Nonparametric Bayesian Methods: Models, Algorithms, and Applications (Part III)

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Electrical Engineering & Computer Science
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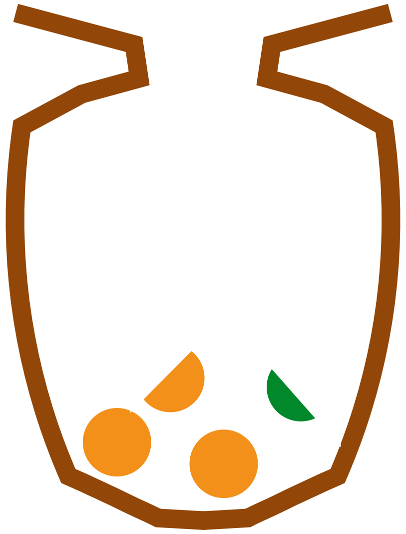
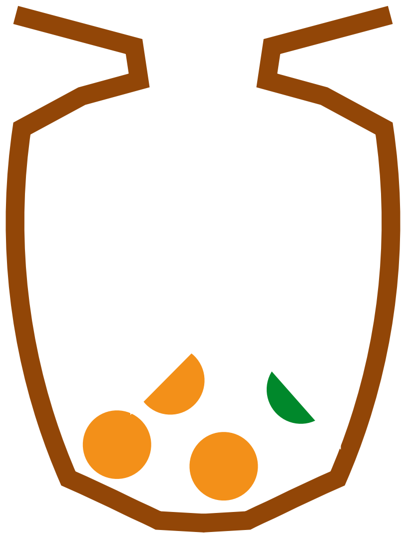
Marginal cluster assignments



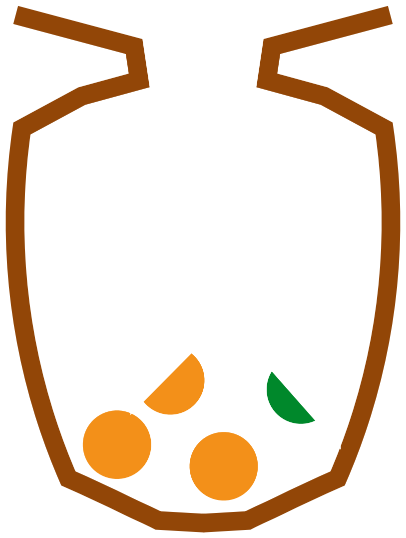
Marginal cluster assignments

- Pólya urn



Marginal cluster assignments

- Pólya urn
  - Choose any ball with prob proportional to its mass
  - Replace and add ball of same color



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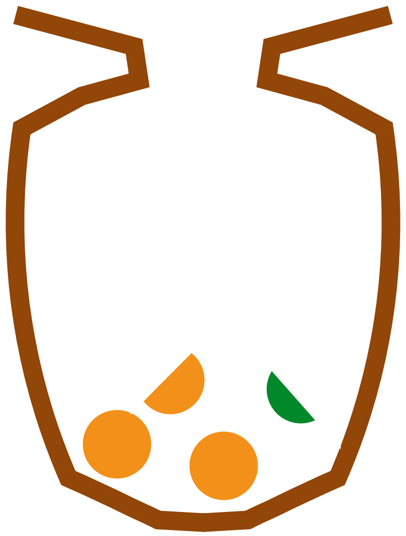
$$\mathrm{PolyaUrn}(a_{\mathrm{orange}}, a_{\mathrm{green}})$$



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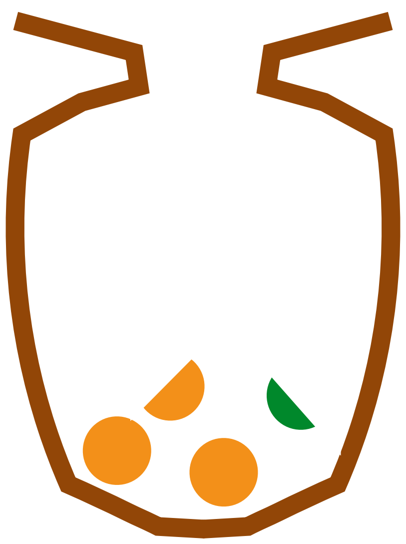


$$\lim_{n\to\infty} \frac{\#\ \mathrm{orange}}{\#\ \mathrm{total}} = \rho_{\mathrm{orange}} \stackrel{d}{=} \mathrm{Beta}(a_{\mathrm{orange}}, a_{\mathrm{green}})$$

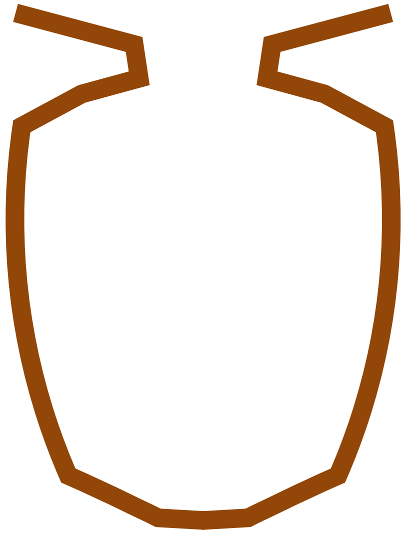
Marginal cluster assignments

- Hoppe urn / Blackwell-MacQueen urn

[Blackwell, MacQueen 1973; Hoppe 1984]

Marginal cluster assignments

- Hoppe urn / Blackwell-MacQueen urn



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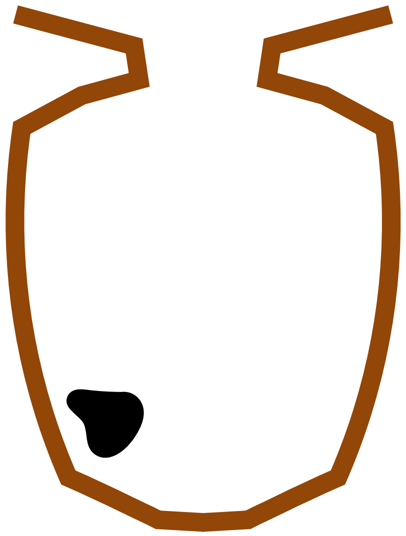

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Marginal cluster assignments

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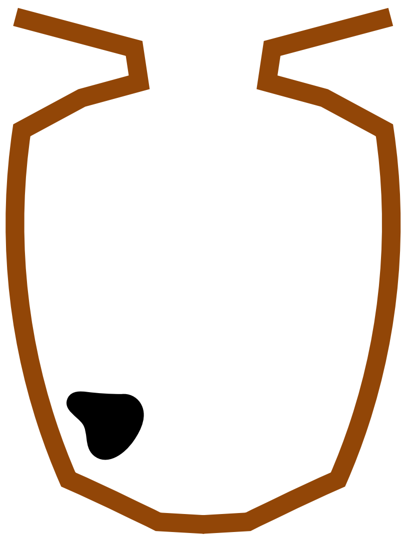

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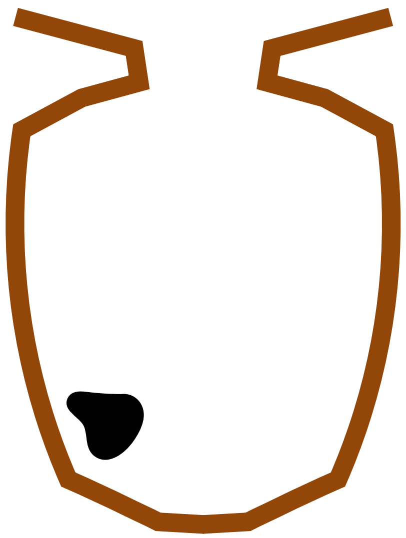

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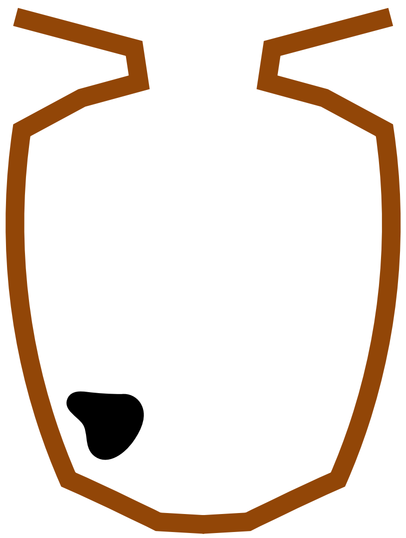

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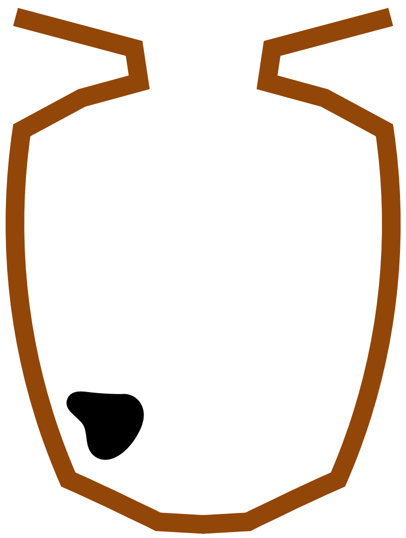

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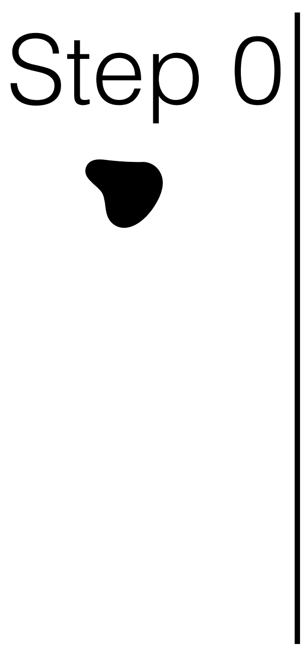

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Marginal cluster assignments

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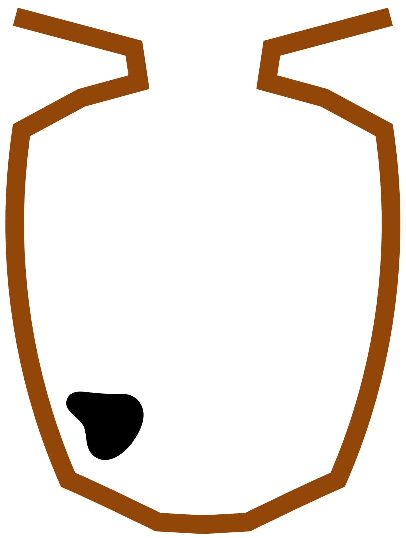

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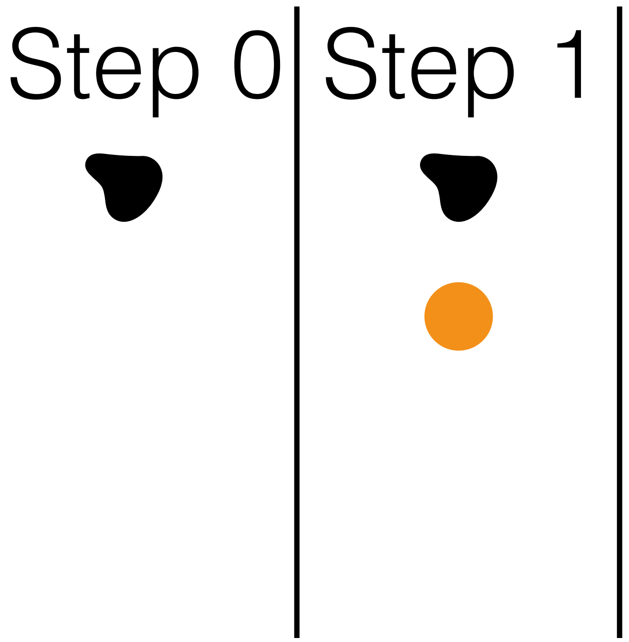

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Marginal cluster assignments

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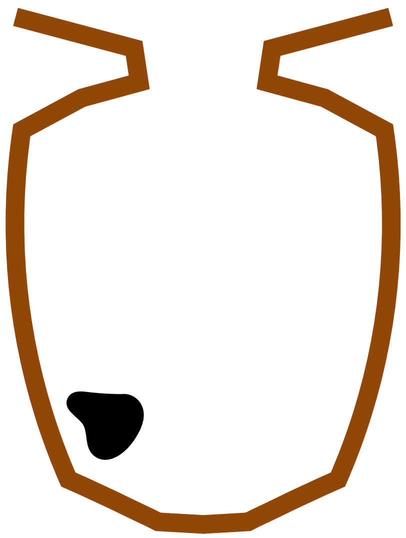

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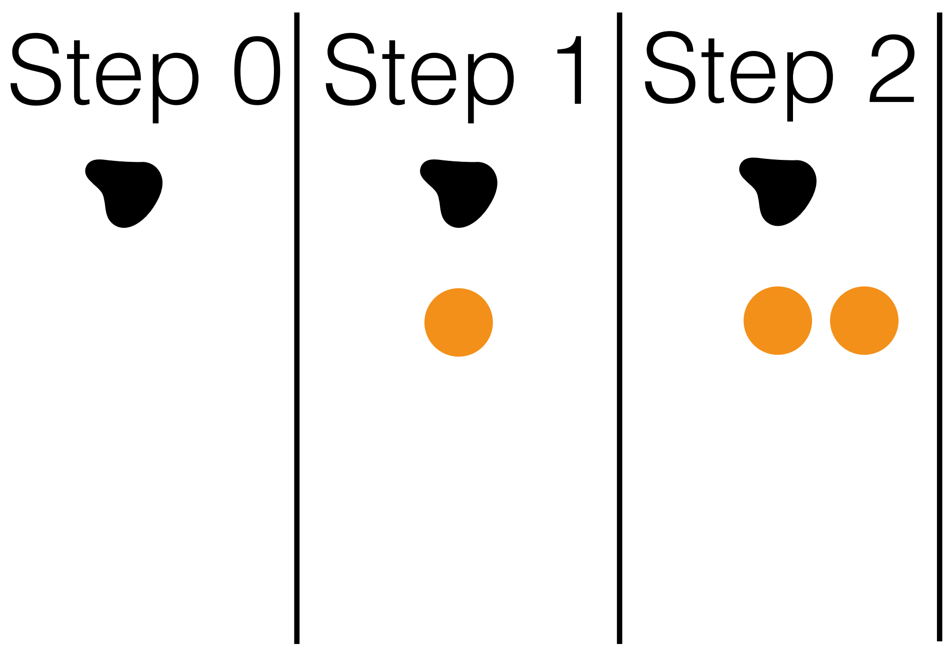

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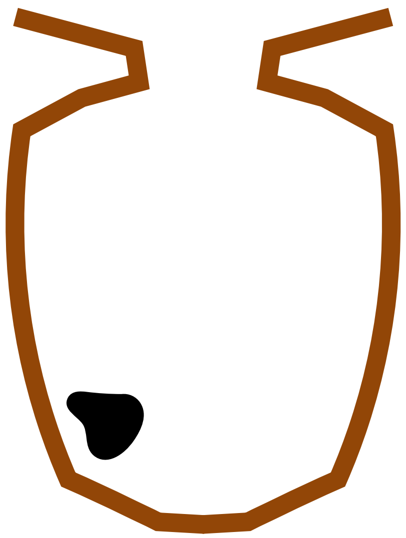

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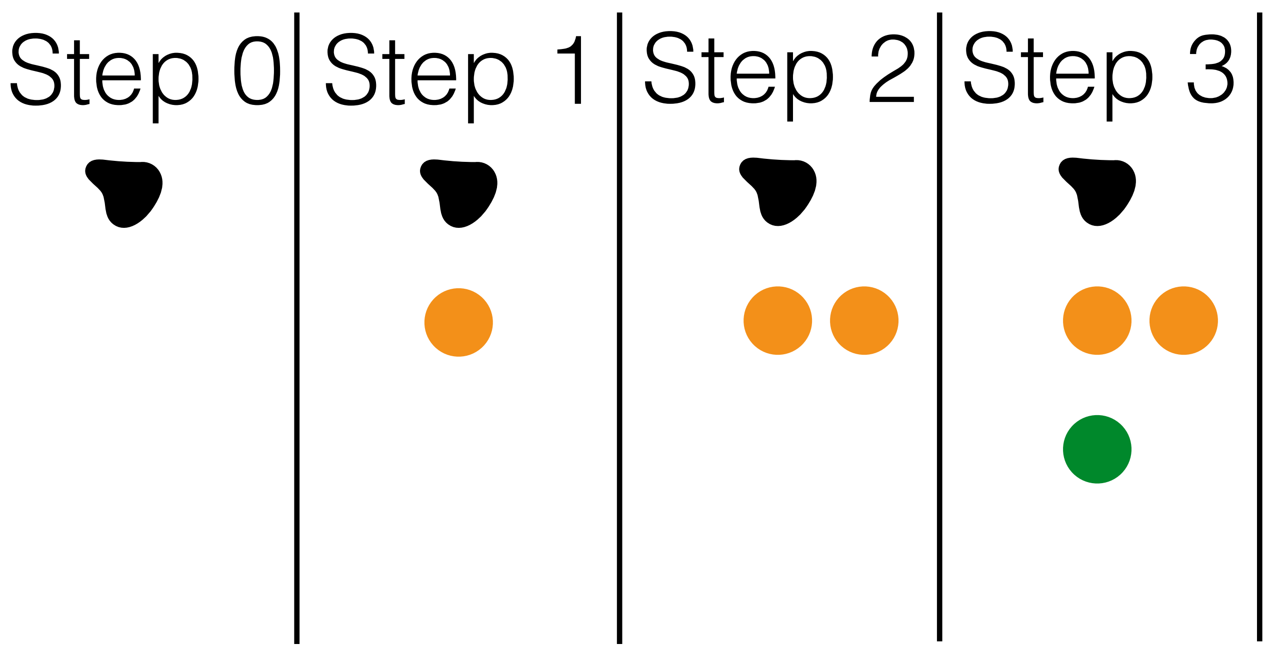

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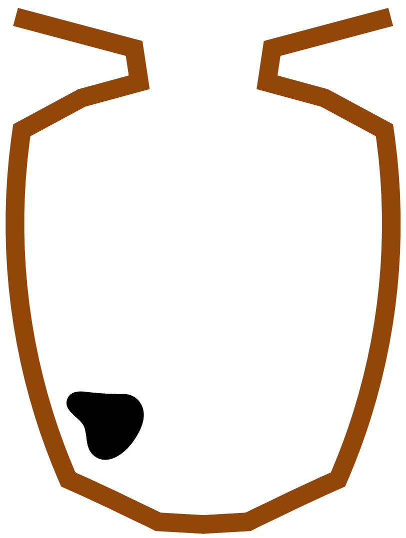

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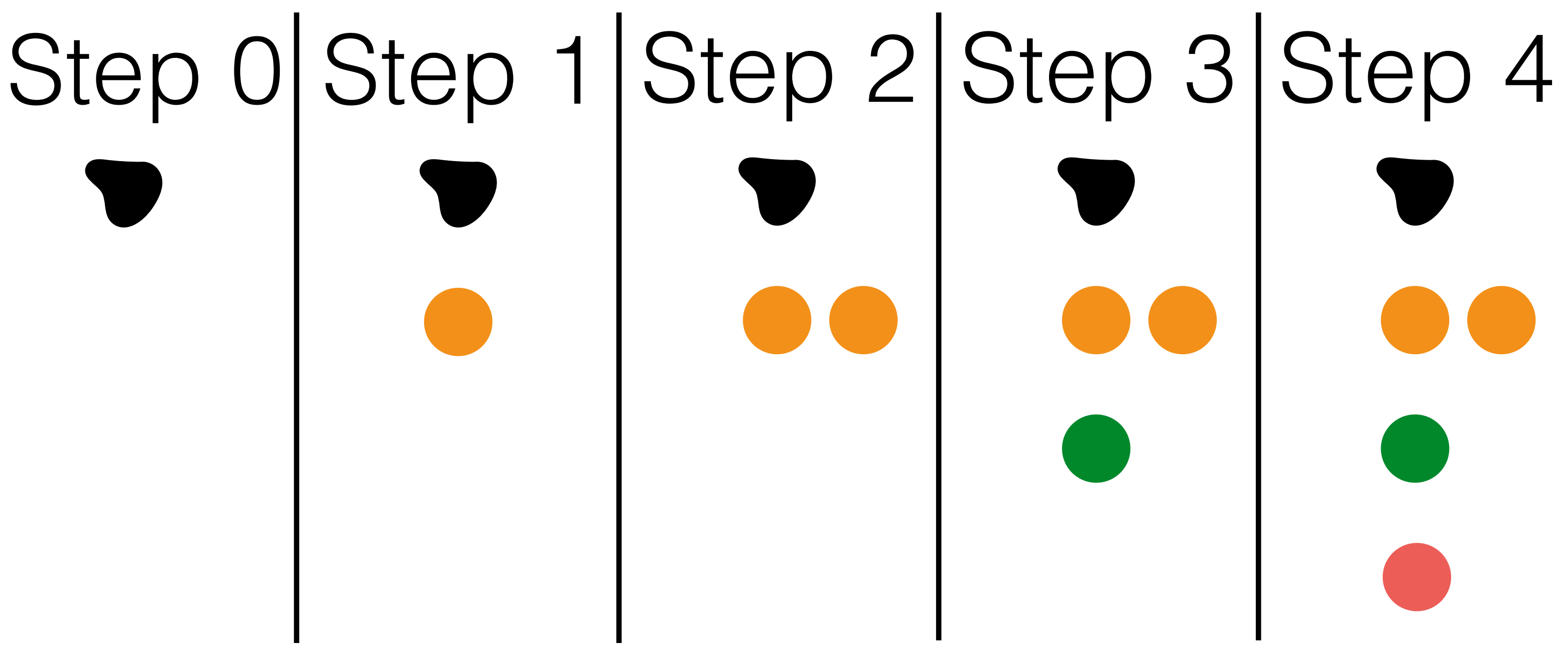

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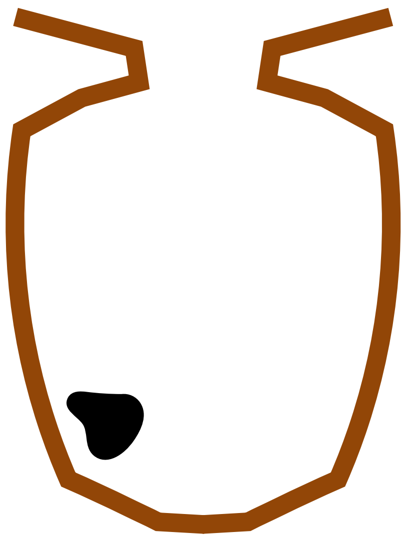

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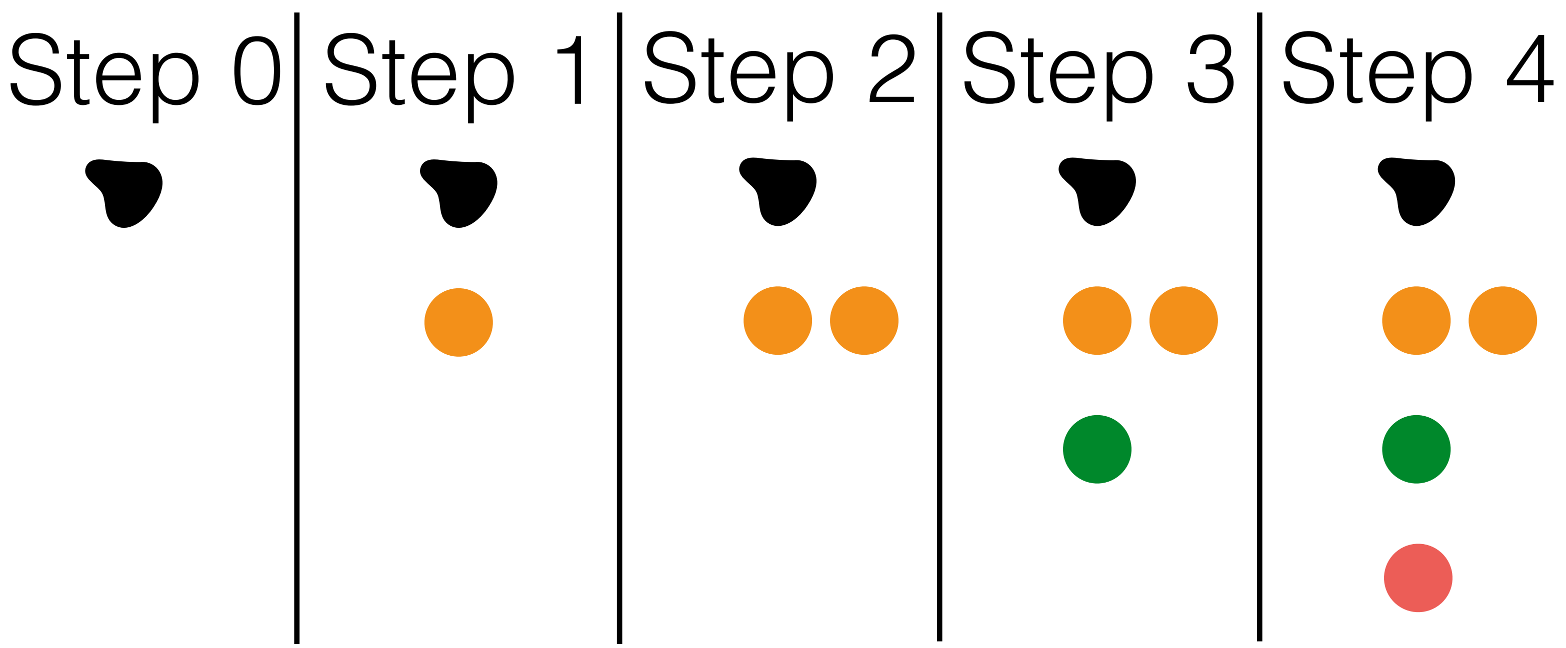
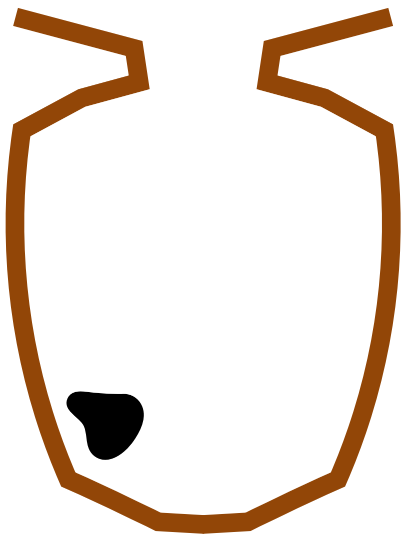


$$(\#\text{orange}, \#\text{other}) = \text{PolyaUrn}(1, \alpha)$$

Marginal cluster assignments

- Hoppe urn / Blackwell-MacQueen urn



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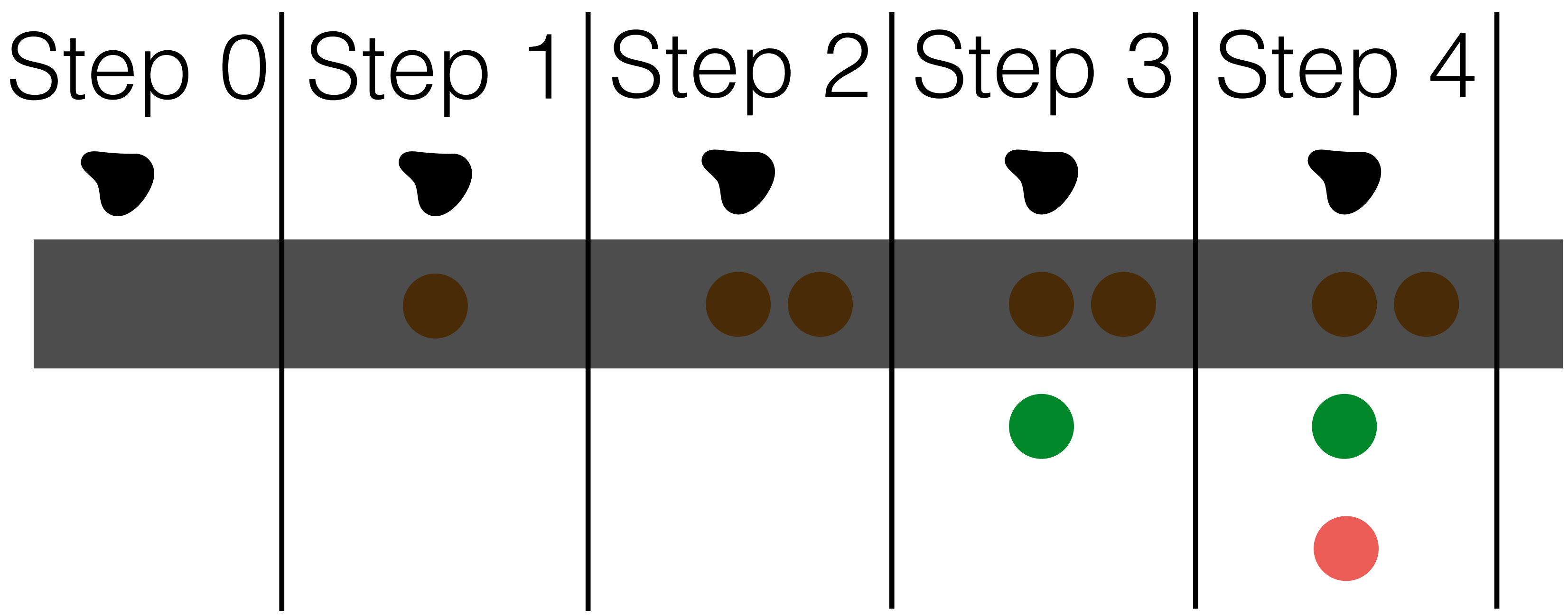

$$(\#\text{orange}, \#\text{other}) = \text{PolyaUrn}(1, \alpha)$$

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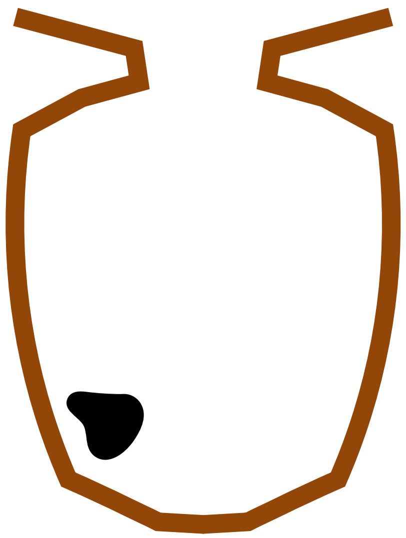

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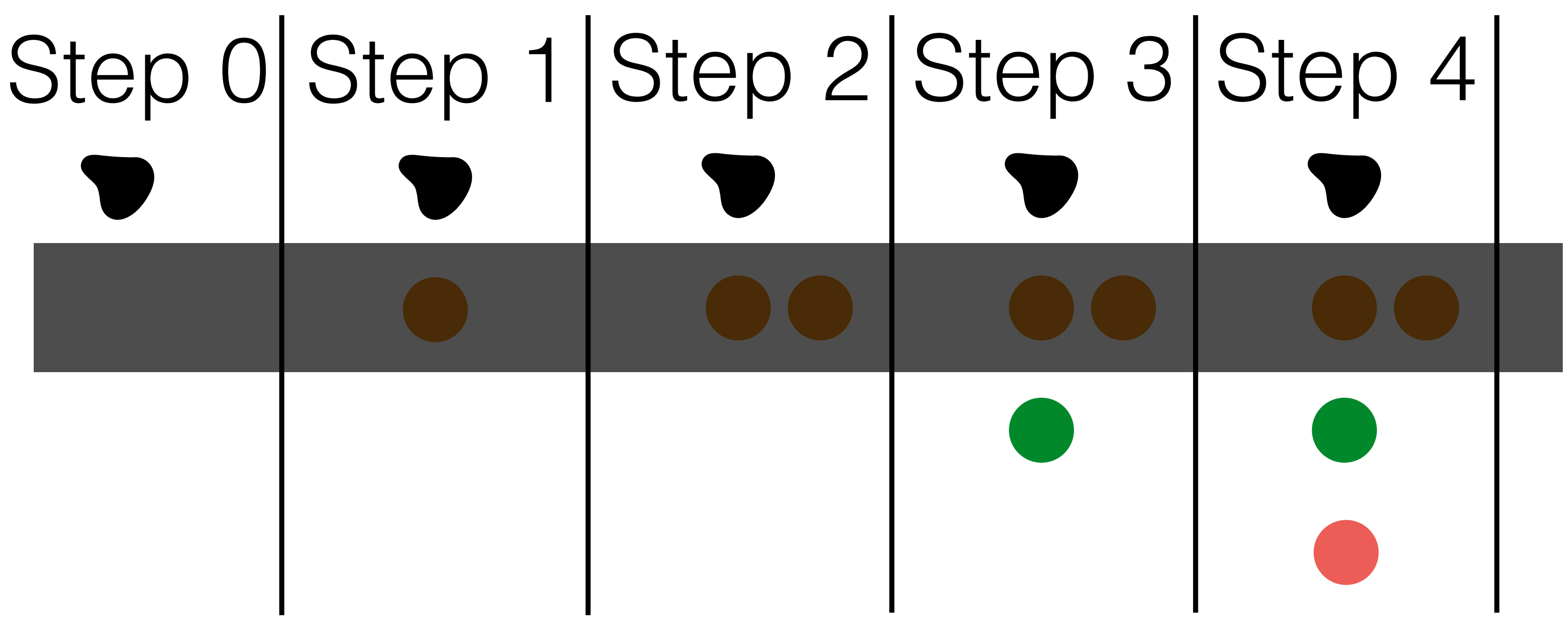

$$(\#\text{orange}, \#\text{other}) = \text{PolyaUrn}(1, \alpha)$$

- not orange: $(\#\text{green}, \#\text{other}) = \text{PolyaUrn}(1, \alpha)$

Marginal cluster assignments

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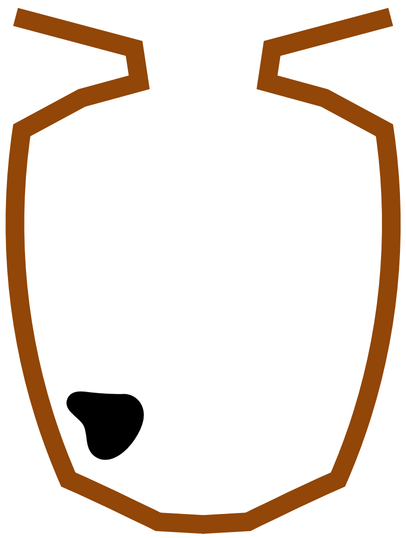

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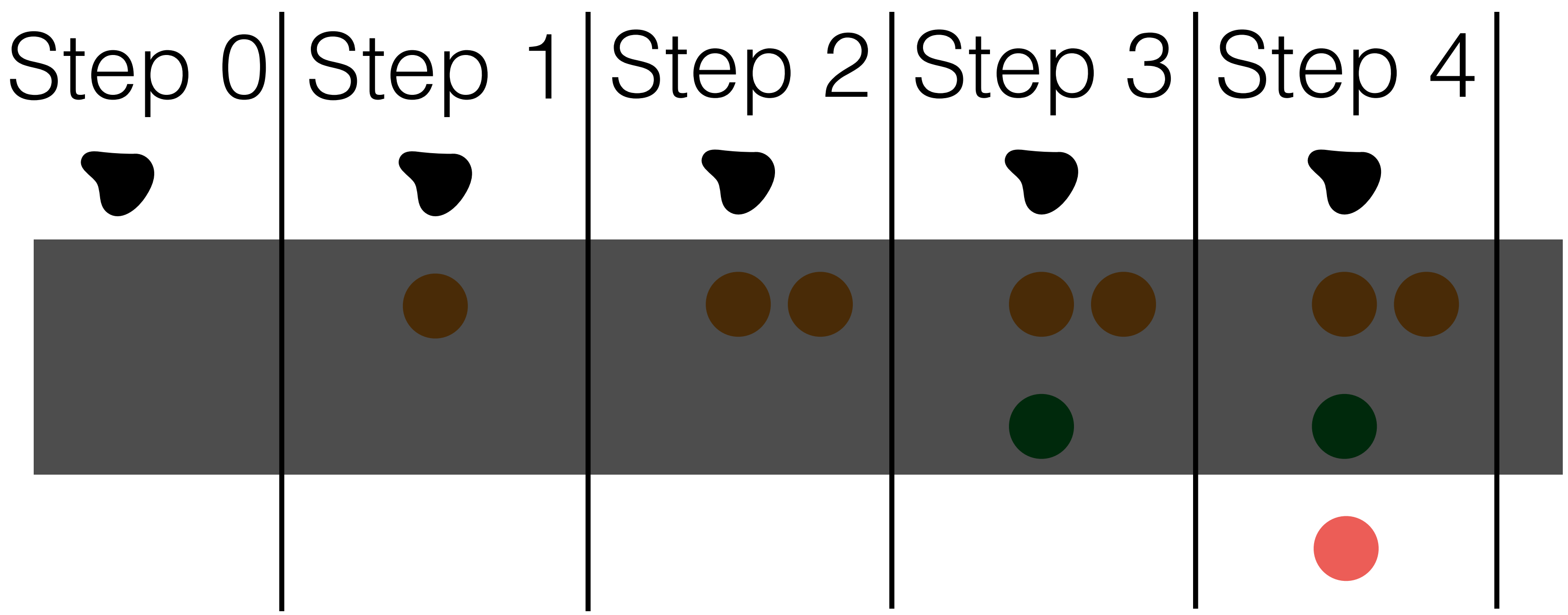

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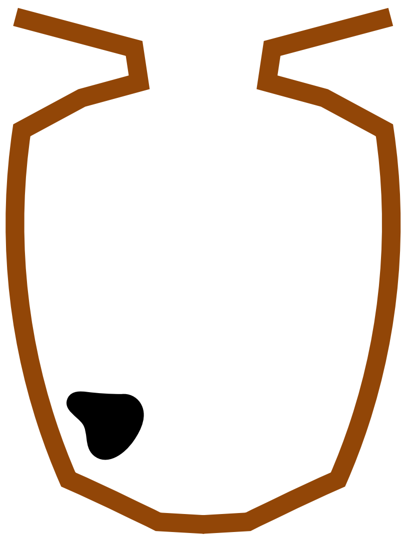

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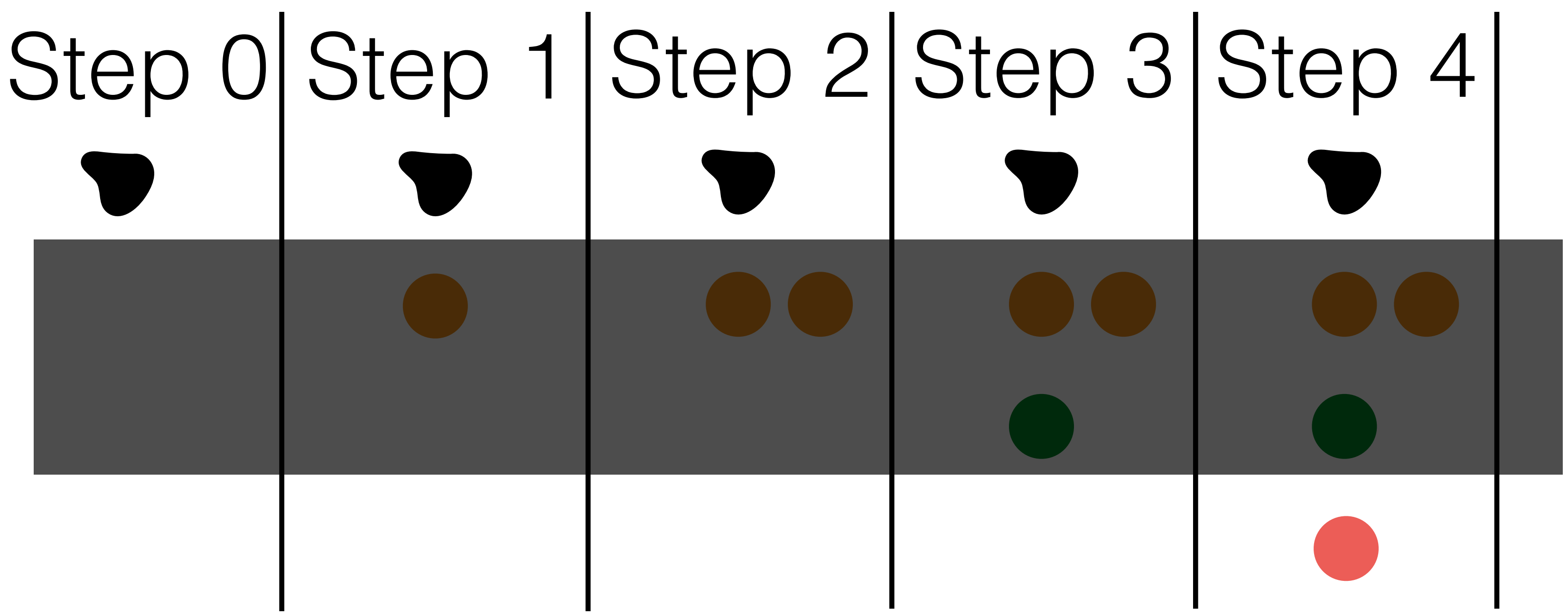

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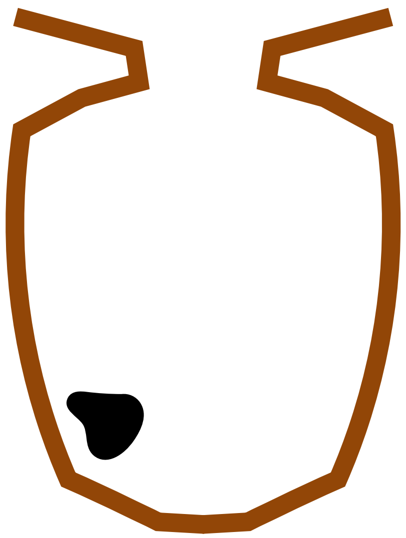

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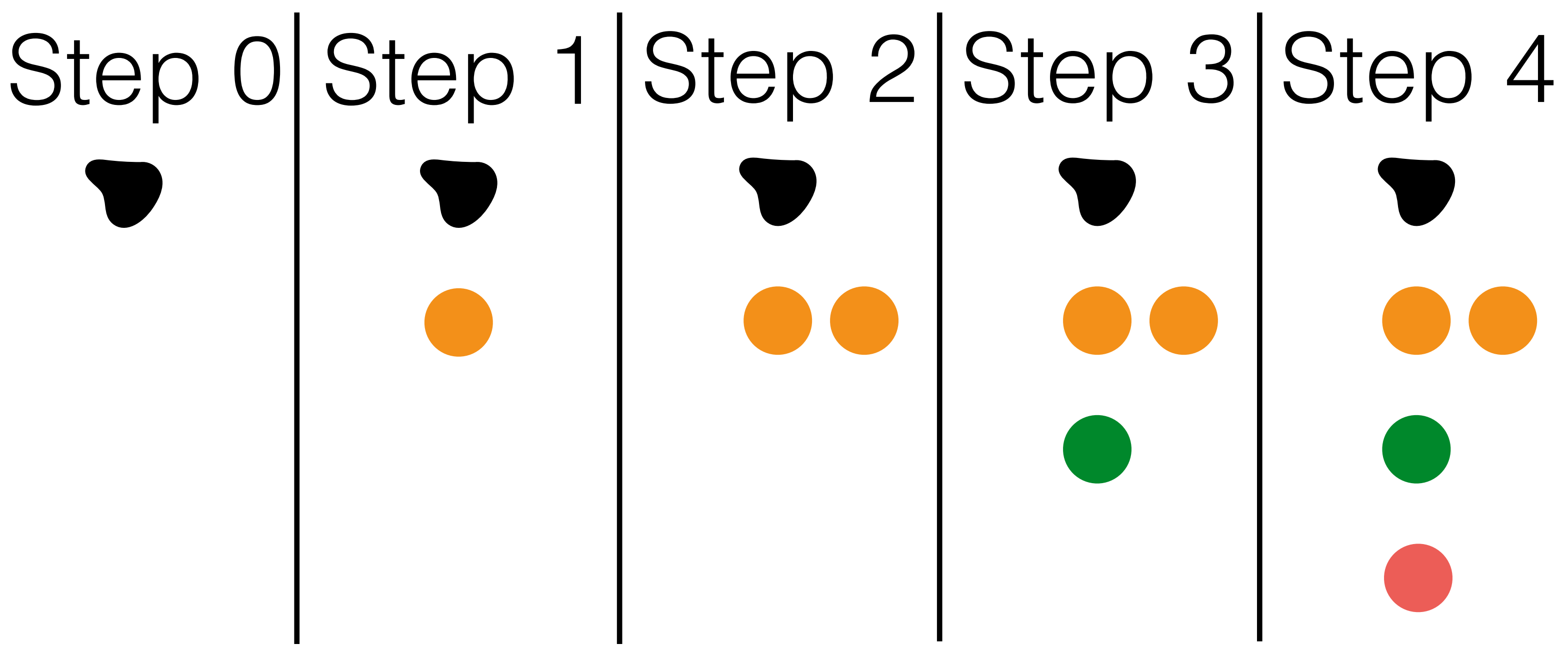

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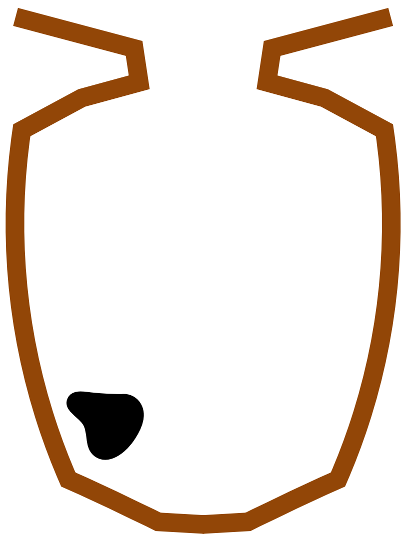

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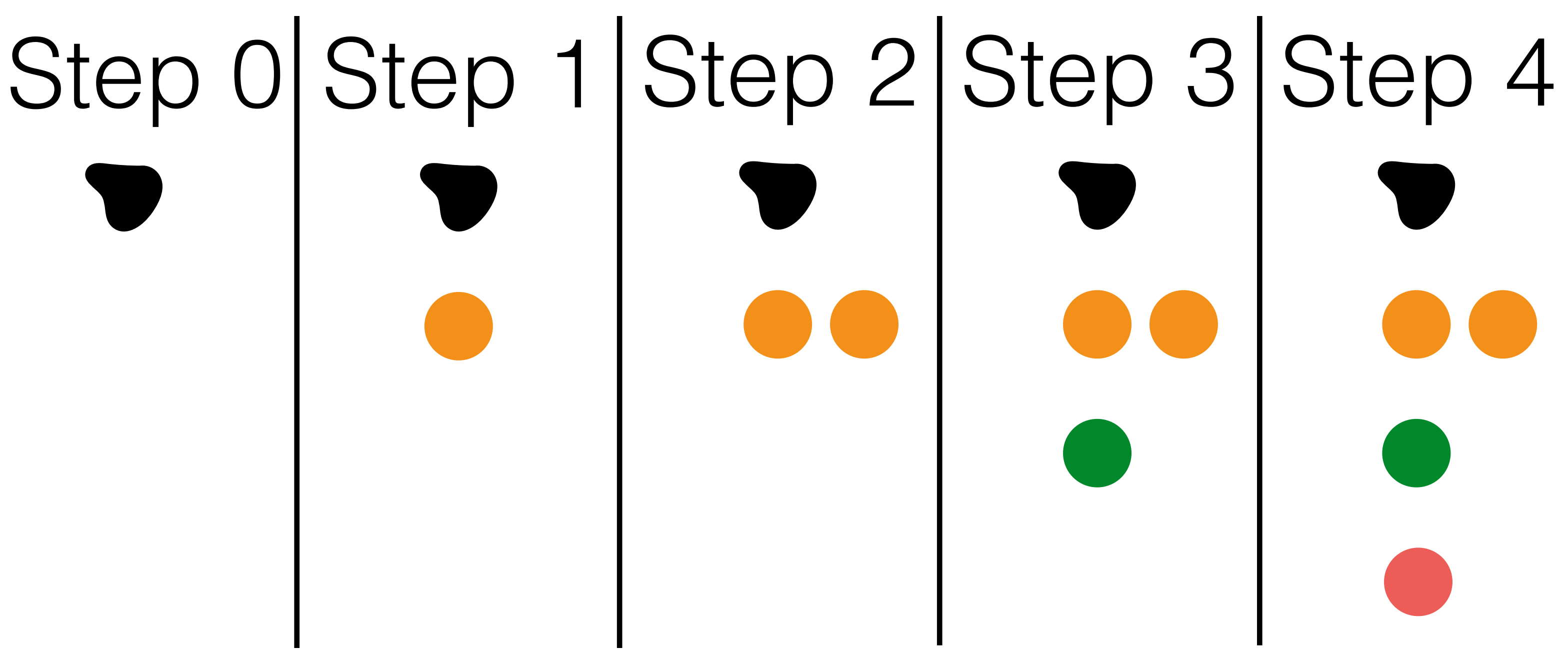

$$V_k \overset{iid}{\sim} \text{Beta}(1, \alpha)$$

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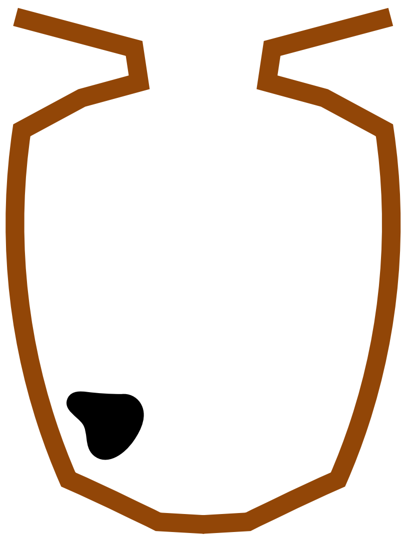

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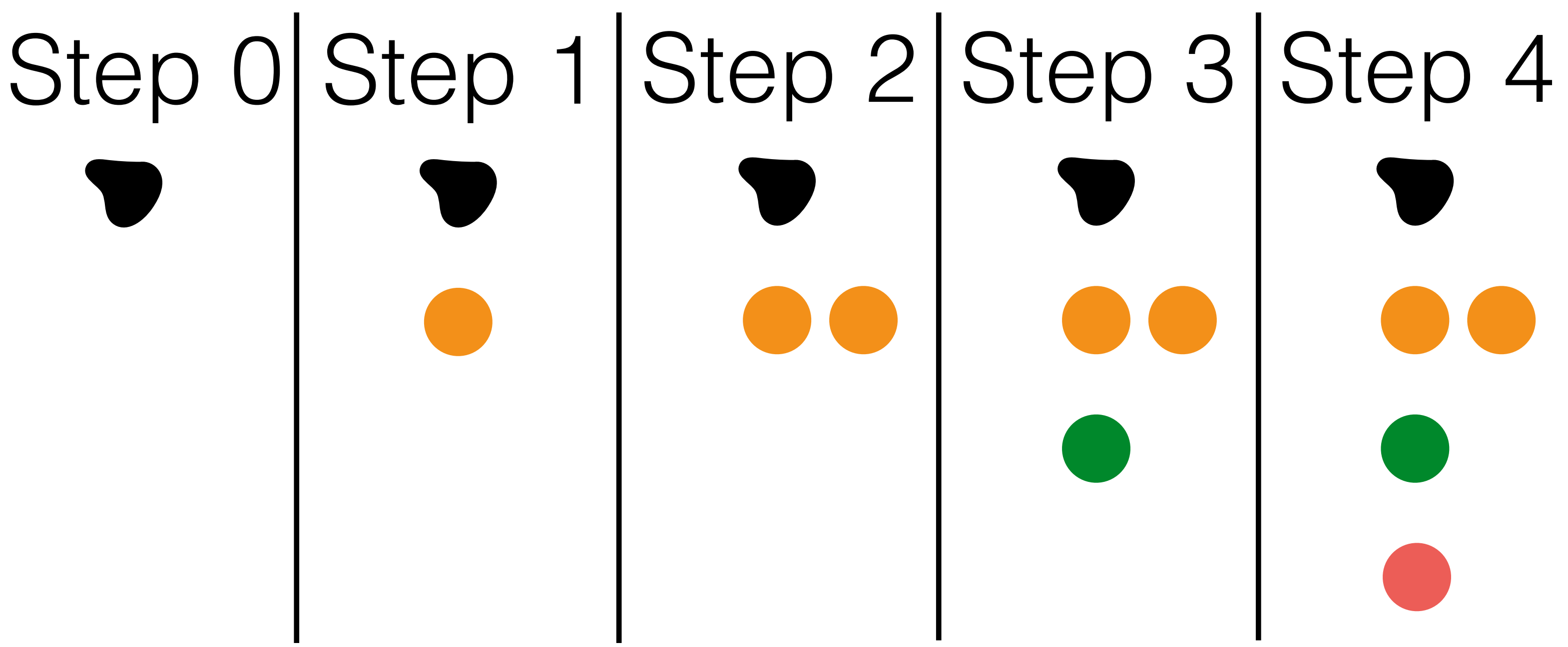

$$V_k \overset{iid}{\sim} \text{Beta}(1, \alpha)$$

$$\rho_1 = V_1$$

$$(\#\text{orange}, \#\text{other}) = \text{PolyaUrn}(1, \alpha)$$

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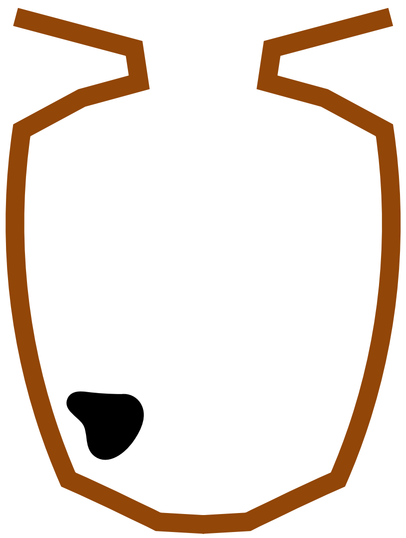

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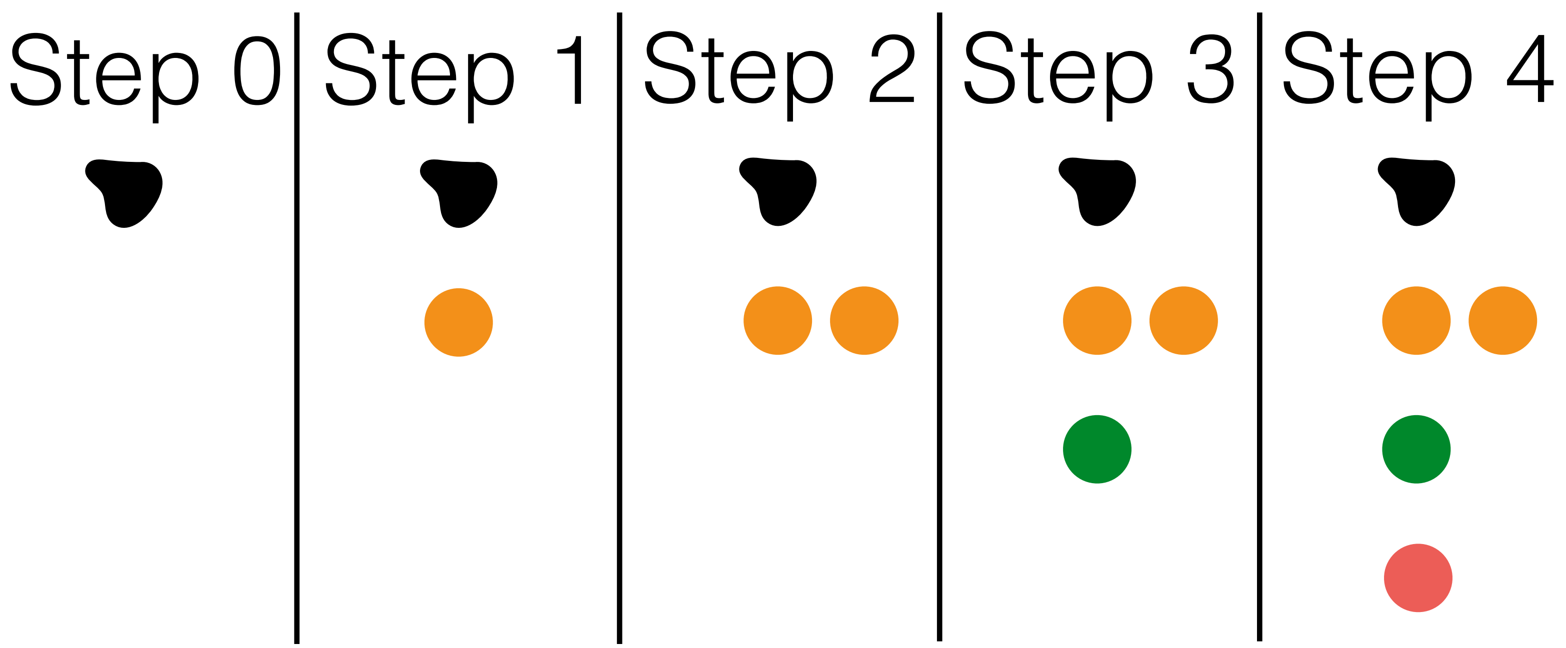

$$V_k \overset{iid}{\sim} \mathrm{Beta}(1, \alpha)$$

$$\rho_1 = V_1$$

$$\rho_2 = (1 - V_1) V_2$$

$$(\#\text{orange}, \#\text{other}) = \text{PolyaUrn}(1, \alpha)$$

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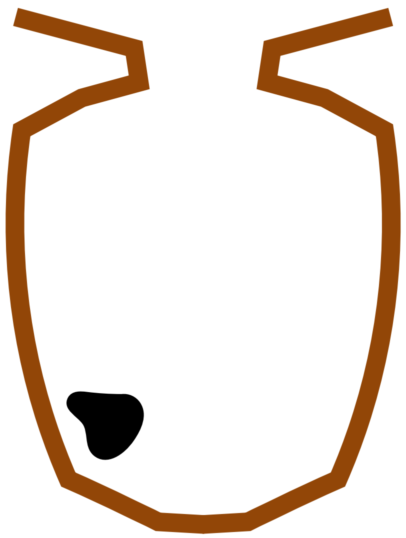

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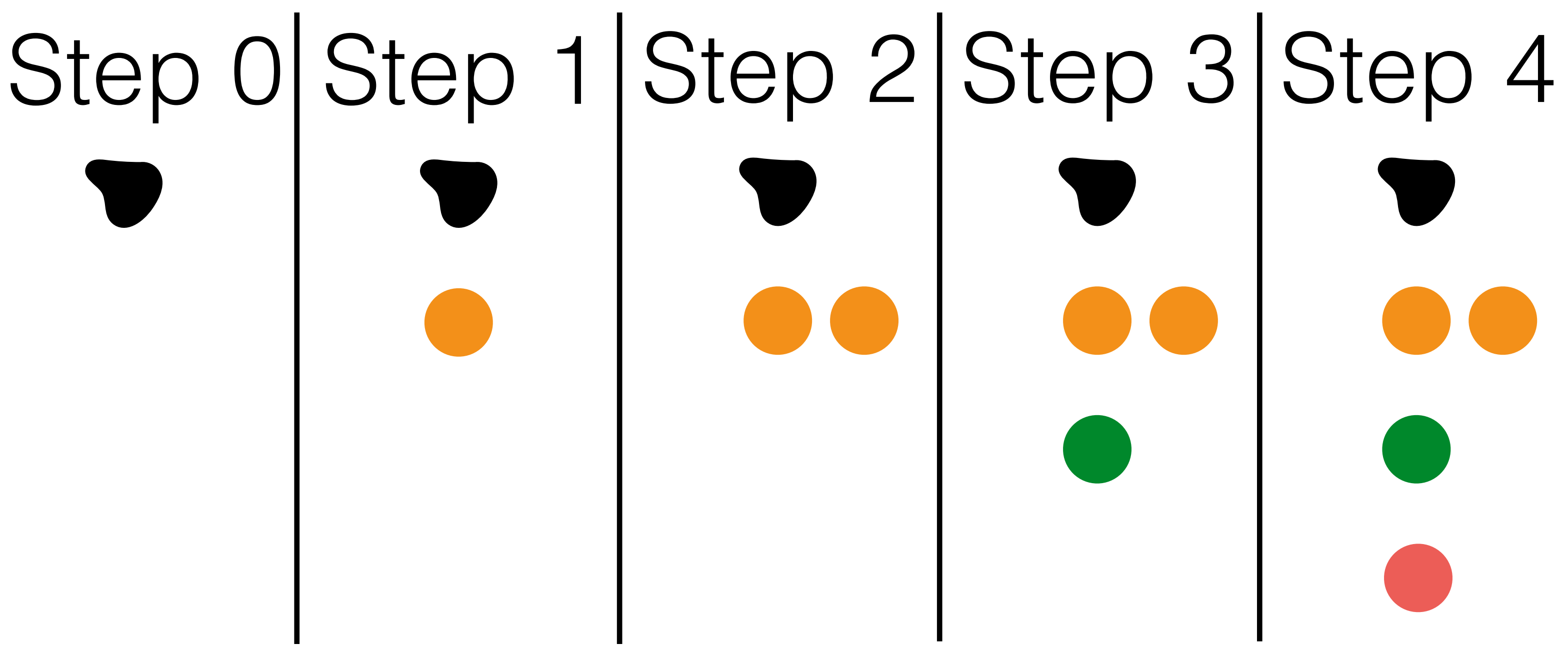

$$V_k \overset{iid}{\sim} \mathrm{Beta}(1, \alpha)$$

$$\rho_1 = V_1$$

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$$\rho_3 = [\prod_{k=1}^{2}(1 - V_k)]V_3$$

$$(\#\text{orange}, \#\text{other}) = \text{PolyaUrn}(1, \alpha)$$

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Chinese restaurant process

Chinese restaurant process

- Same thing we just did

Chinese restaurant process

- Same thing we just did
- Each customer walks into the restaurant

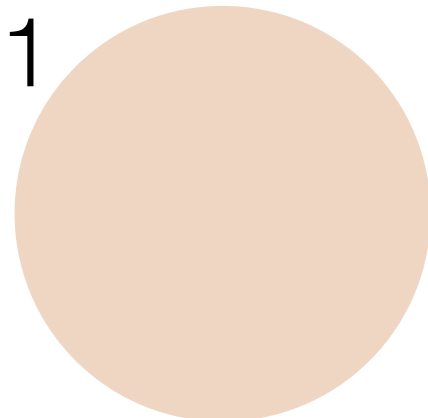
Chinese restaurant process

- Same thing we just did
- Each customer walks into the restaurant
  - Sits at existing table with prob proportional to # people there

Chinese restaurant process

- Same thing we just did
- Each customer walks into the restaurant
  - Sits at existing table with prob proportional to # people there
  - Forms new table with prob proportional to α

Chinese restaurant process



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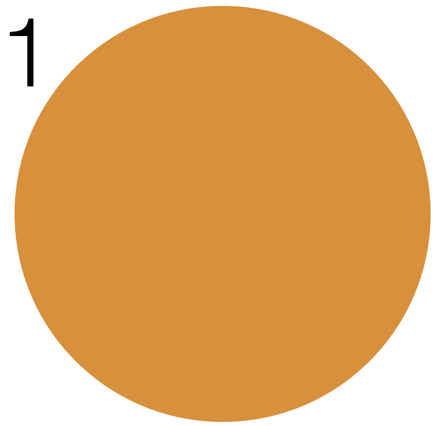
Chinese restaurant process



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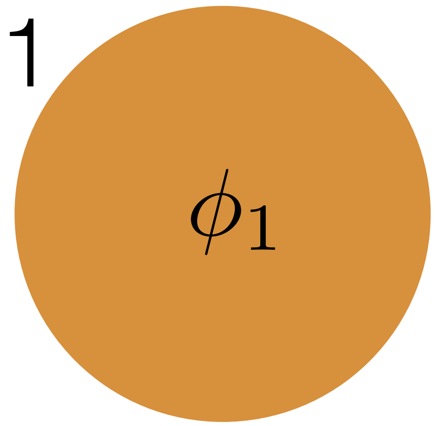

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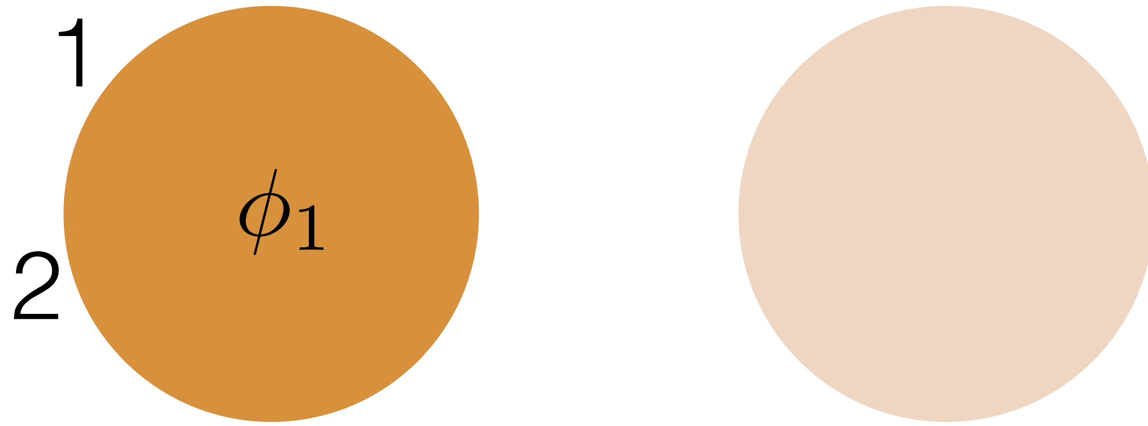

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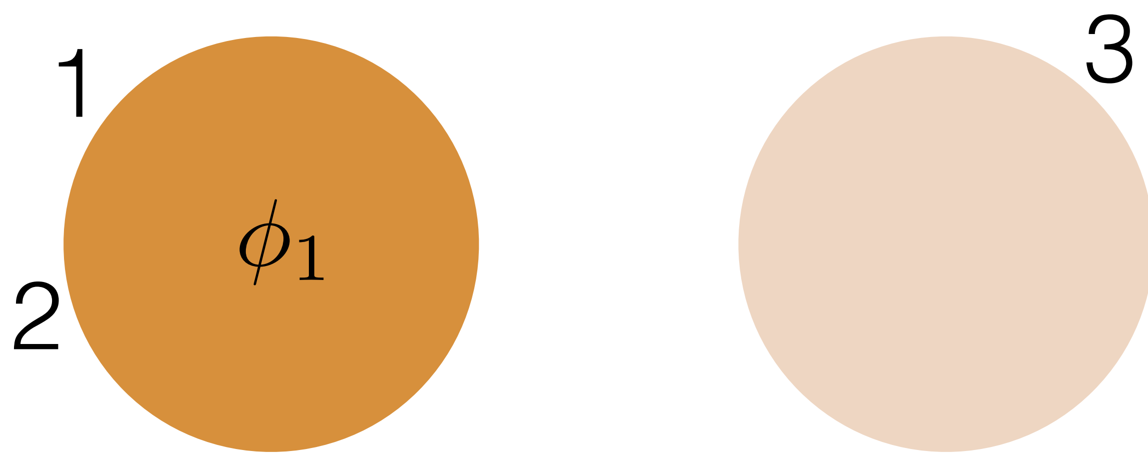

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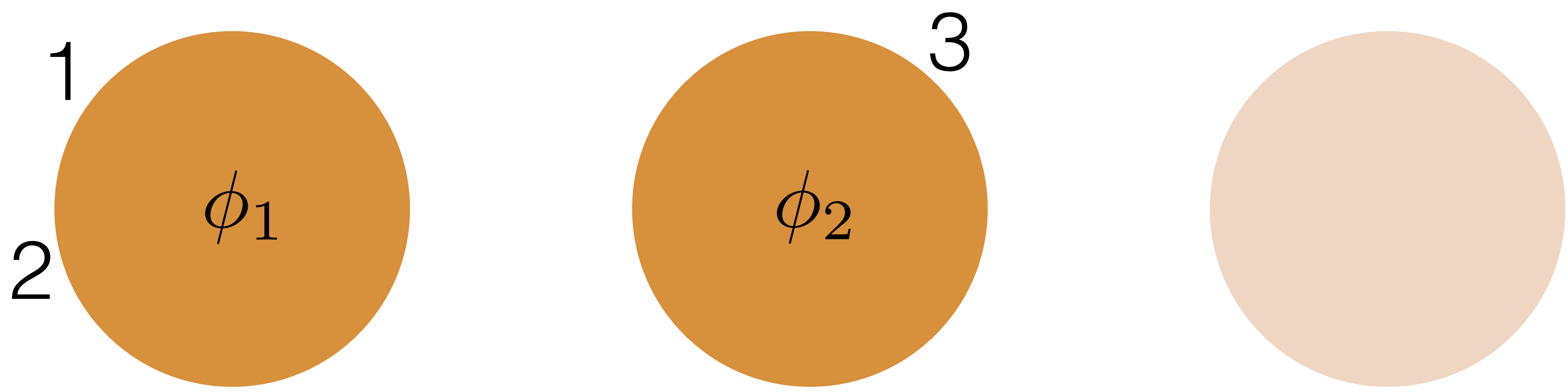

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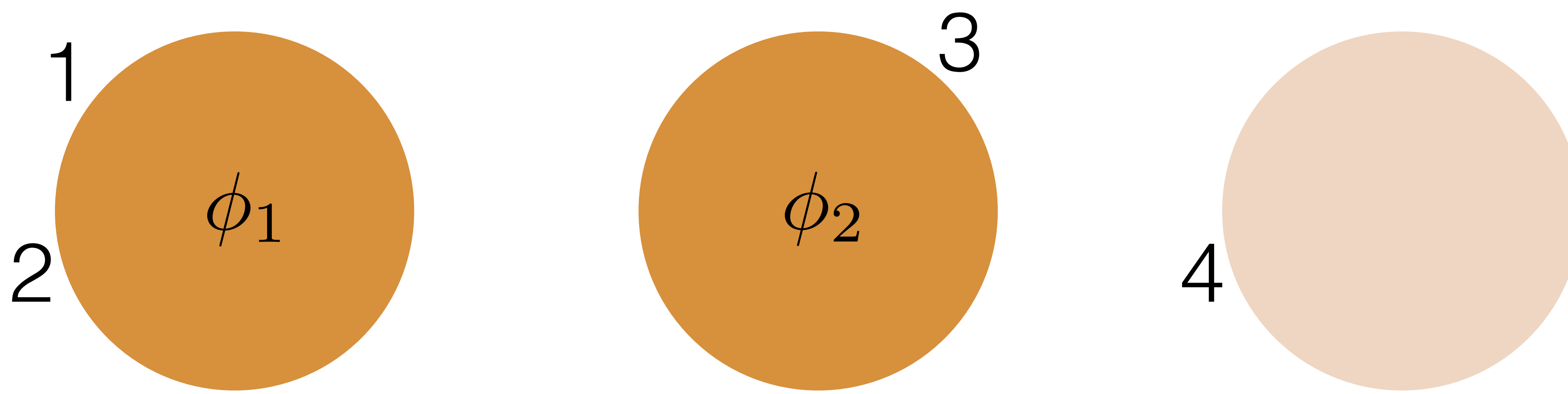

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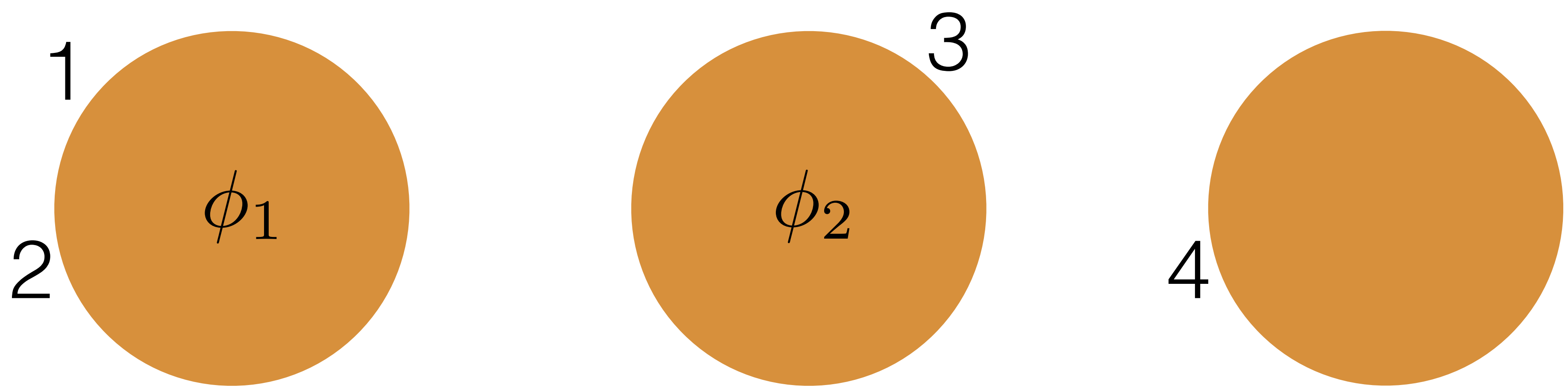

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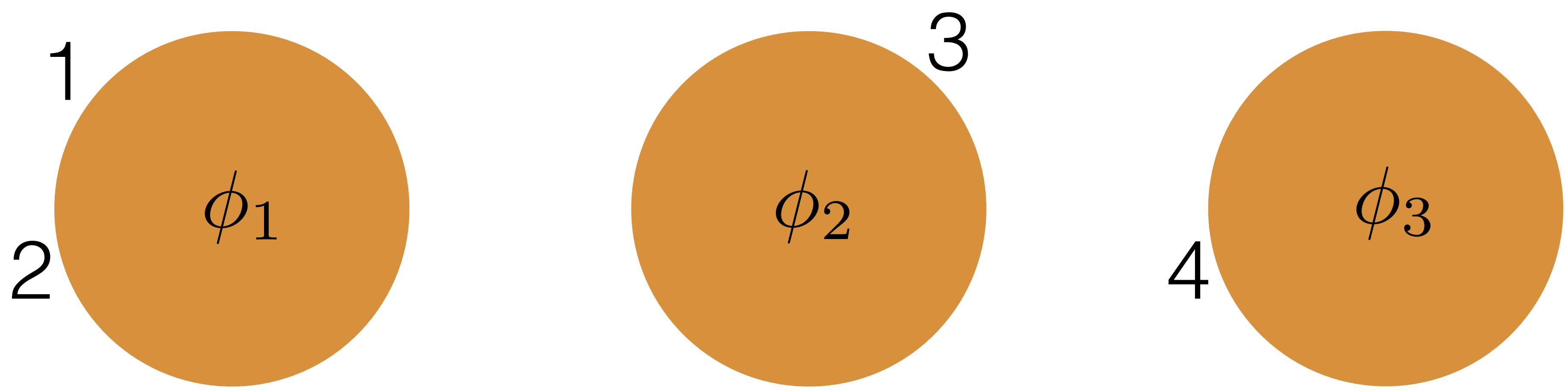
Chinese restaurant process



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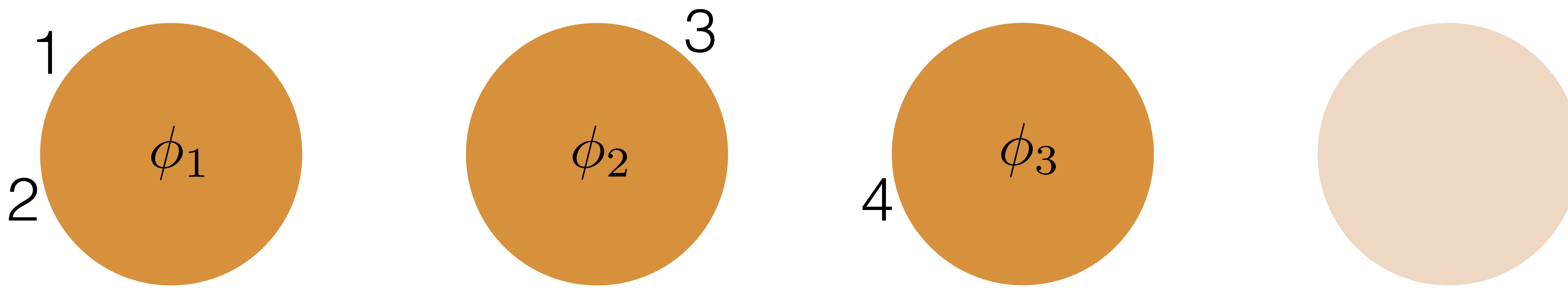

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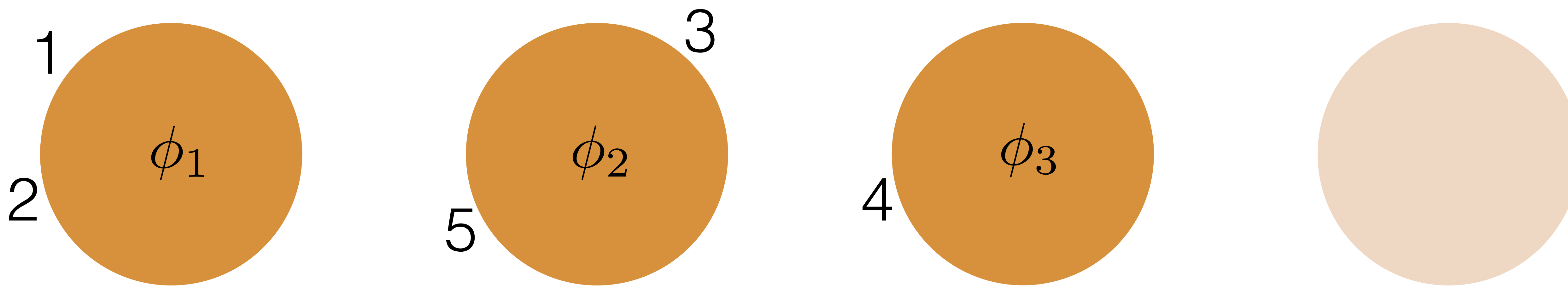

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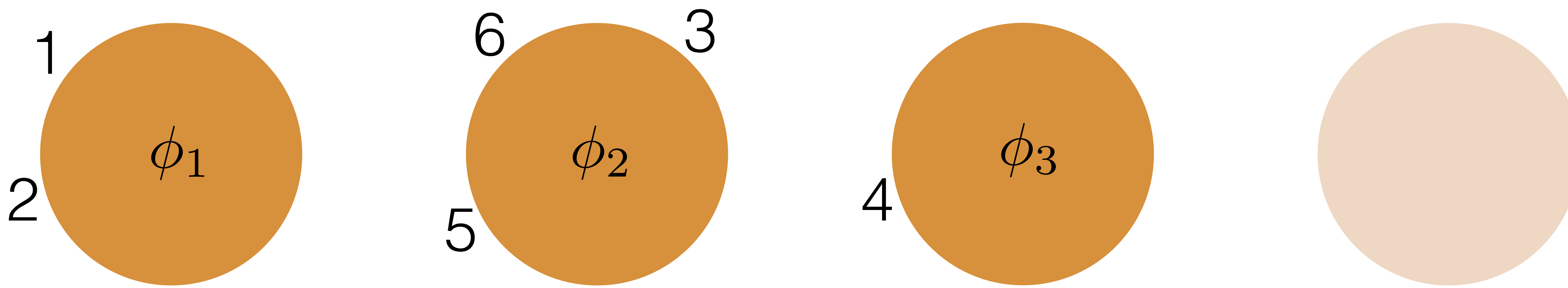

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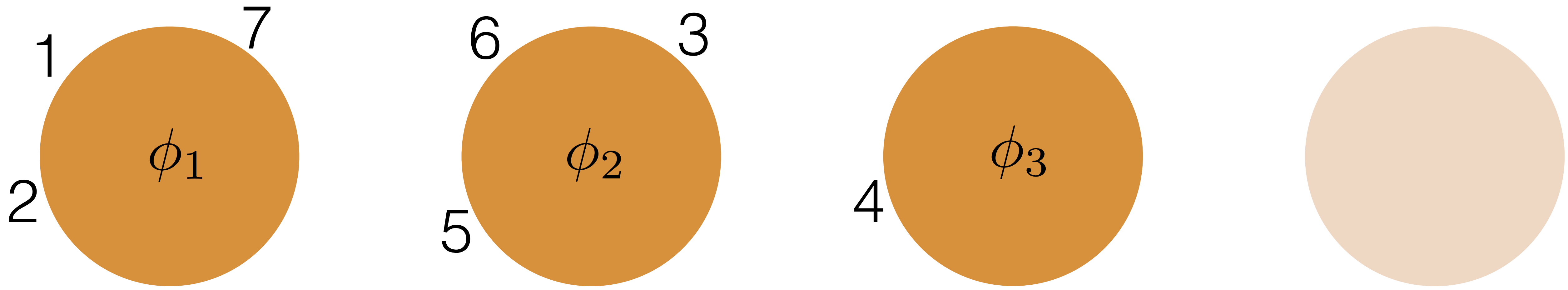

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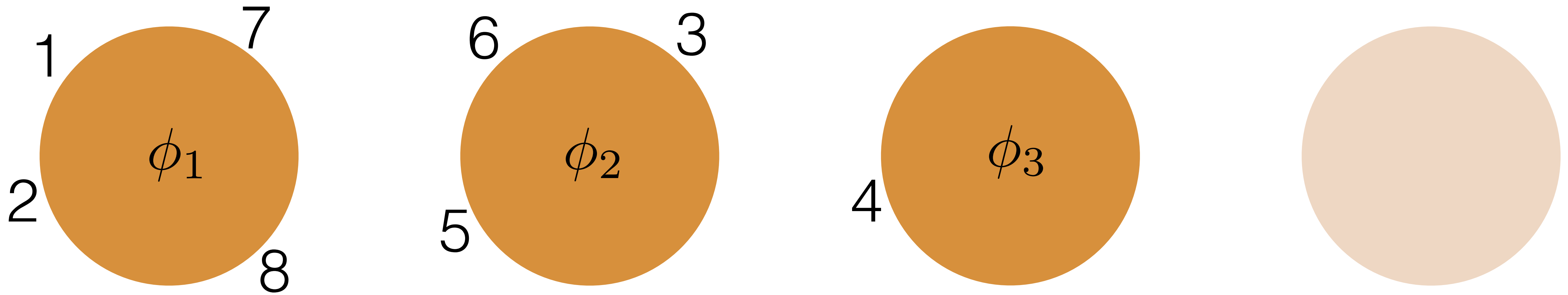

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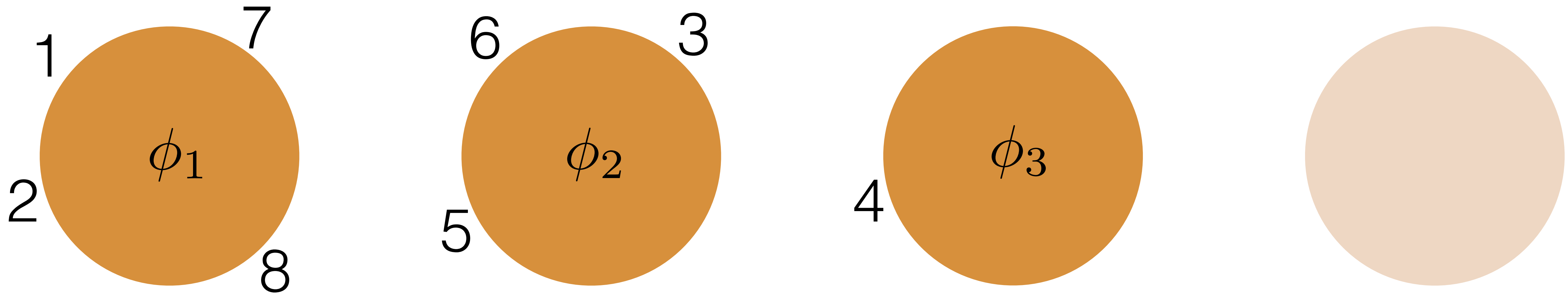

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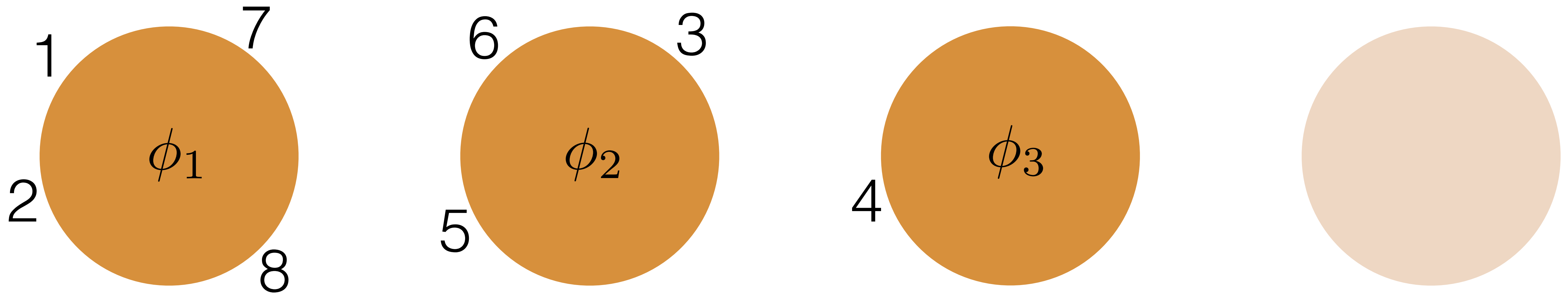
[Aldous 1983]

Chinese restaurant process



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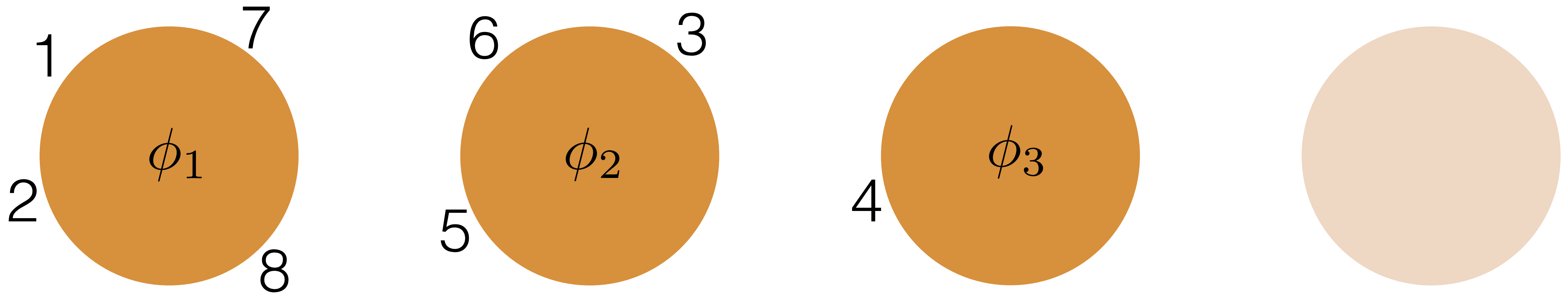

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So far: Dirichlet process, Chinese restaurant process

- Infinity of parameters, growing number of parameters

Roadmap

- Example problem: clustering
- Example NPBayes model: Dirichlet process
- Chinese restaurant process
- Inference
- Venture further into the wild world of Nonparametric Bayes

- Big questions
  - Why NPBayes?
  - What does an infinite/growing number of parameters really mean (in NPBayes)?
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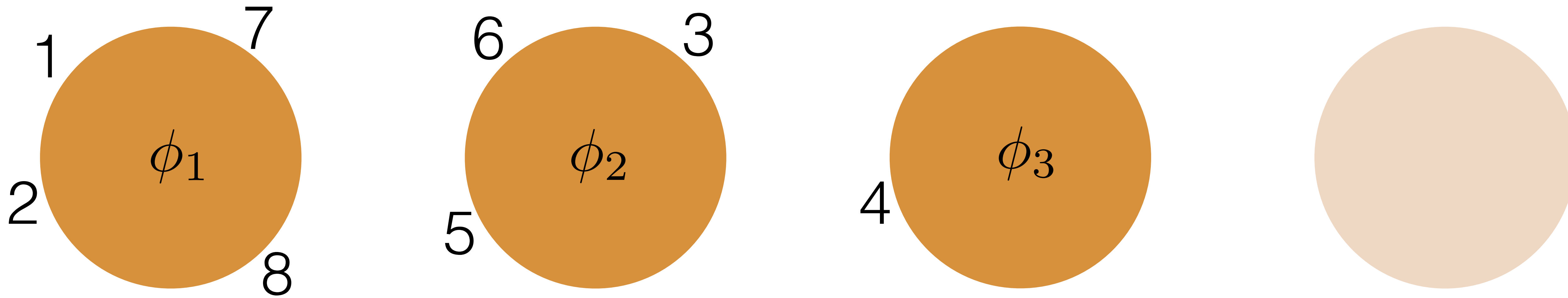
  - Why is NPBayes challenging but practical? Infinite dimensional parameter, but finitely many parameters realized

Chinese restaurant process



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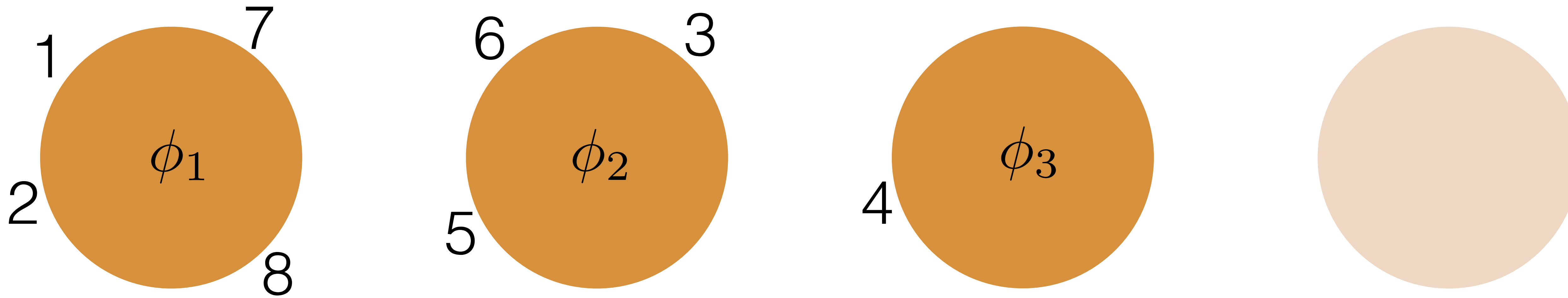

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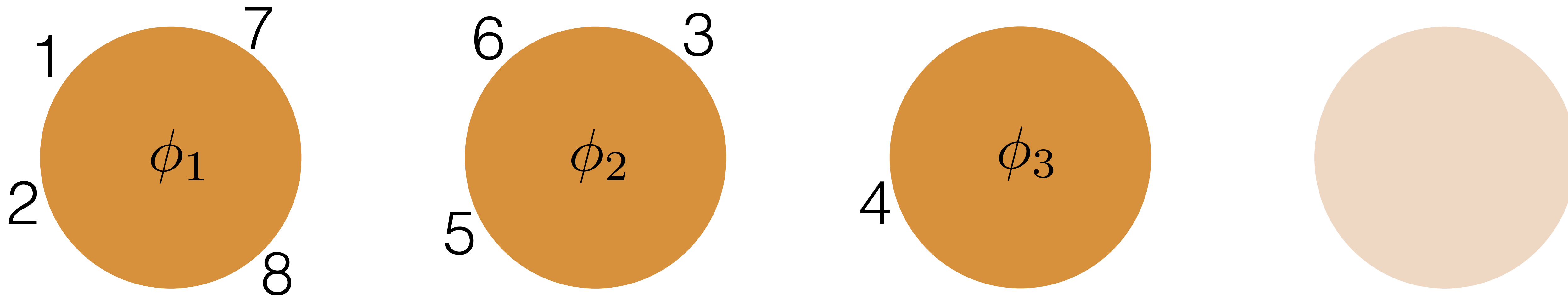
$$z_1 = z_2 = z_7 = z_8 = 1, z_3 = z_5 = z_6 = 2, z_4 = 3$$

Chinese restaurant process



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$$z_1 = z_2 = z_7 = z_8 = 1, z_3 = z_5 = z_6 = 2, z_4 = 3$$
$$\Rightarrow \Pi_8 = \{\{1, 2, 7, 8\}, \{3, 5, 6\}, \{4\}\}$$

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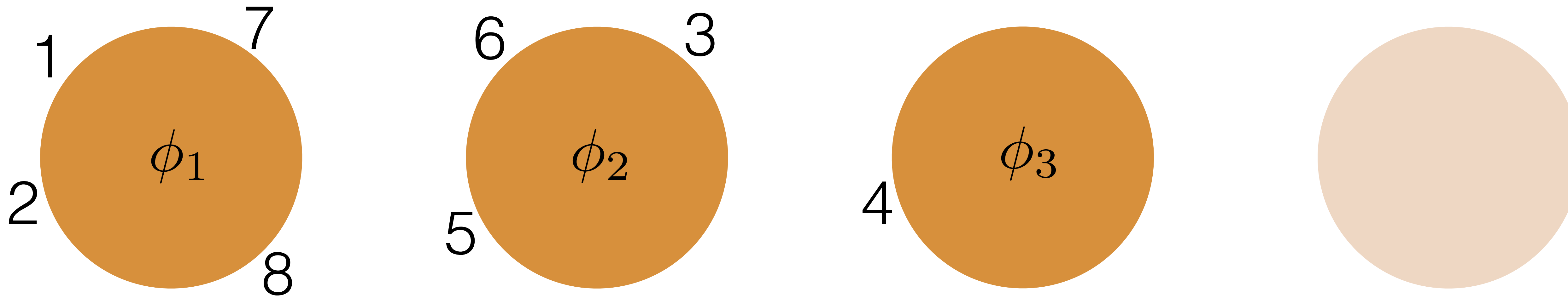

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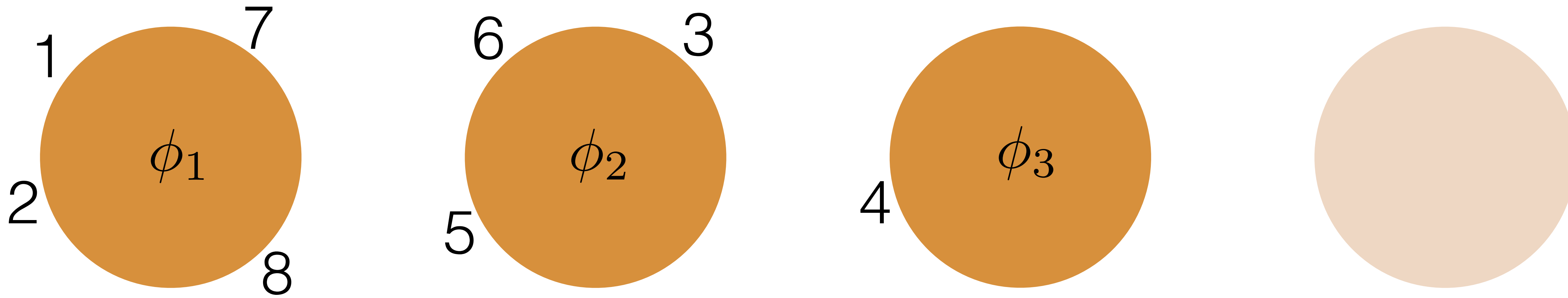
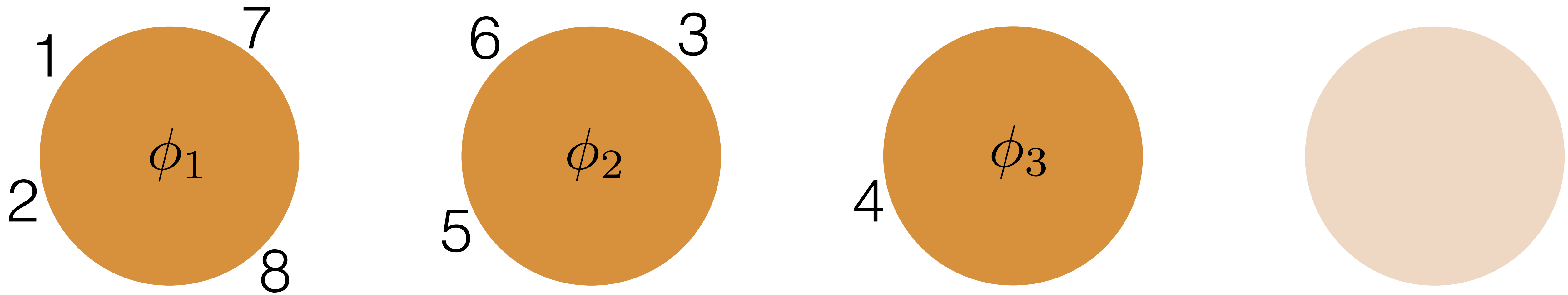
- *Partition of [8]*: set of mutually exclusive & exhaustive sets of $[8] := \{1, \ldots, 8\}$

Chinese restaurant process



- Probability of this seating:

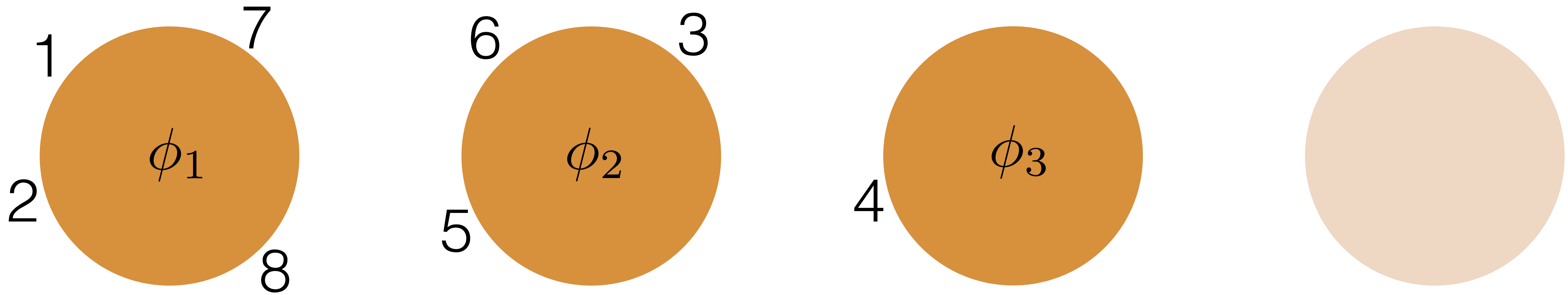
Chinese restaurant process



- Probability of this seating:

$$\frac{\alpha}{\alpha}$$

Chinese restaurant process



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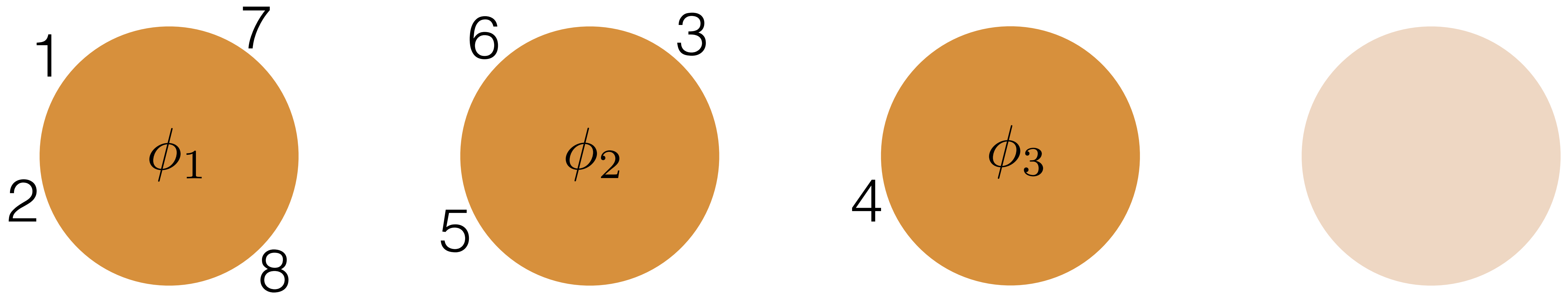
$$\frac{\alpha}{\alpha} \cdot \frac{1}{\alpha + 1}$$

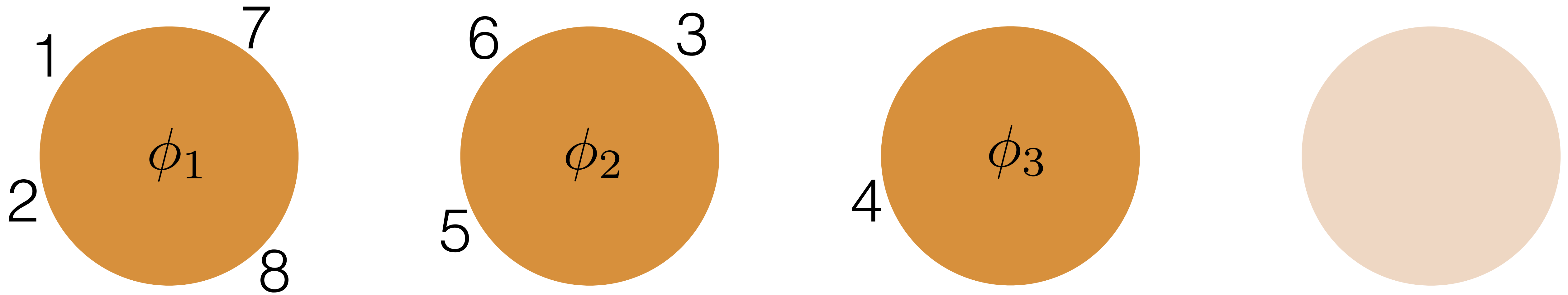
Chinese restaurant process



- Probability of this seating:

$$\frac{\alpha}{\alpha} \cdot \frac{1}{\alpha + 1} \cdot \frac{\alpha}{\alpha + 2}$$

Chinese restaurant process



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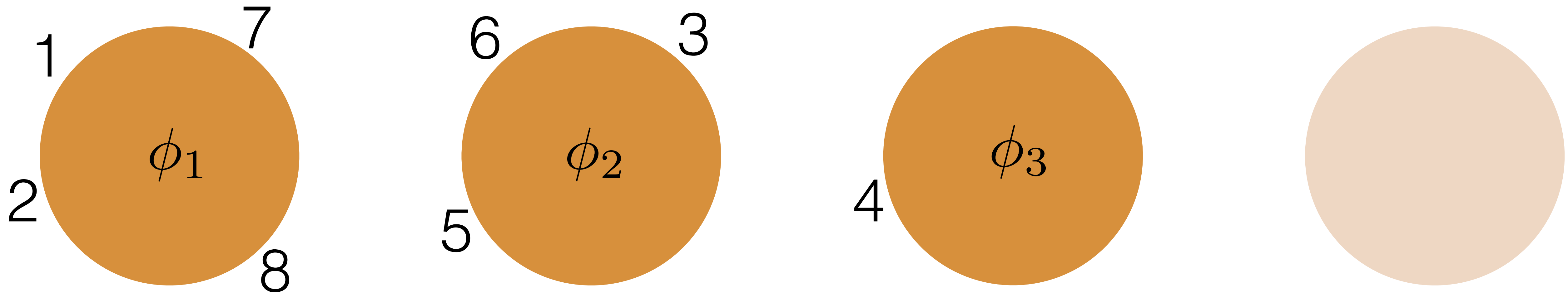
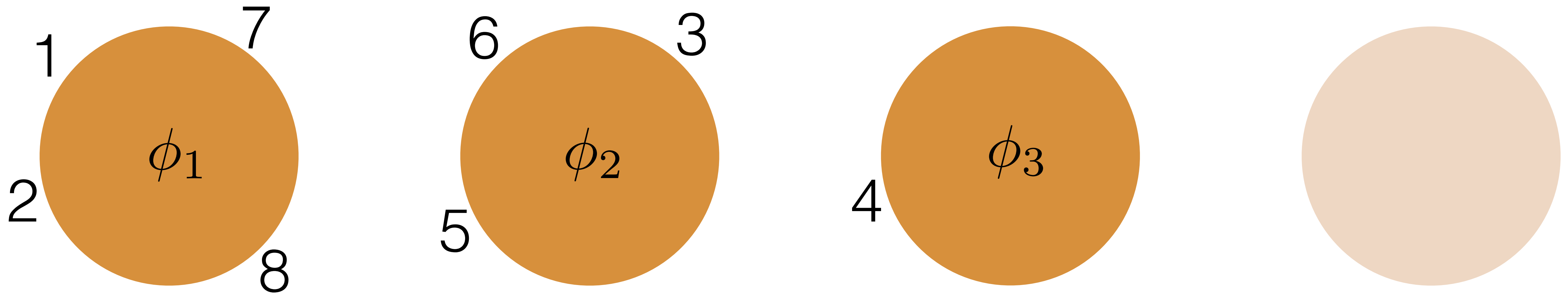
Chinese restaurant process



- Probability of this seating:

$$\frac{\alpha}{\alpha} \cdot \frac{1}{\alpha+1} \cdot \frac{\alpha}{\alpha+2} \cdot \frac{\alpha}{\alpha+3} \cdot \frac{1}{\alpha+4}$$

Chinese restaurant process



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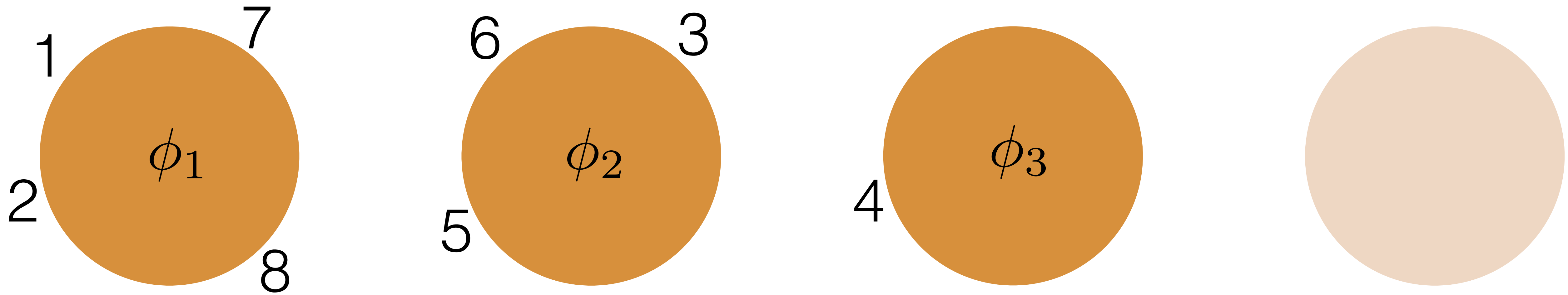
Chinese restaurant process



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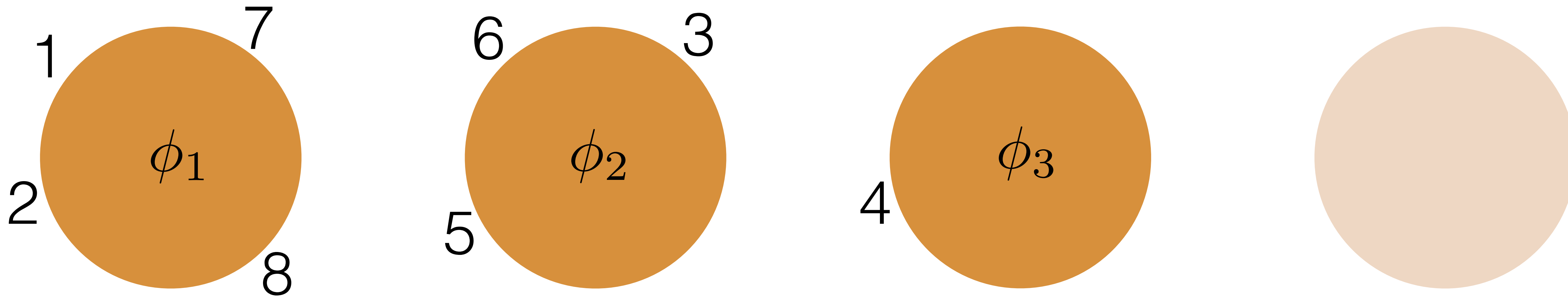

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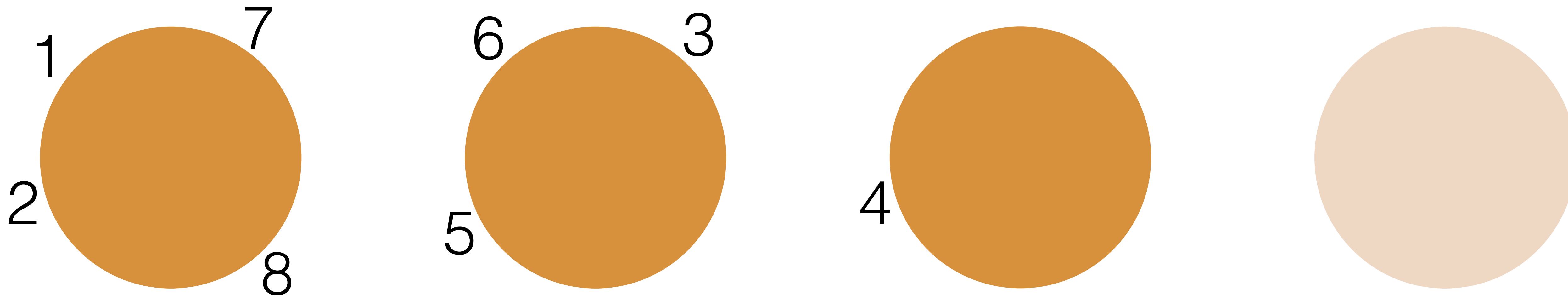
Chinese restaurant process



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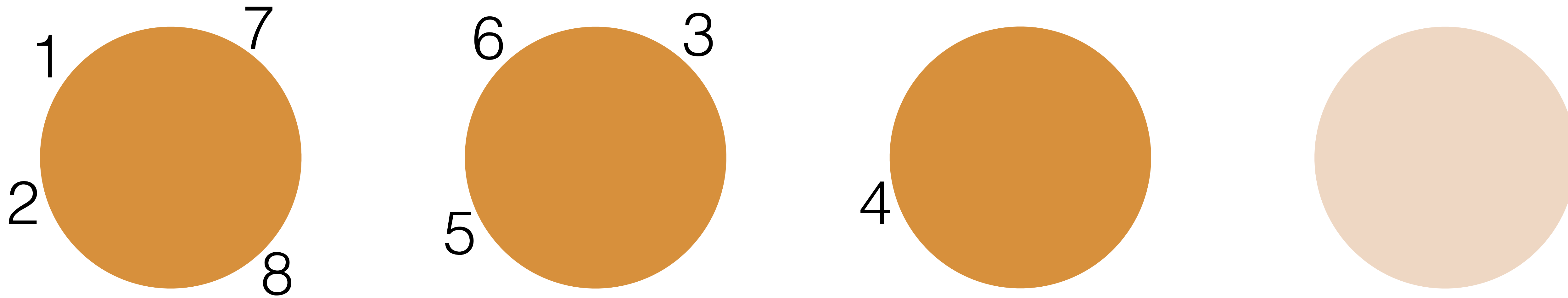
- Probability of *N* customers (K_N tables, n_k at table *k*):

Chinese restaurant process



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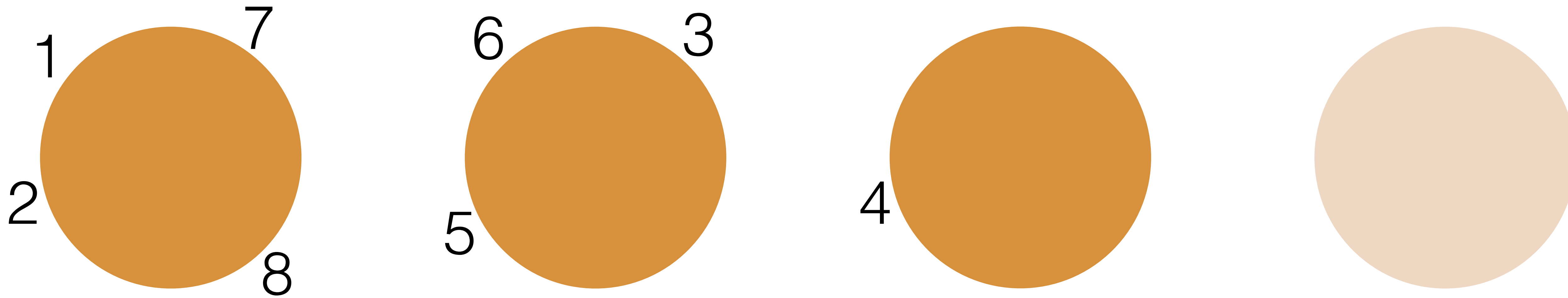

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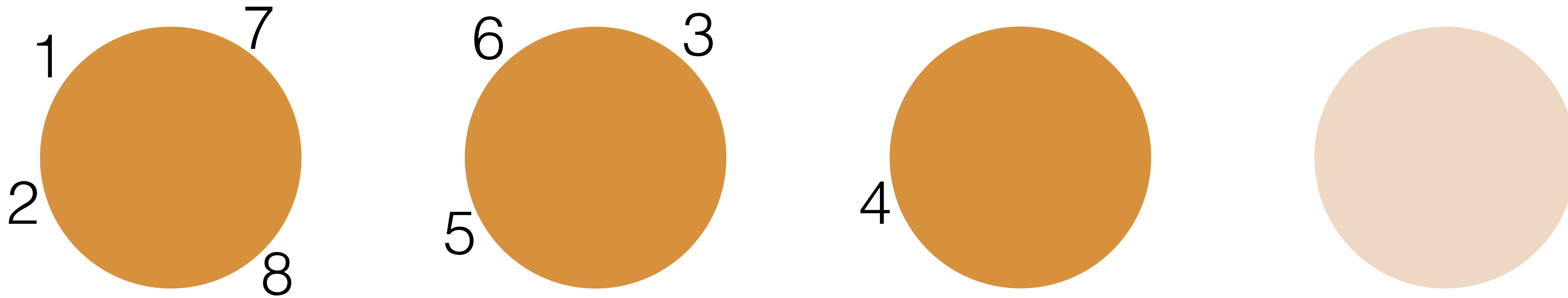
$$\frac{}{\alpha \cdots (\alpha + N - 1)}$$

Chinese restaurant process



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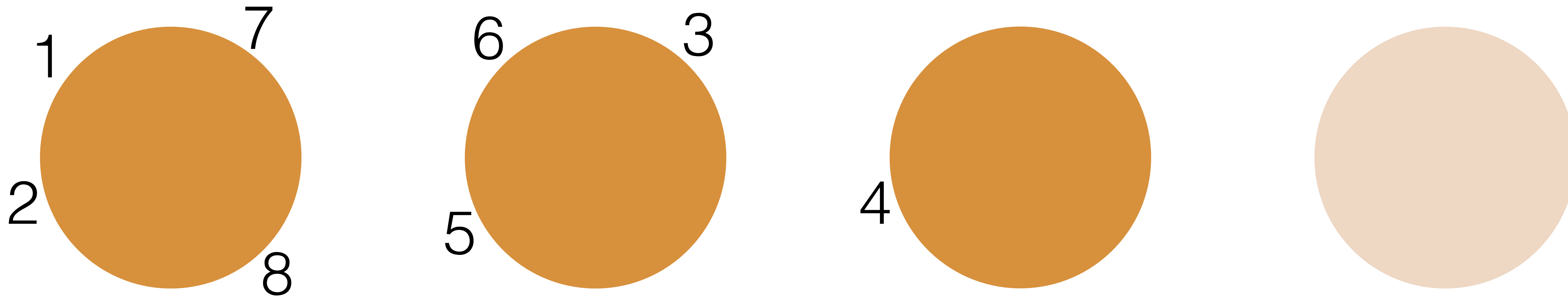

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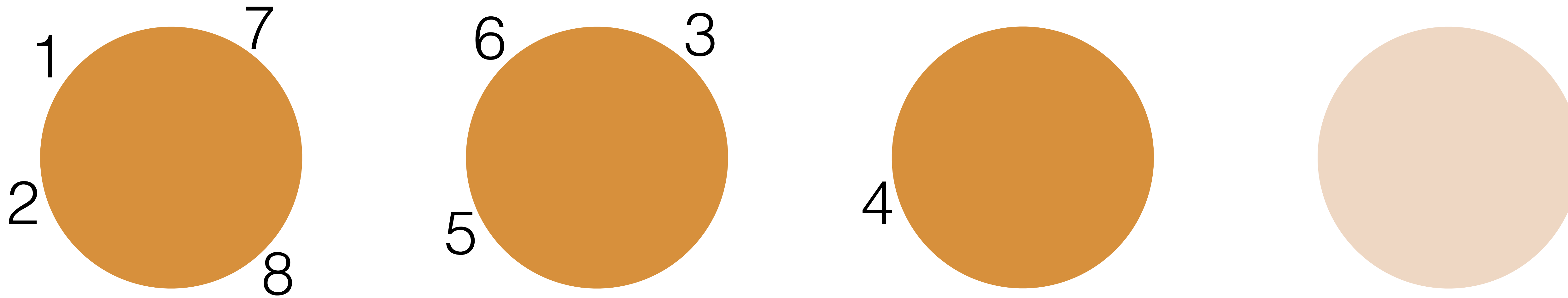

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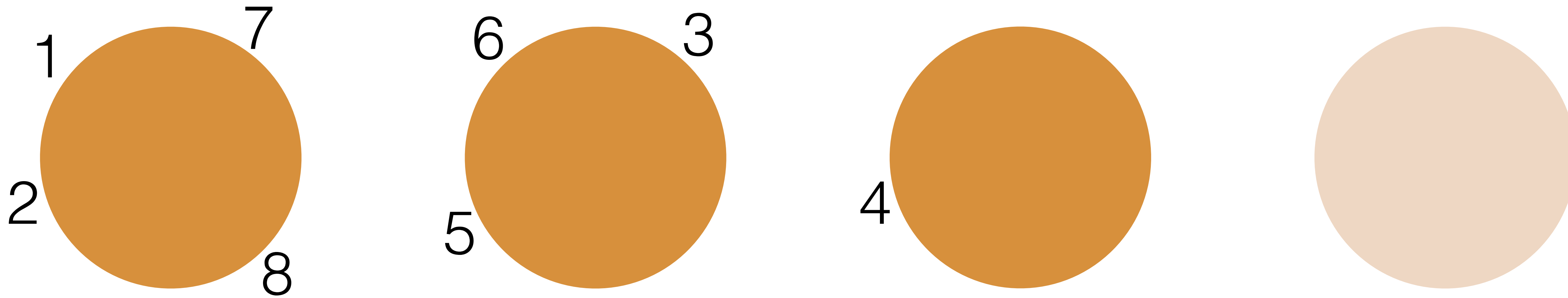

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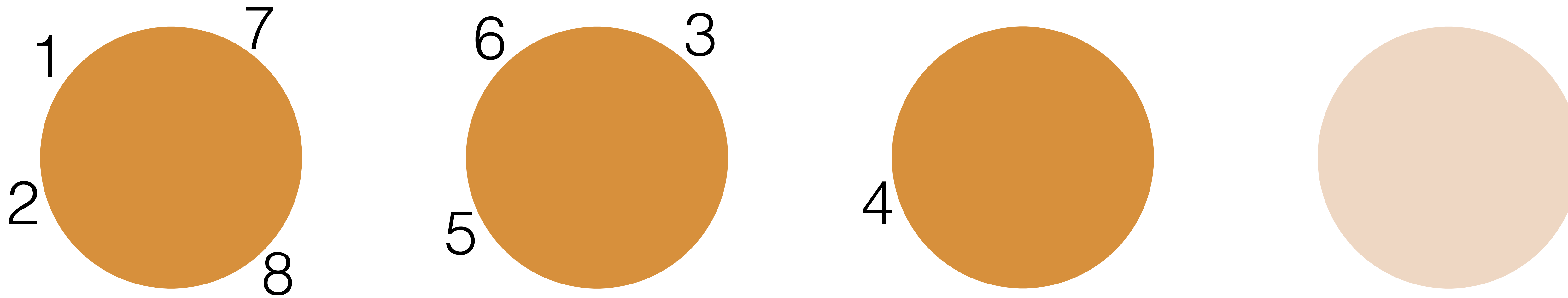

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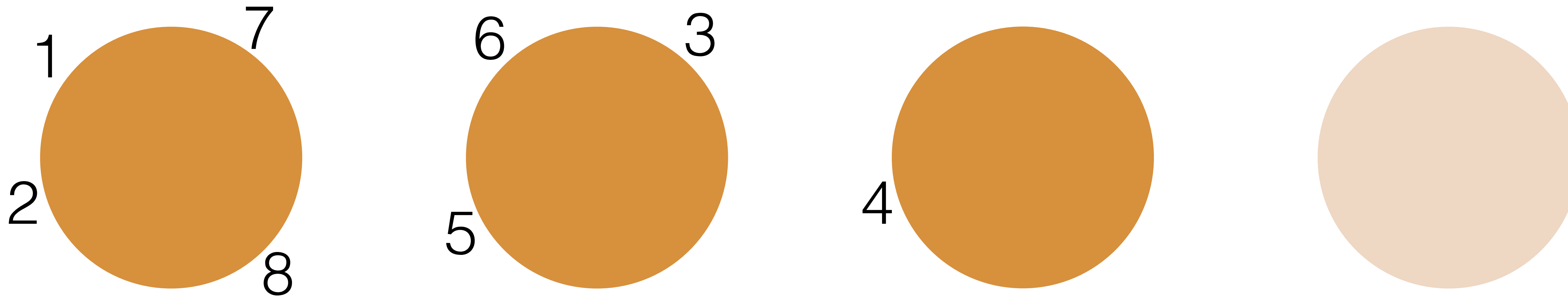

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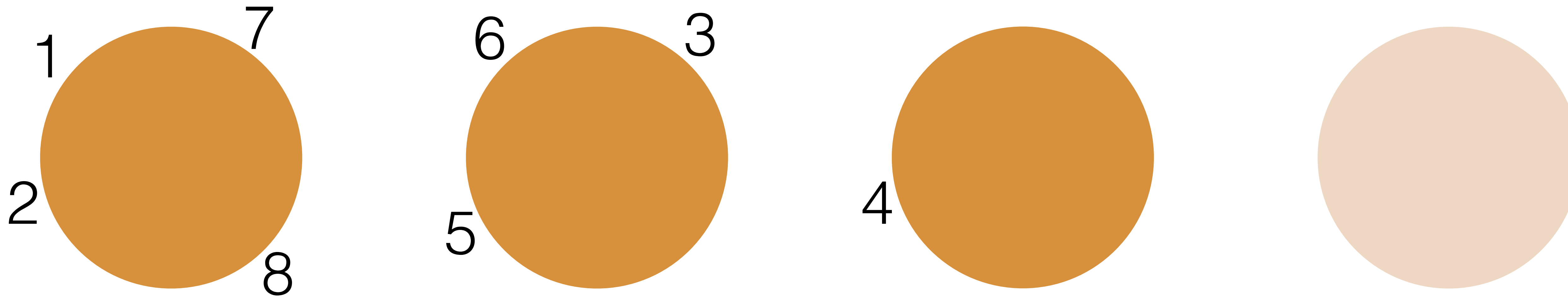

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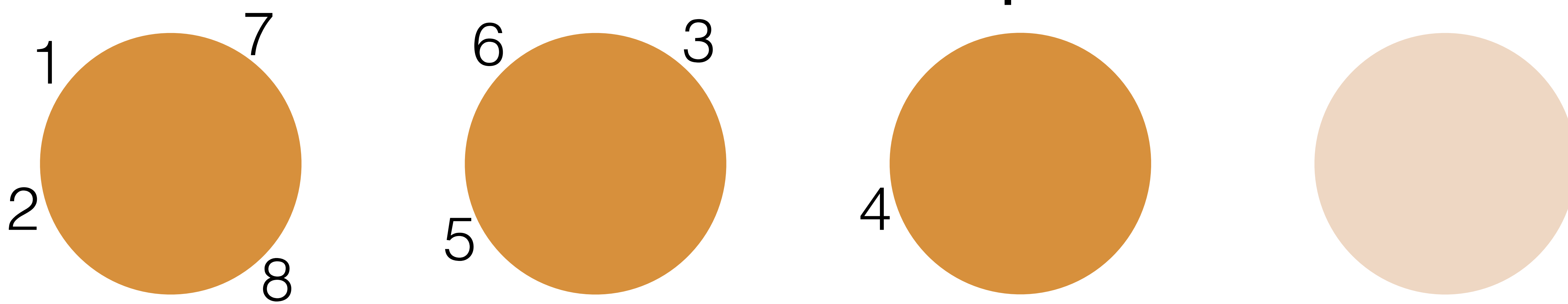
- Probability of *N* customers (K_N tables, #*C* at table *C*):

$$\frac{\alpha^{K_N} \prod_{C \in \Pi_N} (\#C - 1)!}{\alpha \cdots (\alpha + N - 1)}$$

Chinese restaurant process



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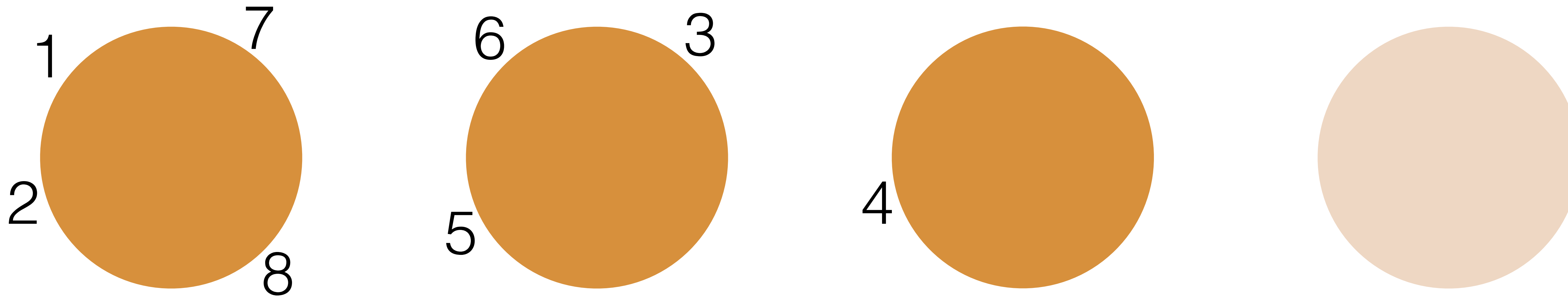
$$\frac{\alpha^{K_N} \prod_{C \in \Pi_N} (\#C - 1)!}{\alpha \cdots (\alpha + N - 1)} = \mathbb{P}(\Pi_N = \pi_N)$$

Chinese restaurant process



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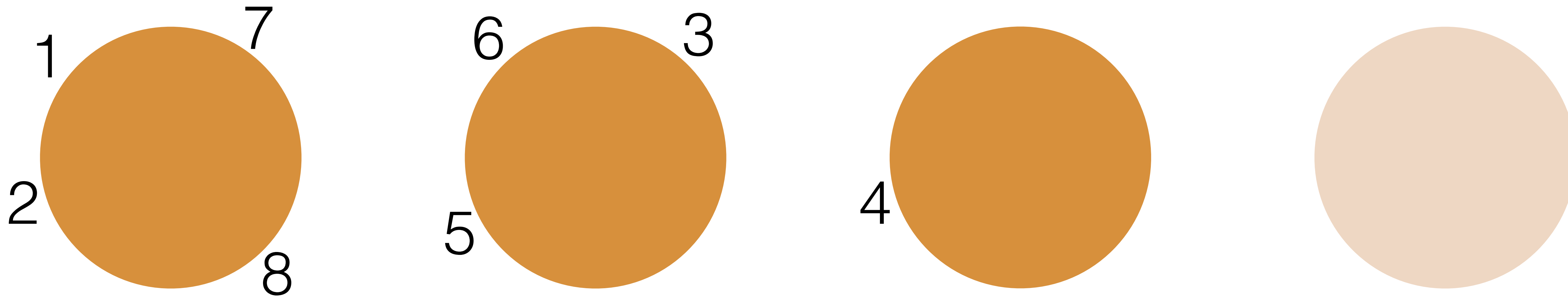
- Prob doesn't depend on customer order: *exchangeable*

Chinese restaurant process



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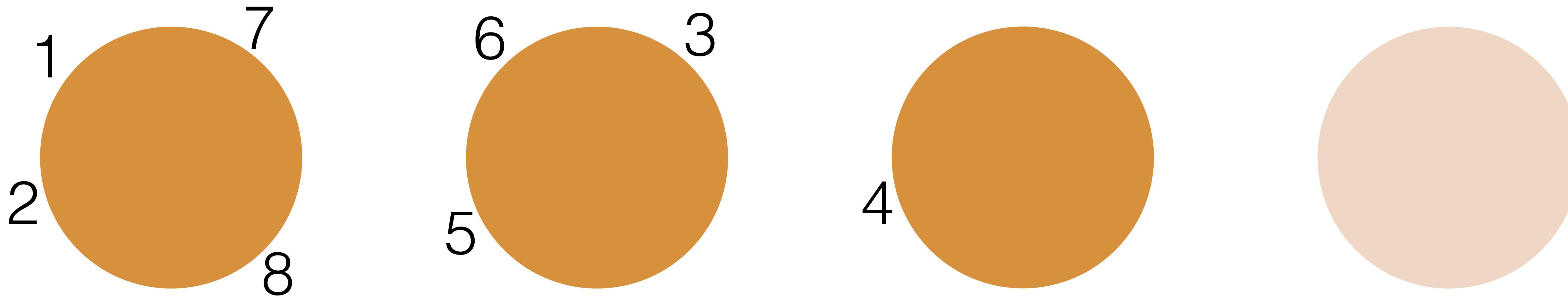
$$\frac{\alpha^{K_N} \prod_{C \in \Pi_N} (\#C - 1)!}{\alpha \cdots (\alpha + N - 1)} = \mathbb{P}(\Pi_N = \pi_N)$$

- Prob doesn't depend on customer order: *exchangeable*

$$\mathbb{P}(\Pi_8 = \{\{1,2,7,8\},\{3,5,6\},\{4\}\}) = \mathbb{P}(\Pi_8 = \{\{2,3,8,1\},\{4,6,7\},\{5\}\})$$

Chinese restaurant process



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$$\frac{\alpha}{\alpha} \cdot \frac{1}{\alpha+1} \cdot \frac{\alpha}{\alpha+2} \cdot \frac{\alpha}{\alpha+3} \cdot \frac{1}{\alpha+4} \cdot \frac{2}{\alpha+5} \cdot \frac{2}{\alpha+6} \cdot \frac{3}{\alpha+7}$$

- Probability of *N* customers (K_N tables, #*C* at table *C*):

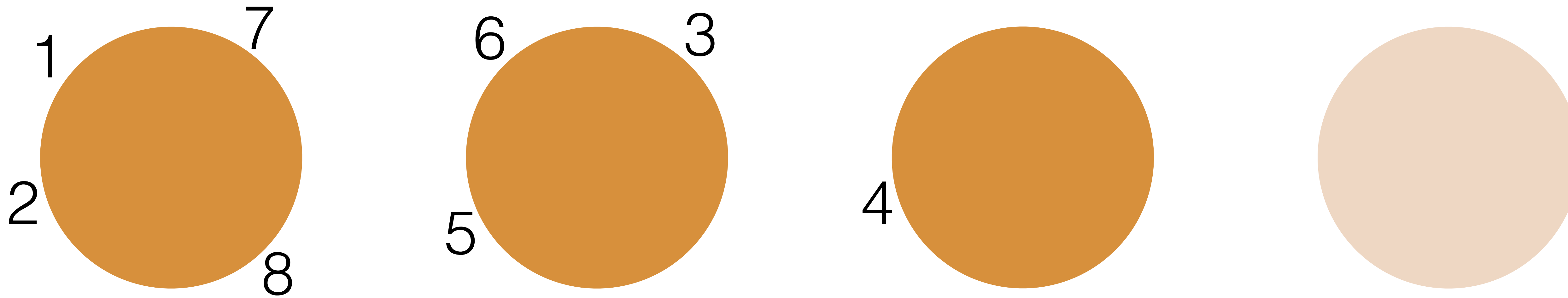
$$\frac{\alpha^{K_N} \prod_{C \in \Pi_N} (\#C - 1)!}{\alpha \cdots (\alpha + N - 1)} = \mathbb{P}(\Pi_N = \pi_N)$$

- Prob doesn't depend on customer order: *exchangeable*

$$\mathbb{P}(\Pi_8 = \{\{1,2,7,8\},\{3,5,6\},\{4\}\}) = \mathbb{P}(\Pi_8 = \{\{2,3,8,1\},\{4,6,7\},\{5\}\})$$

- Can always pretend *n* is the last customer and calculate $p(\Pi_N | \Pi_{N,\text{-}n})$

Chinese restaurant process



- Probability of this seating:

$$\frac{\alpha}{\alpha} \cdot \frac{1}{\alpha+1} \cdot \frac{\alpha}{\alpha+2} \cdot \frac{\alpha}{\alpha+3} \cdot \frac{1}{\alpha+4} \cdot \frac{2}{\alpha+5} \cdot \frac{2}{\alpha+6} \cdot \frac{3}{\alpha+7}$$

- Probability of *N* customers (K_N tables, #*C* at table *C*):

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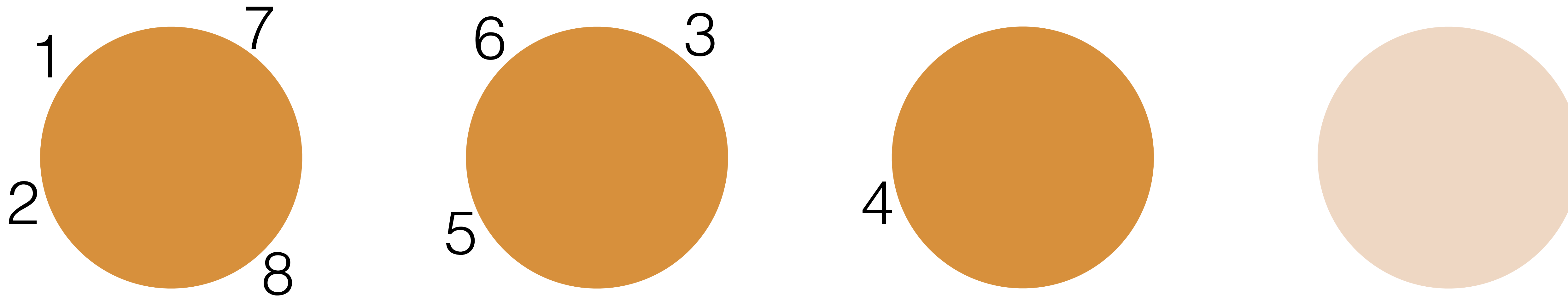
- Can always pretend *n* is the last customer and calculate $p(\Pi_N | \Pi_{N,\text{-}n})$
    - e.g. $\Pi_{8,\text{-}5} = \{\{1,2,7,8\},\{3,6\},\{4\}\}$

Chinese restaurant process



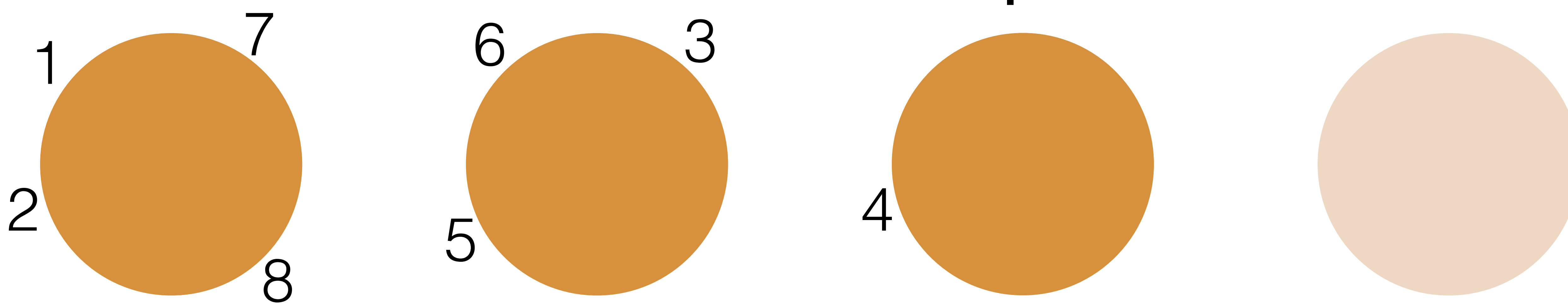
- Probability of N customers (K_N tables, $\#C$ at table C):

$$\frac{\alpha^{K_N} \prod_{C \in \Pi_N} (\#C - 1)!}{\alpha \cdots (\alpha + N - 1)} = \mathbb{P}(\Pi_N = \pi_N)$$

- So:

$$p(\Pi_N | \Pi_{N,-n}) =$$

Chinese restaurant process



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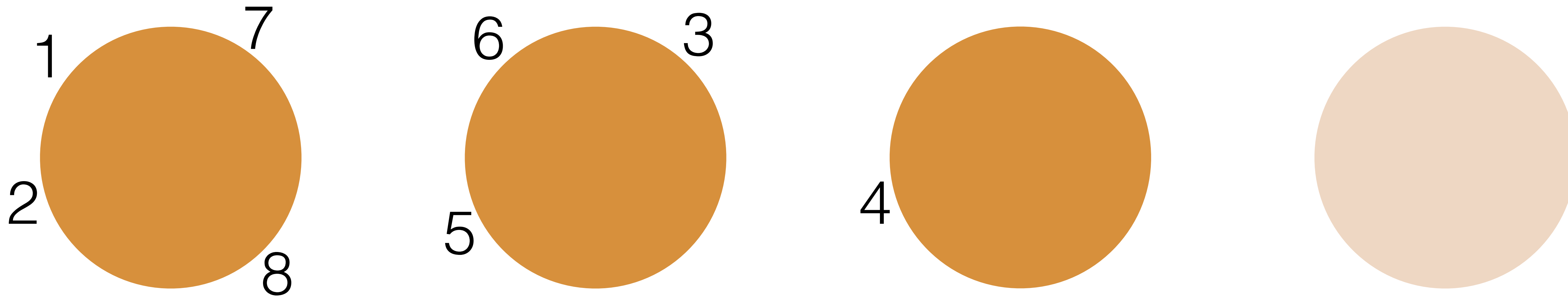
Chinese restaurant process



- Probability of N customers (K_N tables, $\#C$ at table C):

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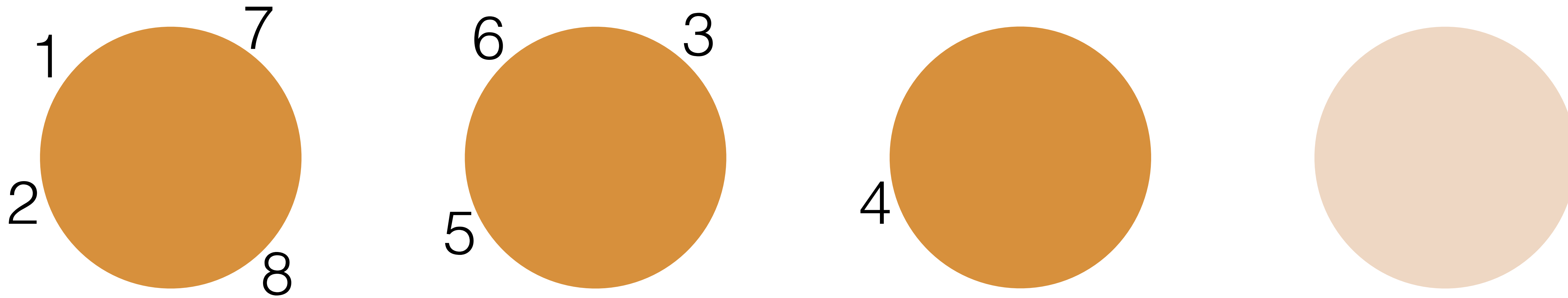
- So:

$$p(\Pi_N | \Pi_{N,-n}) = \begin{cases} & \text{if } n \text{ joins cluster } C \\ & \text{if } n \text{ starts a new cluster} \end{cases}$$

Chinese restaurant process



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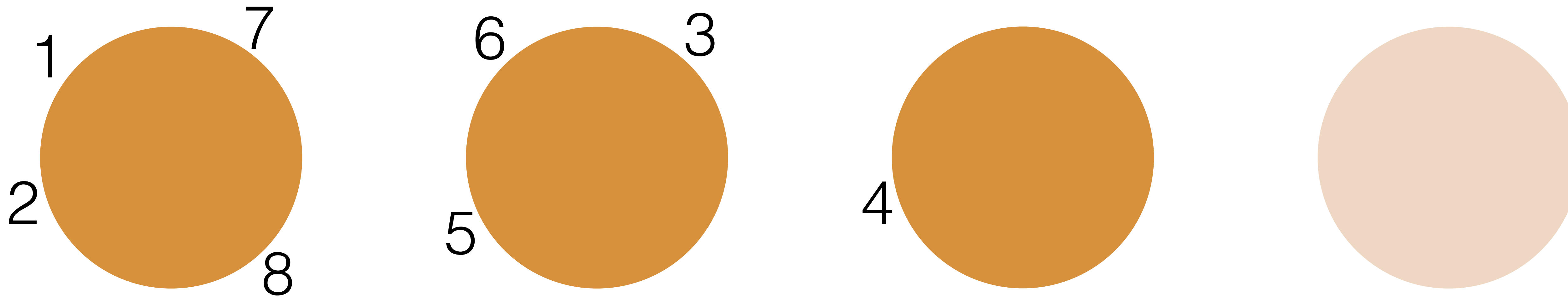
$$p(\Pi_N | \Pi_{N,-n}) = \begin{cases} \frac{\#C}{\alpha+N-1} & \text{if } n \text{ joins cluster } C \\ & \text{if } n \text{ starts a new cluster} \end{cases}$$

Chinese restaurant process



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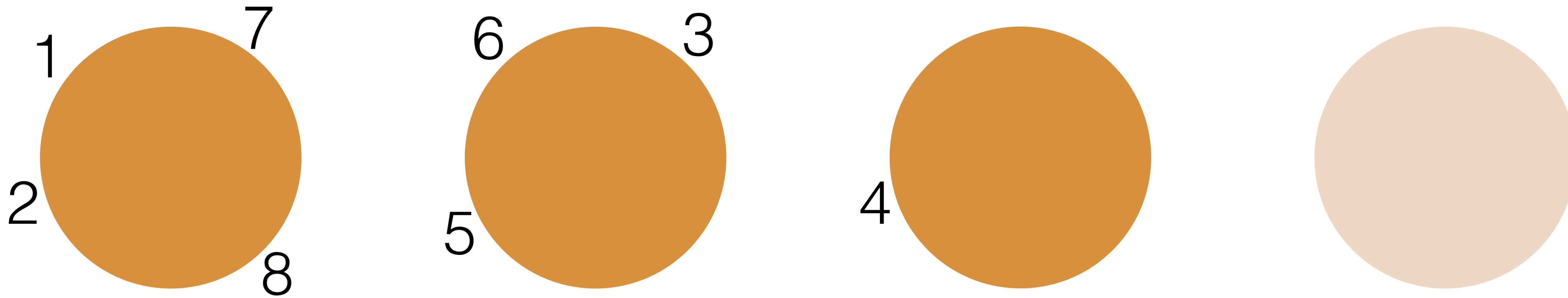
Chinese restaurant process



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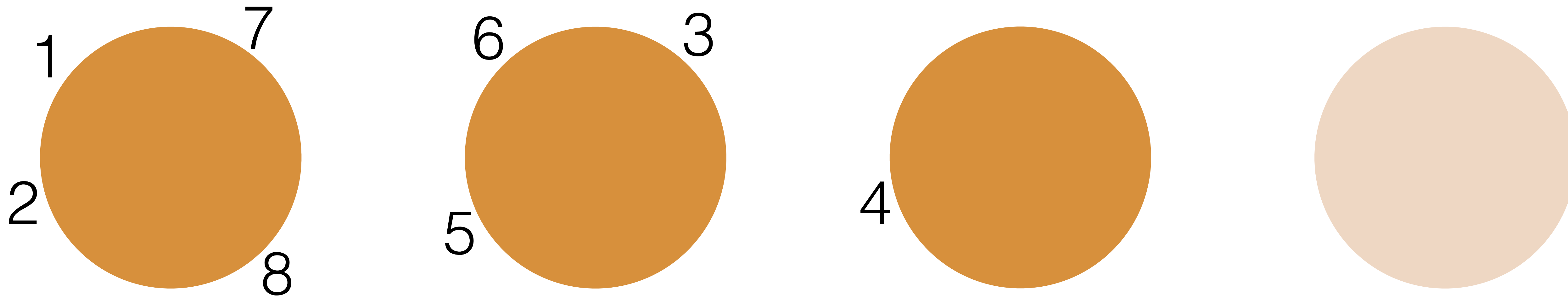
- Gibbs sampling review:

Chinese restaurant process



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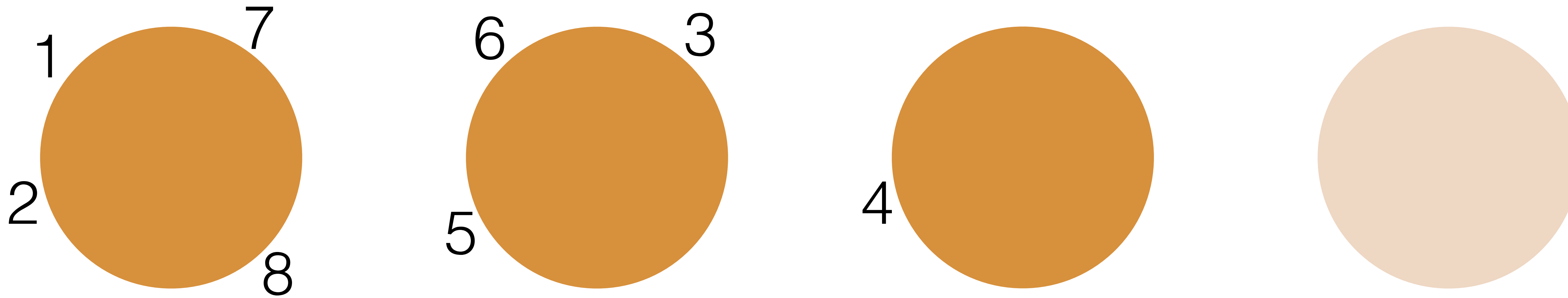
- Gibbs sampling review: target distribution $p(v_1, v_2, v_3)$

Chinese restaurant process



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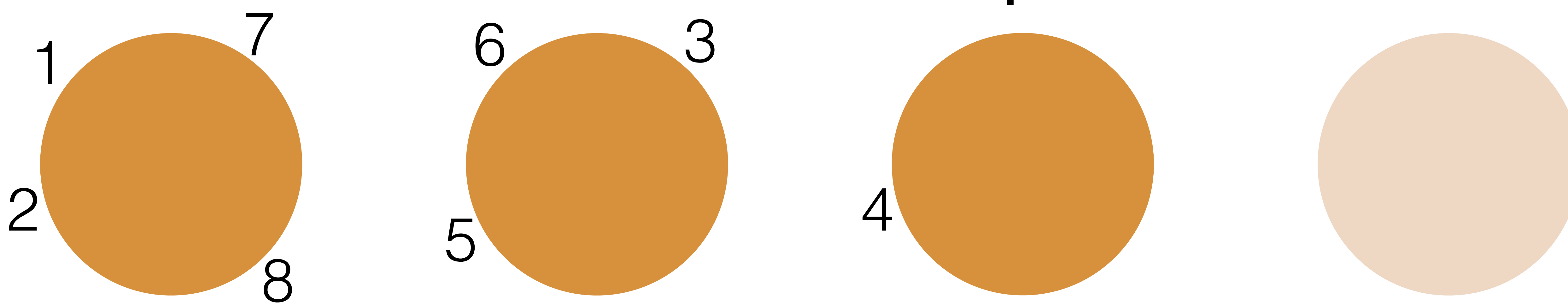
- Gibbs sampling review: target distribution $p(v_1, v_2, v_3)$
  - Start: $v_1^{(0)}, v_2^{(0)}, v_3^{(0)}$

Chinese restaurant process



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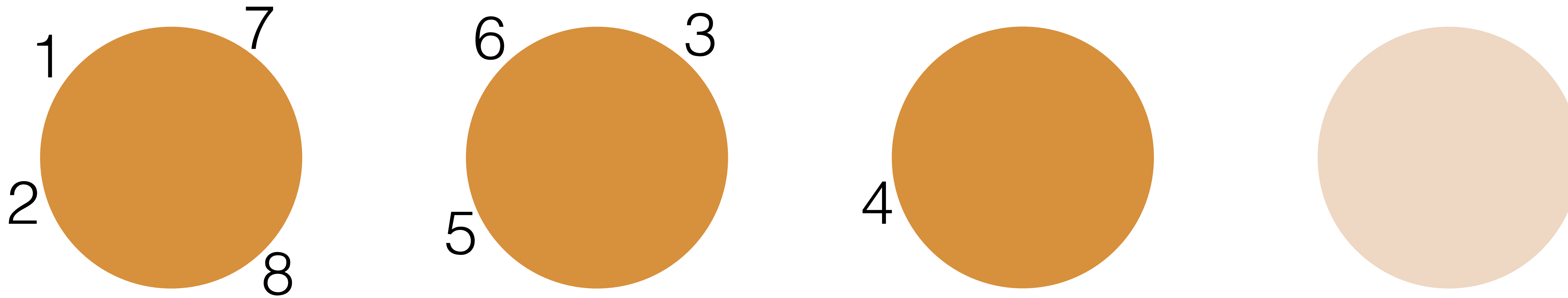
- Gibbs sampling review: target distribution $p(v_1, v_2, v_3)$
  - Start: $v_1^{(0)}, v_2^{(0)}, v_3^{(0)}$
  - t th step: $v_1^{(t)} \sim p(v_1 | v_2^{(t-1)}, v_3^{(t-1)})$

Chinese restaurant process



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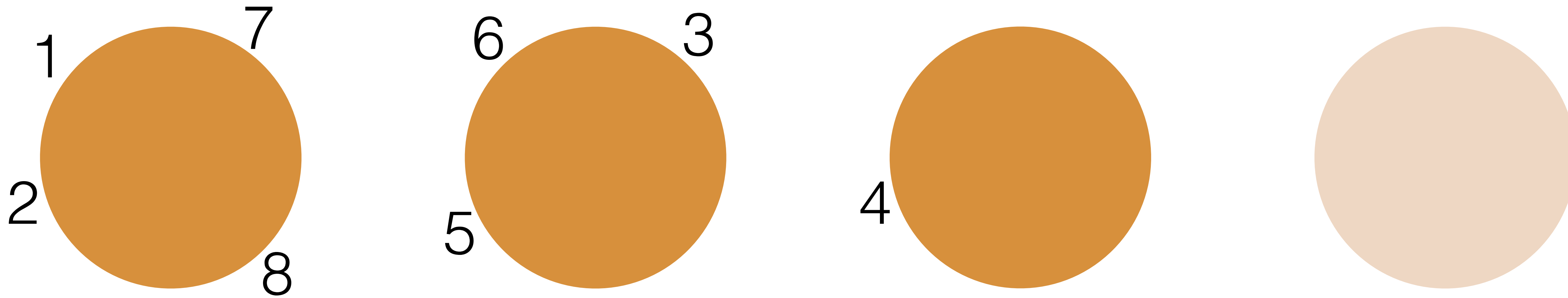
$$v_2^{(t)} \sim p(v_2 | v_1^{(t)}, v_3^{(t-1)})$$

Chinese restaurant process



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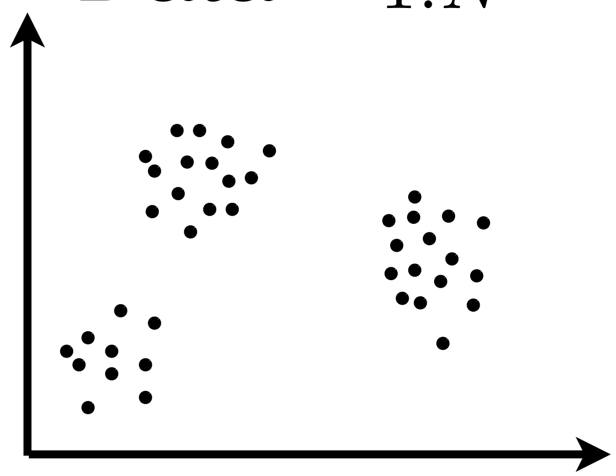
    $v_3^{(t)} \sim p(v_3 | v_1^{(t)}, v_2^{(t)})$

CRP mixture model: inference

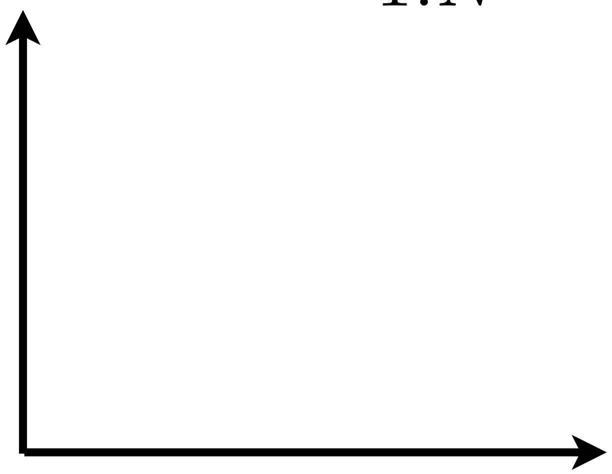
CRP mixture model: inference

- Data $x_{1:N}$



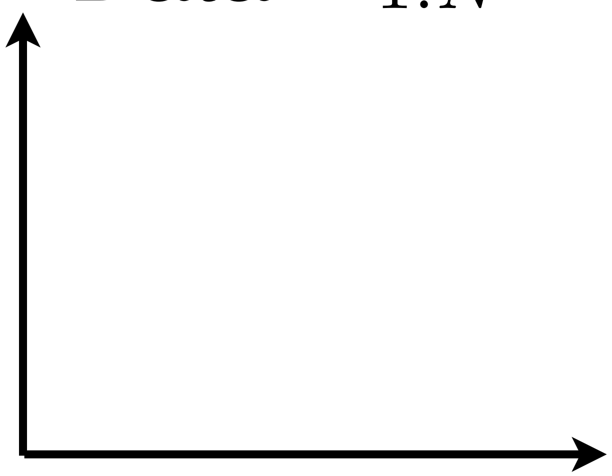
CRP mixture model: inference

- Data $x_{1:N}$



CRP mixture model: inference

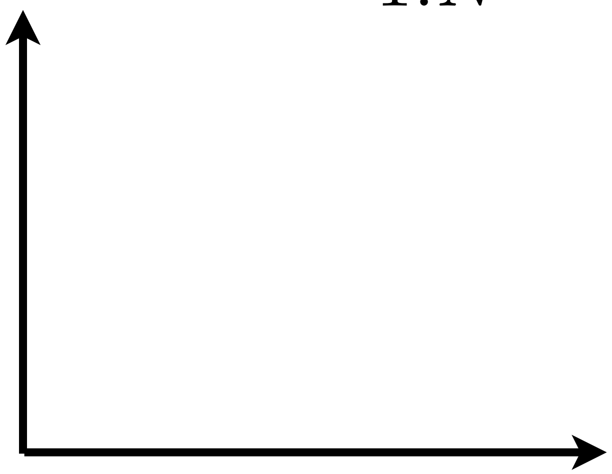
- Data $x_{1:N}$



- Generative model

CRP mixture model: inference

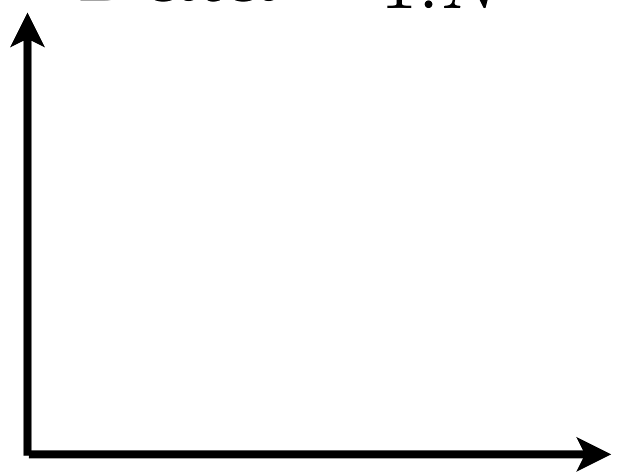
- Data $x_{1:N}$



- Generative model
  $\Pi_N \sim \mathrm{CRP}(N, \alpha)$

CRP mixture model: inference

- Data $x_{1:N}$



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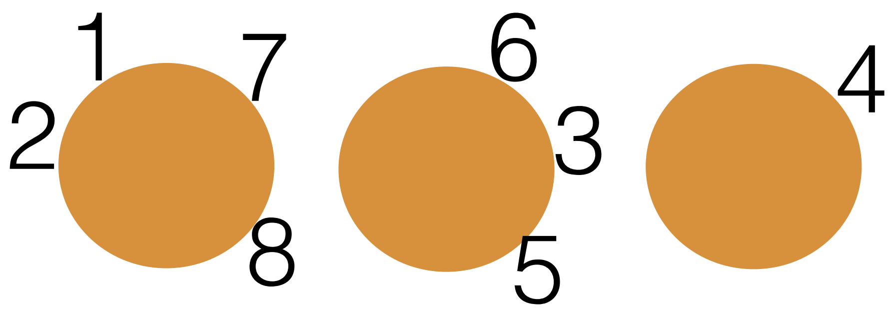
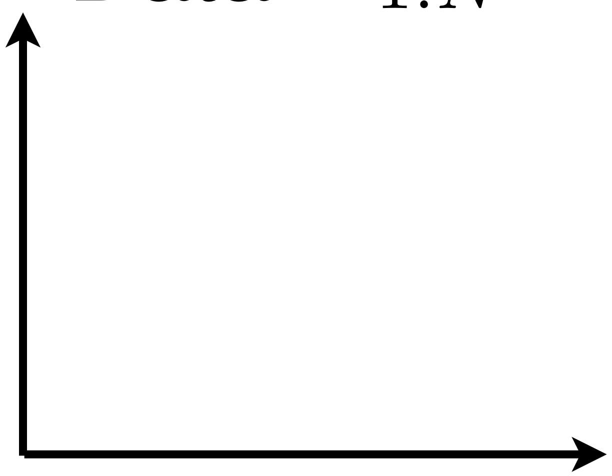


CRP mixture model: inference

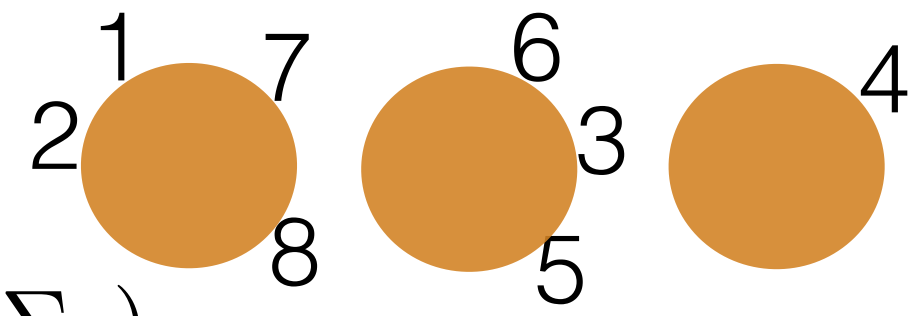
- Data $x_{1:N}$



- Generative model

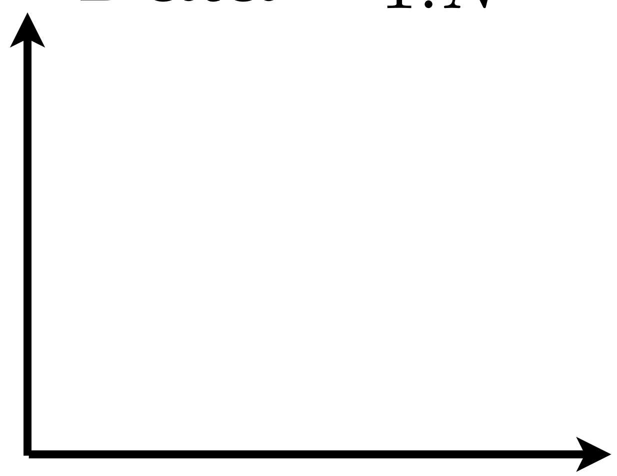
$\Pi_N \sim \mathrm{CRP}(N, \alpha)$

$\forall C \in \Pi_N, \mu_C \stackrel{iid}{\sim} \mathcal{N}(\mu_0, \Sigma_0)$



CRP mixture model: inference

- Data $x_{1:N}$



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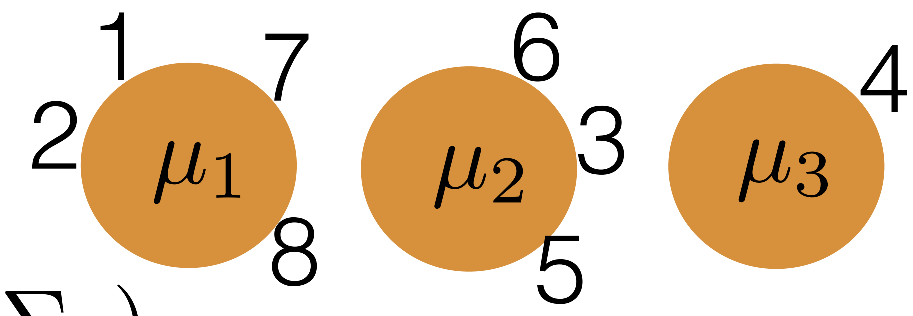
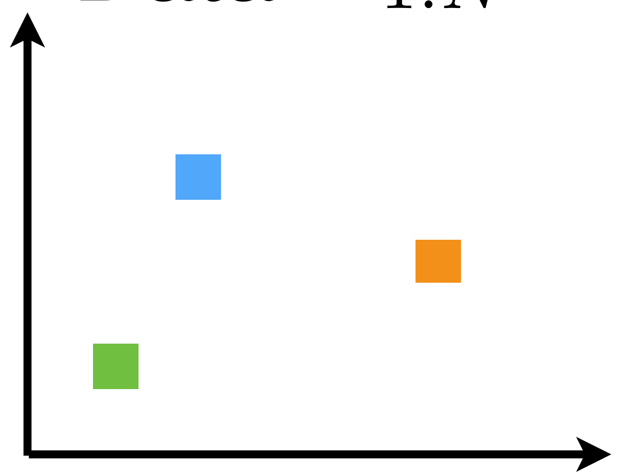


CRP mixture model: inference

- Data $x_{1:N}$



- Generative model

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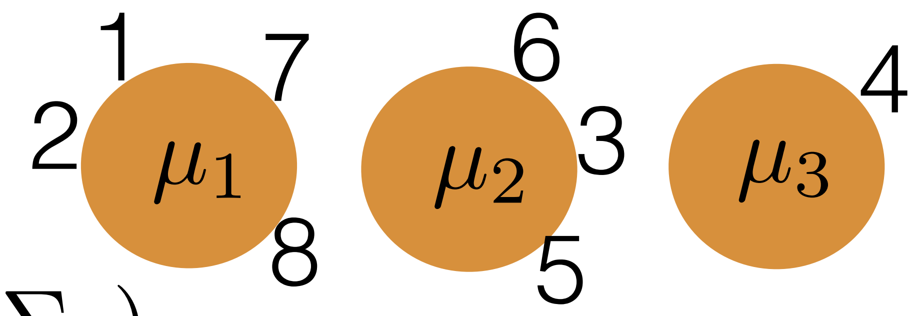

CRP mixture model: inference

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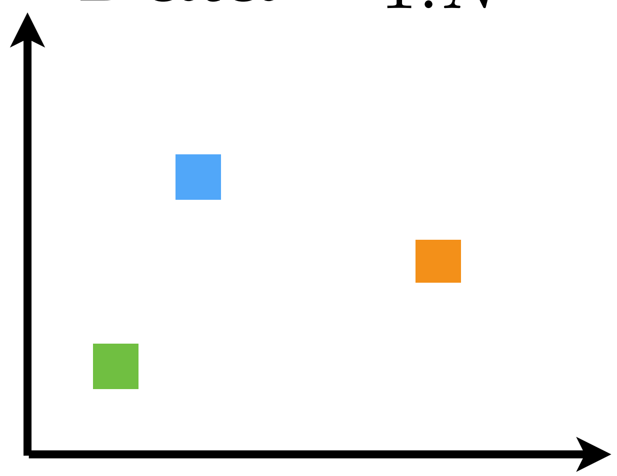

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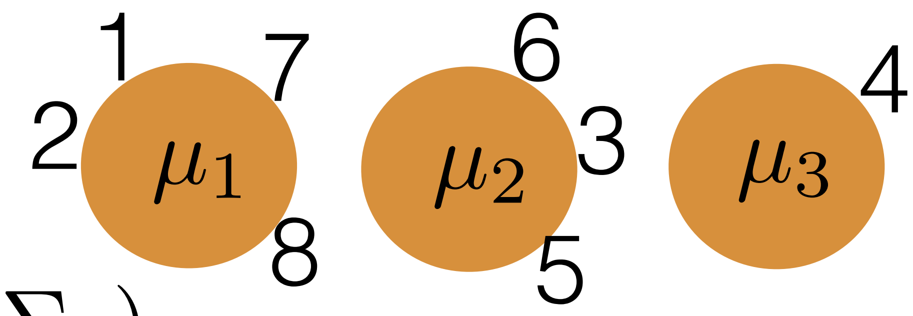
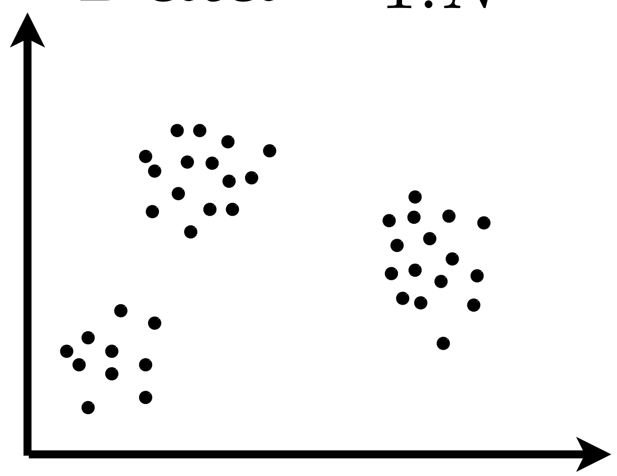


CRP mixture model: inference

- Data $x_{1:N}$



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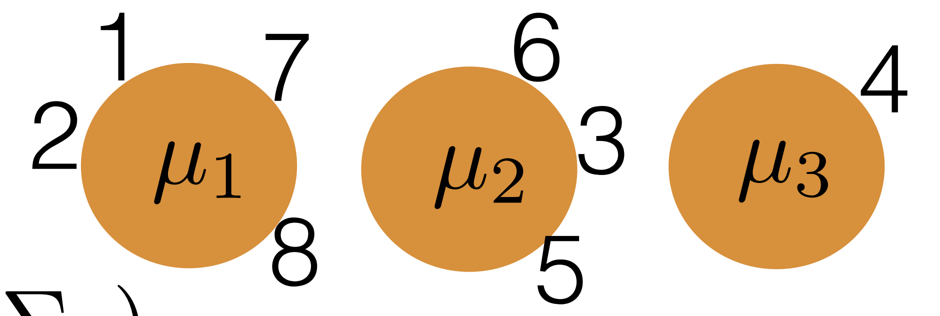
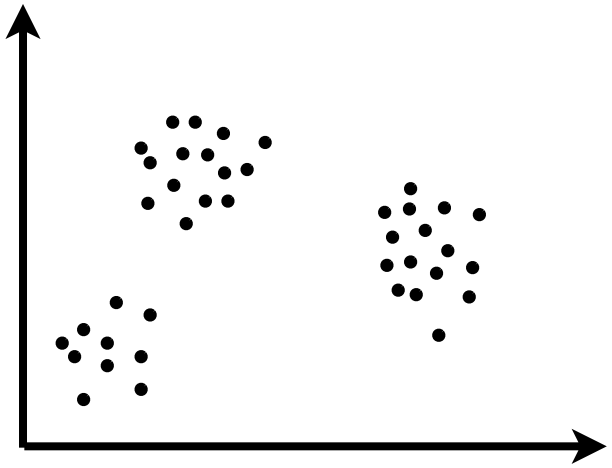


CRP mixture model: inference

- Data $x_{1:N}$



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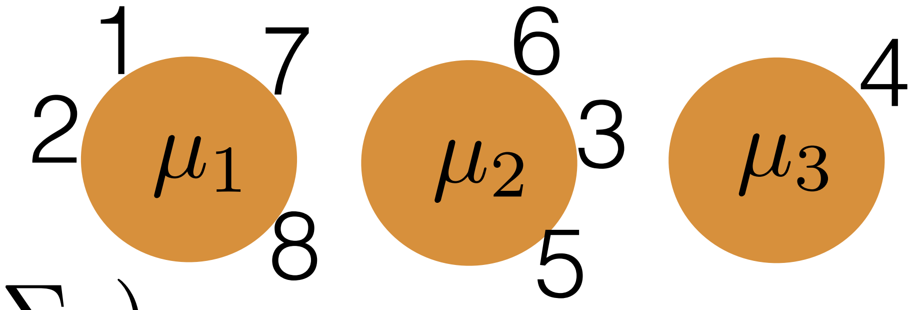
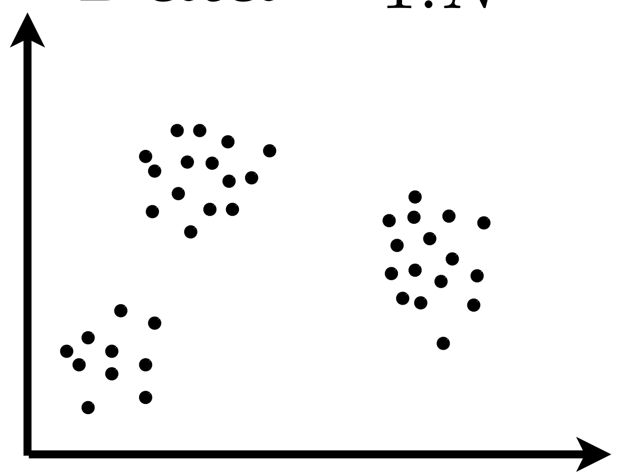


- Want: posterior

CRP mixture model: inference

- Data $x_{1:N}$



- Generative model

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$$\forall C \in \Pi_N, \mu_C \overset{iid}{\sim} \mathcal{N}(\mu_0, \Sigma_0)$$
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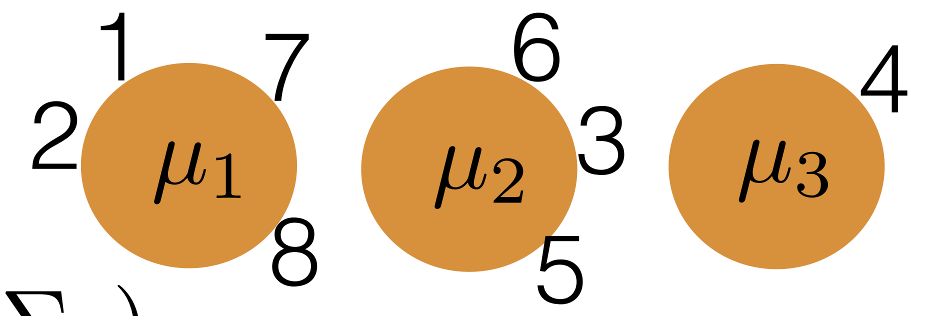
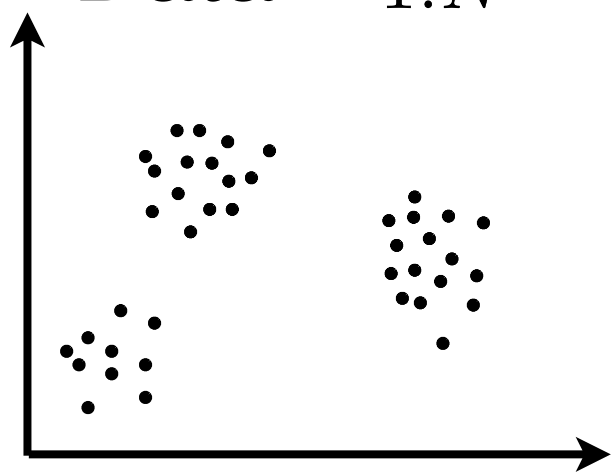


- Want: posterior $p(\Pi_N | x_{1:N})$

CRP mixture model: inference

- Data $x_{1:N}$



- Generative model

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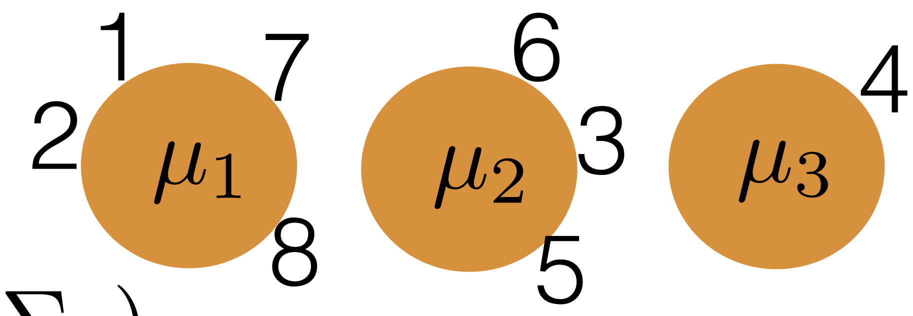
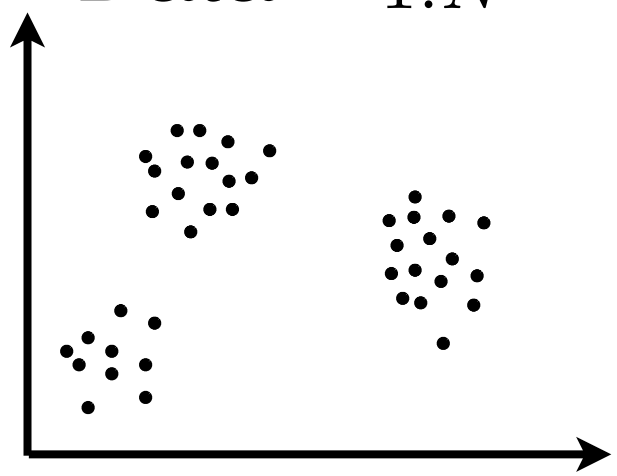


- Want: posterior $p(\Pi_N | x_{1:N})$

- Gibbs sampler:

CRP mixture model: inference

- Data $x_{1:N}$



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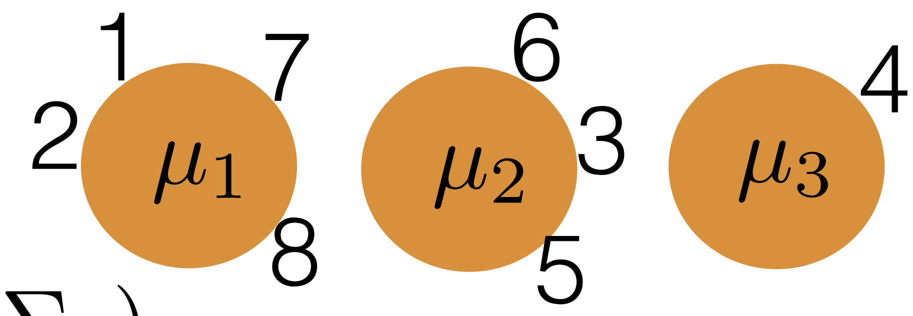


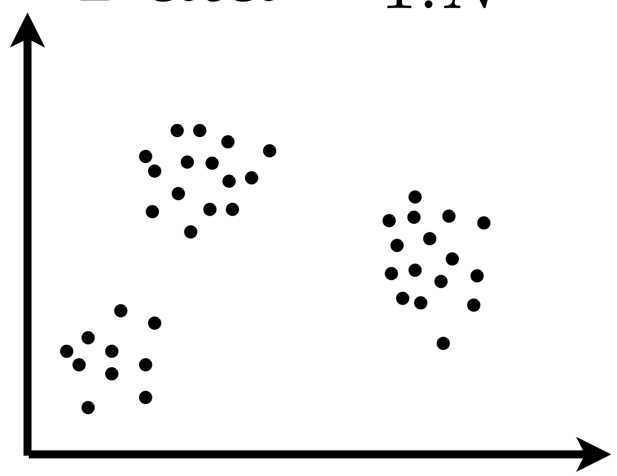
- Want: posterior $p(\Pi_N | x_{1:N})$

- Gibbs sampler:

$$p(\Pi_N | \Pi_{N,\text{-}n}, x)$$

CRP mixture model: inference

- Data $x_{1:N}$



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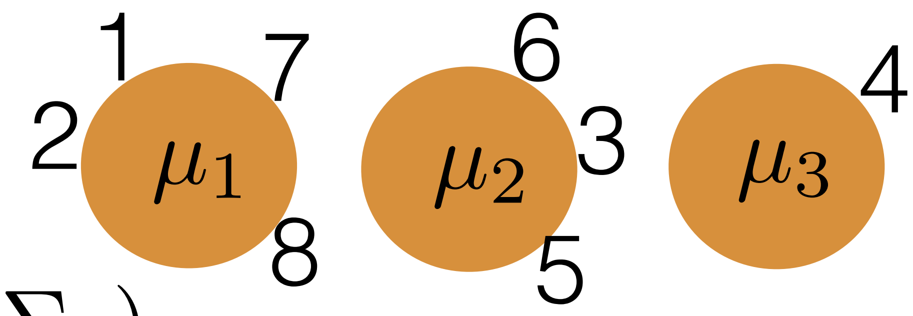


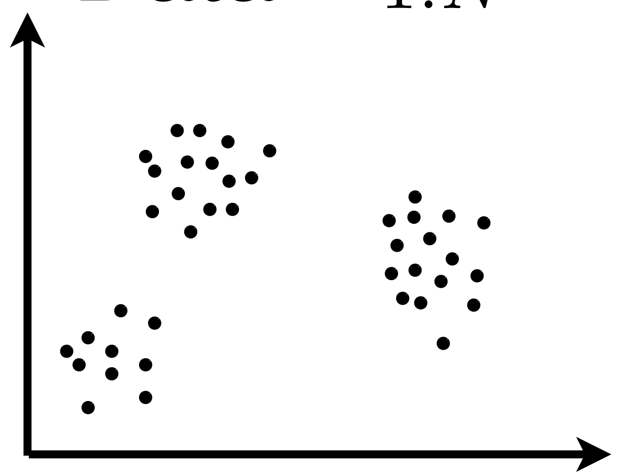
- Want: posterior $p(\Pi_N | x_{1:N})$

- Gibbs sampler:

$$p(\Pi_N | \Pi_{N,\text{-}n}, x) \propto \left\{ \right.$$

CRP mixture model: inference

- Data $x_{1:N}$



- Generative model

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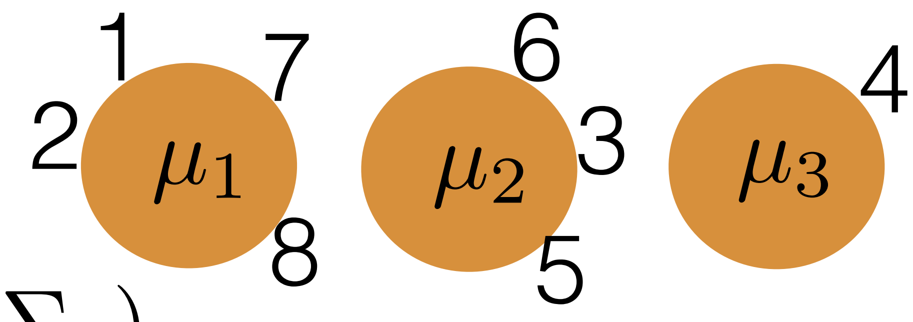

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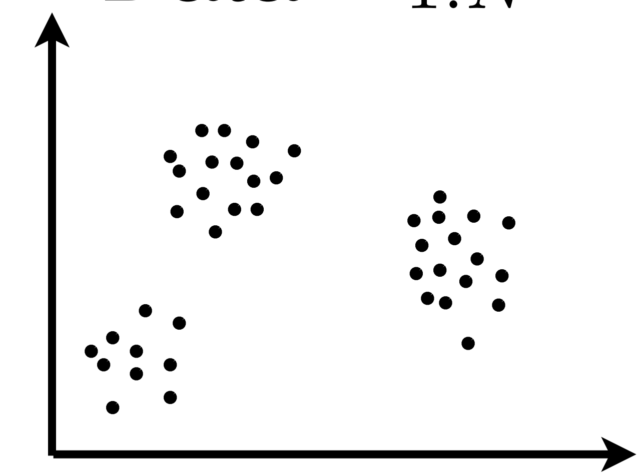

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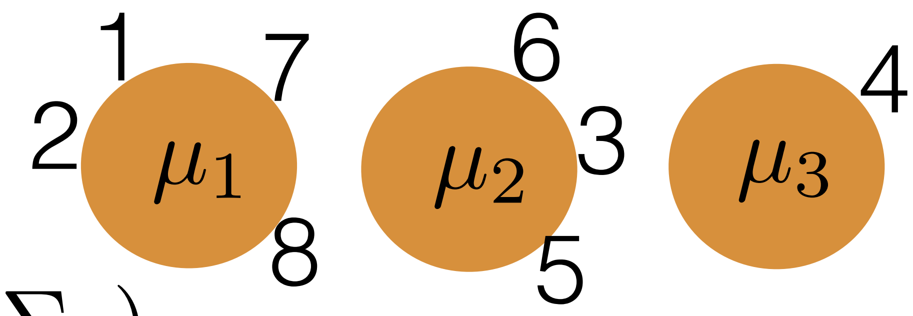

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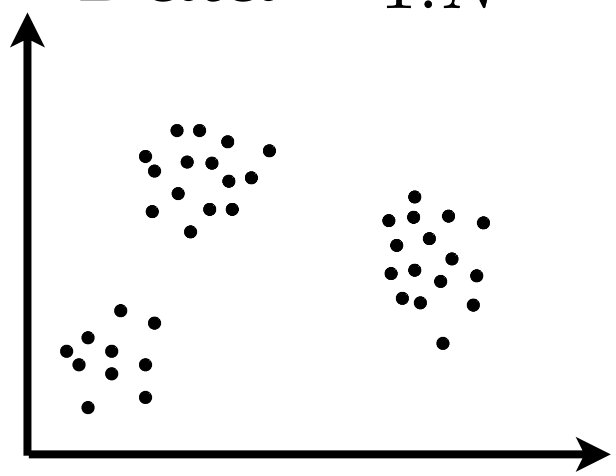

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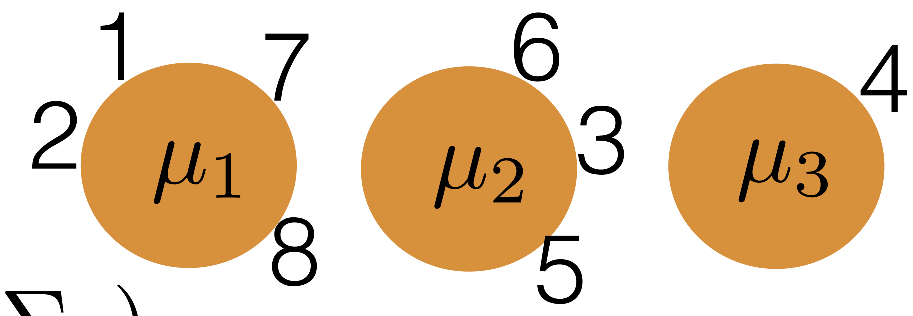

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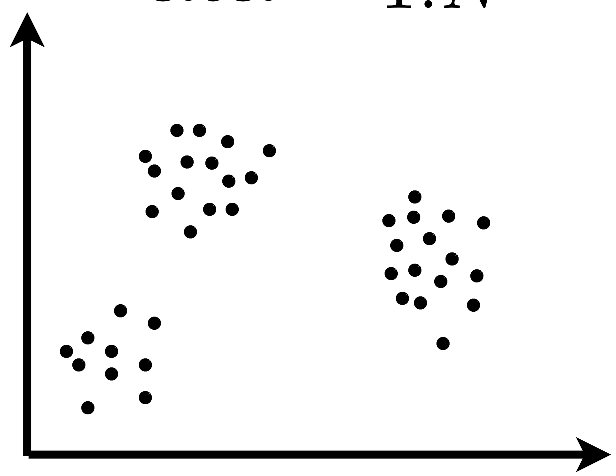

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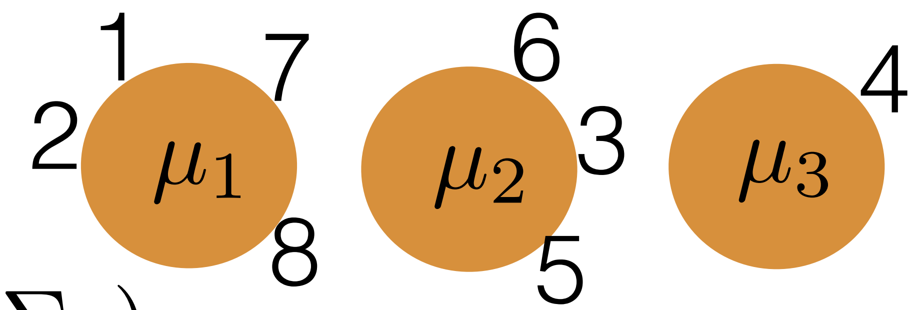

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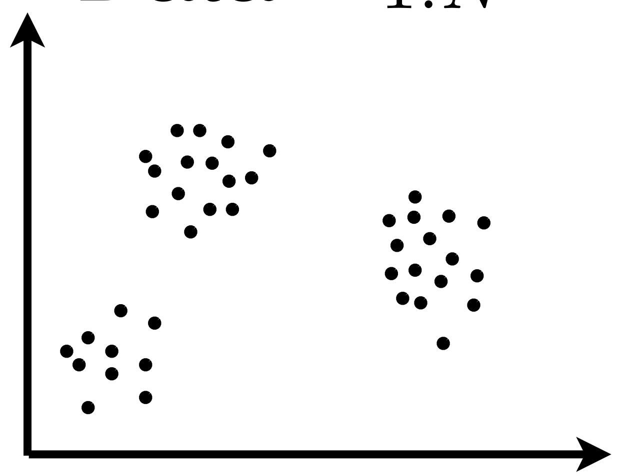

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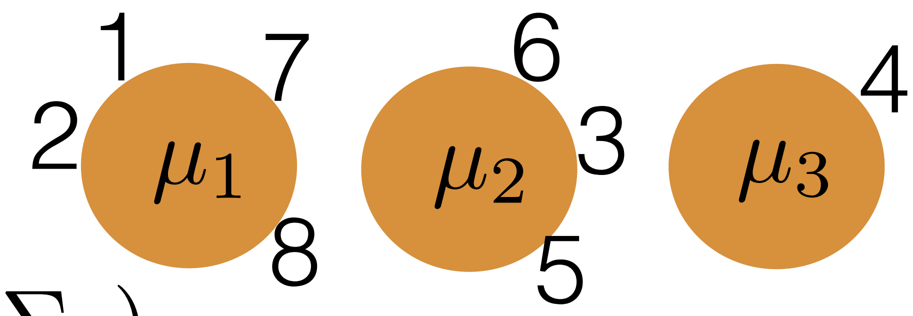

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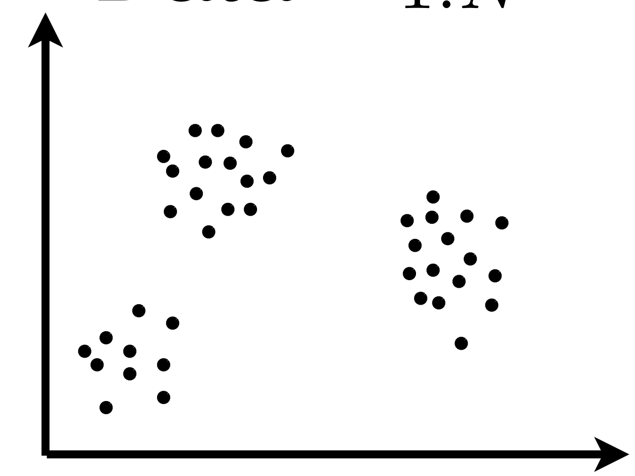

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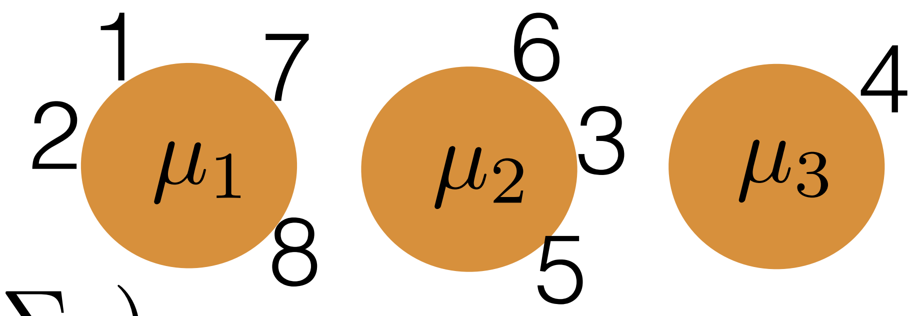

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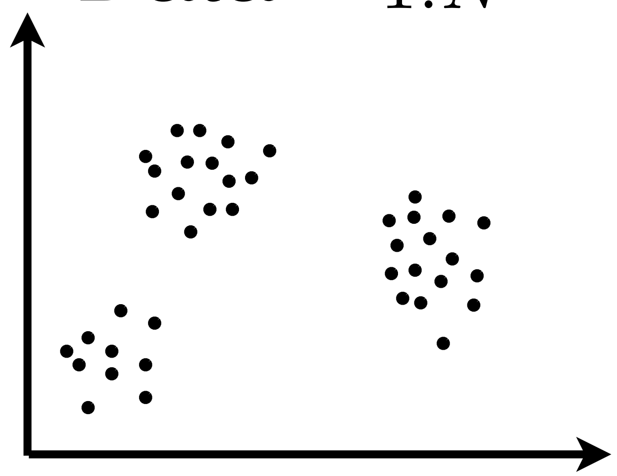

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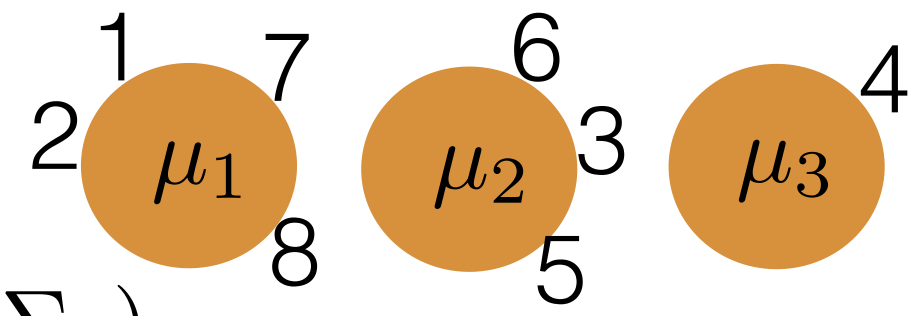

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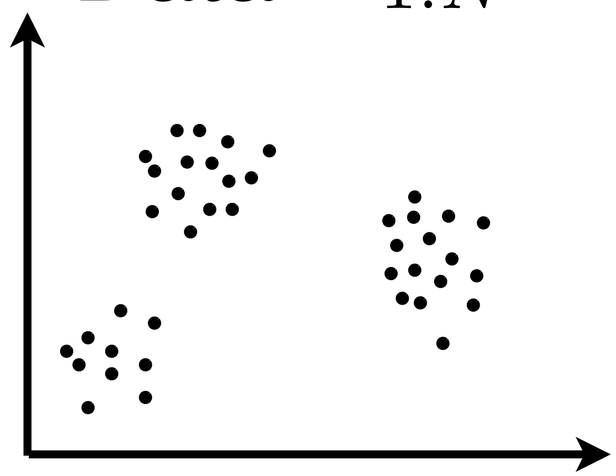

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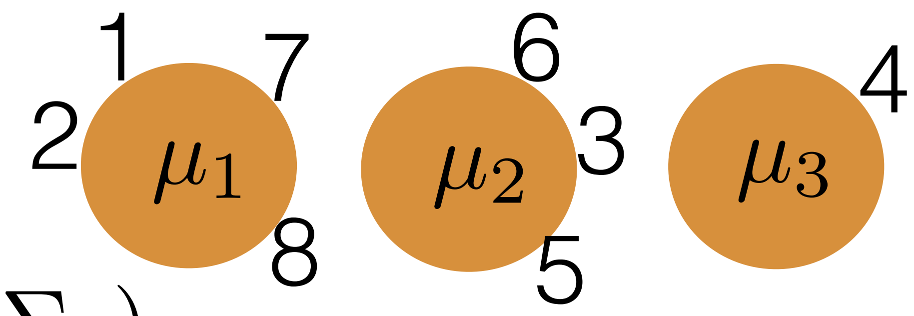

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[MacEachern 1994; Neal 1992; Neal 2000]

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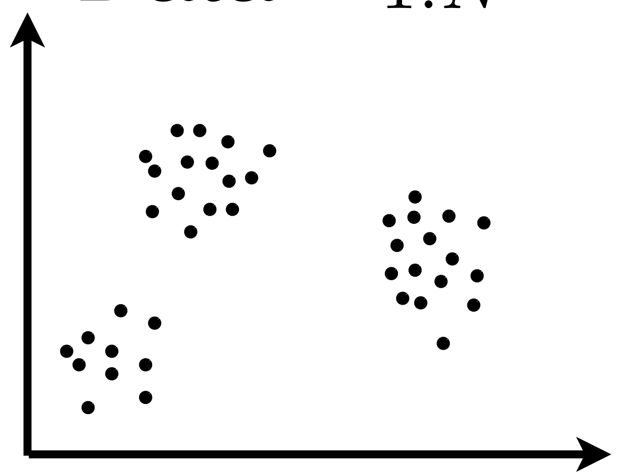

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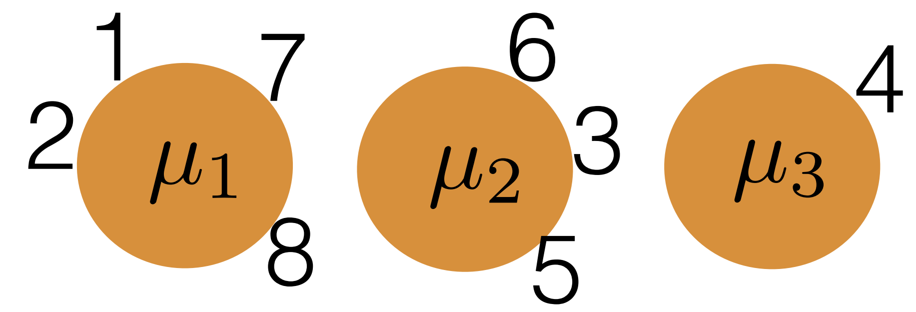

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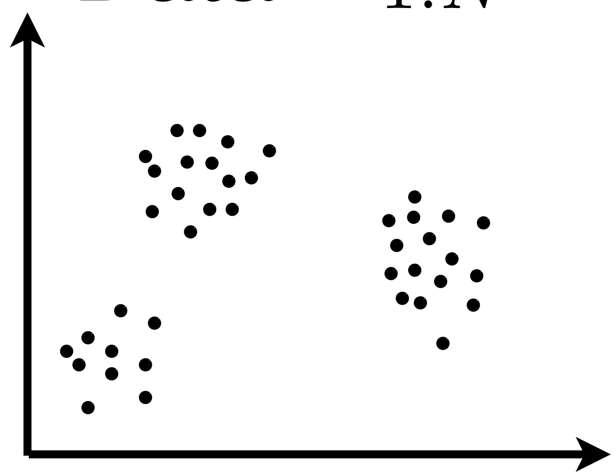

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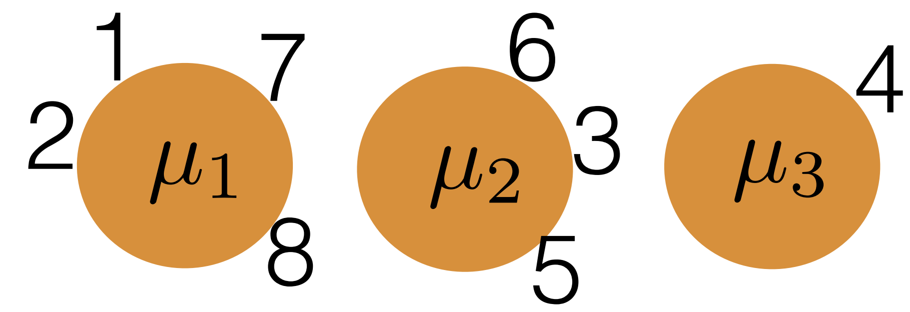

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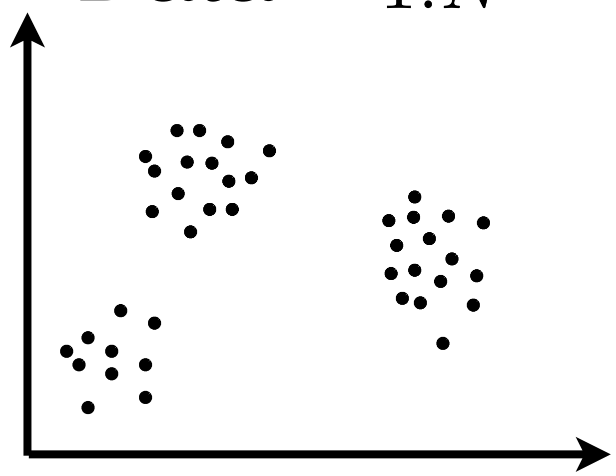

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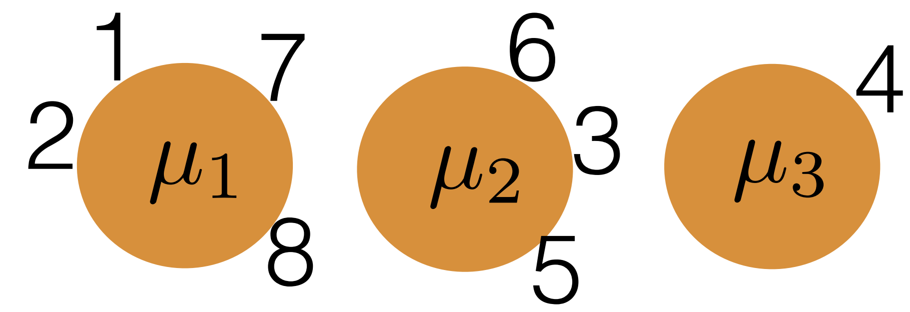

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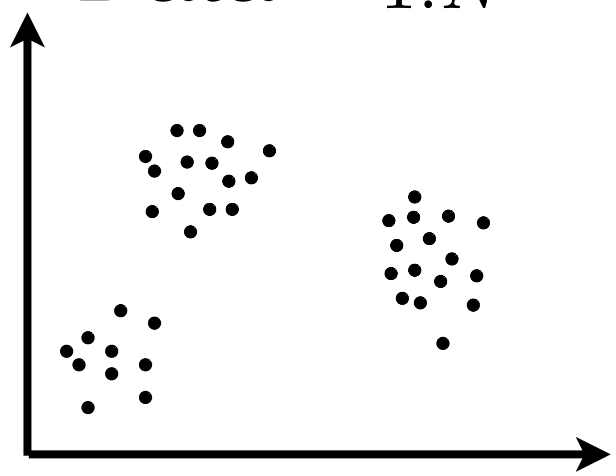

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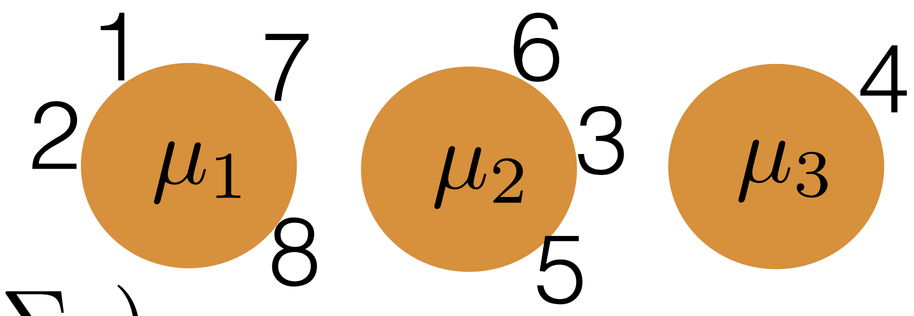

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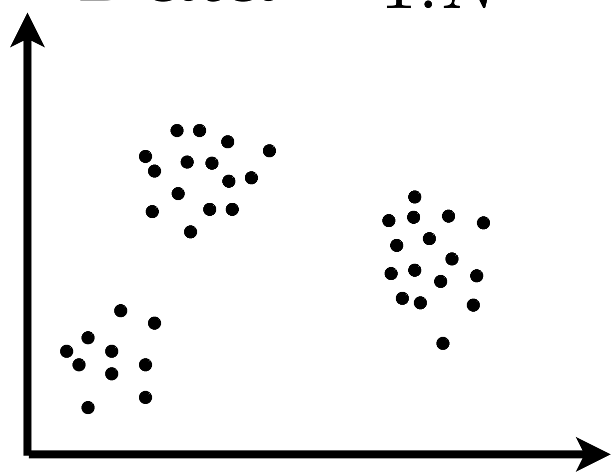

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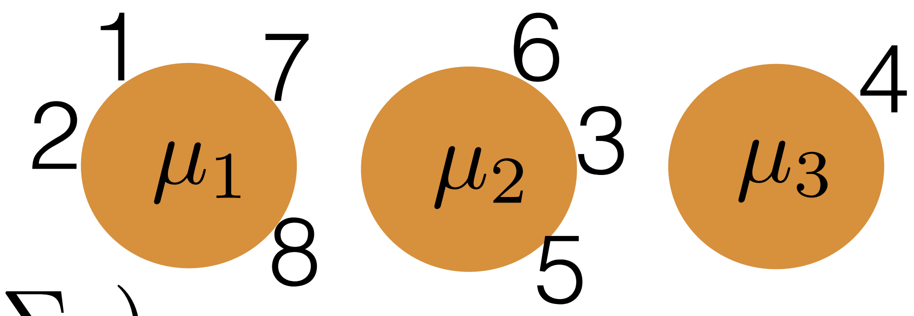

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[demo]

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More Markov Chain Monte Carlo

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- Slice sampling

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  - auxiliary variable ➔ finite conditionals

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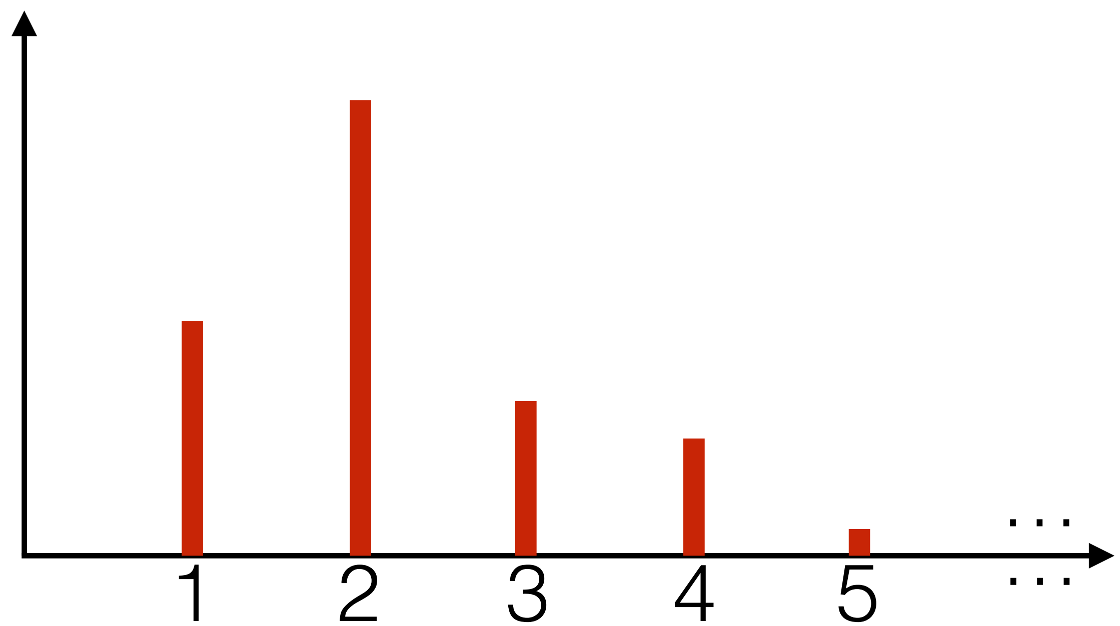

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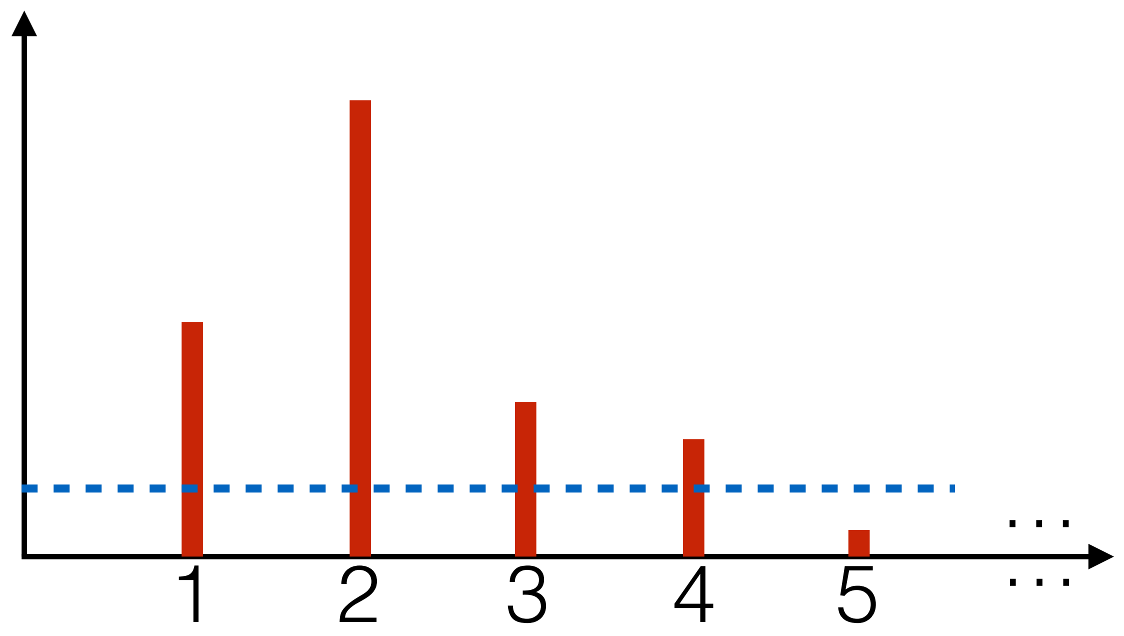

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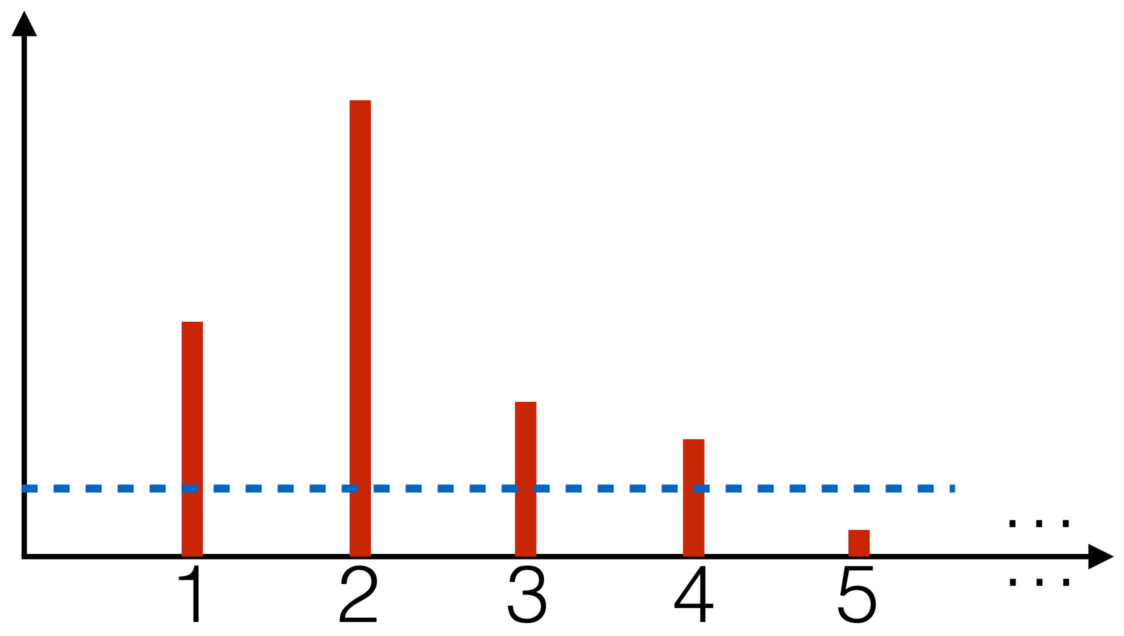

- Approximate with truncated distribution

[Ishwaran, James 2001; Campbell*, Huggins*, Broderick 2016]

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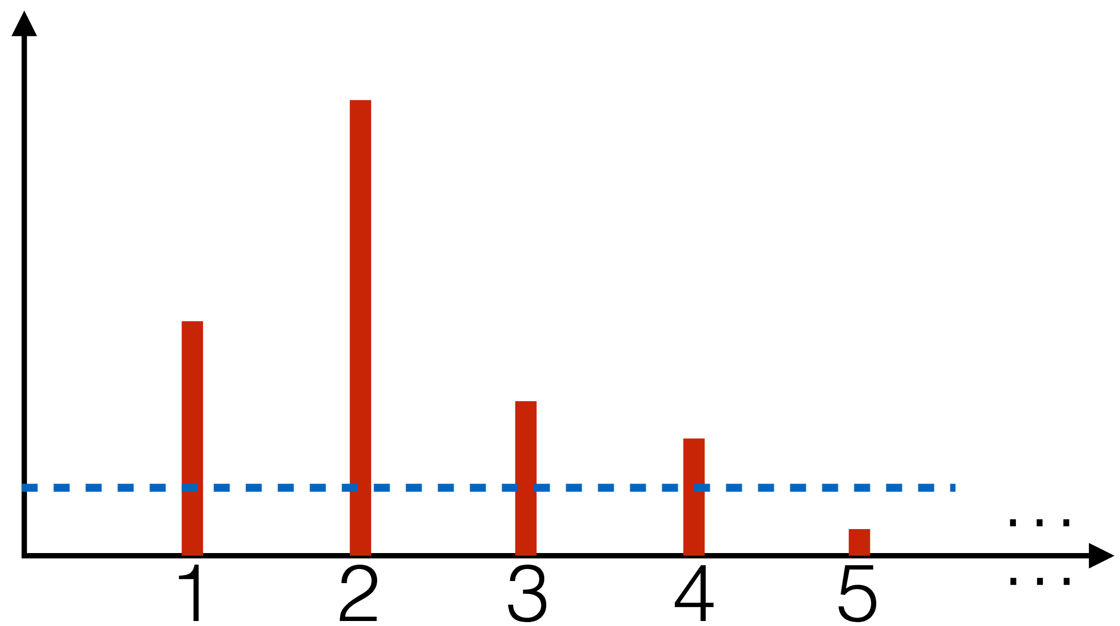

- Approximate with truncated distribution
  - E.g., Hamiltonian Monte Carlo

[Ishwaran, James 2001; Campbell*, Huggins*, Broderick 2016]

Variational Bayes

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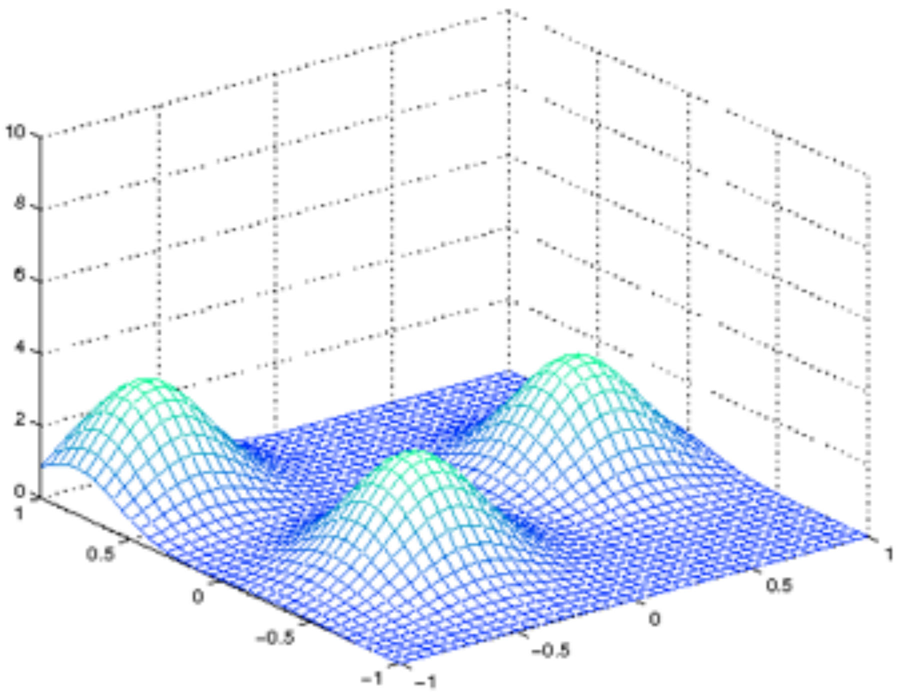

- Variational Bayes (VB)

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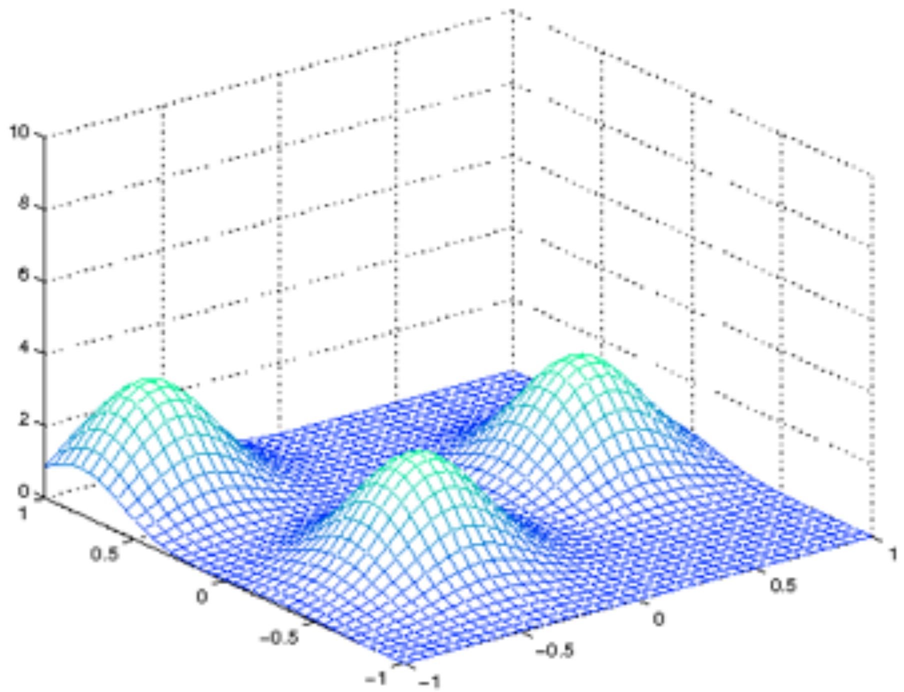

- Variational Bayes (VB)
  - Approximation $q^*(\theta)$ for posterior $p(\theta|x)$

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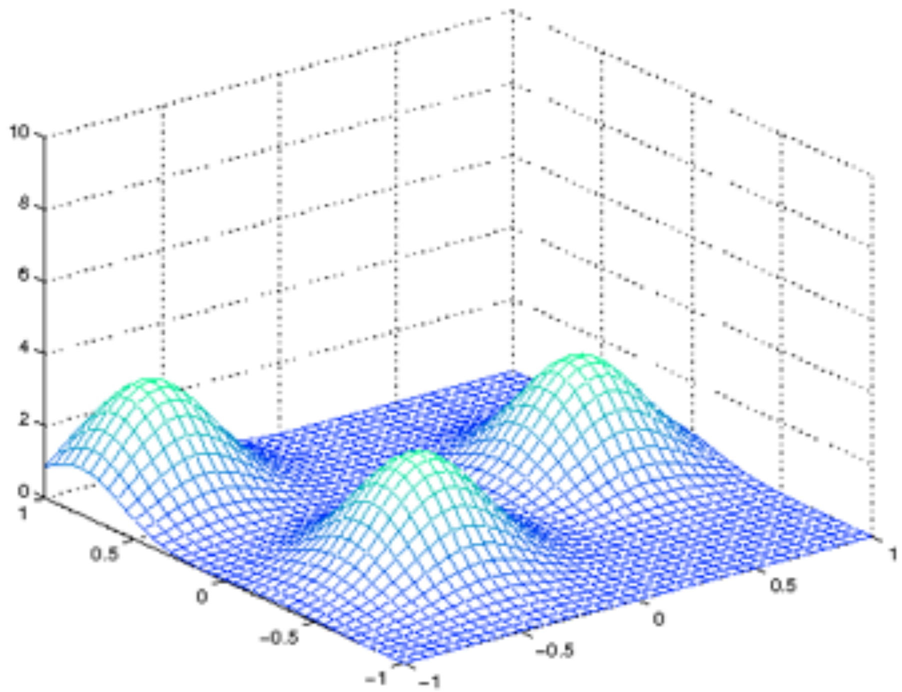

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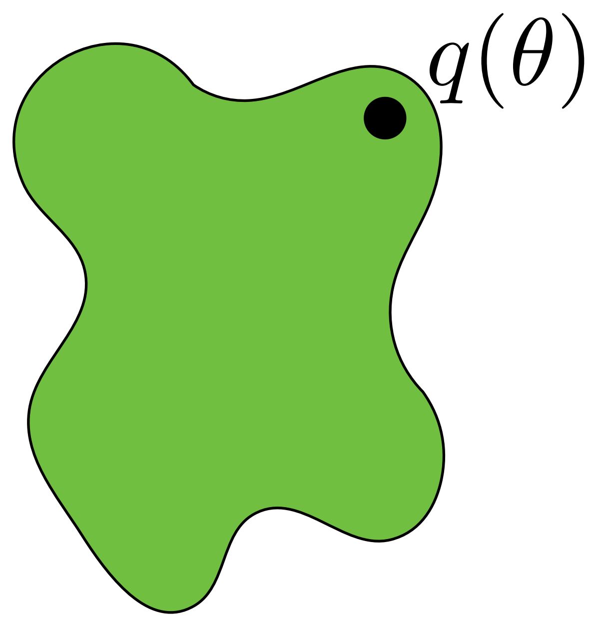

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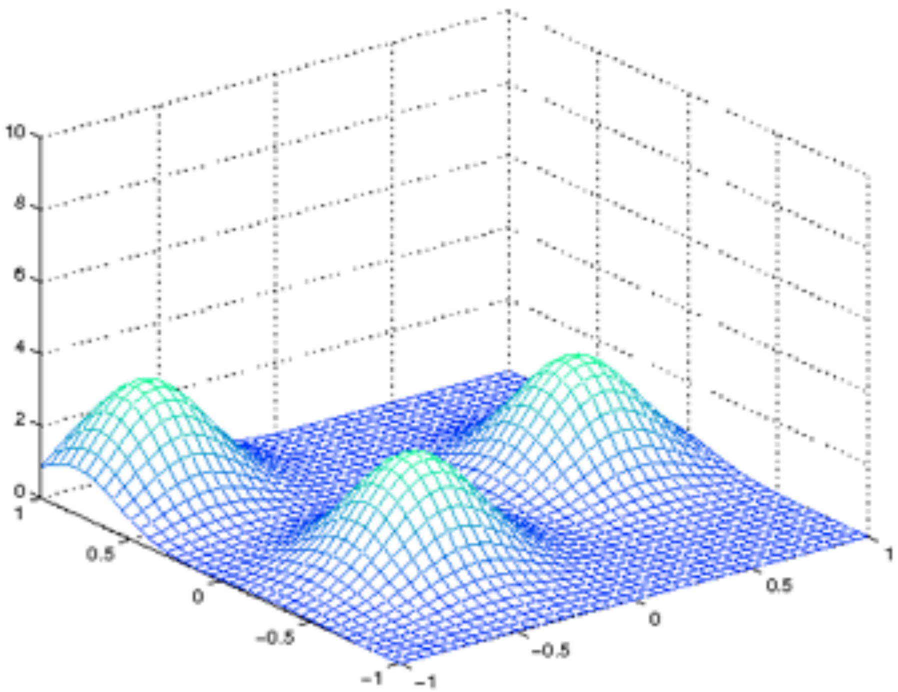
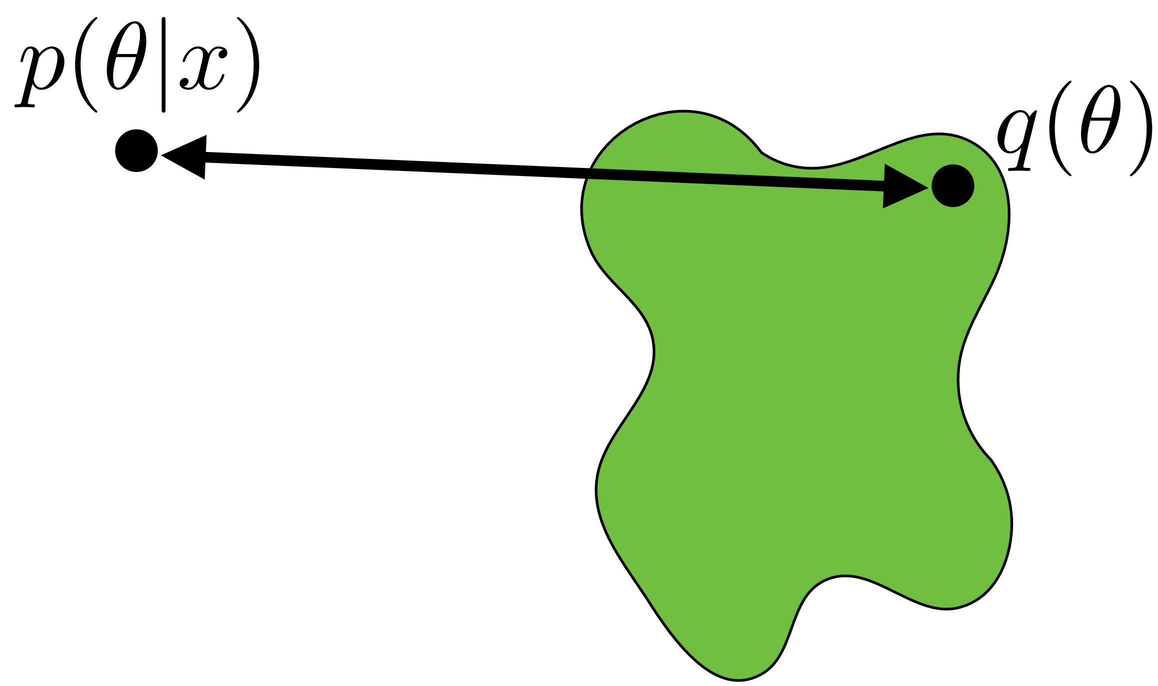
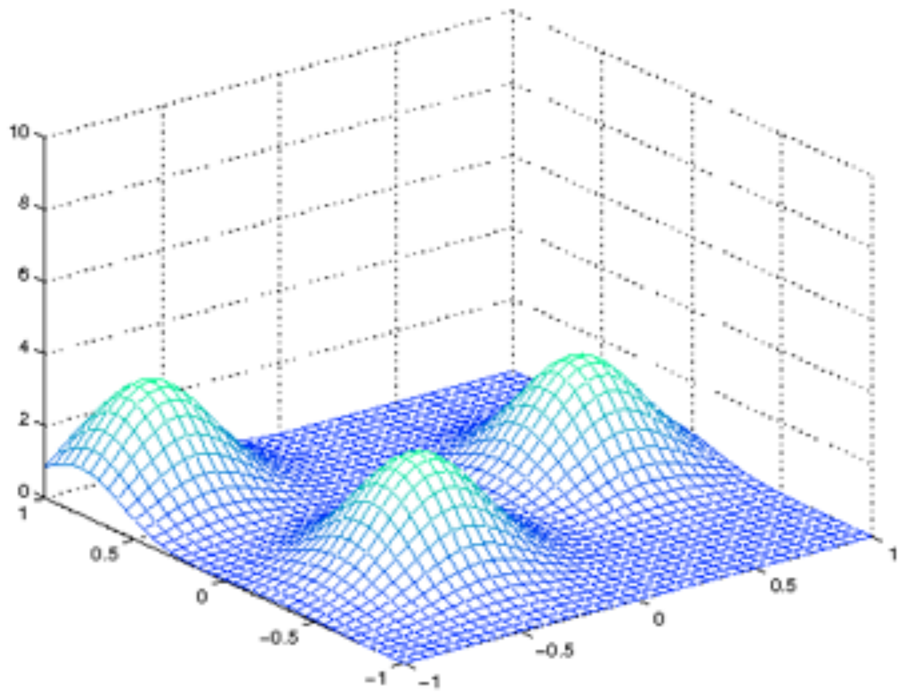
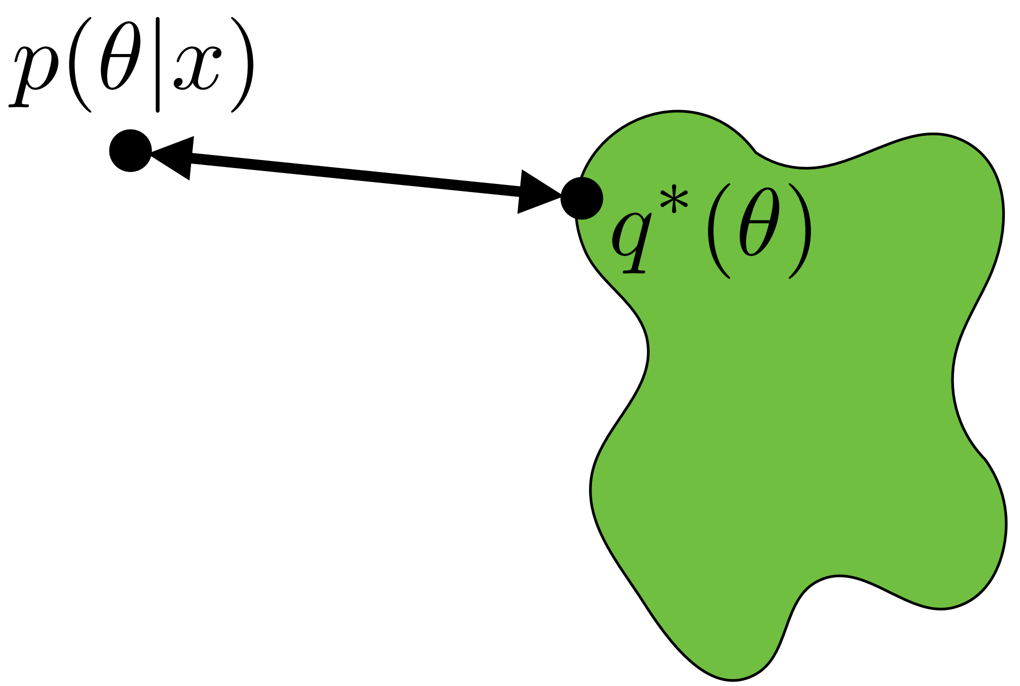




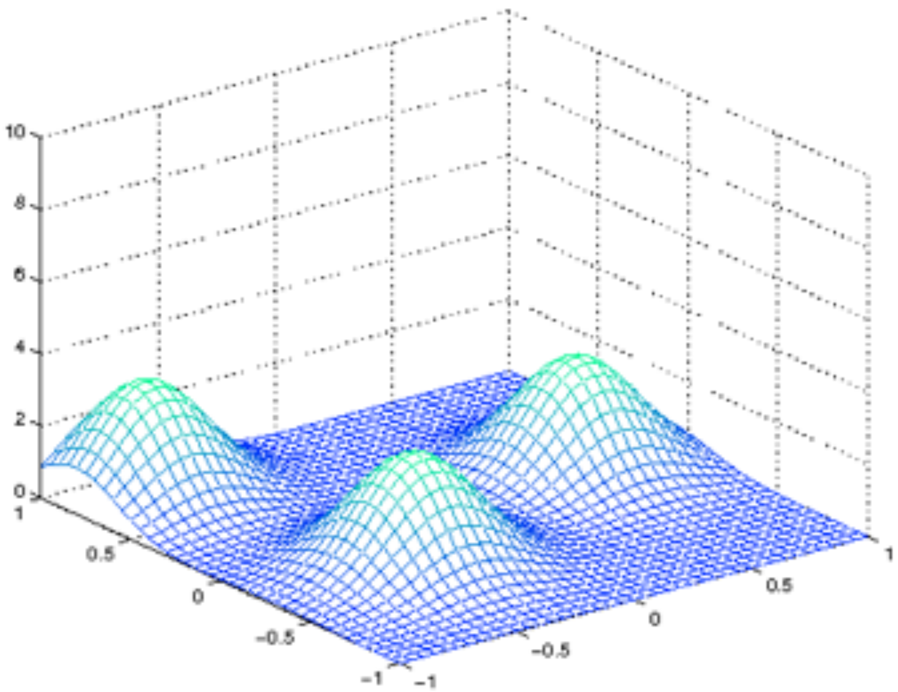
Variational Bayes

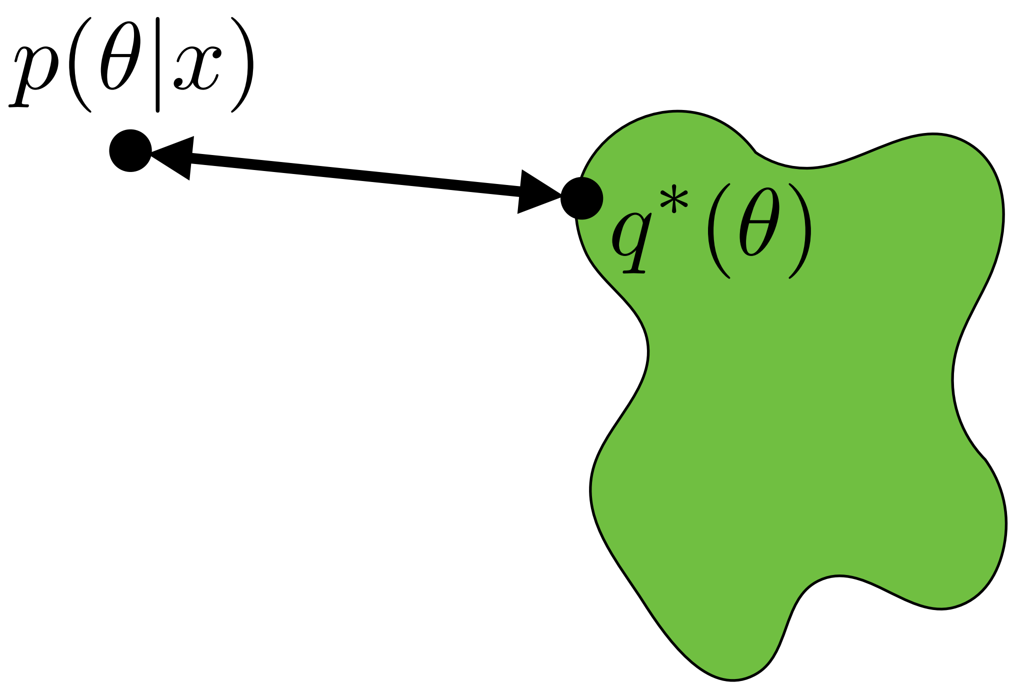




- Variational Bayes (VB)
  - Approximation $q^*(\theta)$ for posterior $p(\theta|x)$

Variational Bayes

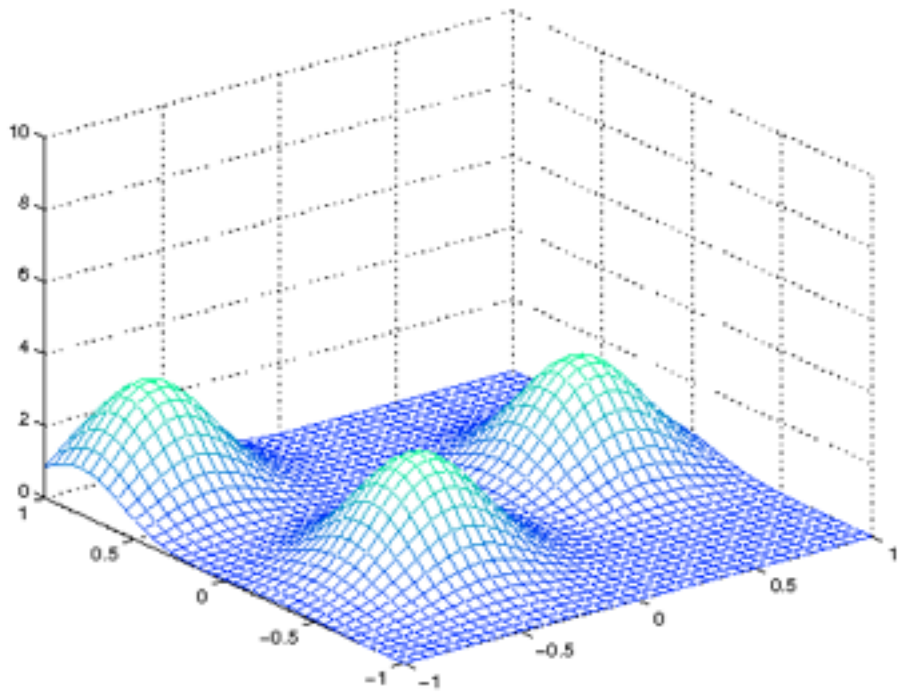




- Variational Bayes (VB)
  - Approximation $q^*(\theta)$ for posterior $p(\theta|x)$
  - "Close": Minimize Kullback-Liebler (KL) divergence:

$$KL(q\|p(\cdot|x))$$

Variational Bayes



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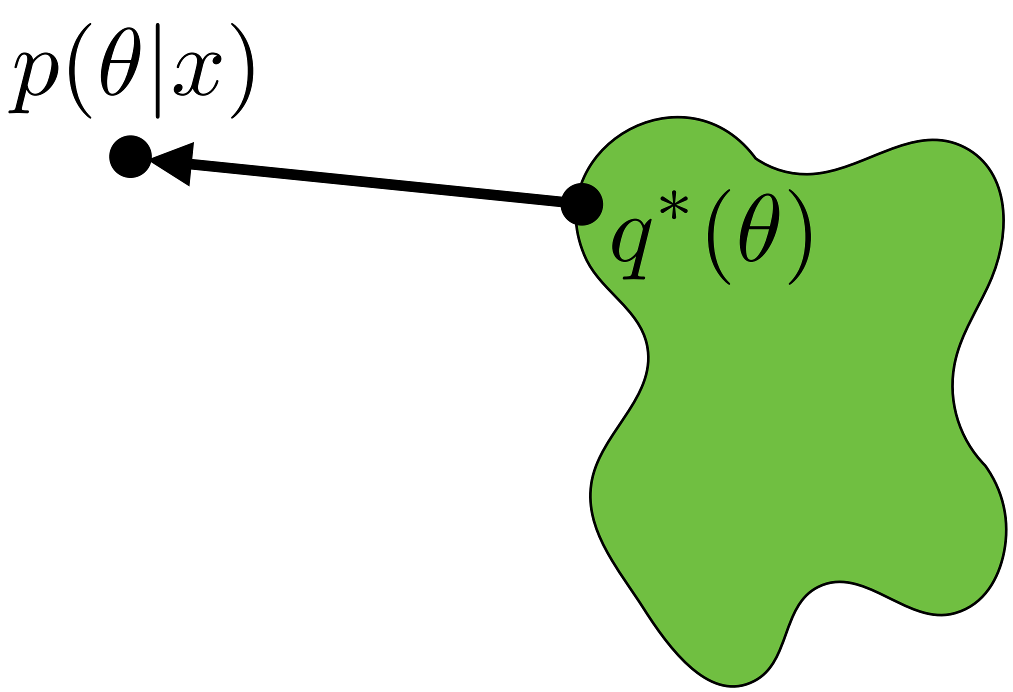
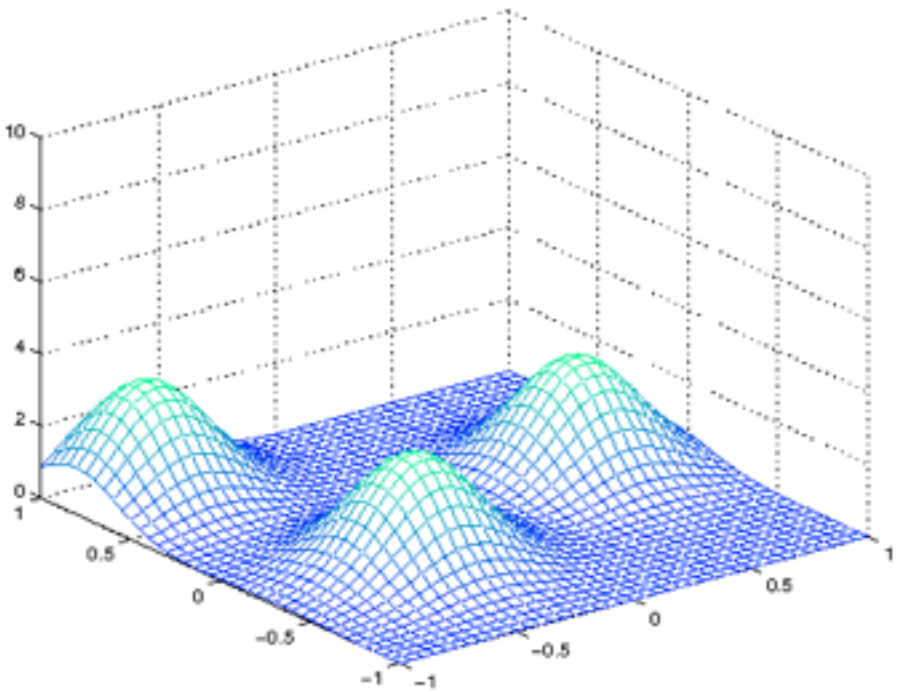
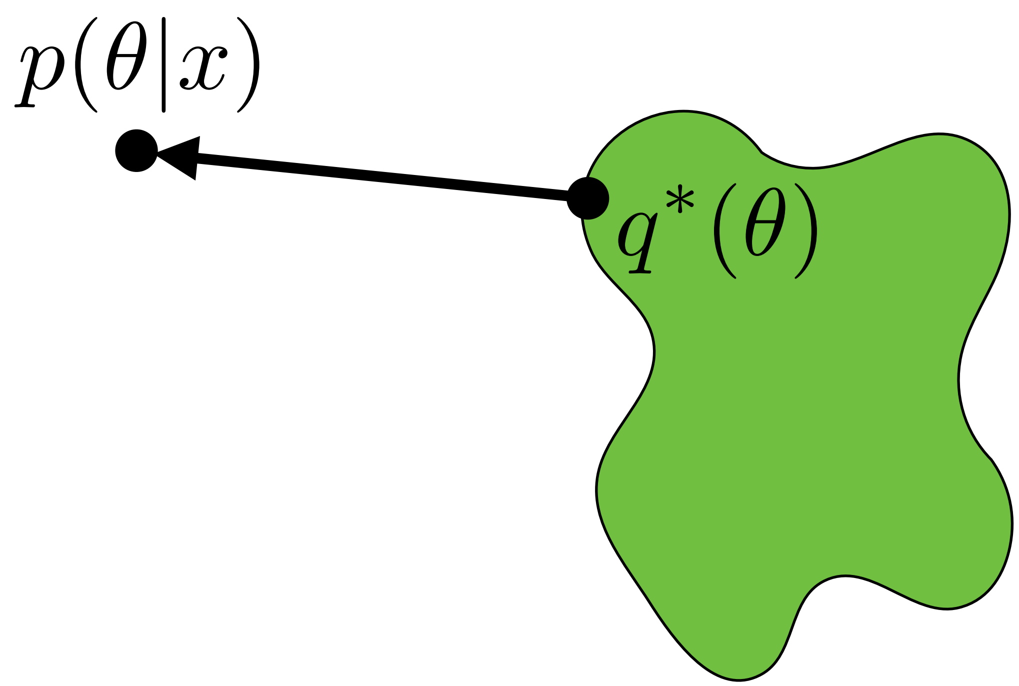


Variational Bayes





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  - "Close": Minimize Kullback-Liebler (KL) divergence:
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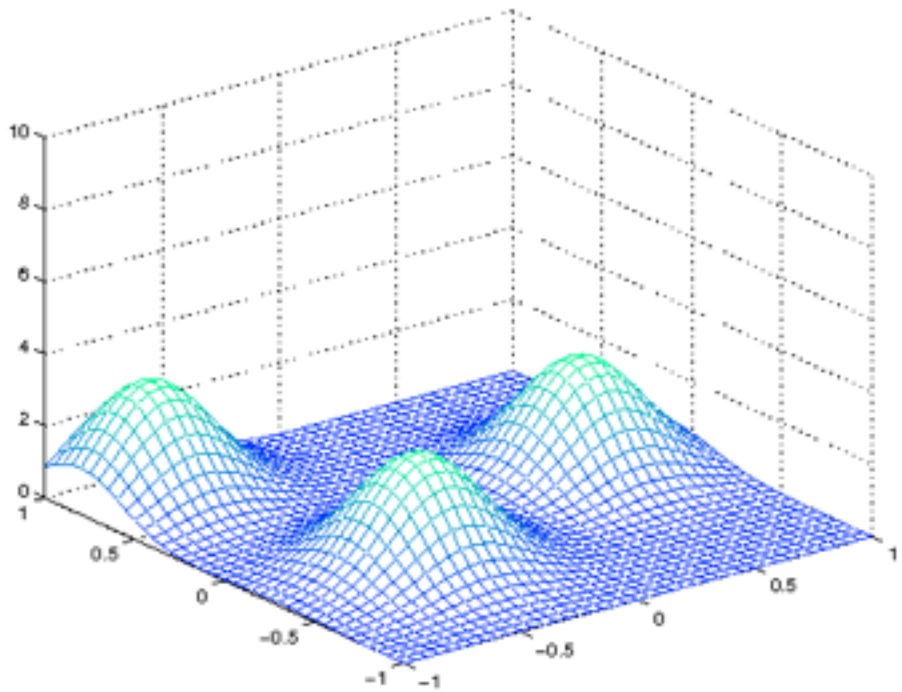
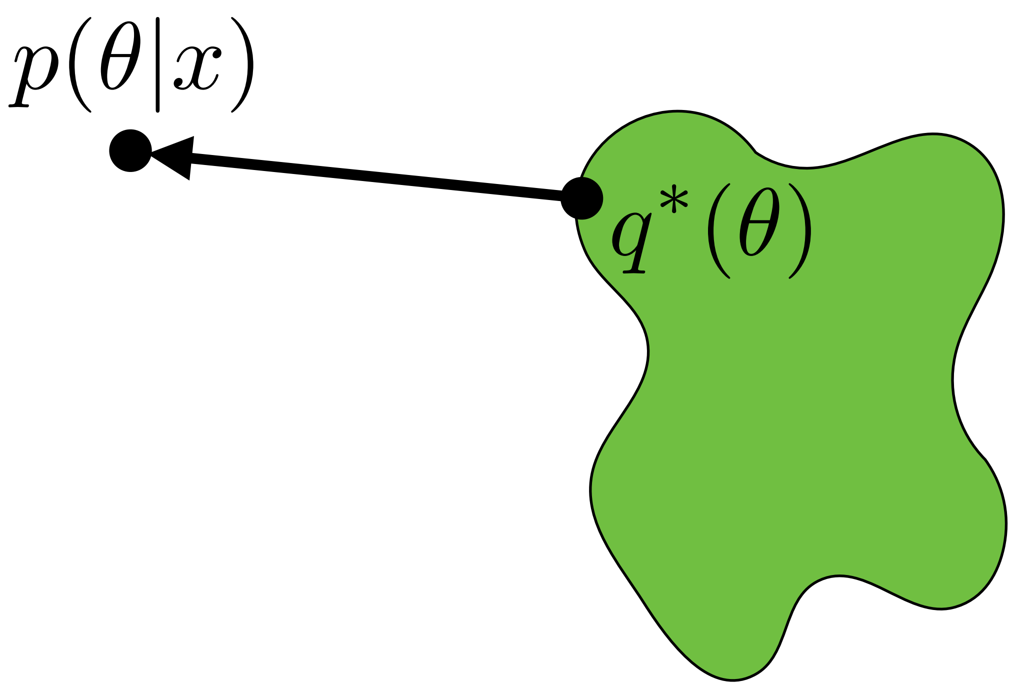
  - "Nice": factorizes, exponential family, truncation

Variational Bayes





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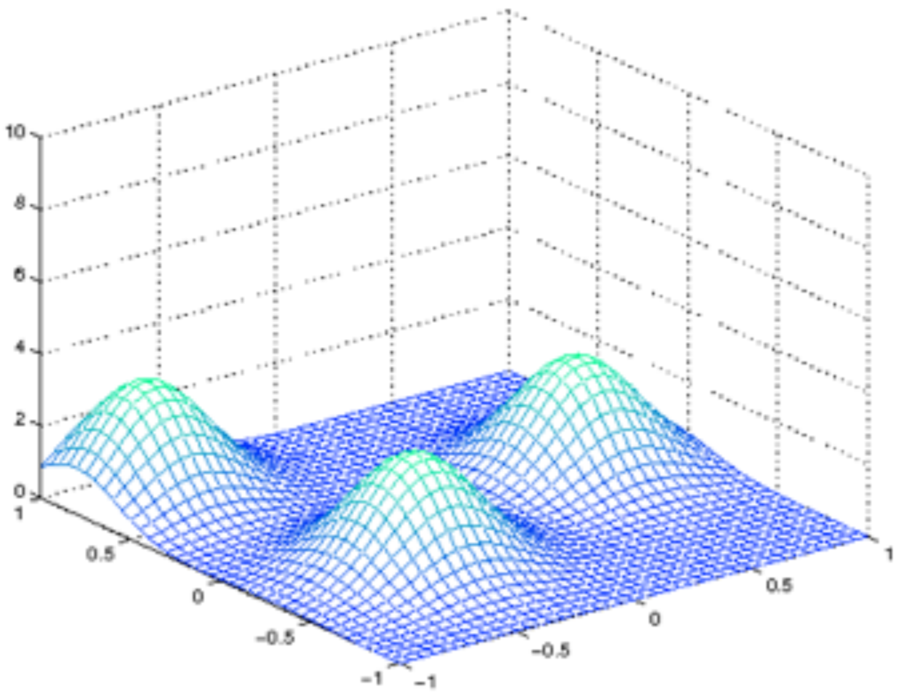
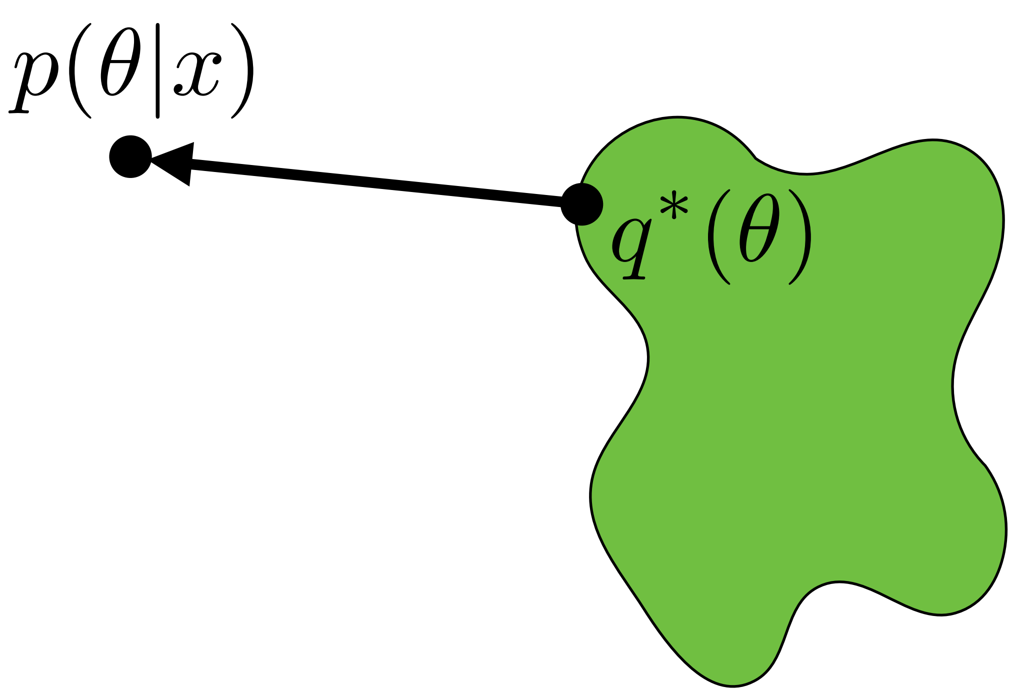
- VB practical success

Variational Bayes





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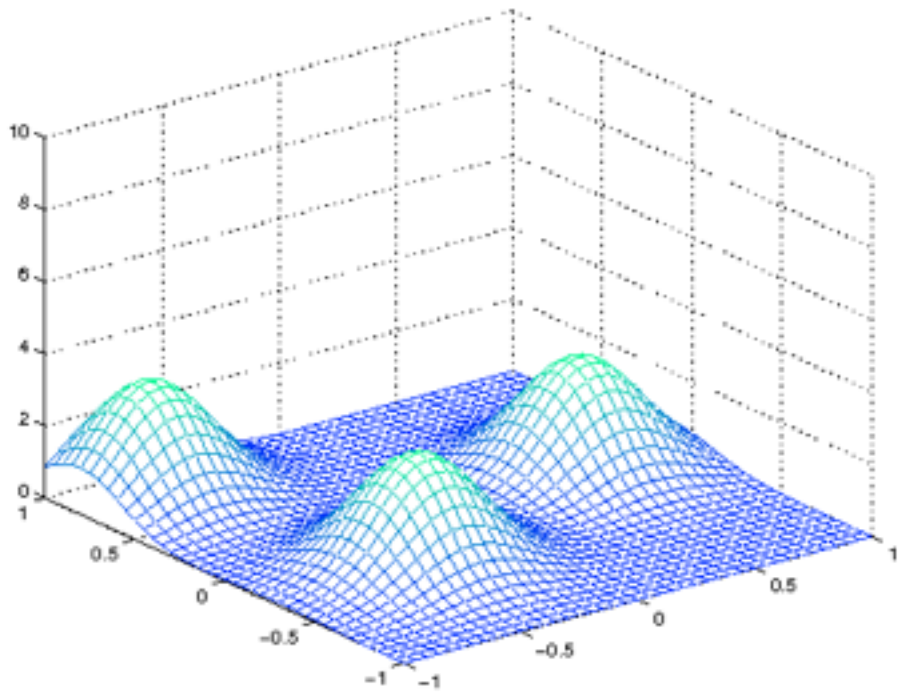
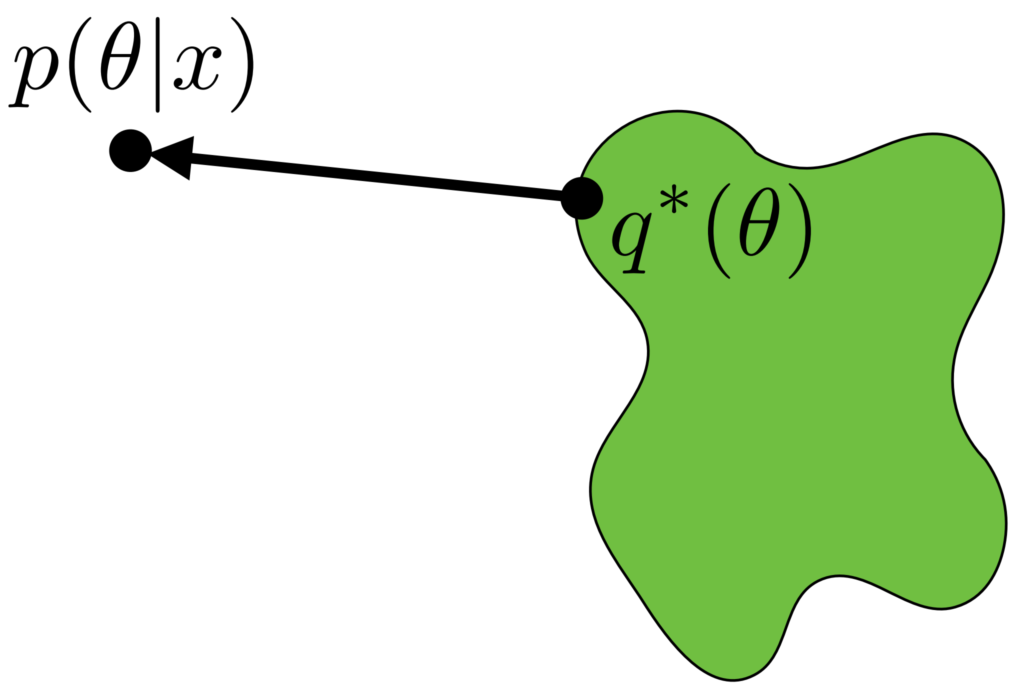
Variational Bayes





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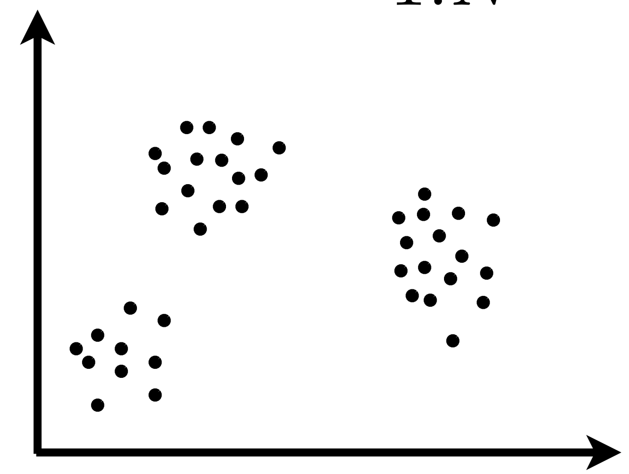
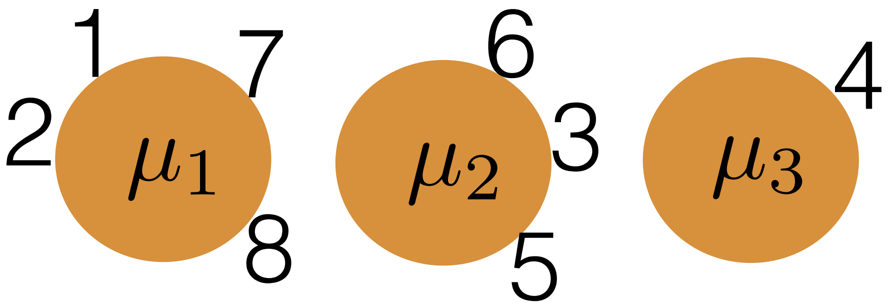
  - "Nice": factorizes, exponential family, truncation
- VB practical success
  - point estimates and prediction
  - fast, streaming, distributed

[Broderick, Boyd, Wibisono, Wilson, Jordan 2013]

Exercises

- Data $x_{1:N}$

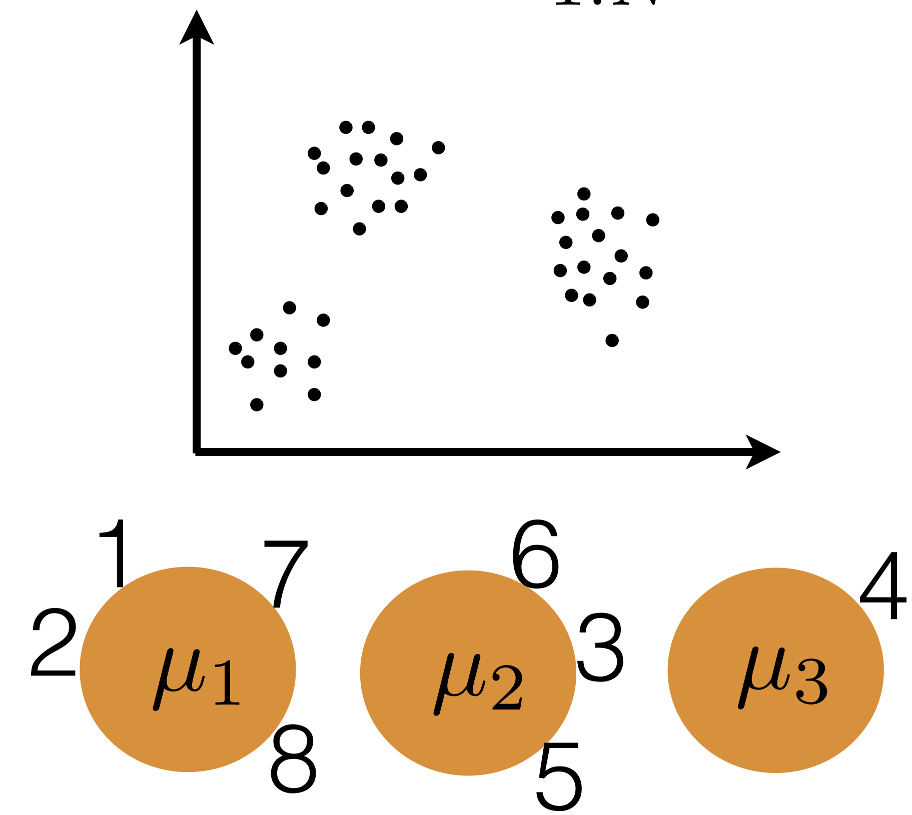




Exercises

- Code a CRP mixture model simulator

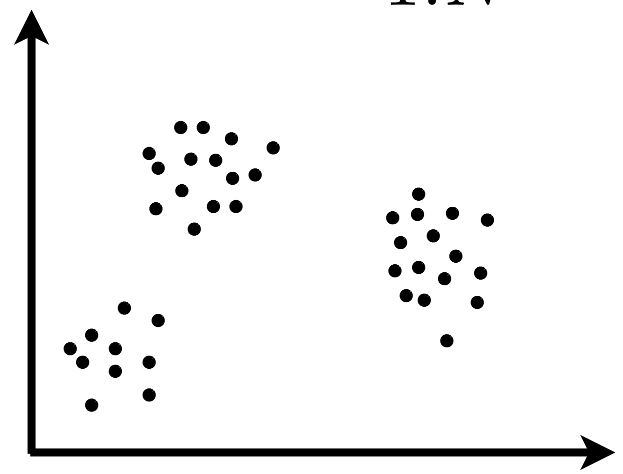
- Data $x_{1:N}$

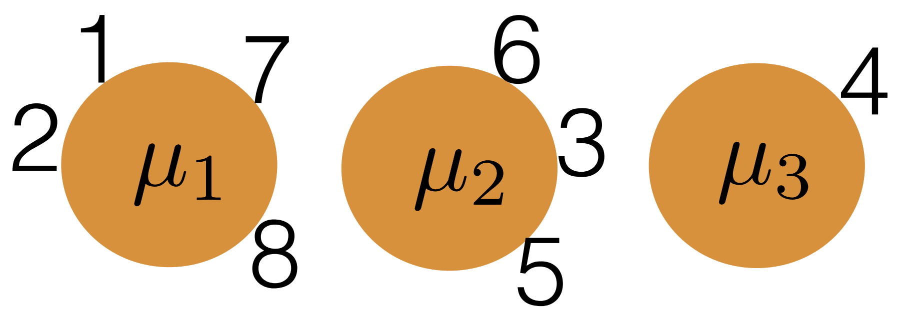


Exercises

- Code a CRP mixture model simulator
- Derive the CRP mixture model Gibbs sampler in the slides; derive $p(x_{C \cup \{n\}} | x_C)$ explicitly for a Gaussian mixture

- Data $x_{1:N}$





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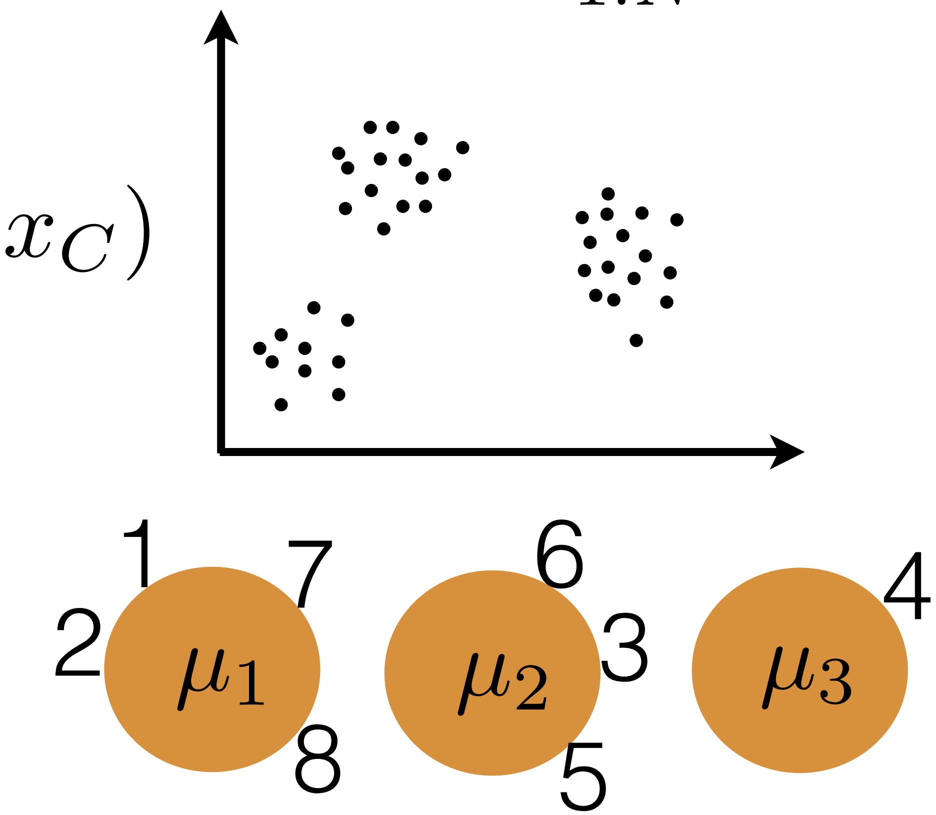

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- Extend the CRP mixture model Gibbs sampler in the slides to sample the cluster-specific parameters as well
- Review Gibbs sampling, slice sampling [Neal 2003], variational Bayes [Bishop 2006]

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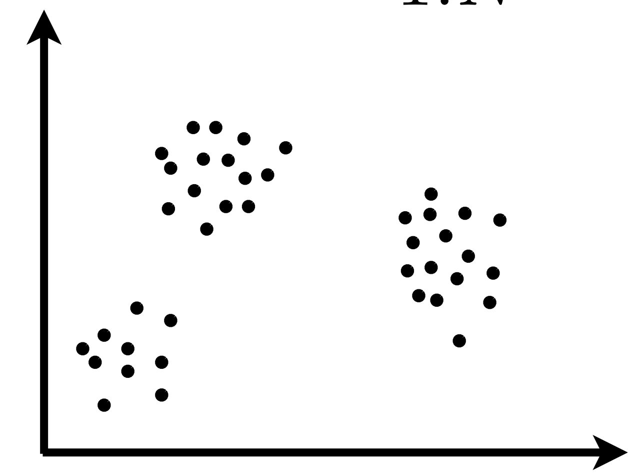

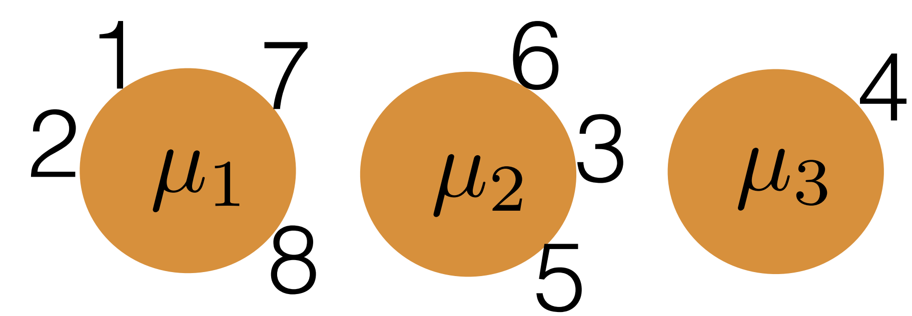

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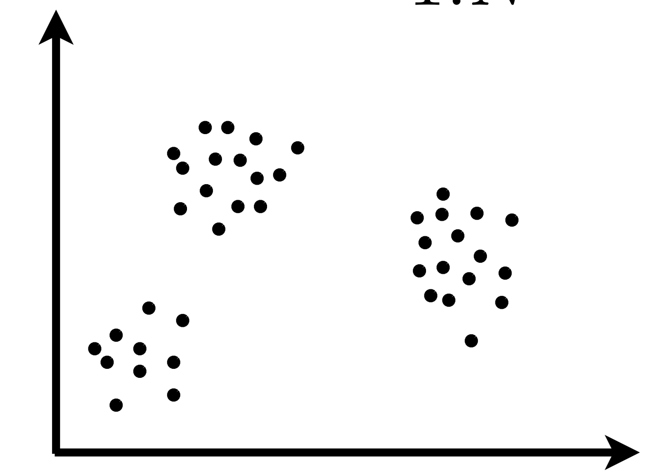

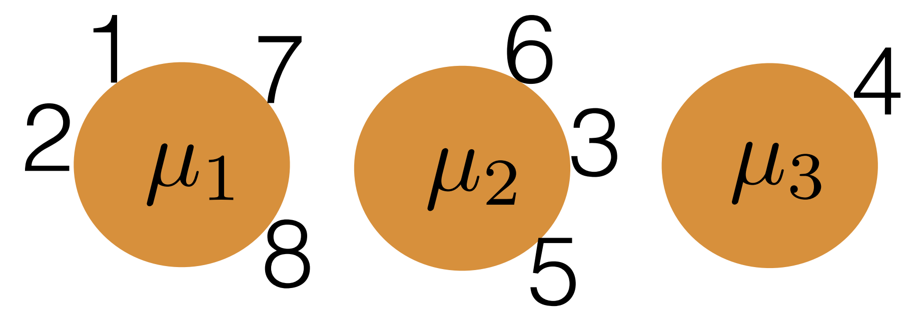

- Review Gibbs sampling, slice sampling [Neal 2003], variational Bayes [Bishop 2006]
- Read [Neal 2000] and code a DPMM Gibbs sampler

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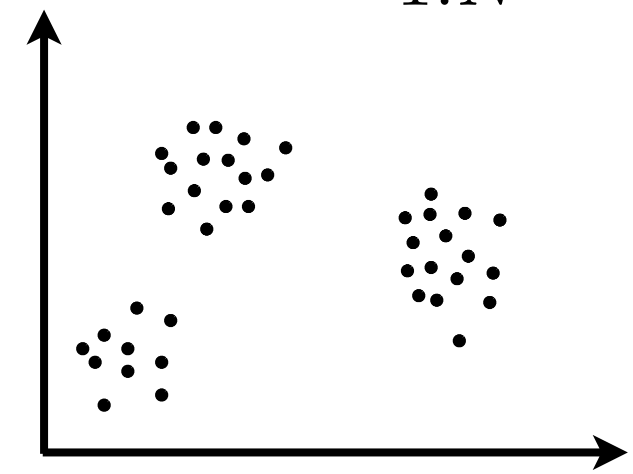

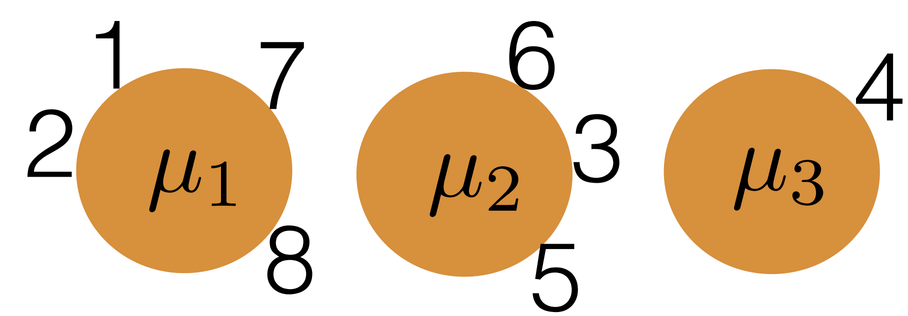

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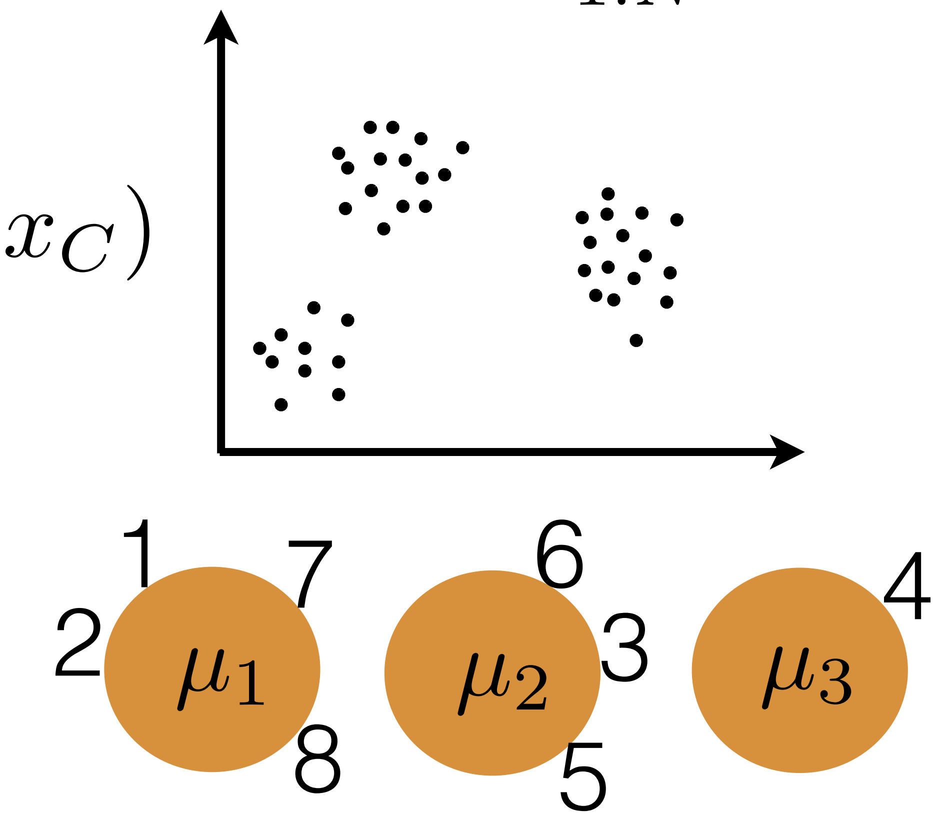

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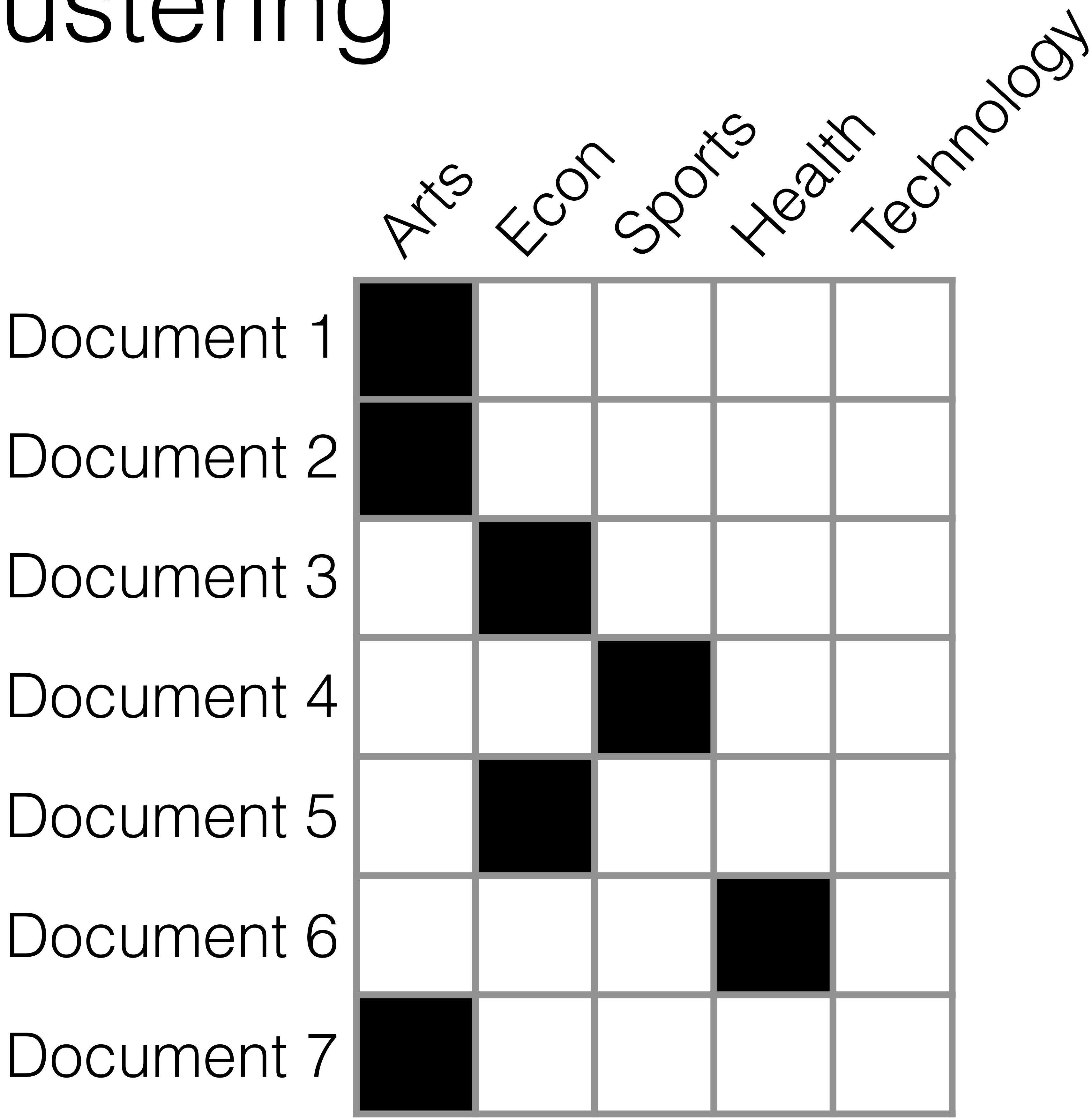
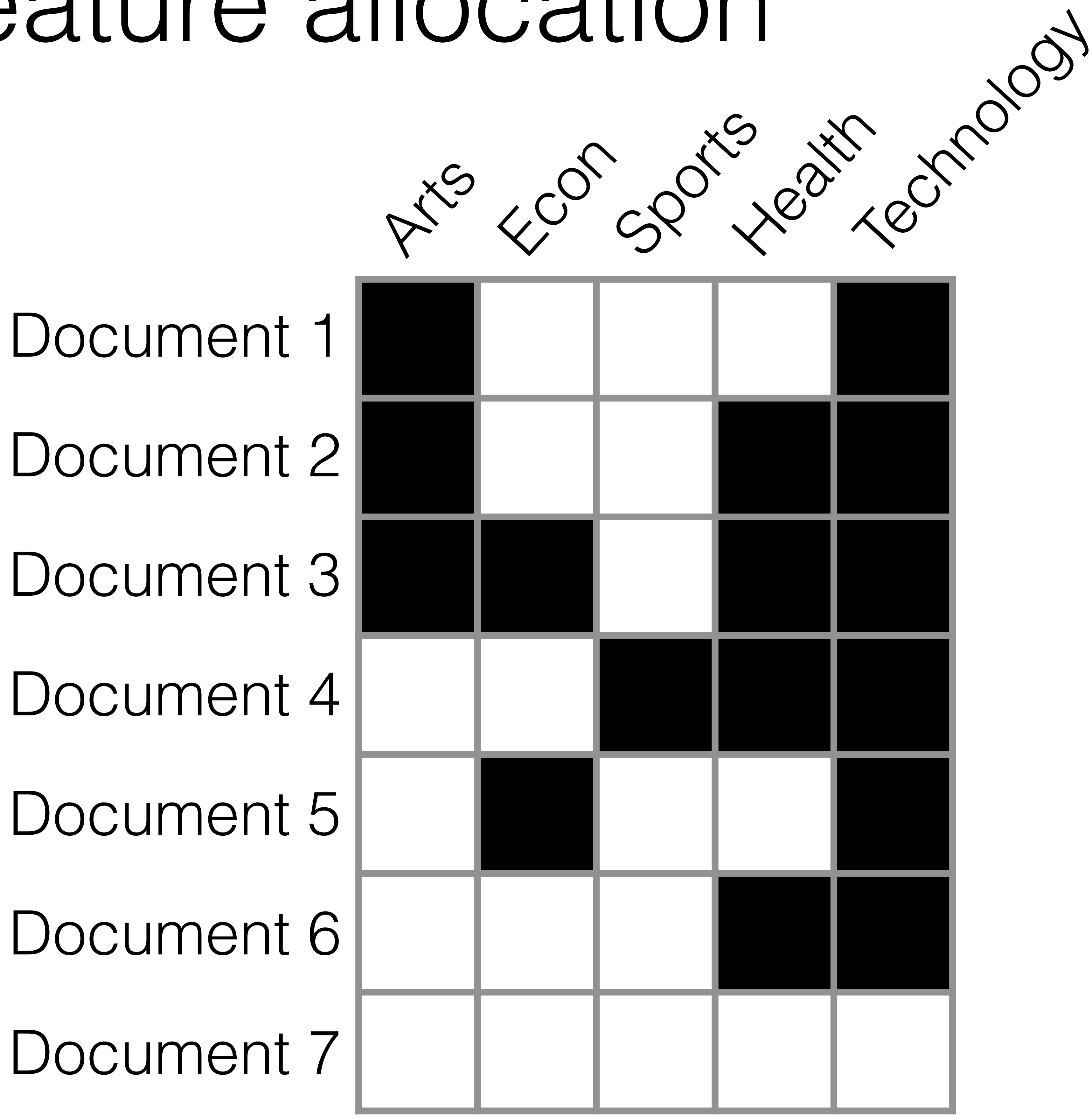
- Read [Blei, Jordan 2006] and code variational inference for the DPMM

[Tiia Monto, https://commons.wikimedia.org/wiki/File:Trail_and_mountain.jpg]

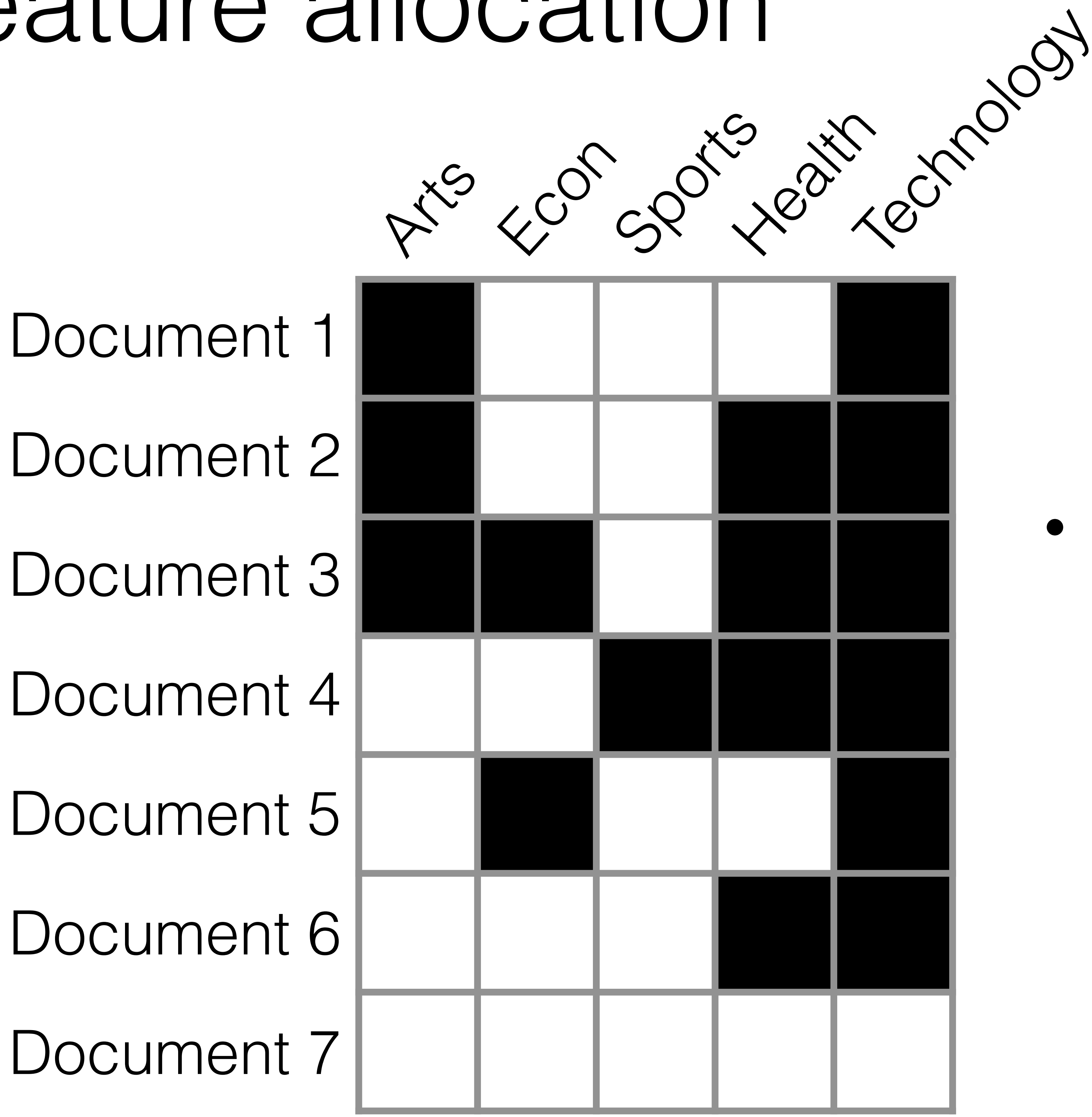
Clustering


Arts
Econ
Sports
Health
Technology
Document 1
Document 2
Document 3
Document 4
Document 5
Document 6
Document 7

Feature allocation

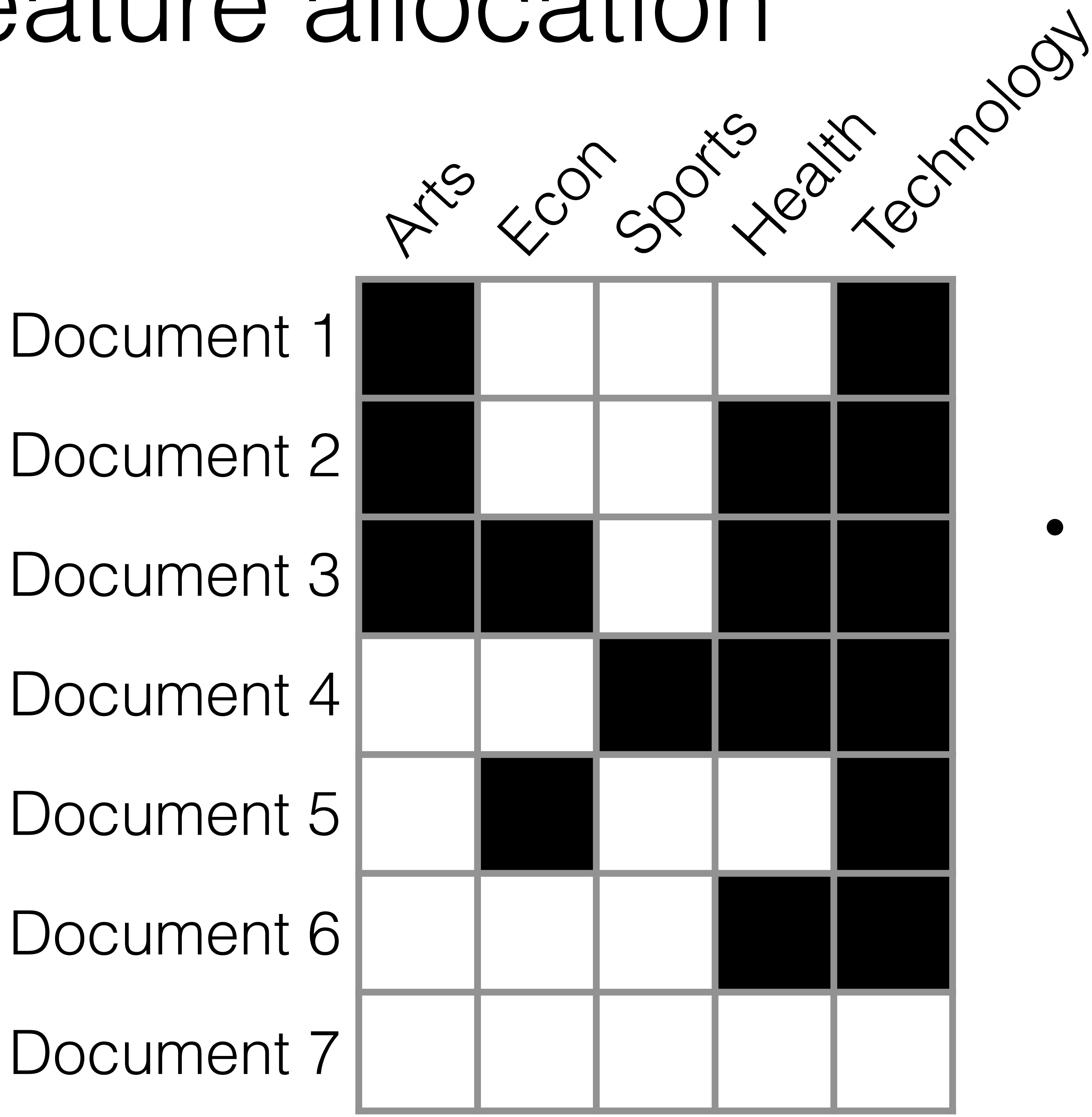


Feature allocation



- Indian buffet process

Feature allocation



- Indian buffet process

 [Griffiths, Ghahramani 2005

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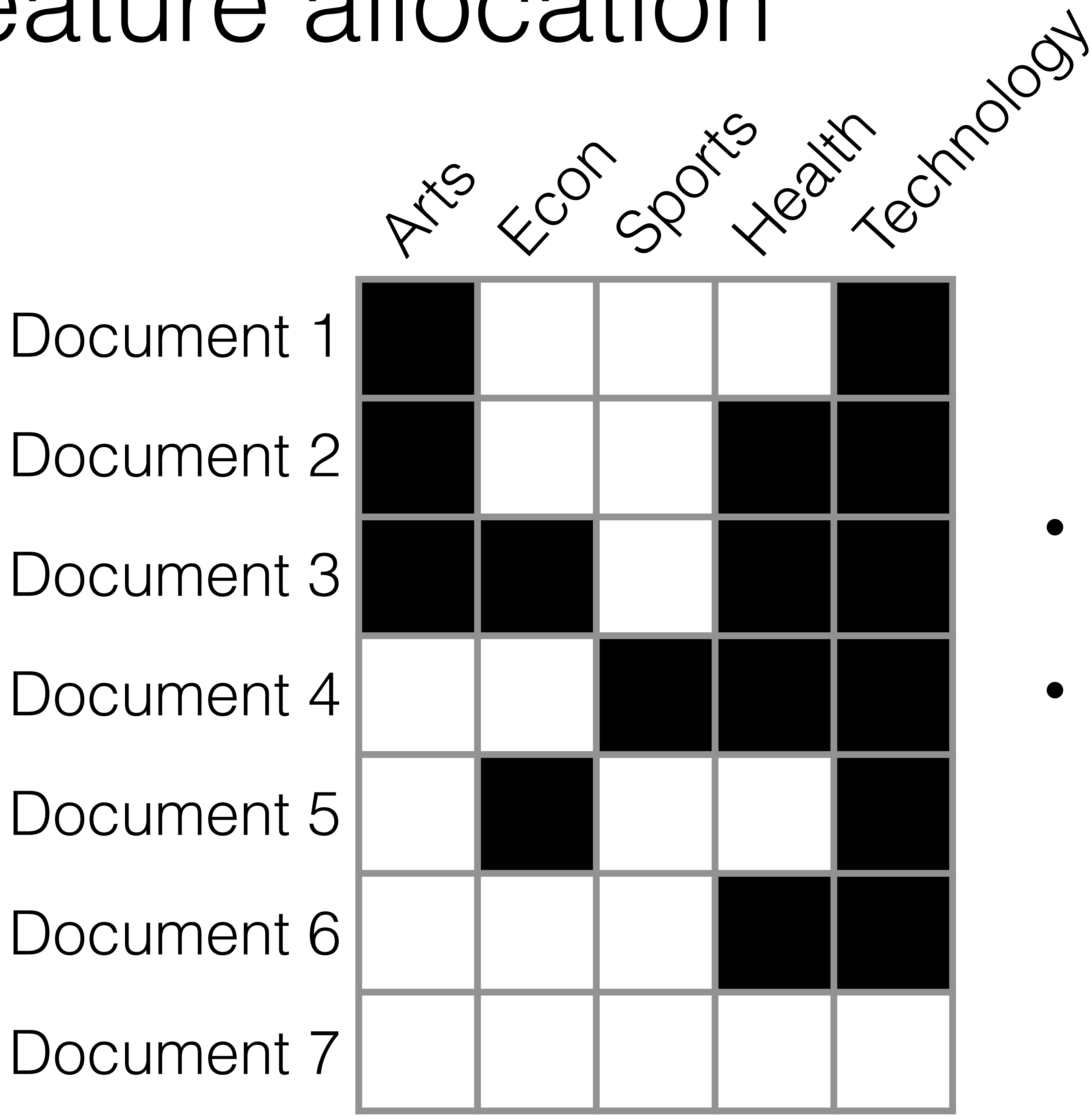
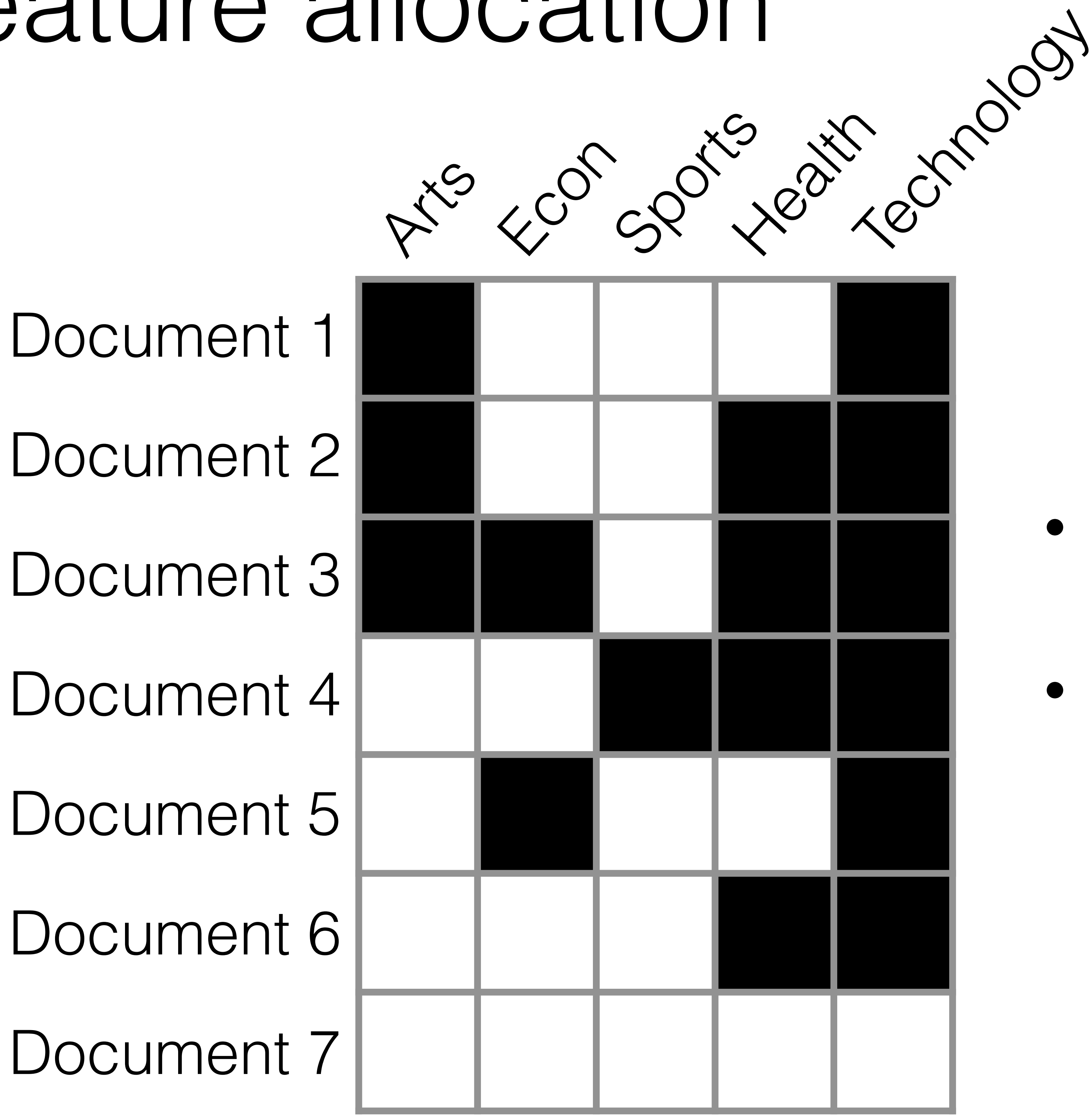


- Indian buffet process
- Beta process

[Griffiths, Ghahramani 2005

Feature allocation



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[Griffiths, Ghahramani 2005, Hjort 1990

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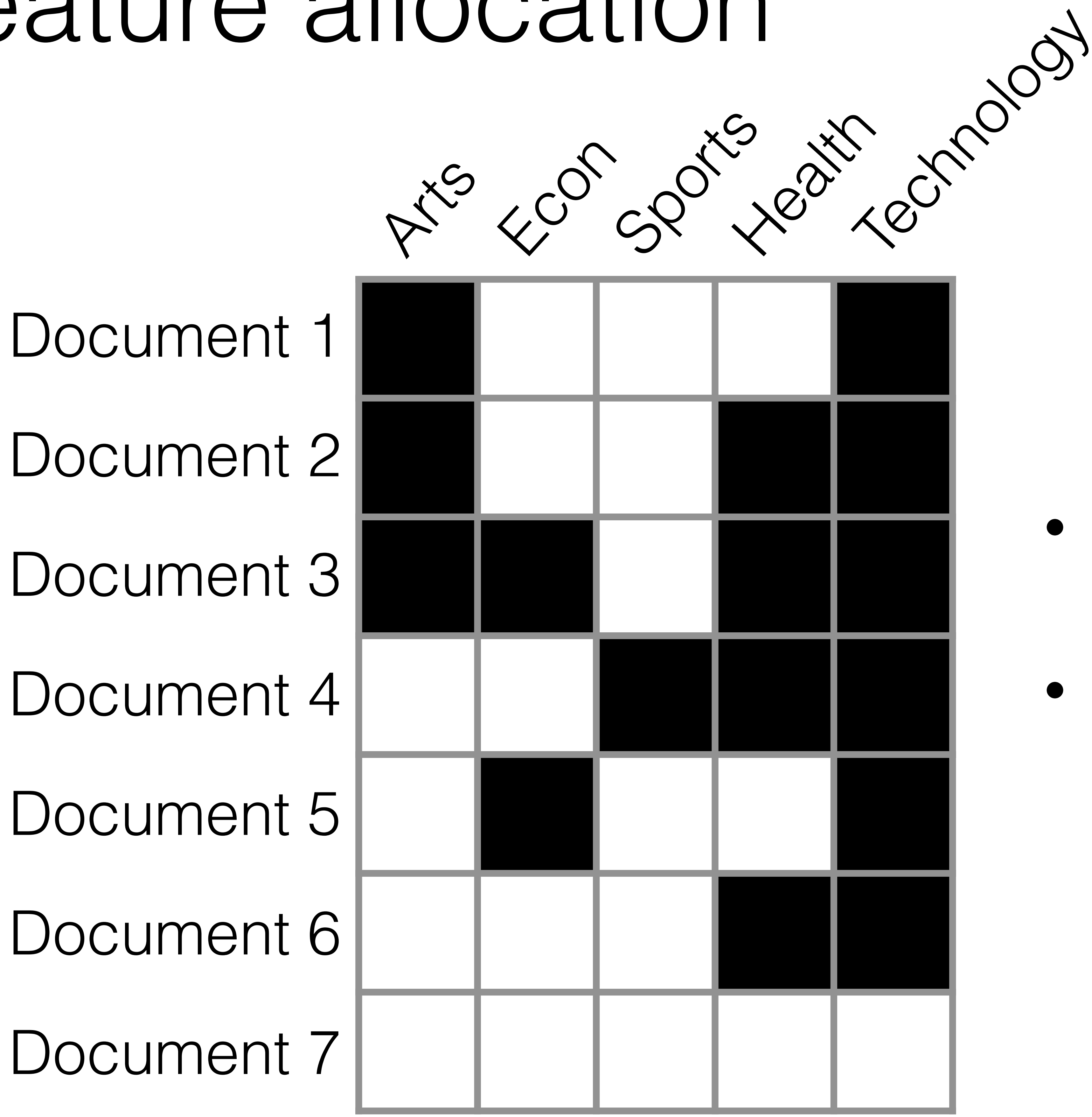

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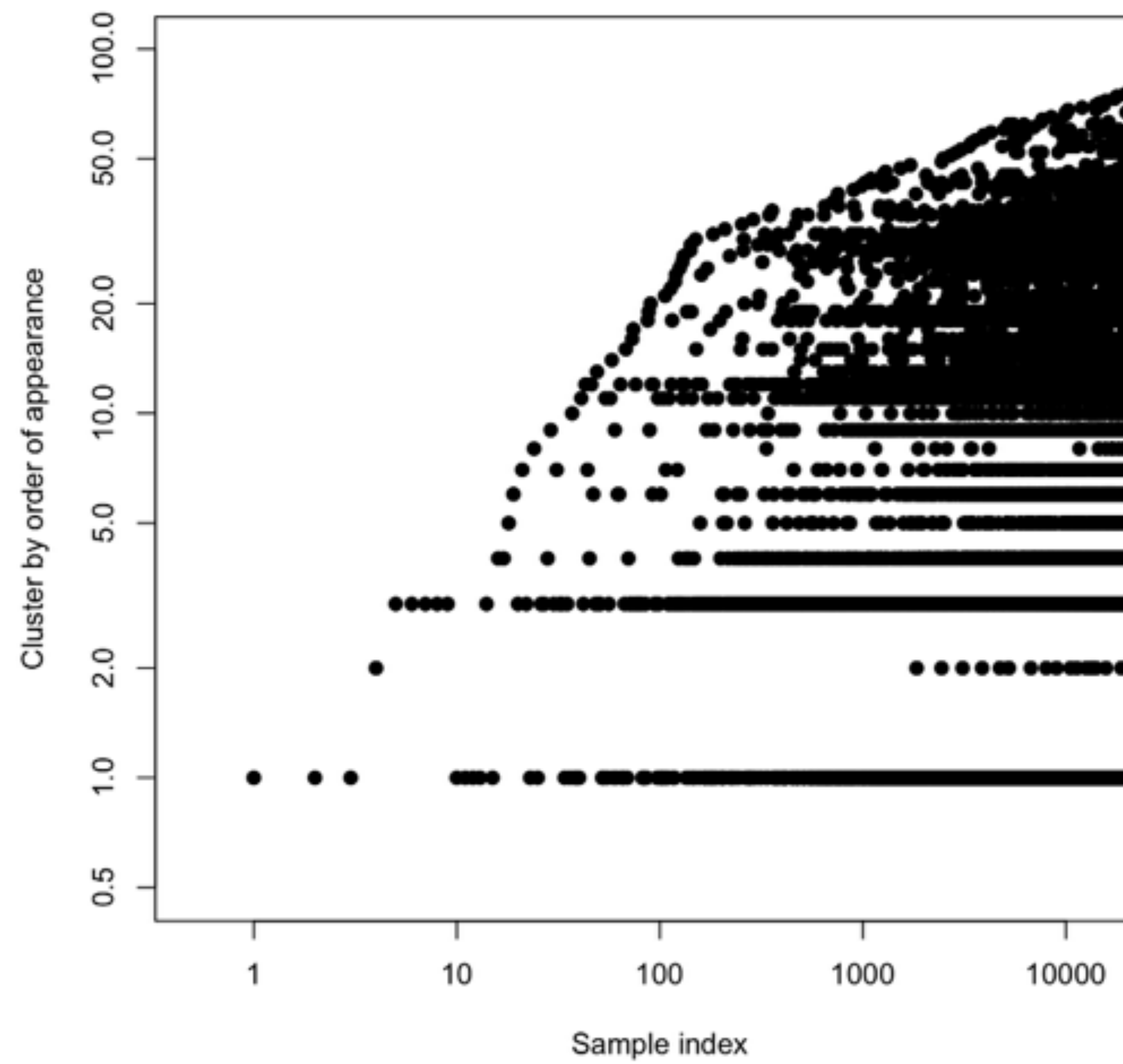
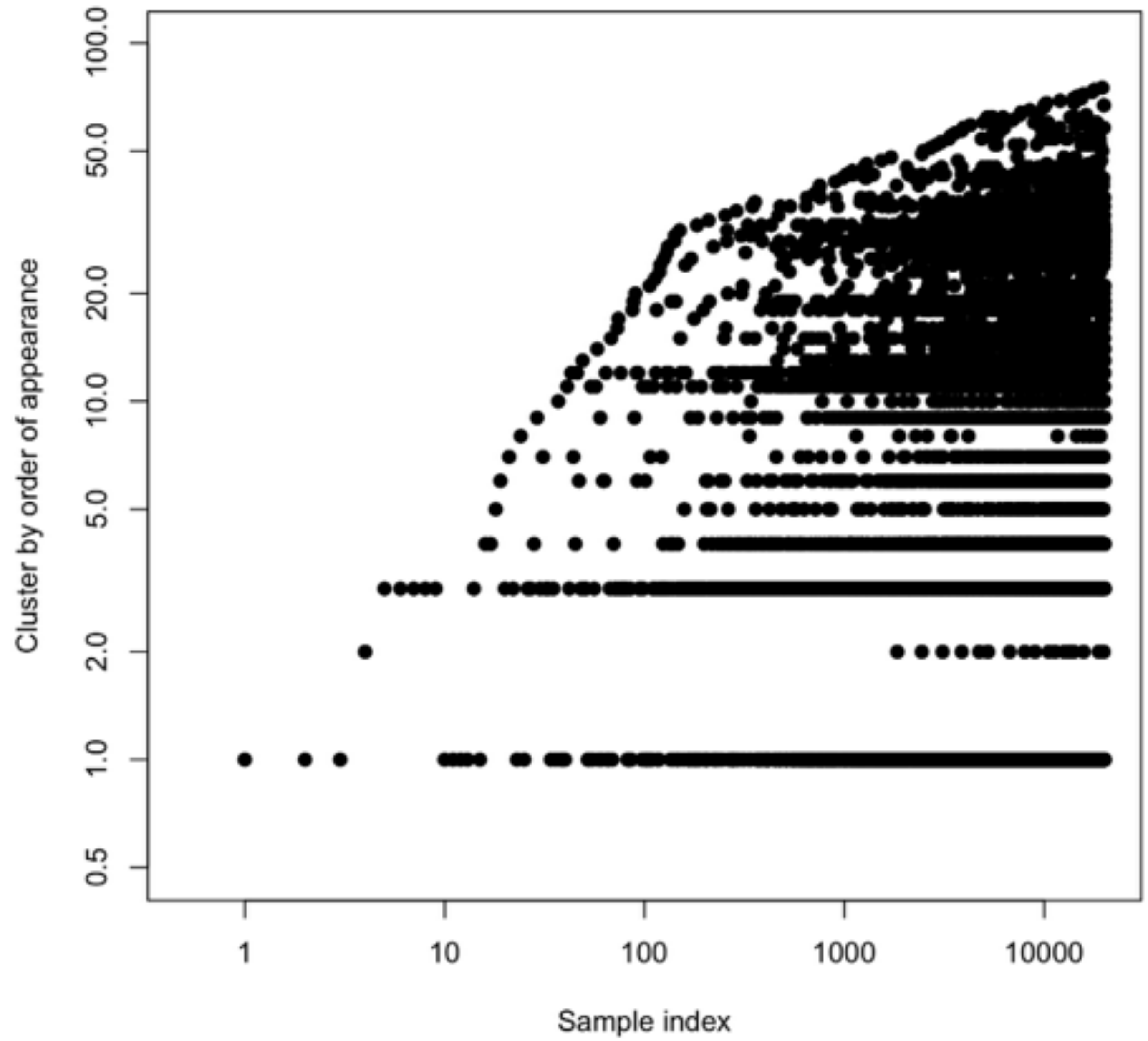
[Griffiths, Ghahramani 2005, Hjort 1990, Kim 1999, Thibaux, Jordan 2007, Broderick, Jordan, Pitman 2013]

Power laws


Cluster by order of appearance
Sample index
100.0
50.0
20.0
10.0
5.0
2.0
1.0
0.5
1
10
100
1000
10000

Power laws

- K_N := # clusters occupied by N data points



Power laws

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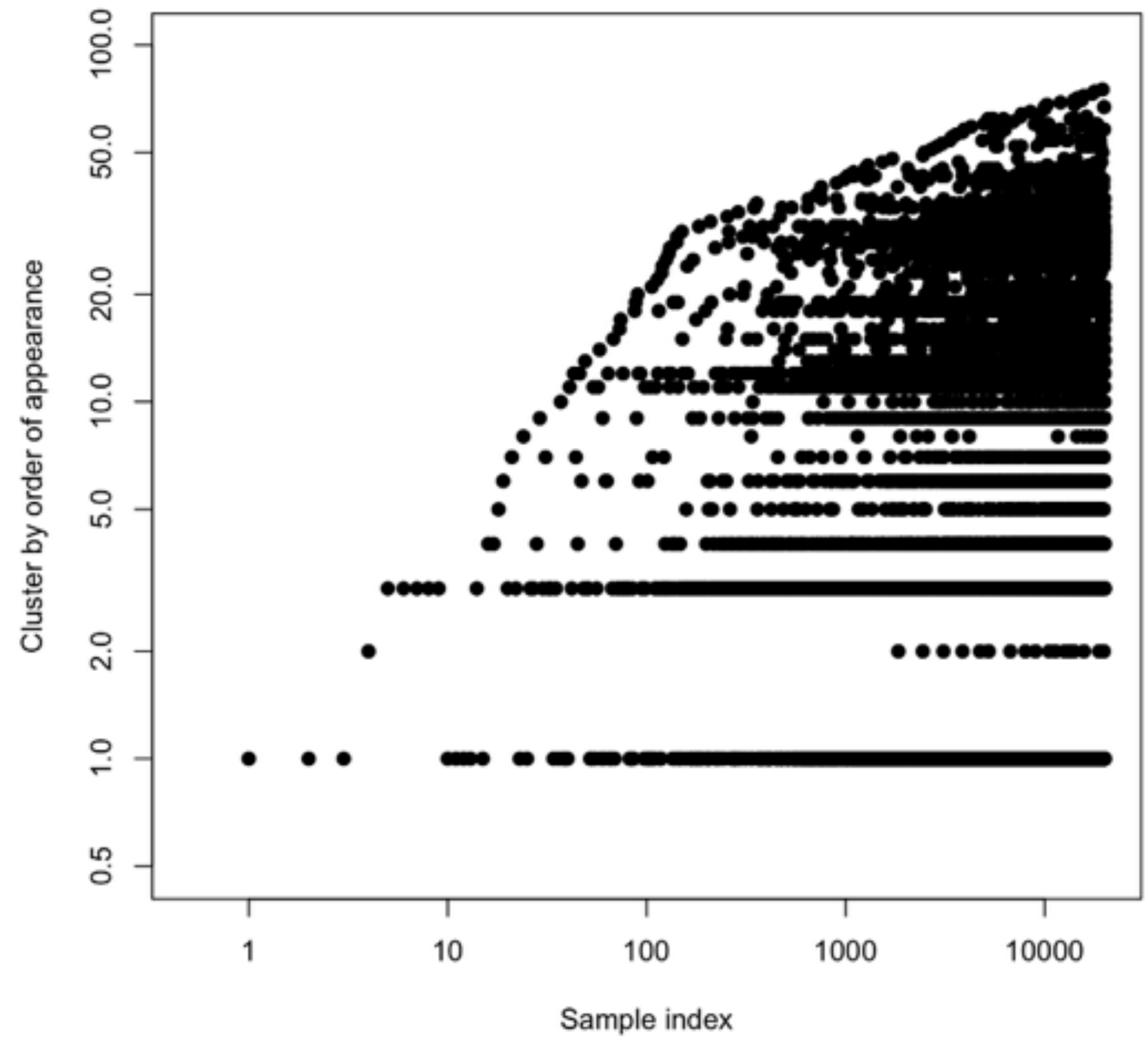

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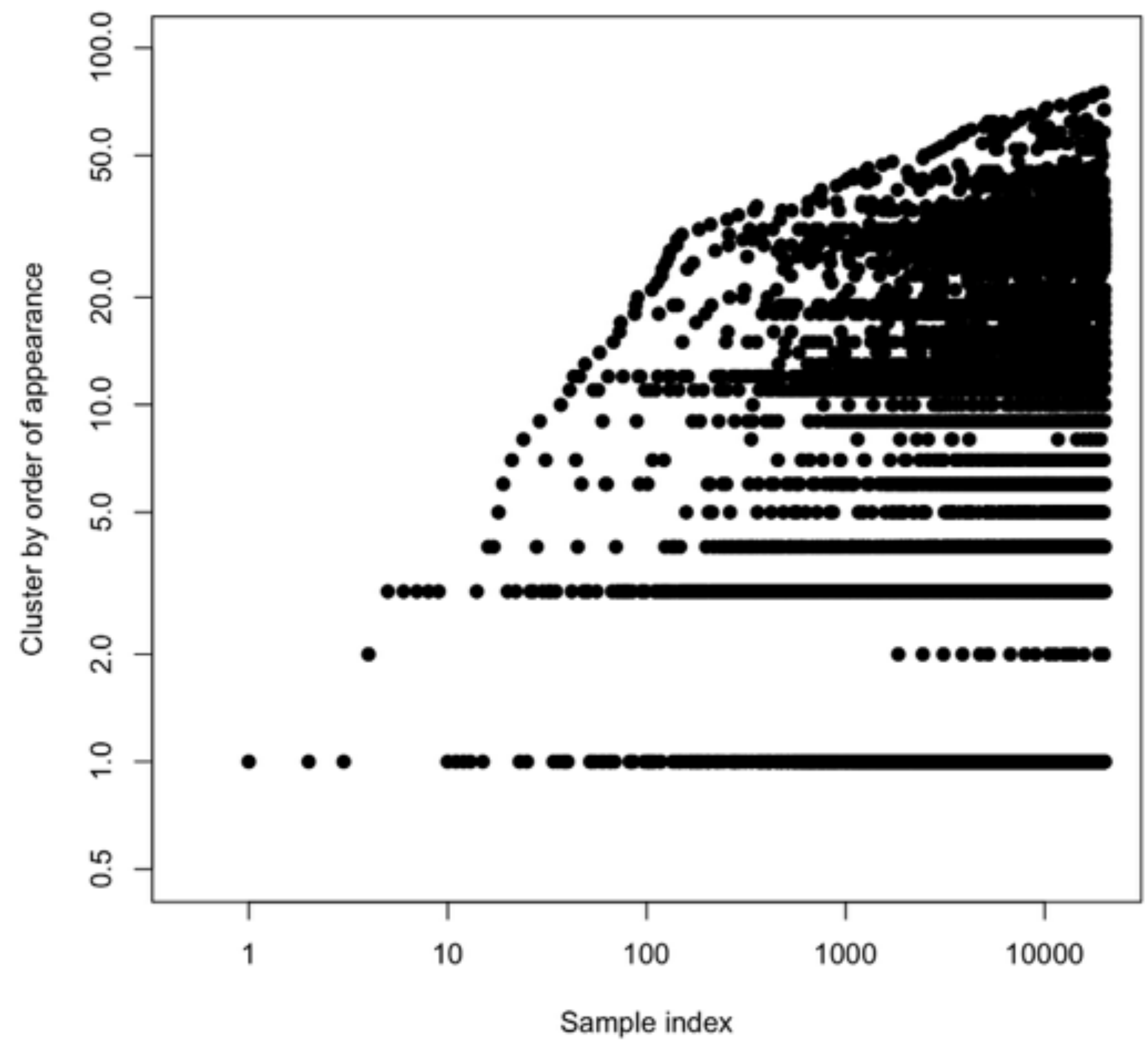
    - vs. Heaps' law, Herdan's law, etc



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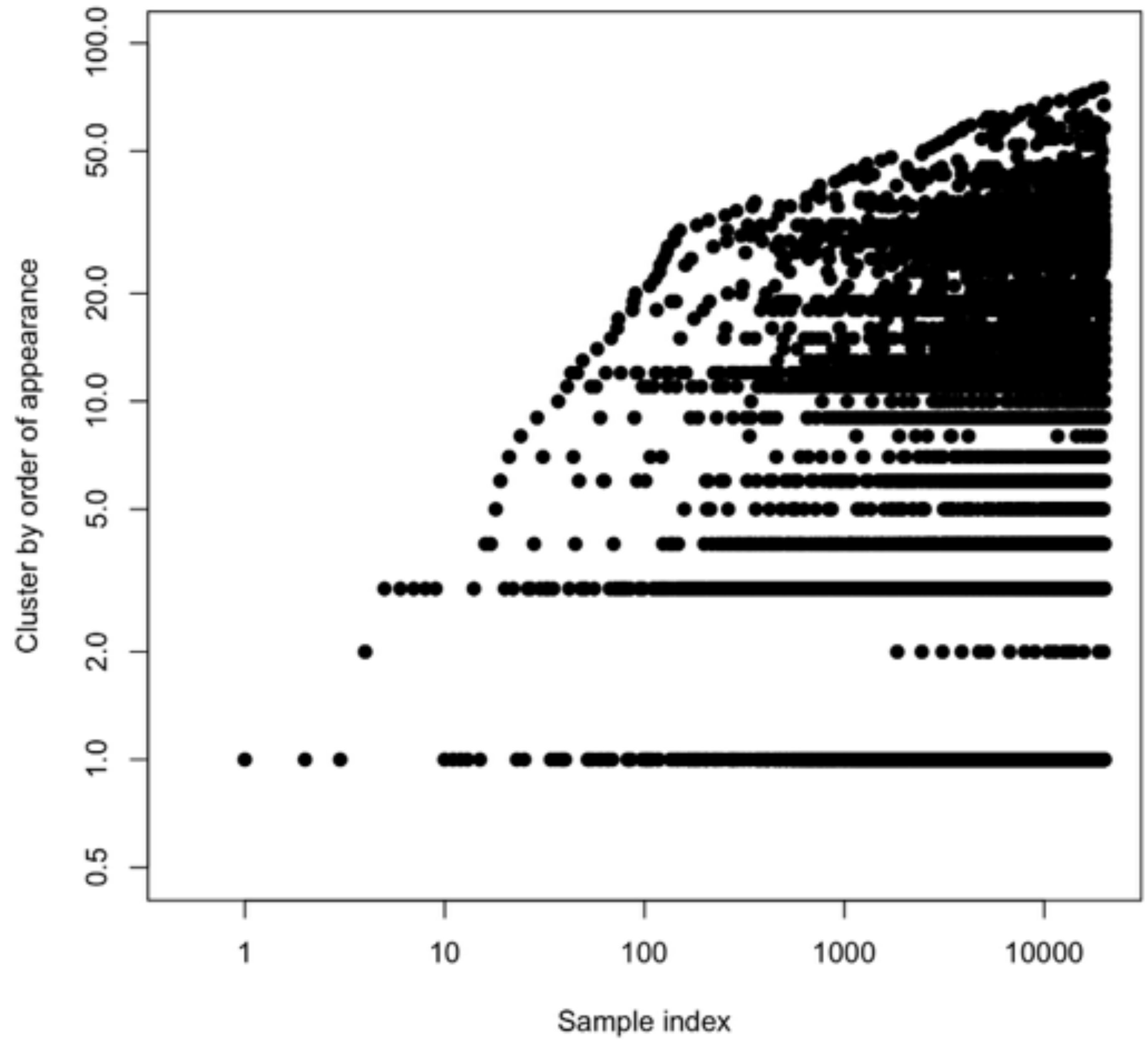

[Gnedin, et al 2007

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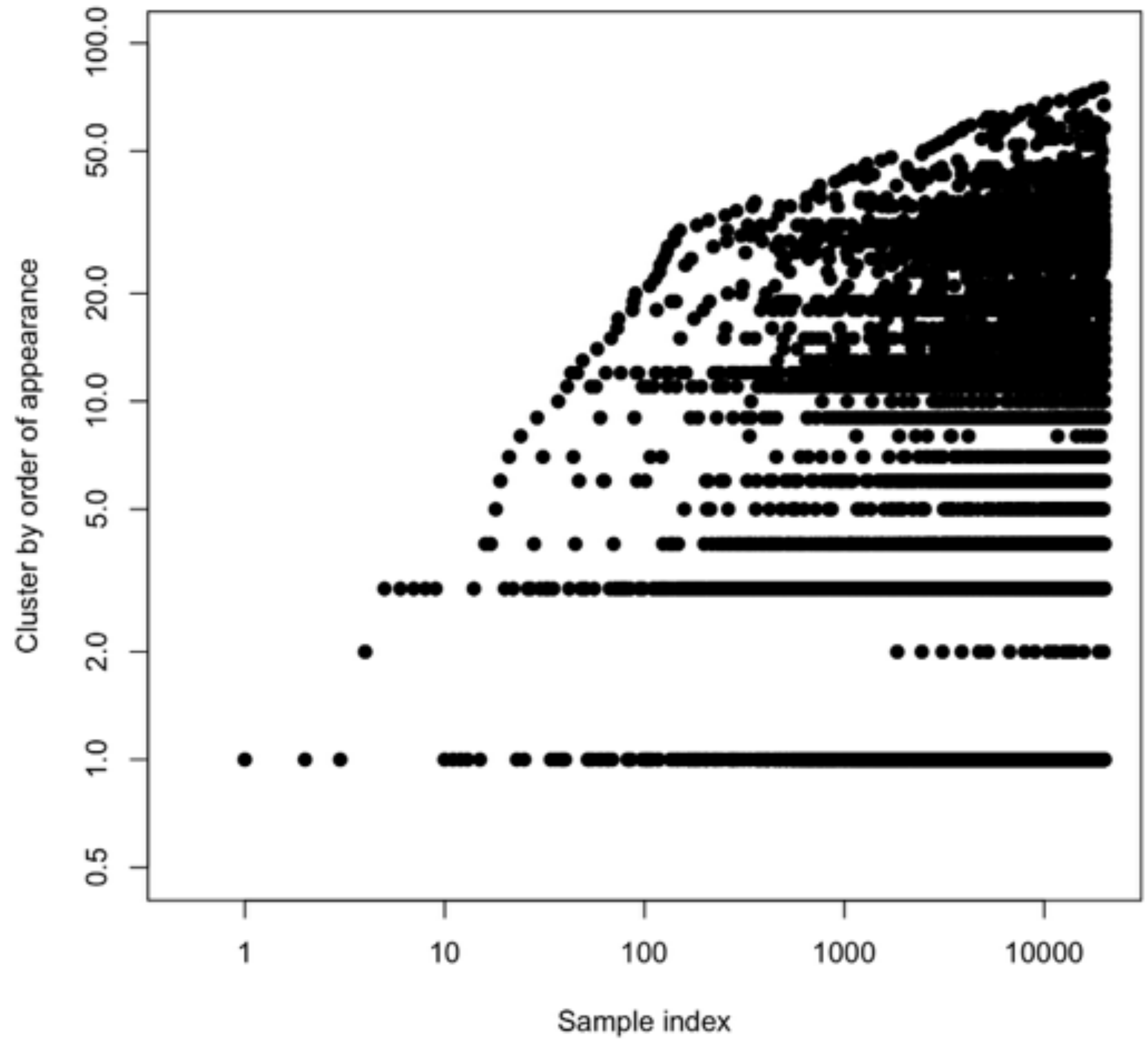
- Pitman-Yor process:



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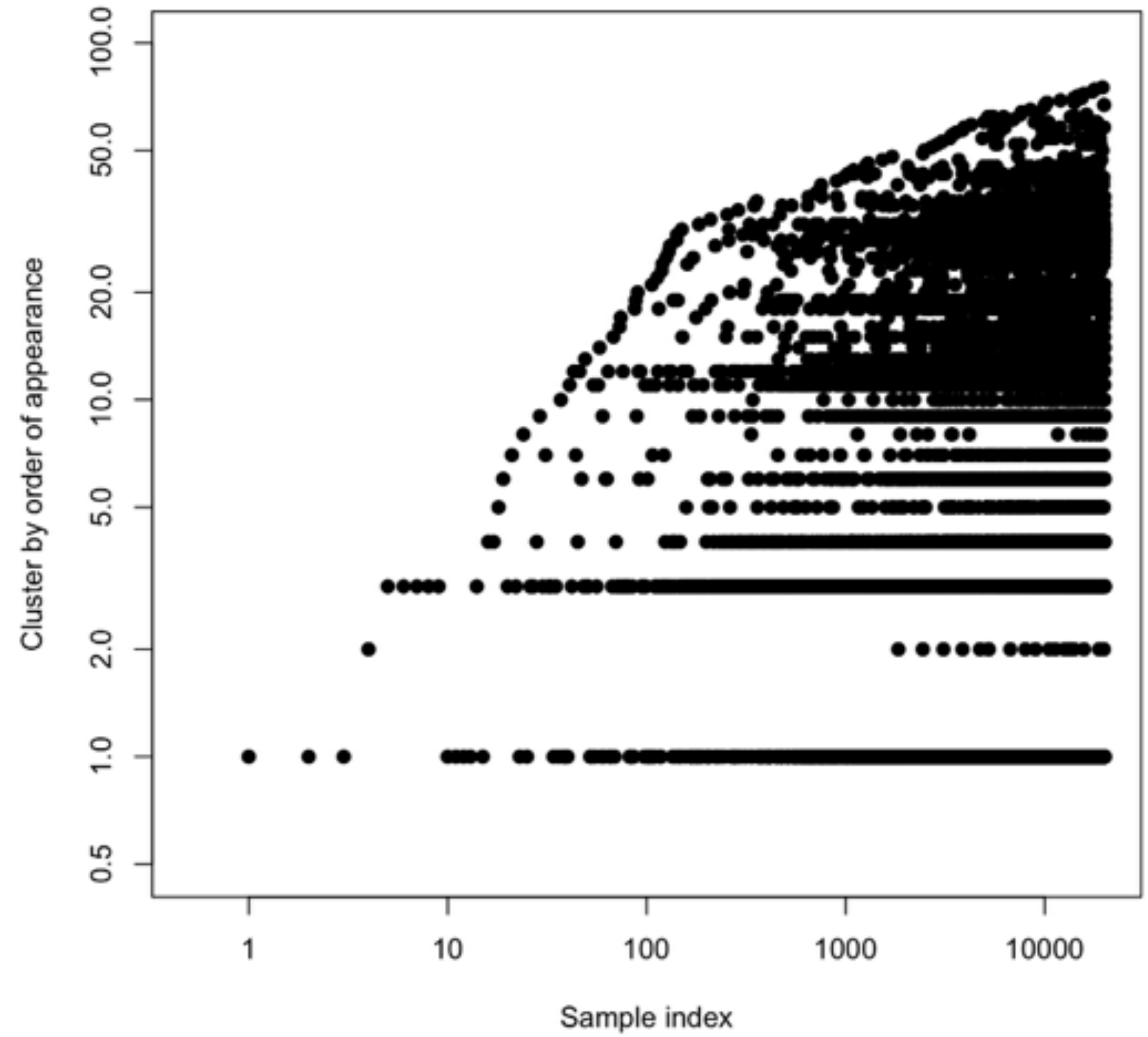

[Gnedin, et al 2007, Pitman, Yor 1997

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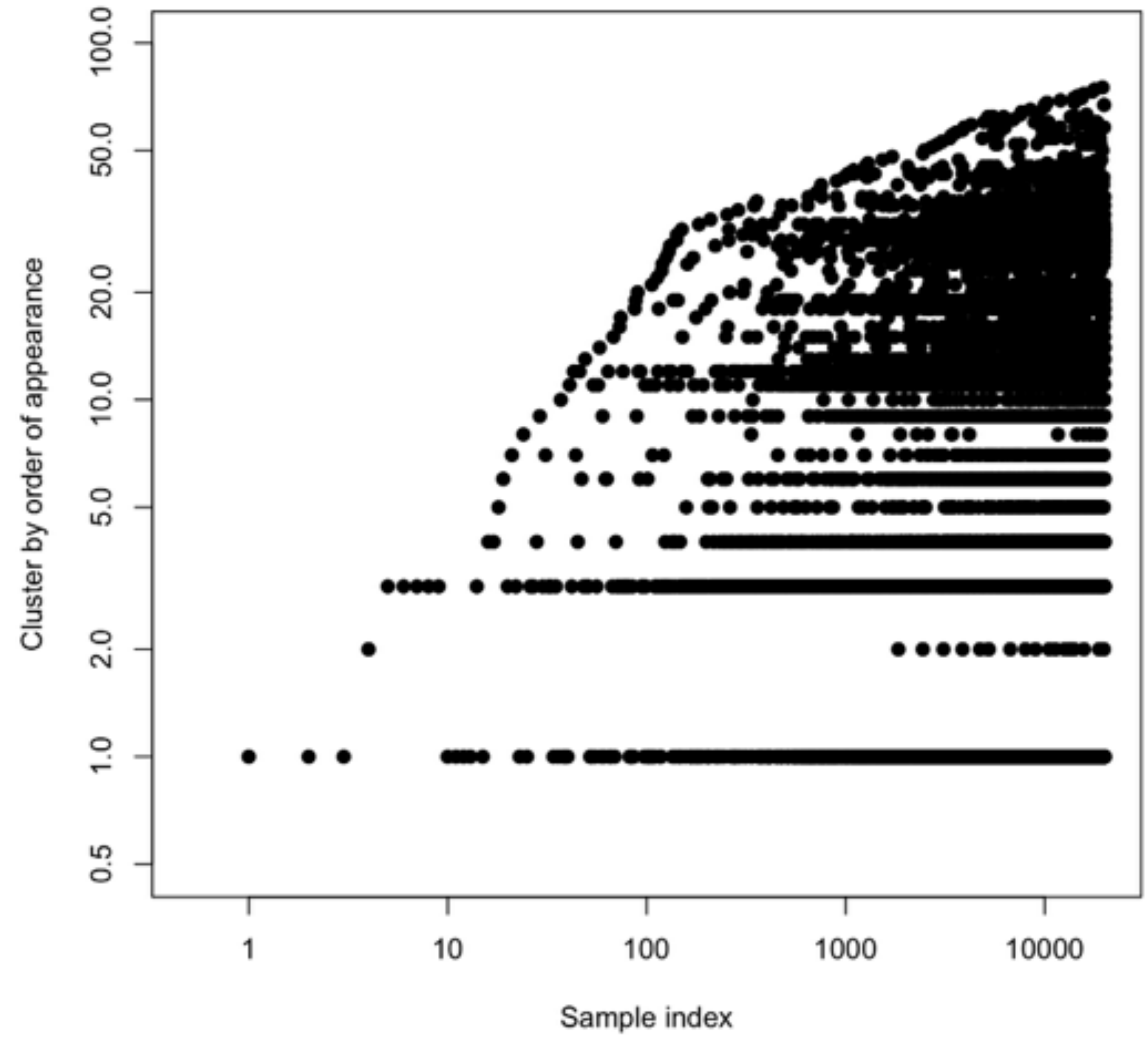

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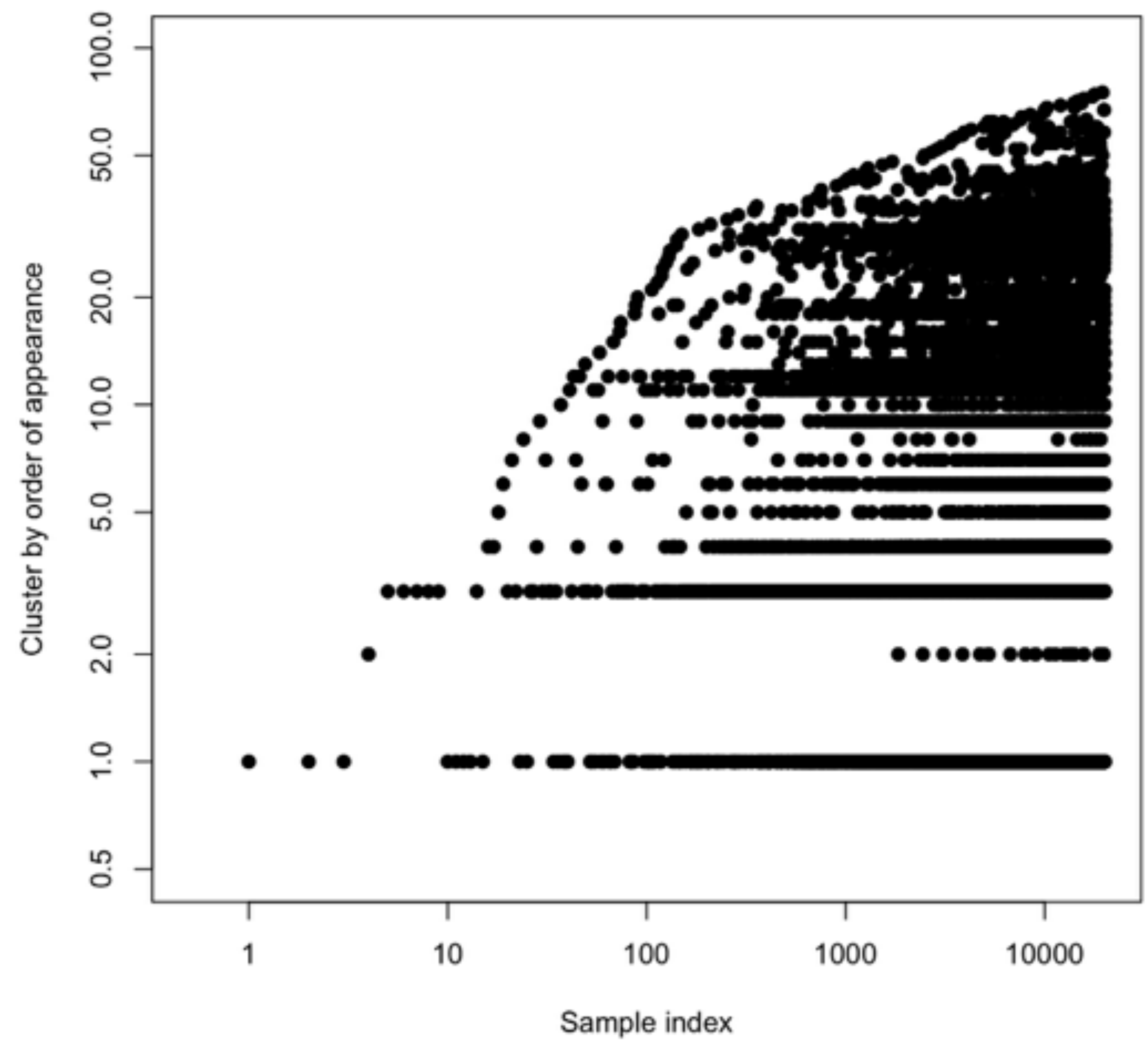

[Gnedin, et al 2007, Pitman, Yor 1997, Goldwater et al 2005, Teh 2006

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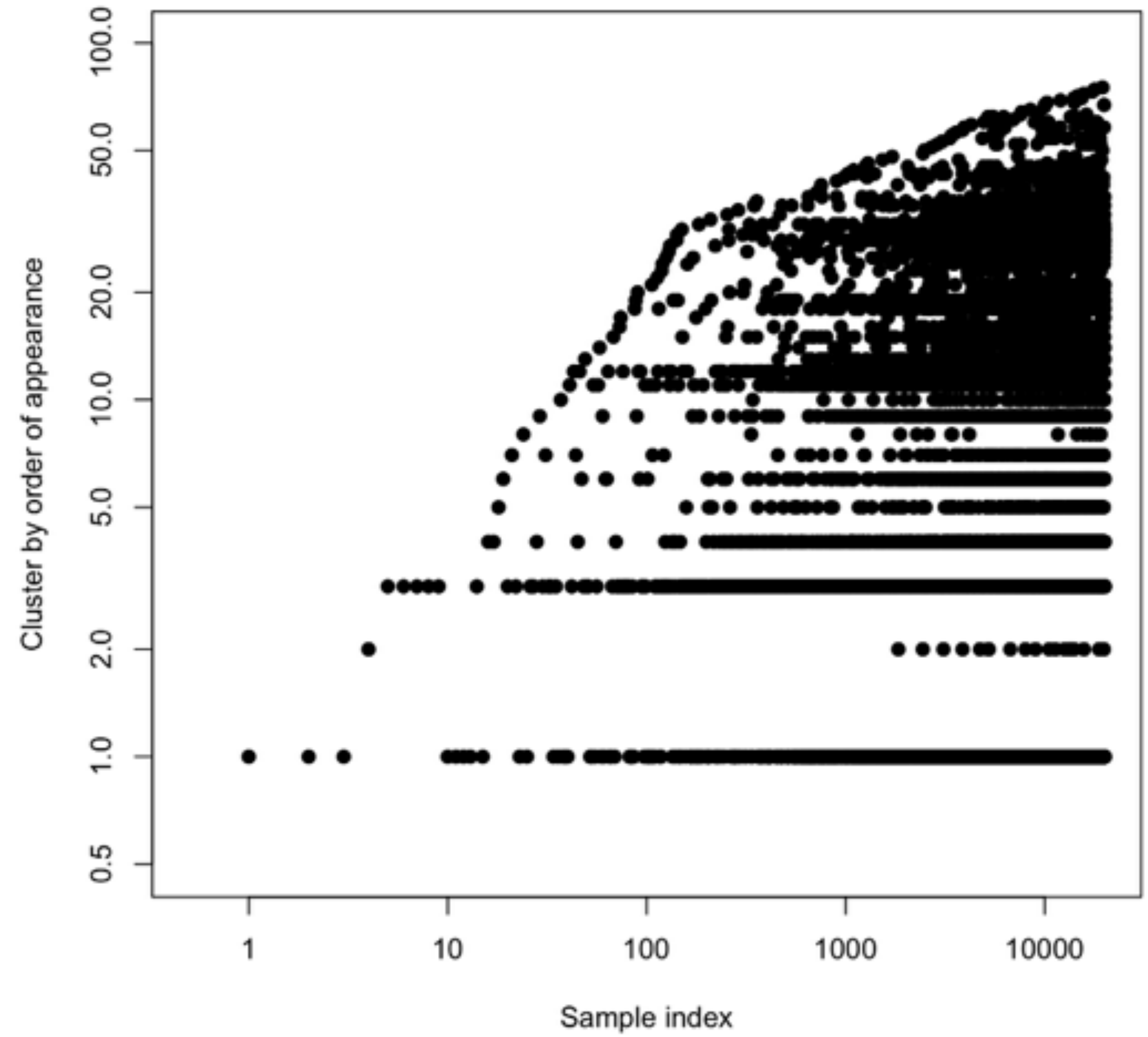
    - related to Zipf's law (ranked frequencies)



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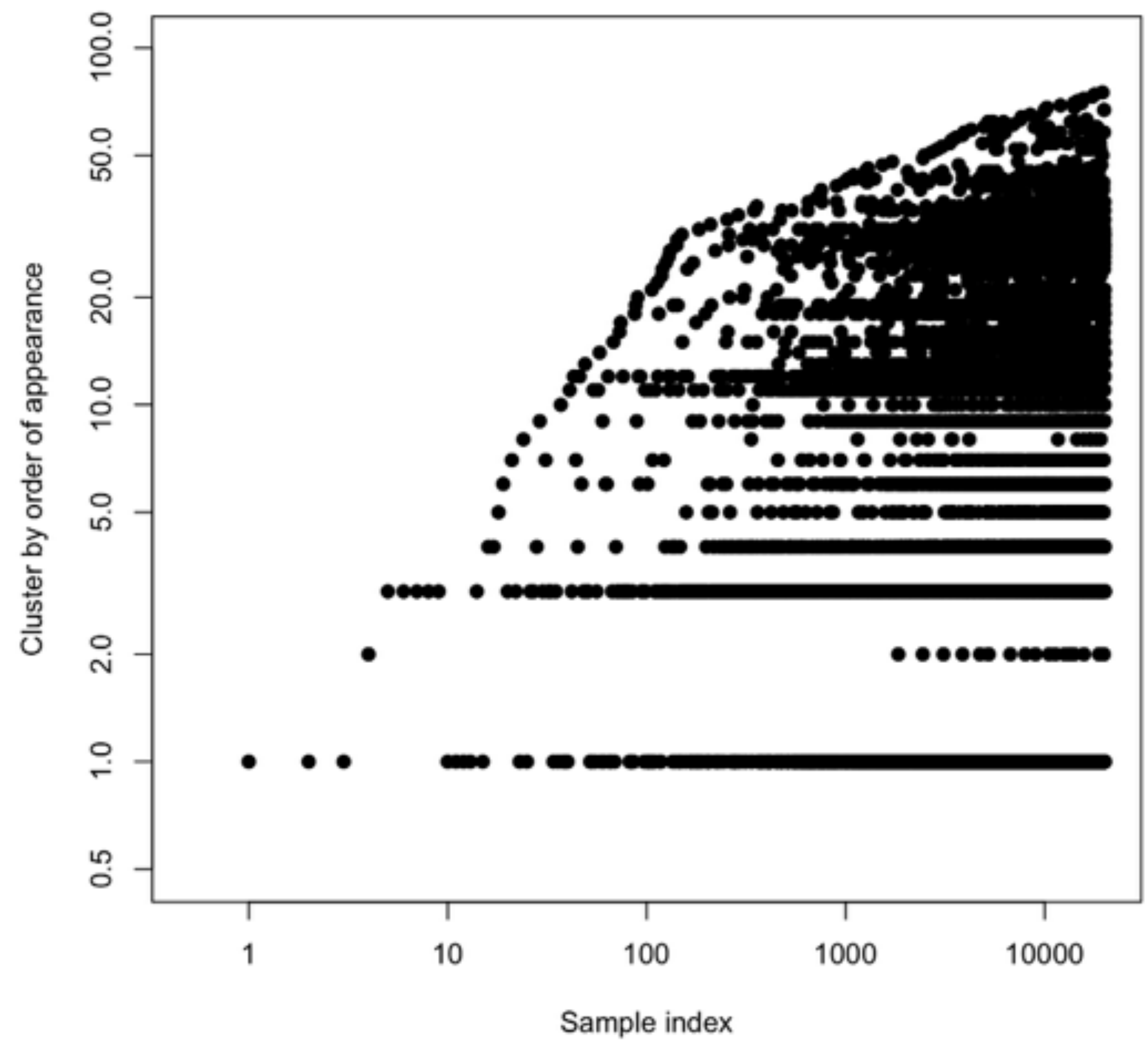
  - related to Zipf's law (ranked frequencies)
- Not just clusters



[Gnedin, et al 2007, Pitman, Yor 1997, Goldwater et al 2005, Teh 2006, Broderick et al 2012]

Hierarchies

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- Hierarchical Dirichlet process

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[Teh et al 2006, Rodríguez et al 2008

Hierarchies

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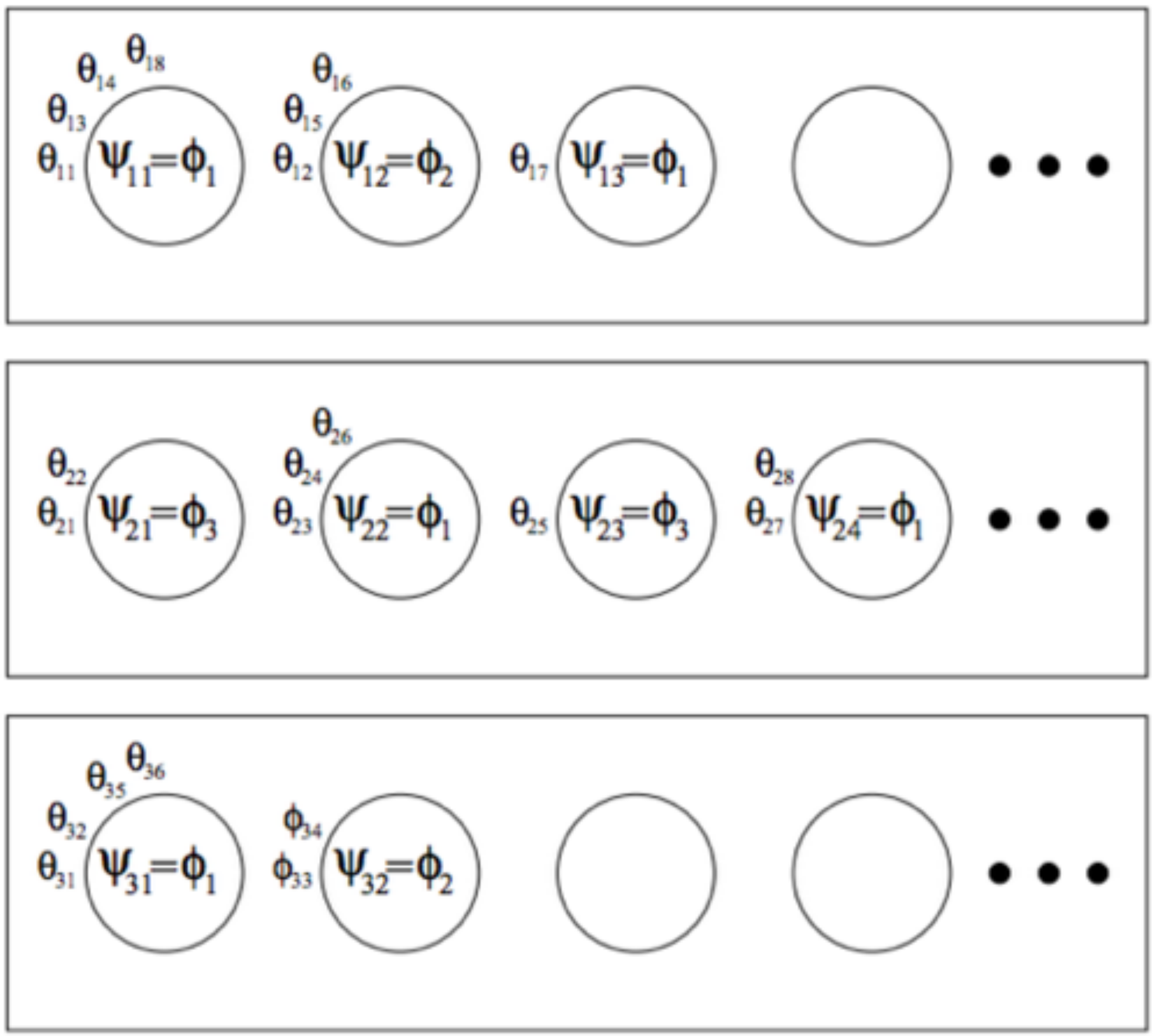
[Teh et al 2006, Rodríguez et al 2008

Hierarchies



[Teh et al 2006]

- Hierarchical Dirichlet process
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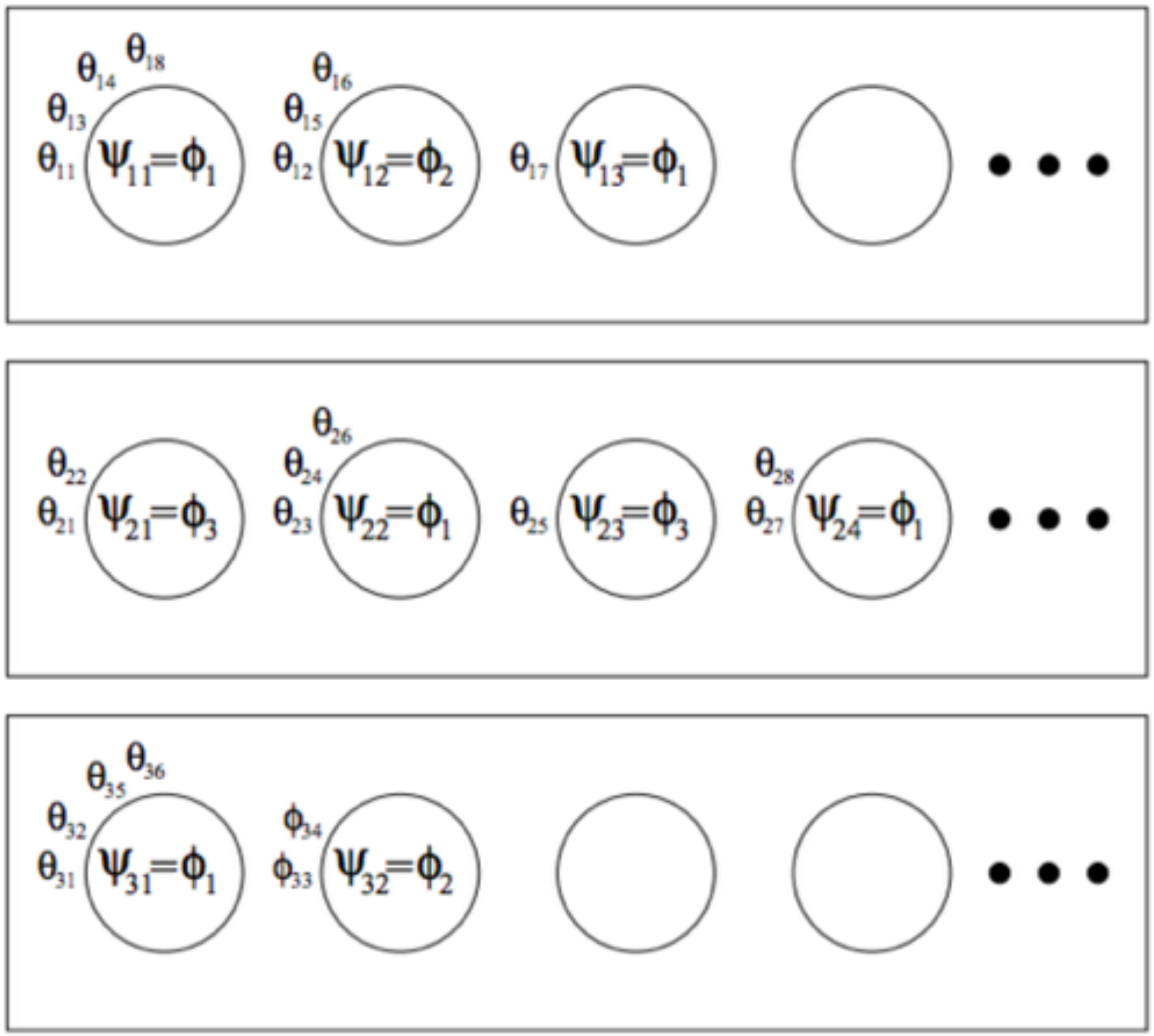
Hierarchies



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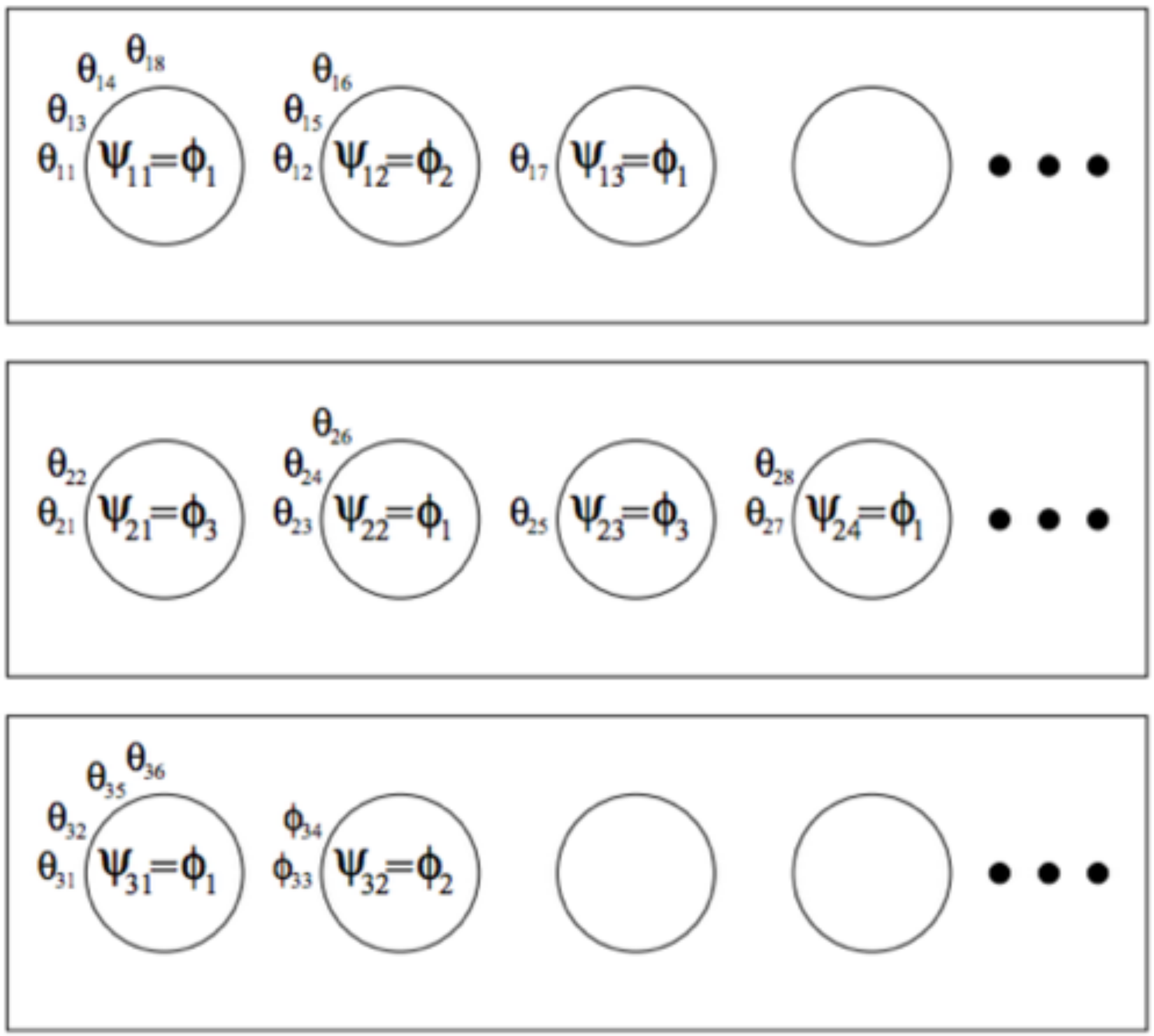

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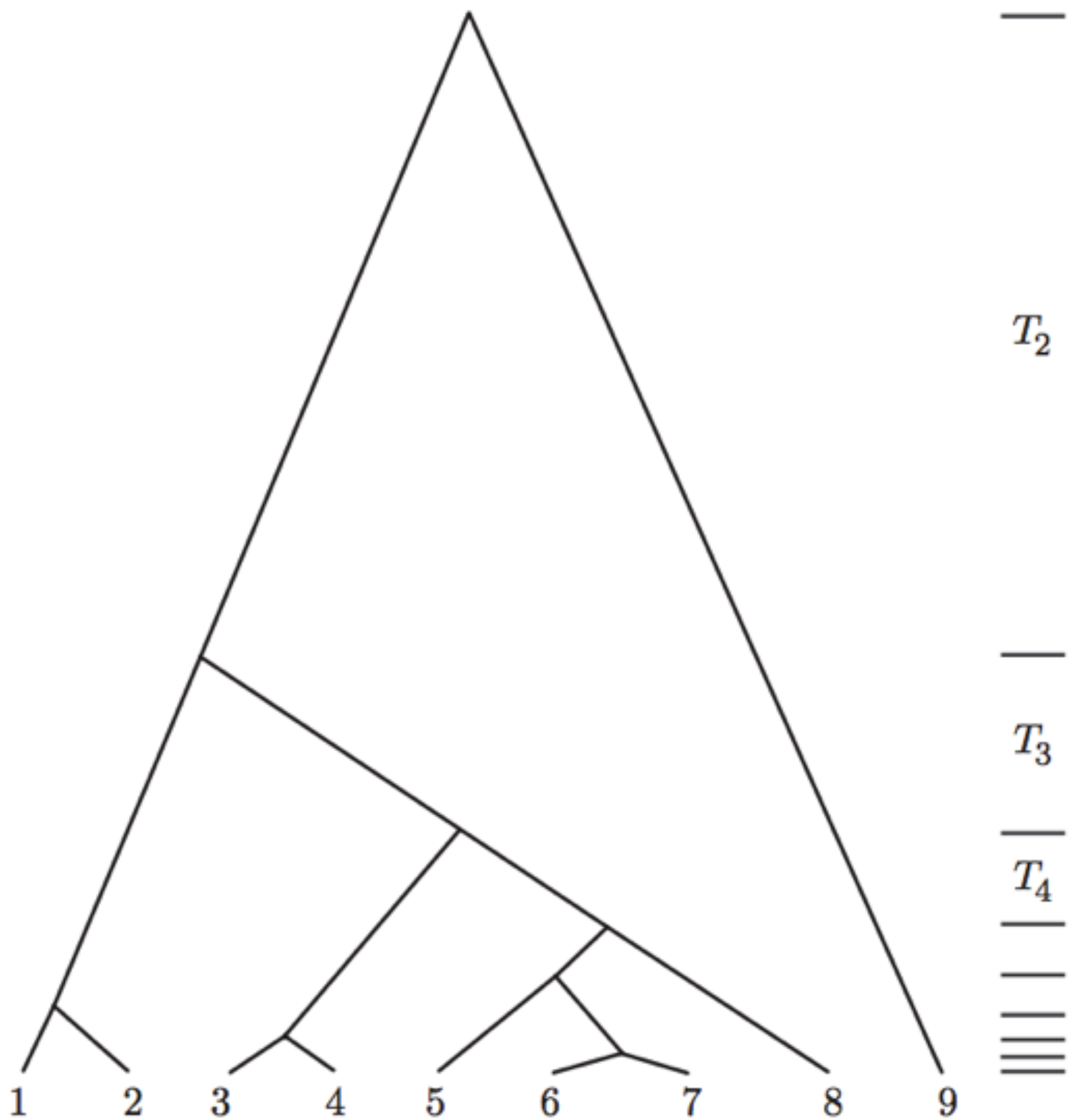
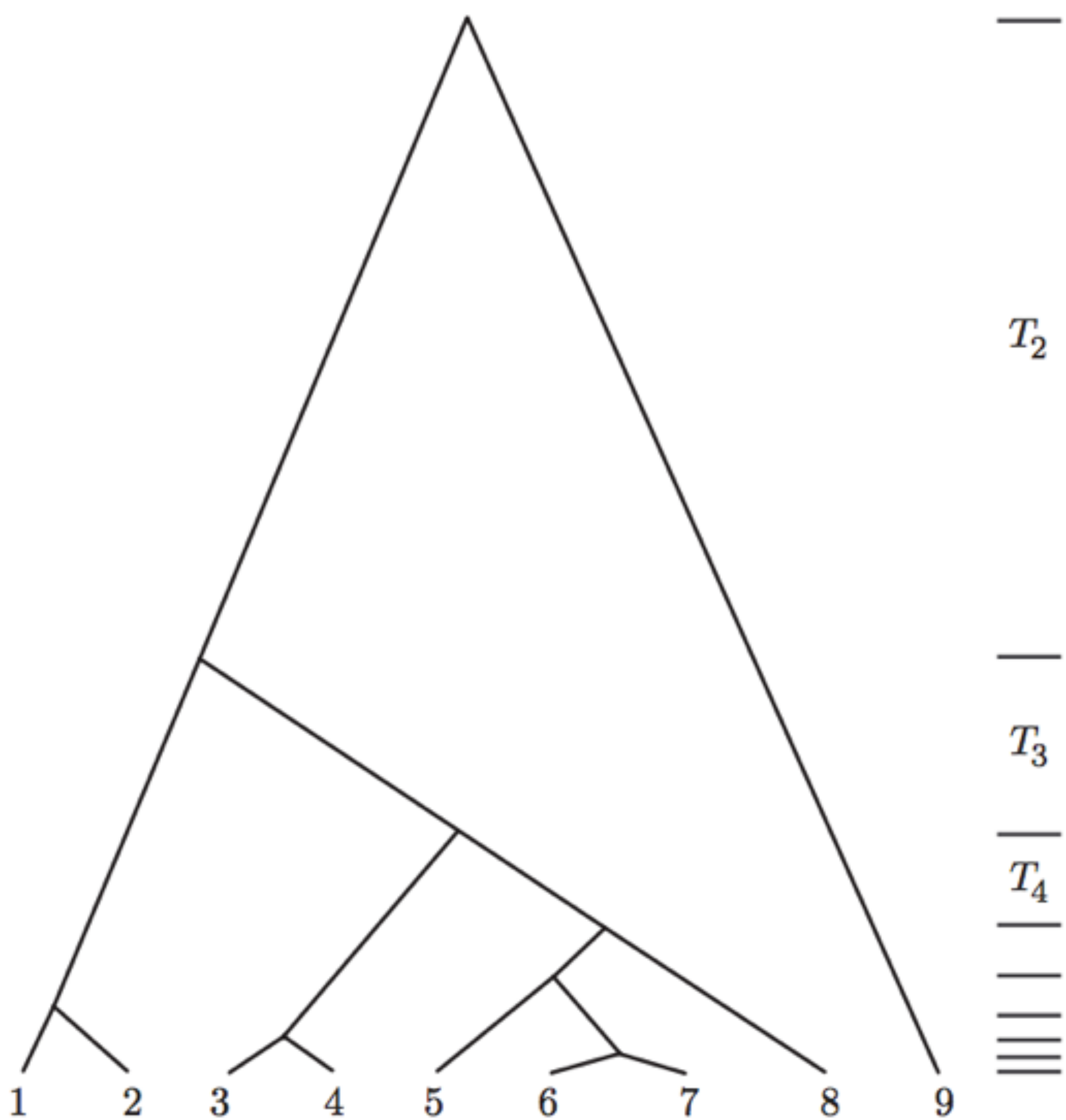
Genealogy, trees, beyond trees



[Wakeley 2008]

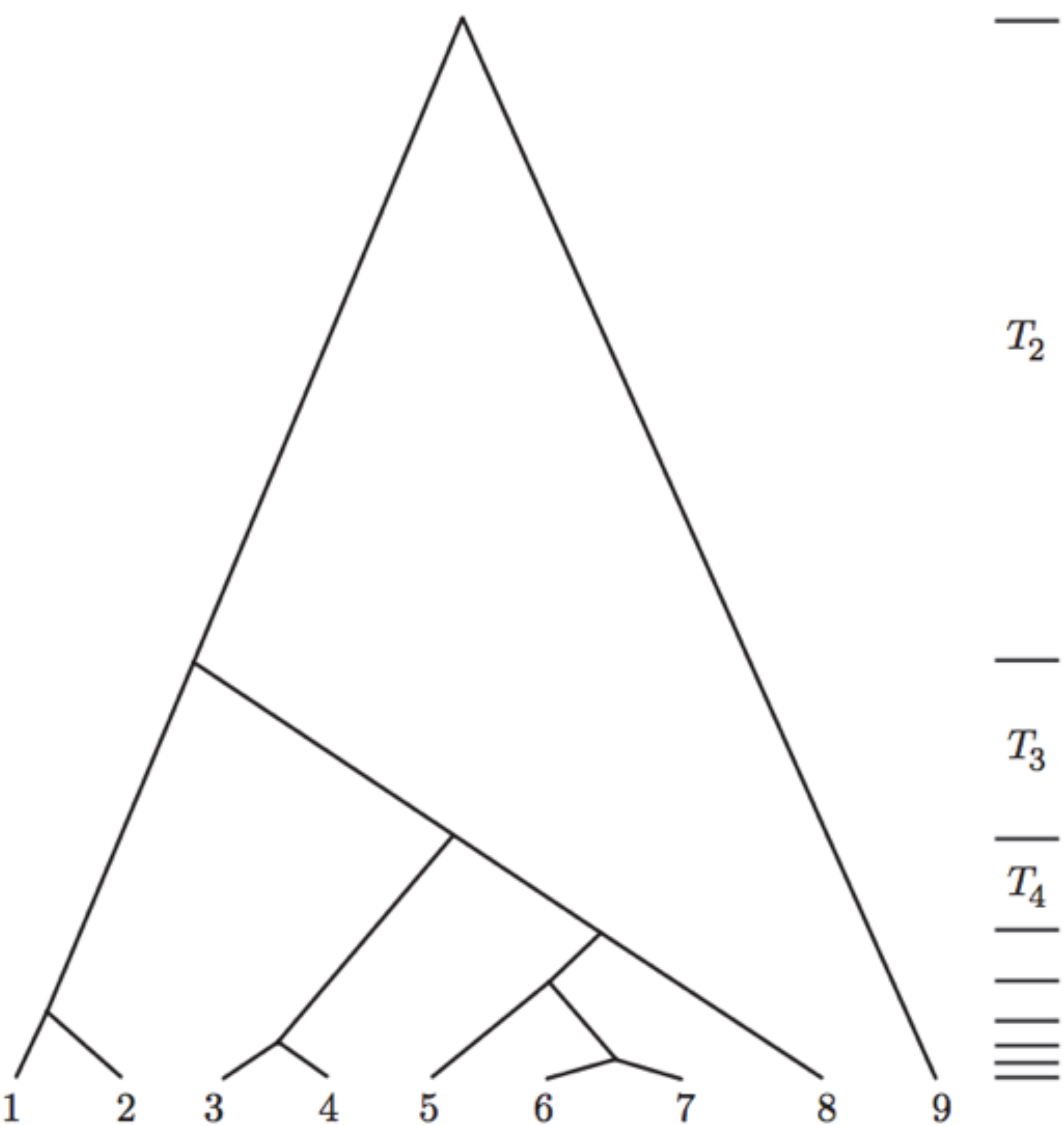
Genealogy, trees, beyond trees



[Wakeley 2008]

- Kingman coalescent

Genealogy, trees, beyond trees

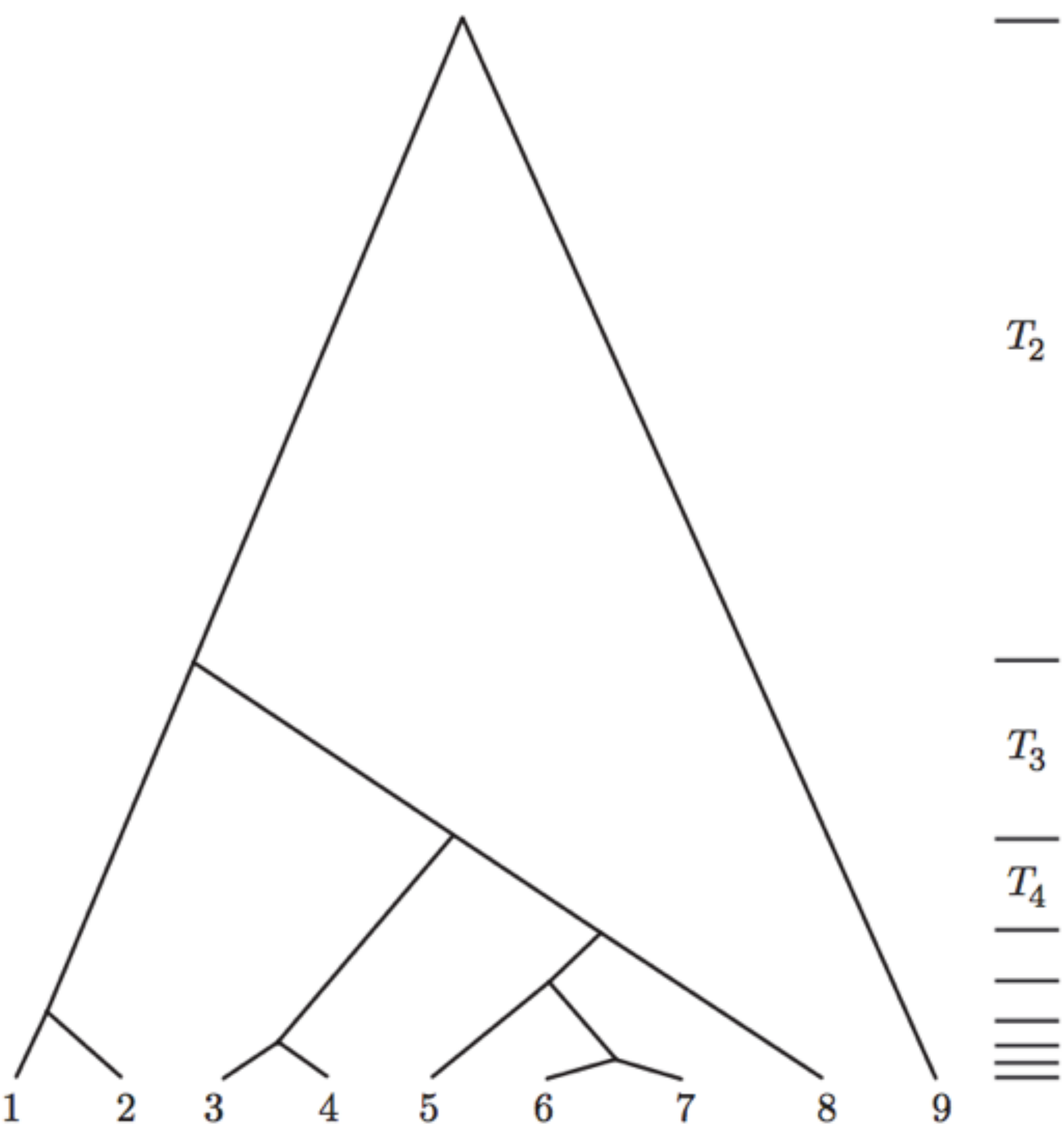


[Wakeley 2008]

- Kingman coalescent

[Kingman 1982

Genealogy, trees, beyond trees



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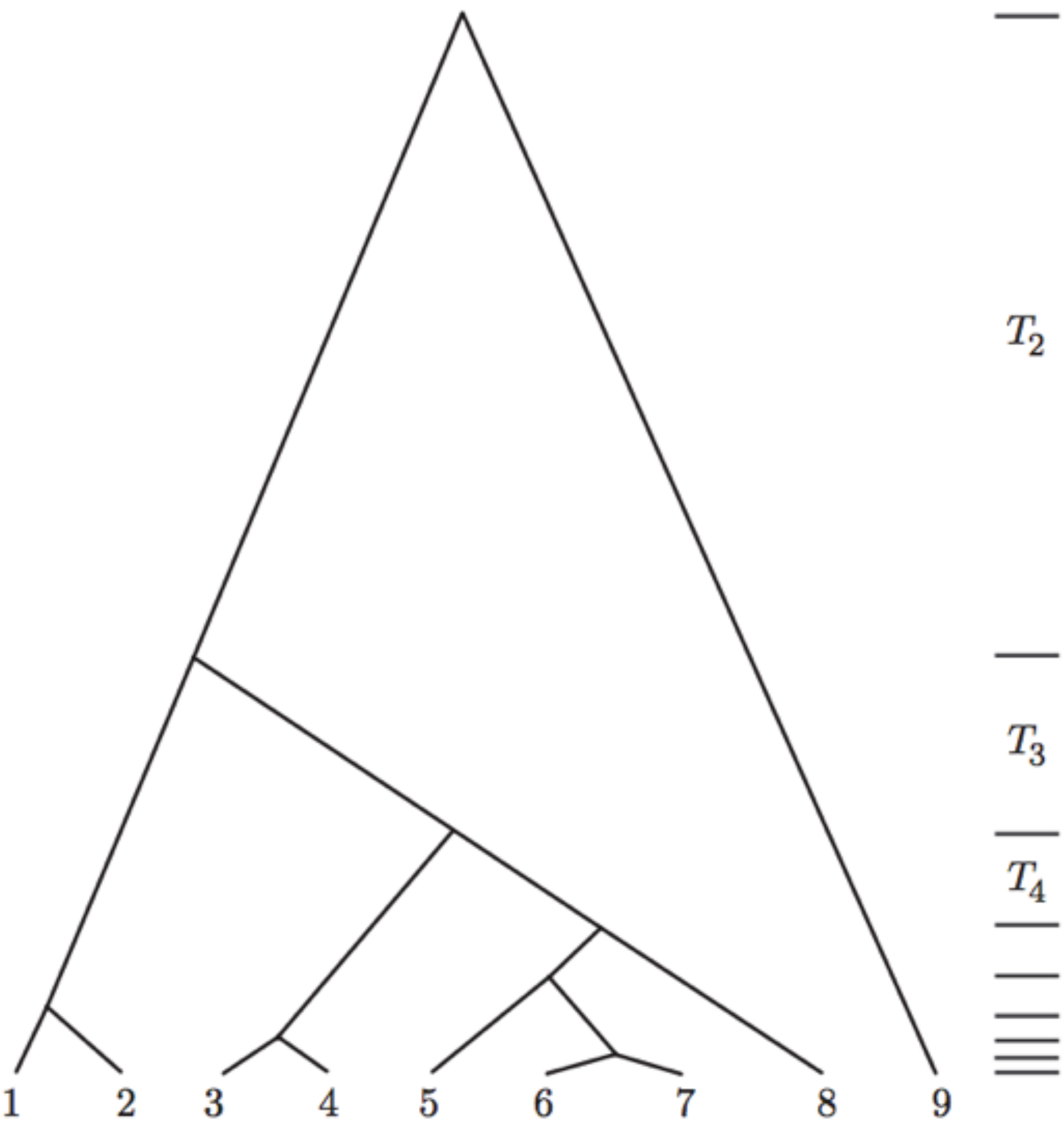

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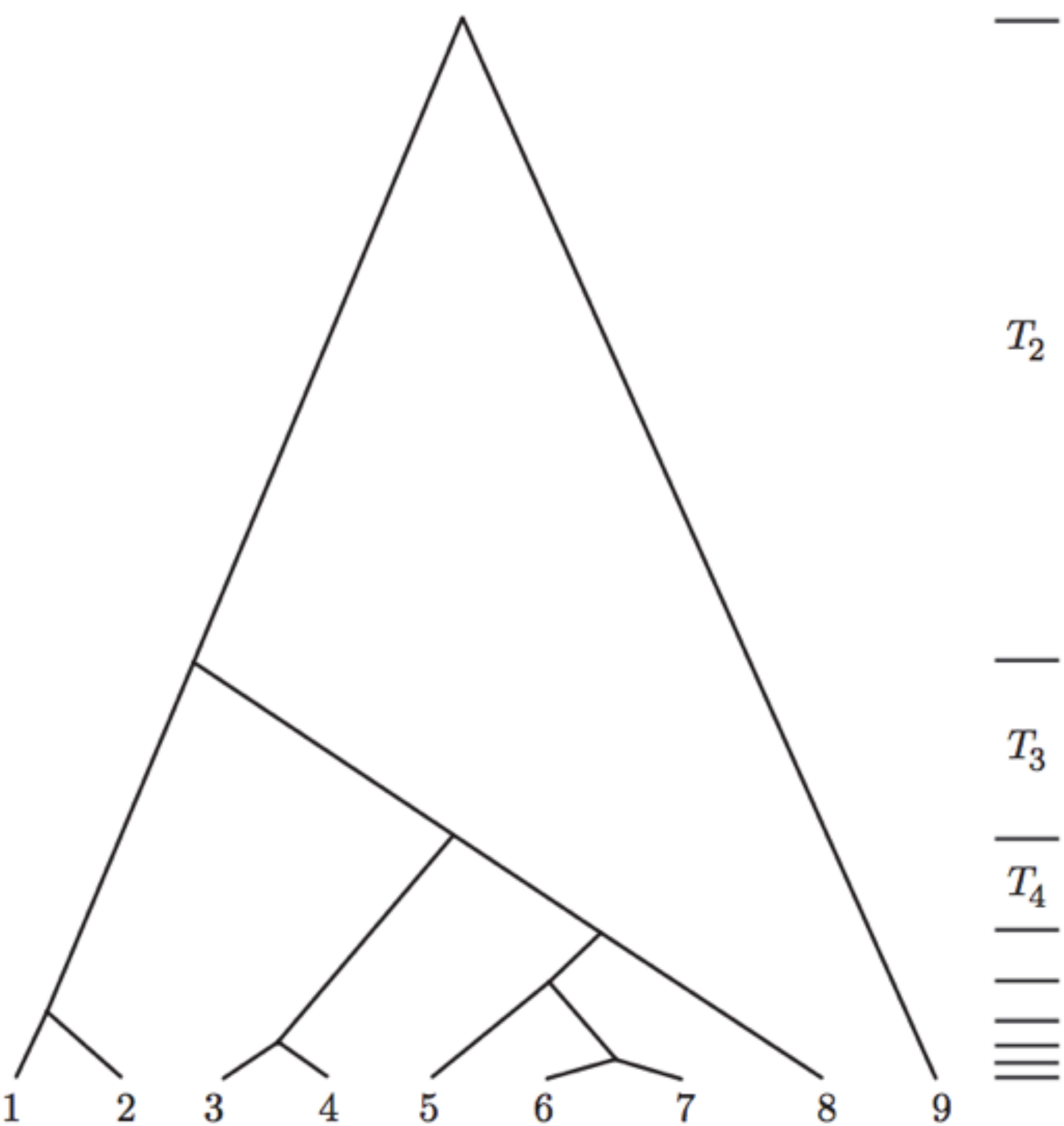
Genealogy, trees, beyond trees



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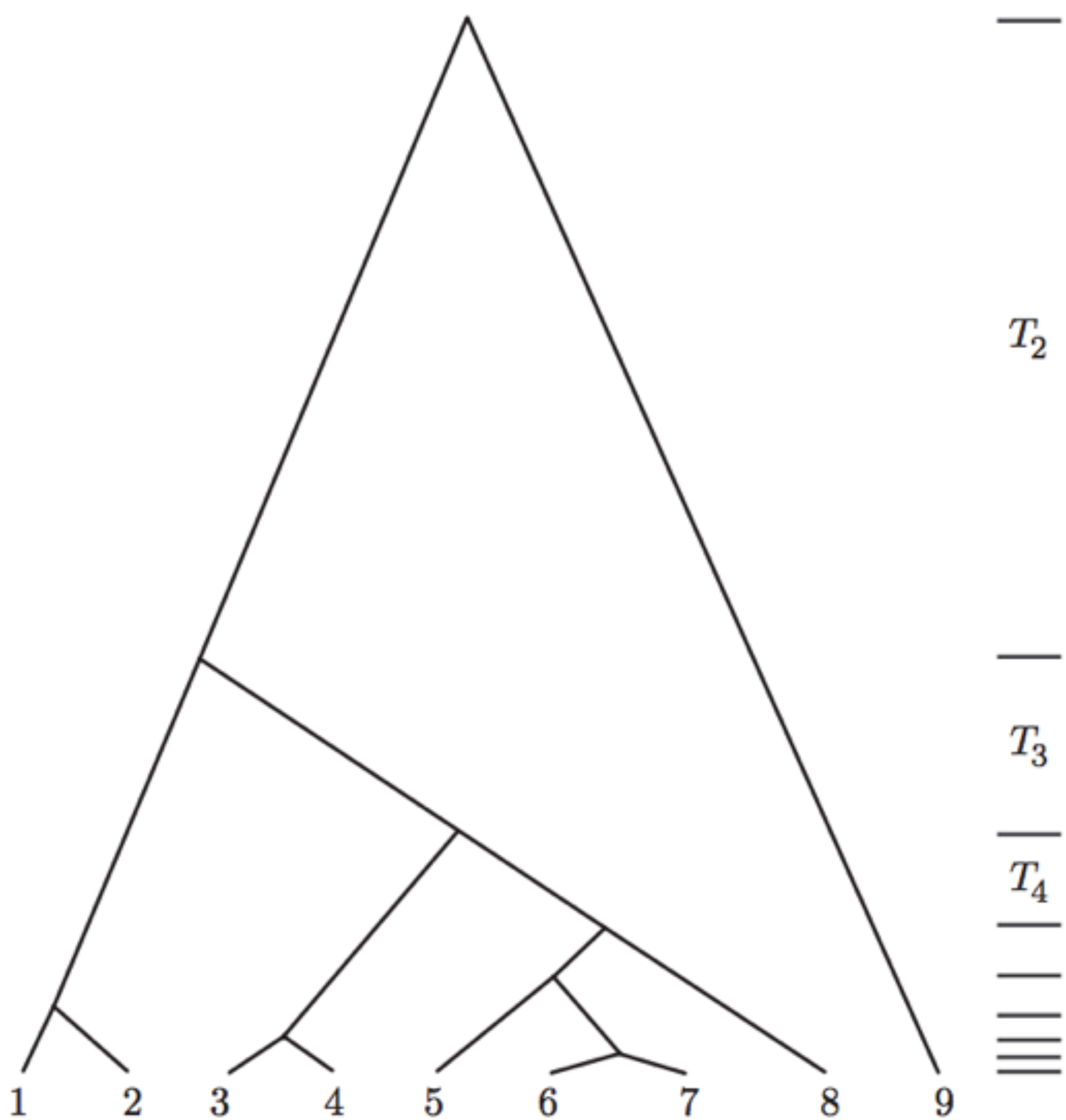

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Conjugacy & Poisson point processes

Orbanz 2009, Orbanz 2010

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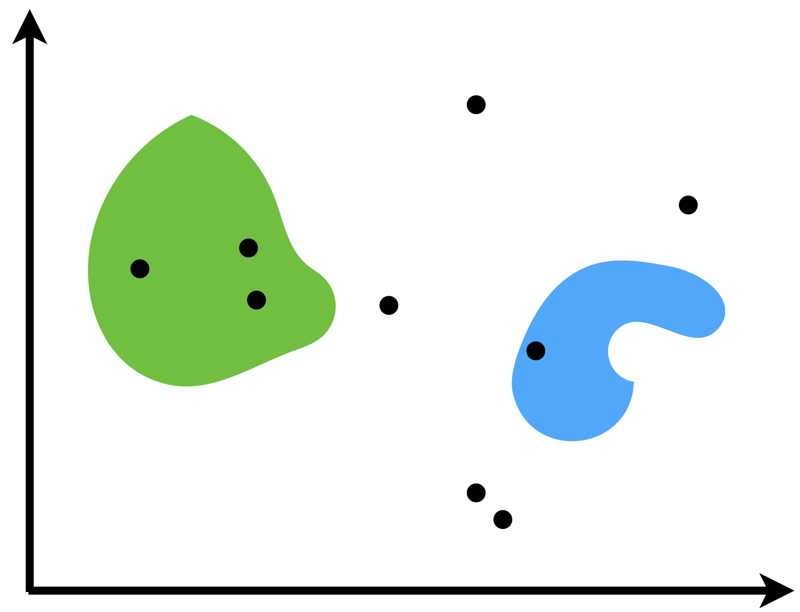

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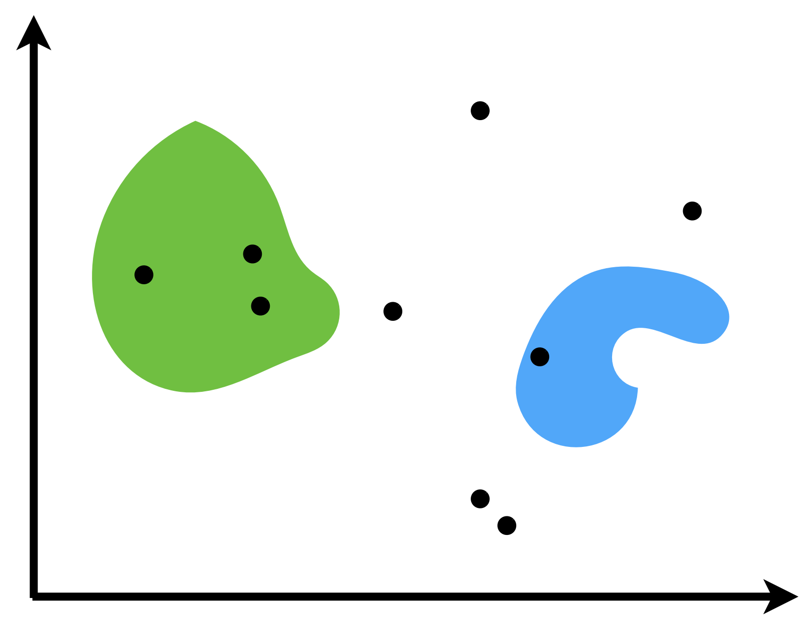

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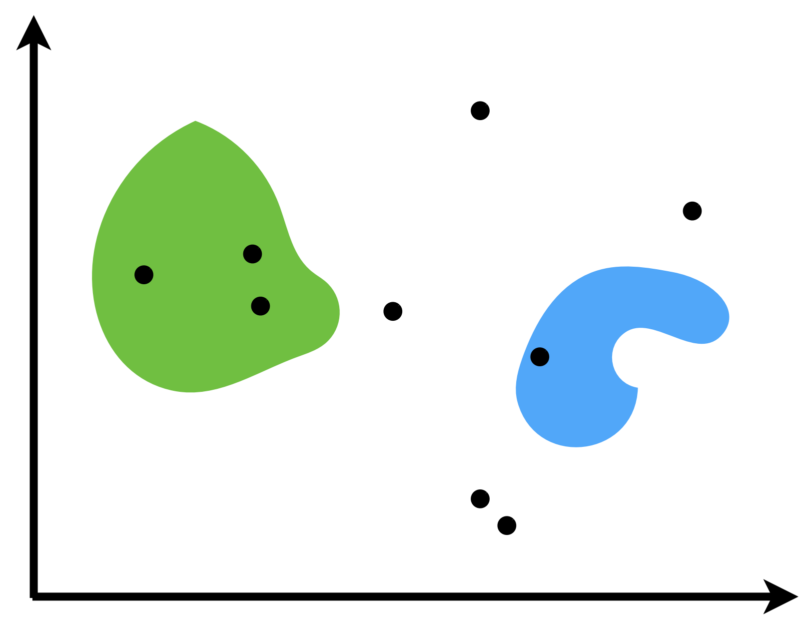

- Posteriors, conjugacy, and exponential families for completely random measures

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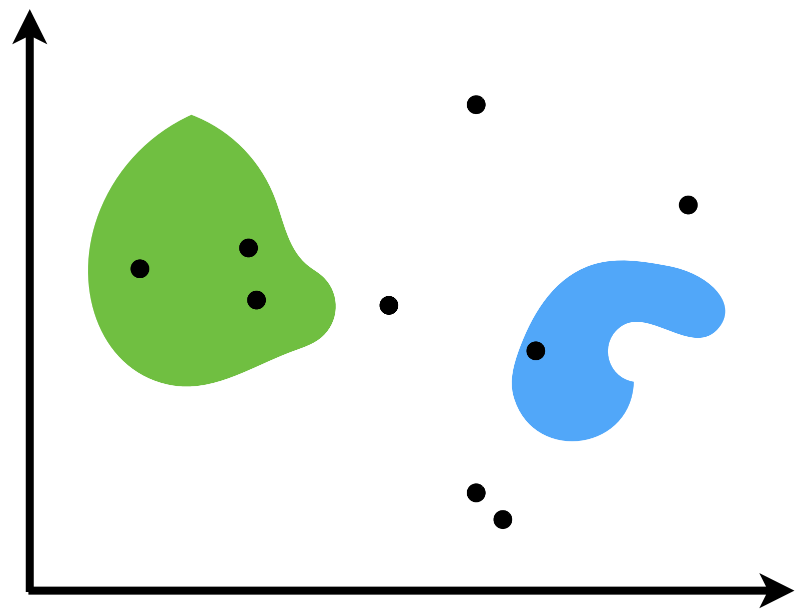

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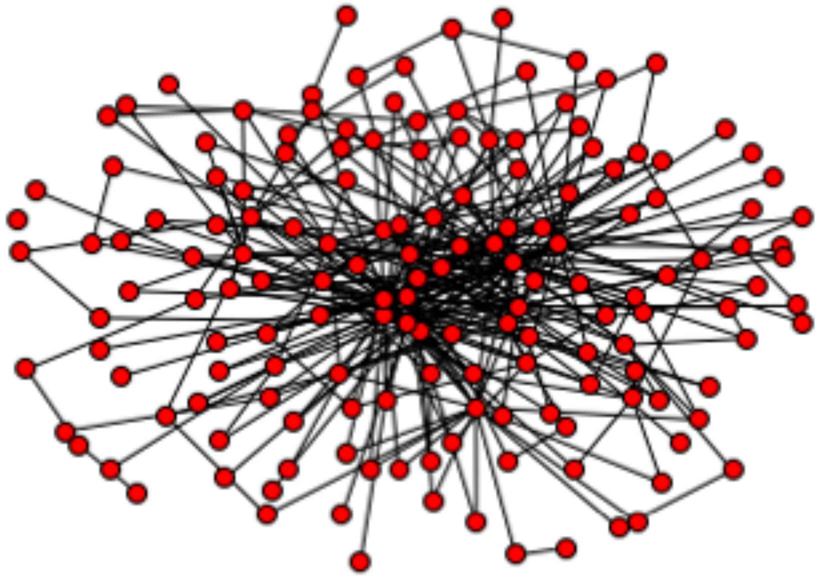
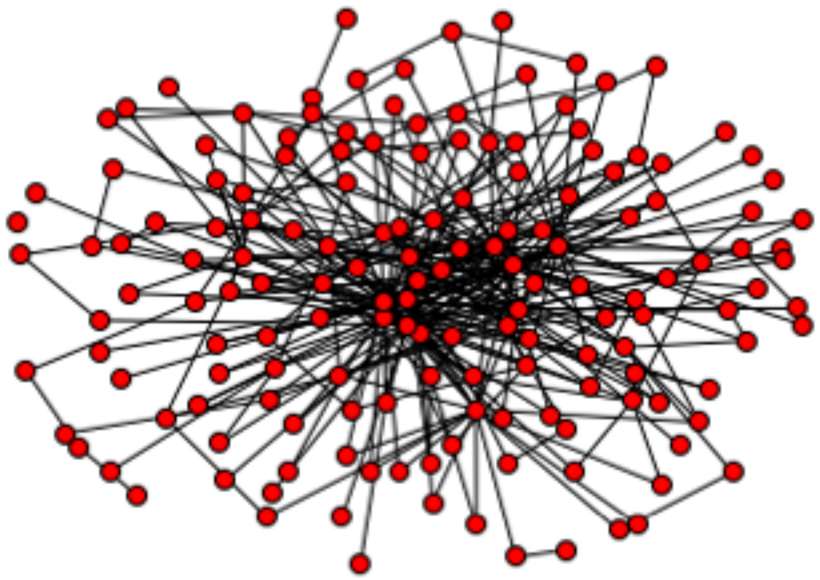
[Kingman 1992, Orbanz 2009, Orbanz 2010, Broderick et al 2014, James 2014]



[Holland et al 1983; Kemp et al 2006; Xu et al 2007; Airoldi et al 2008; Lloyd et al 2012]



social: *Facebook, Twitter, email*
biological: *ecological, protein, gene*
transportation: *roads, railways*

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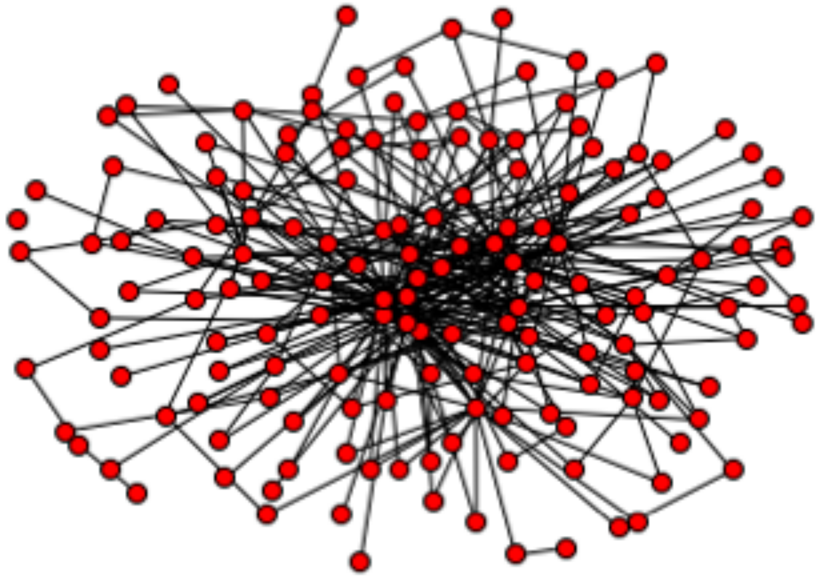
Probabilistic models for graphs

$p($  $)$

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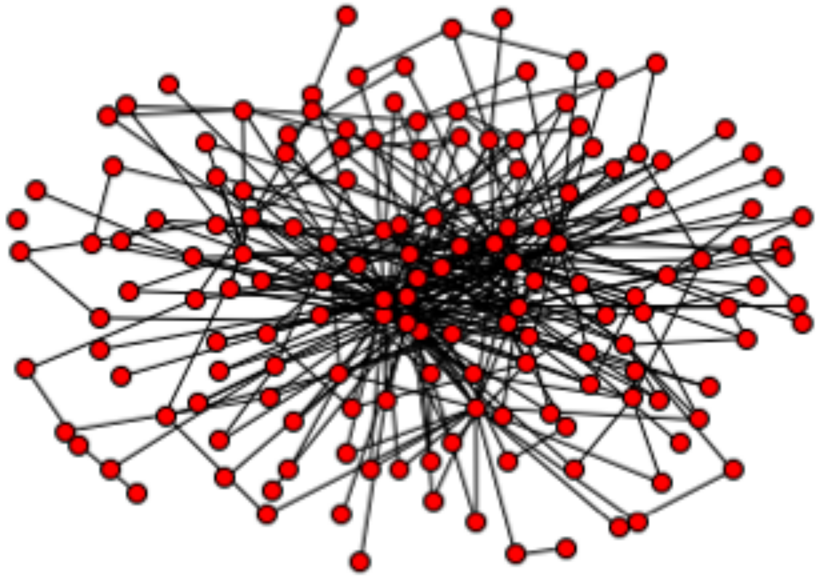
$p\Big($  $\Big)$

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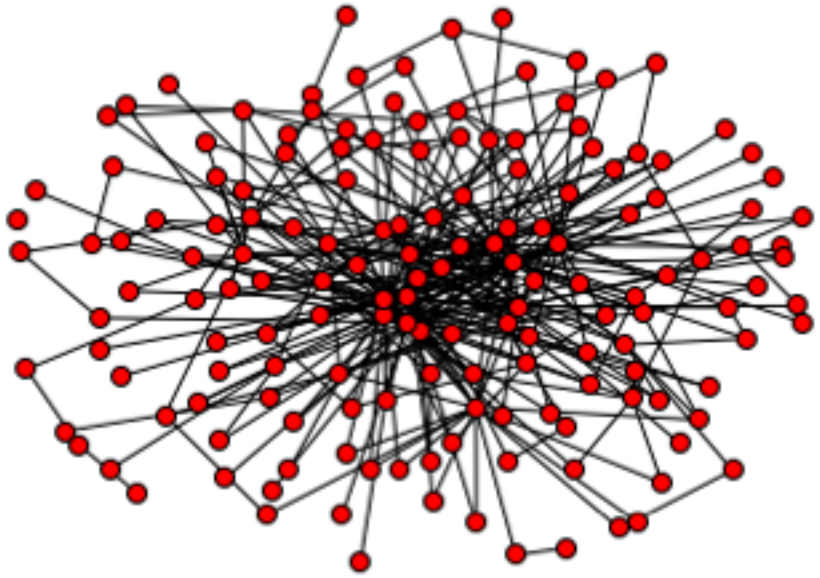
$p($  $)$

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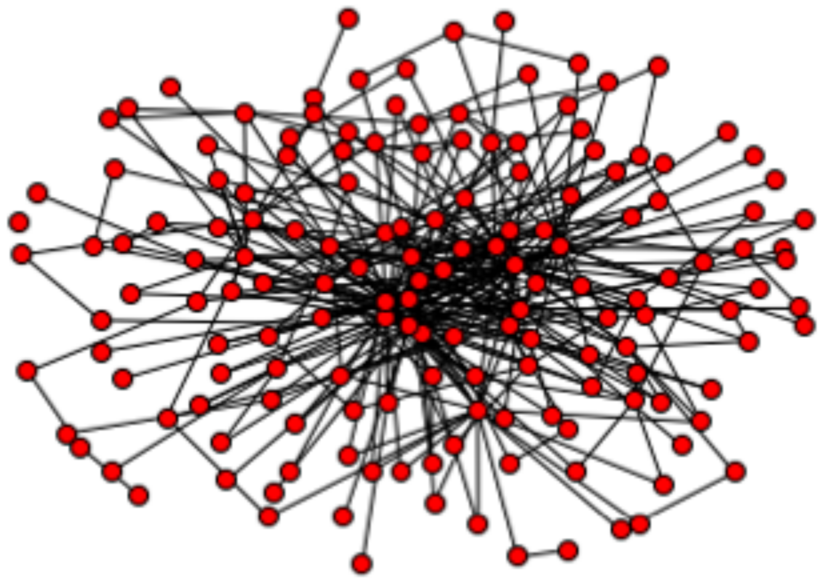
$p($  $)$

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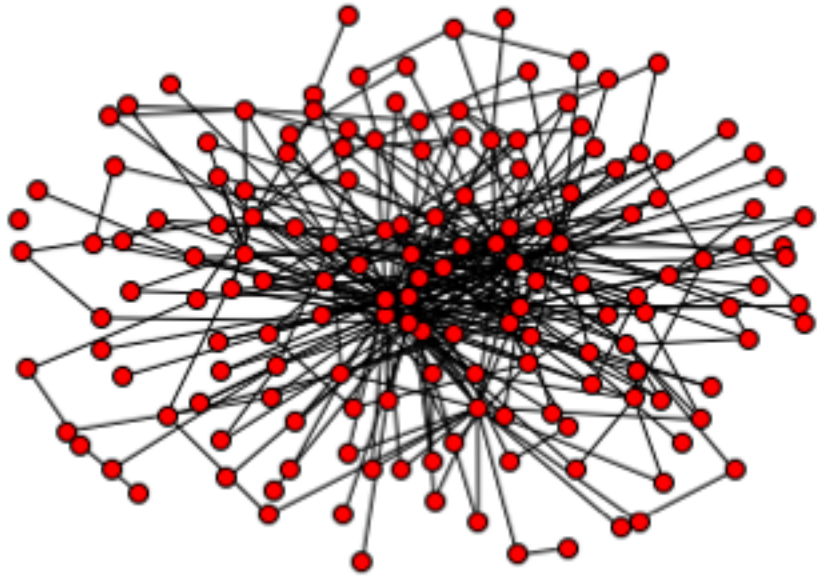
$p($  $)$

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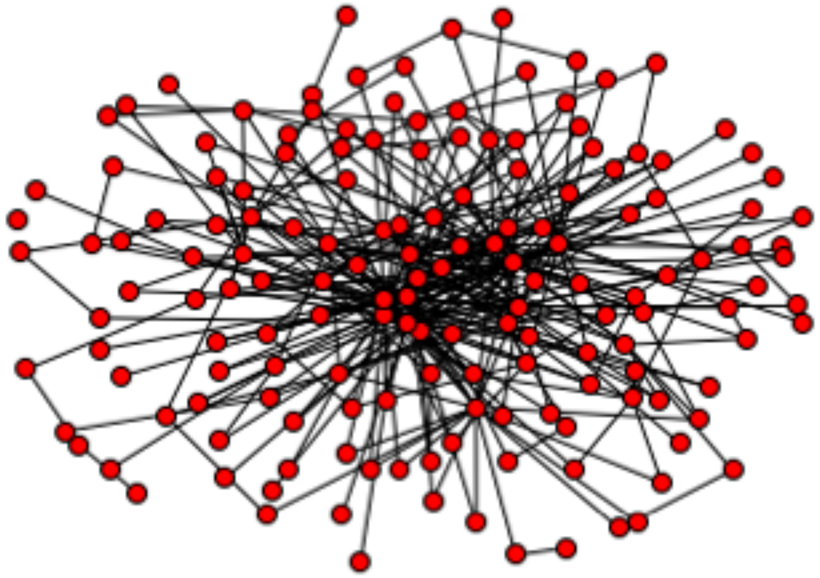
$p($  $)$

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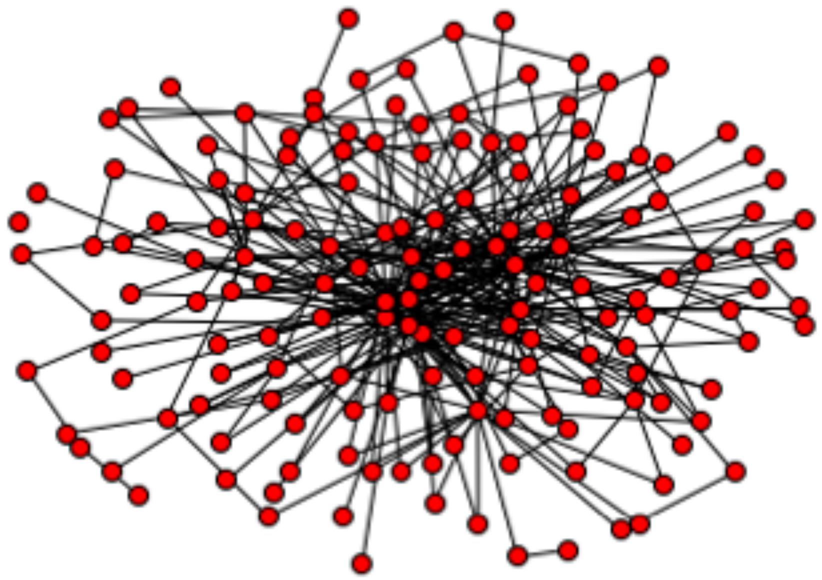
$p($  $)$

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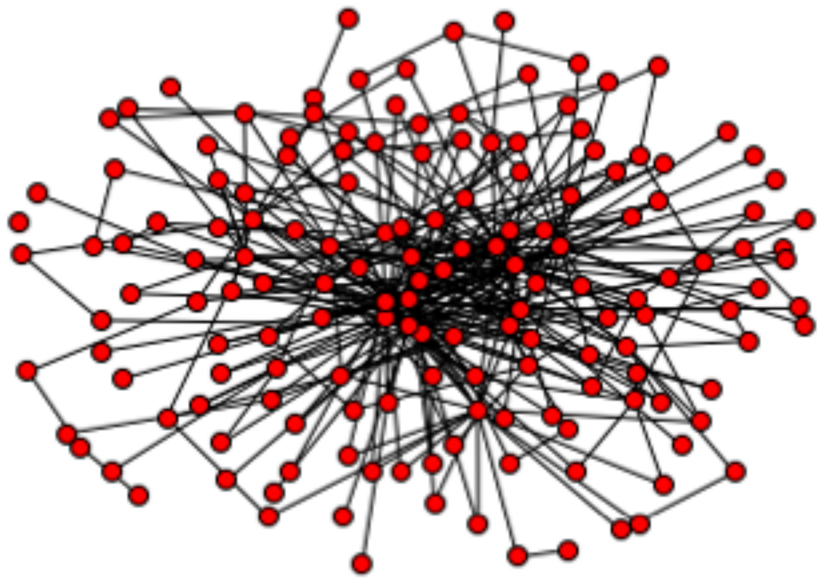
$p($  $)$

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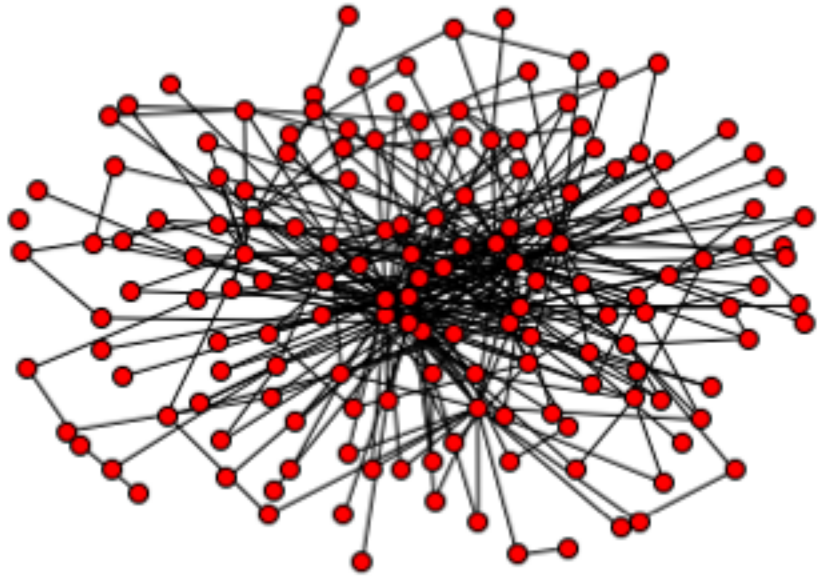
$p($  $)$

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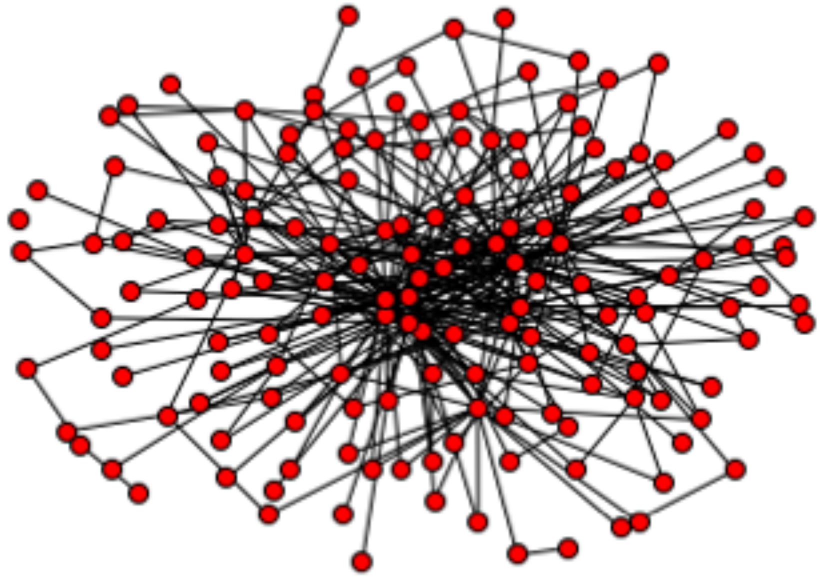
Probabilistic models for graphs

$p($  $)$

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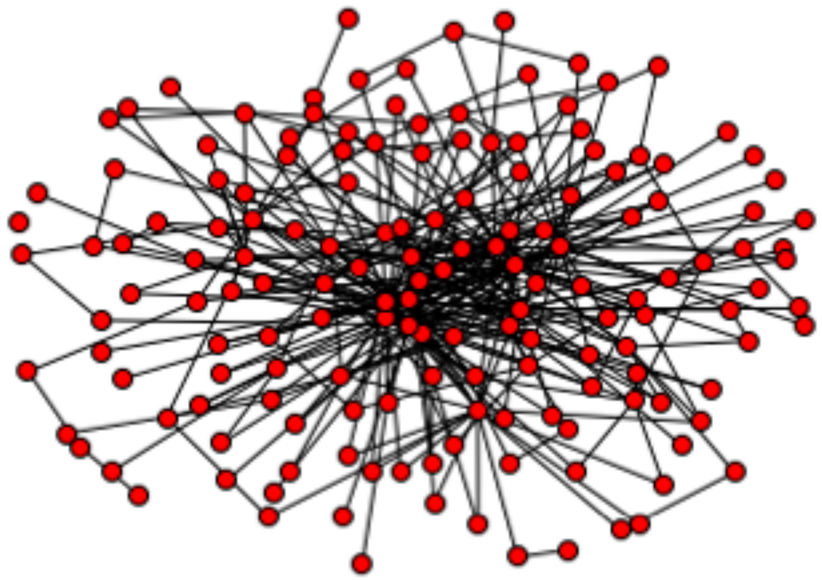
[Holland et al 1983; Kemp et al 2006; Xu et al 2007; Airoldi et al 2008; Lloyd et al 2012]

Probabilistic models for graphs

$p($  $)$

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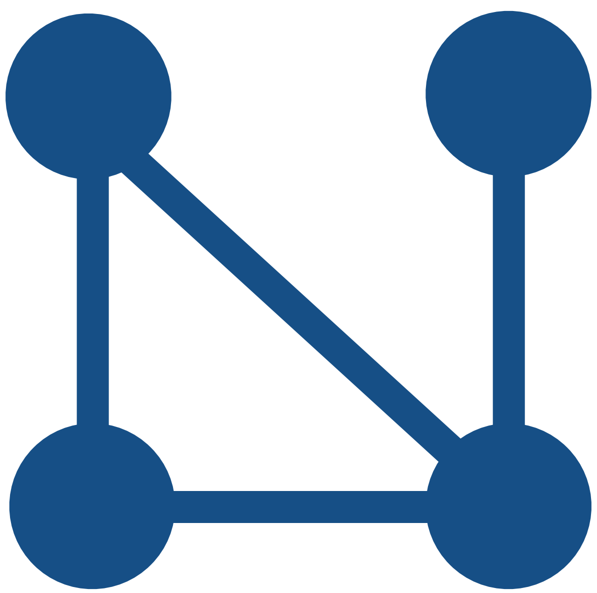
    - Concurrent & independent graphs work by Crane & Dempsey

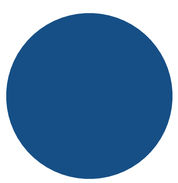
 [B, Cai 2015; Cai, B 2015a,b; Crane, Dempsey 2015a,b,16a,b; Cai, Campbell, B 2016; Campbell, Cai, B 2016]

Sequence of graphs

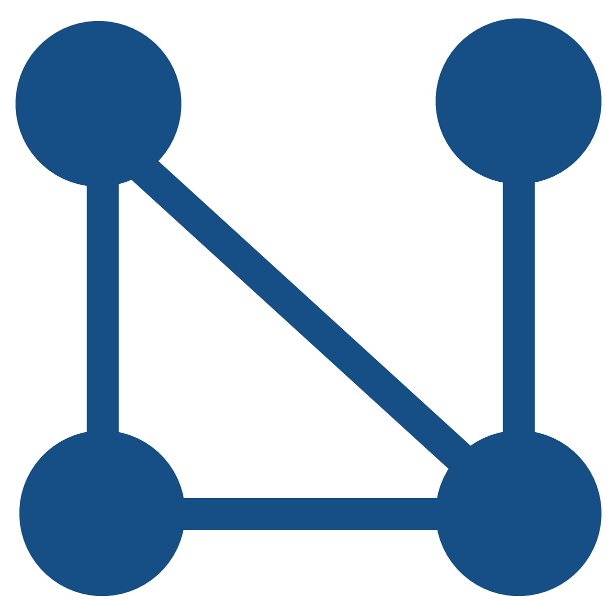


G

Sequence of graphs



G_1



G

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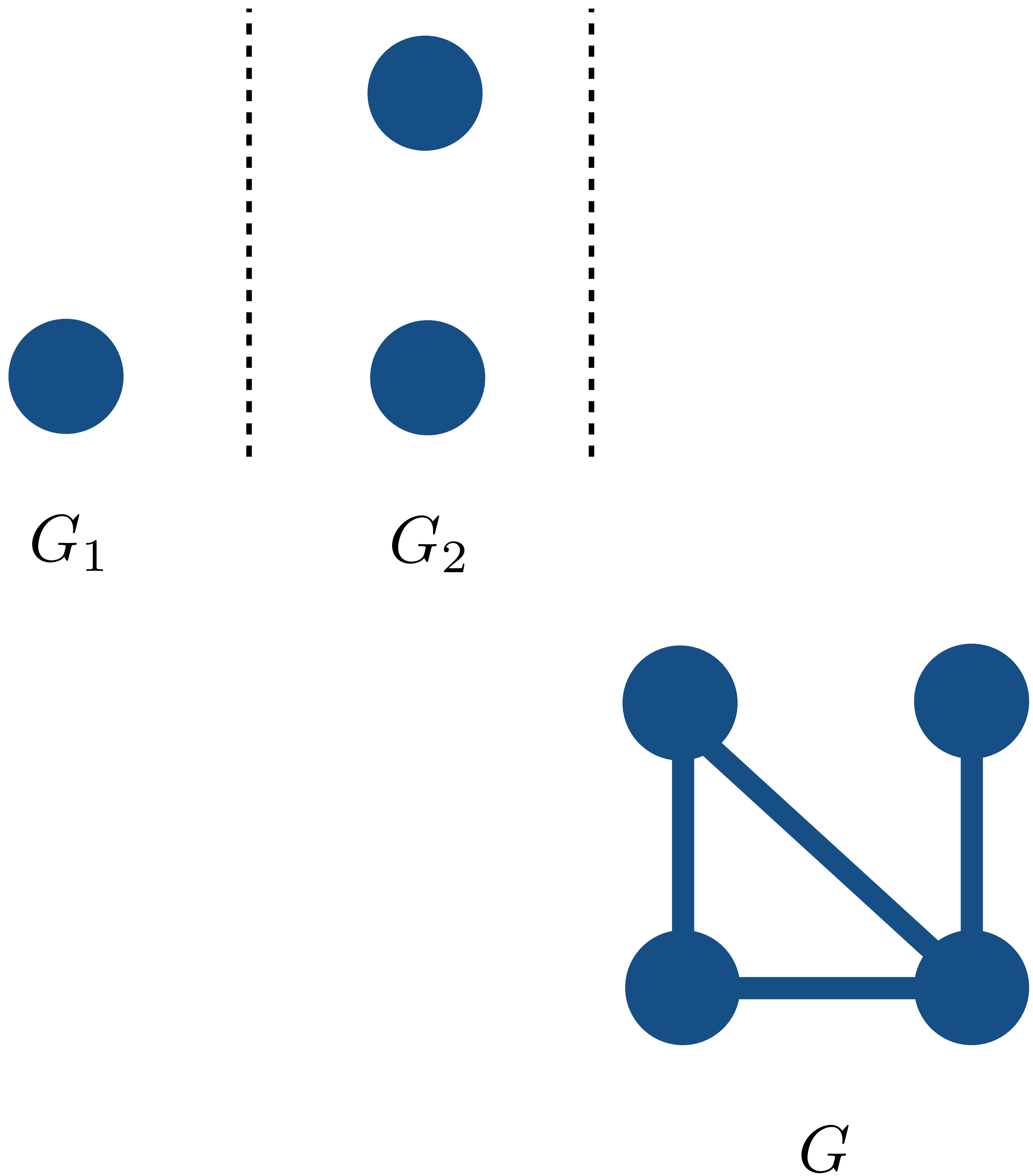

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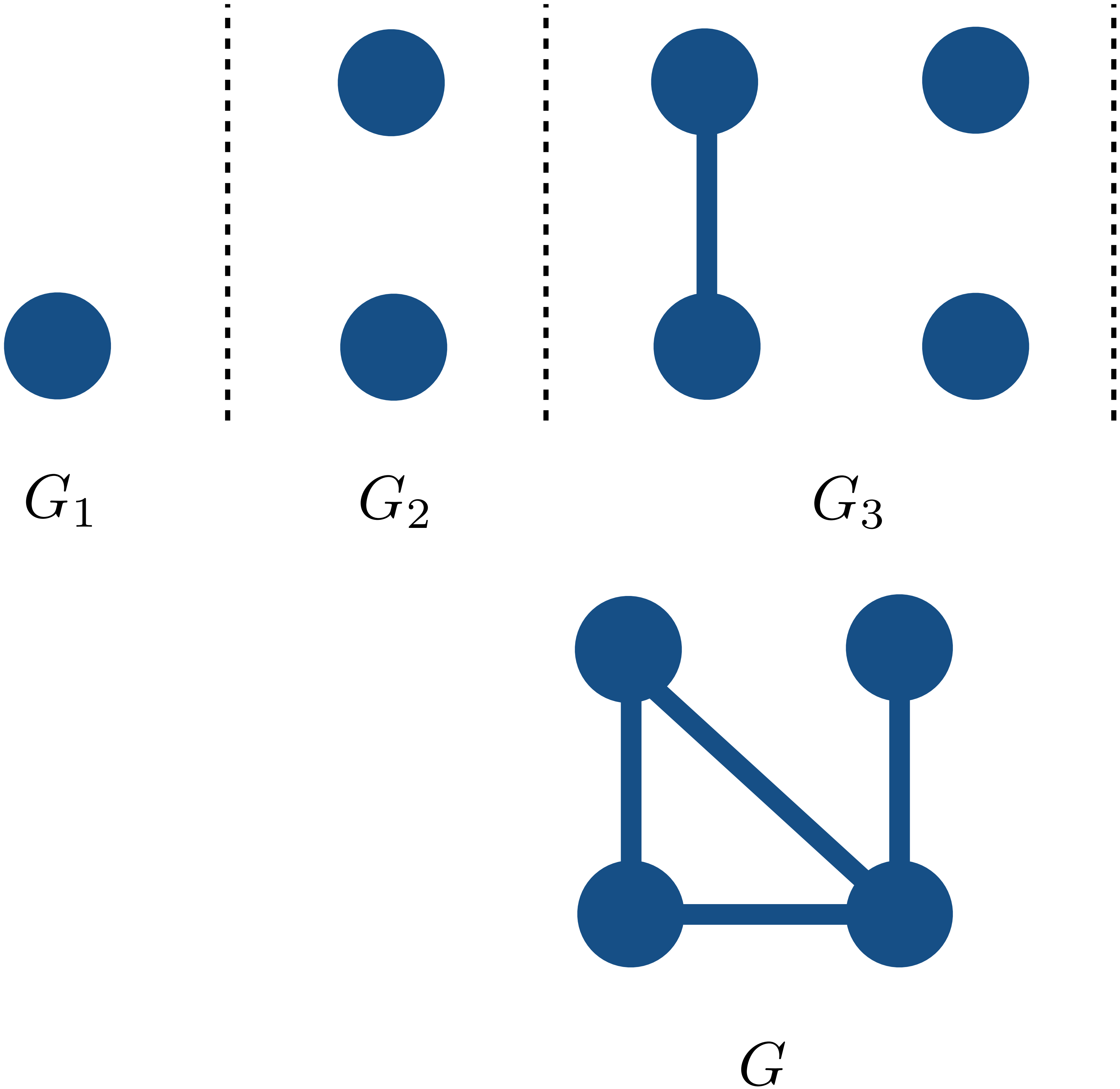
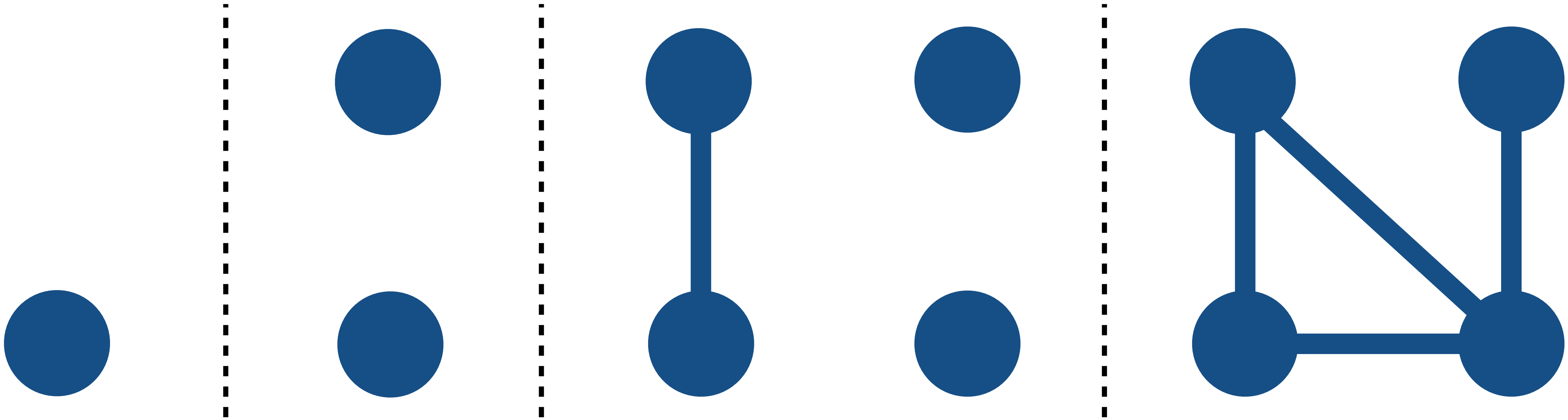
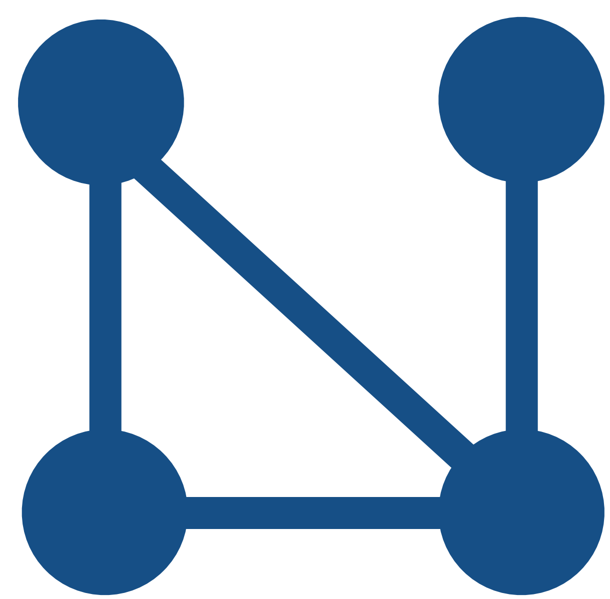


Sequence of graphs



G_1 G_2 G_3 G_4



G

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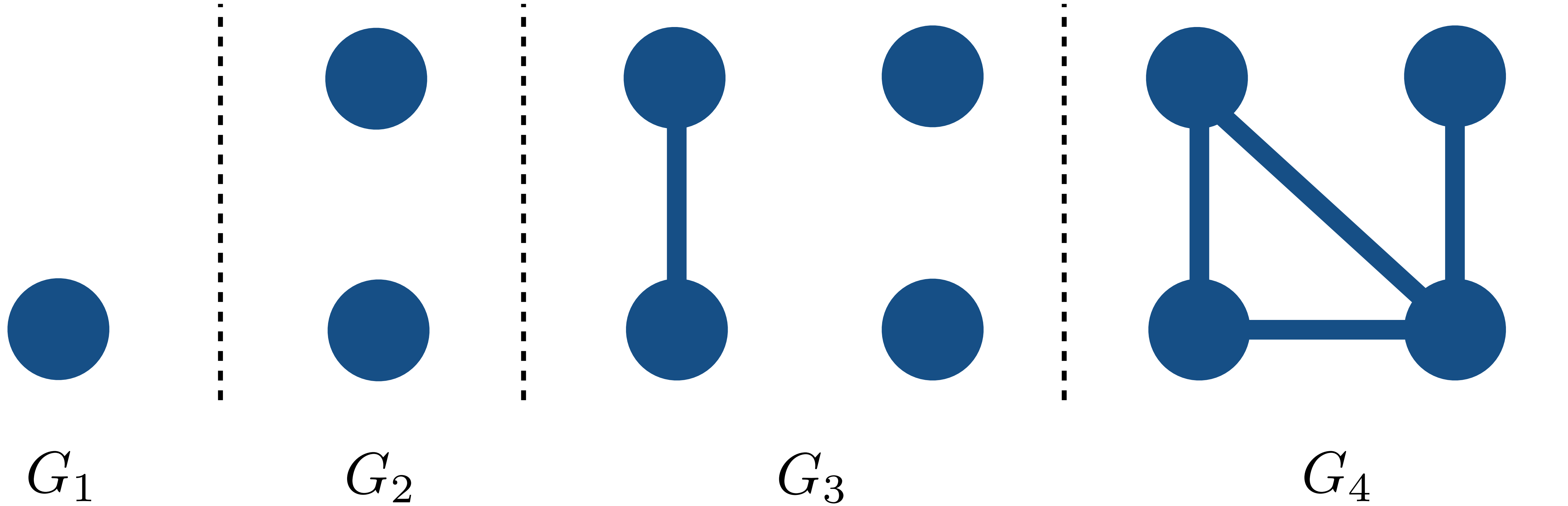
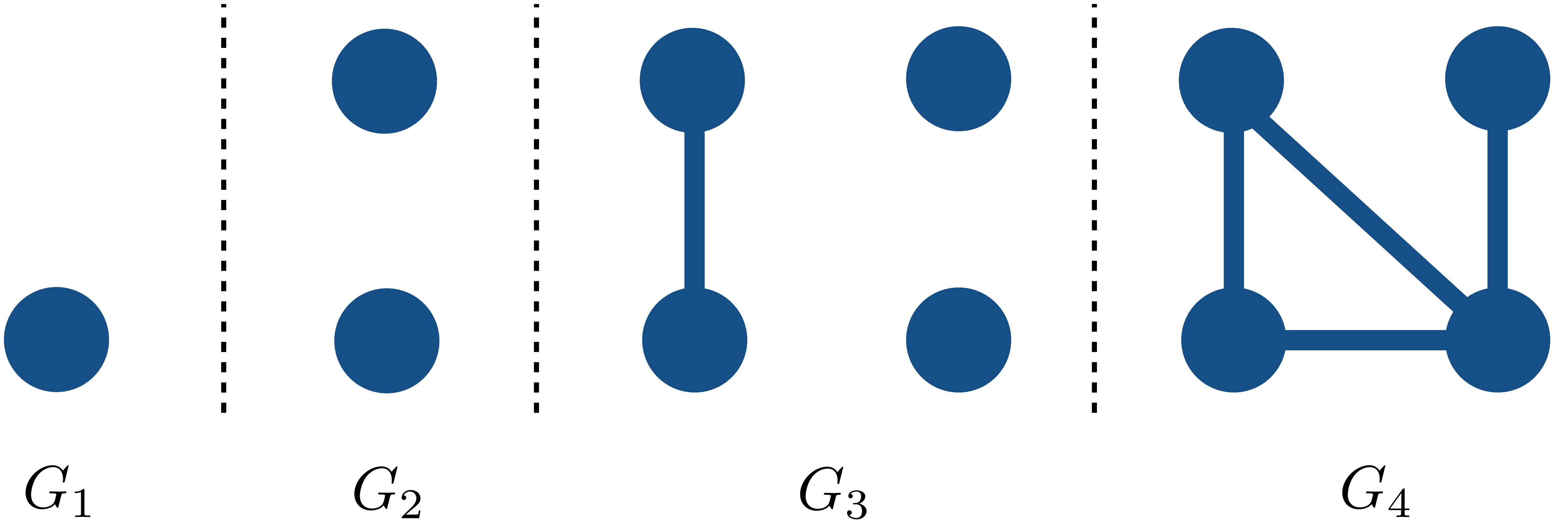


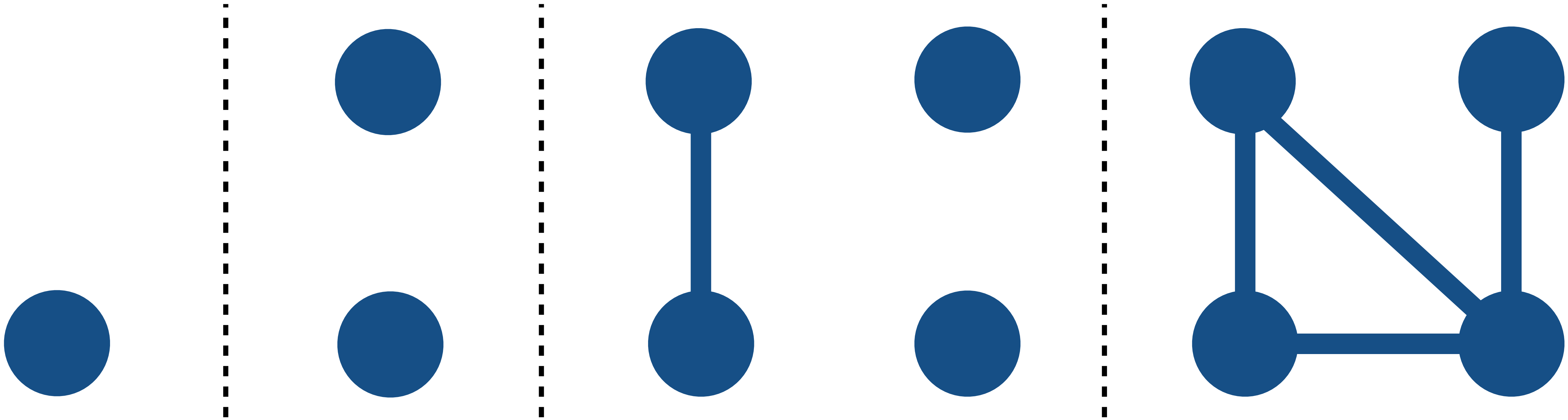
...

Sequence of graphs



- *Dense* graph sequence $\#\text{edges}(G_n) \geq c \cdot [\#\text{nodes}(G_n)]^2$

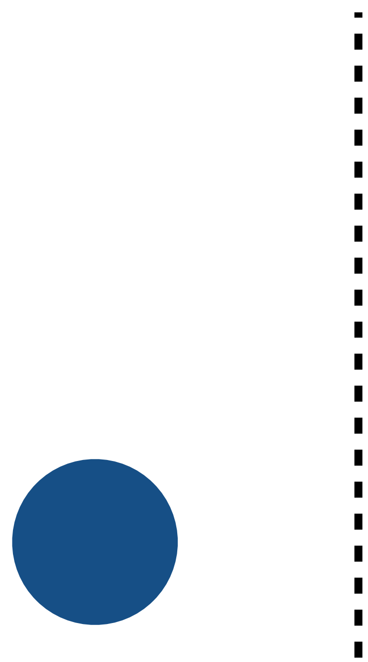
Sequence of graphs



G_1 G_2 G_3 G_4

- *Dense* graph sequence $\#\mathrm{edges}(G_n) \geq c \cdot [\#\mathrm{nodes}(G_n)]^2$
- *Sparse* graph sequence $\#\mathrm{edges}(G_n) \in o([\#\mathrm{nodes}(G_n)]^2)$

The Old Way: Nodes



G_1

[Hoover 1979, Aldous 1981]

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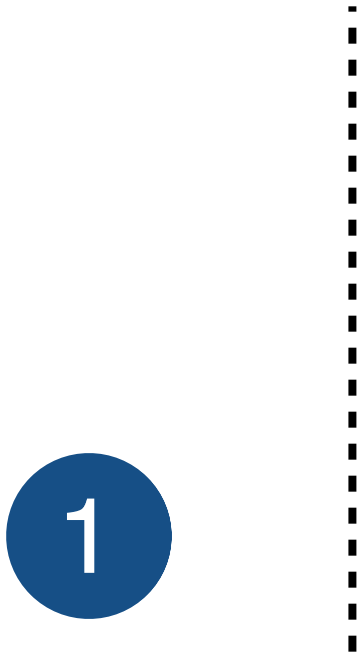

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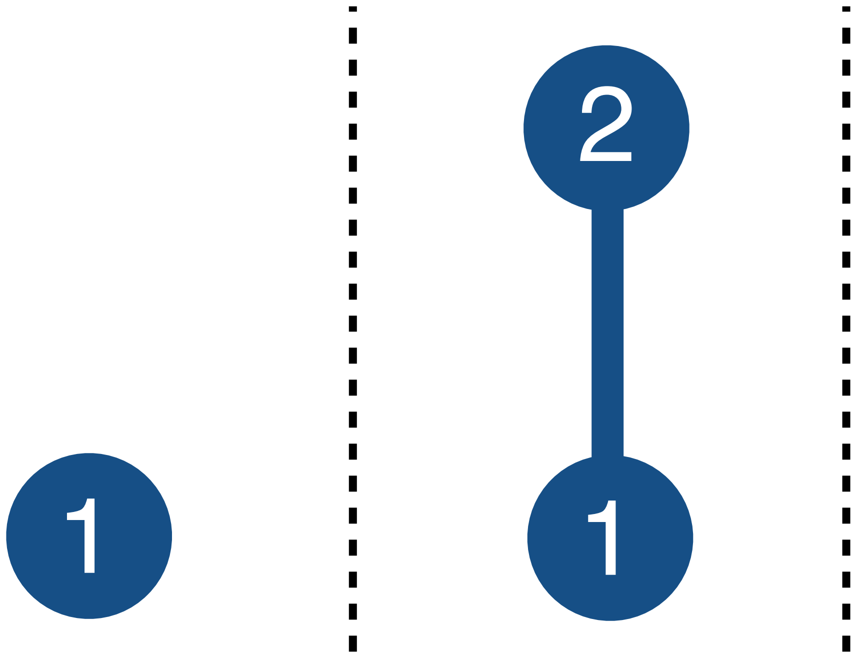

G_1 G_2

[Hoover 1979, Aldous 1981]

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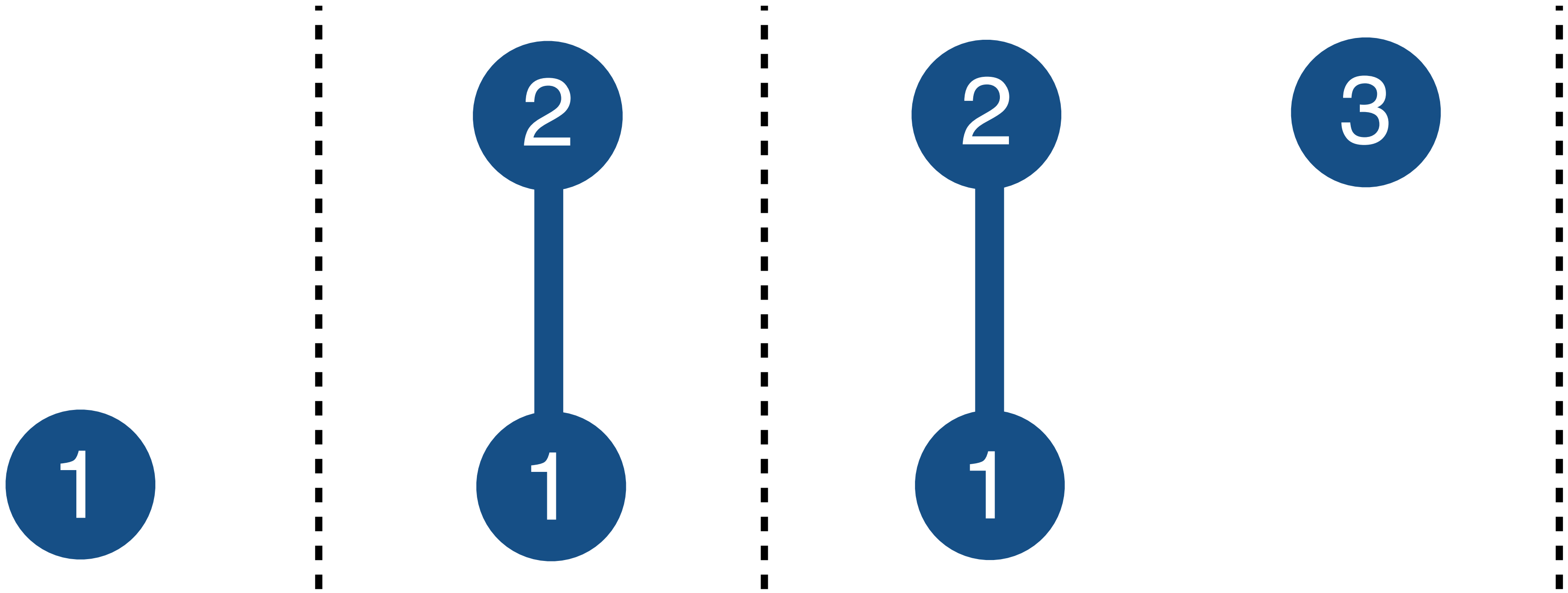

G_1 G_2 G_3

[Hoover 1979, Aldous 1981]

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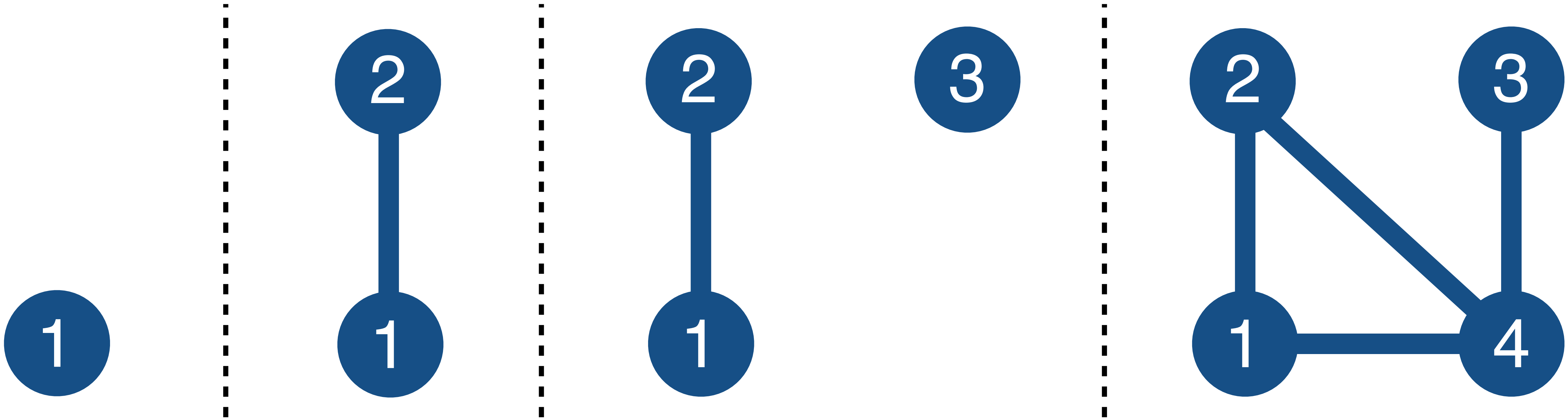
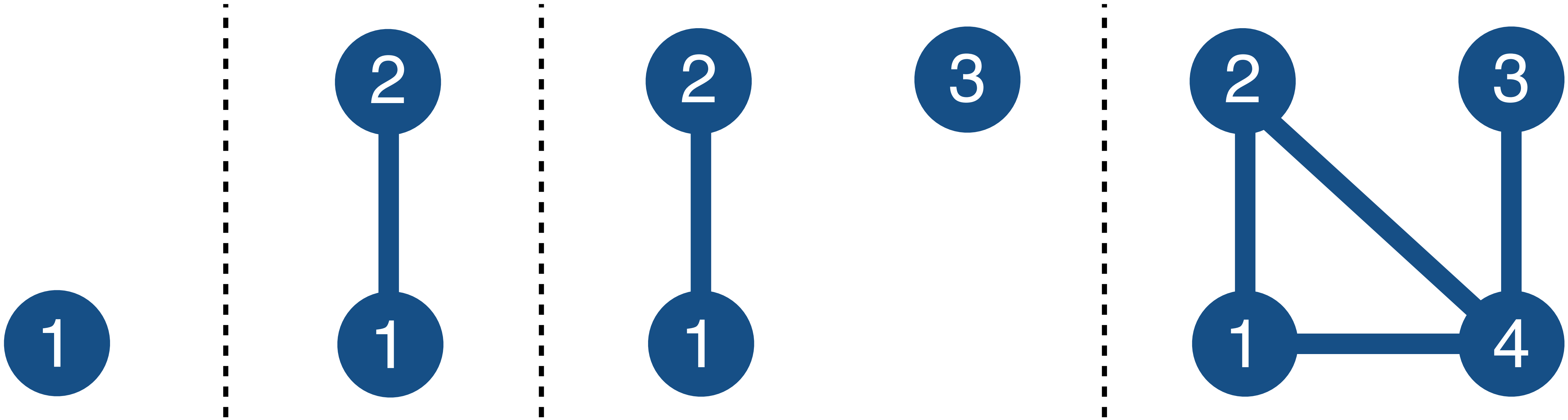


G_1 G_2 G_3 G_4

[Hoover 1979, Aldous 1981]

Exchangeability



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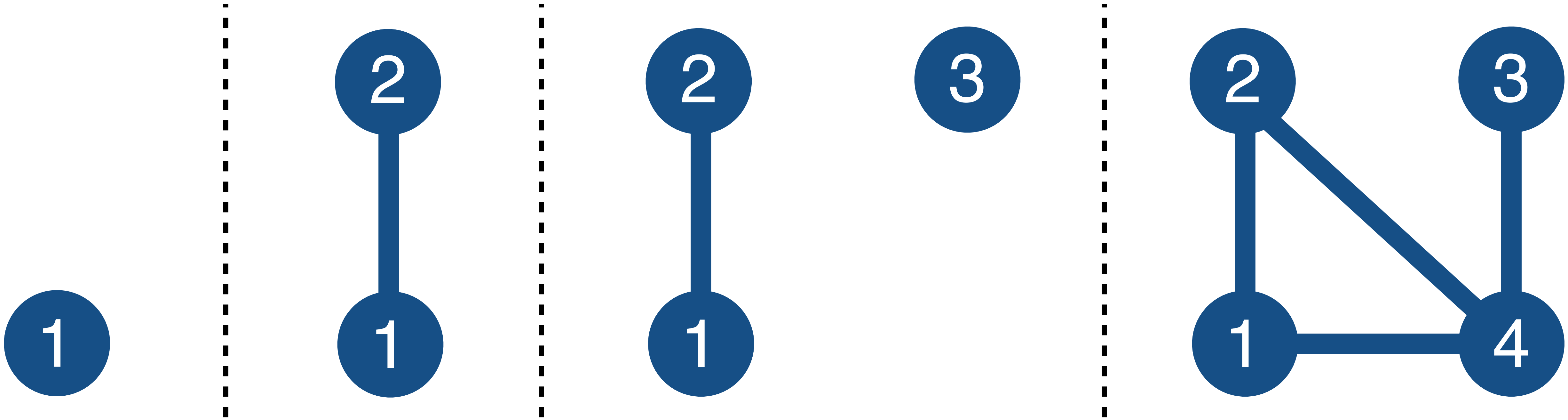

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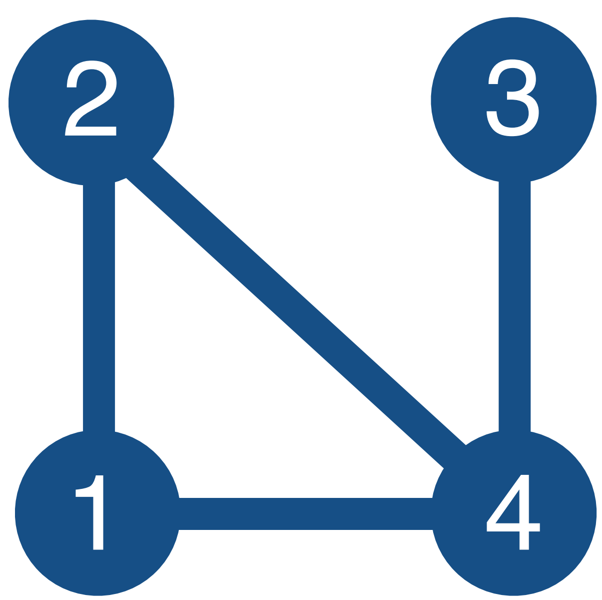

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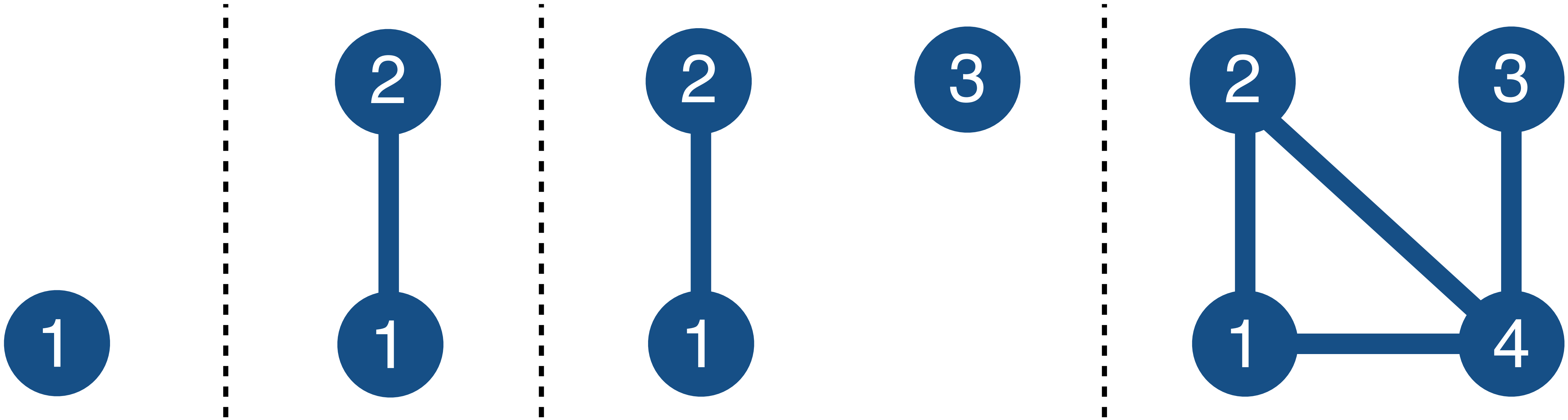

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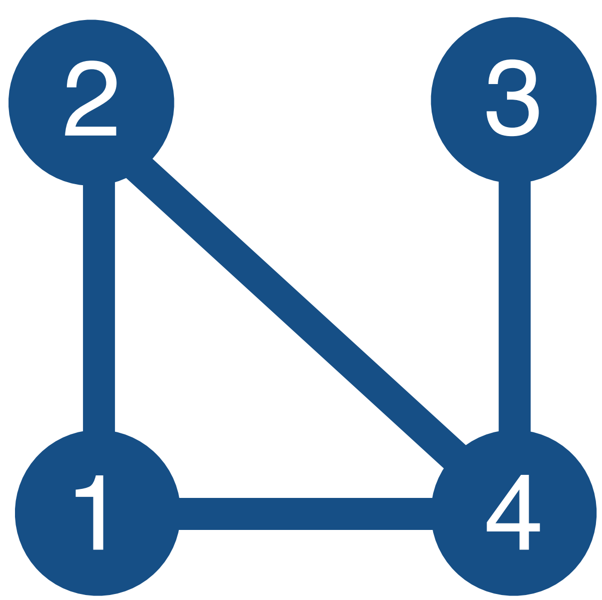
$p($  $)$

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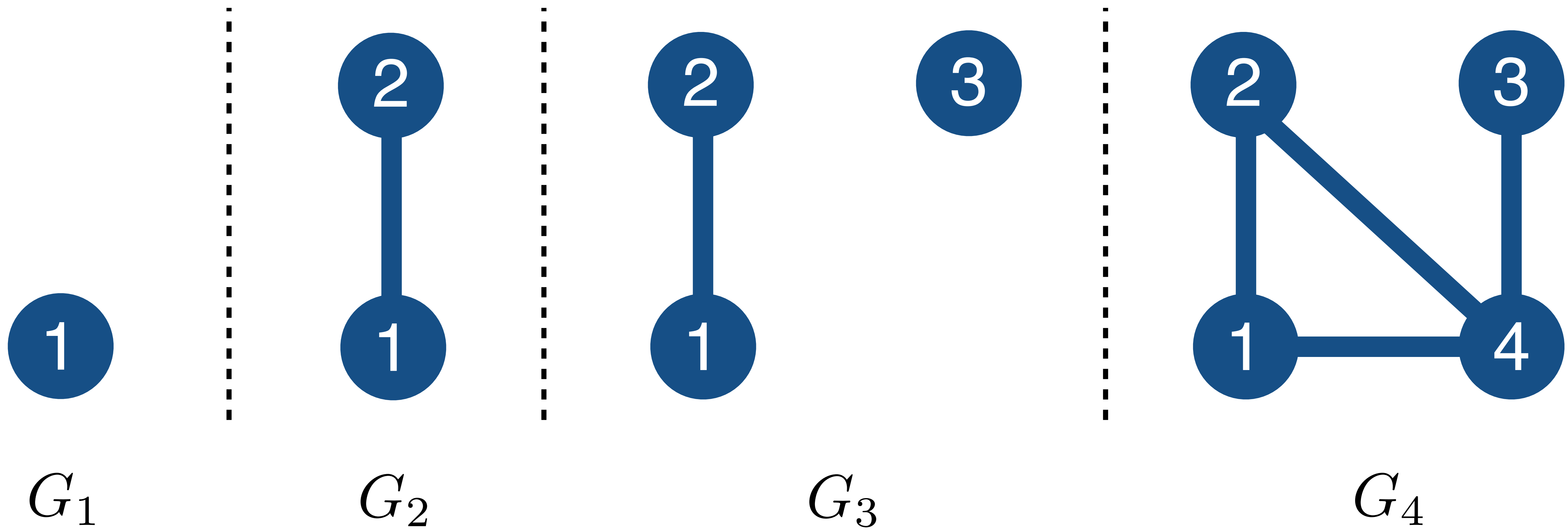
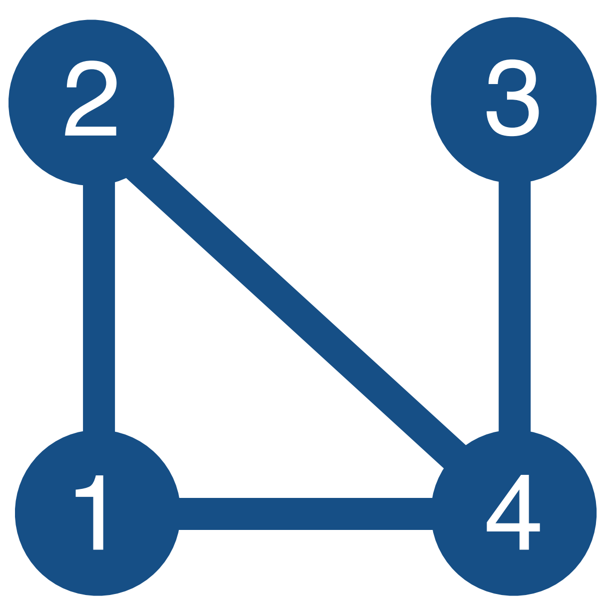
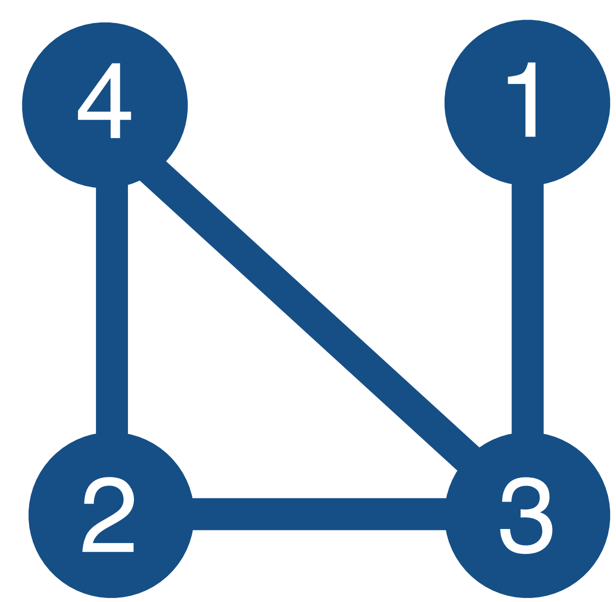
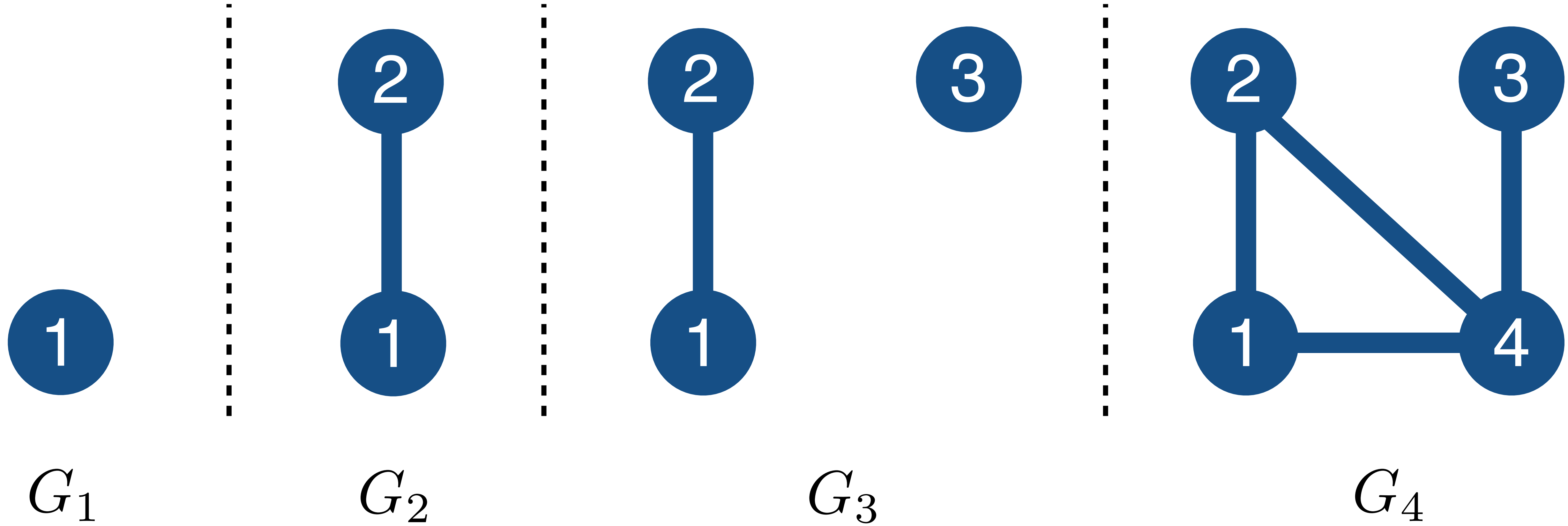
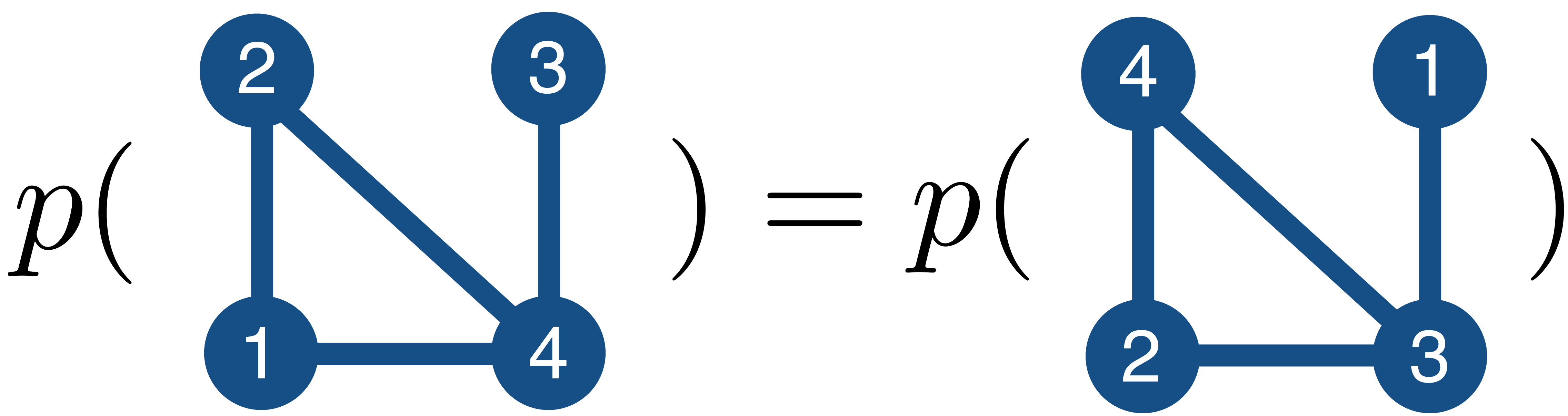


$p($  $) = p($  $)$

[Hoover 1979, Aldous 1981]

Node exchangeability





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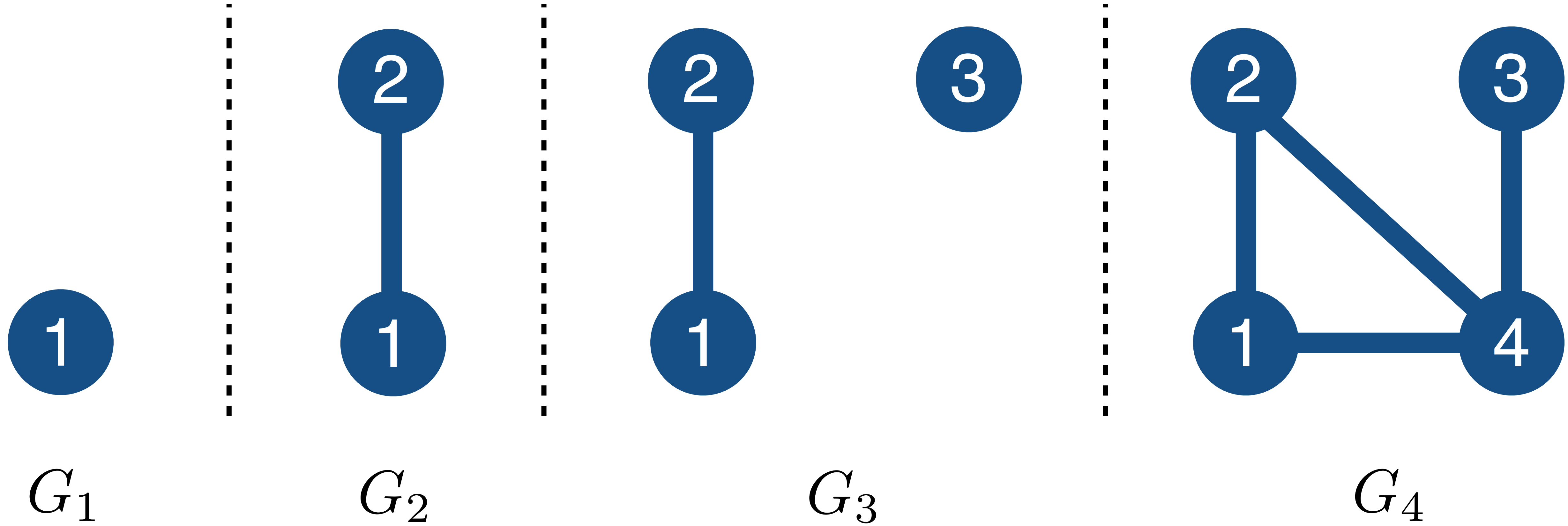
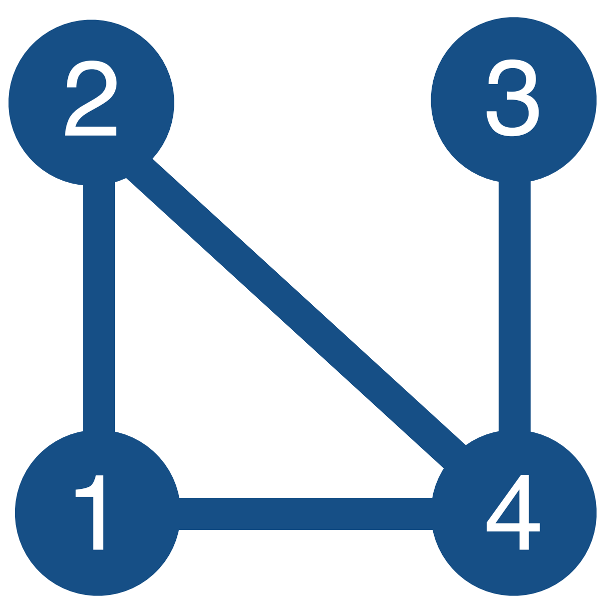
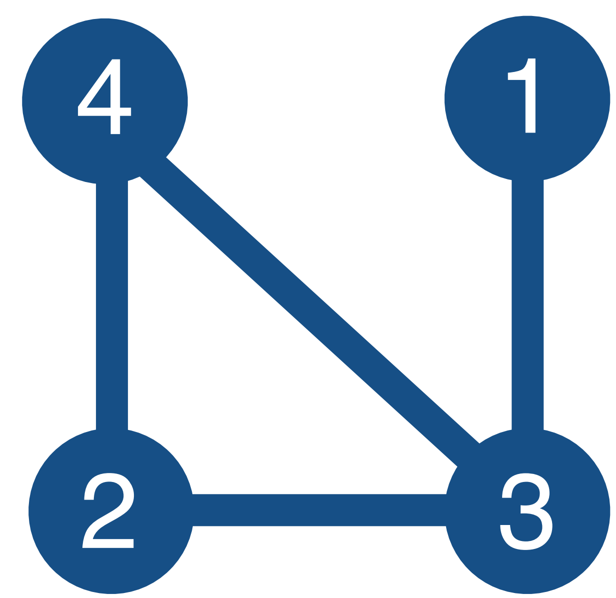
The Old Way: Node exchangeability



$p($ 

$) = p($ 

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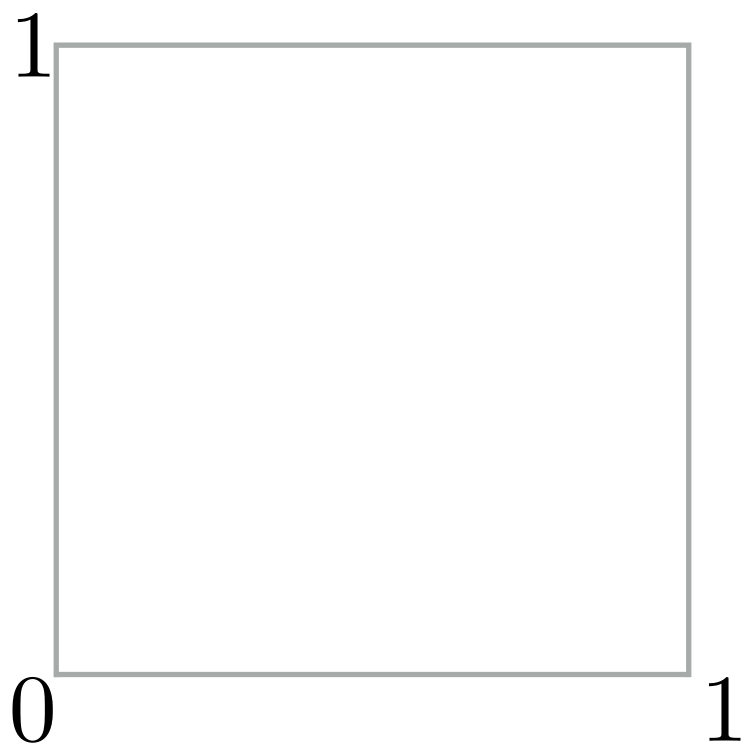
Aldous-Hoover



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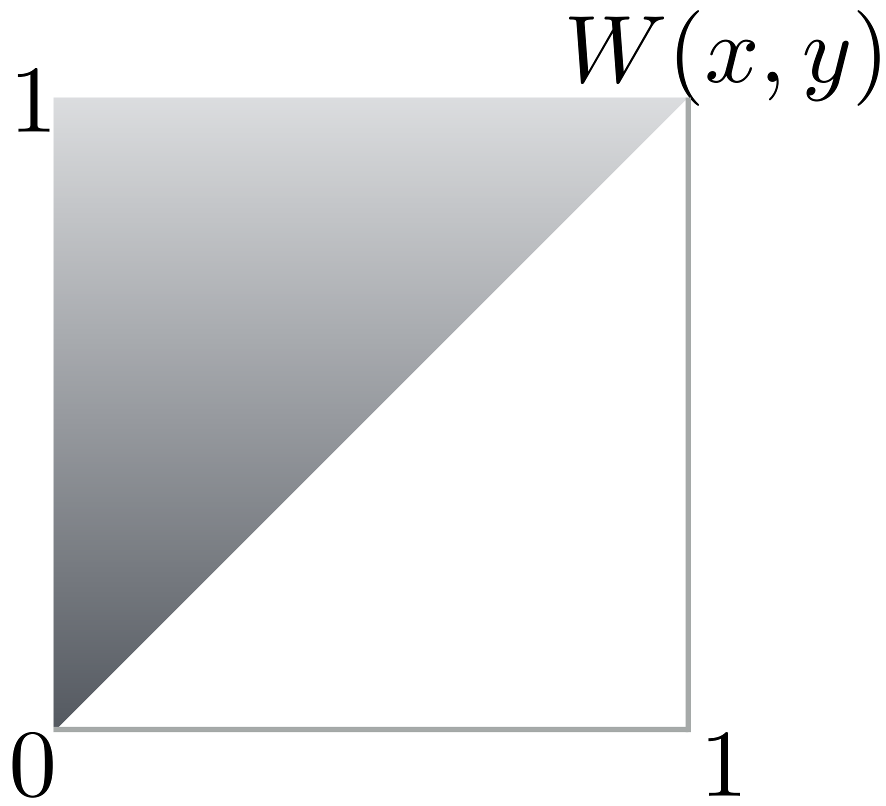
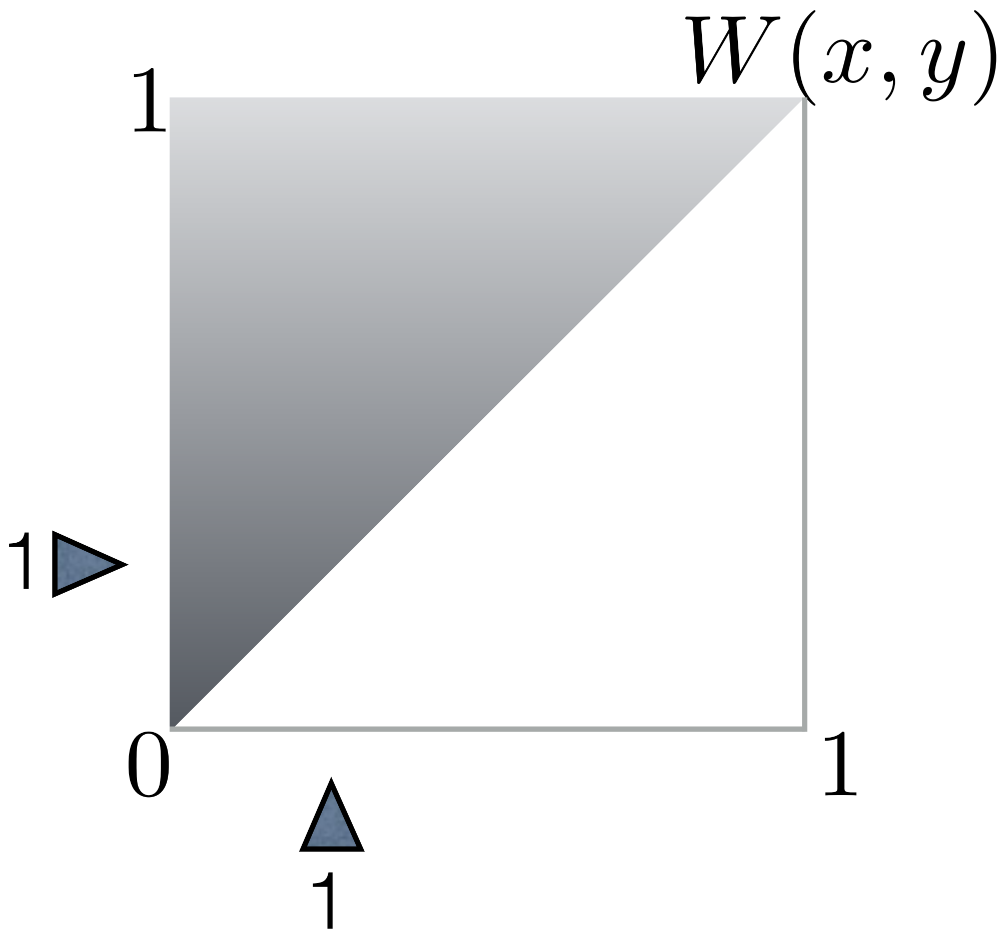
Aldous-Hoover



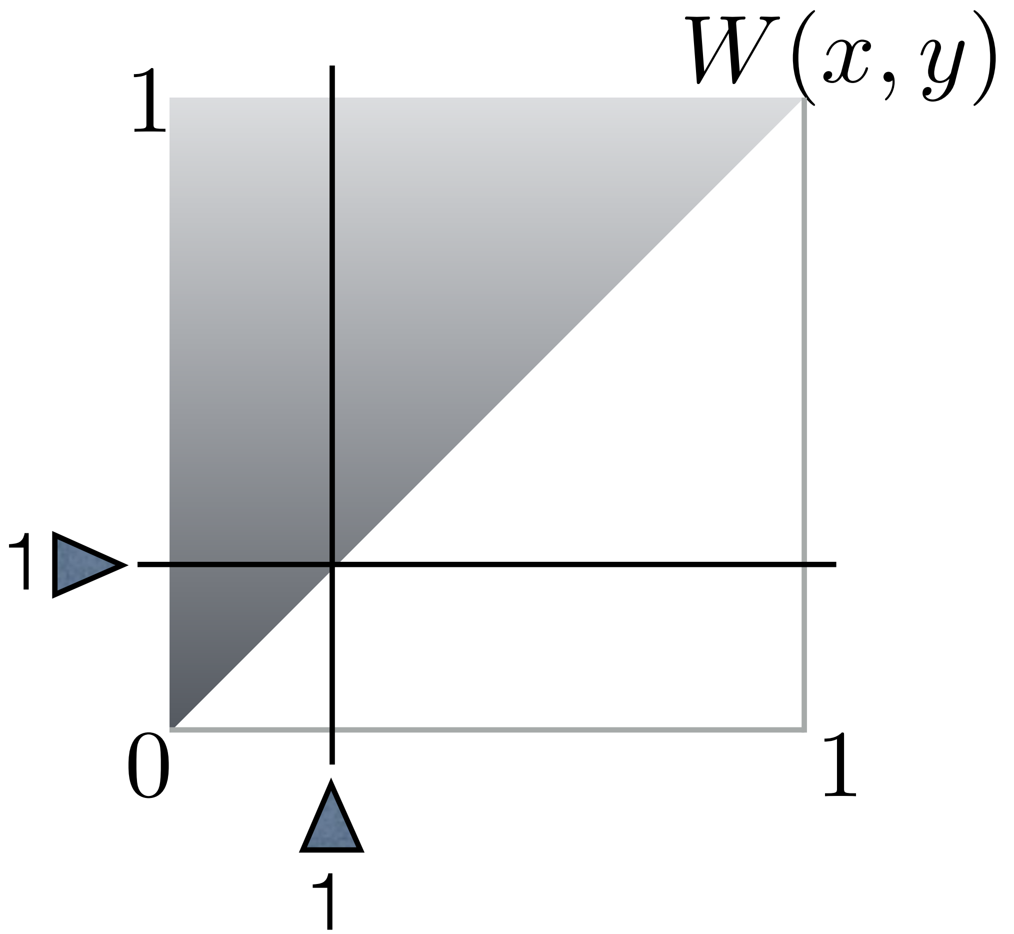
[Hoover 1979, Aldous 1981]

Aldous-Hoover



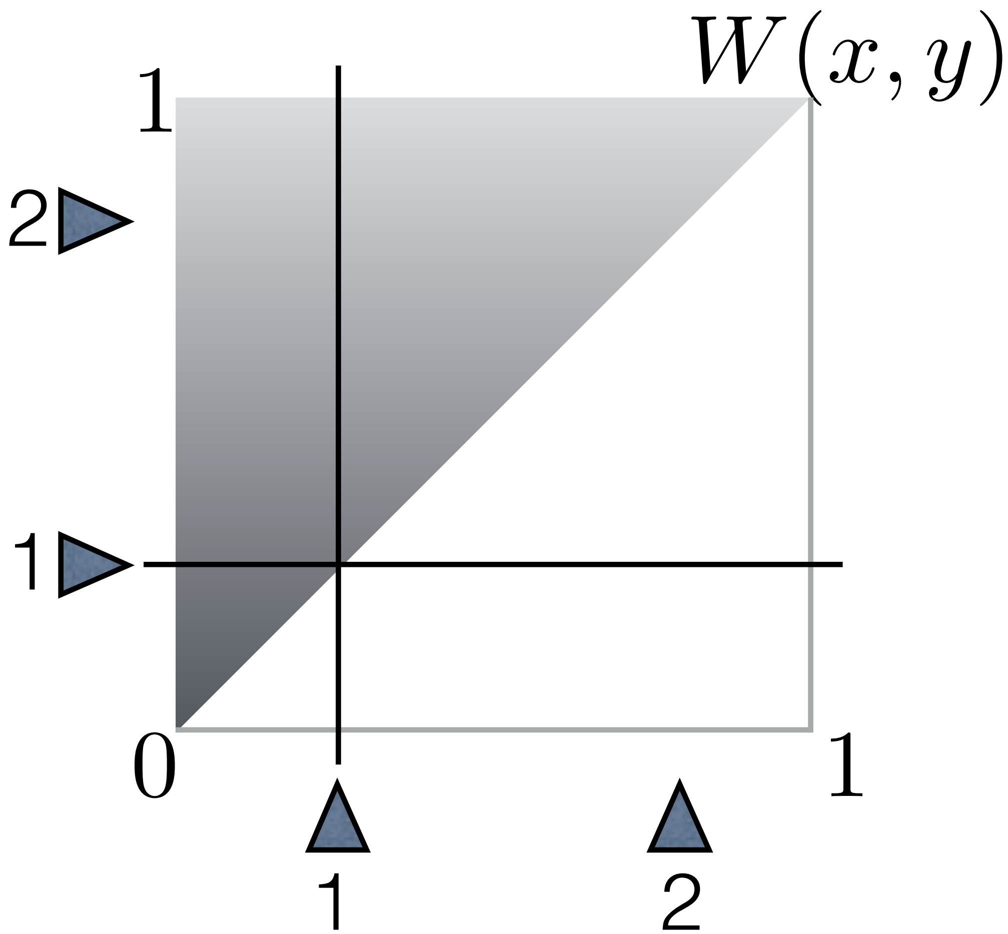
[Hoover 1979, Aldous 1981]

Aldous-Hoover



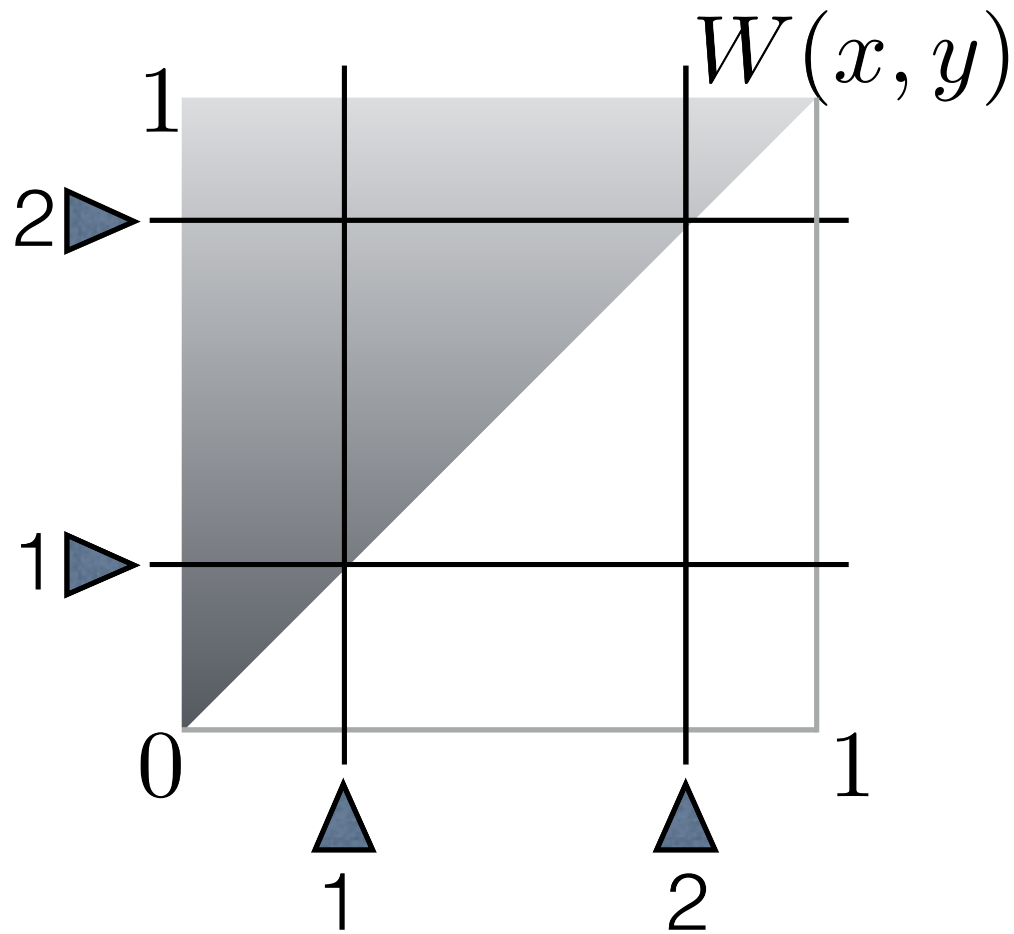
[Hoover 1979, Aldous 1981]

Aldous-Hoover



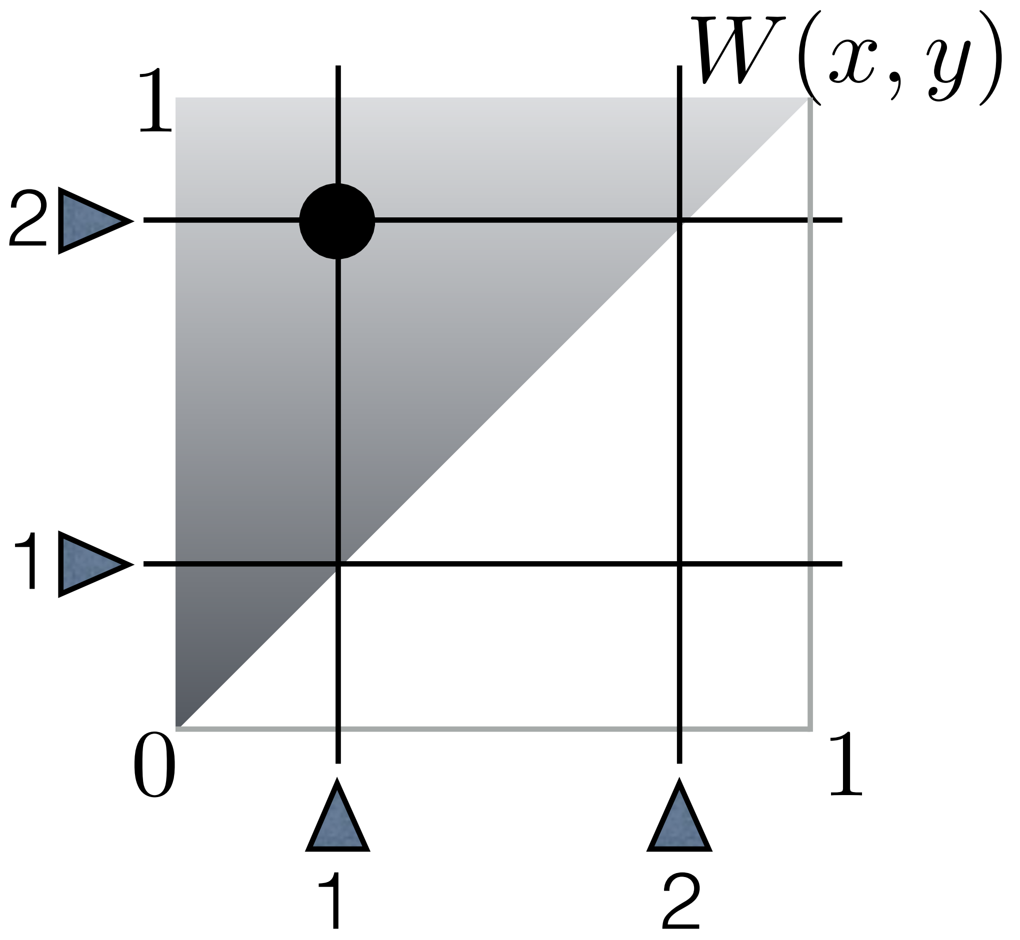
[Hoover 1979, Aldous 1981]

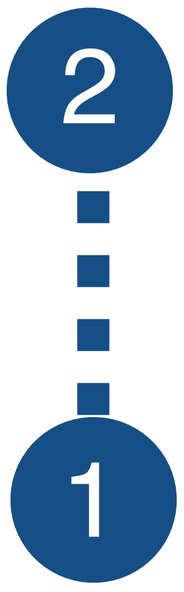
Aldous-Hoover



[Hoover 1979, Aldous 1981]

Aldous-Hoover





[Hoover 1979, Aldous 1981]

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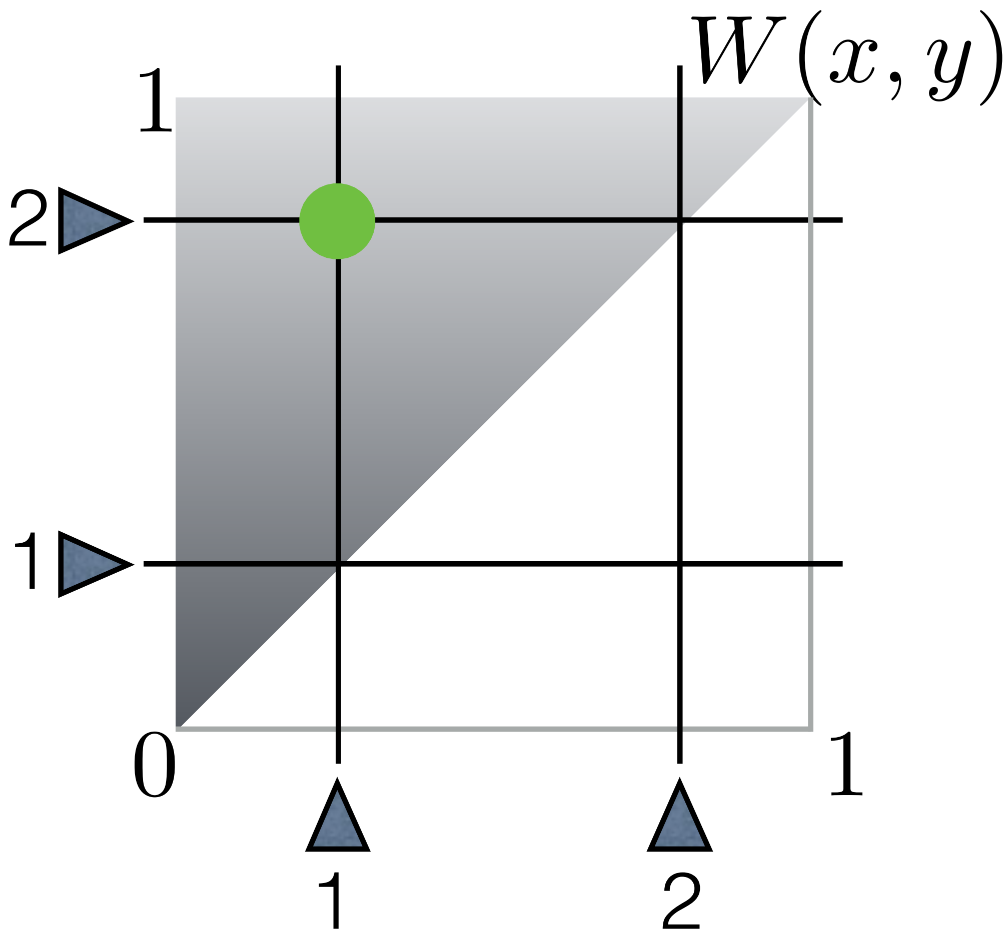
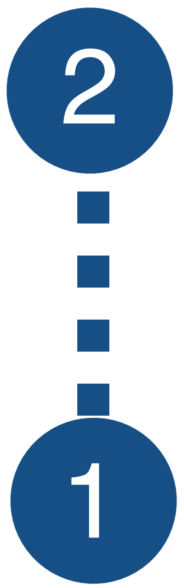
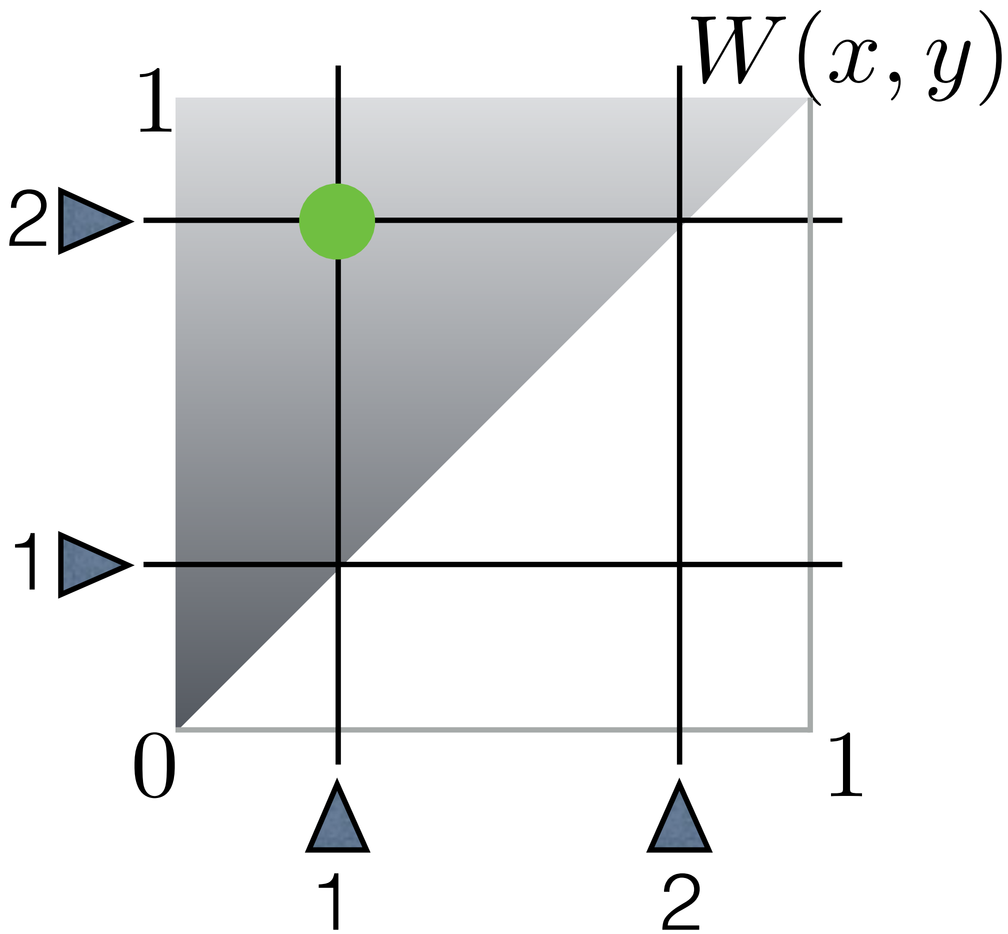




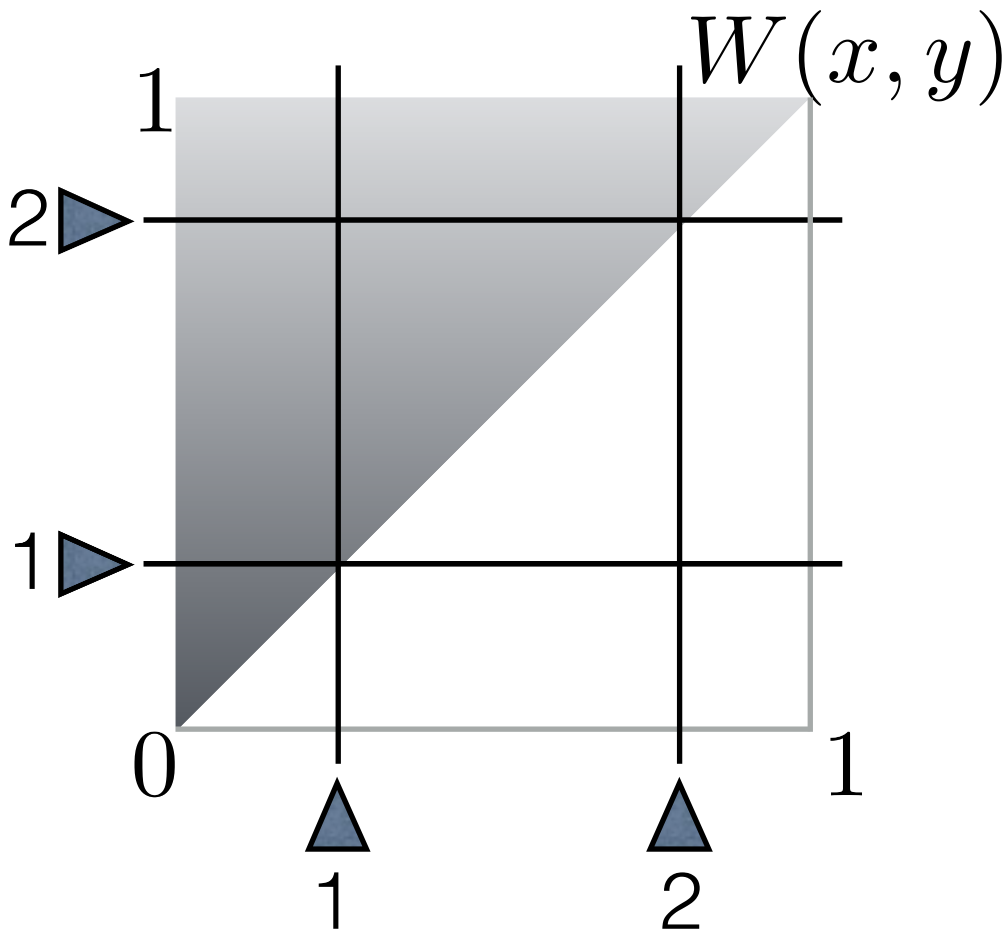
[Hoover 1979, Aldous 1981]

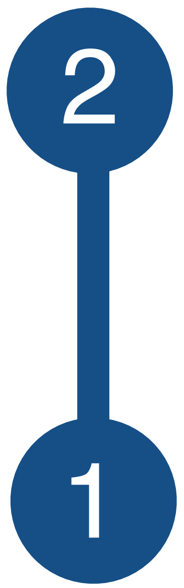
Aldous-Hoover



[Hoover 1979, Aldous 1981]

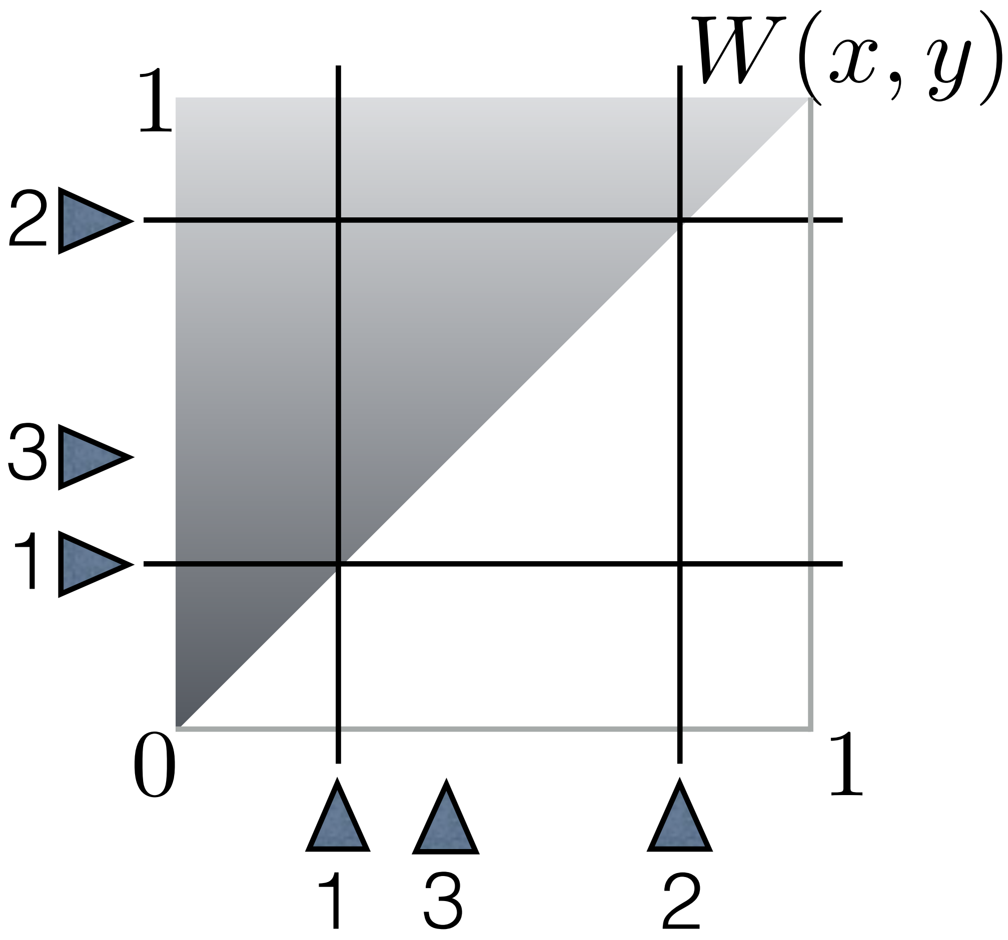
Aldous-Hoover

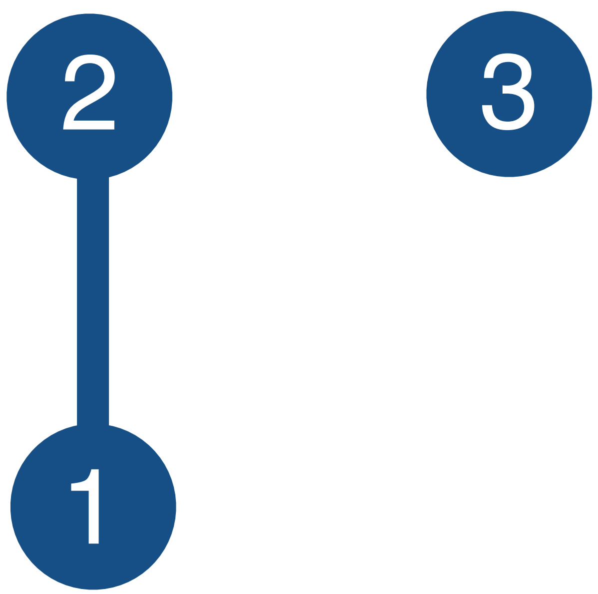




[Hoover 1979, Aldous 1981]

Aldous-Hoover





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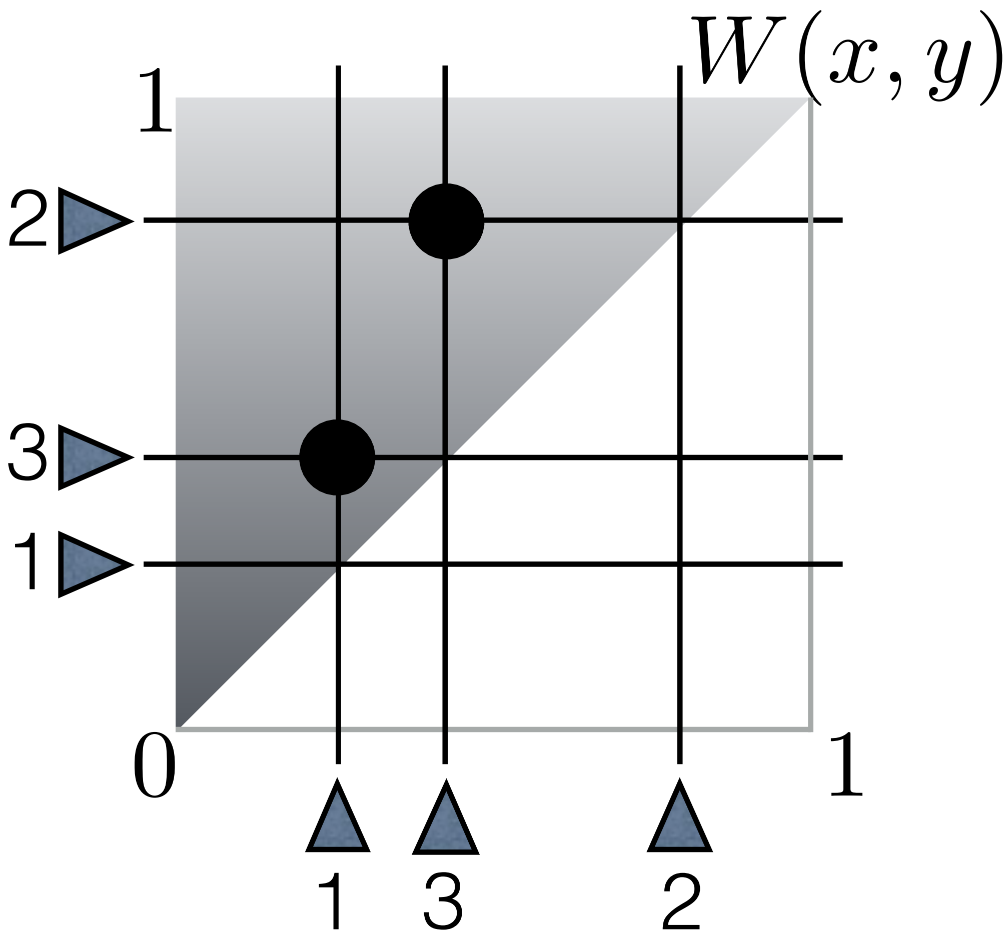
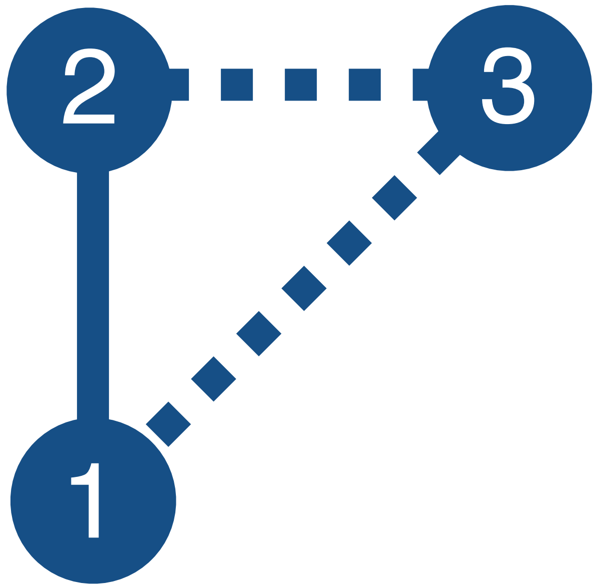
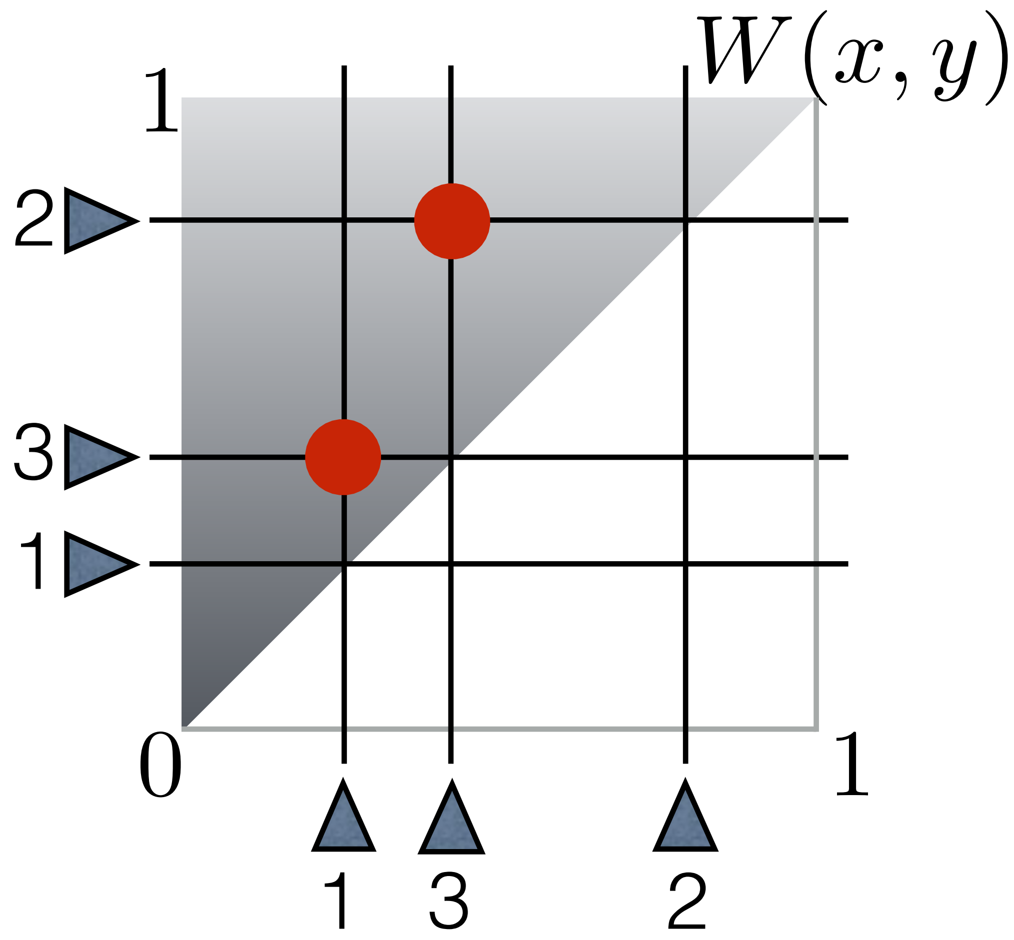
Aldous-Hoover

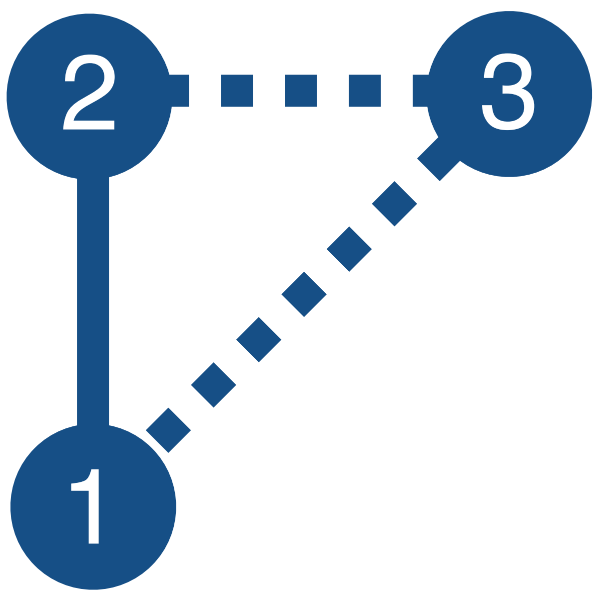




[Hoover 1979, Aldous 1981]

Aldous-Hoover





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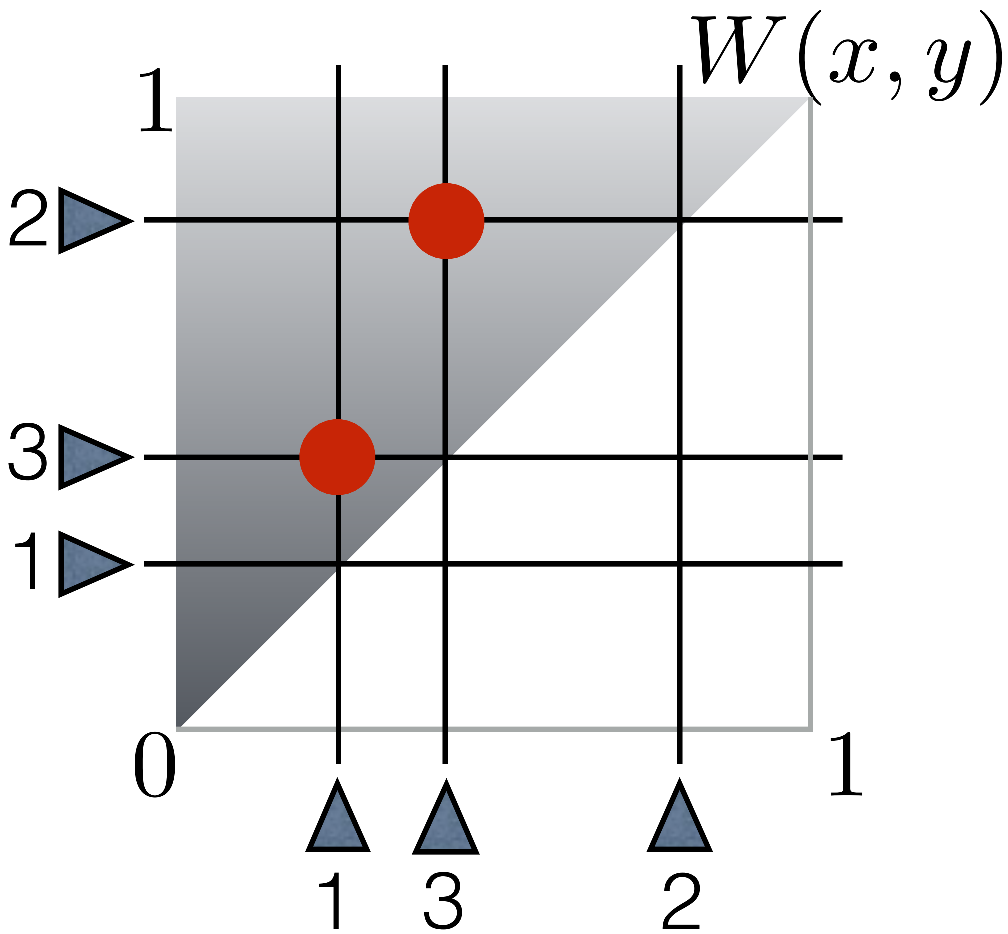
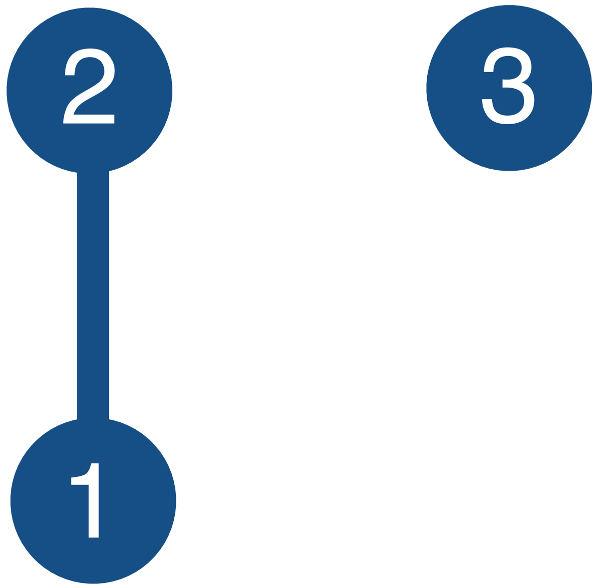
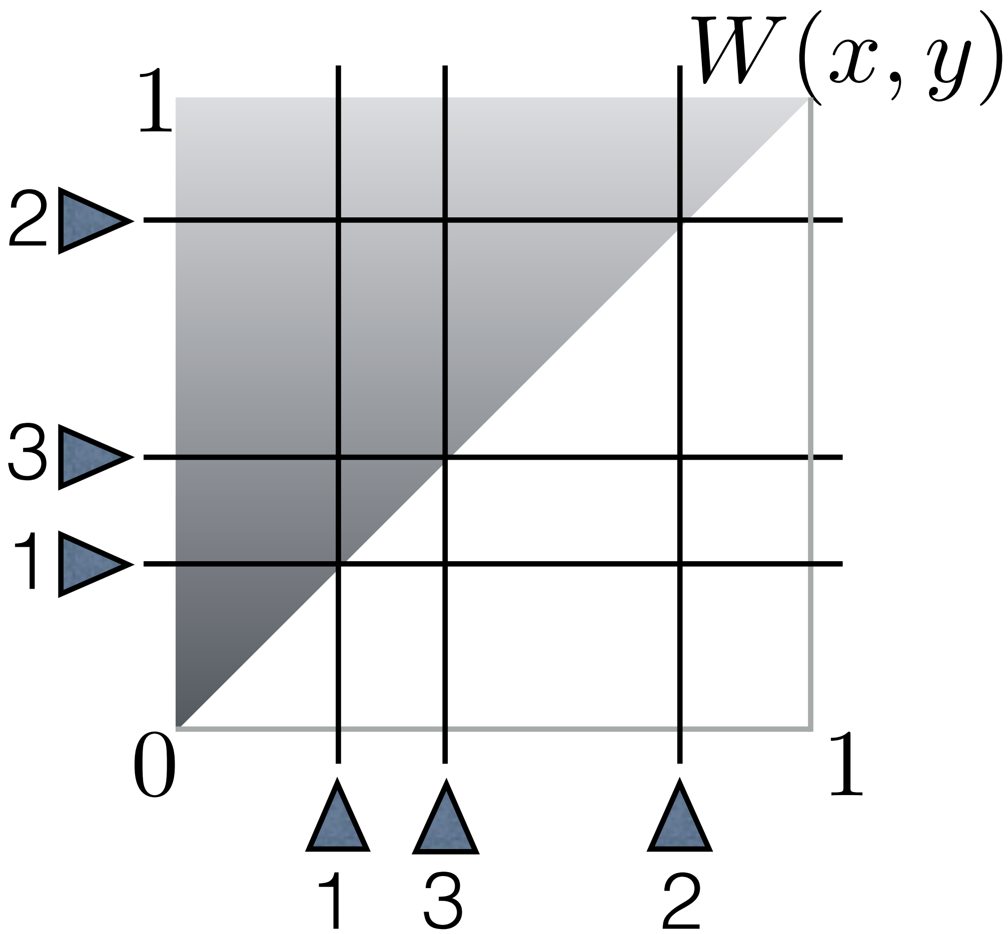
Aldous-Hoover

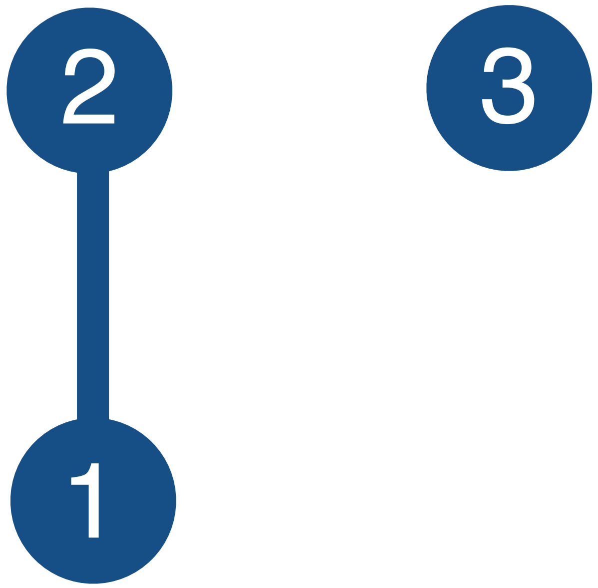




[Hoover 1979, Aldous 1981]

Aldous-Hoover





[Hoover 1979, Aldous 1981]

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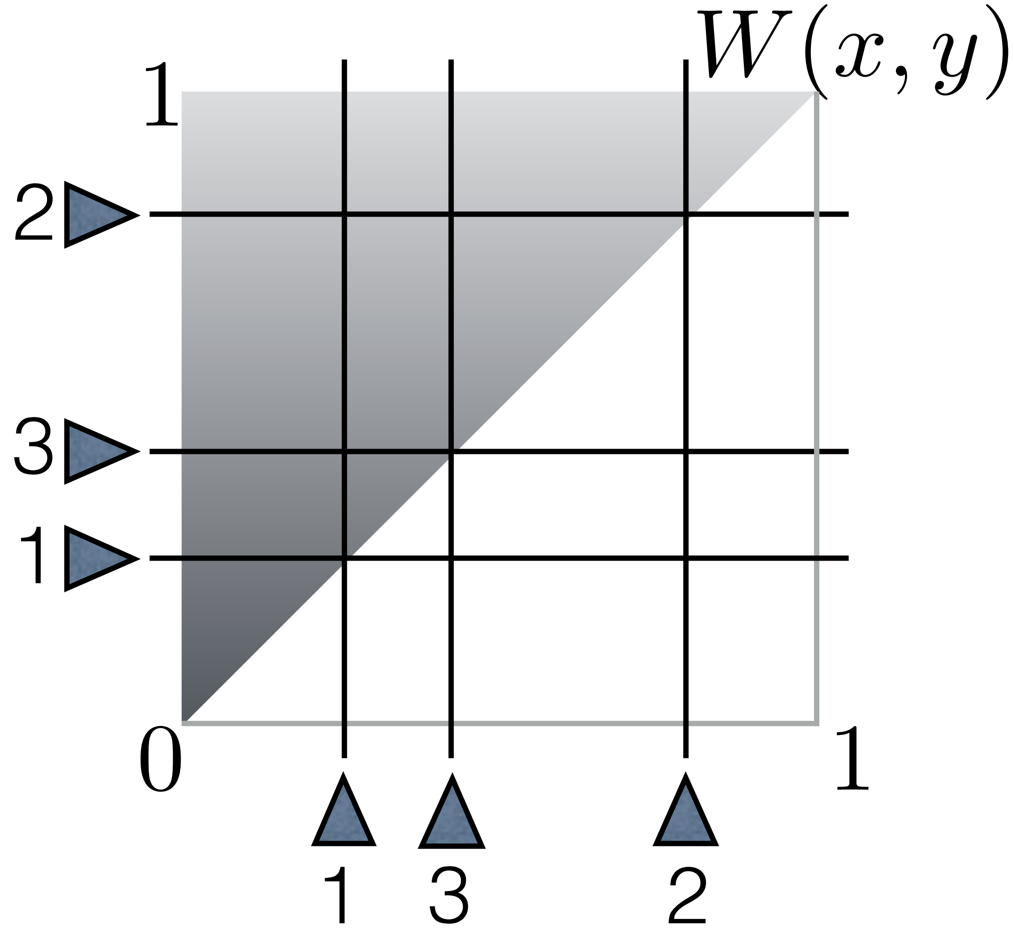
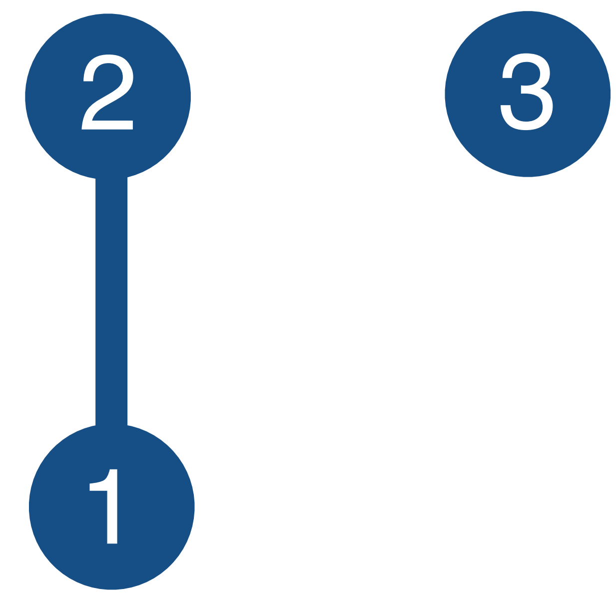
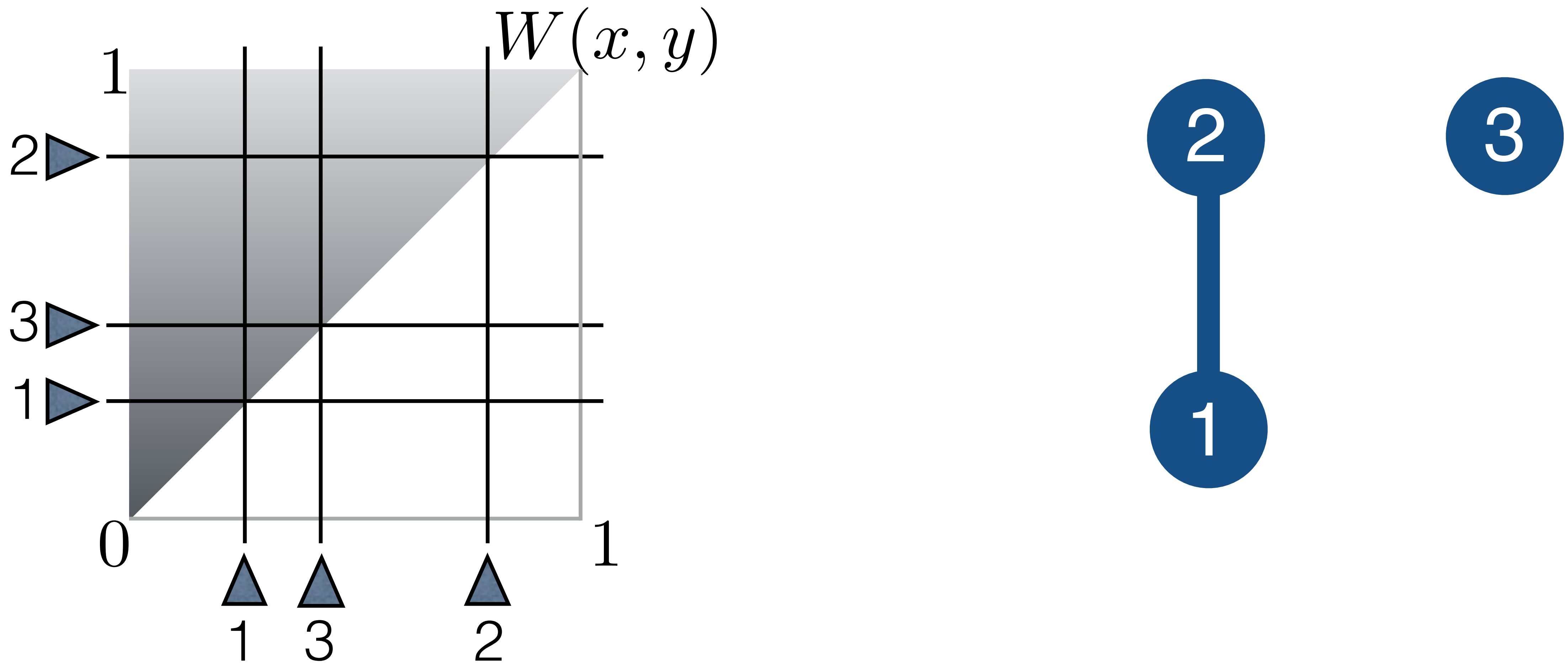




Every node-exchangeable graph has a *graphon* rep

[Hoover 1979, Aldous 1981]

Aldous-Hoover

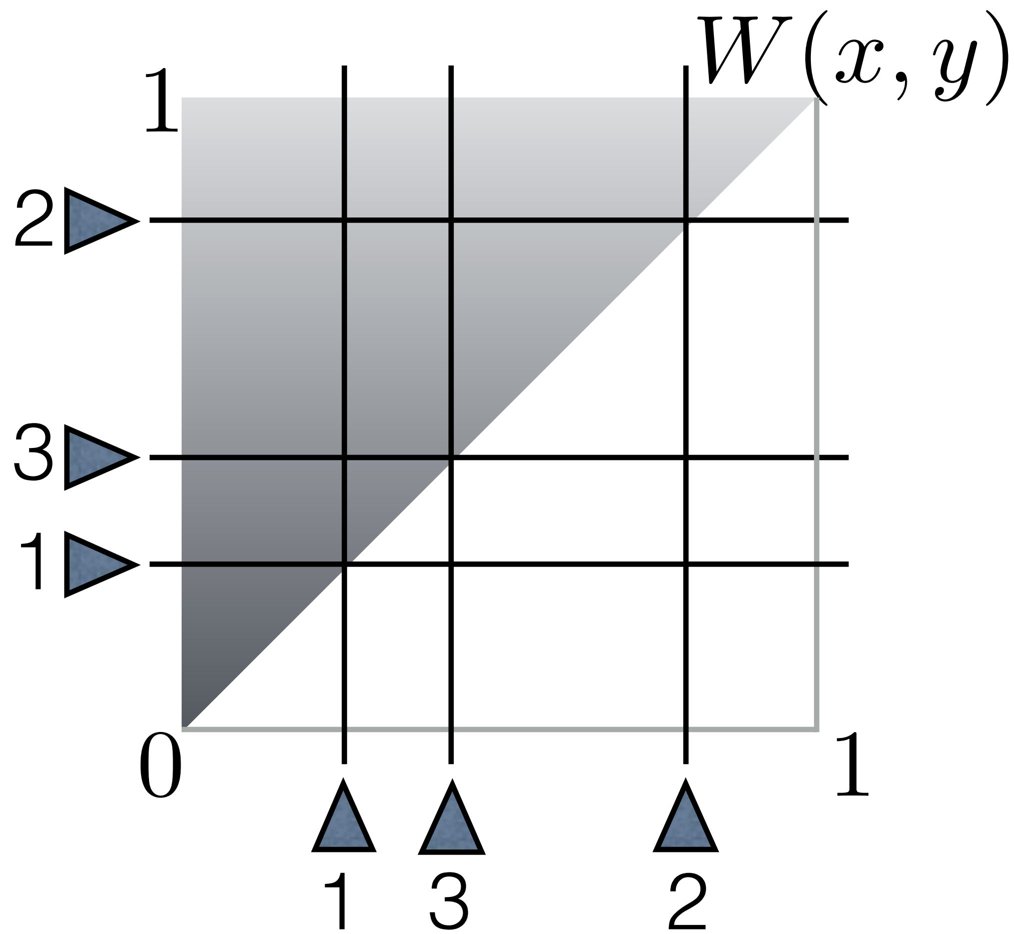


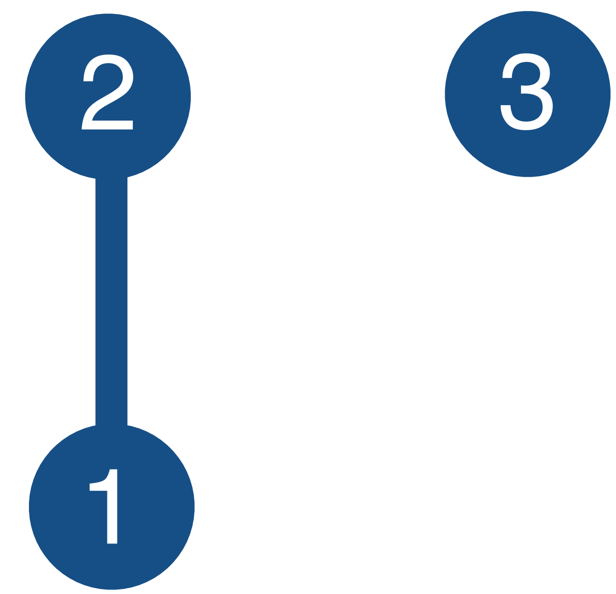
Every node-exchangeable graph has a graphon rep

$$\mathbb{E}[\#\text{edges}(G_n)]$$

[Hoover 1979, Aldous 1981, Orbanz, Roy 2015]

Aldous-Hoover



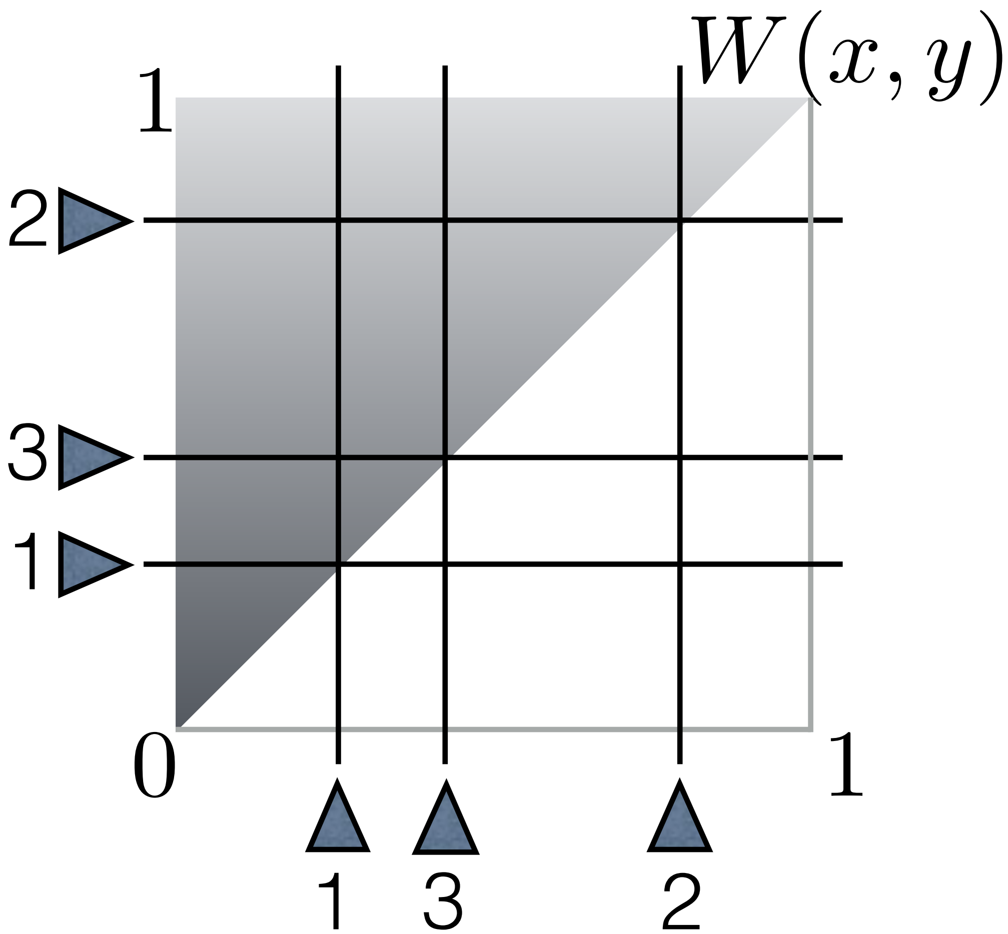


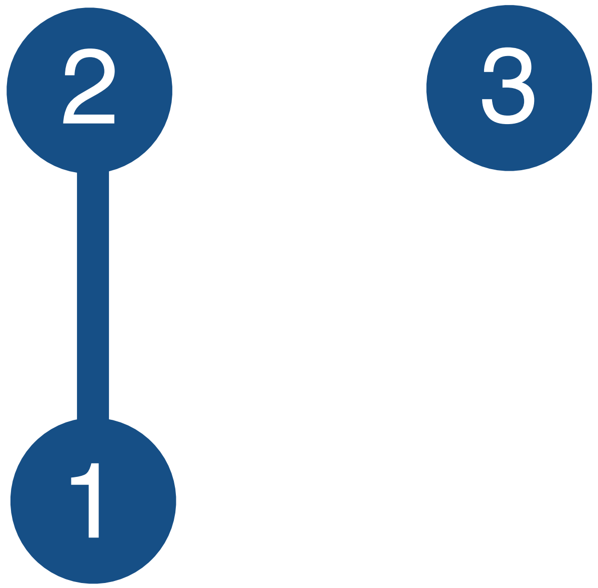
Every node-exchangeable graph has a graphon rep

$$\mathbb{E}[\#\text{edges}(G_n)] = \mathbb{E}\left[\binom{n}{2}\frac{1}{2}\int_0^1\int_0^1 W(x,y)\,dx\,dy\right]$$

[Hoover 1979, Aldous 1981, Orbanz, Roy 2015]

Aldous-Hoover





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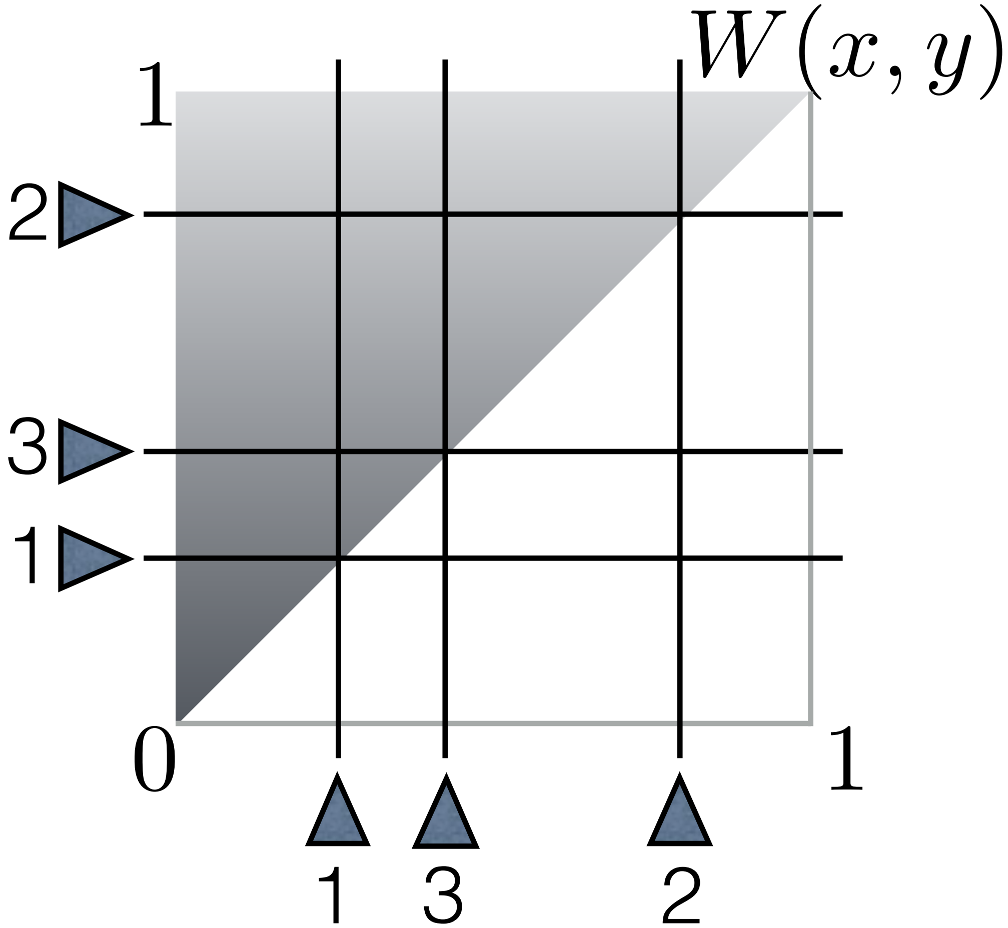
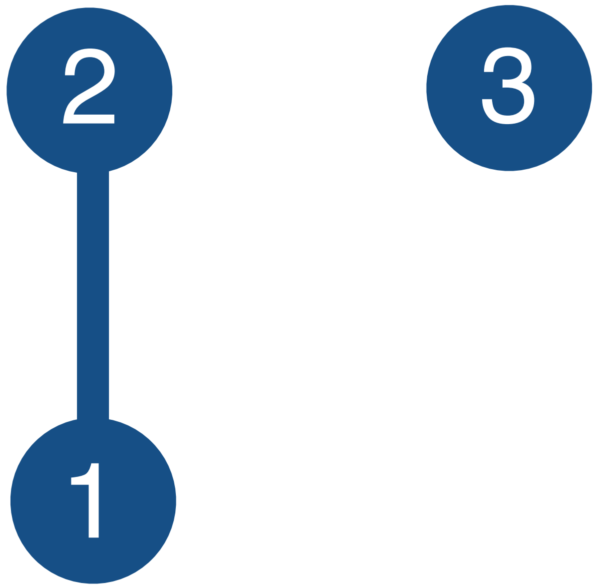
$$\sim cn^2$$

[Hoover 1979, Aldous 1981, Orbanz, Roy 2015]

Aldous-Hoover





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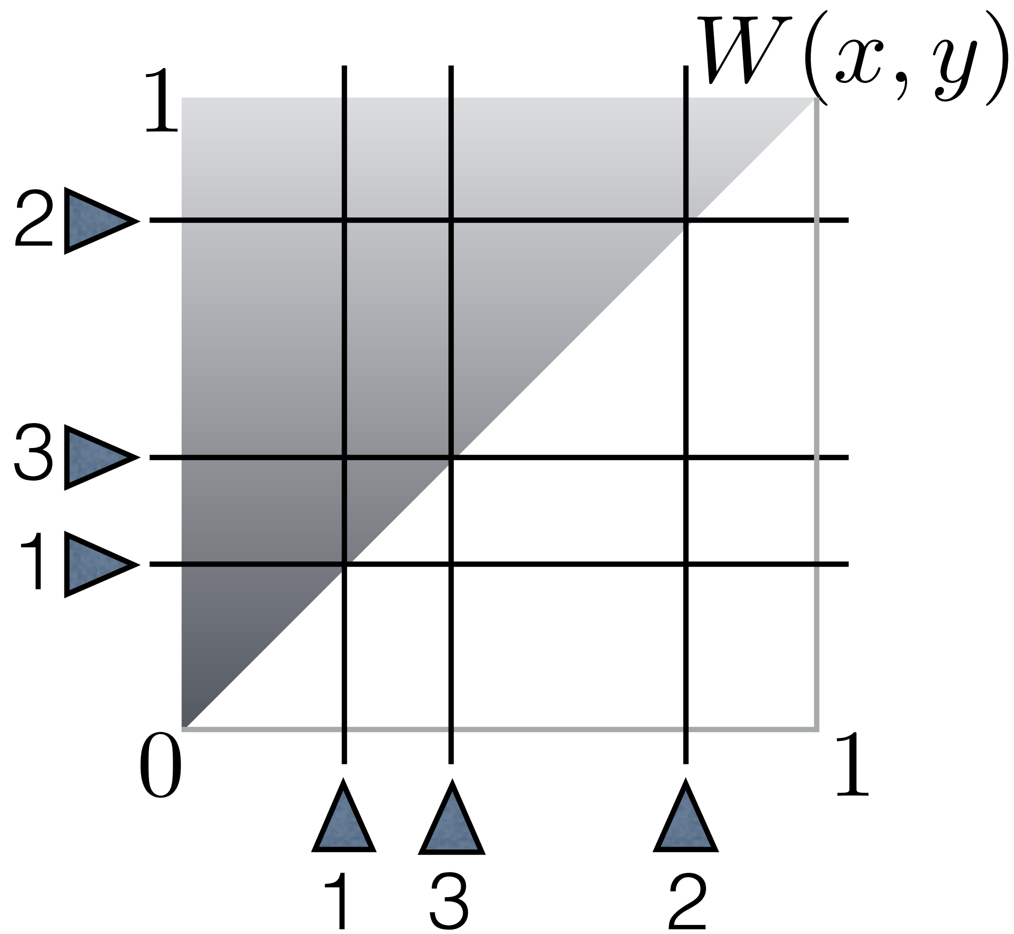
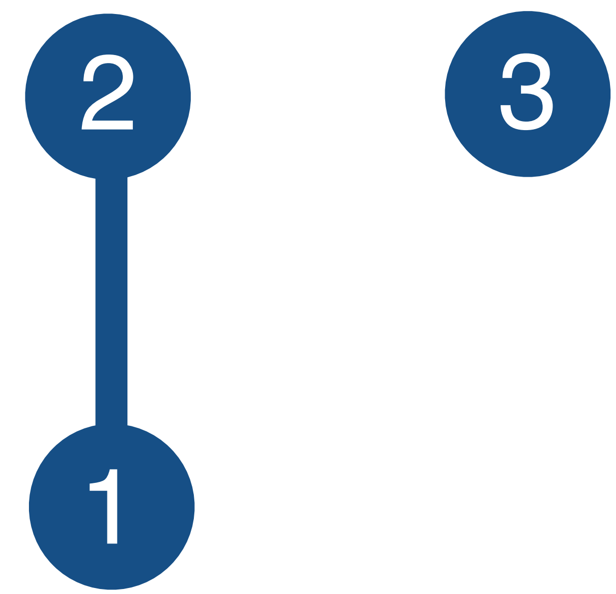
$$\sim cn^2 = c \cdot [\#\text{nodes}(G_n)]^2$$

[Hoover 1979, Aldous 1981, Orbanz, Roy 2015]

Aldous-Hoover





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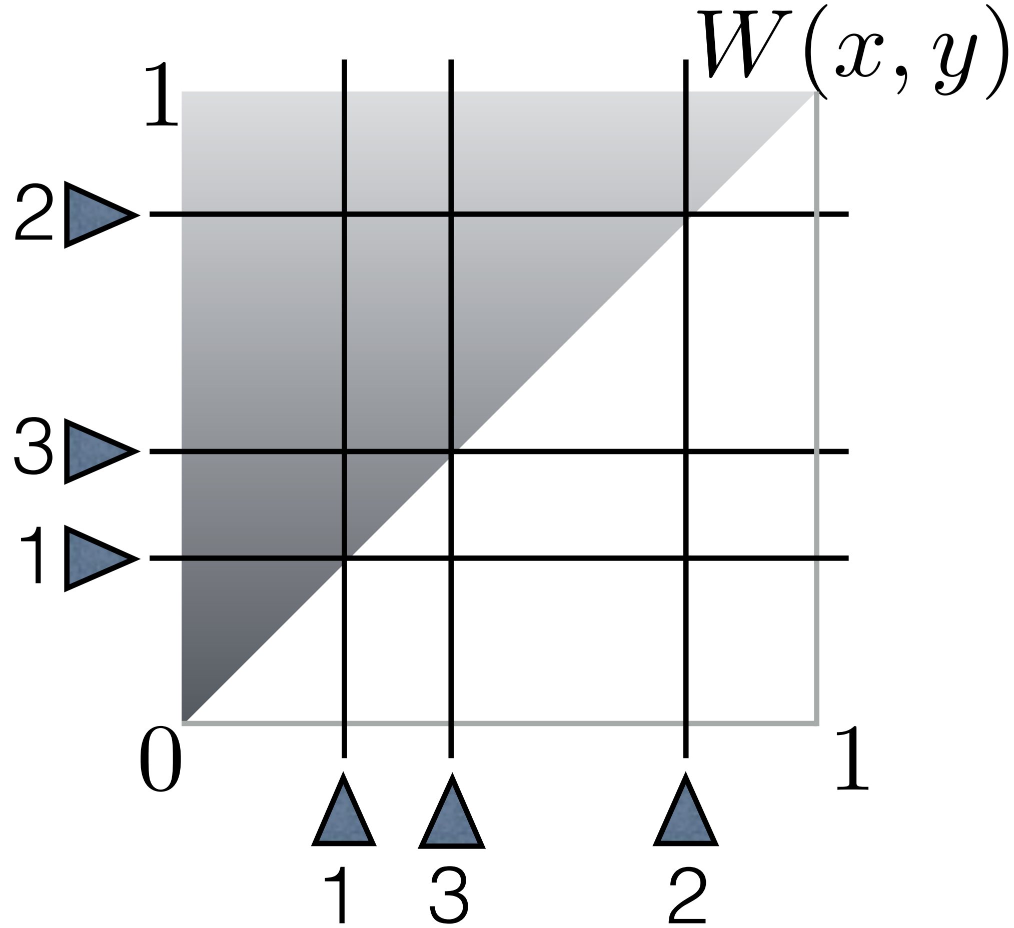
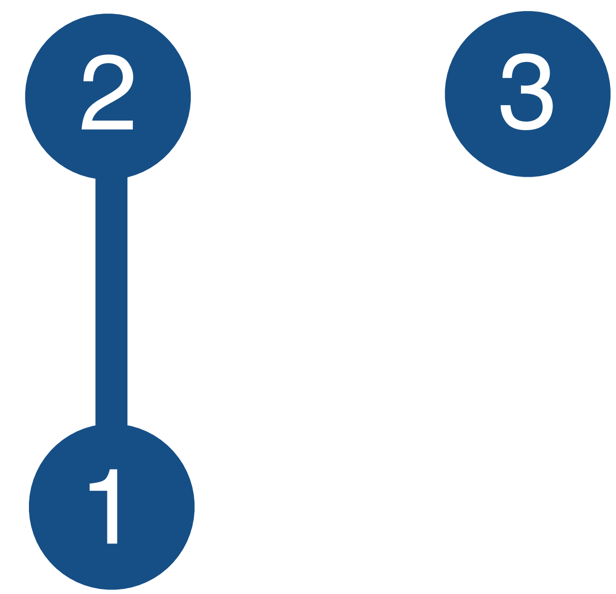
$$\sim cn^2 = c \cdot [\#\text{nodes}(G_n)]^2$$

Every node-exch graph sequence is dense (or empty)

[Hoover 1979, Aldous 1981, Orbanz, Roy 2015]

Aldous-Hoover





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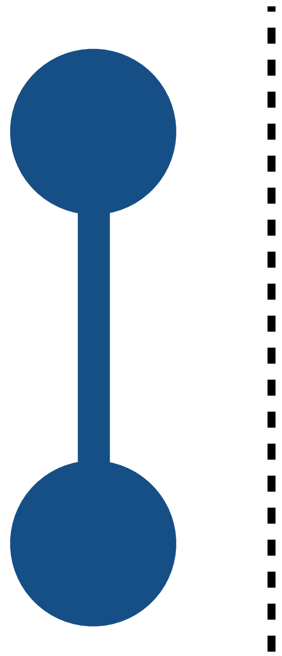
Every node-exch graph sequence is dense (or empty)

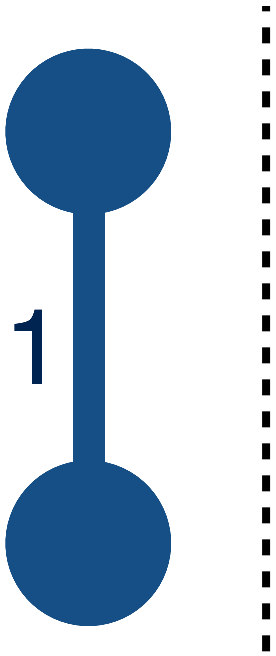
[Hoover 1979, Aldous 1981, Orbanz, Roy 2015; Caron, Fox 2014; Veitch, Roy 2015; Borgs, Chayes, Cohn, Holden 2016; Broderick, Cai 2015; Crane, Dempsey 2015; Crane, Dempsey 2016; Cai, Campbell, Broderick 2016]

A New Way: Edges



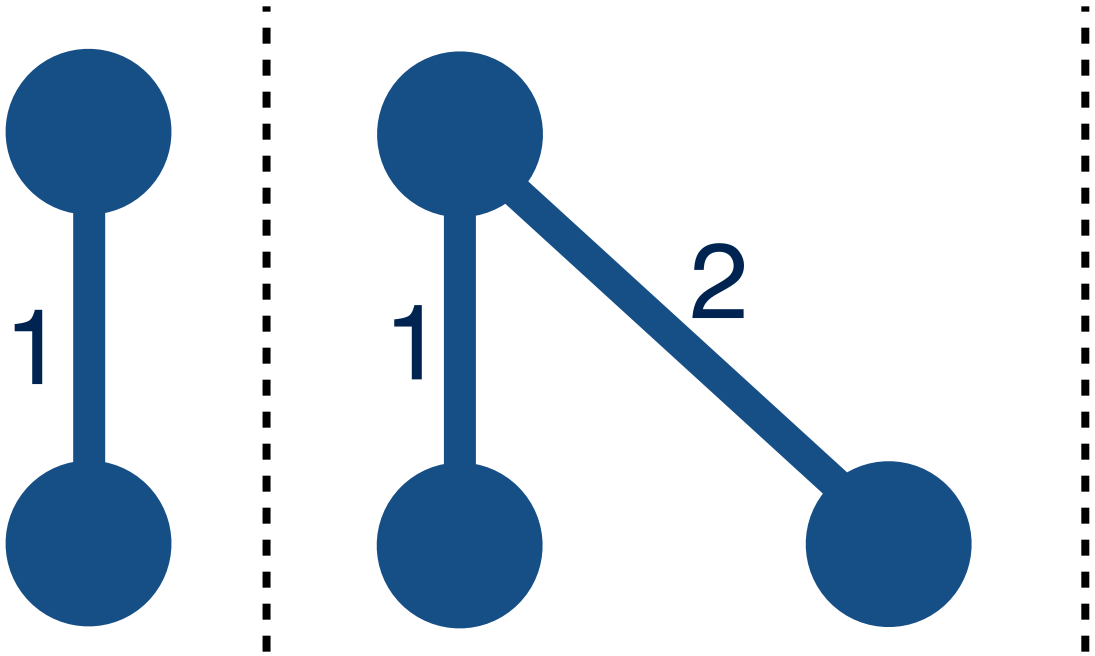
G_1

A New Way: Edges



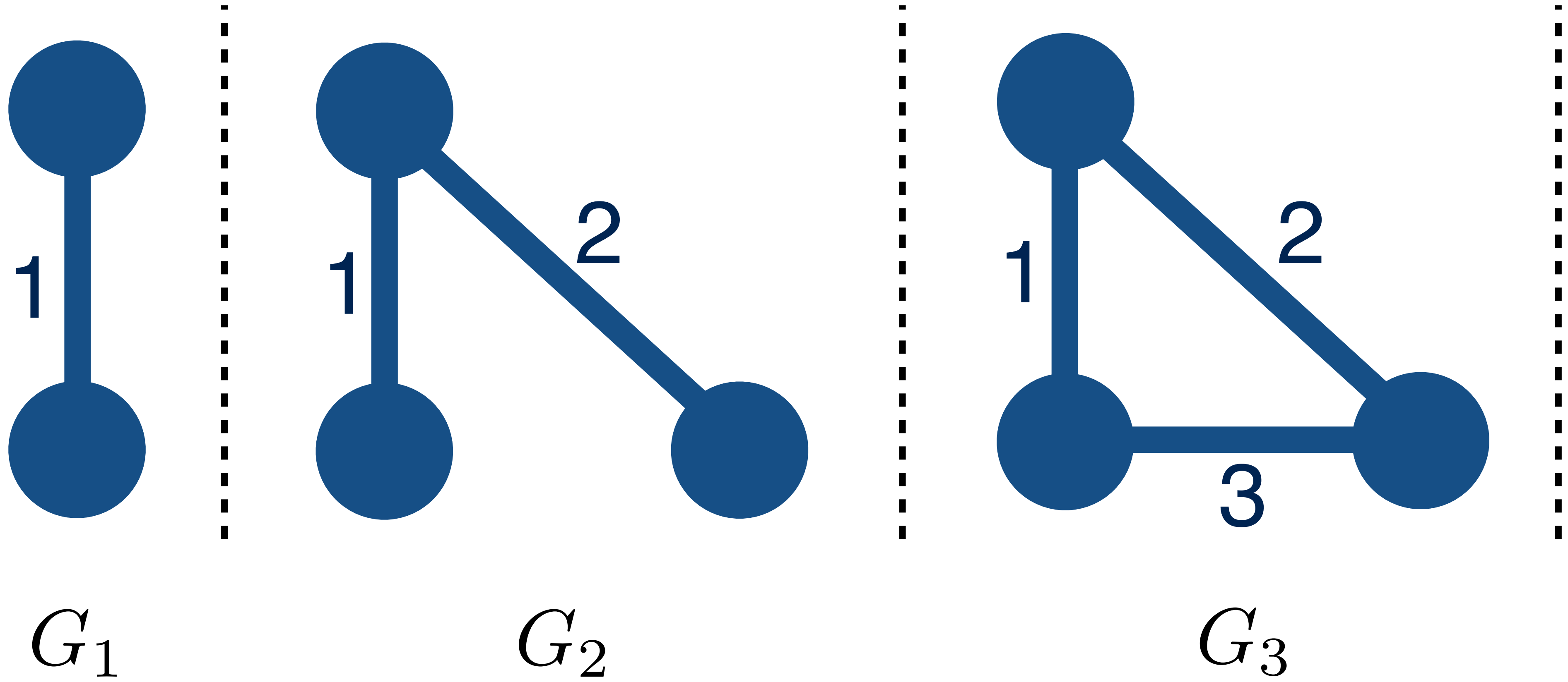
G_1

A New Way: Edges

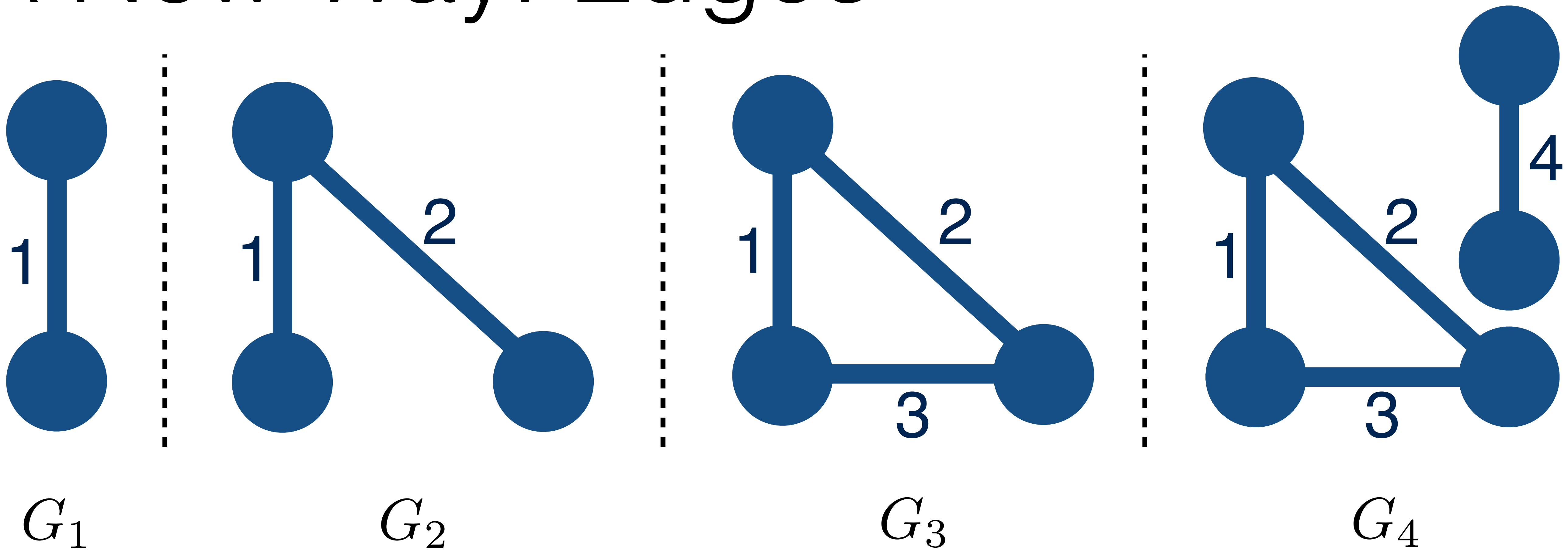


G_1 G_2

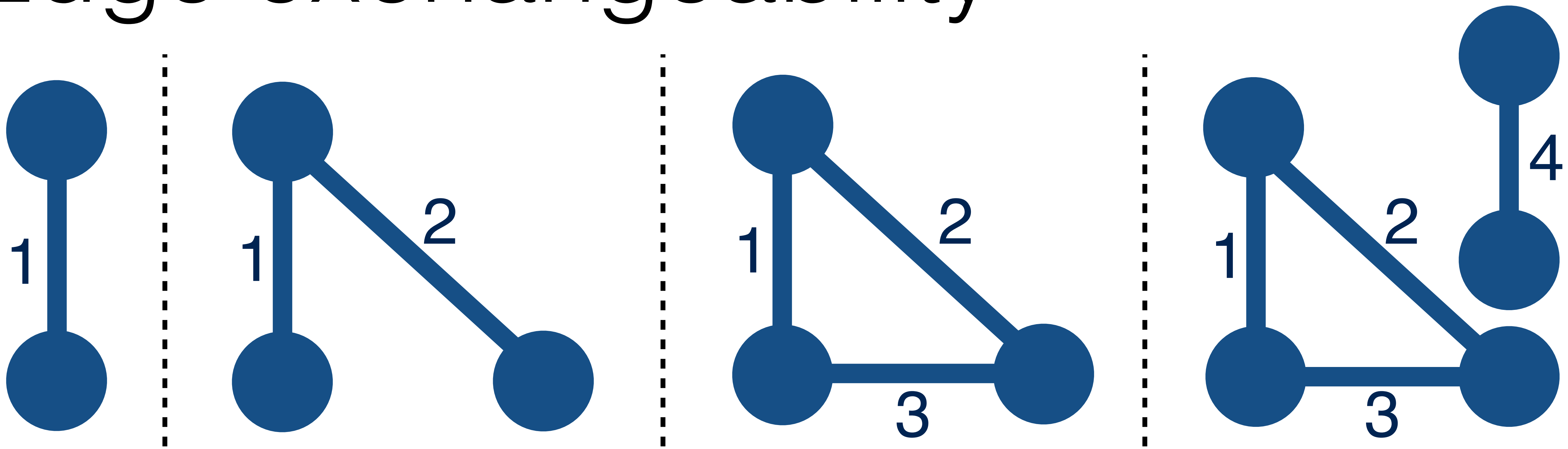
A New Way: Edges



A New Way: Edges

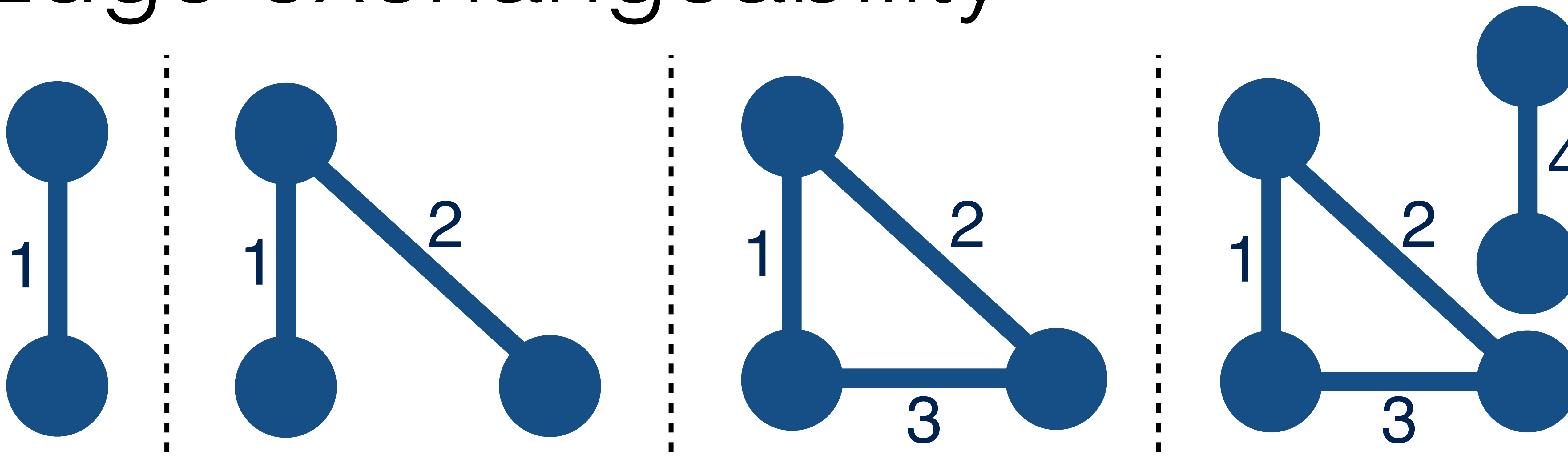


Edge exchangeability

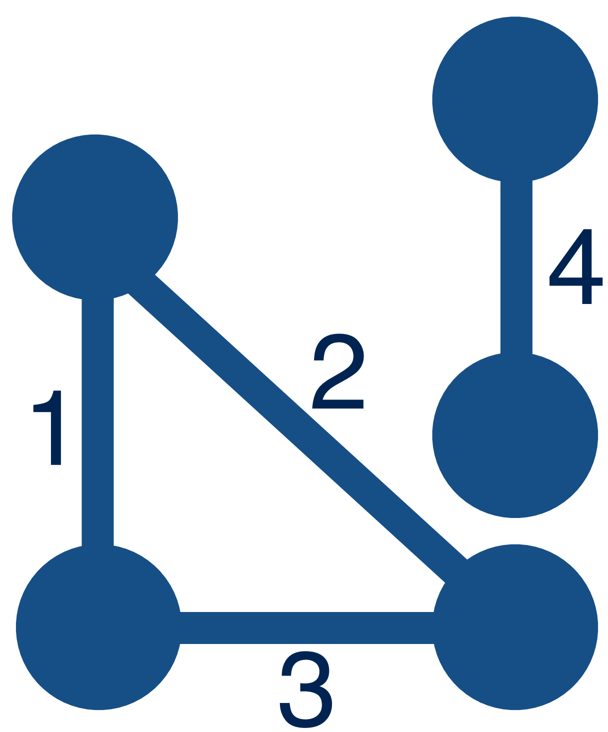


G_1 G_2 G_3 G_4

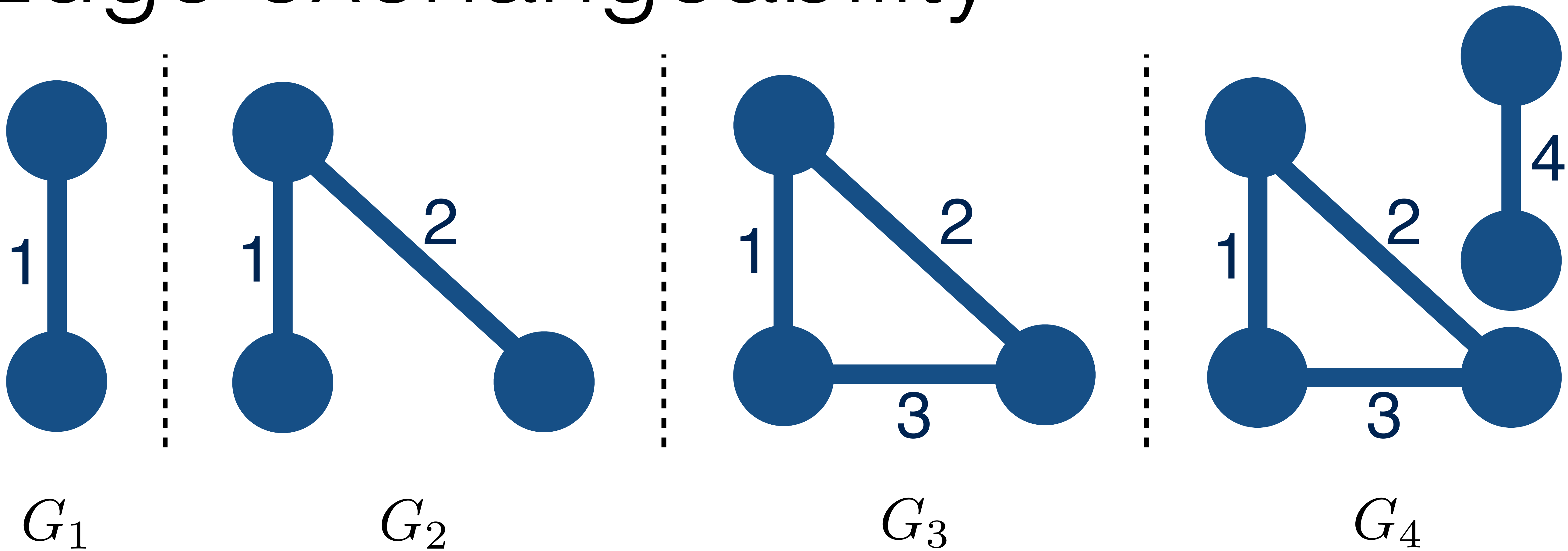
Edge exchangeability

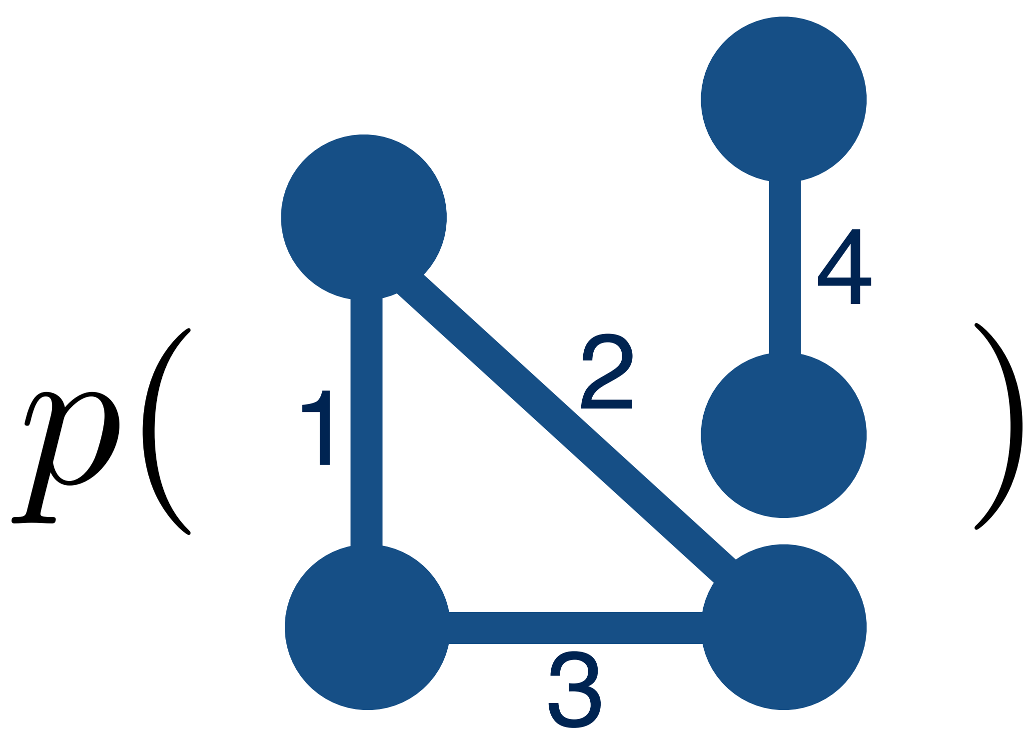


G_1 G_2 G_3 G_4

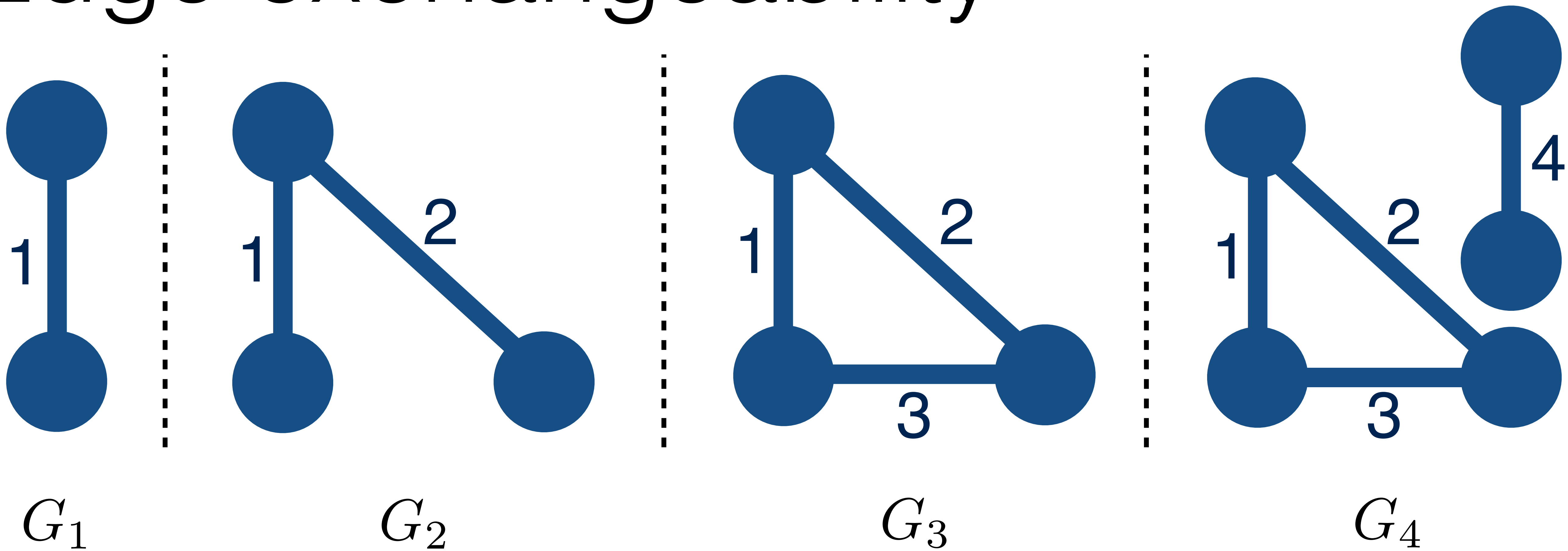


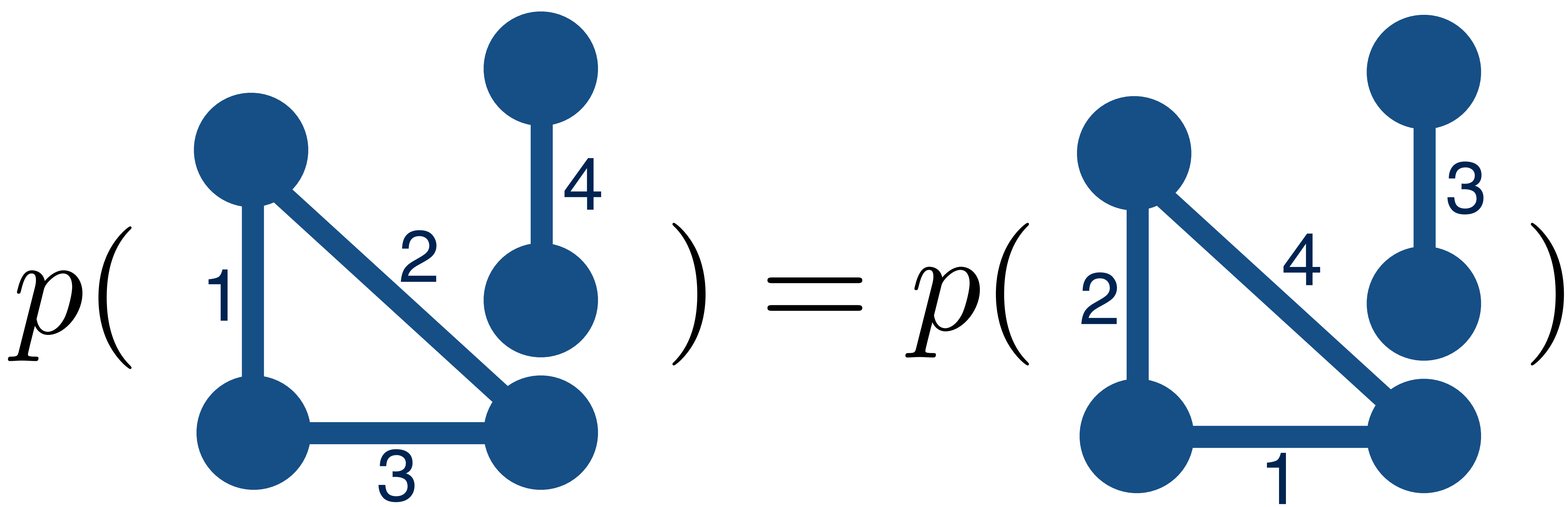
Edge exchangeability



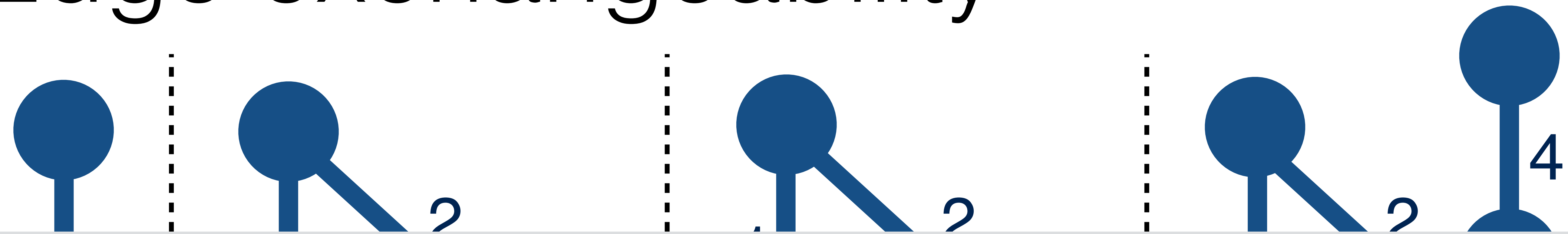


Edge exchangeability



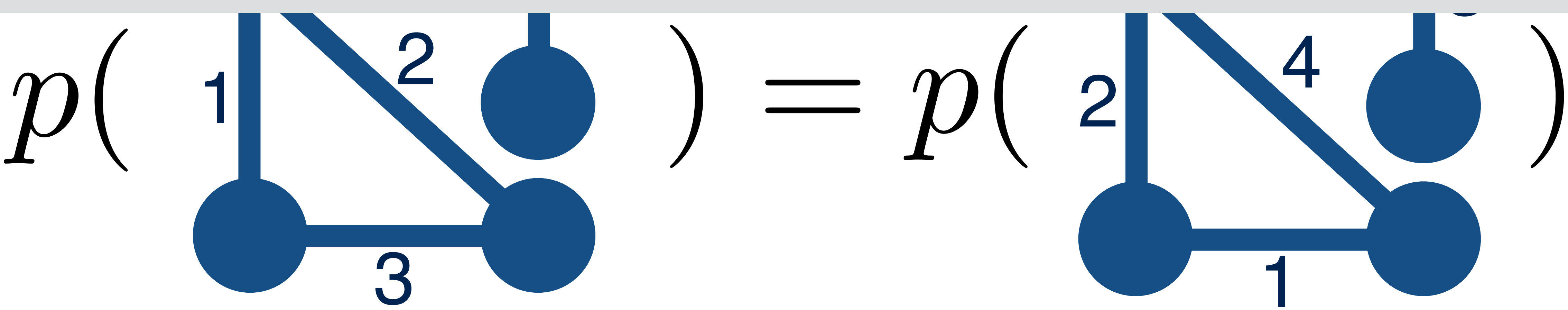


Edge exchangeability



Thm. A wide range of edge-exchangeable graph sequences are sparse

Thm. A paintbox-style characterization for edge-exchangeable graph sequences

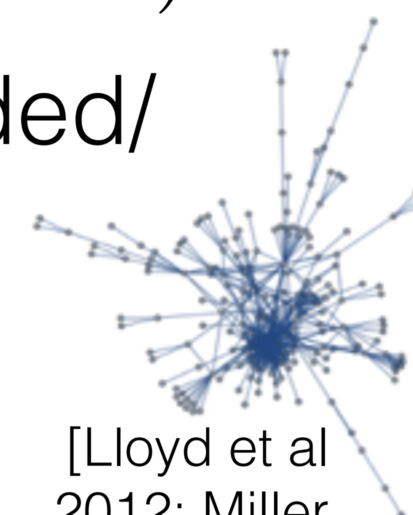


Nonparametric Bayes

- Bayesian methods that are not parametric
- Bayesian

$$\mathbb{P}(\text{parameters}|\text{data}) \propto \mathbb{P}(\text{data}|\text{parameters})\mathbb{P}(\text{parameters})$$

- Not parametric (i.e. not finite parameter, unbounded/growing/infinite number of parameters)



[Lloyd et al 2012; Miller et al, 2010]

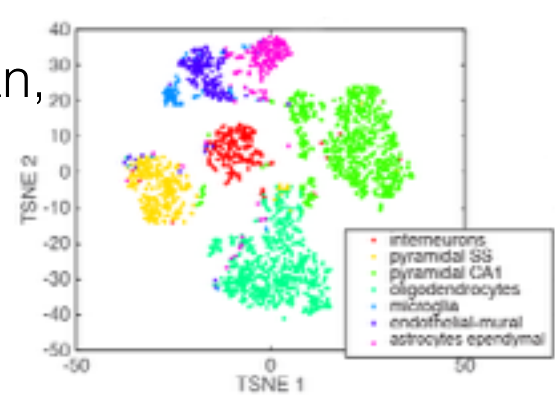
[wikipedia.org]

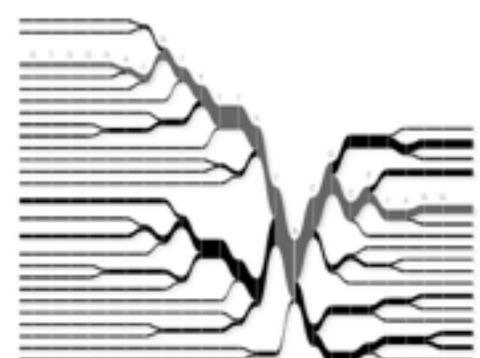
[Ed Bowlby, NOAA]

[Fox, et al 2014]

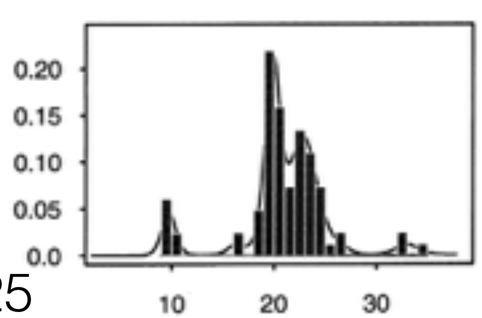
[Sudderth, Jordan 2009]

[Prabhakaran, Azizi, Carr, Pe'er 2016]



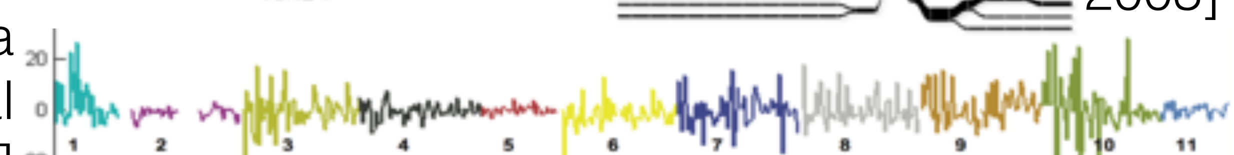


[Ewens, 1972; Hartl, Clark 2003]



[Escobar, West 1995; Ghosal, et al 1999]

[Saria et al 2010]



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