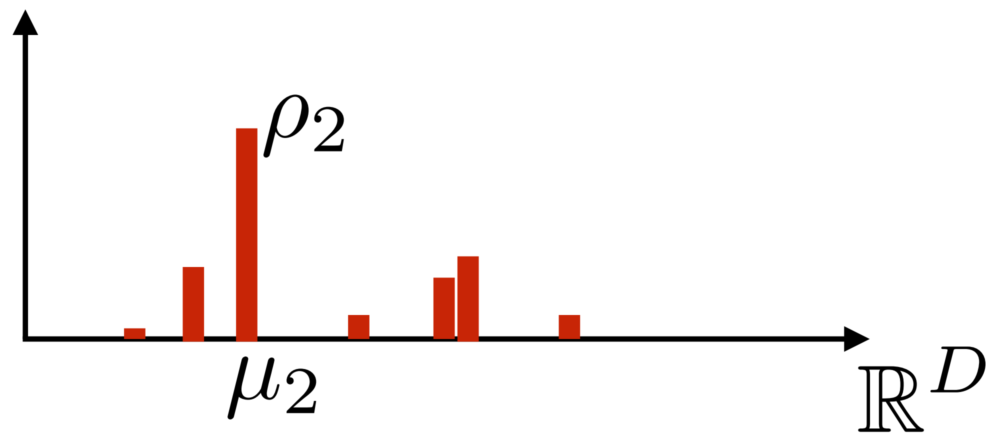

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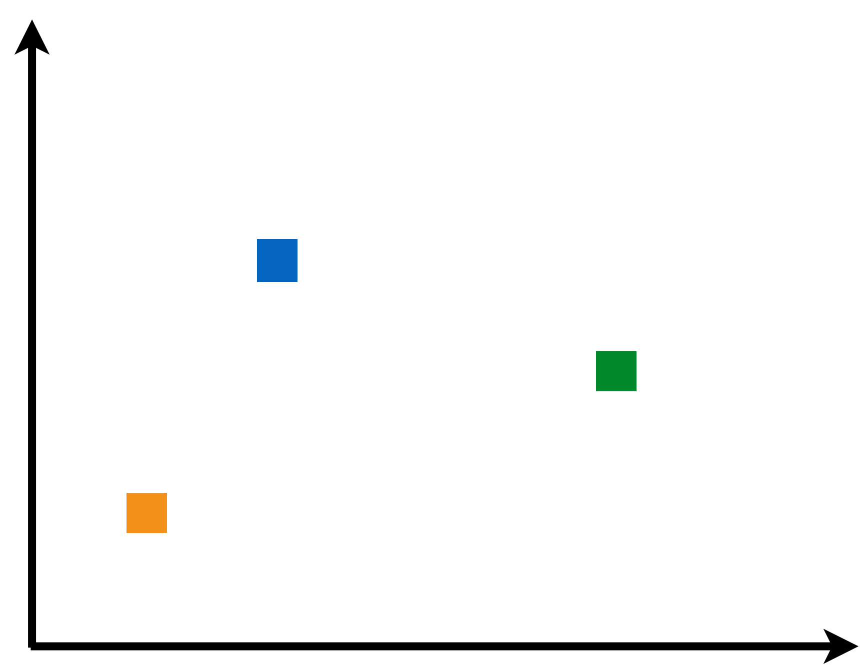

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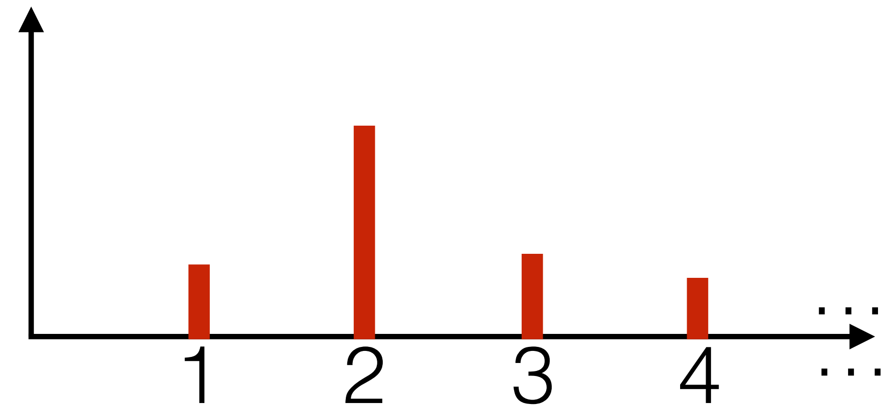

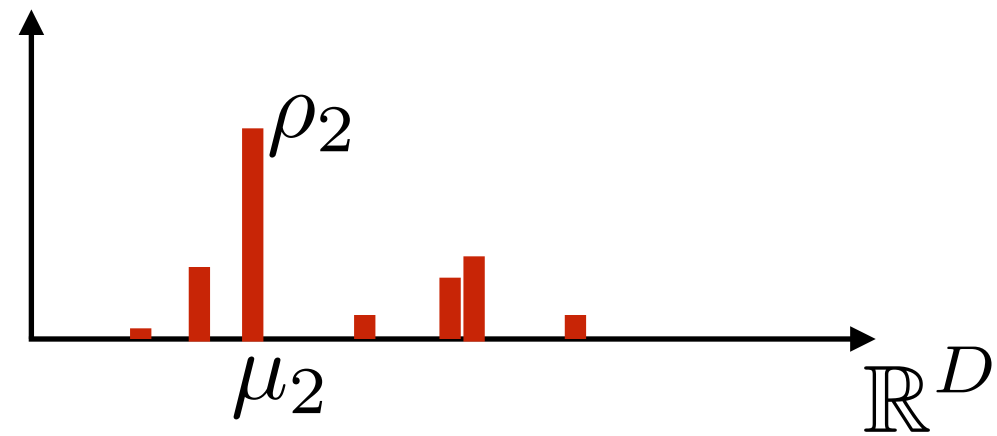

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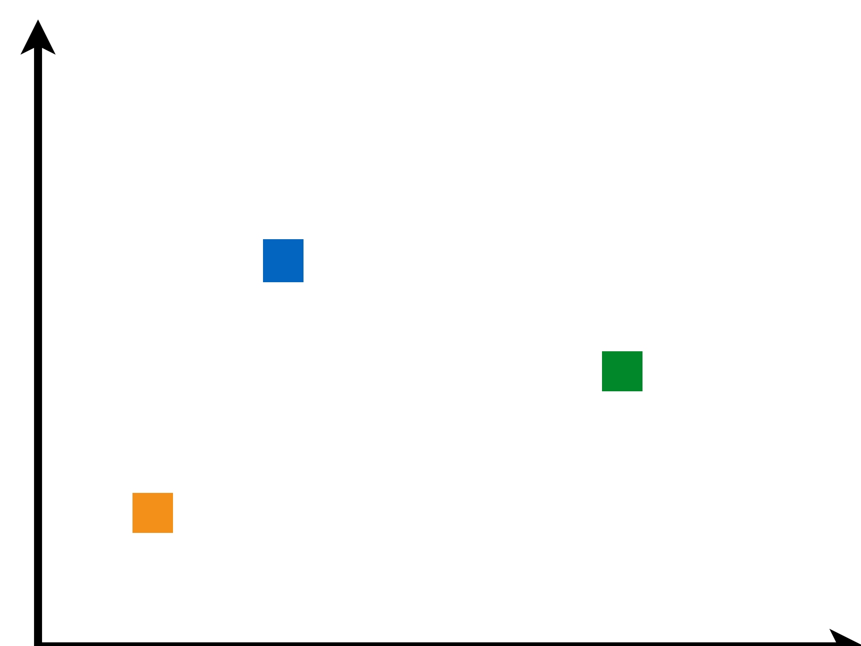

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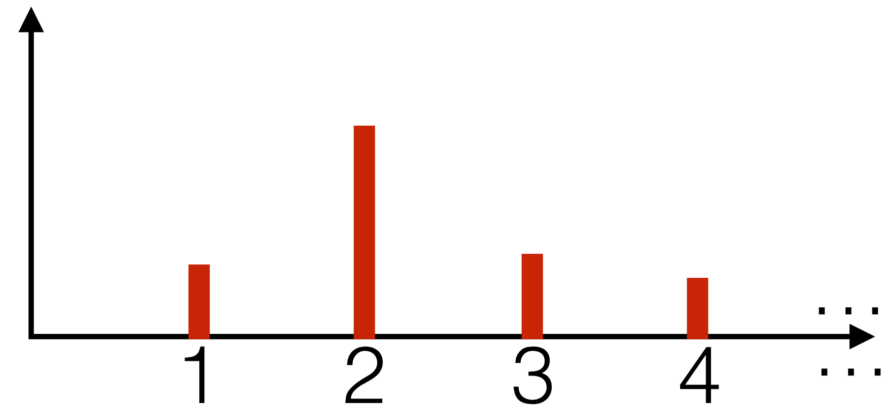

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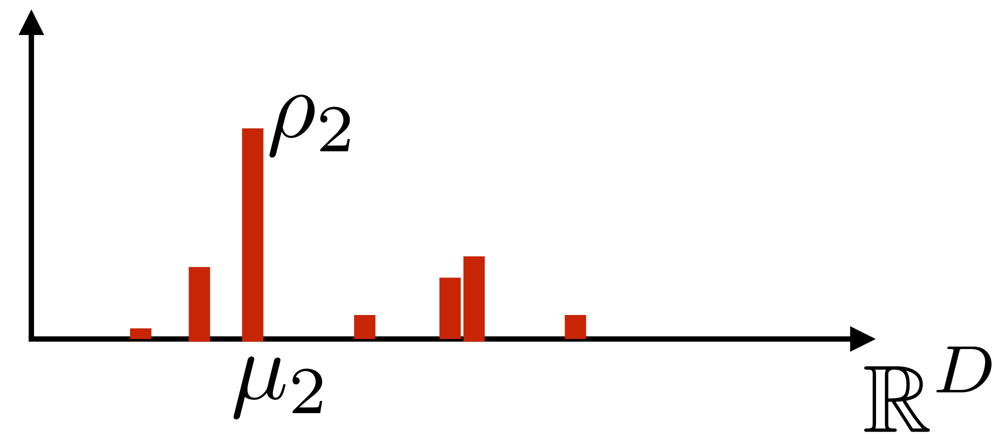

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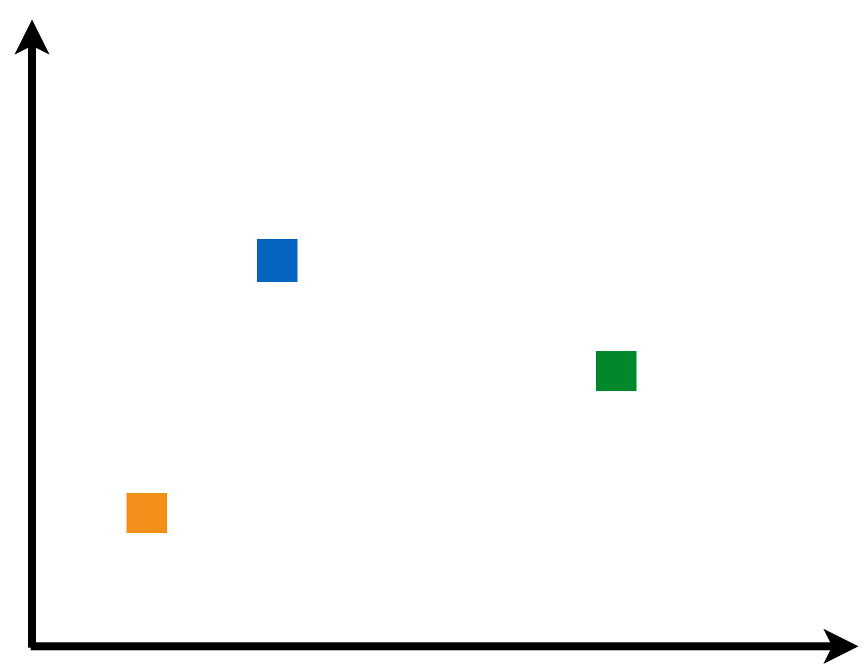

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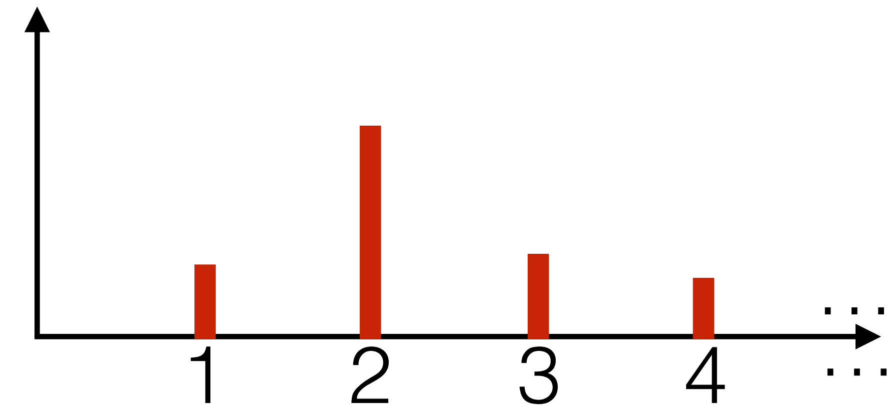

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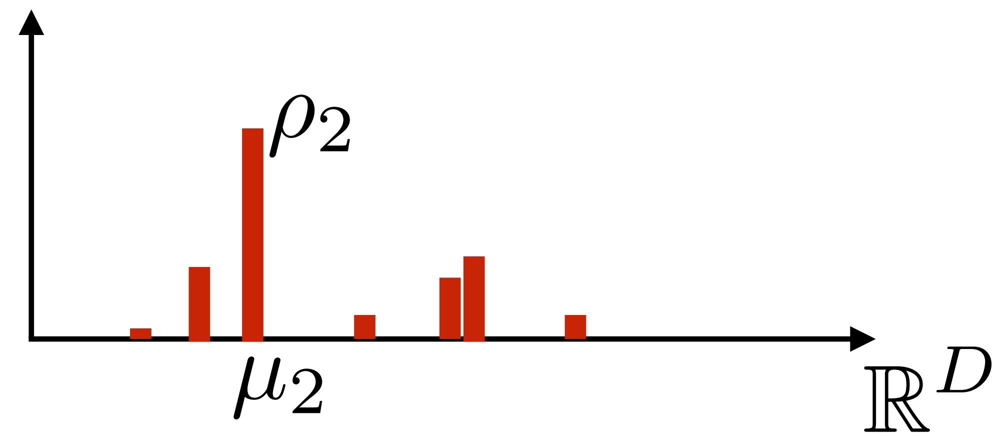

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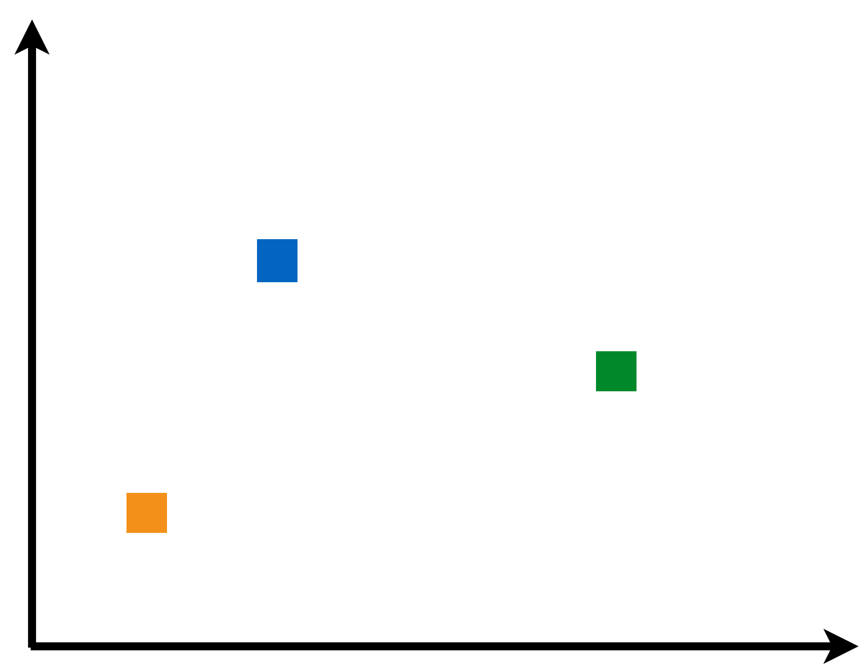

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Dirichlet process mixture model

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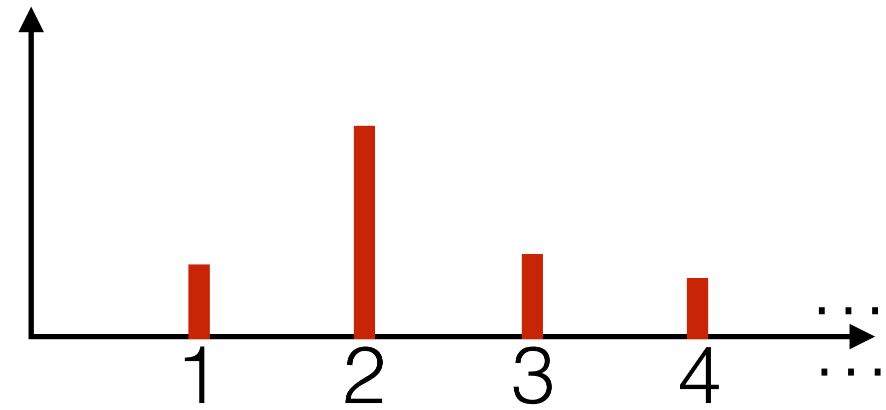

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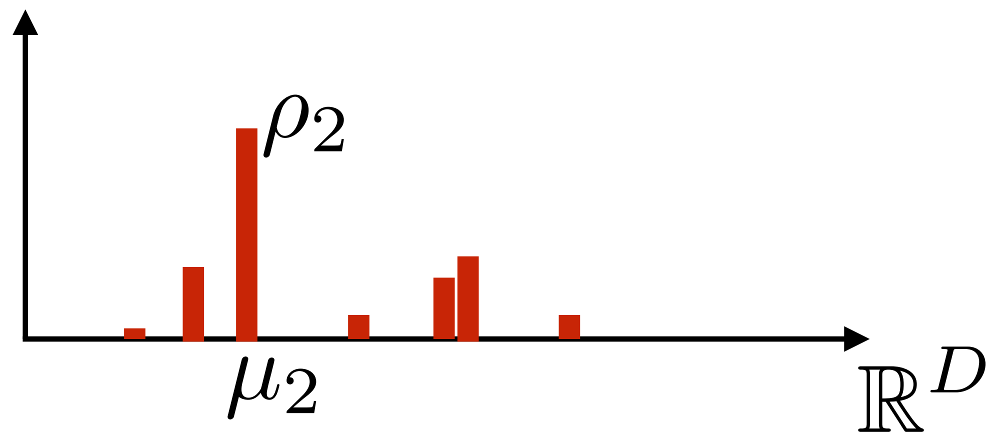

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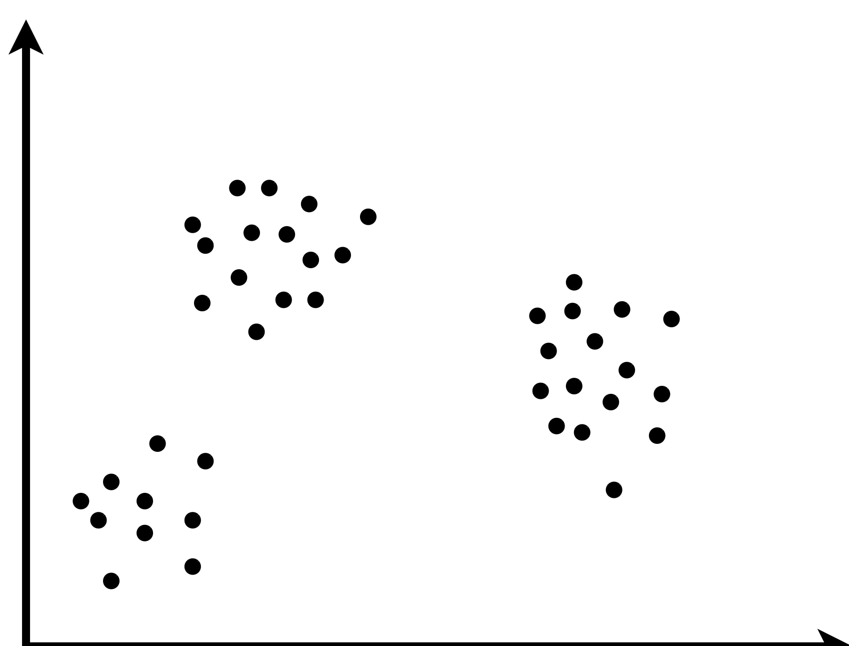

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Dirichlet process mixture model

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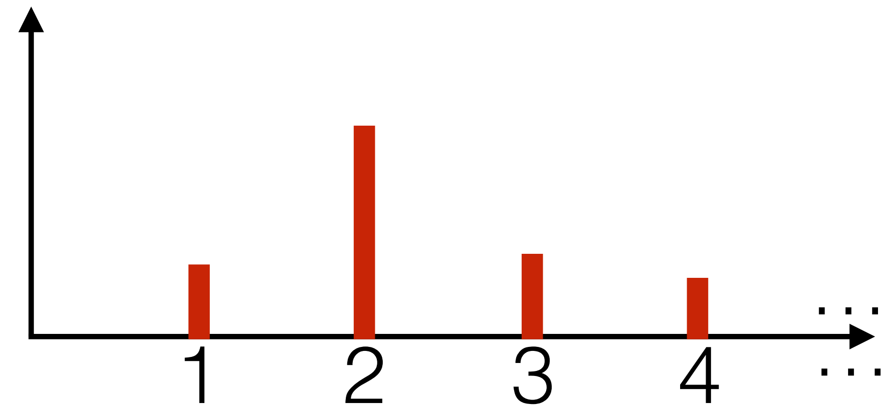

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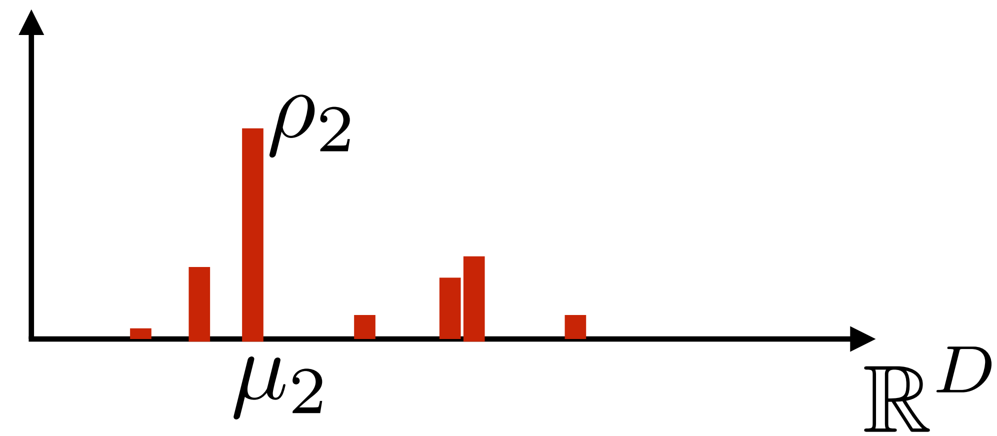

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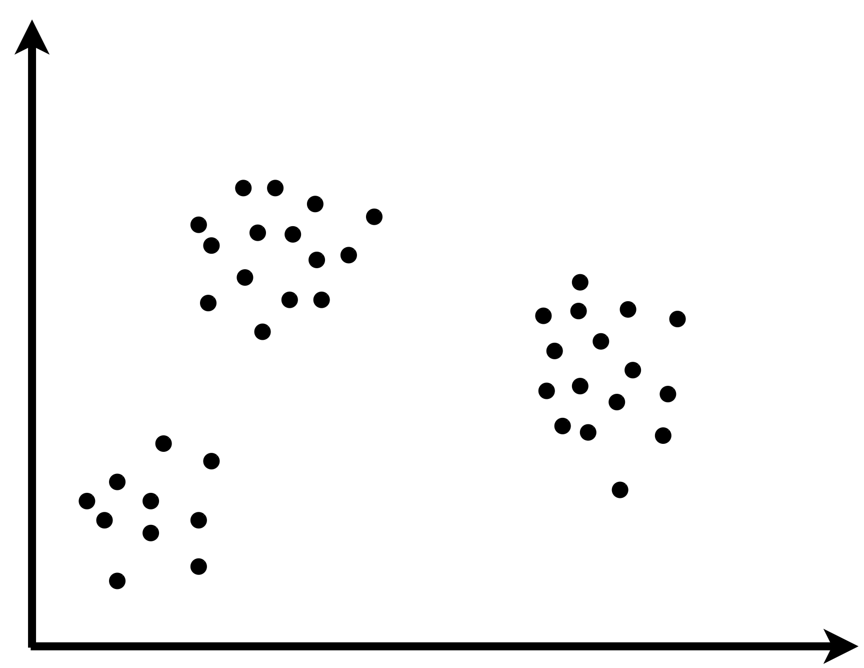

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[demo]

Dirichlet process mixture model

- More generally

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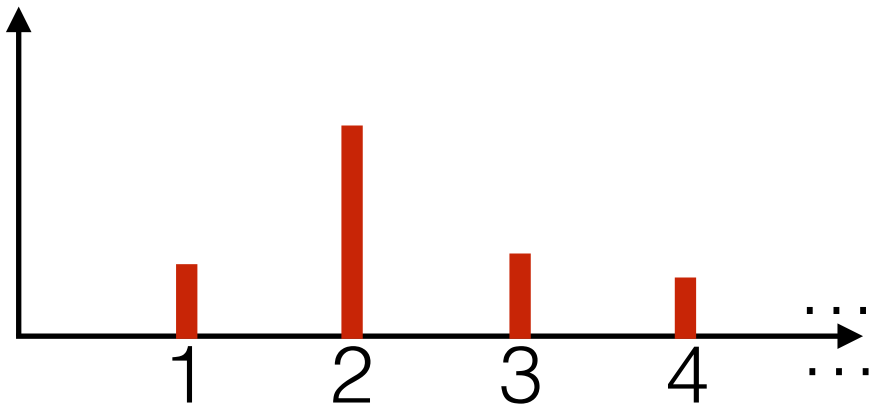

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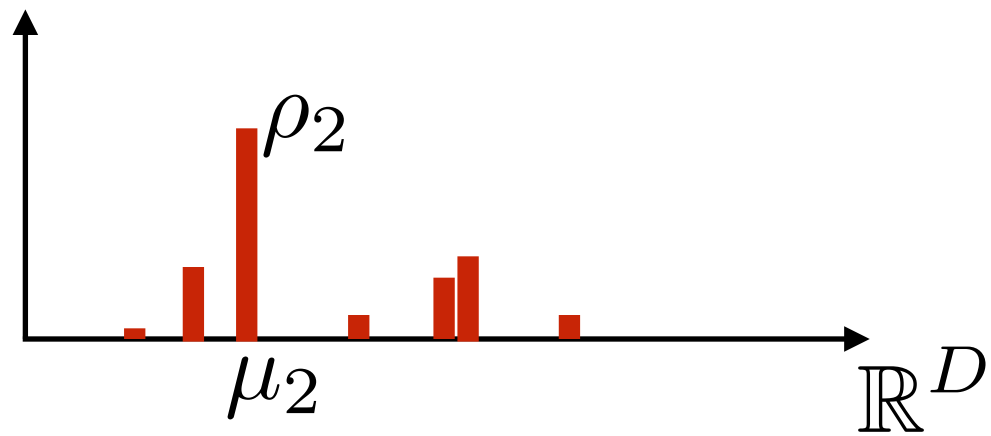

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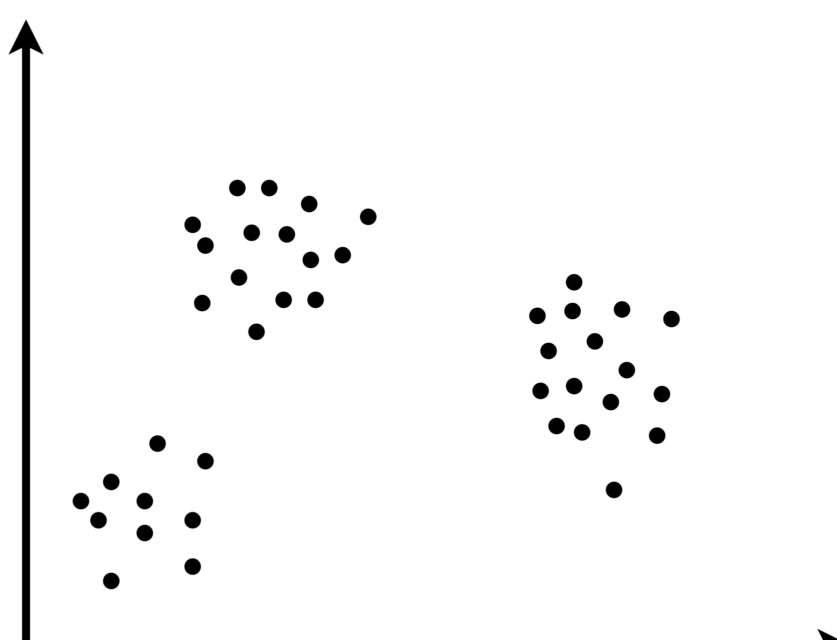

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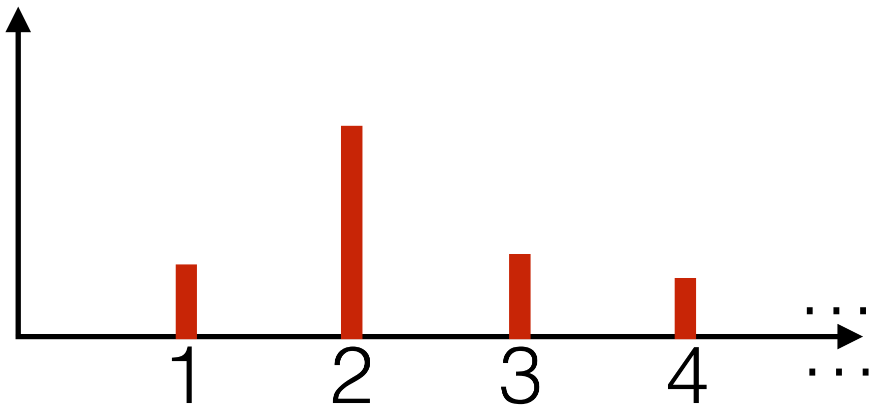

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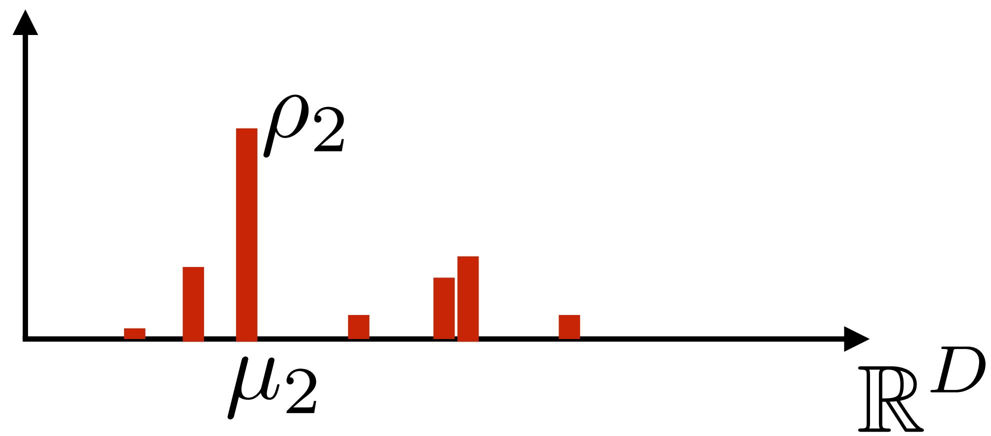

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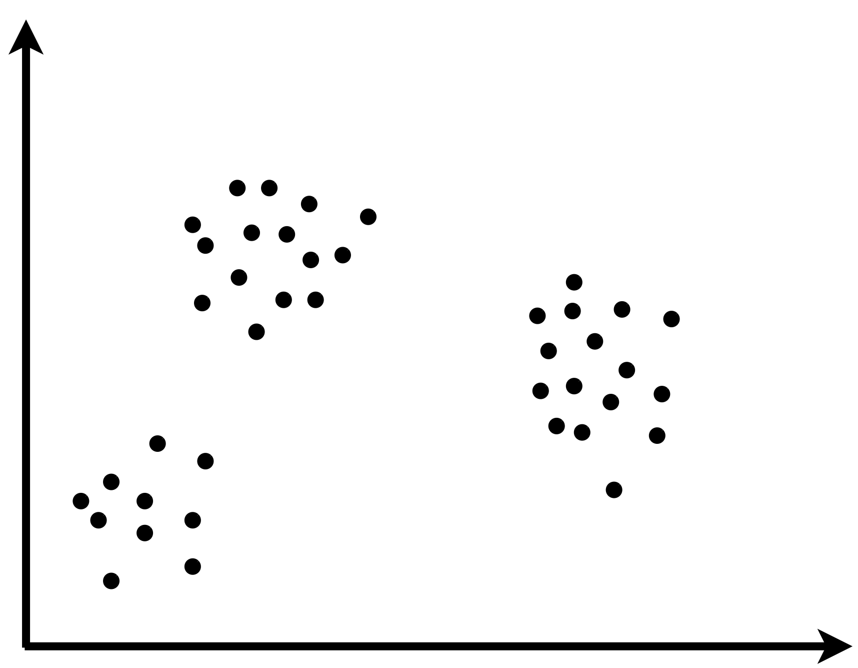
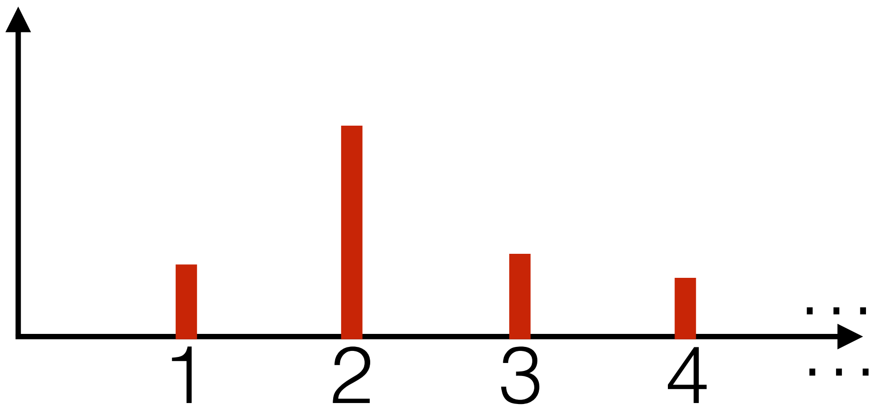


Dirichlet process mixture model

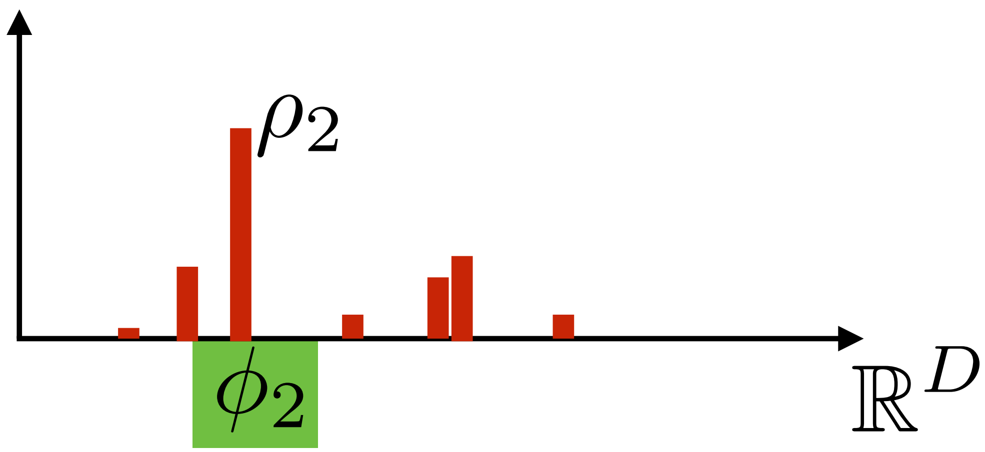
- More generally

$$\rho = (\rho_1, \rho_2, \ldots) \sim \text{GEM}(\alpha)$$

$$\phi_k \overset{iid}{\sim} G_0 \qquad k = 1, 2, \ldots$$



- i.e. $G = \sum_{k=1}^{\infty} \rho_k \delta_{\mu_k} \overset{d}{=} \text{DP}(\alpha, \mathcal{N}(\mu_0, \Sigma_0))$



$$z_n \overset{iid}{\sim} \text{Categorical}(\rho)$$

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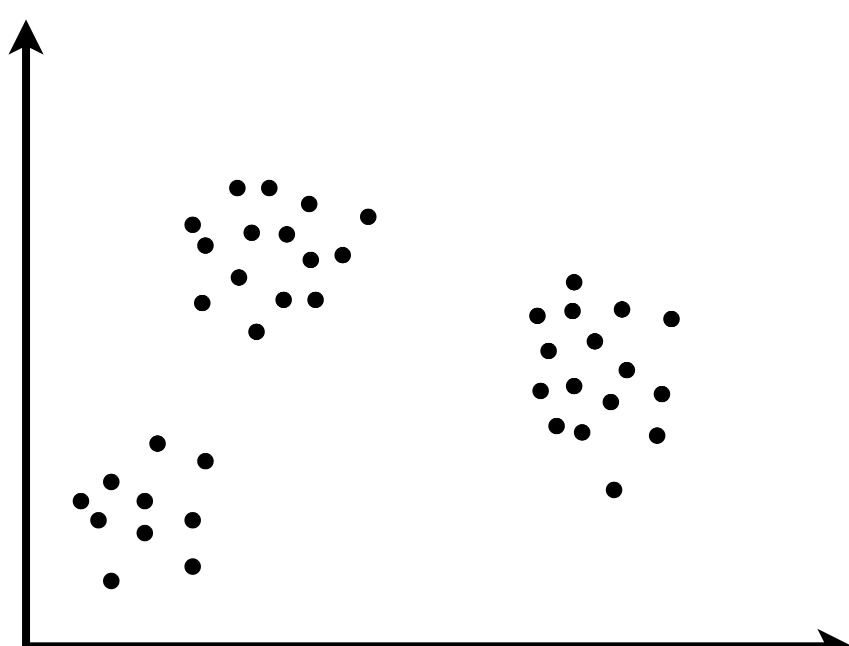

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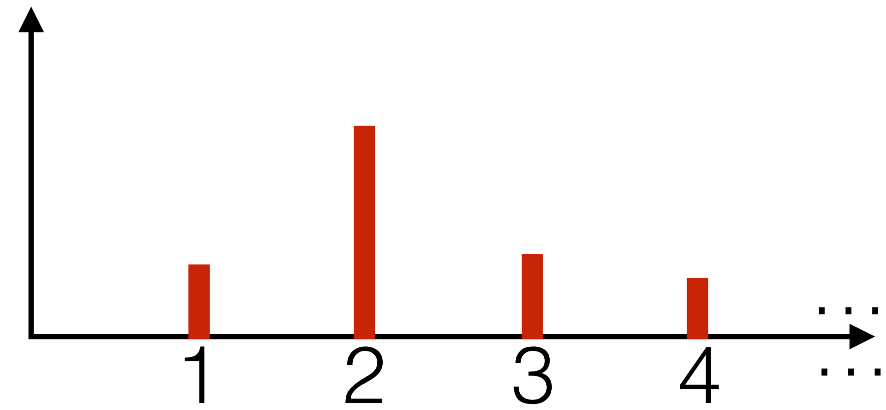

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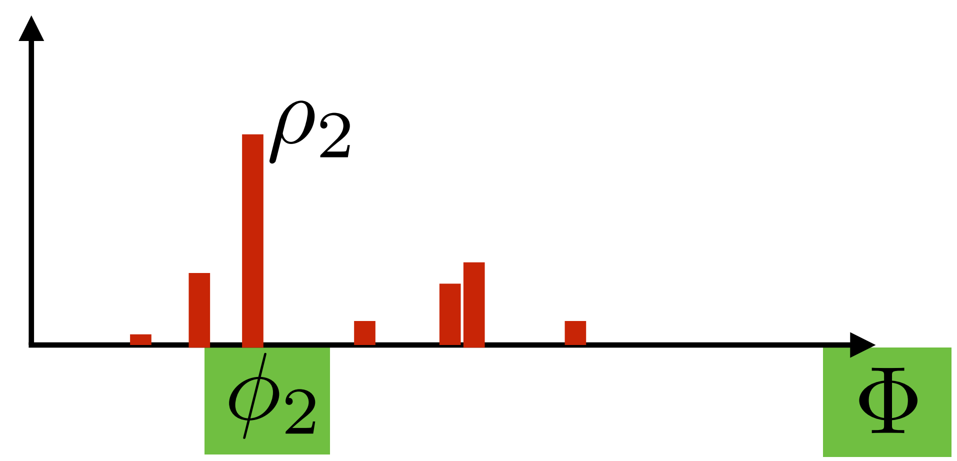

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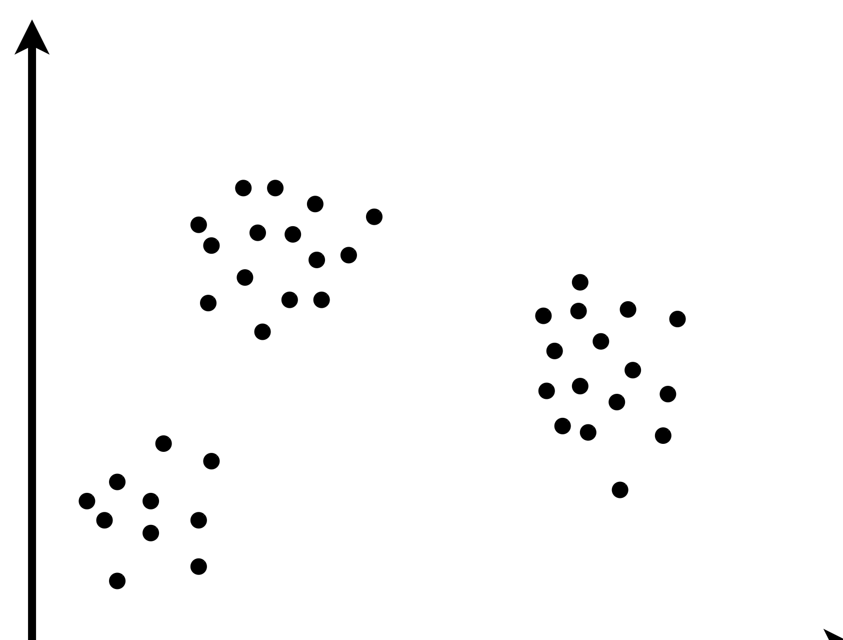

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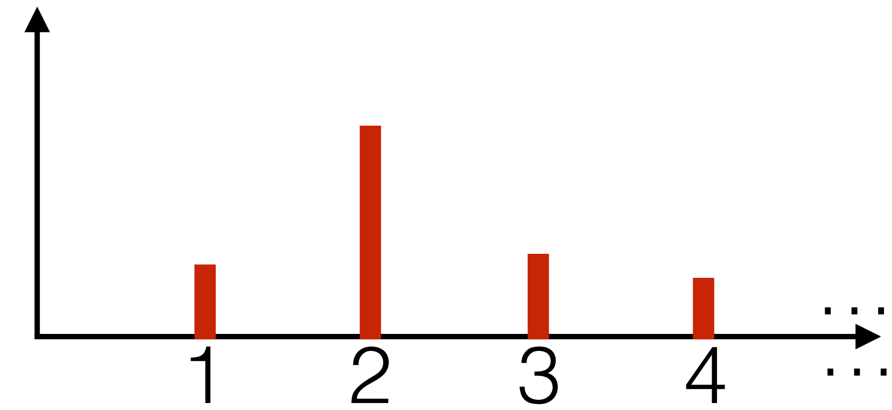

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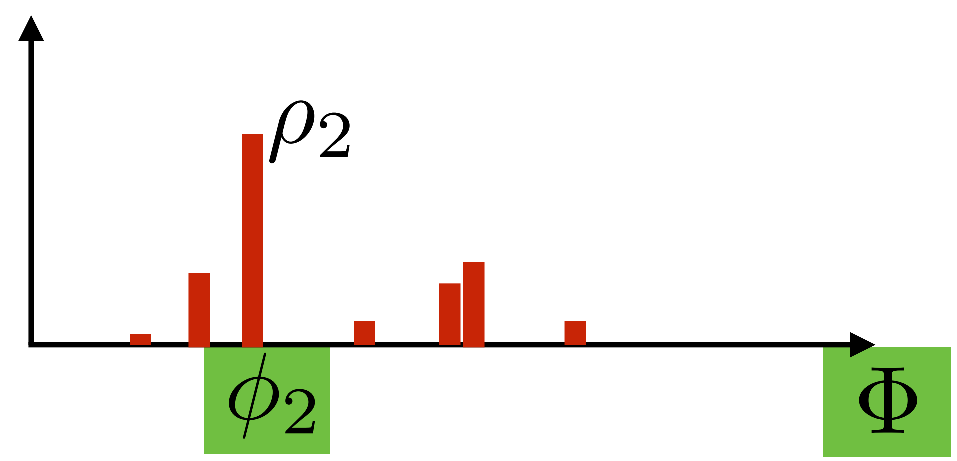

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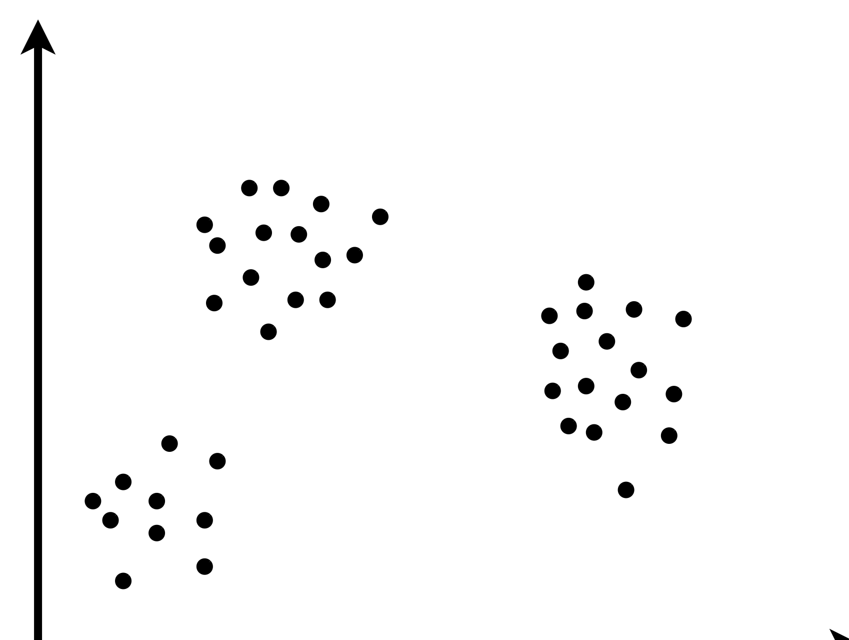

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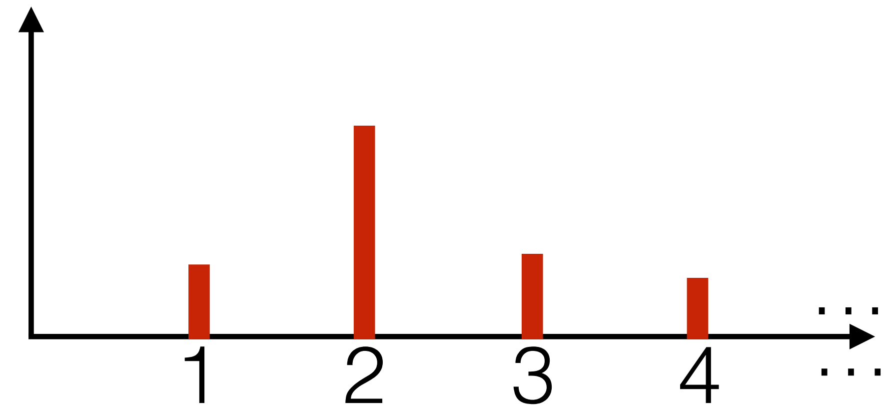

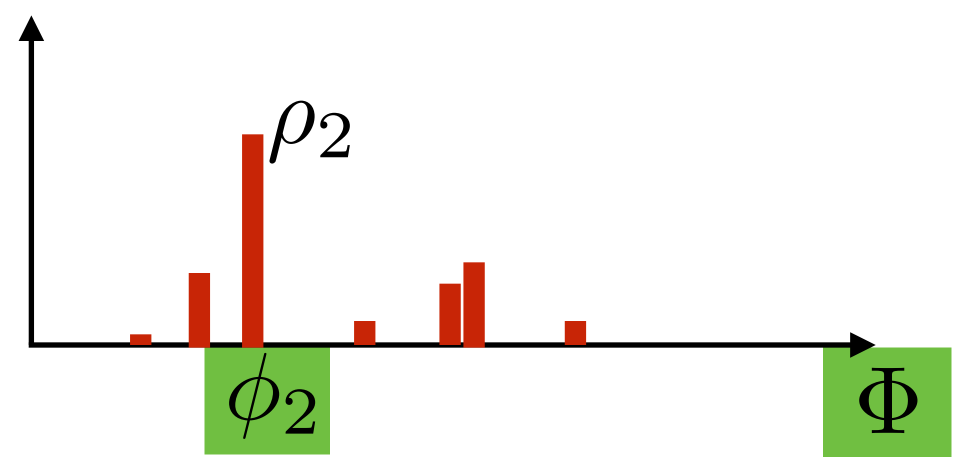

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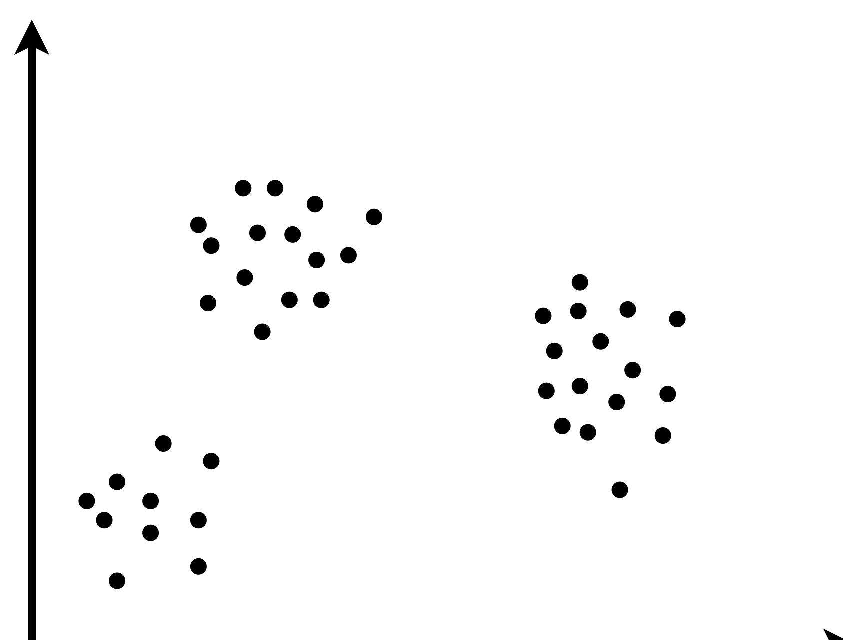

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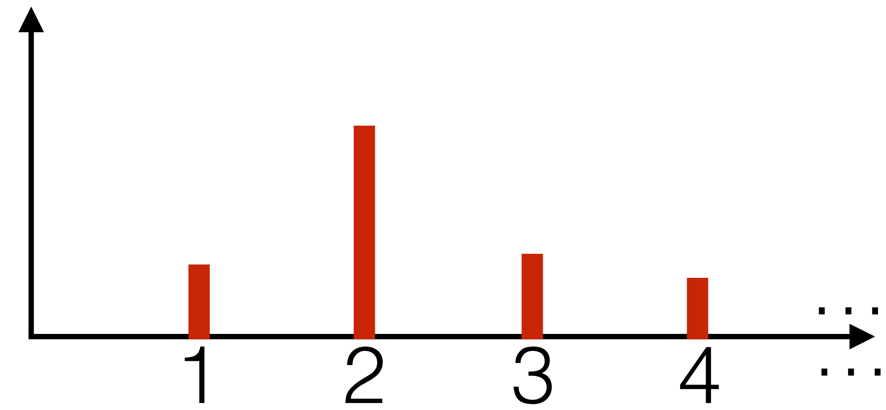

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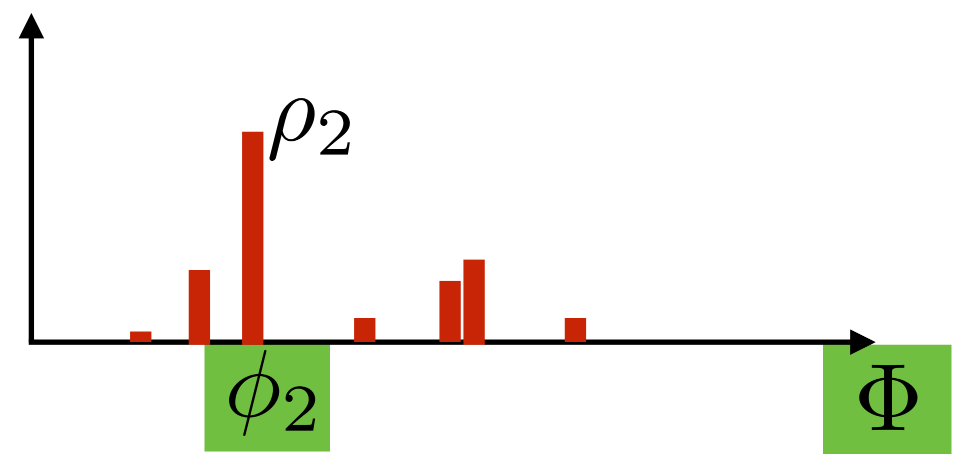

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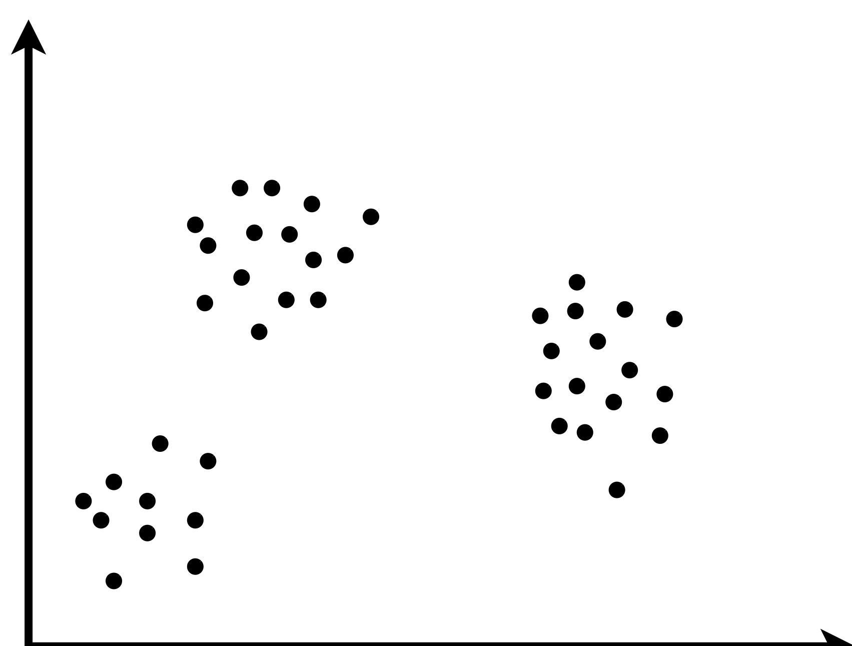

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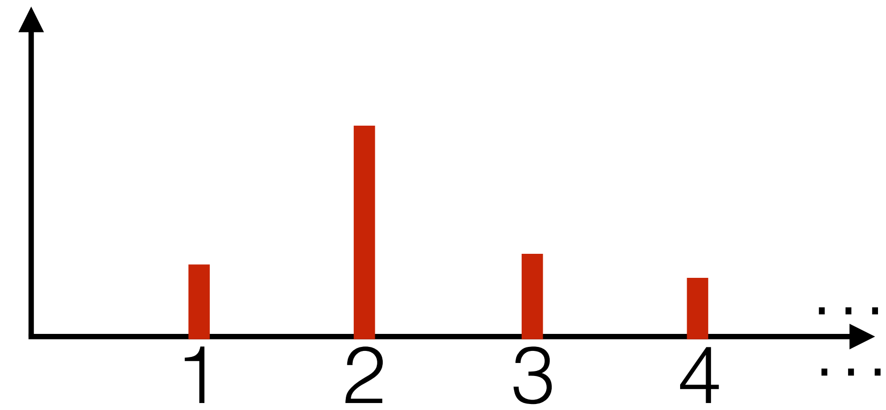

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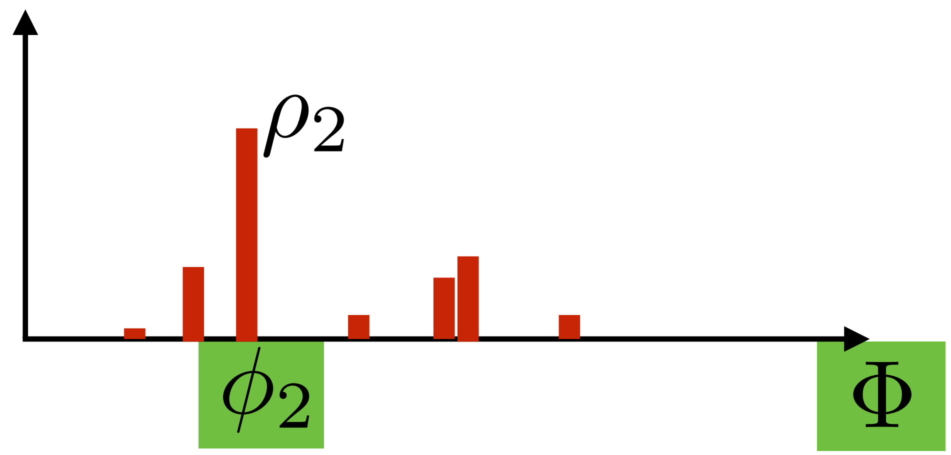

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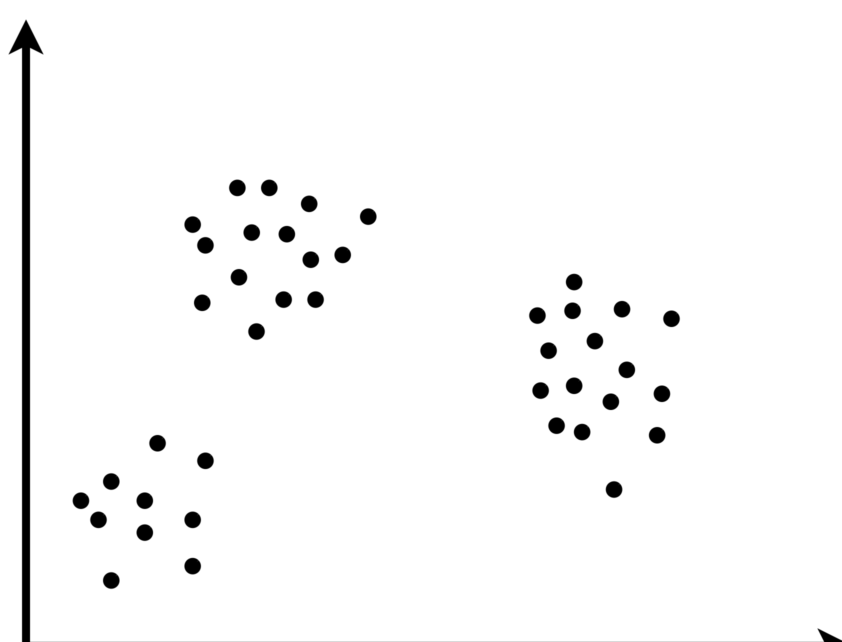
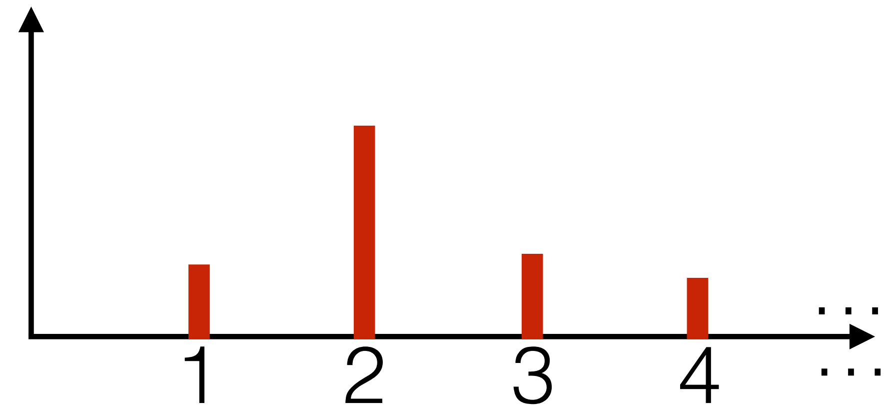


Dirichlet process mixture model

- More generally



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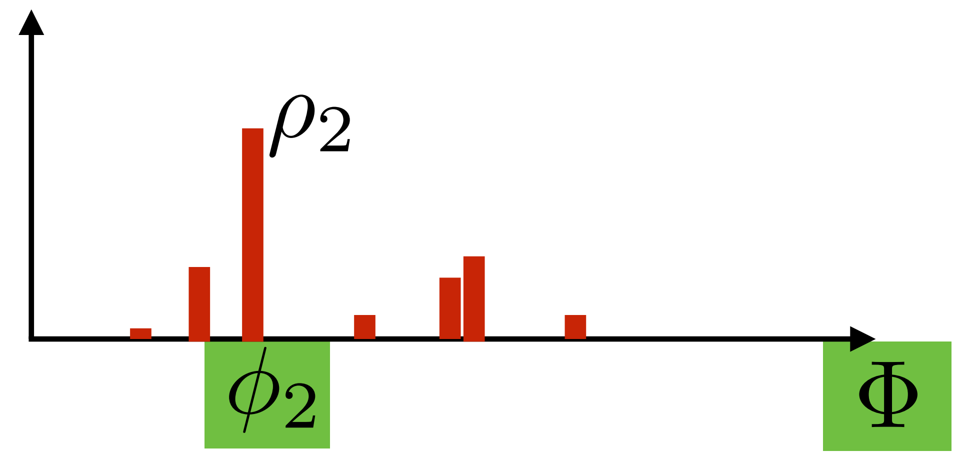

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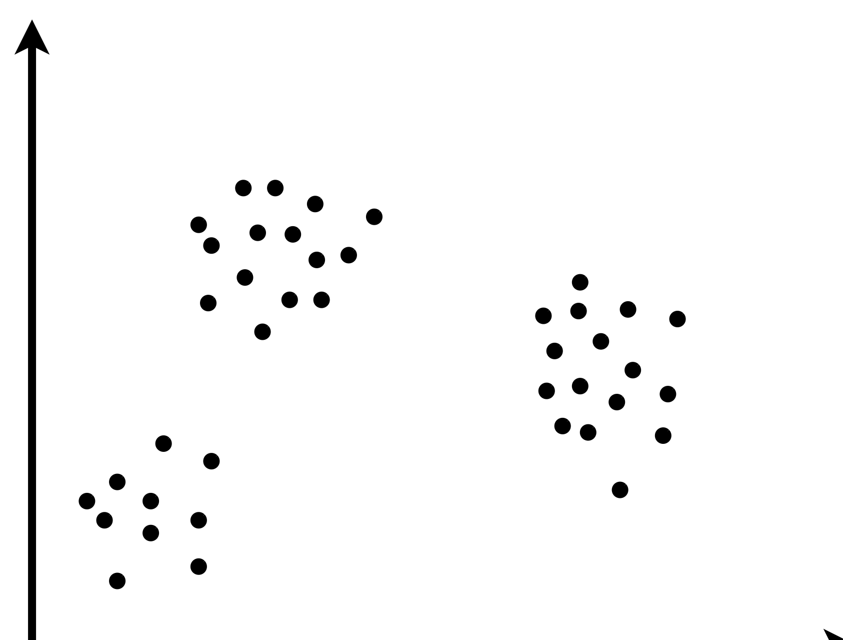

$$x_n \overset{indep}{\sim} F(\theta_n)$$

Dirichlet process mixture model

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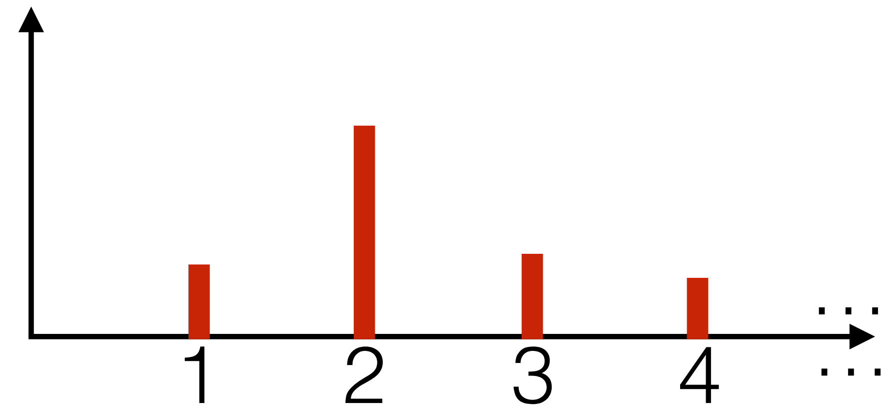

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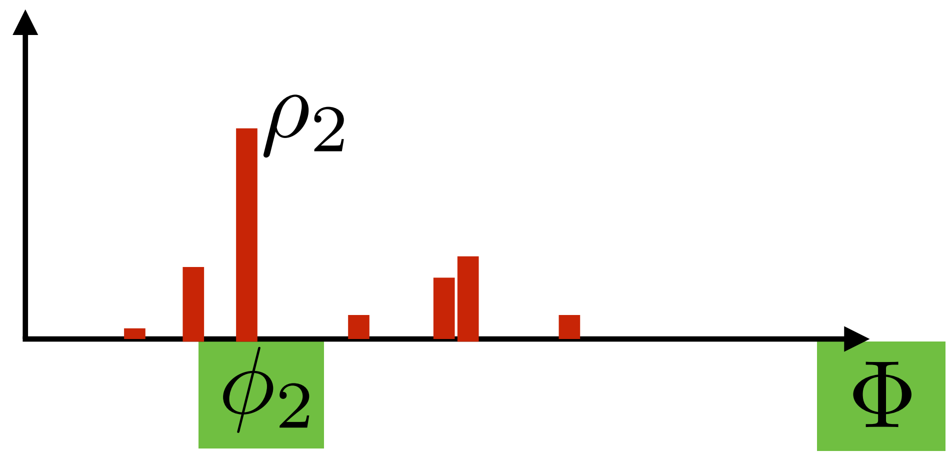

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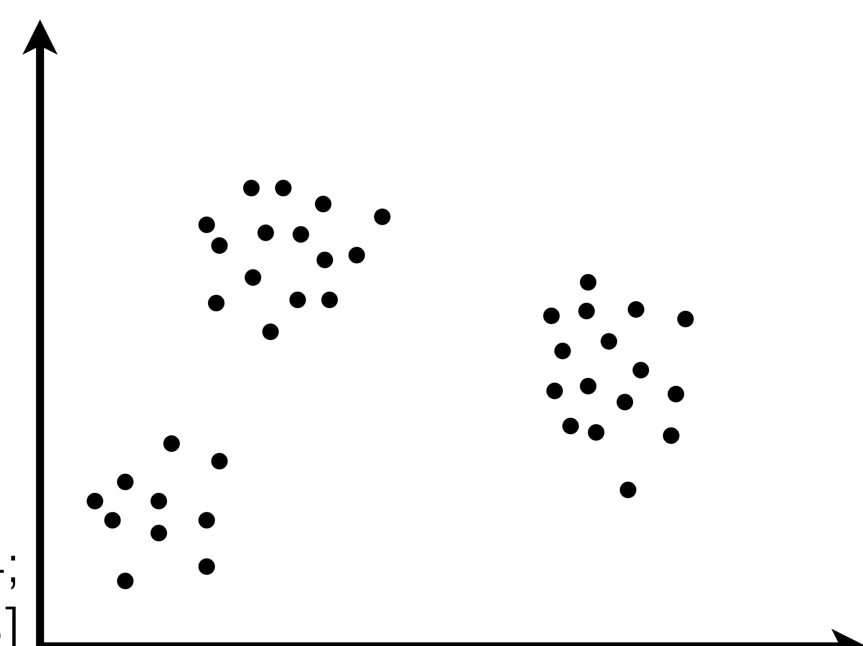
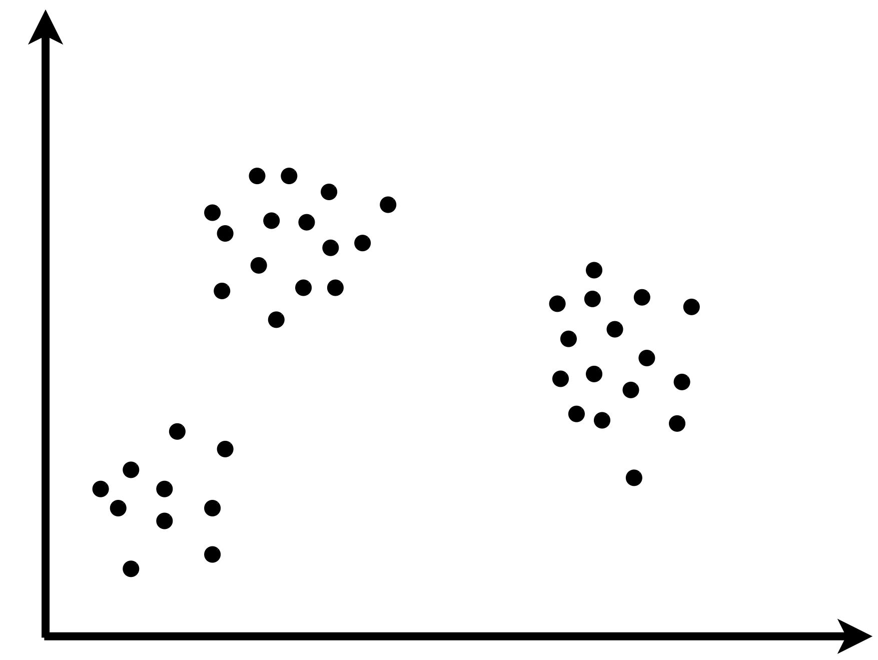


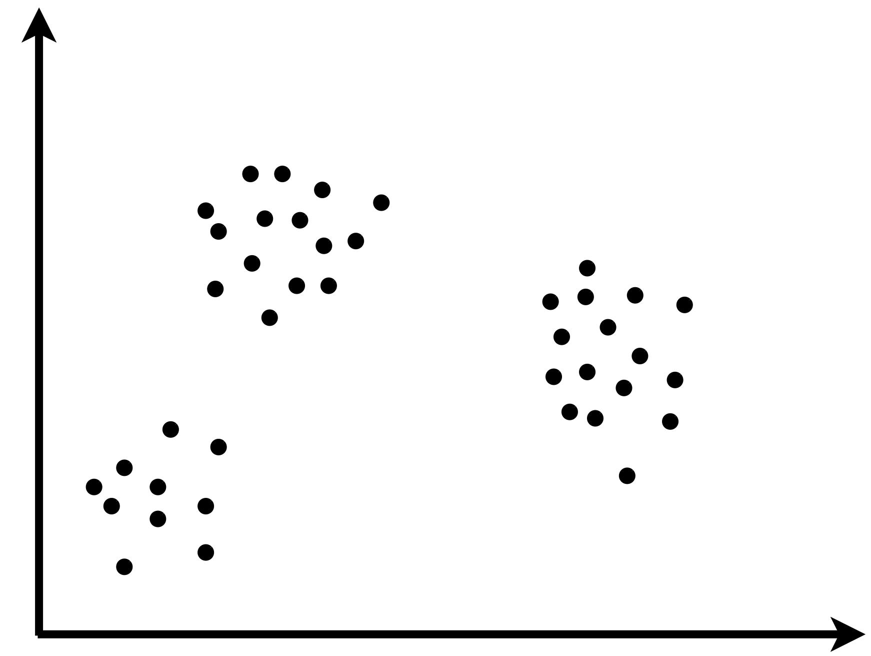
[Antoniak 1974; Ferguson 1983; West, Müller, Escobar 1994; Escobar, West 1995; MacEachern, Müller 1998]

DP or not DP, that is the question



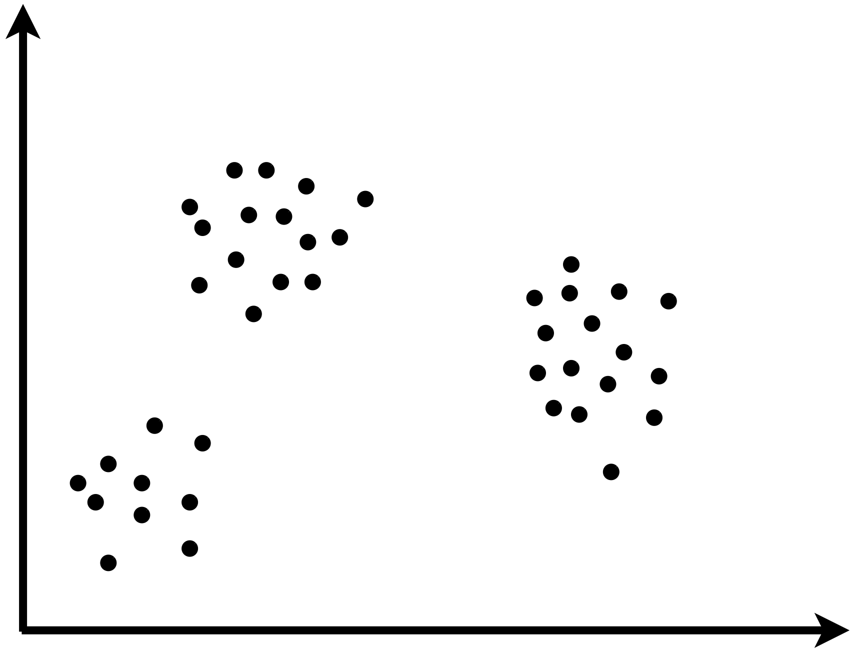
DP or not DP, that is the question

- GEM: 



DP or not DP, that is the question

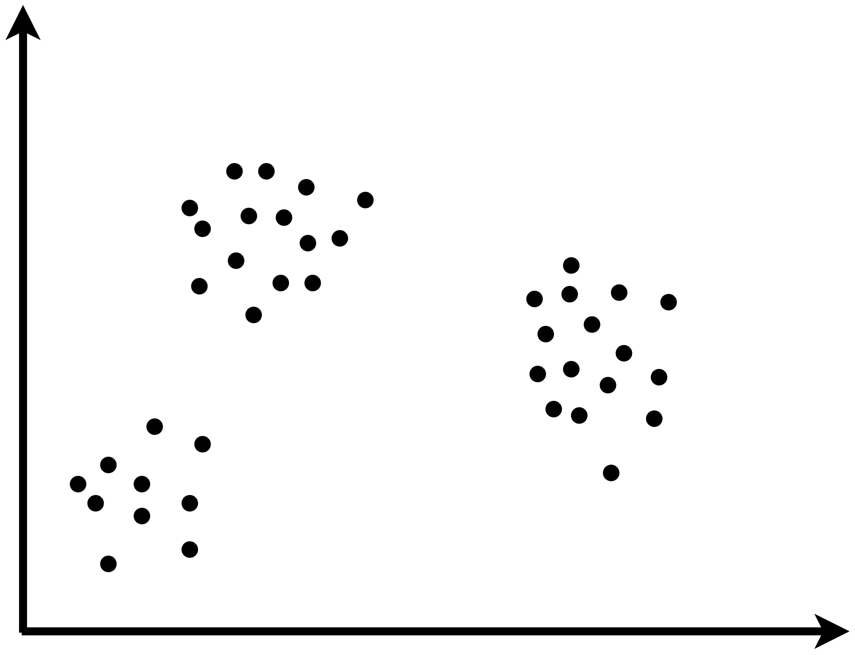
- GEM: 
- Compare to:



DP or not DP, that is the question

- GEM: 

- Compare to:
  - Finite (small *K*) mixture model



DP or not DP, that is the question

- GEM: 
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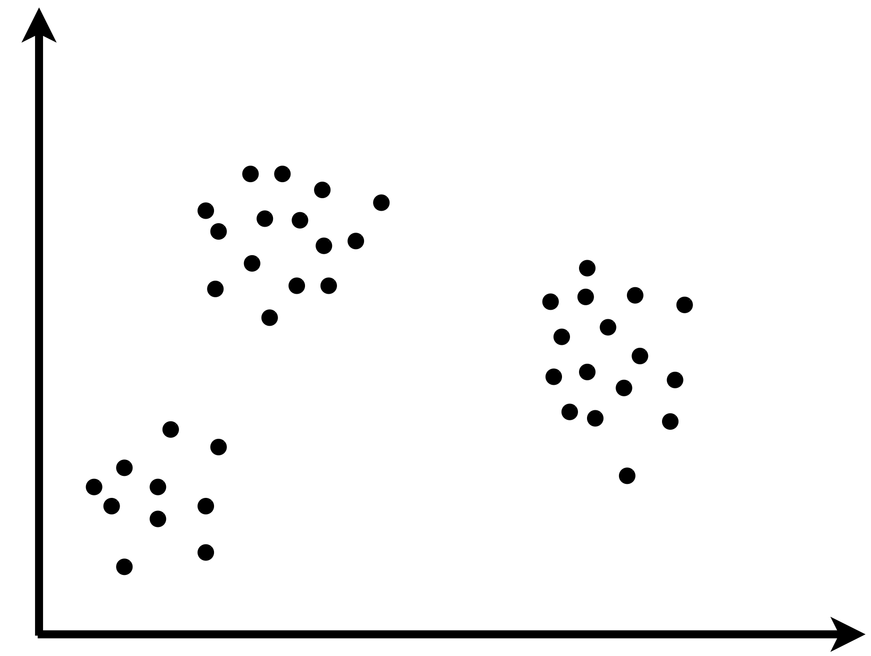

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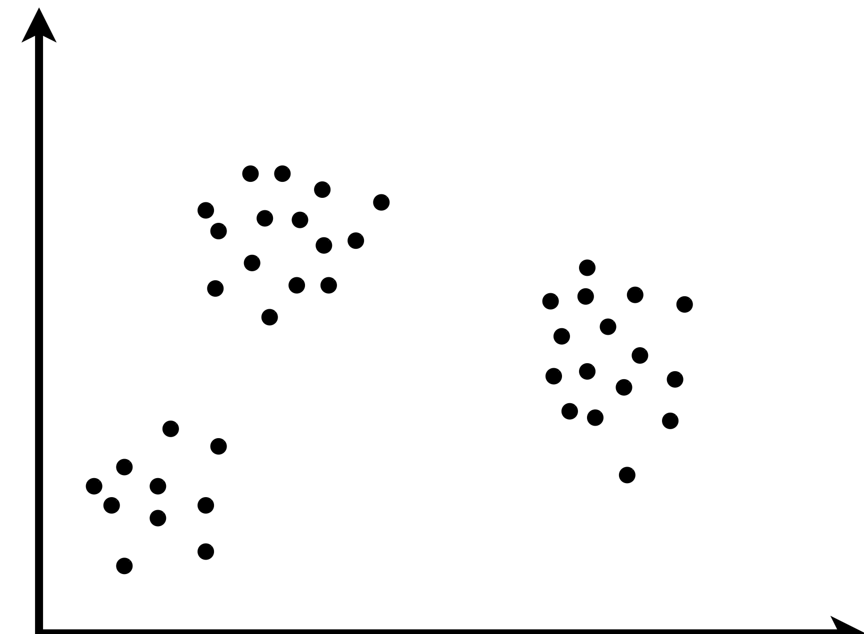
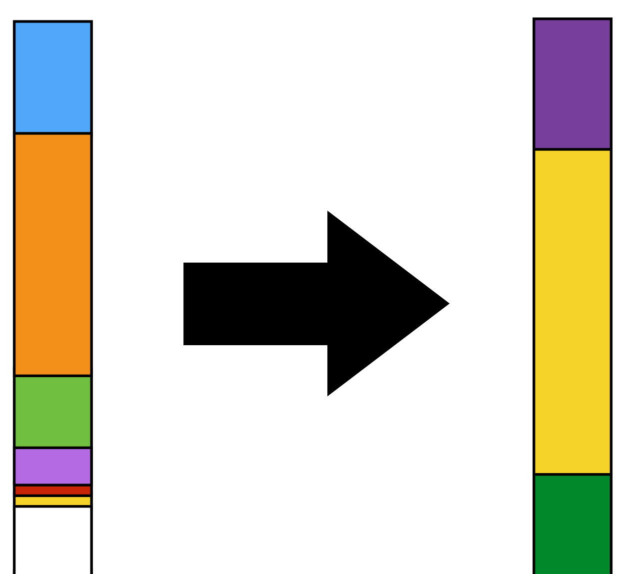


- Compare to:
  - Finite (small *K*) mixture model

  - Finite (large *K*) mixture model

  - Time series



Roadmap

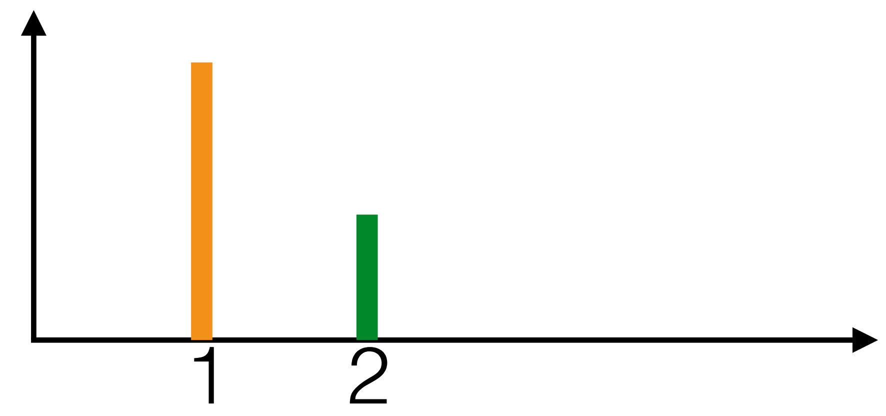
- Example problem: clustering
- Example NPBayes model: Dirichlet process
- Chinese restaurant process
- Inference
- Venture further into the wild world of Nonparametric Bayes

- Big questions
  - Why NPBayes? Learn more as acquire more data
  - What does an infinite/growing number of parameters really mean (in NPBayes)? Components vs. clusters; latent vs. realized
  - Why is NPBayes challenging but practical? Infinite dimensional parameter; more on this today!

Marginal cluster assignments

Marginal cluster assignments

$$\rho_1 \sim \mathrm{Beta}(a_1, a_2), z_n \overset{iid}{\sim} \mathrm{Cat}(\rho_1, \rho_2)$$



Marginal cluster assignments

- Integrate out the frequencies

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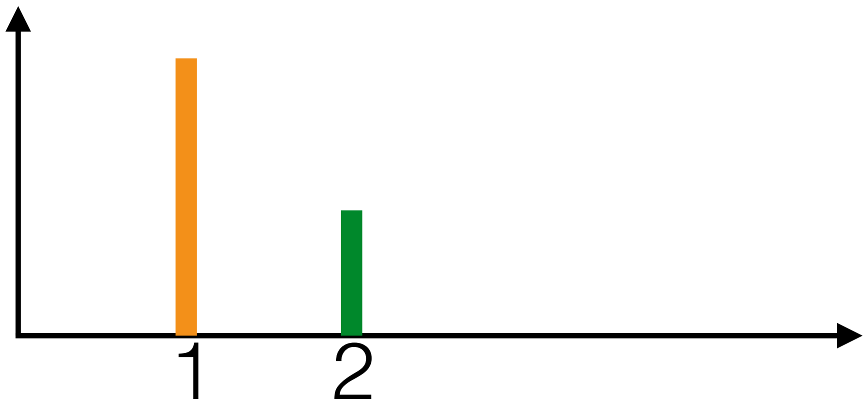

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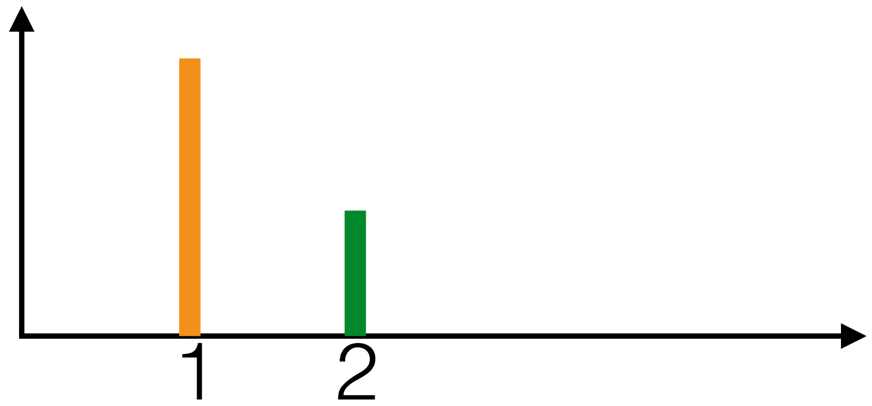

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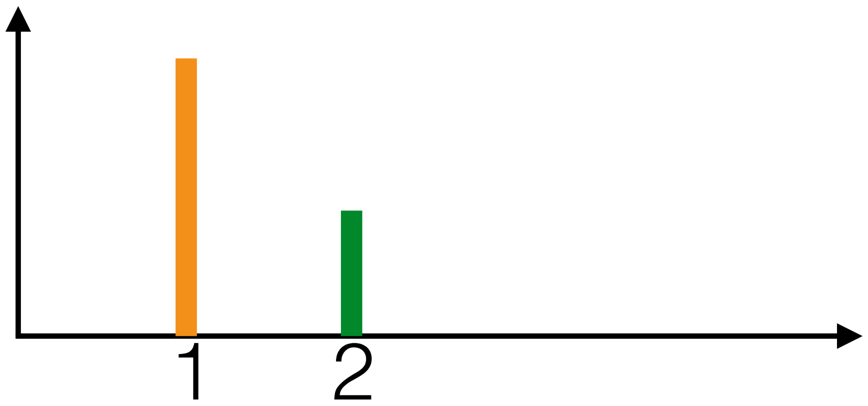

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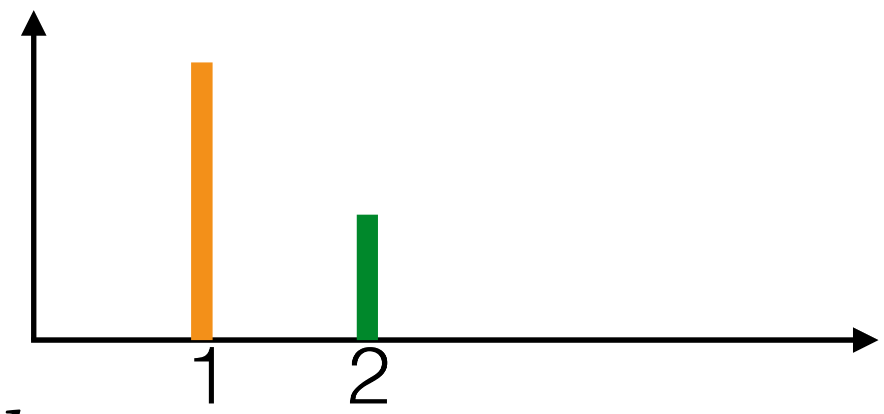

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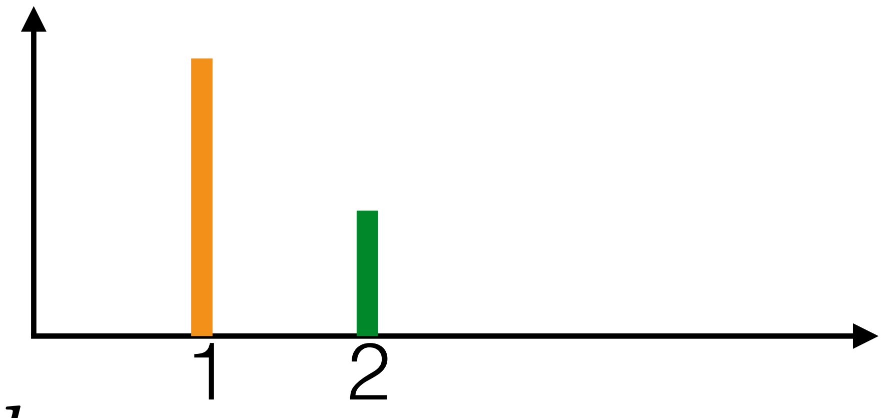

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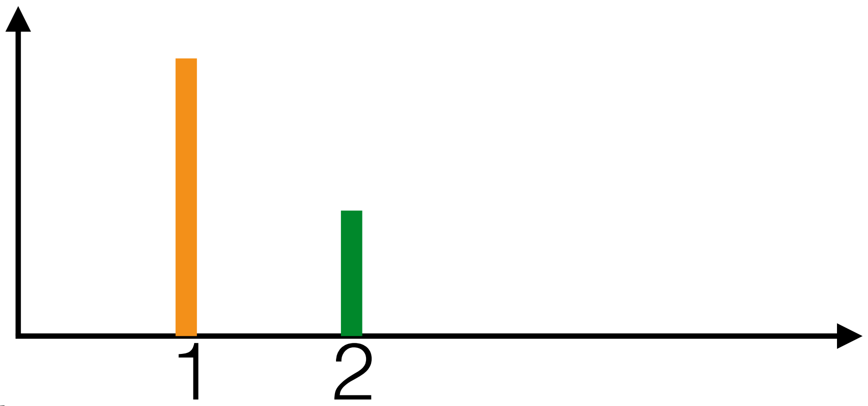

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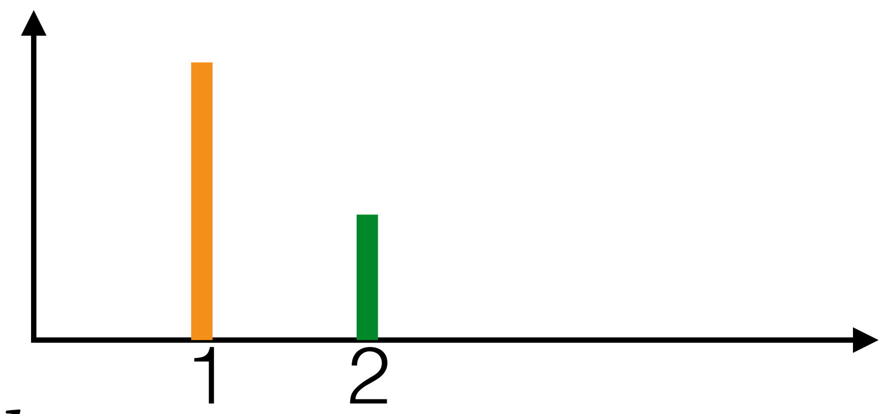

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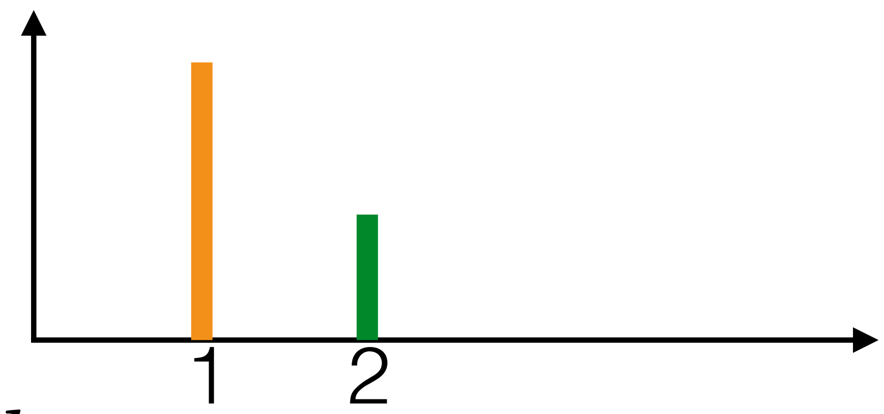

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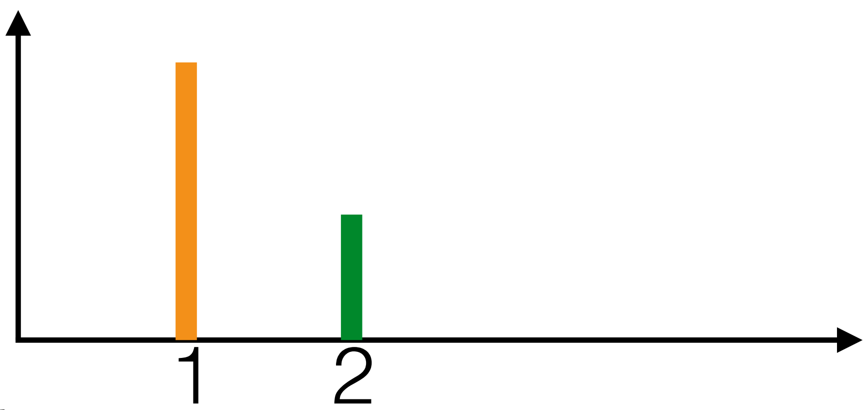

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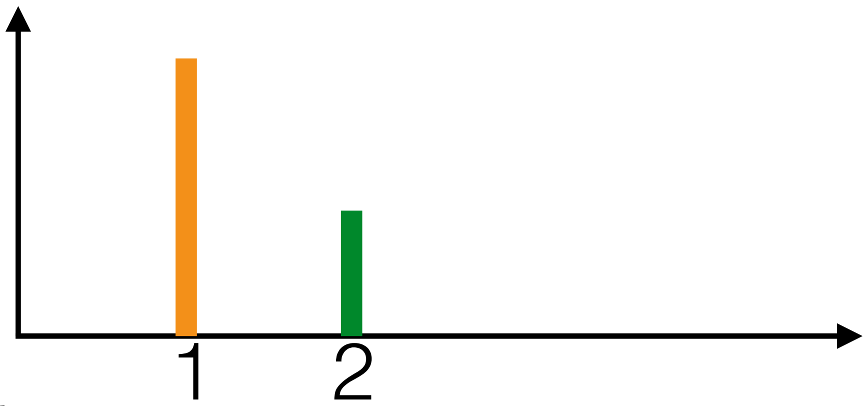

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$$a_{1,n} := a_1 + \sum_{m=1}^{n-1} \mathbf{1}\{z_m = 1\}, a_{2,n} = a_2 + \sum_{m=1}^{n-1} \mathbf{1}\{z_m = 2\}$$

Marginal cluster assignments

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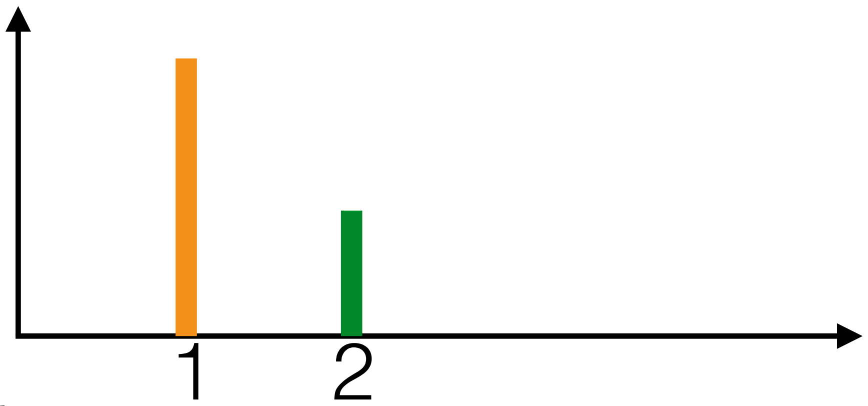

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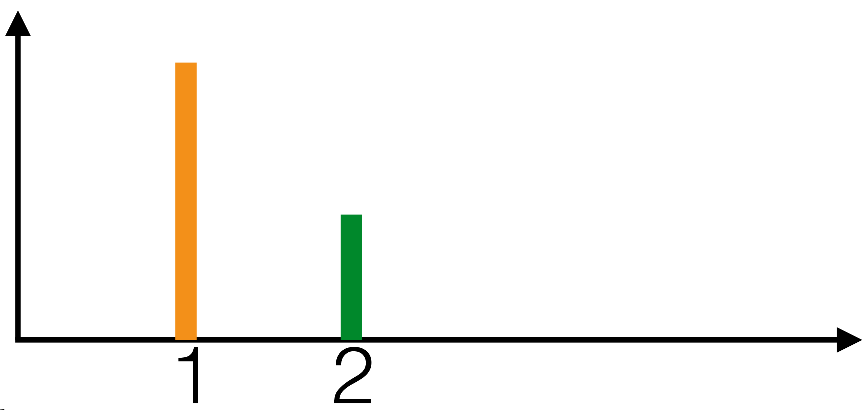
$$= \int \rho_1 \frac{\Gamma(a_{1,n} + a_{2,n})}{\Gamma(a_{1,n})\Gamma(a_{2,n})} \rho_1^{a_{1,n}-1} (1 - \rho_1)^{a_{2,n}-1} d\rho_1$$

Marginal cluster assignments

- Integrate out the frequencies



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Marginal cluster assignments

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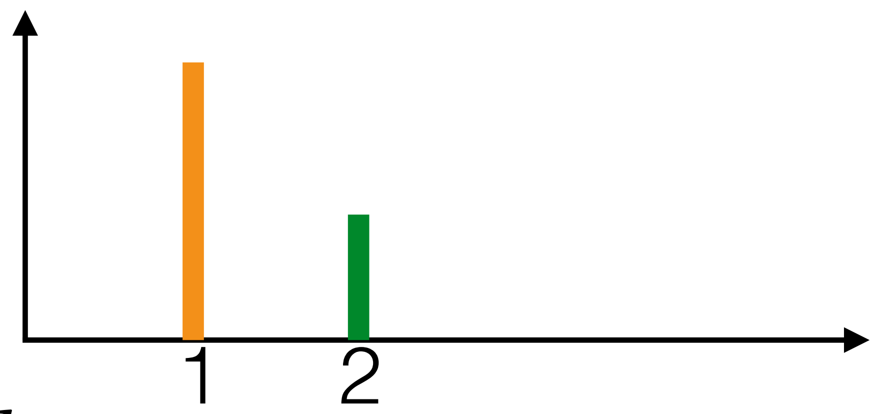

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Recall
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Marginal cluster assignments

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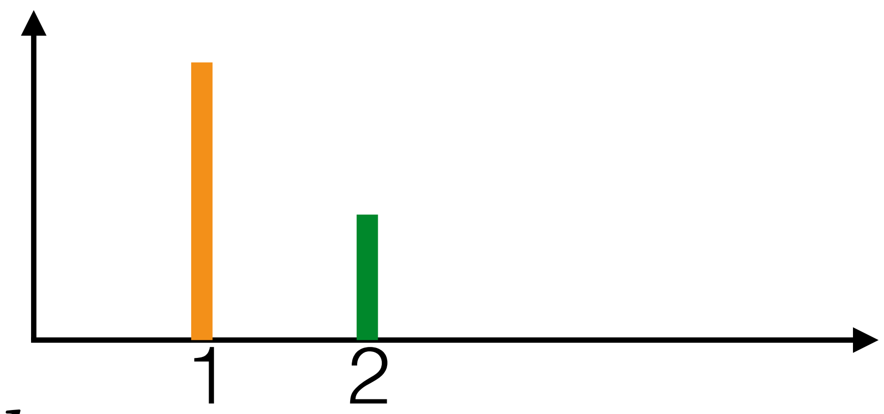

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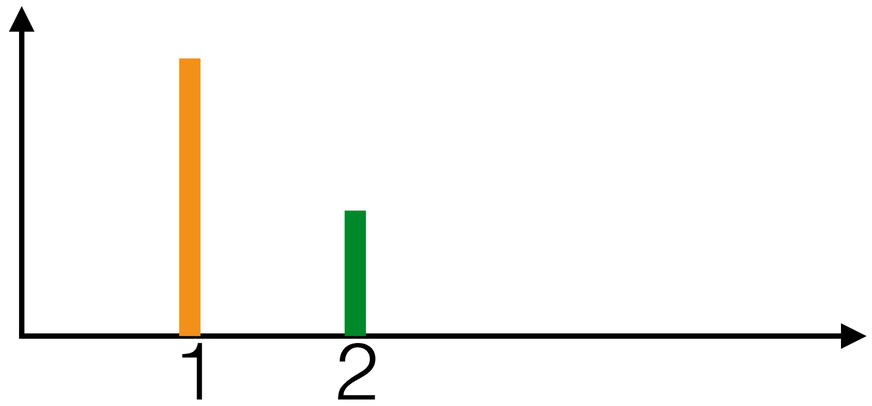

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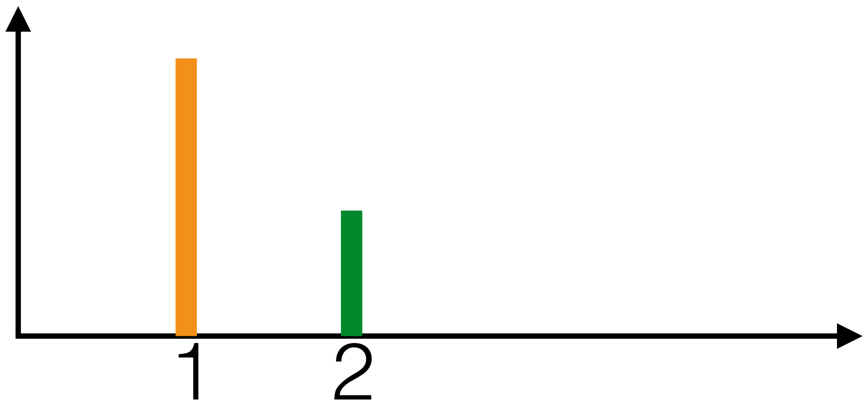

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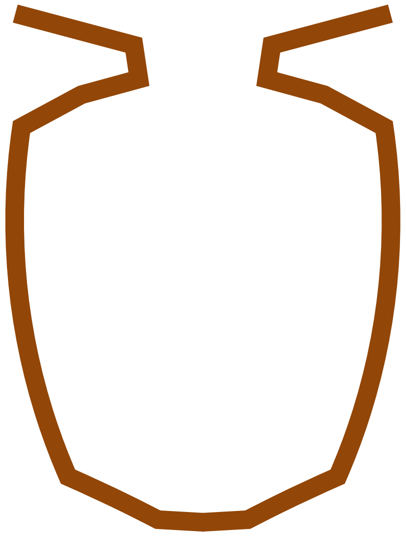

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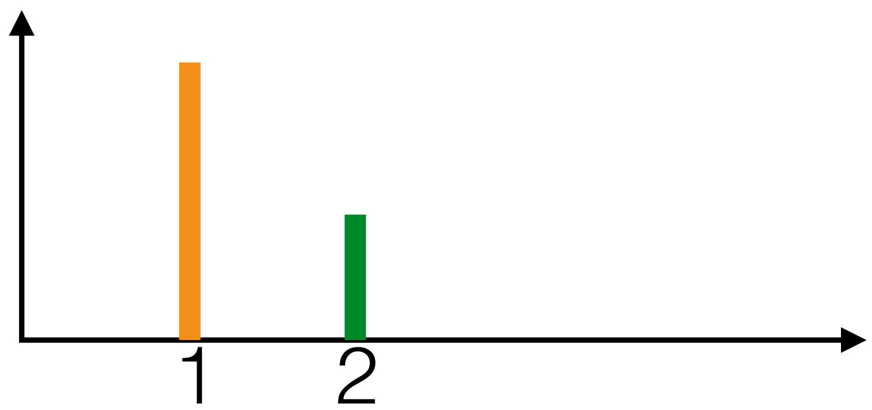

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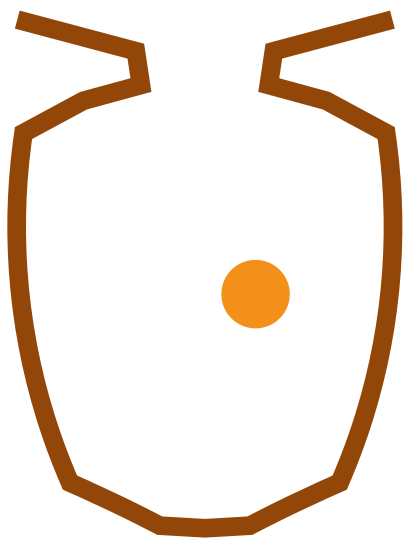

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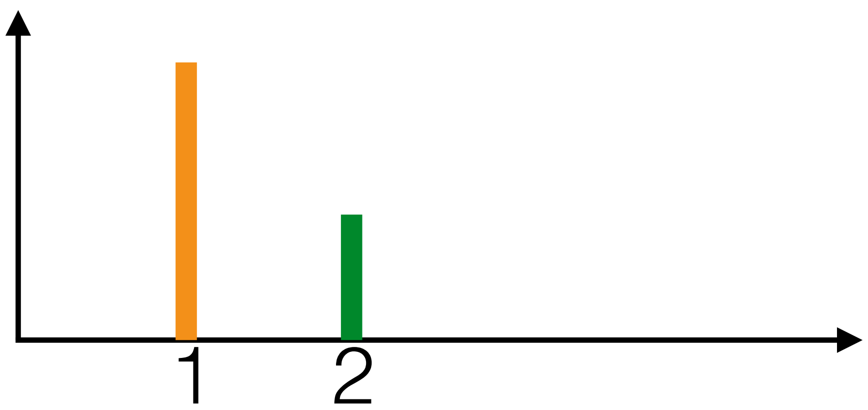

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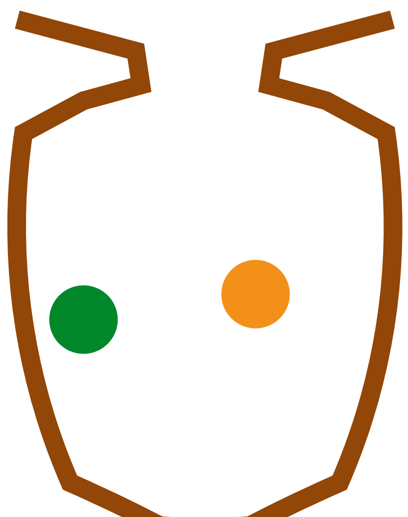

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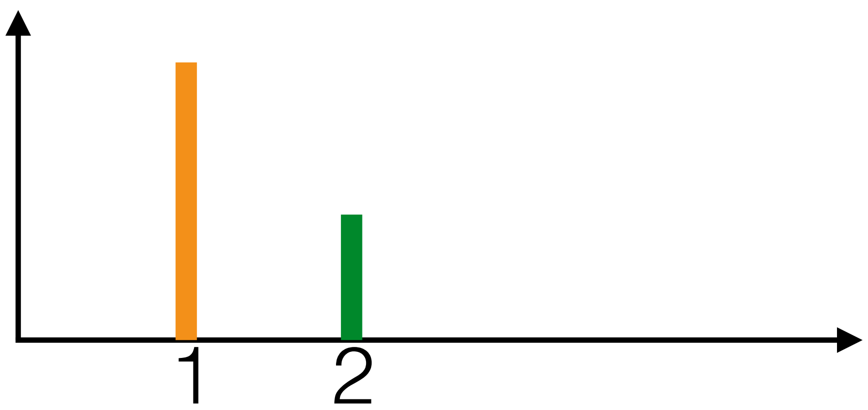

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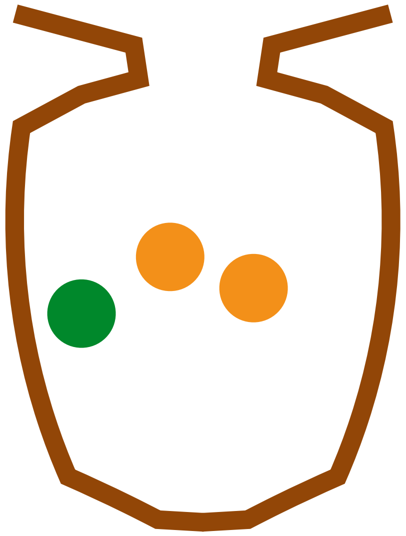

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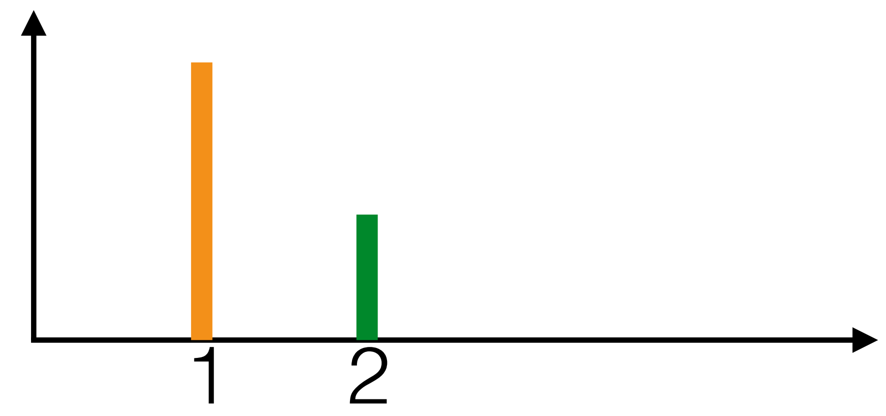

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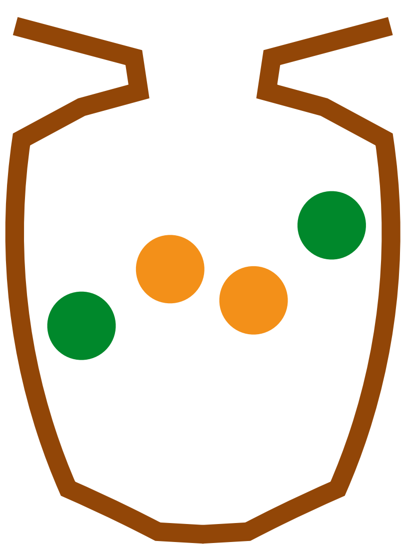

Marginal cluster assignments

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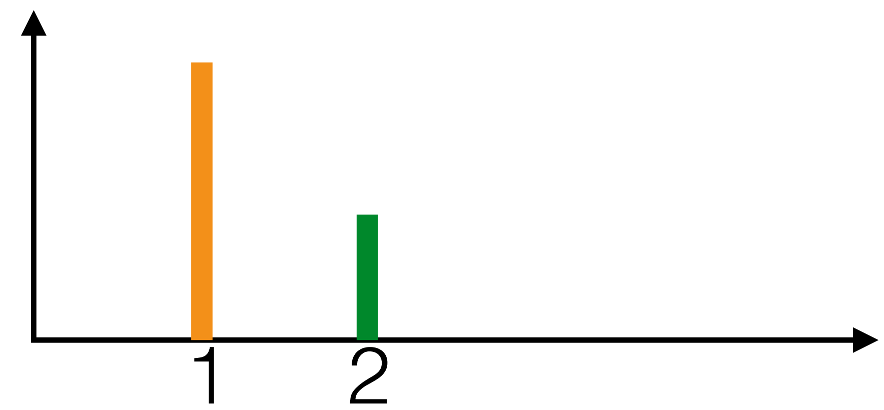

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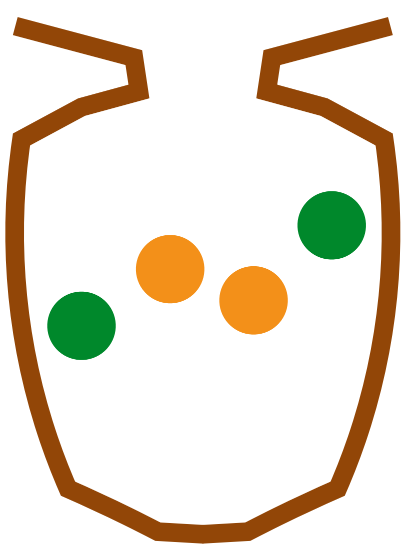

$$\lim_{n \to \infty} \frac{\#\ \text{orange}}{\#\ \text{total}}$$

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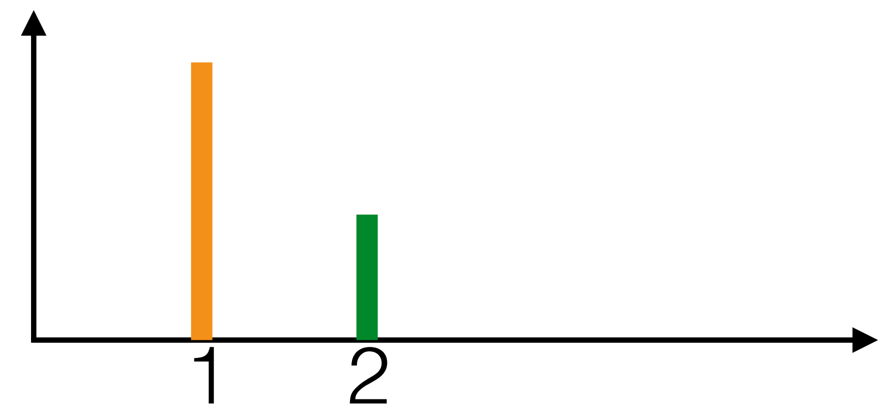

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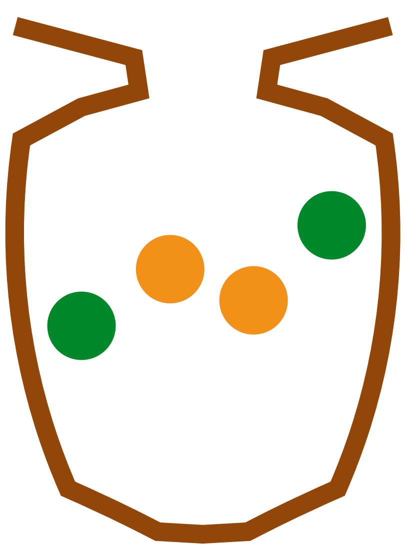

$$\lim_{n \to \infty} \frac{\#\ \text{orange}}{\#\ \text{total}} = \rho_{\text{orange}}$$

Marginal cluster assignments

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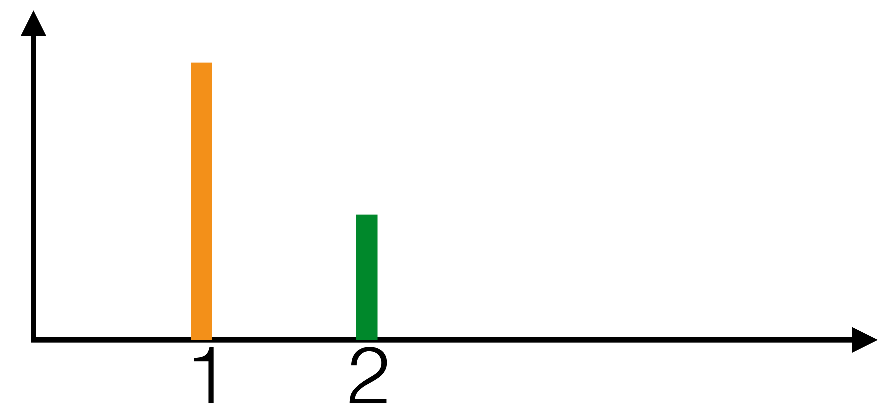

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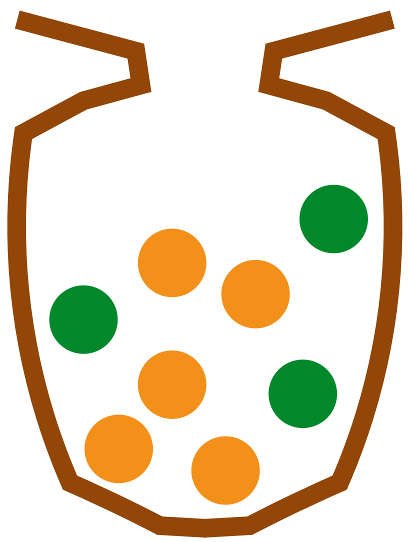

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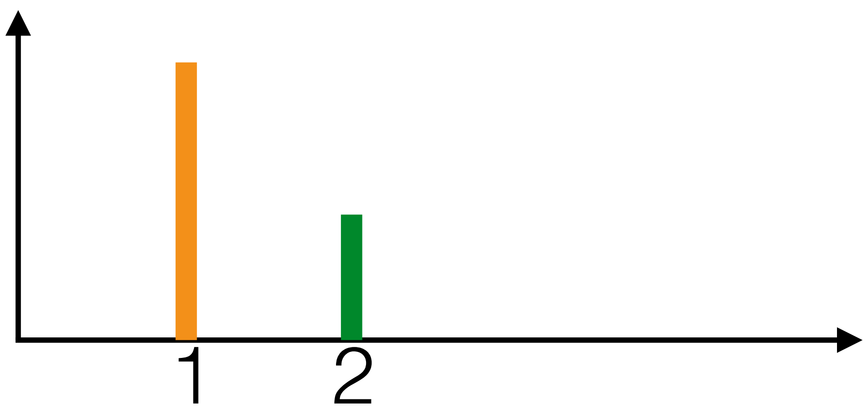

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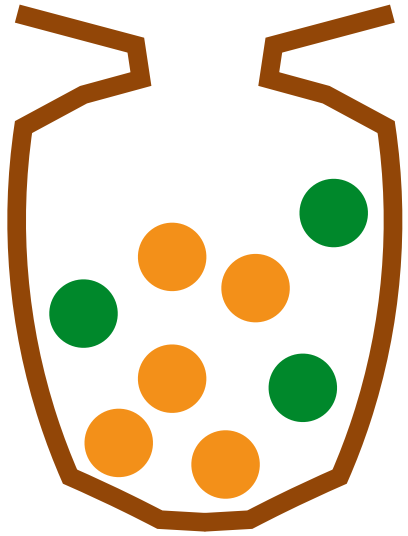

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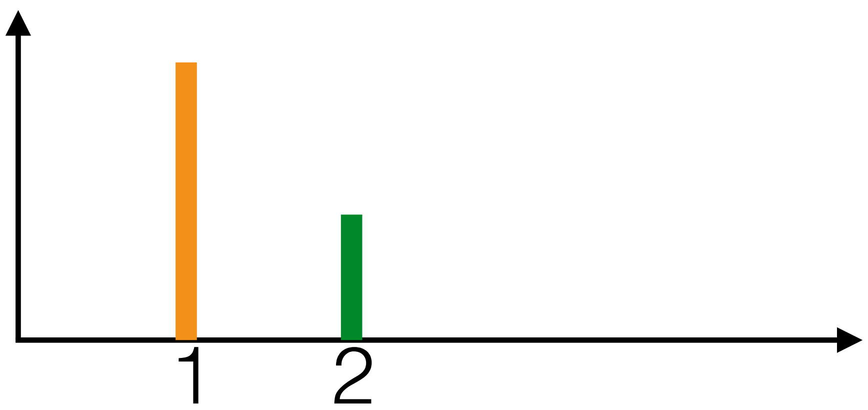

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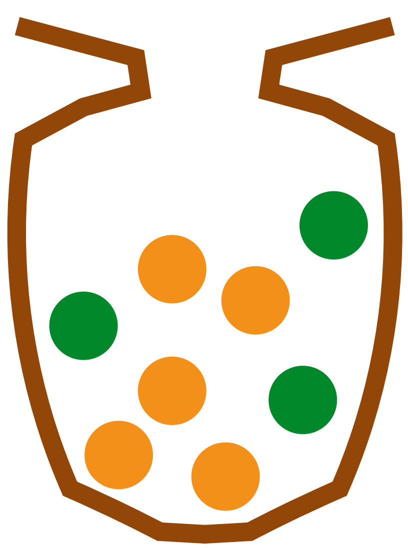

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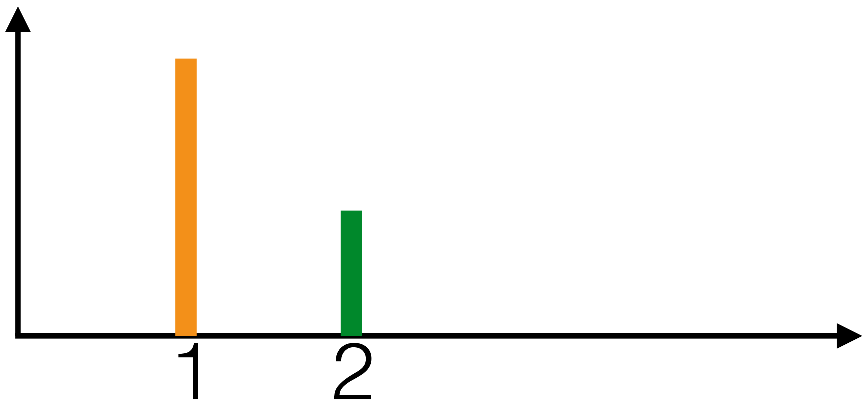

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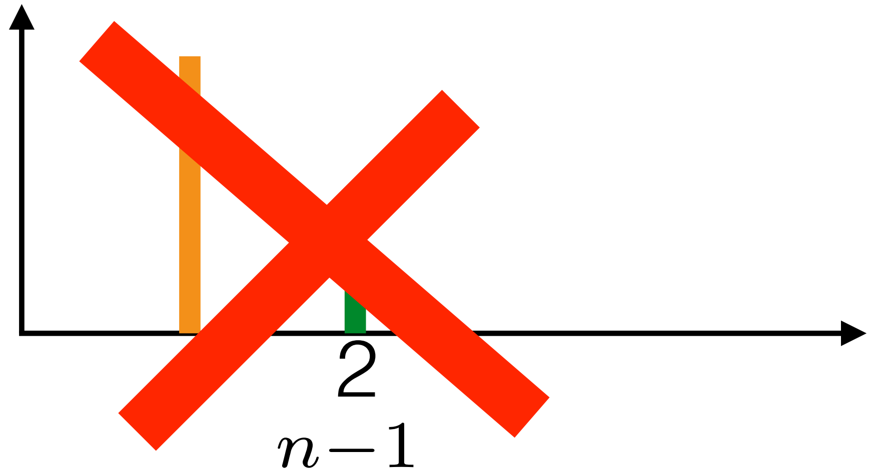

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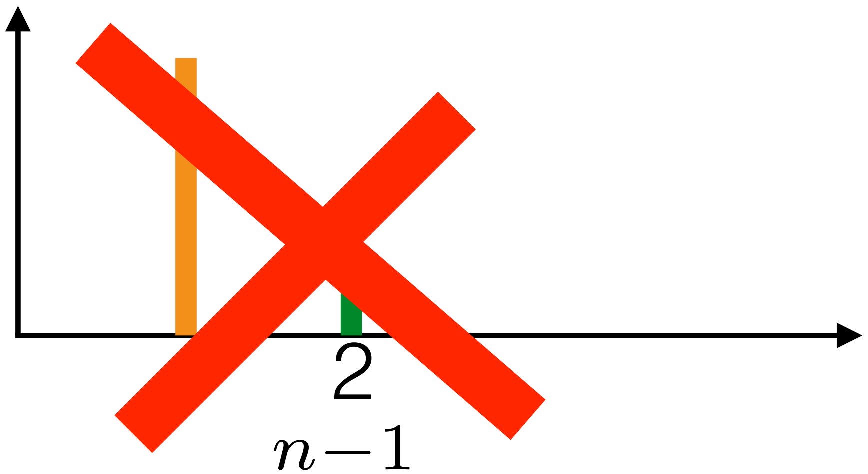

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- Pólya urn

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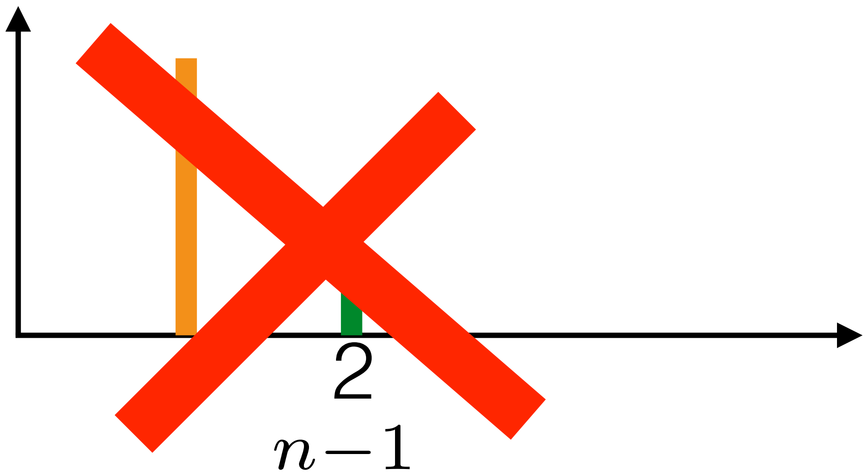

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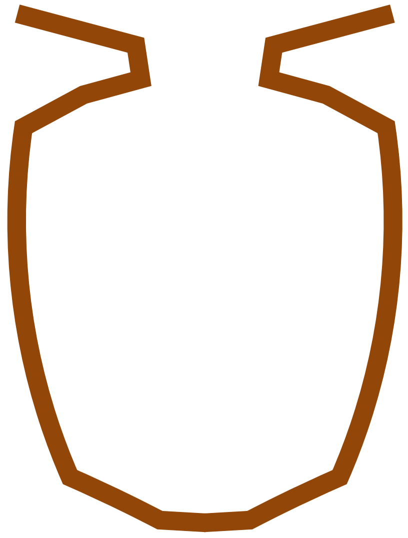

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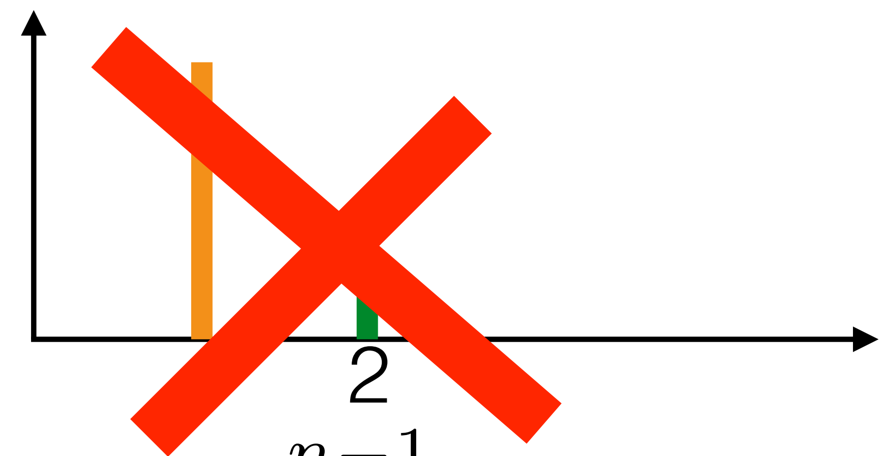

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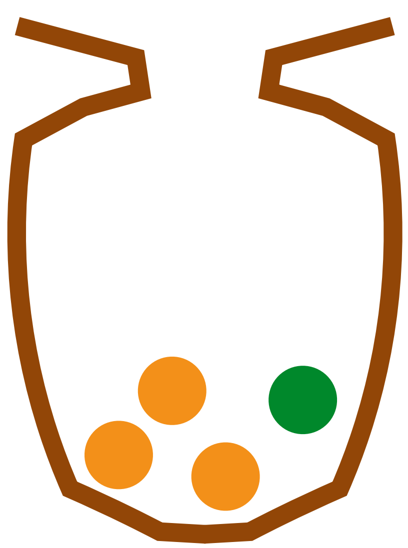

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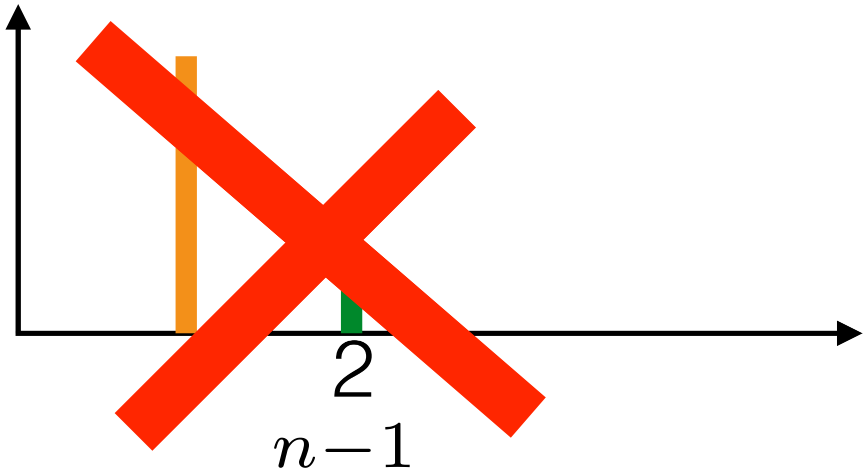
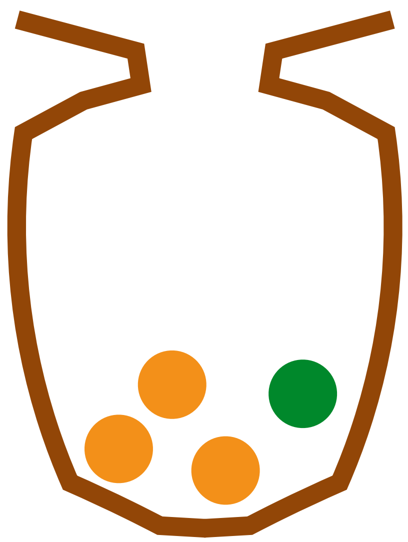


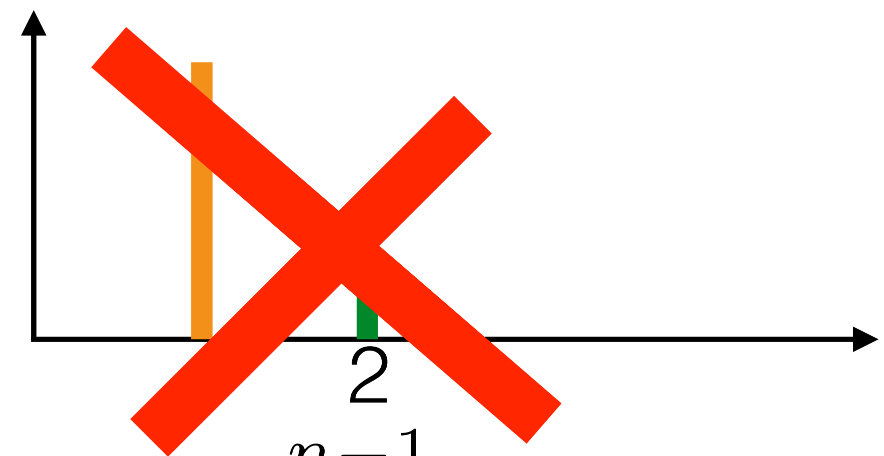
- Pólya urn
  - Choose any ball with equal probability



Marginal cluster assignments

- Integrate out the frequencies

$$\rho_1 \sim \text{Beta}(a_1, a_2),\ z_n \overset{iid}{\sim} \text{Cat}(\rho_1, \rho_2)$$



$$p(z_n = 1 | z_1, \ldots, z_{n-1}) = \frac{a_{1,n}}{a_{1,n} + a_{2,n}}$$

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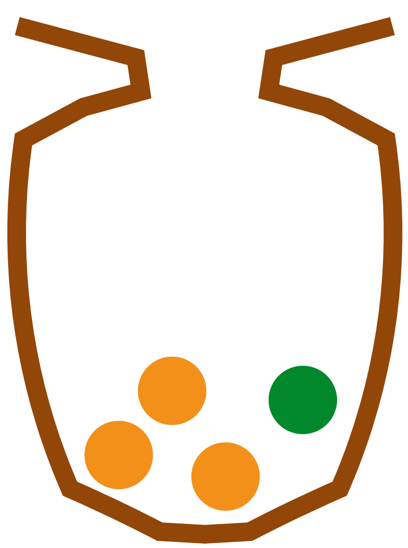

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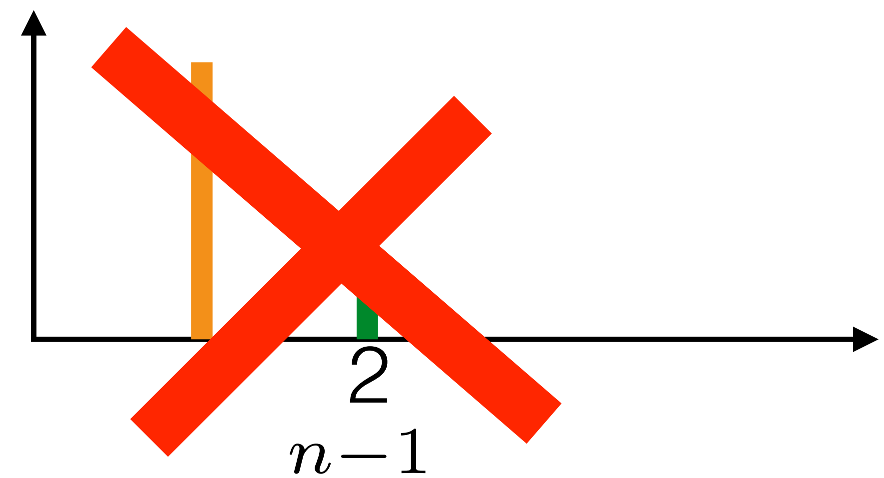

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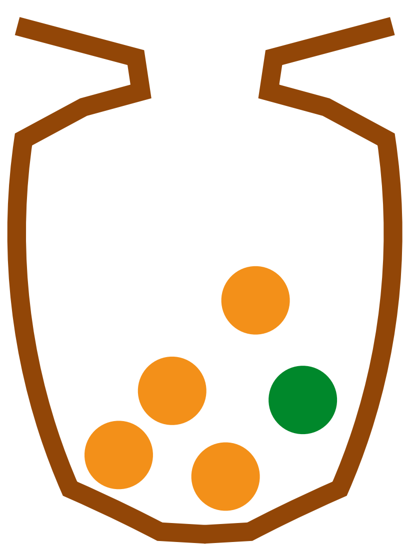

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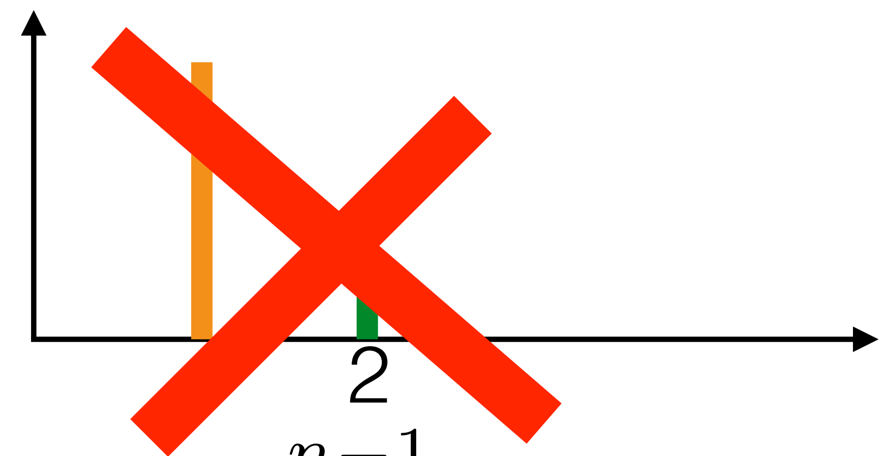

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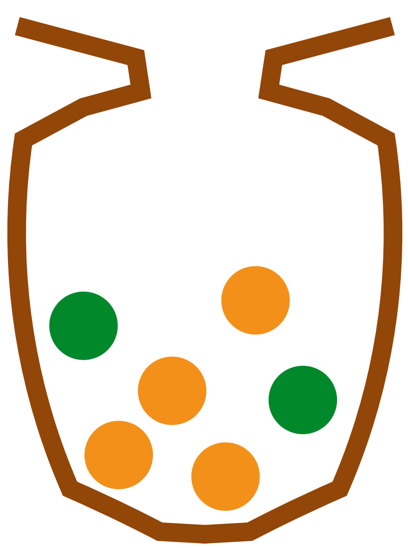

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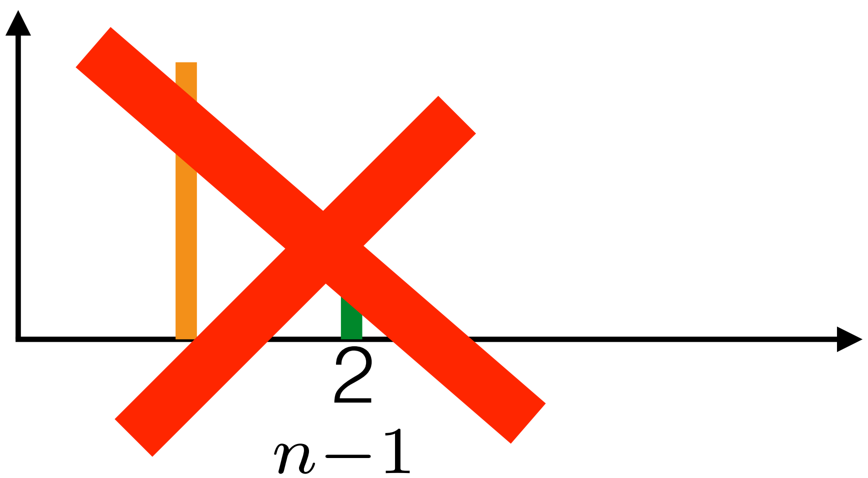

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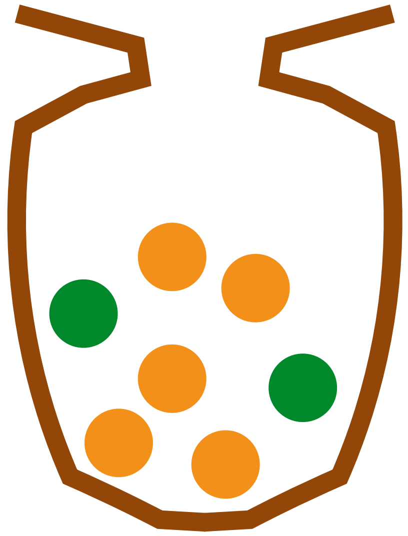

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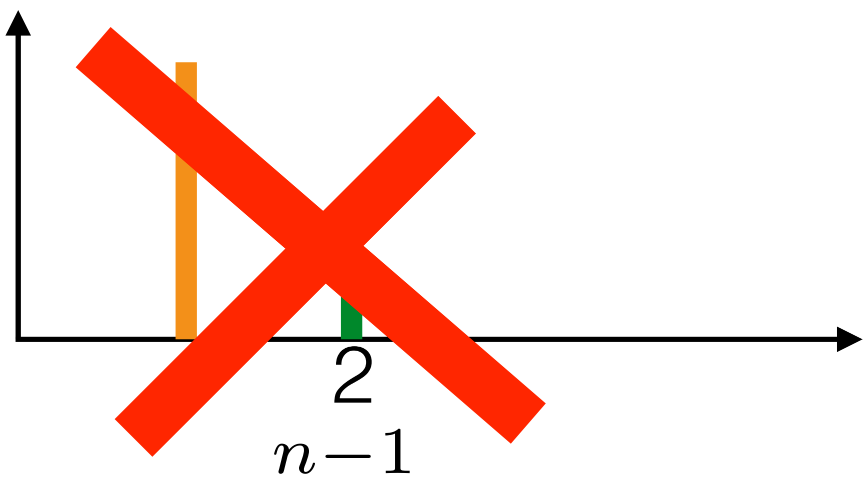

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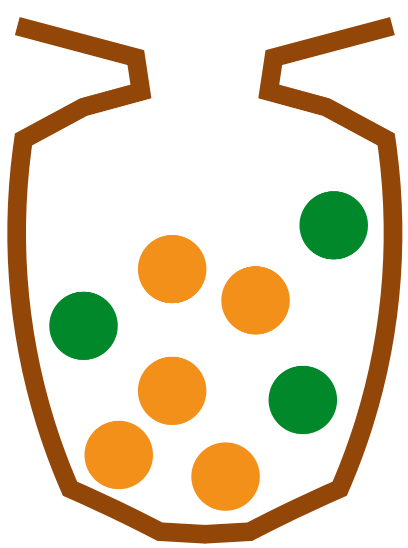

Marginal cluster assignments

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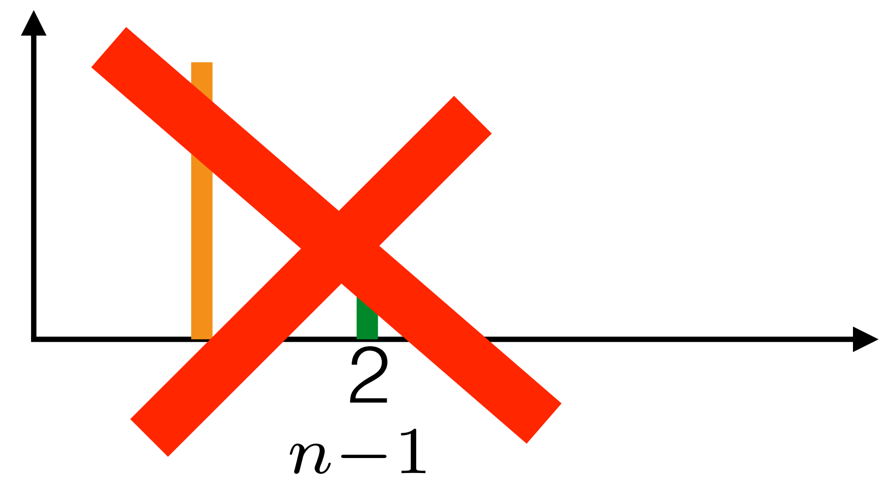

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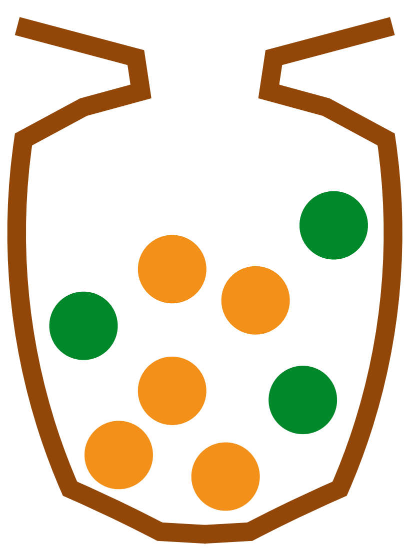

$$\lim_{n \to \infty} \frac{\#\ \text{orange}}{\#\ \text{total}}$$

Marginal cluster assignments

- Integrate out the frequencies

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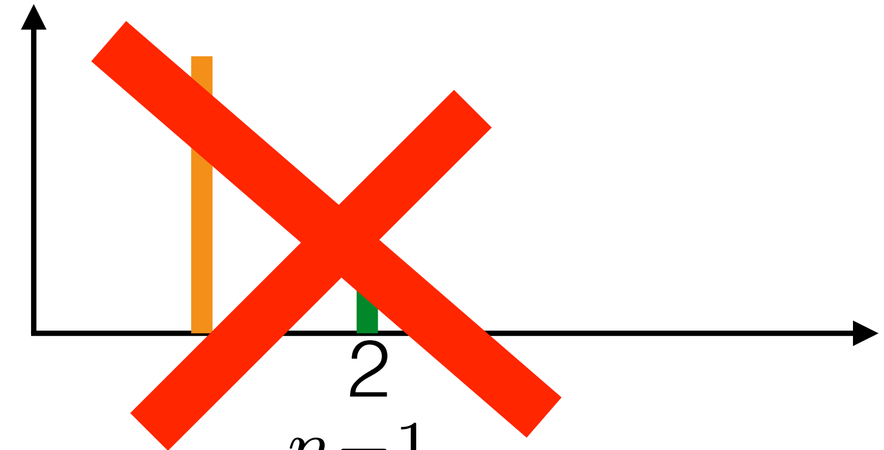

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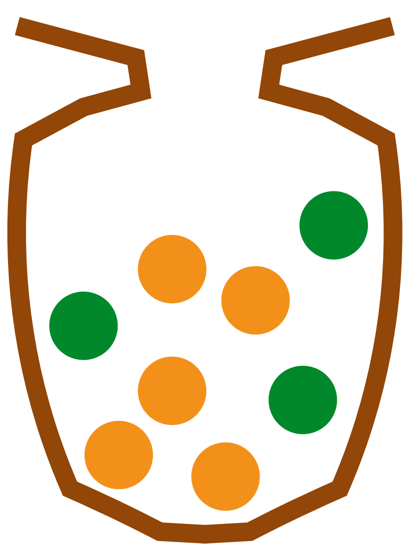

$$\lim_{n \to \infty} \frac{\#\ \text{orange}}{\#\ \text{total}} = \rho_{\text{orange}}$$

Marginal cluster assignments

- Integrate out the frequencies

$$\rho_1 \sim \text{Beta}(a_1, a_2),\ z_n \overset{iid}{\sim} \text{Cat}(\rho_1, \rho_2)$$

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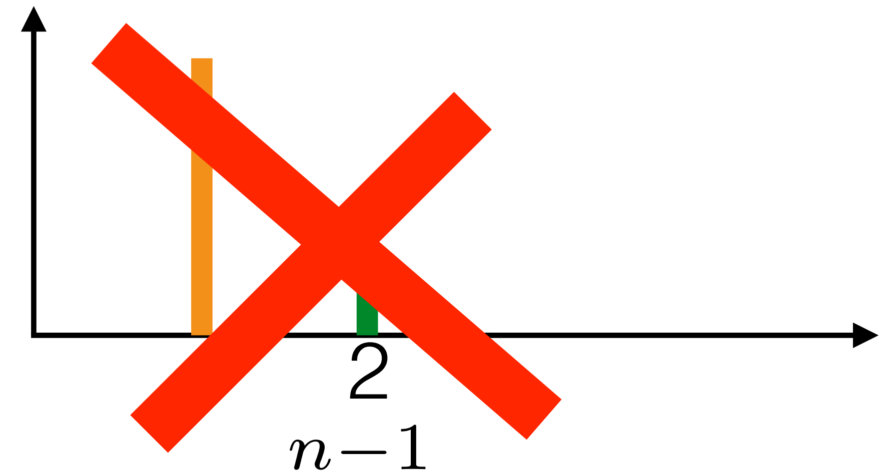

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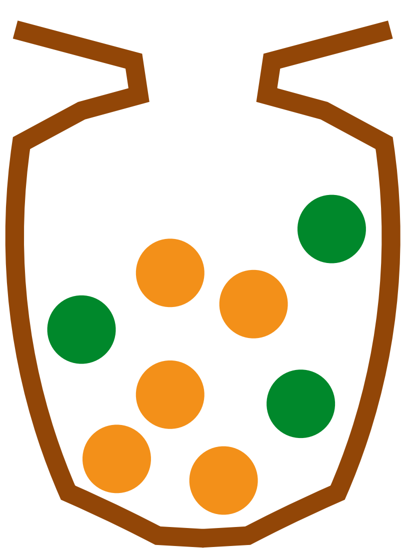

$$\lim_{n \to \infty} \frac{\#\ \text{orange}}{\#\ \text{total}} = \rho_{\text{orange}} \overset{d}{=} \text{Beta}(a_{\text{orange}}, a_{\text{green}})$$

Marginal cluster assignments

- Integrate out the frequencies

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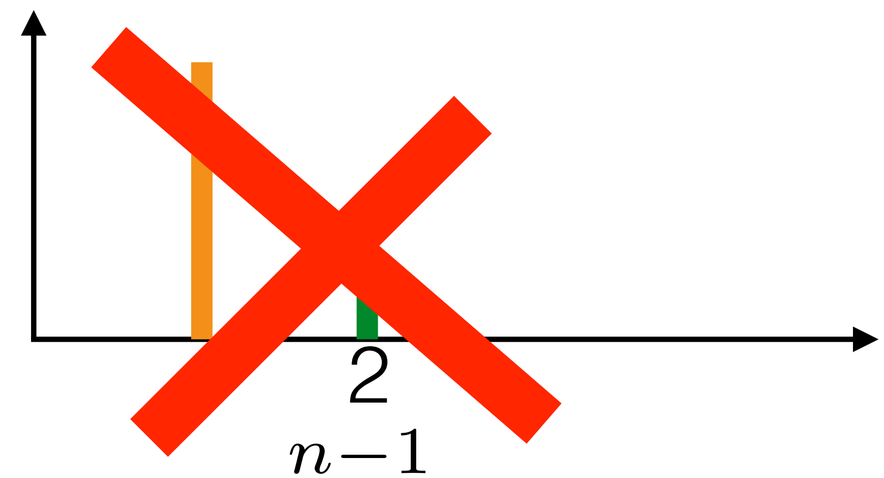

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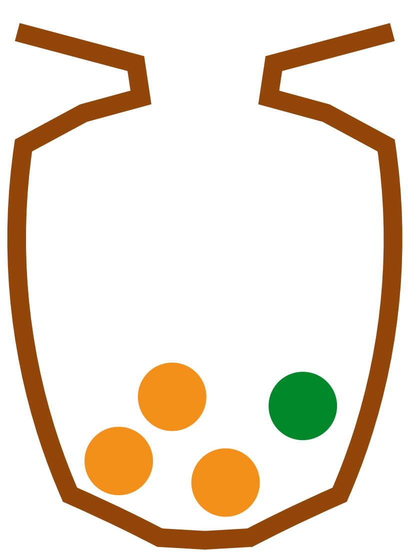

$$\lim_{n \to \infty} \frac{\#\ \text{orange}}{\#\ \text{total}} = \rho_{\text{orange}} \overset{d}{=} \mathrm{Beta}(a_{\text{orange}}, a_{\text{green}})$$

Marginal cluster assignments

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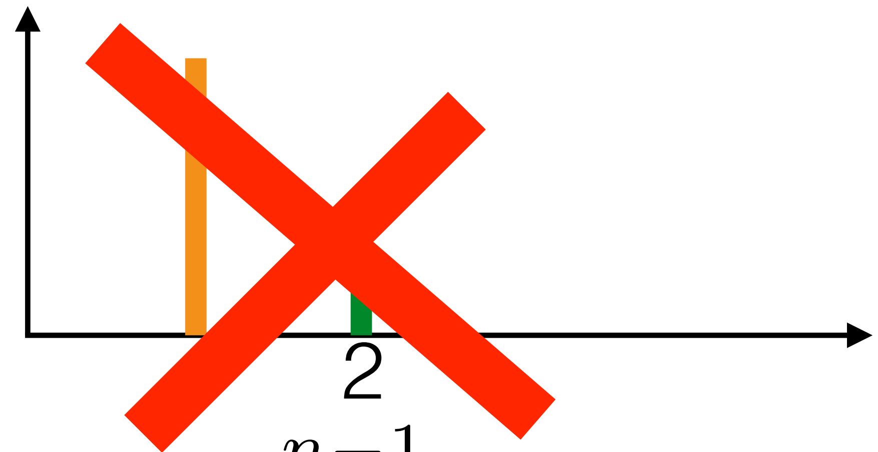

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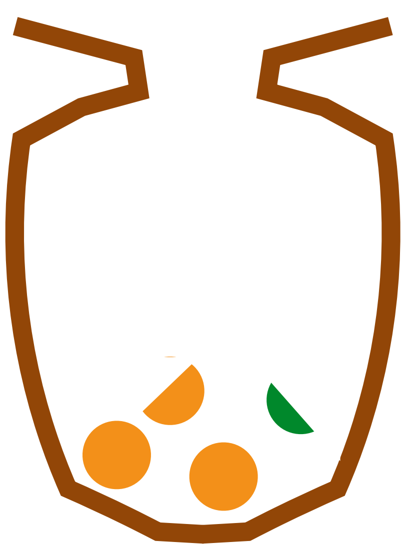

$$\lim_{n\to\infty} \frac{\#\text{ orange}}{\#\text{ total}} = \rho_{\text{orange}} \overset{d}{=} \text{Beta}(a_{\text{orange}}, a_{\text{green}})$$

Marginal cluster assignments

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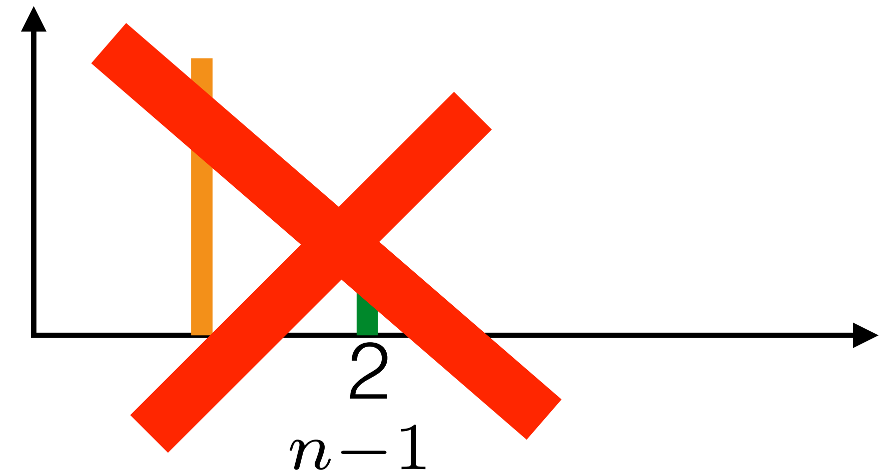

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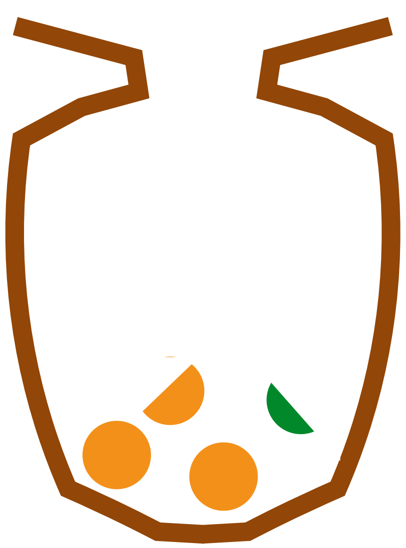

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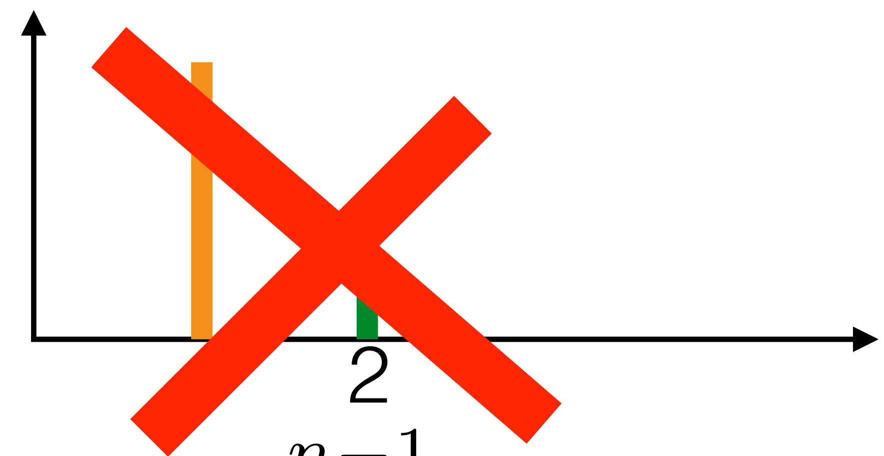

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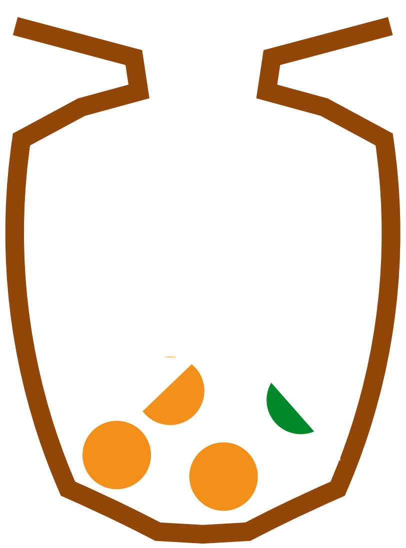

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$$\text{PolyaUrn}(a_{\text{orange}}, a_{\text{green}})$$

Marginal cluster assignments

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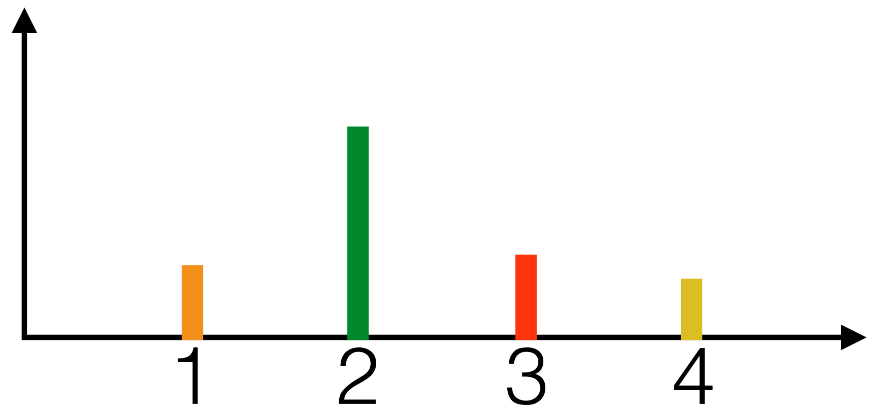
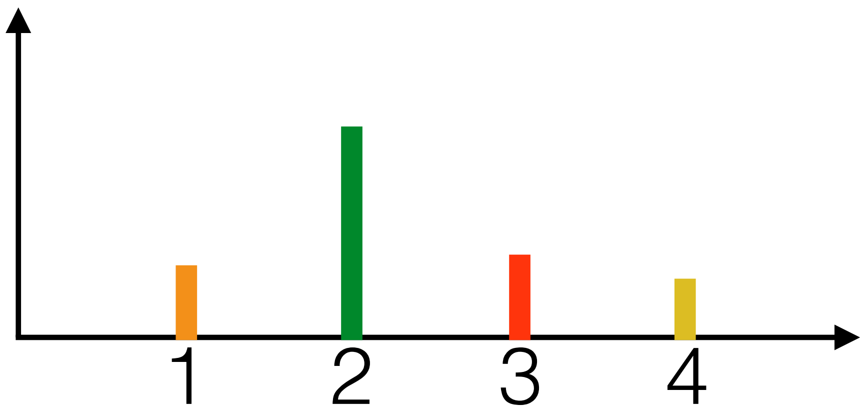


Marginal cluster assignments

- Integrate out the frequencies

$$\rho_{1:K} \sim \mathrm{Dirichlet}(a_{1:K}), z_n \overset{iid}{\sim} \mathrm{Cat}(\rho_{1:K})$$



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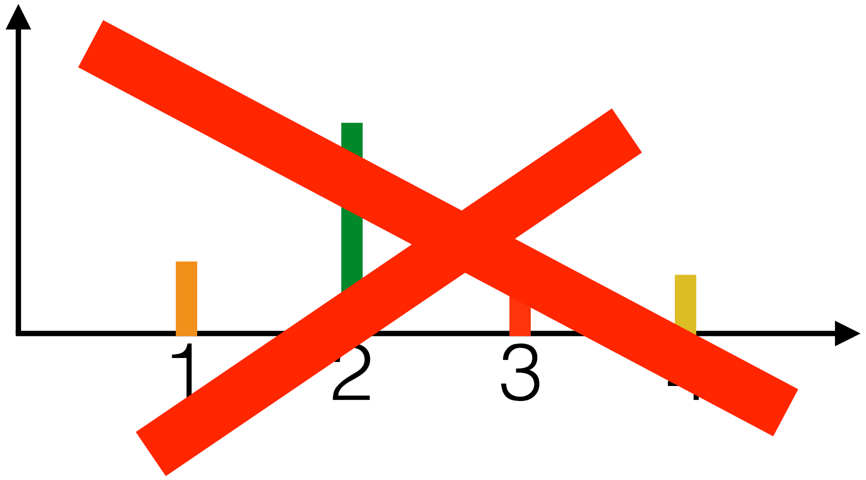

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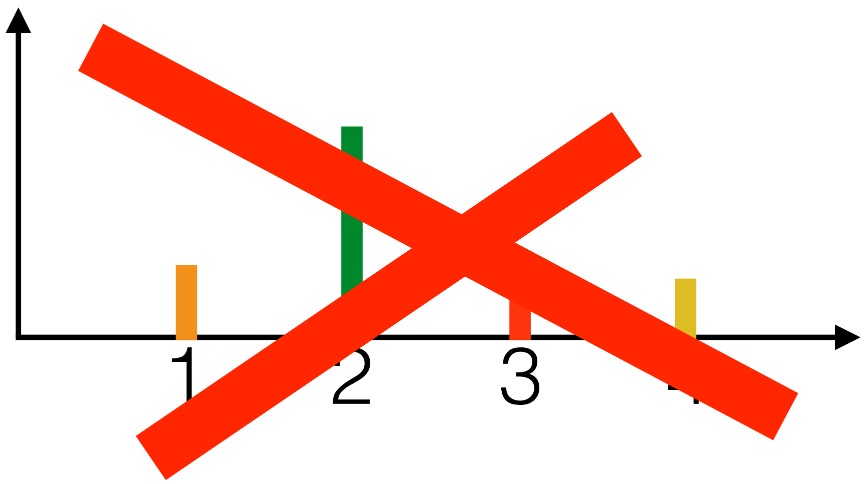

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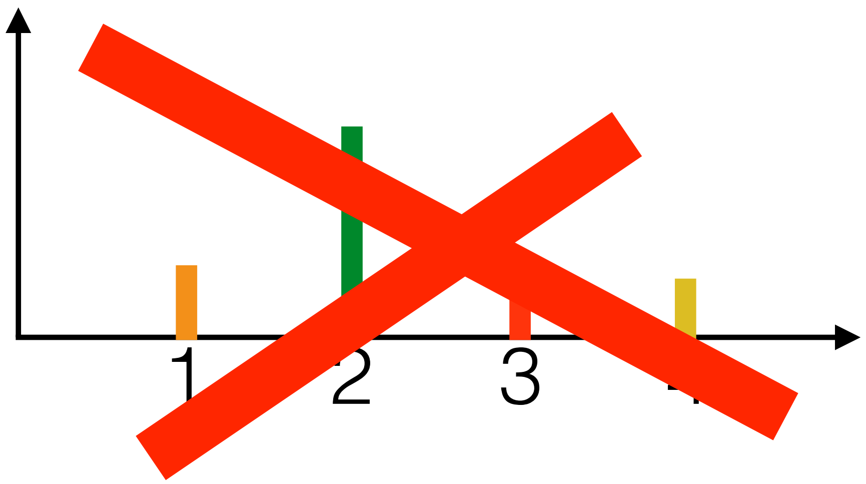

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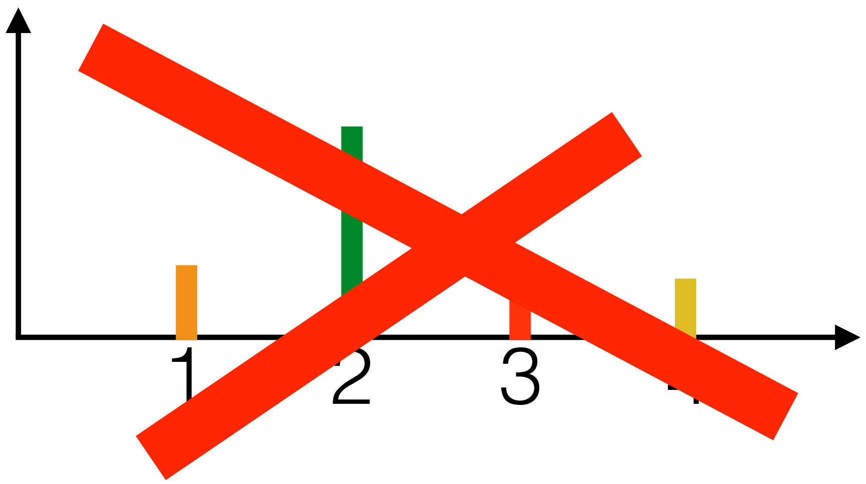

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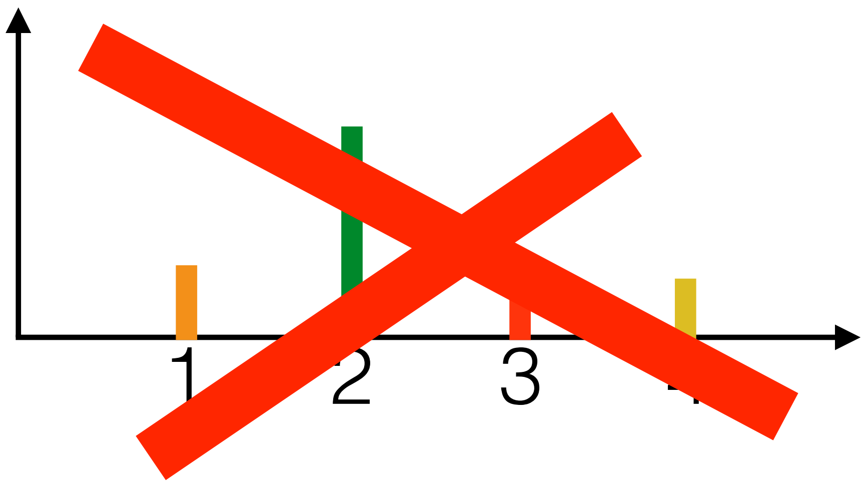

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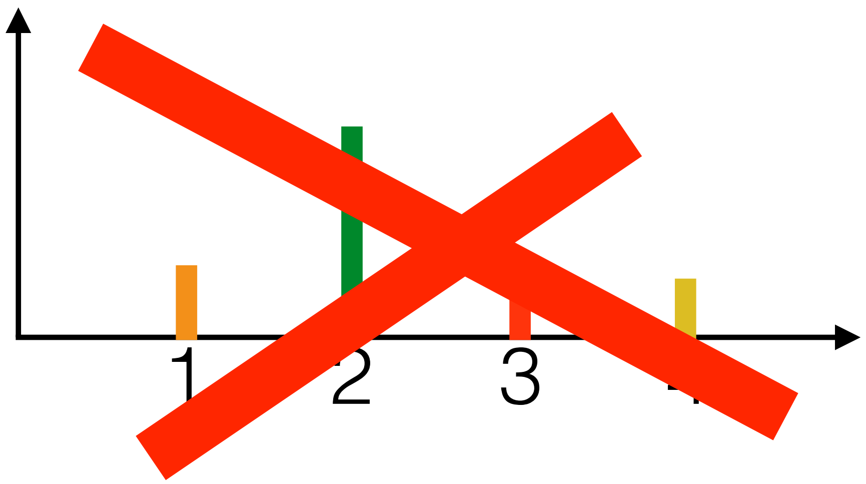

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  - Choose any ball with prob proportional to its mass

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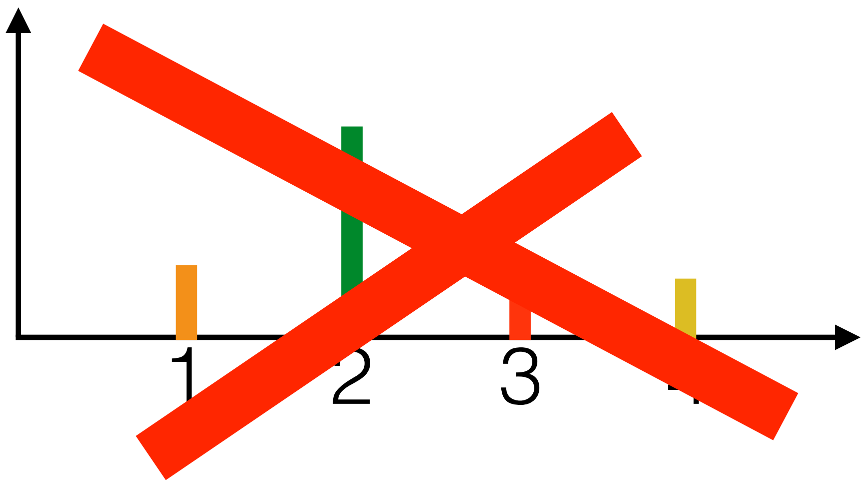

- multivariate Pólya urn
  - Choose any ball with prob proportional to its mass
  - Replace and add ball of same color

Marginal cluster assignments

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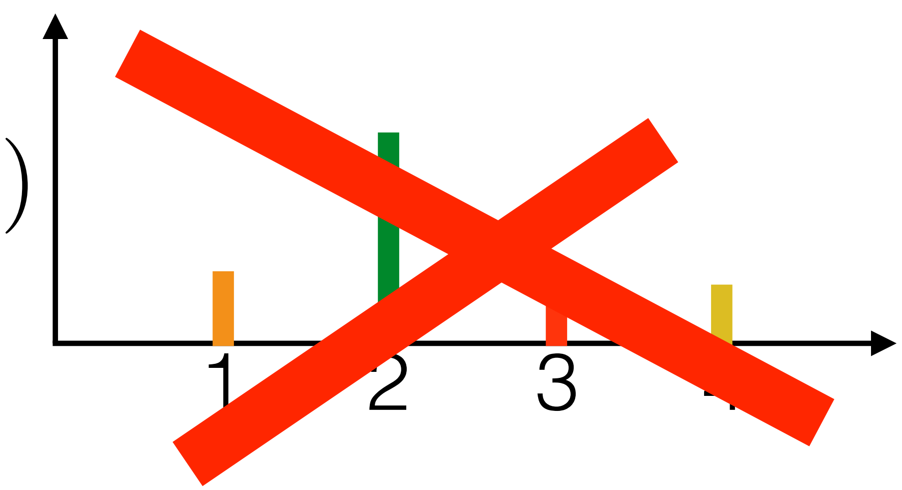

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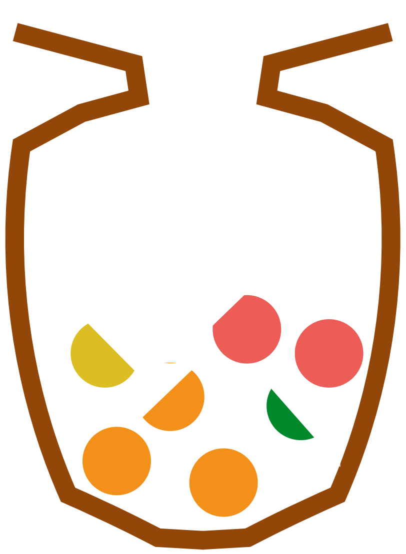
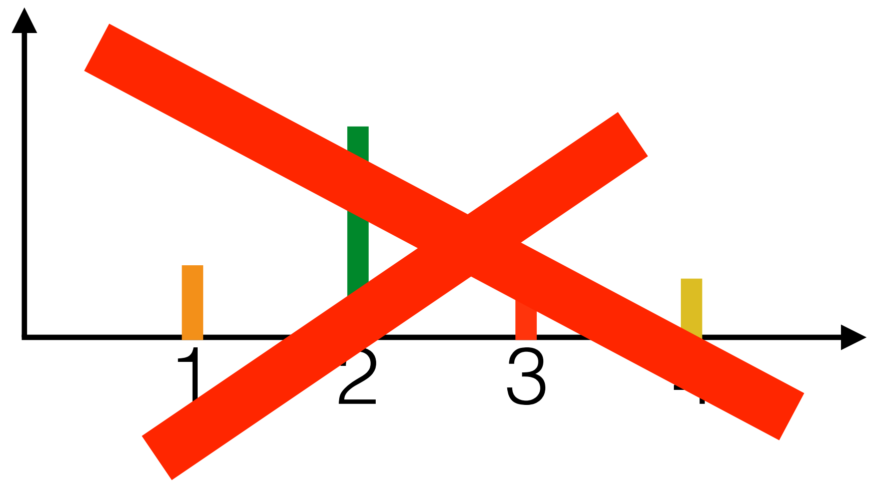


$$\lim_{n \to \infty} \frac{(\#\text{ orange}, \#\text{ green}, \#\text{ red}, \#\text{ yellow})}{\#\text{ total}}$$

Marginal cluster assignments

- Integrate out the frequencies

$$\rho_{1:K} \sim \text{Dirichlet}(a_{1:K}), z_n \overset{iid}{\sim} \text{Cat}(\rho_{1:K})$$



$$p(z_n = k | z_1, \ldots, z_{n-1}) = \frac{a_{k,n}}{\sum_{j=1}^{K} a_{j,n}}$$

$$a_{k,n} := a_k + \sum_{m=1}^{n-1} \mathbf{1}\{z_m = k\}$$

- multivariate Pólya urn
  - Choose any ball with prob proportional to its mass
  - Replace and add ball of same color

$$\lim_{n \to \infty} \frac{(\#\text{ orange}, \#\text{ green}, \#\text{ red}, \#\text{ yellow})}{\#\text{ total}}$$

$$\to (\rho_{\text{orange}}, \rho_{\text{green}}, \rho_{\text{red}}, \rho_{\text{yellow}})$$

Marginal cluster assignments

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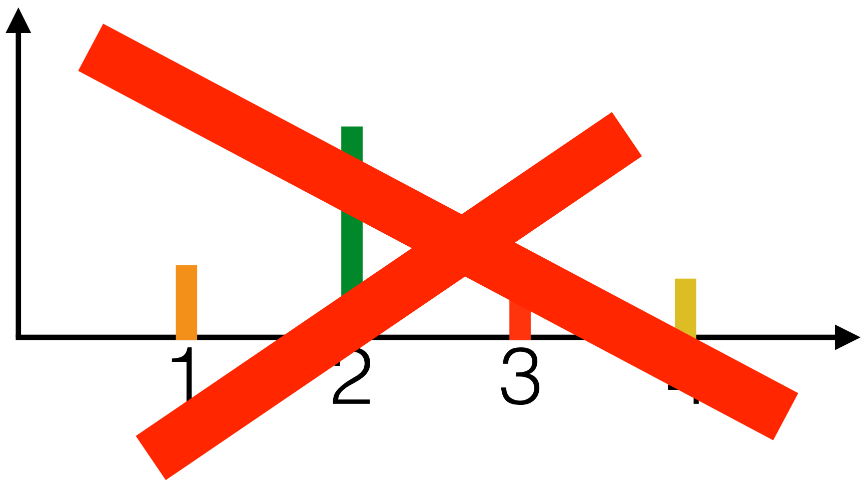

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Marginal cluster assignments

- Hoppe urn / Blackwell-MacQueen urn

[Blackwell, MacQueen 1973; Hoppe 1984]

Marginal cluster assignments

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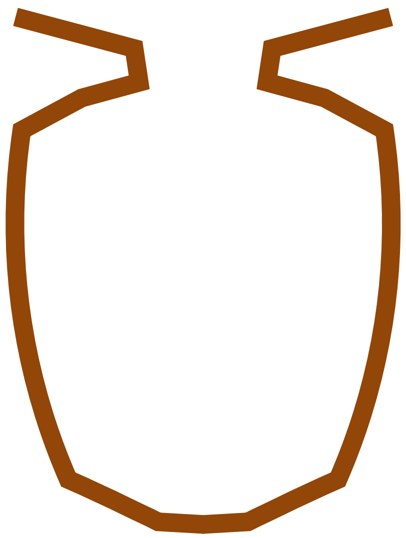

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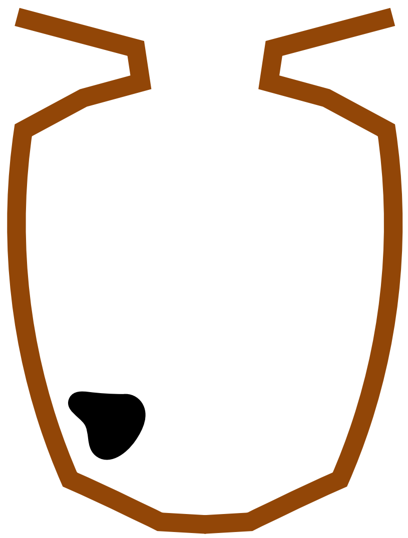

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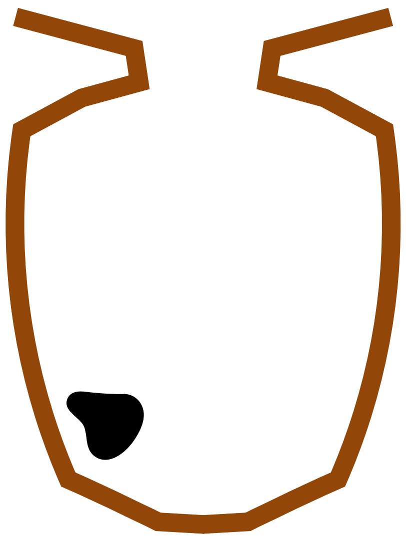

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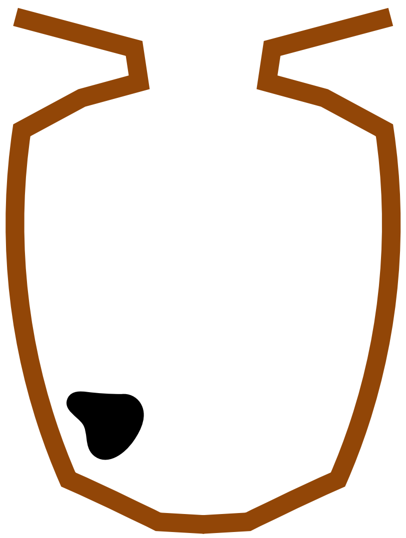

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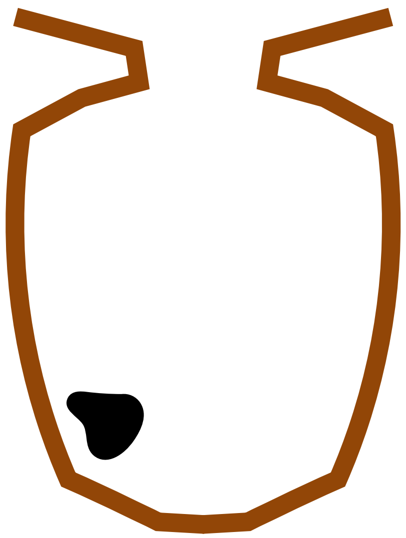

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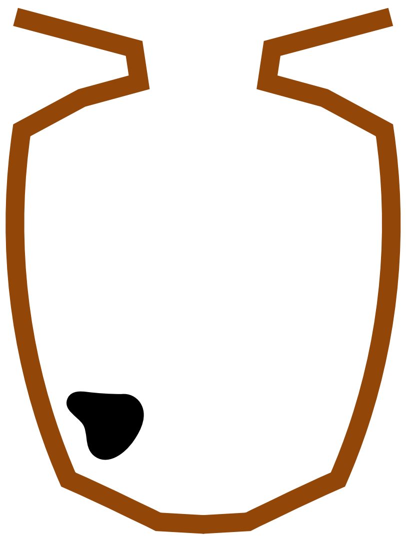

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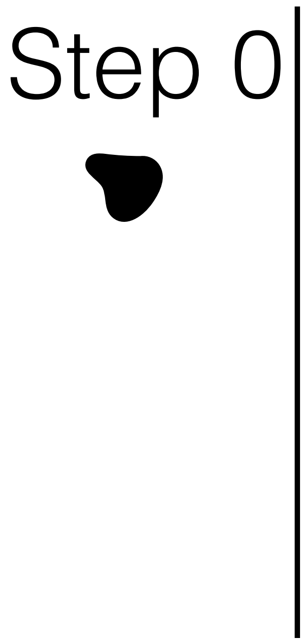

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Marginal cluster assignments

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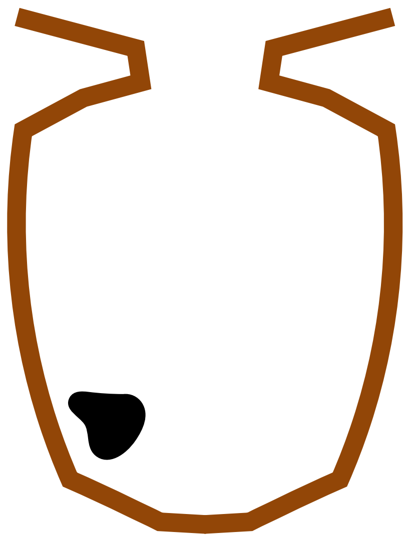

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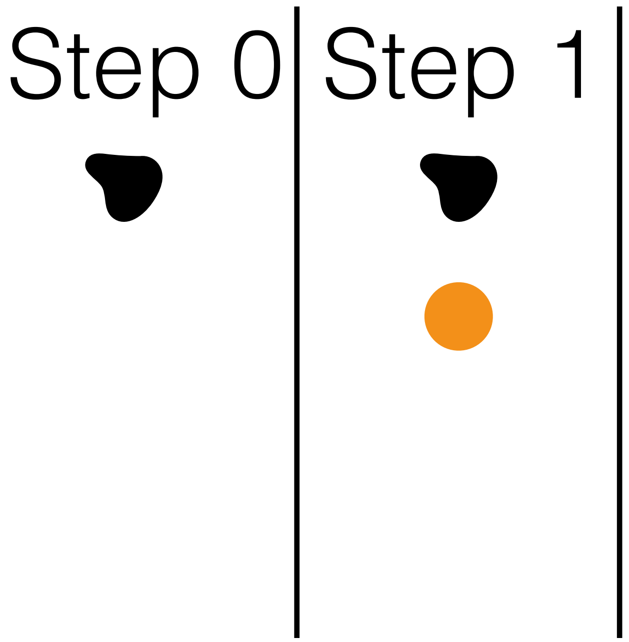

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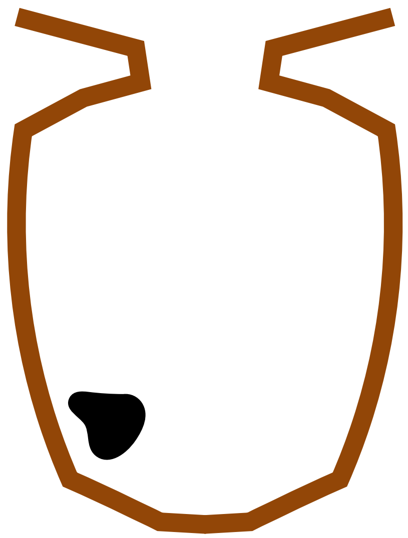

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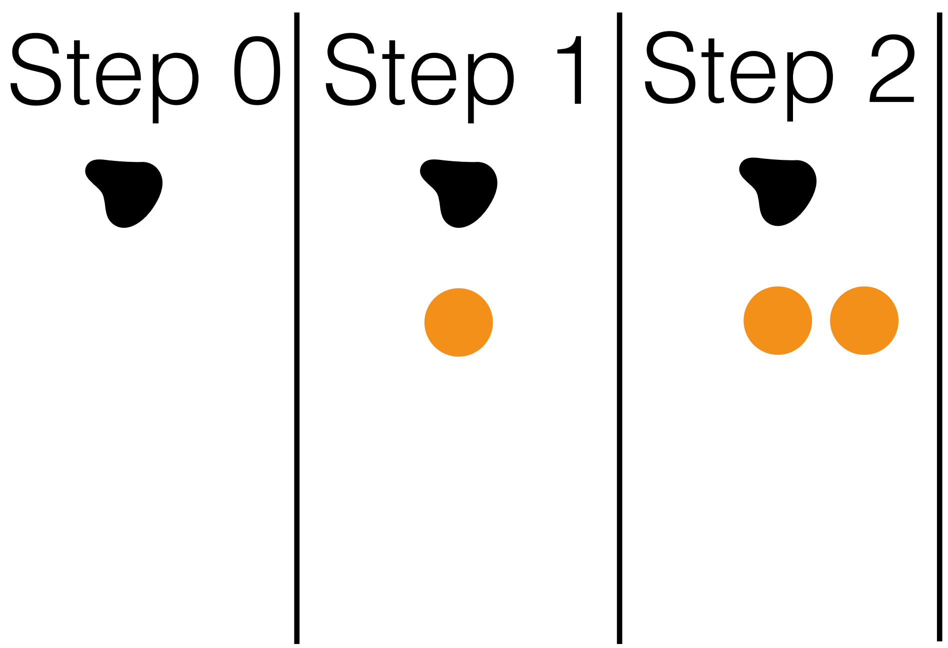

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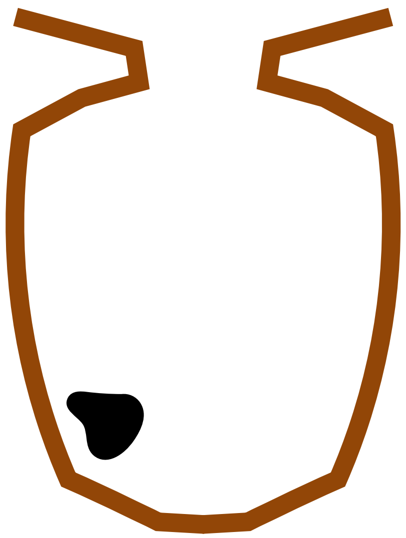

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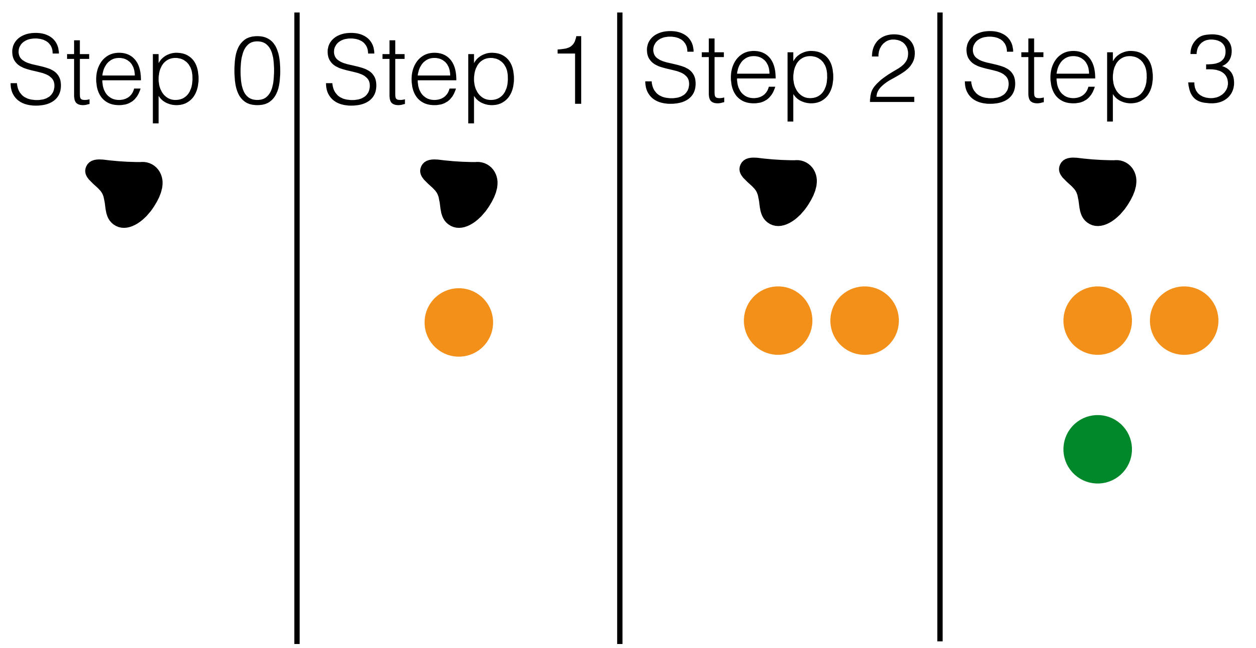

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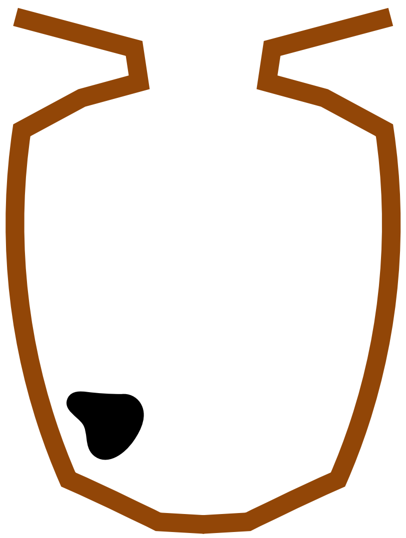

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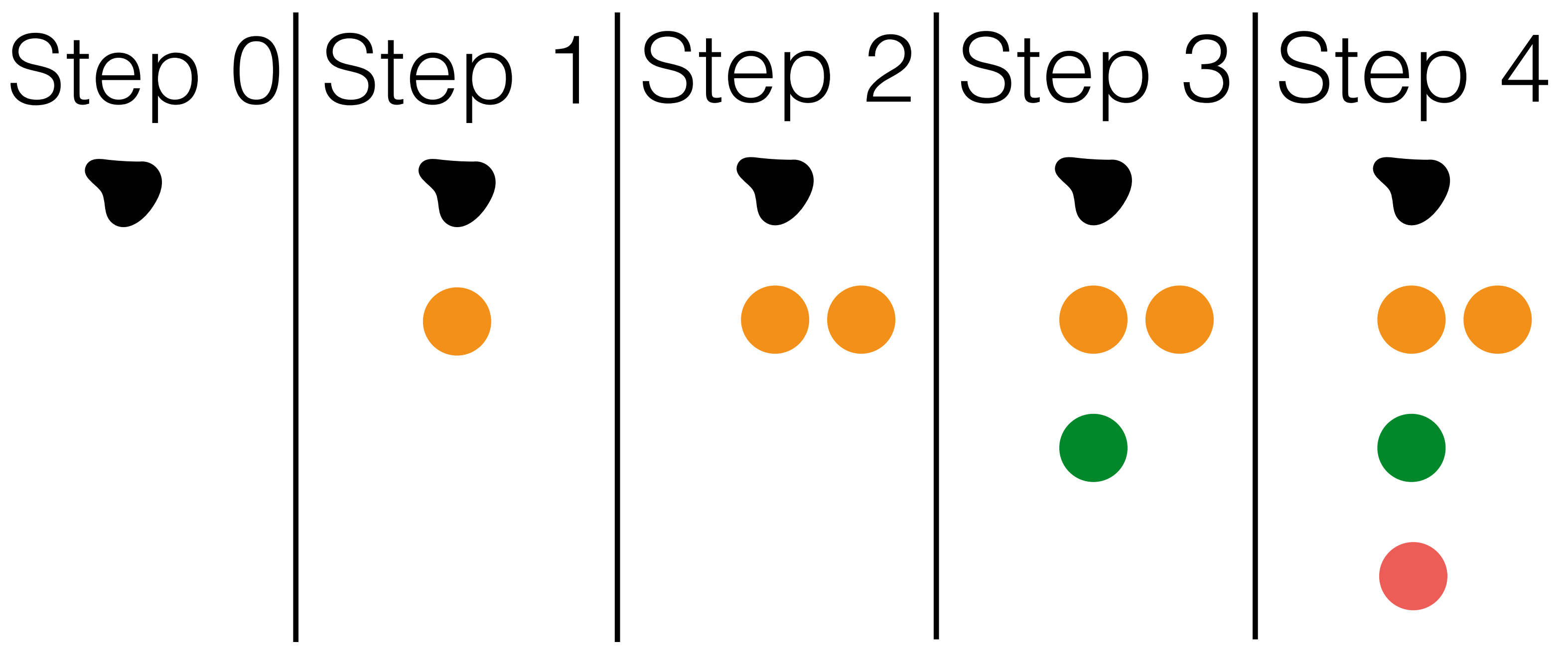

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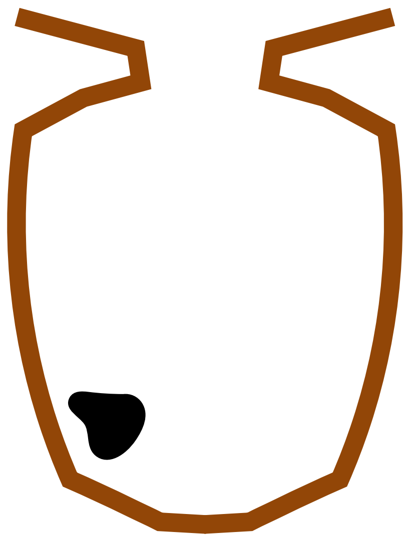

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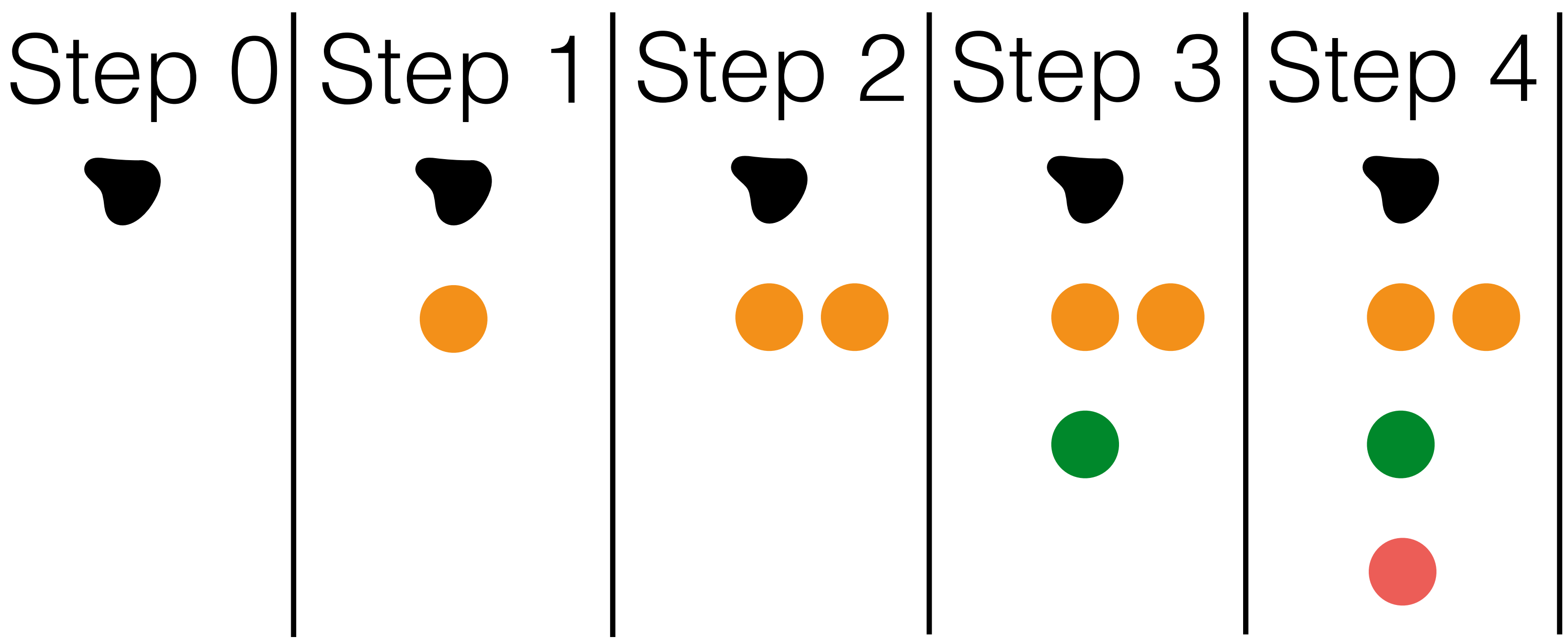

$$(\#\text{orange}, \#\text{other}) = \text{PolyaUrn}(1, \alpha)$$

Marginal cluster assignments

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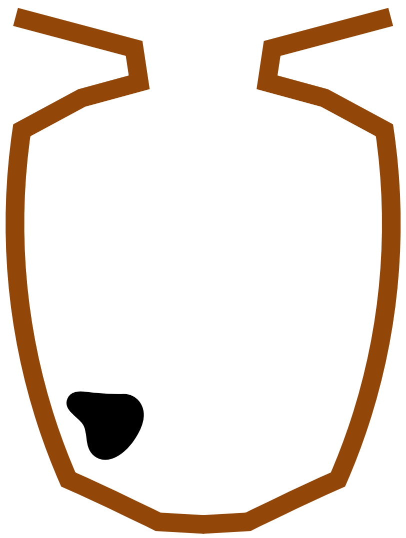

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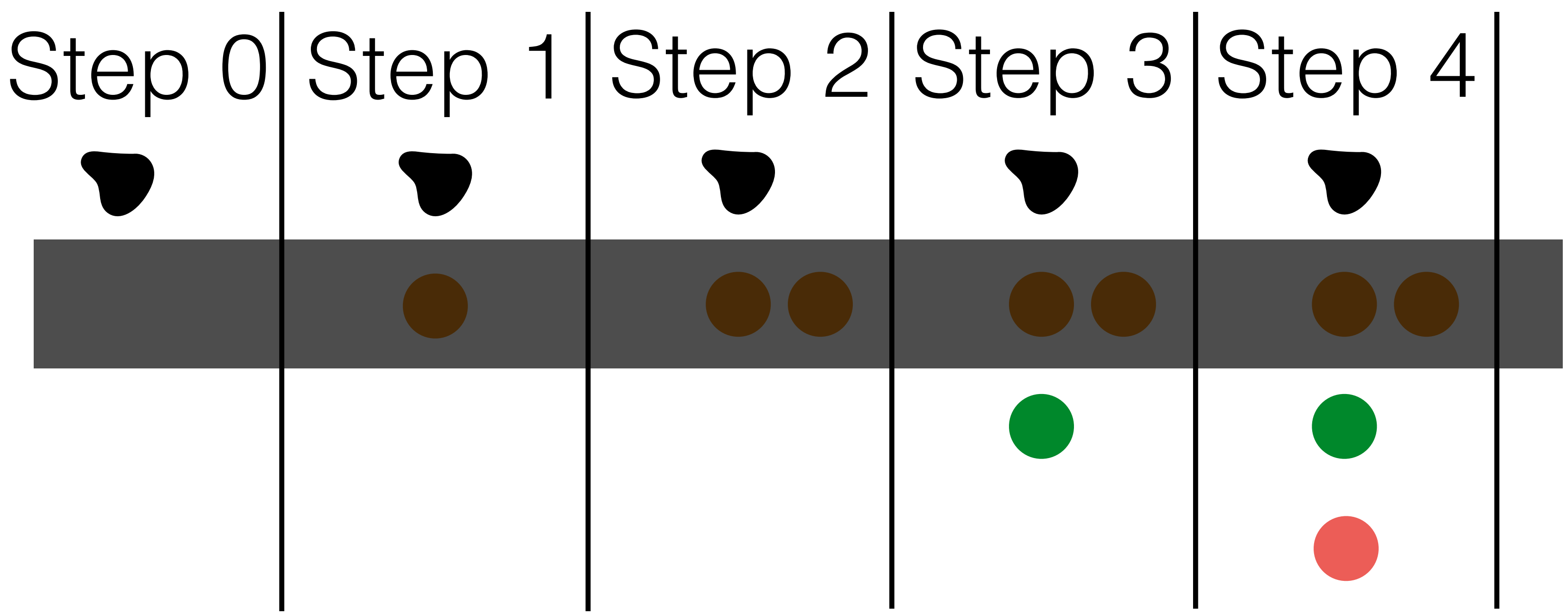

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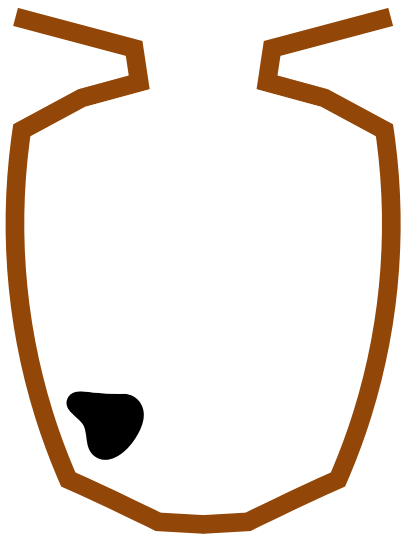

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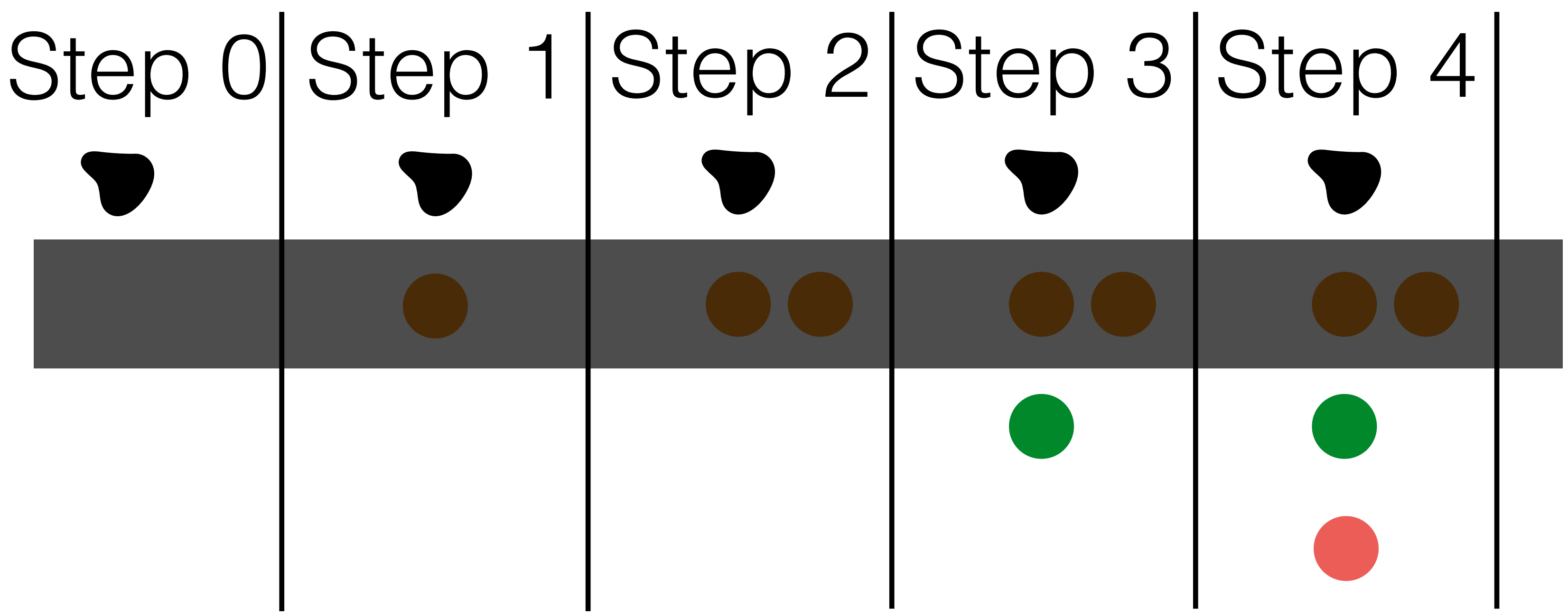

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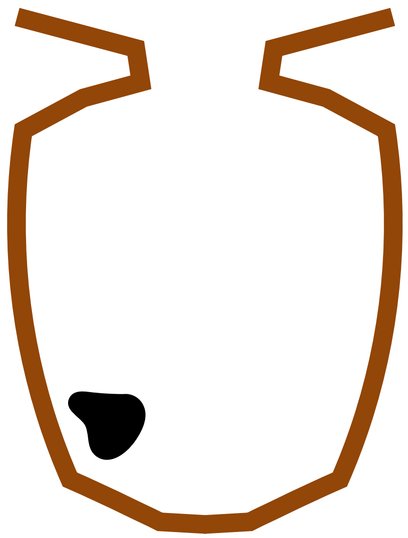

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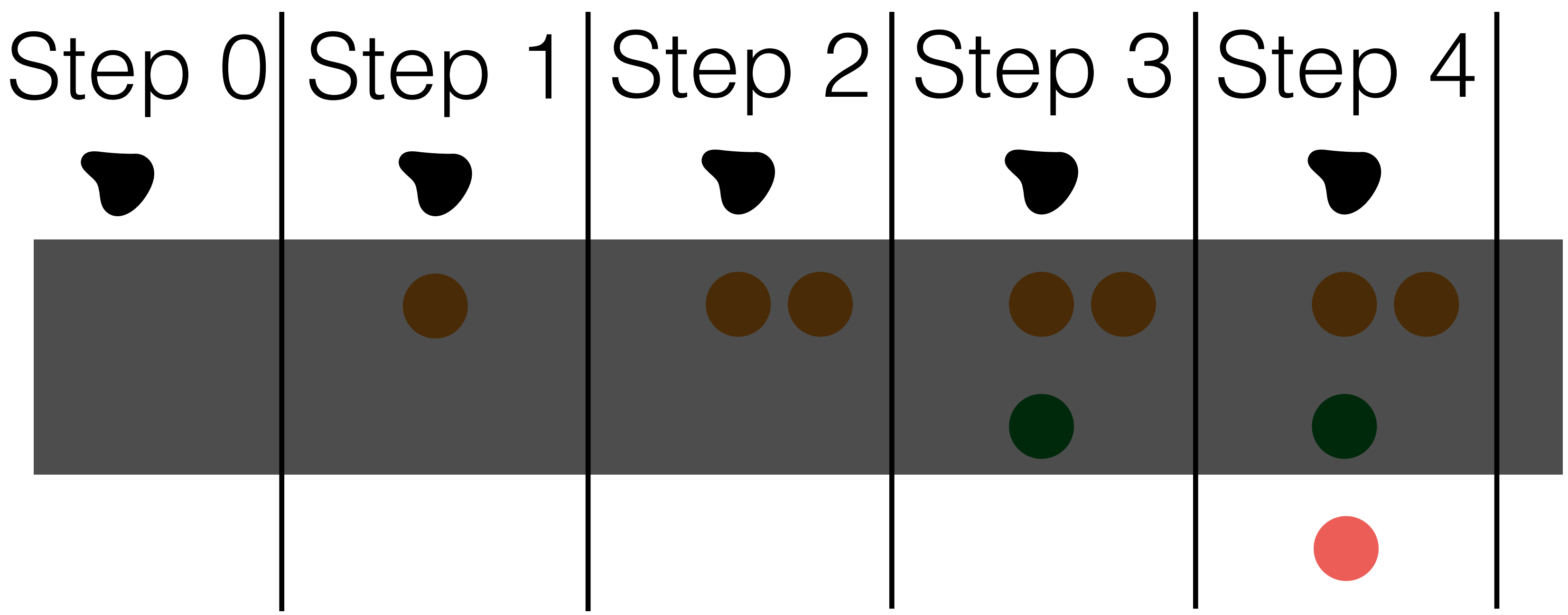

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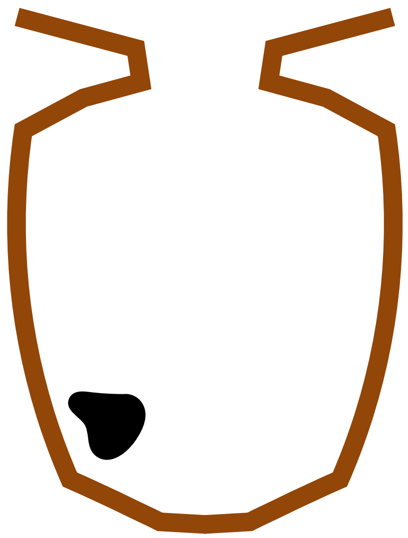

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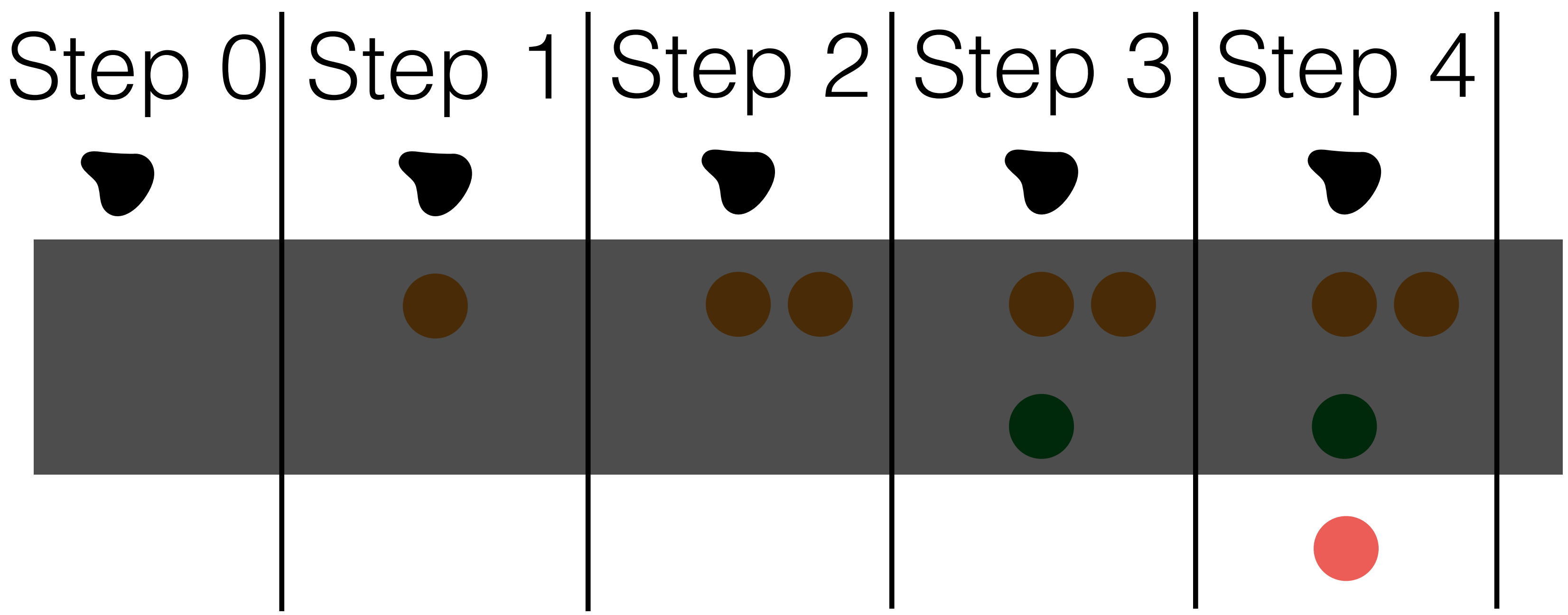

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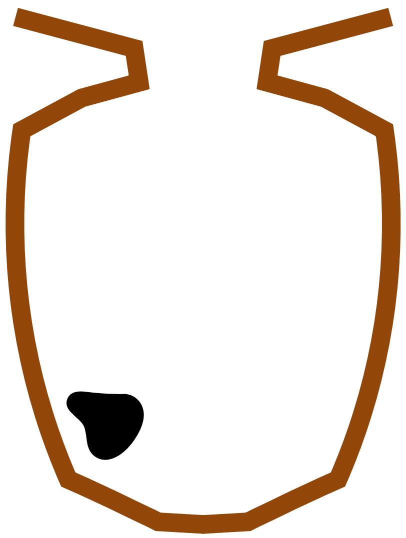

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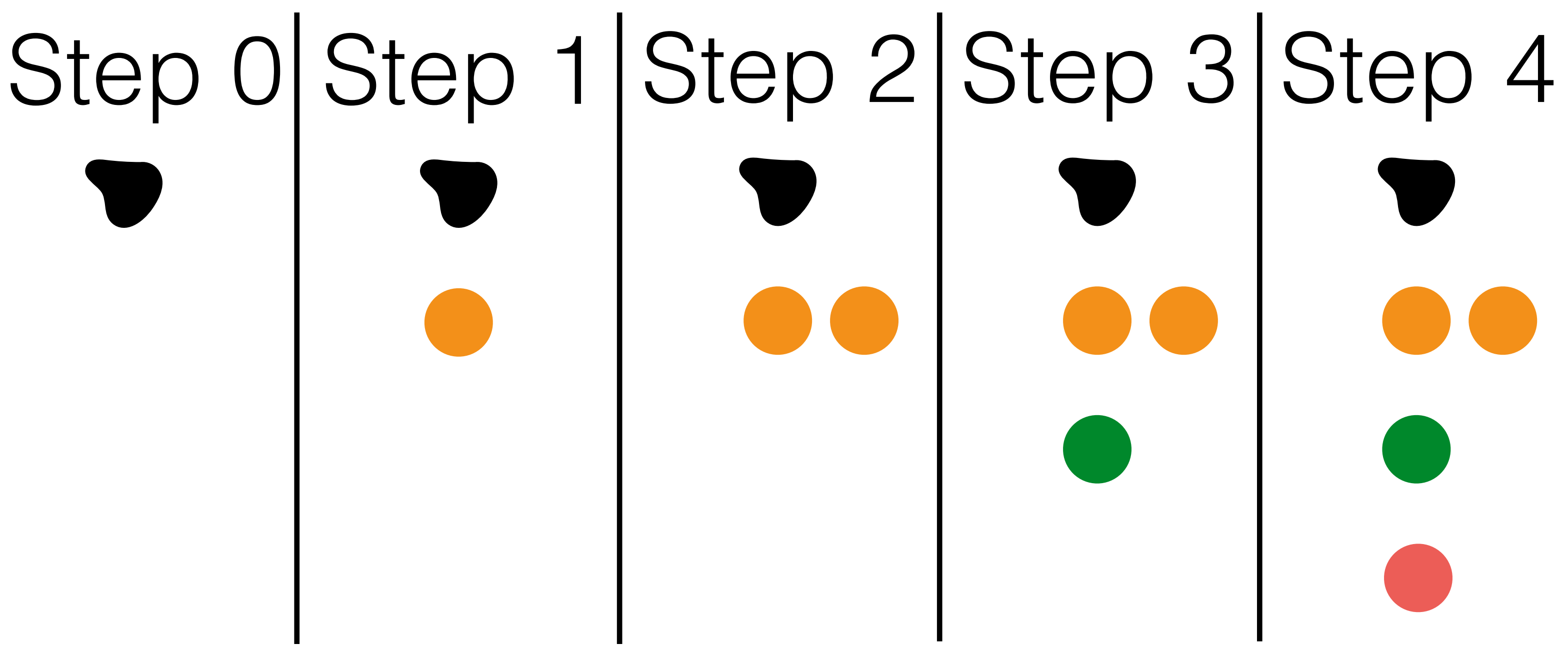

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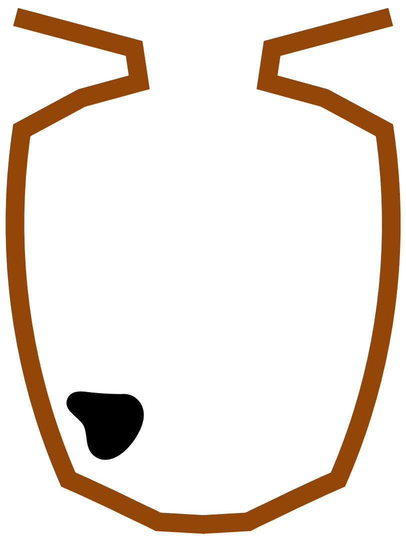

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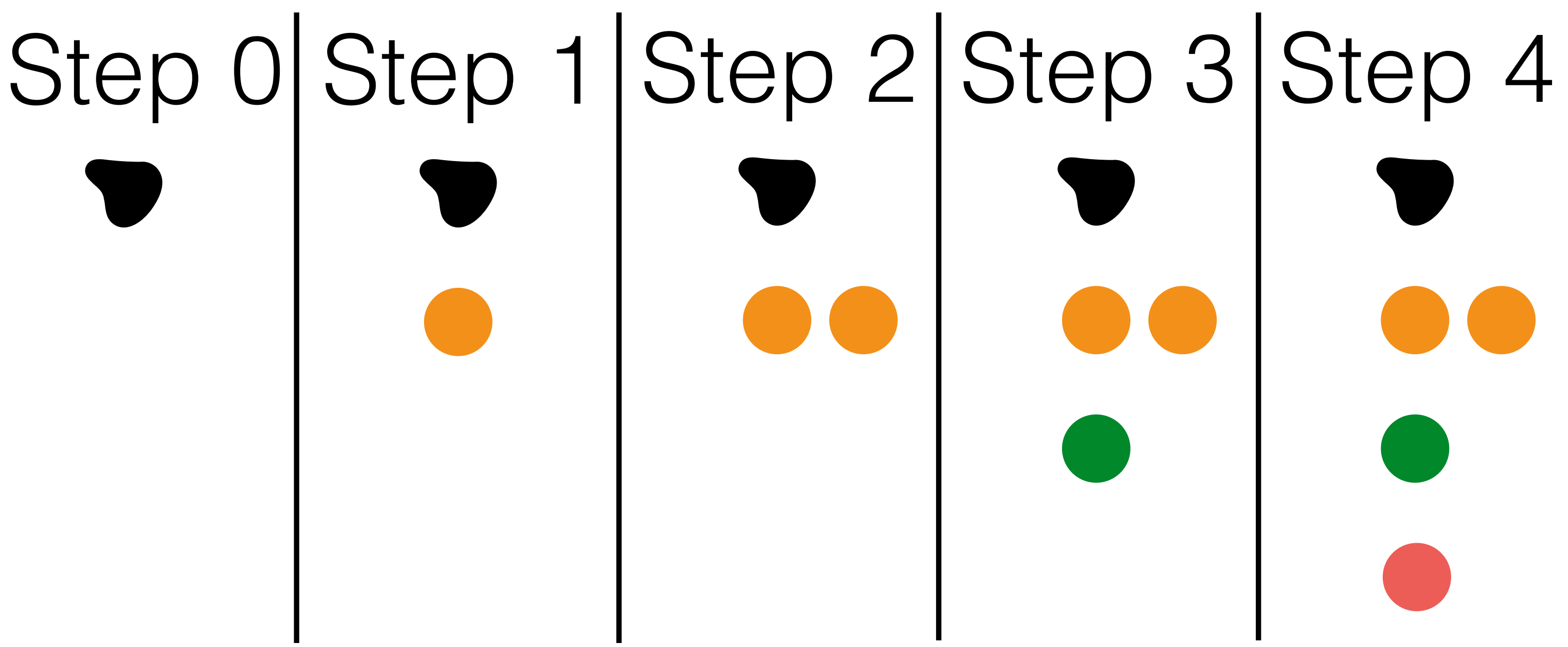

$V_k \stackrel{iid}{\sim} \mathrm{Beta}(1, \alpha)$

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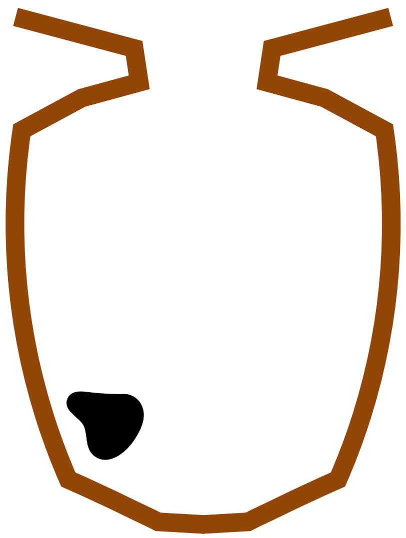
Marginal cluster assignments

- Hoppe urn / Blackwell-MacQueen urn



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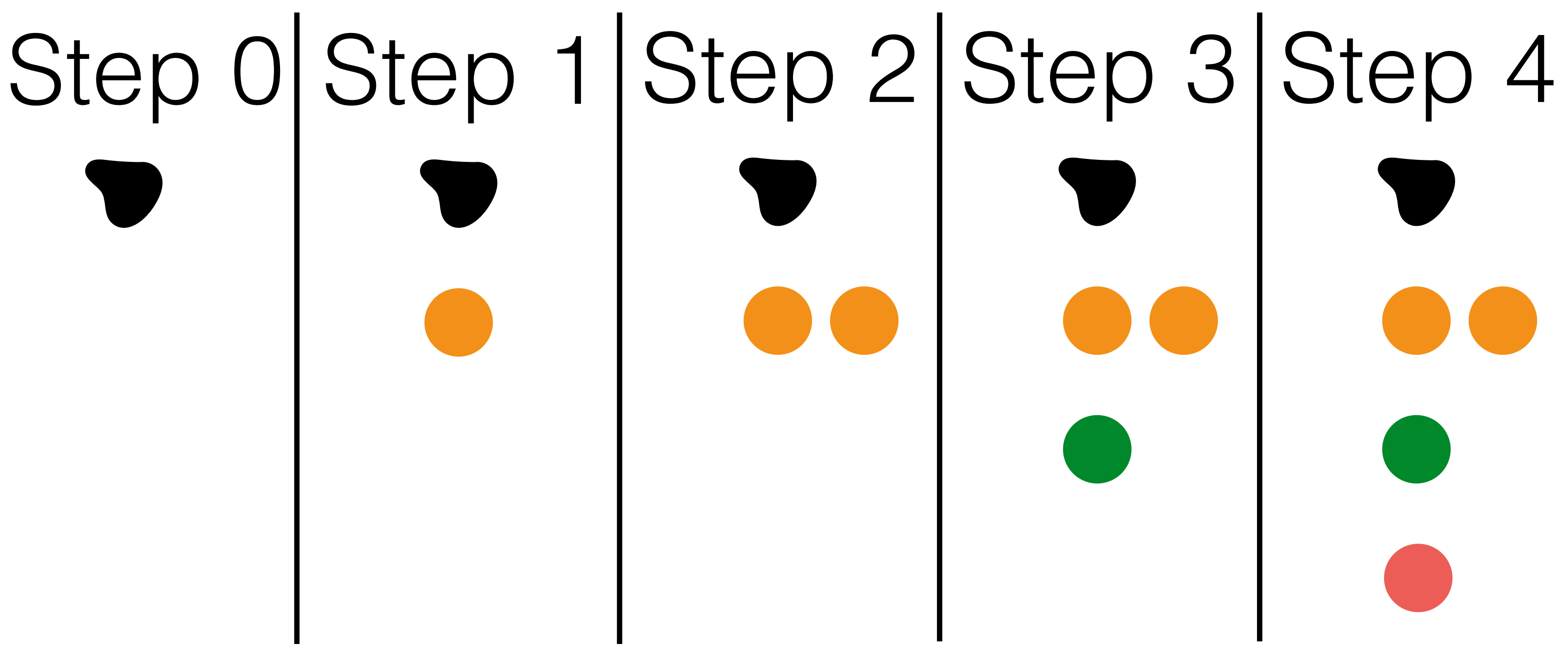

$$V_k \overset{iid}{\sim} \mathrm{Beta}(1, \alpha)$$

$$\rho_1 = V_1$$

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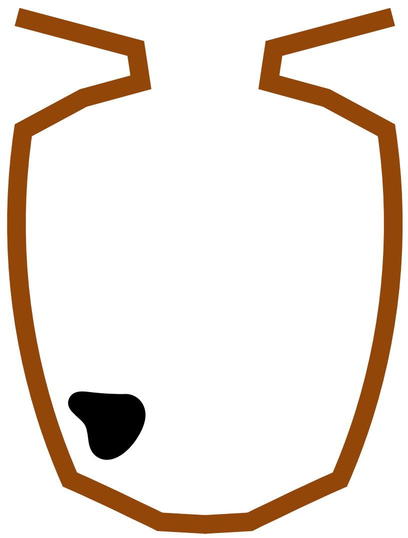

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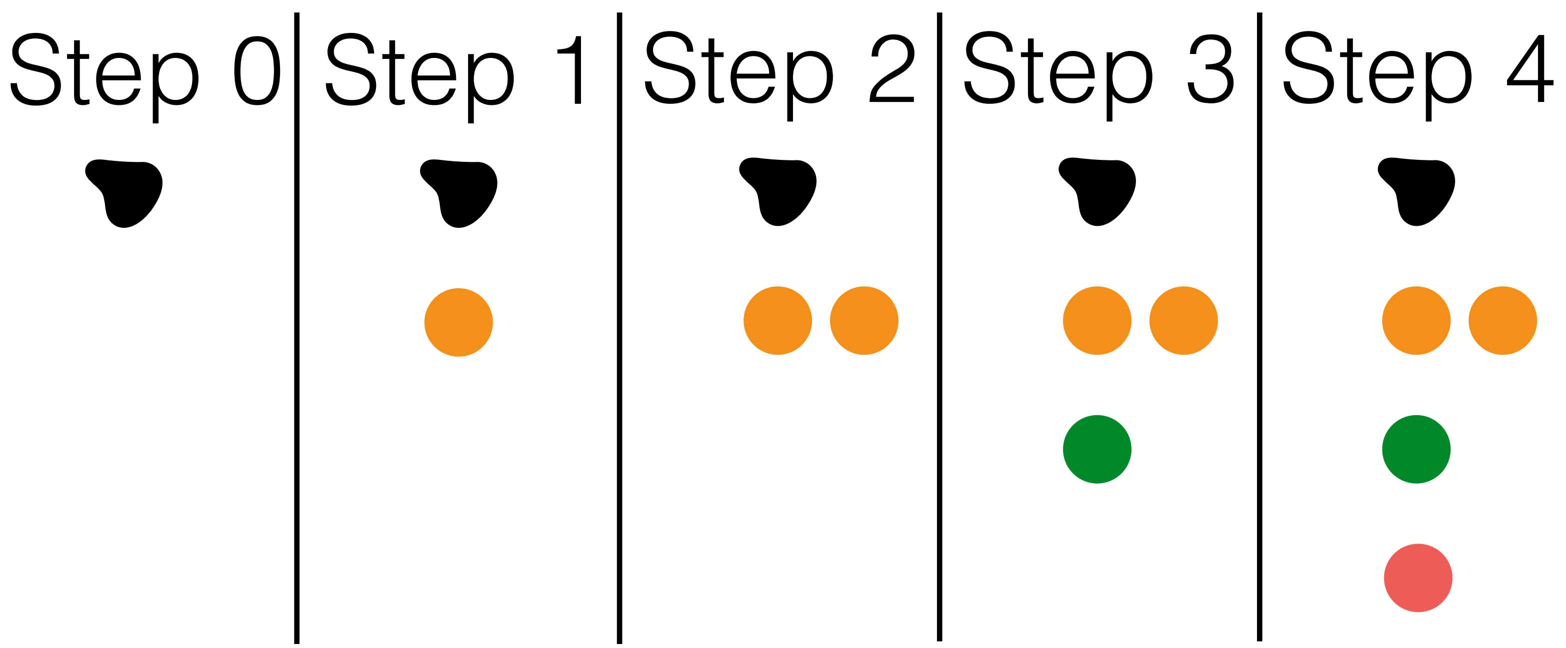

$$V_k \overset{iid}{\sim} \text{Beta}(1, \alpha)$$

$$\rho_1 = V_1$$

$$\rho_2 = (1 - V_1)V_2$$

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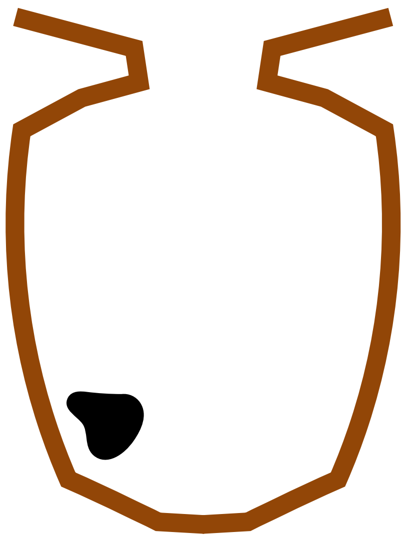

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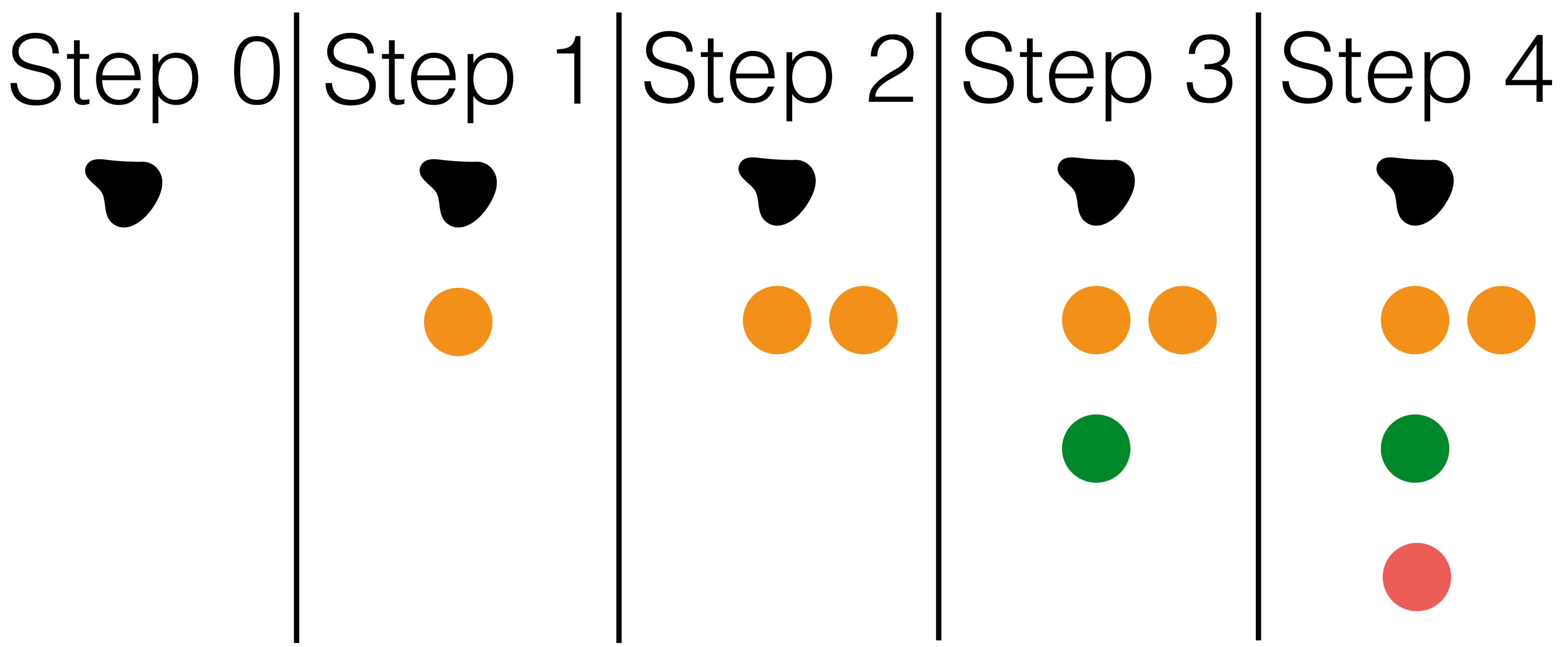

$$V_k \stackrel{iid}{\sim} \mathrm{Beta}(1, \alpha)$$

$$\rho_1 = V_1$$

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$$\rho_3 = [\prod_{k=1}^{2}(1 - V_k)]V_3$$

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- What do you think about the answer to the previous question when it comes to real-life data modeling?
- Code a Hoppe/Blackwell-MacQ urn simulator. Examine the empirical distribution of the # clusters after N customers

References

A full reference list is provided at the end of the “Part III” slides.