# Nonparametric Bayesian Methods: Models, Algorithms, and Applications

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ITT Career Development Assistant Professor
Electrical Engineering & Computer Science
MIT

http://www.tamarabroderick.com/tutorials.html

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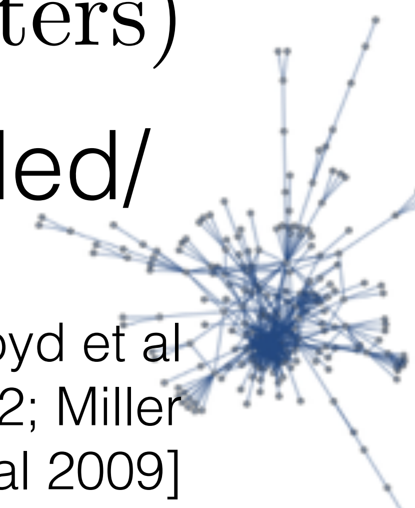

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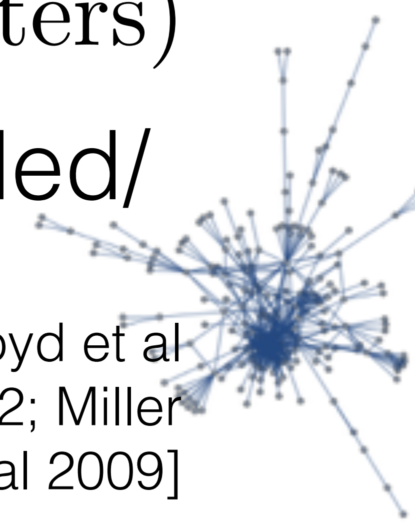

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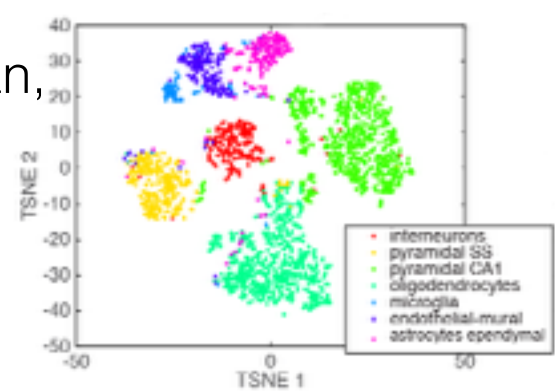

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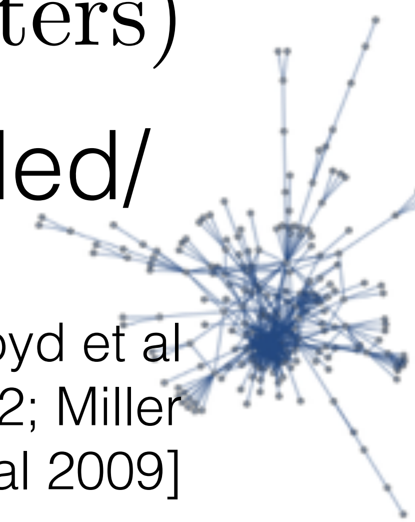

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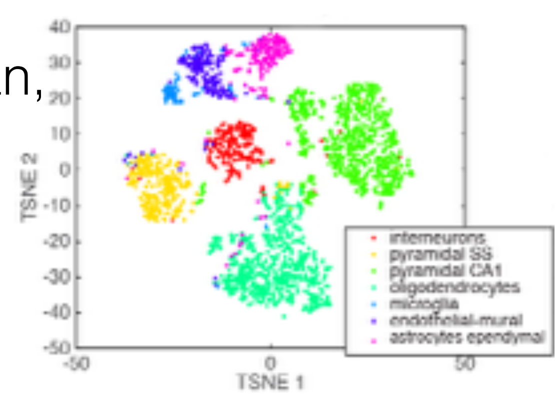


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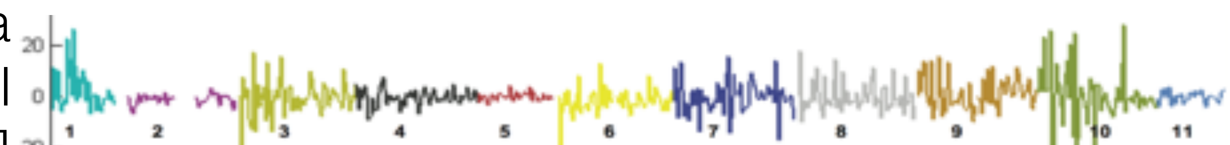

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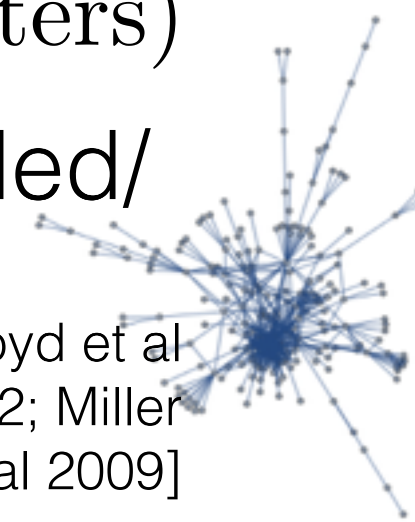

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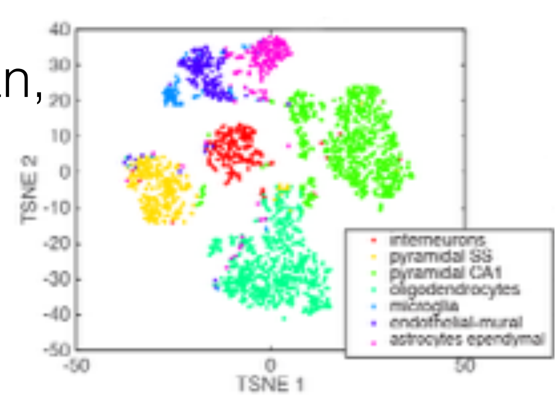


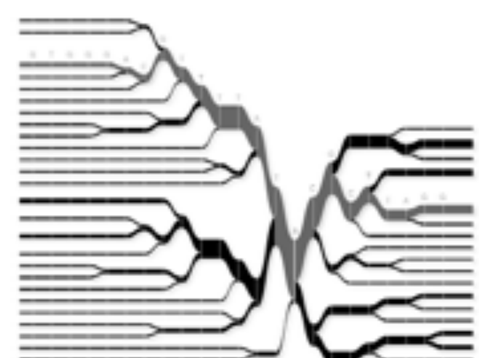

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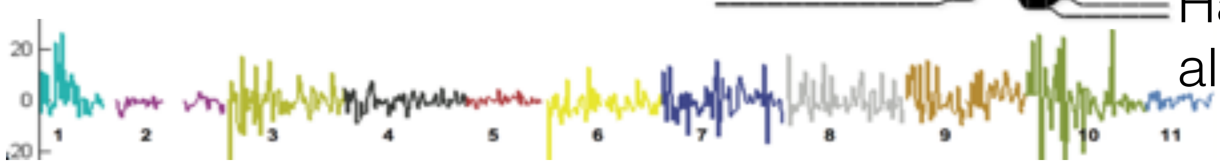

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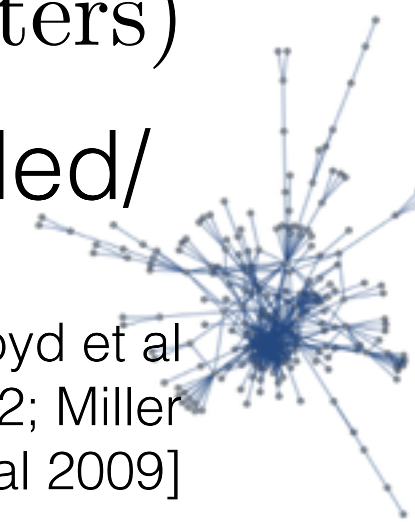

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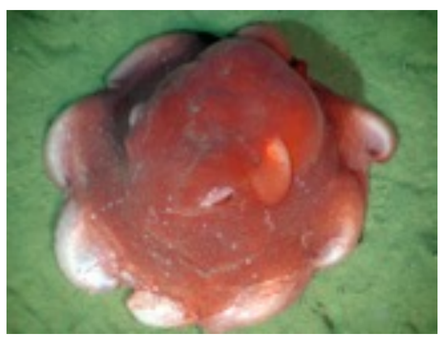

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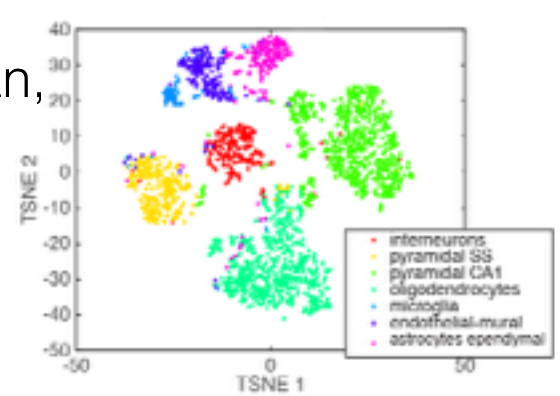


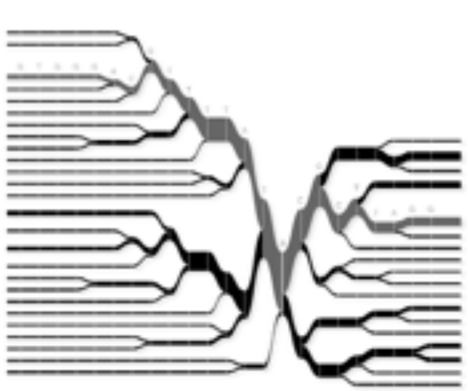

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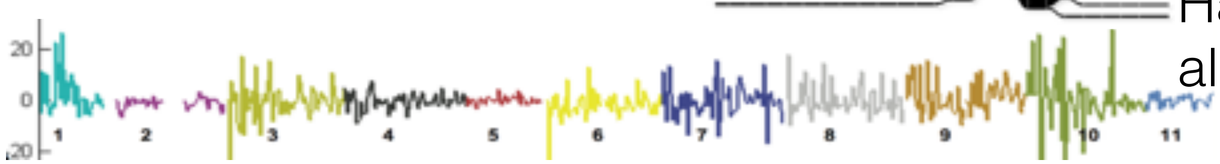

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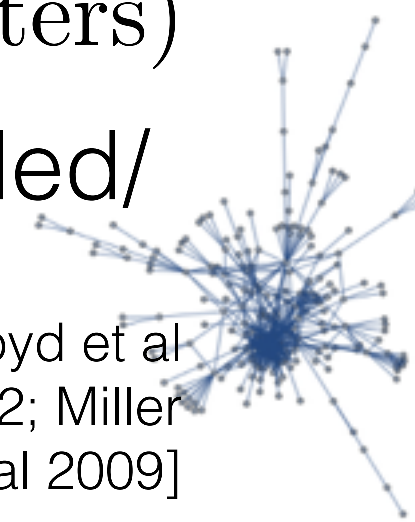

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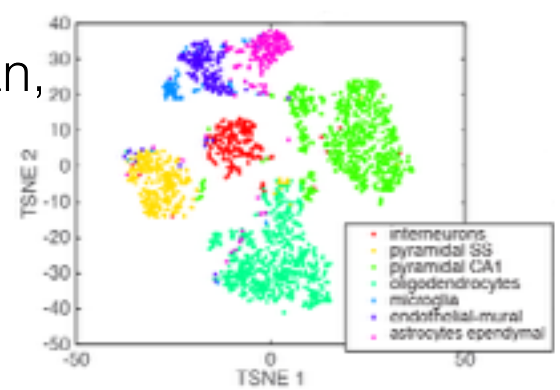


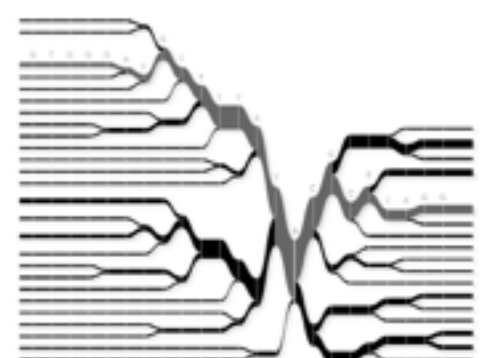

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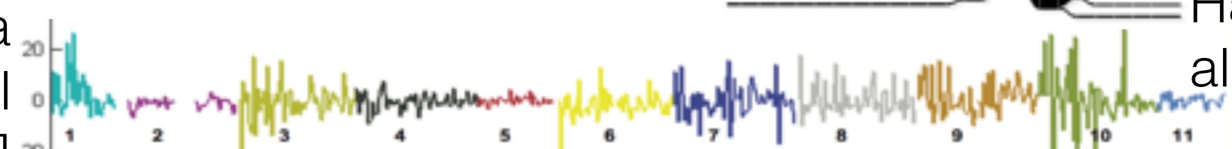

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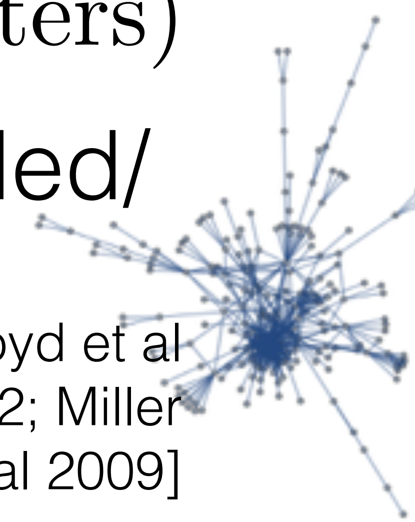

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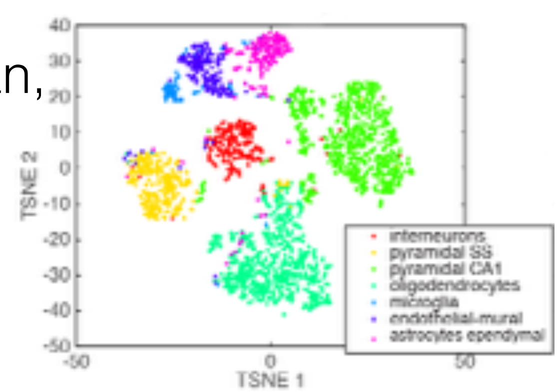


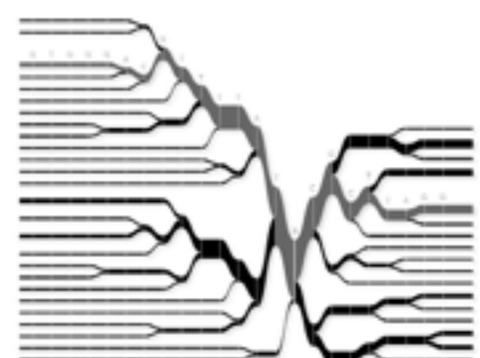

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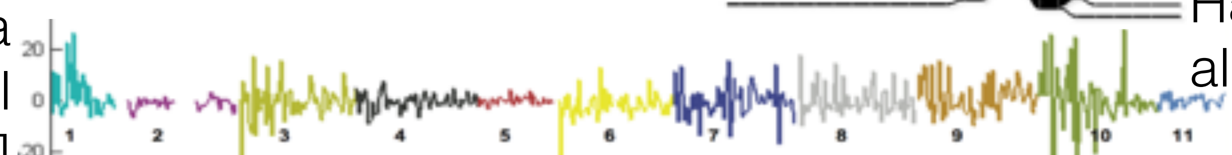

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    - Why is NPBayes challenging but practical?

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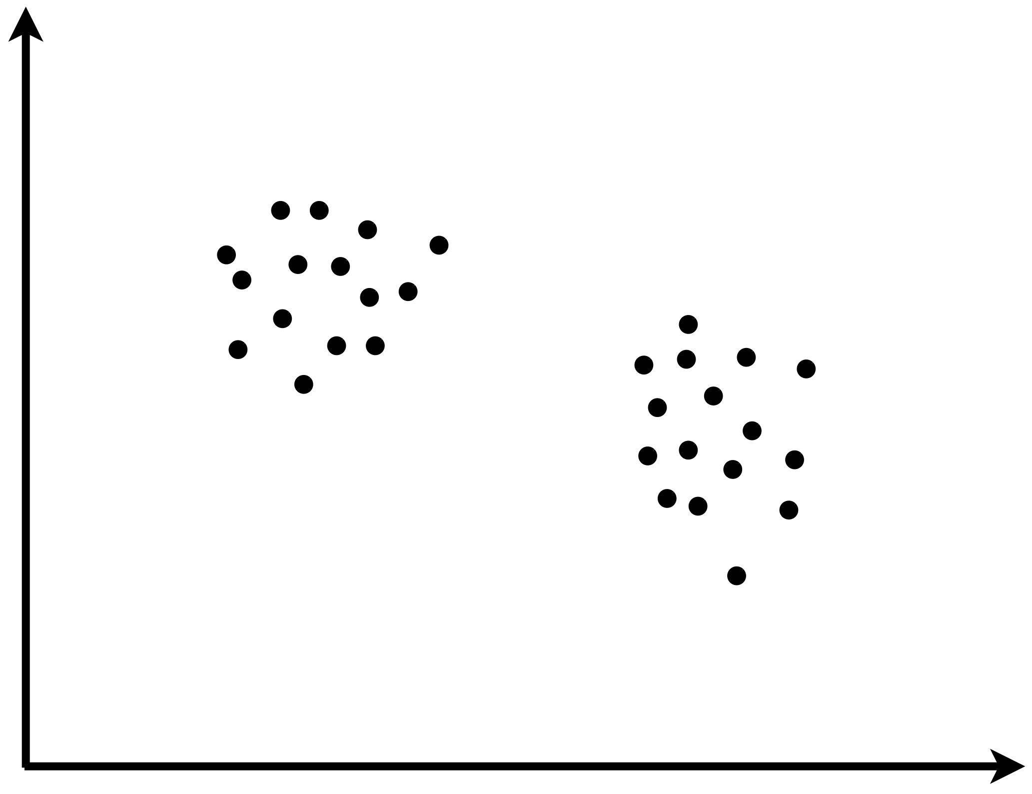

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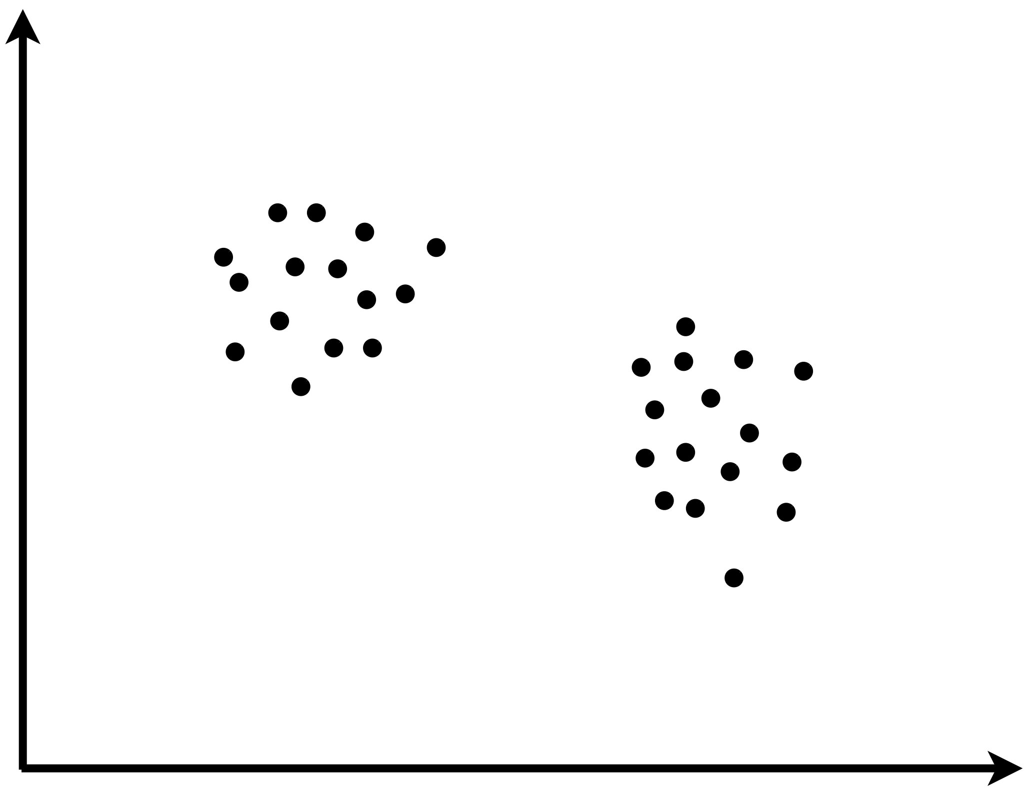

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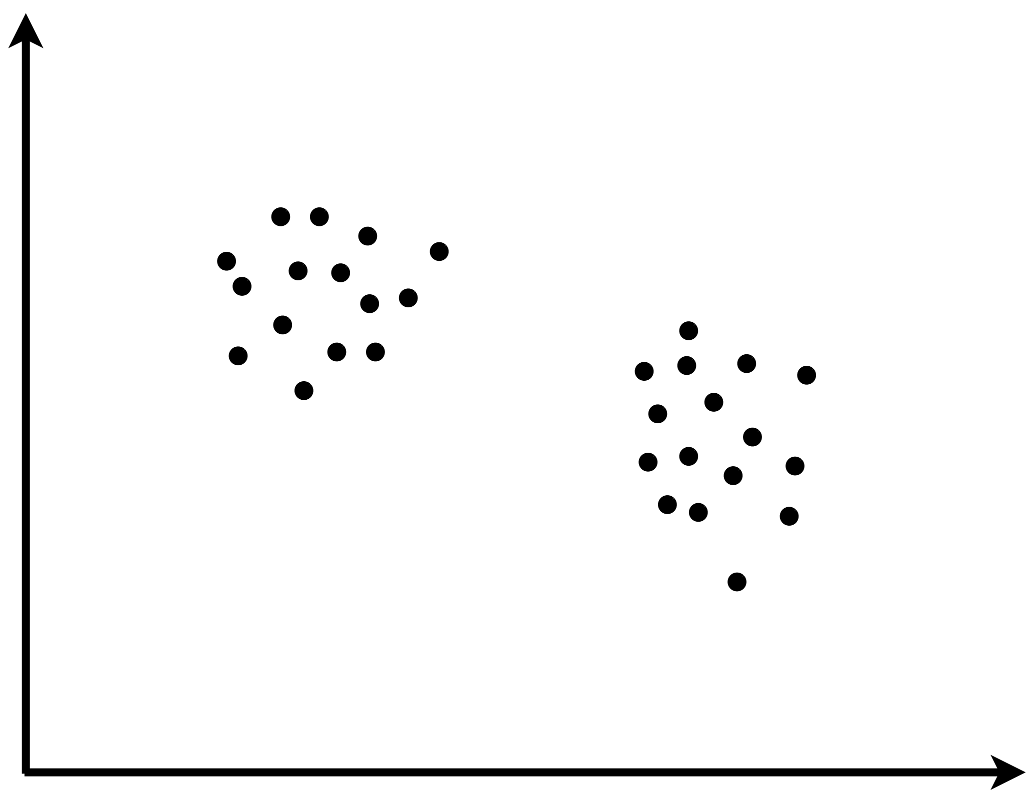

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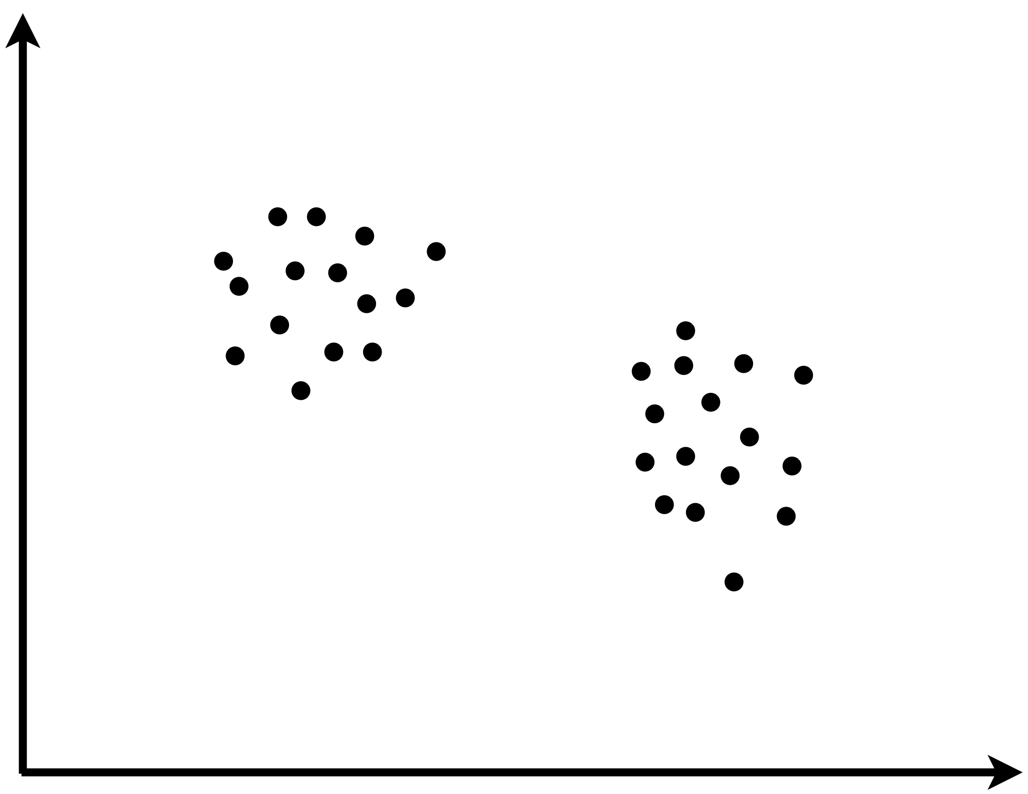

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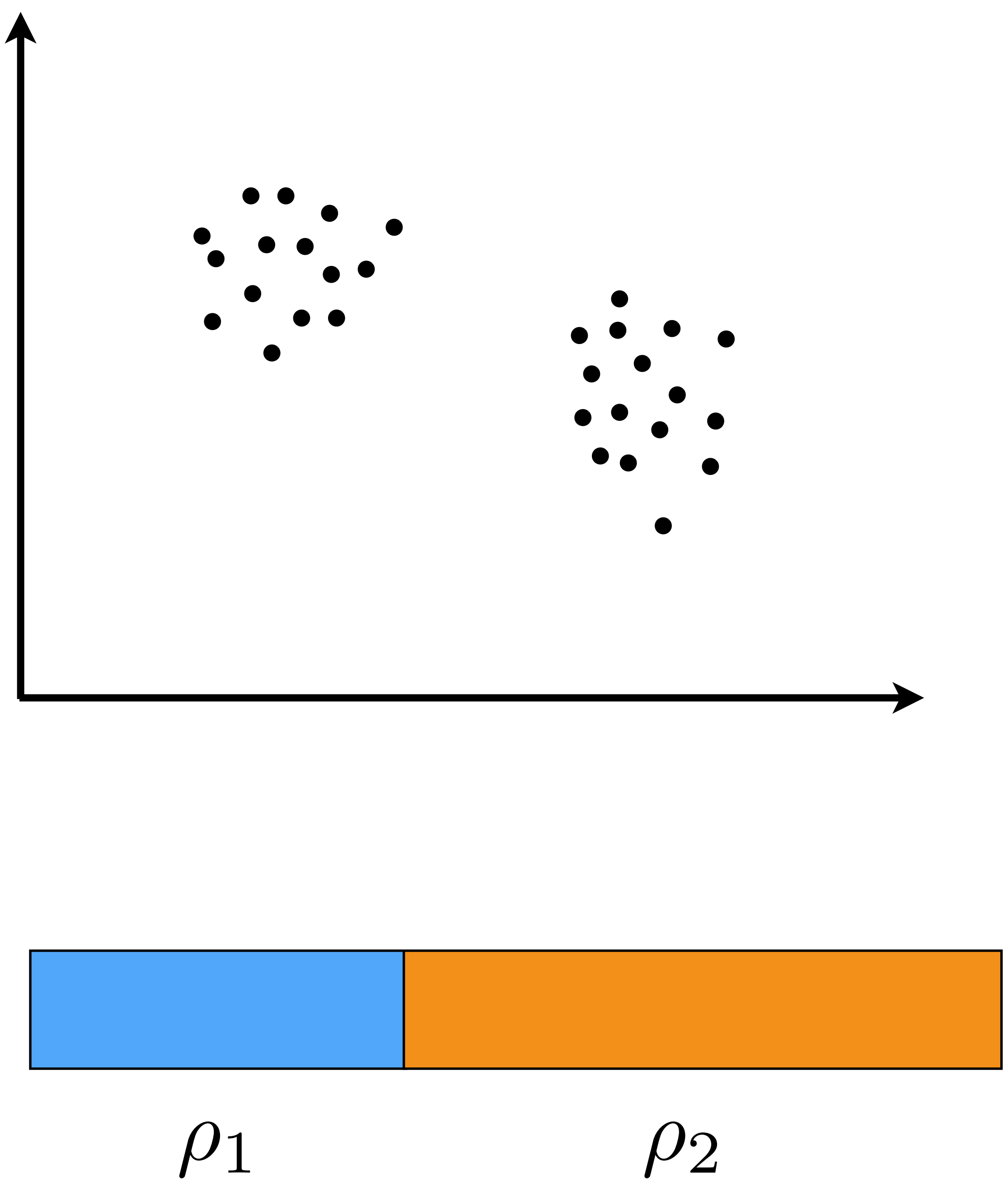

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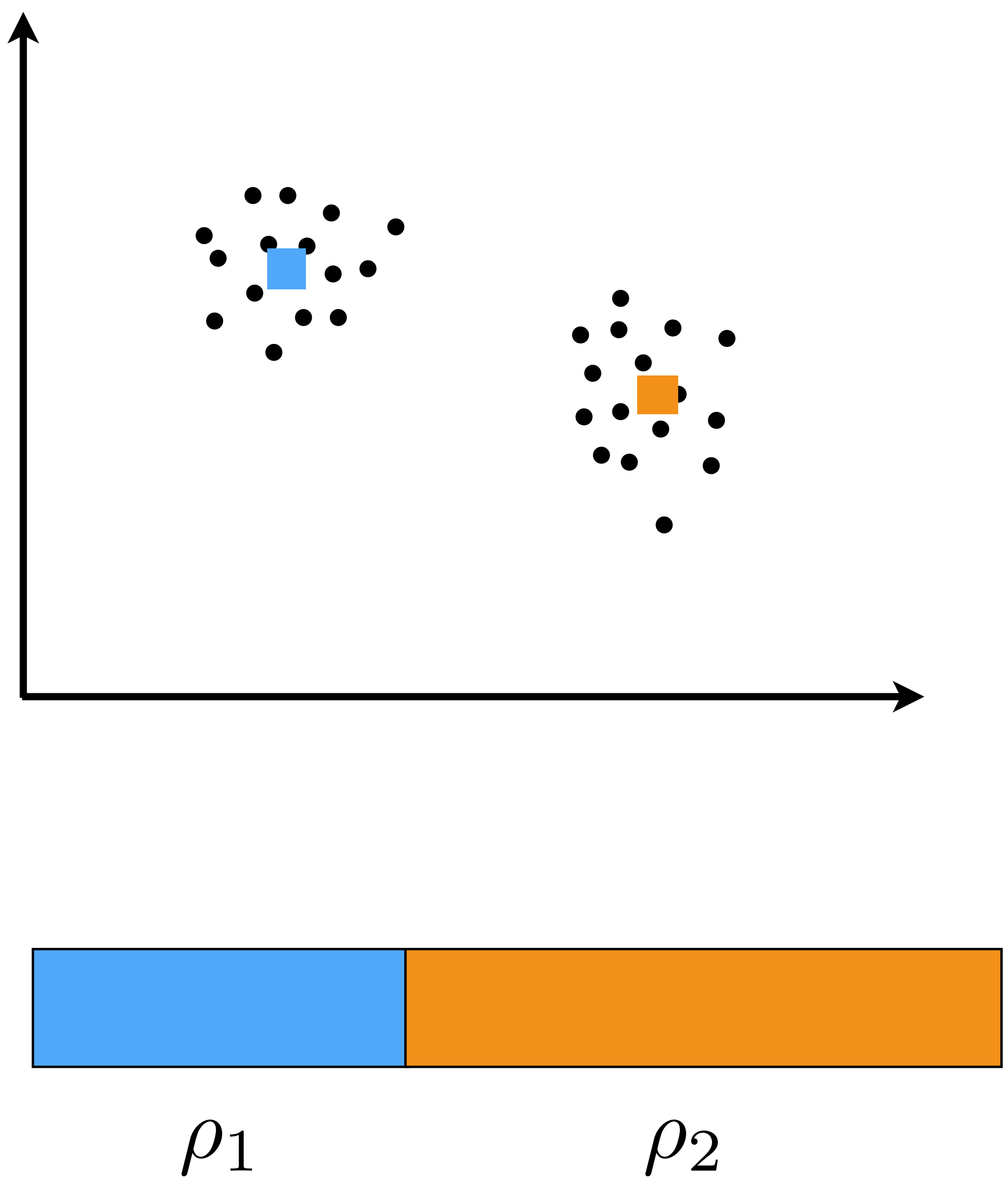


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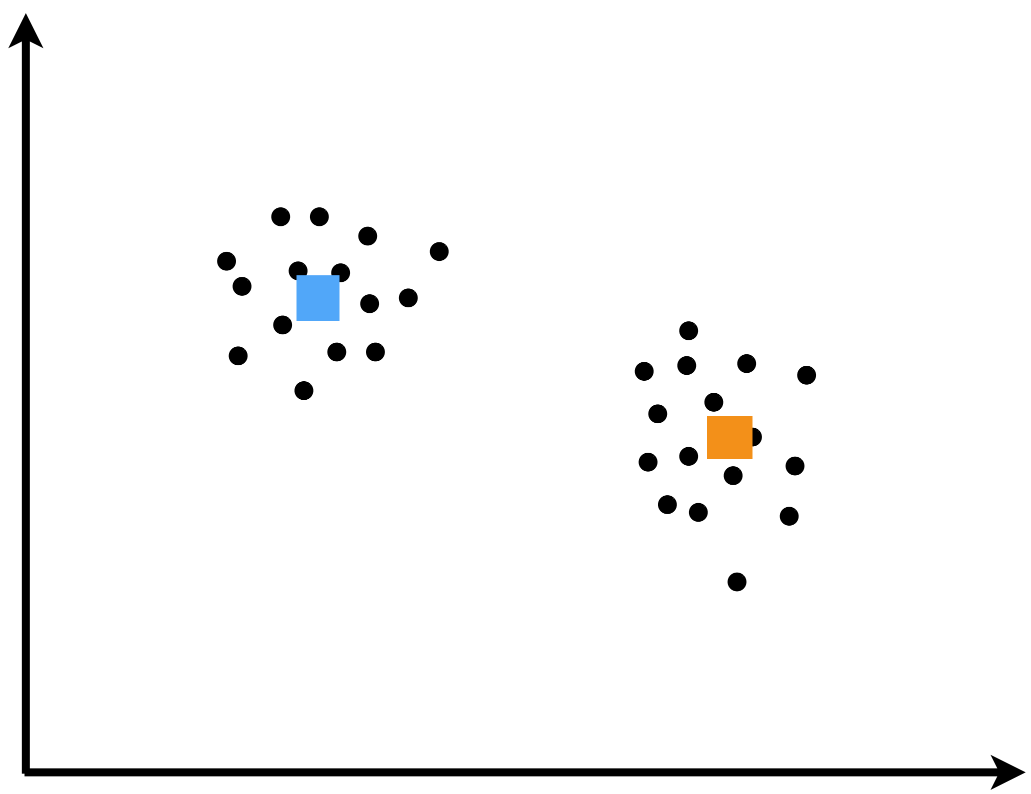

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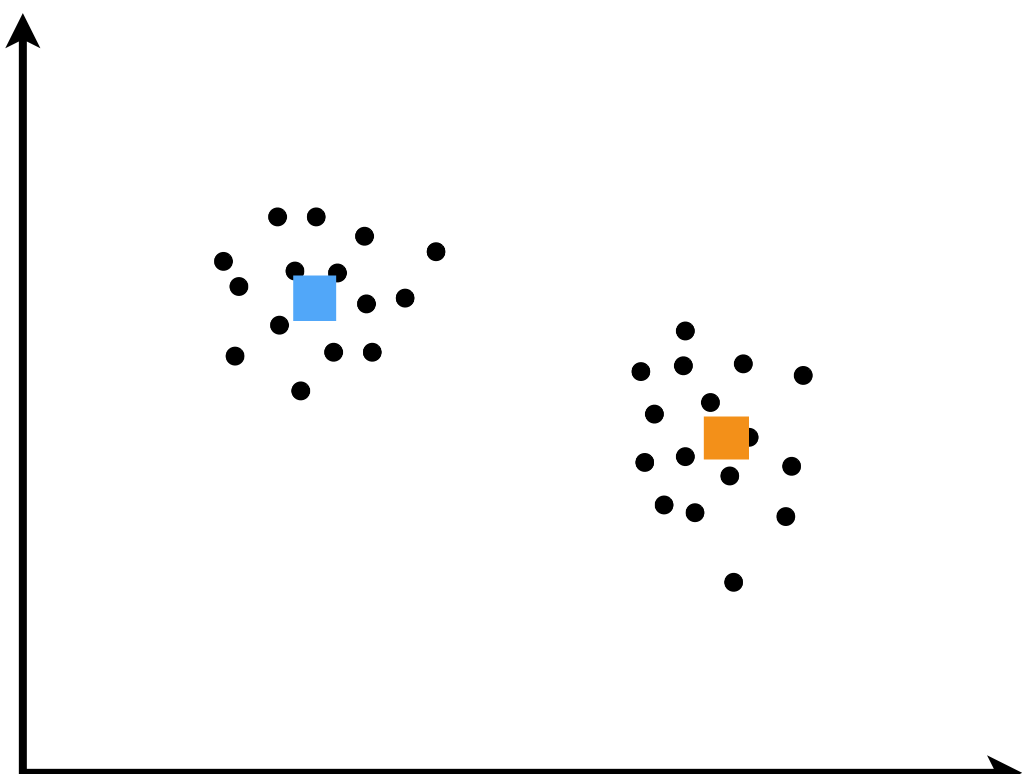

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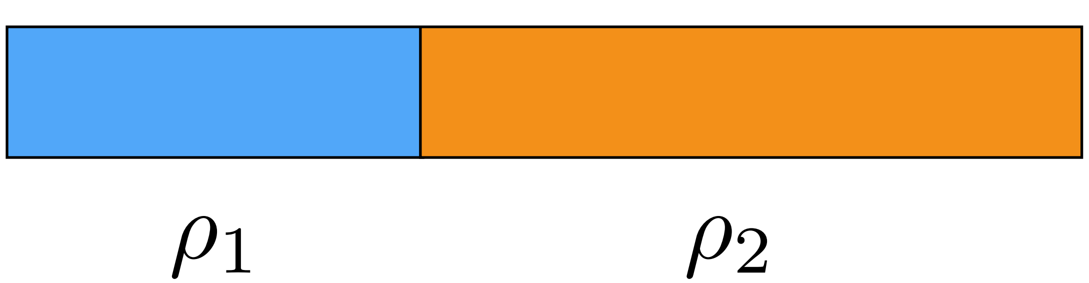

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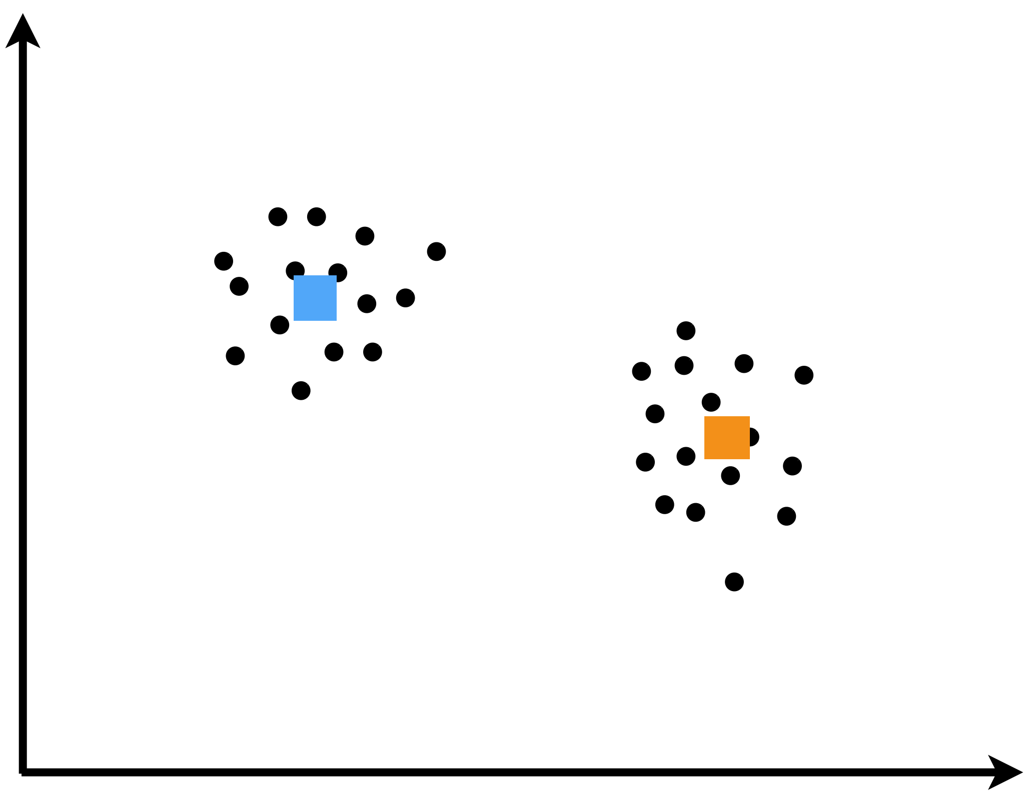

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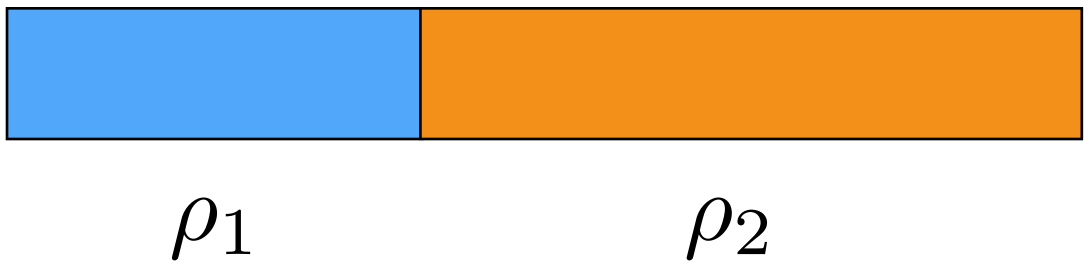

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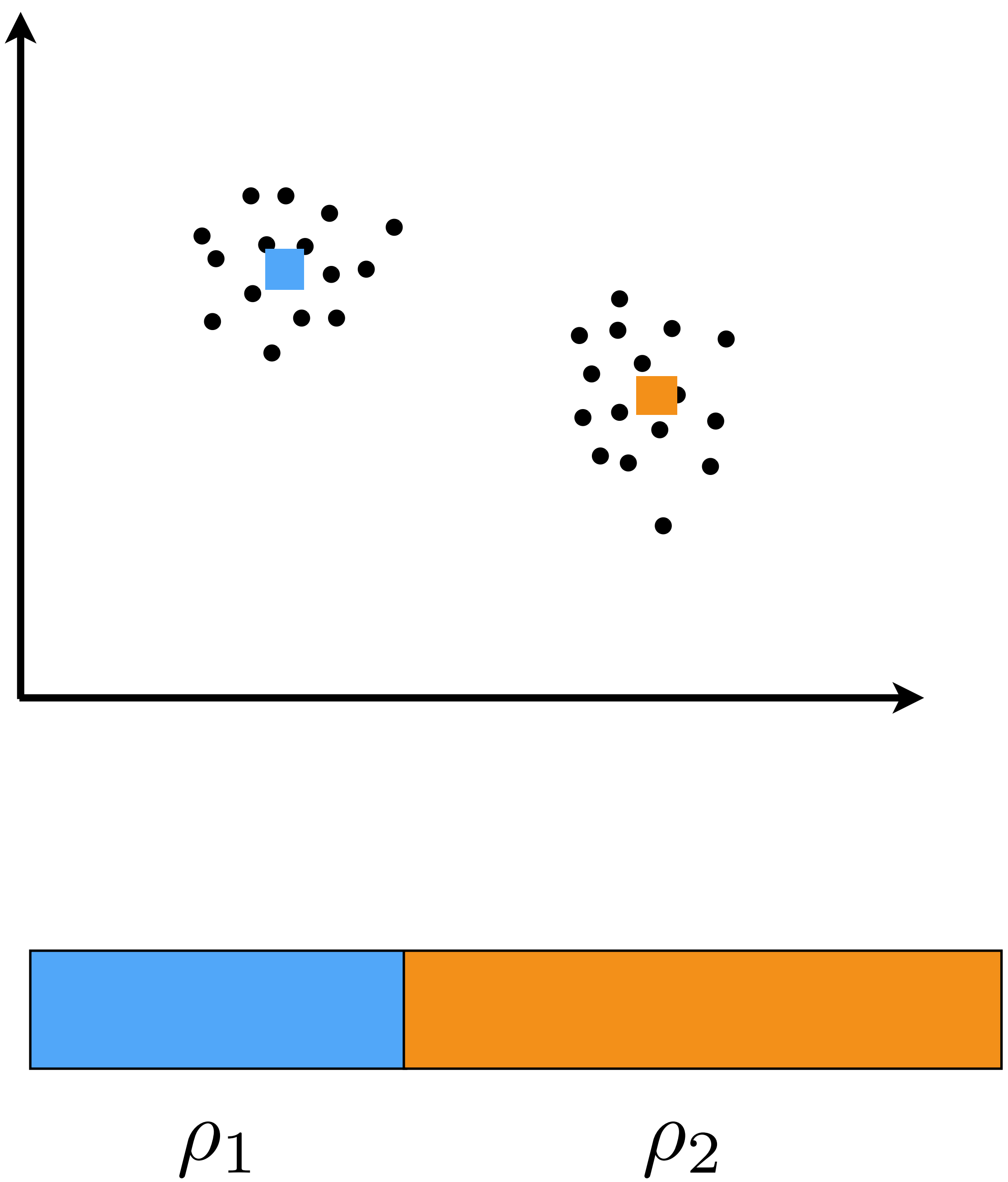


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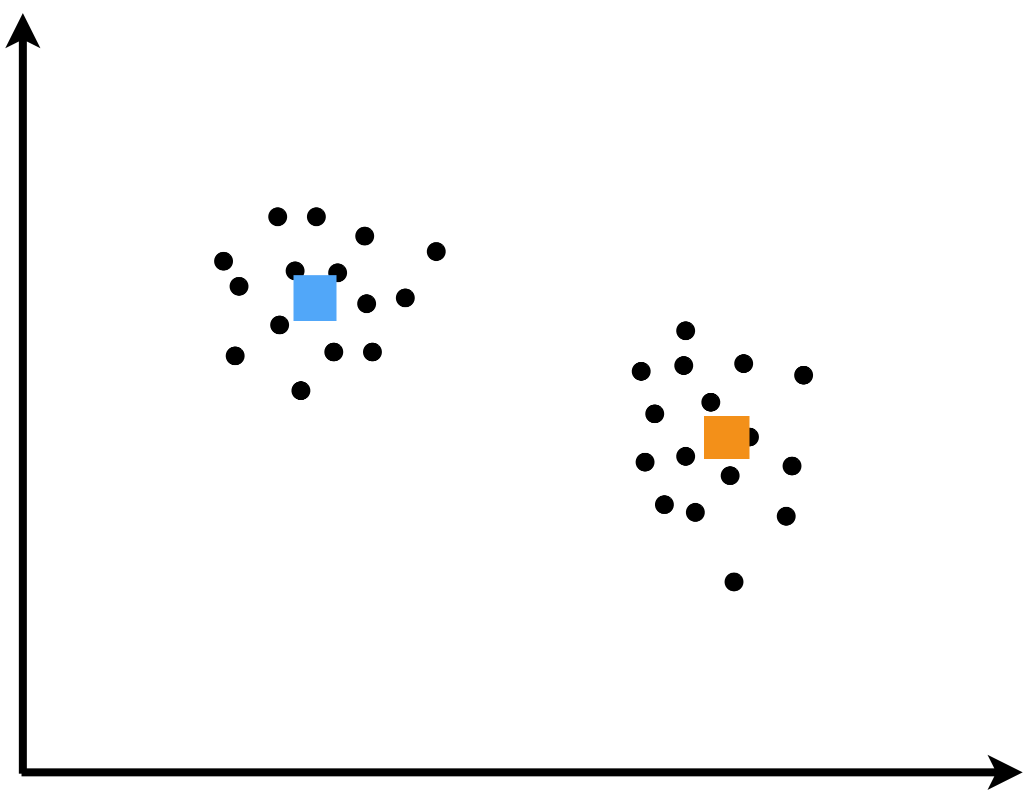

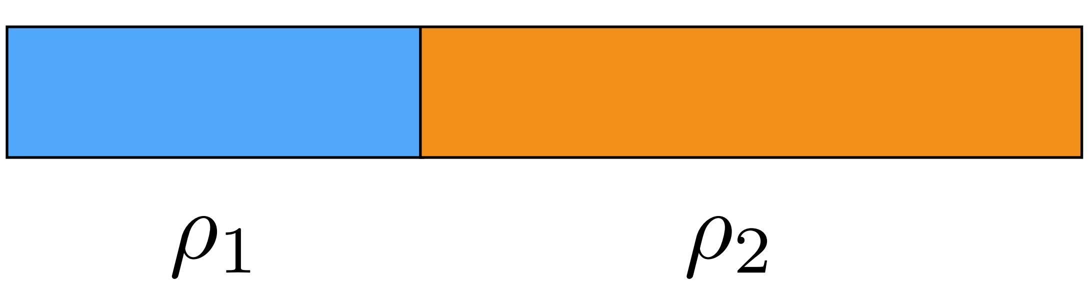


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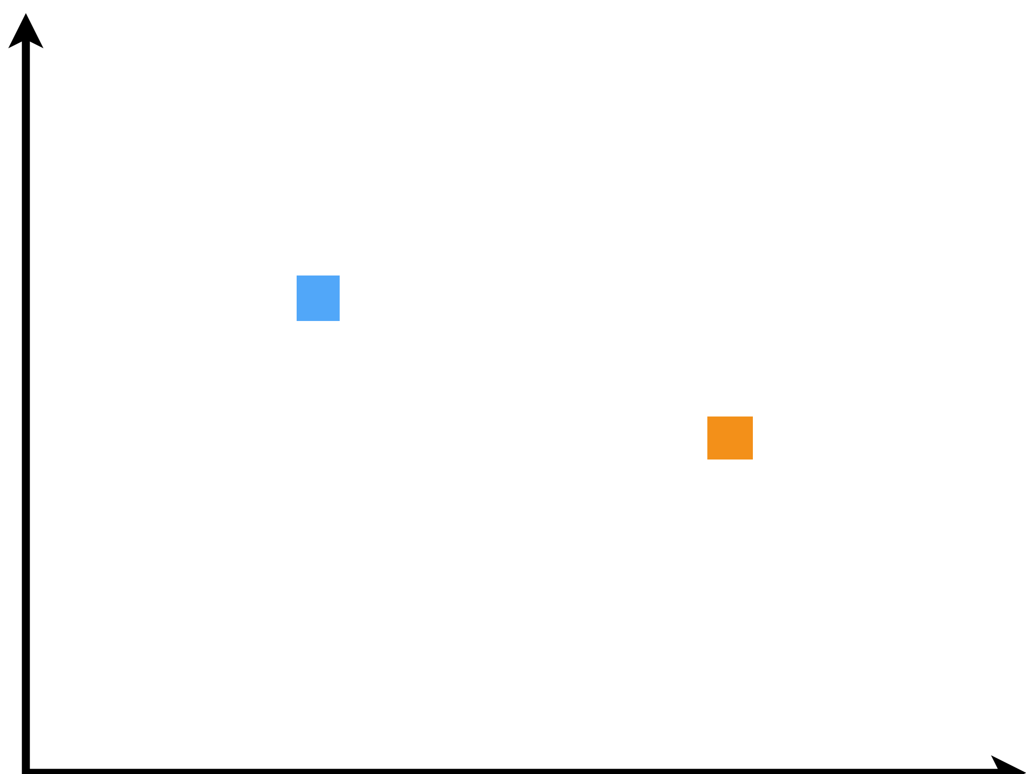

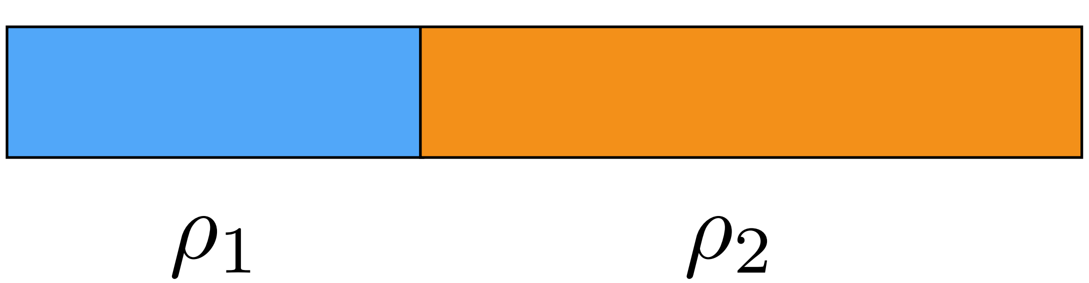


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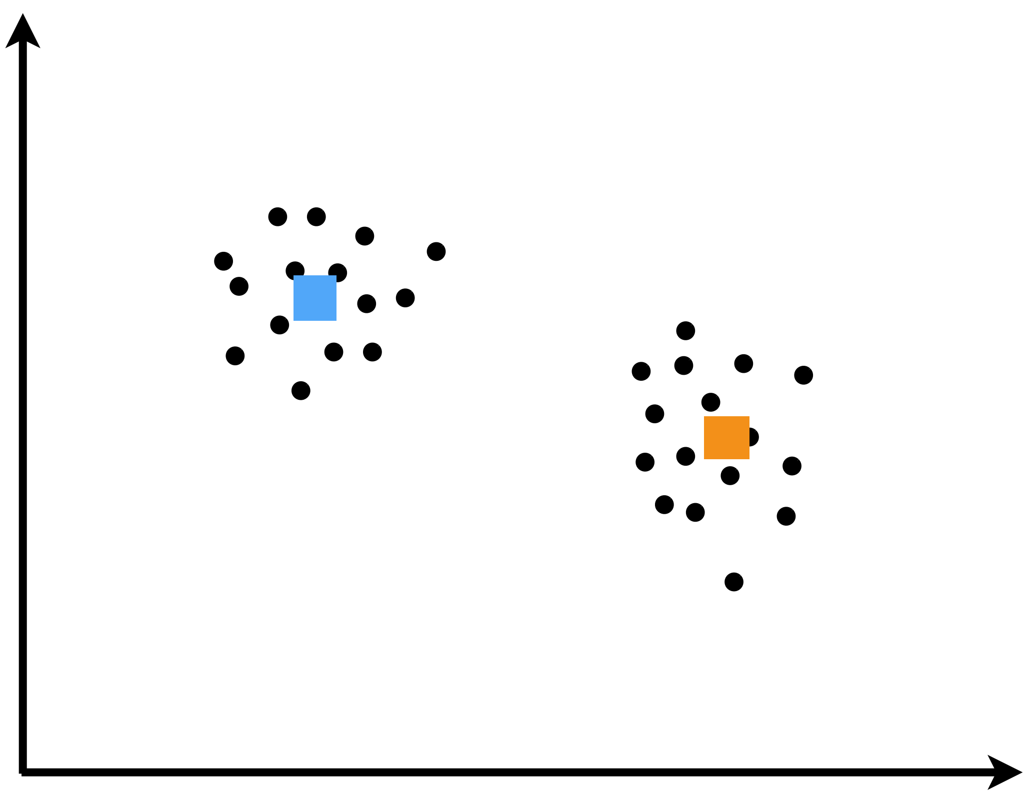

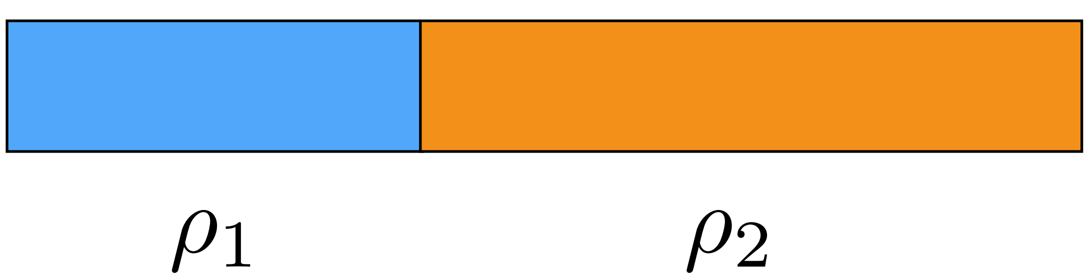


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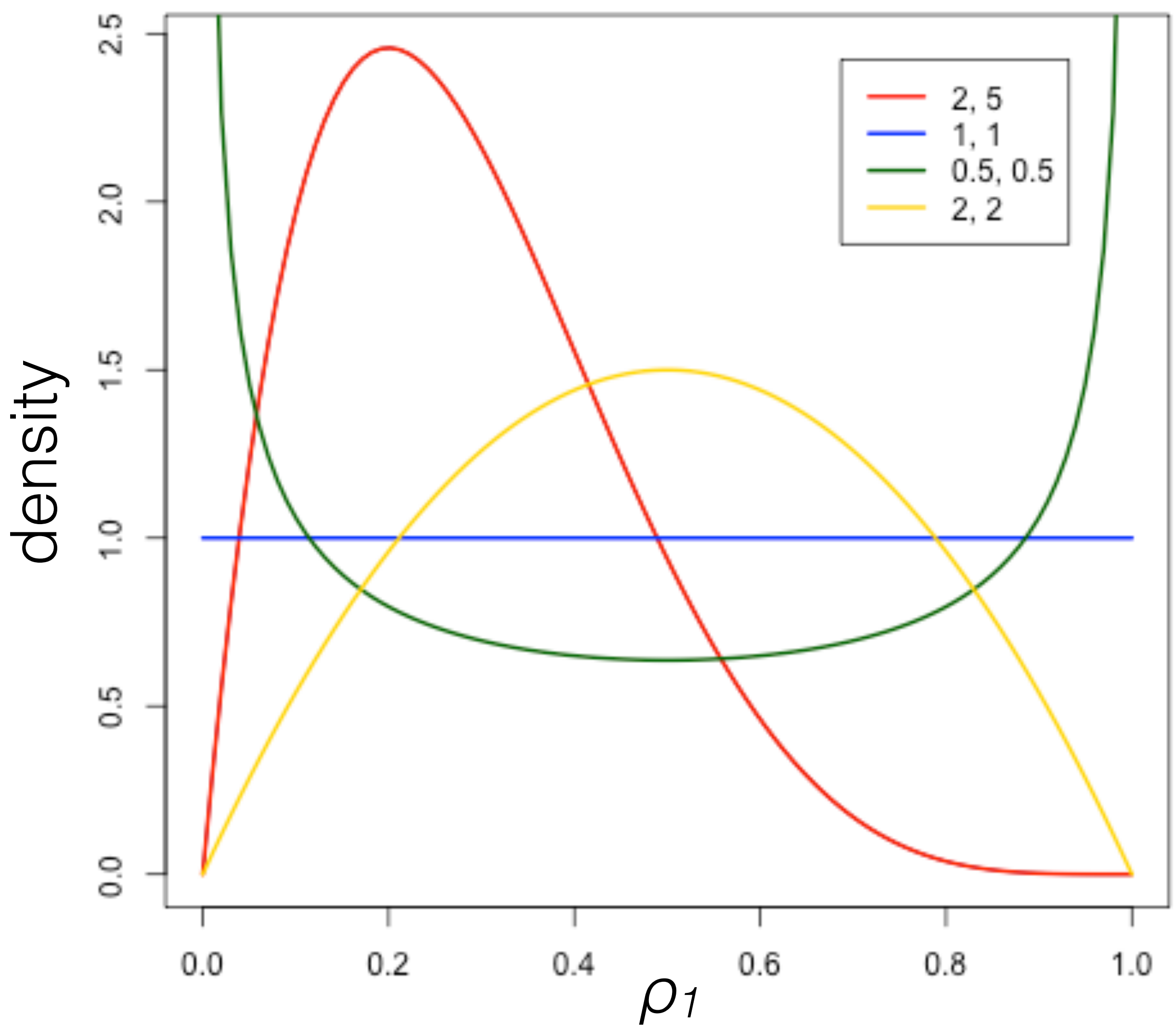


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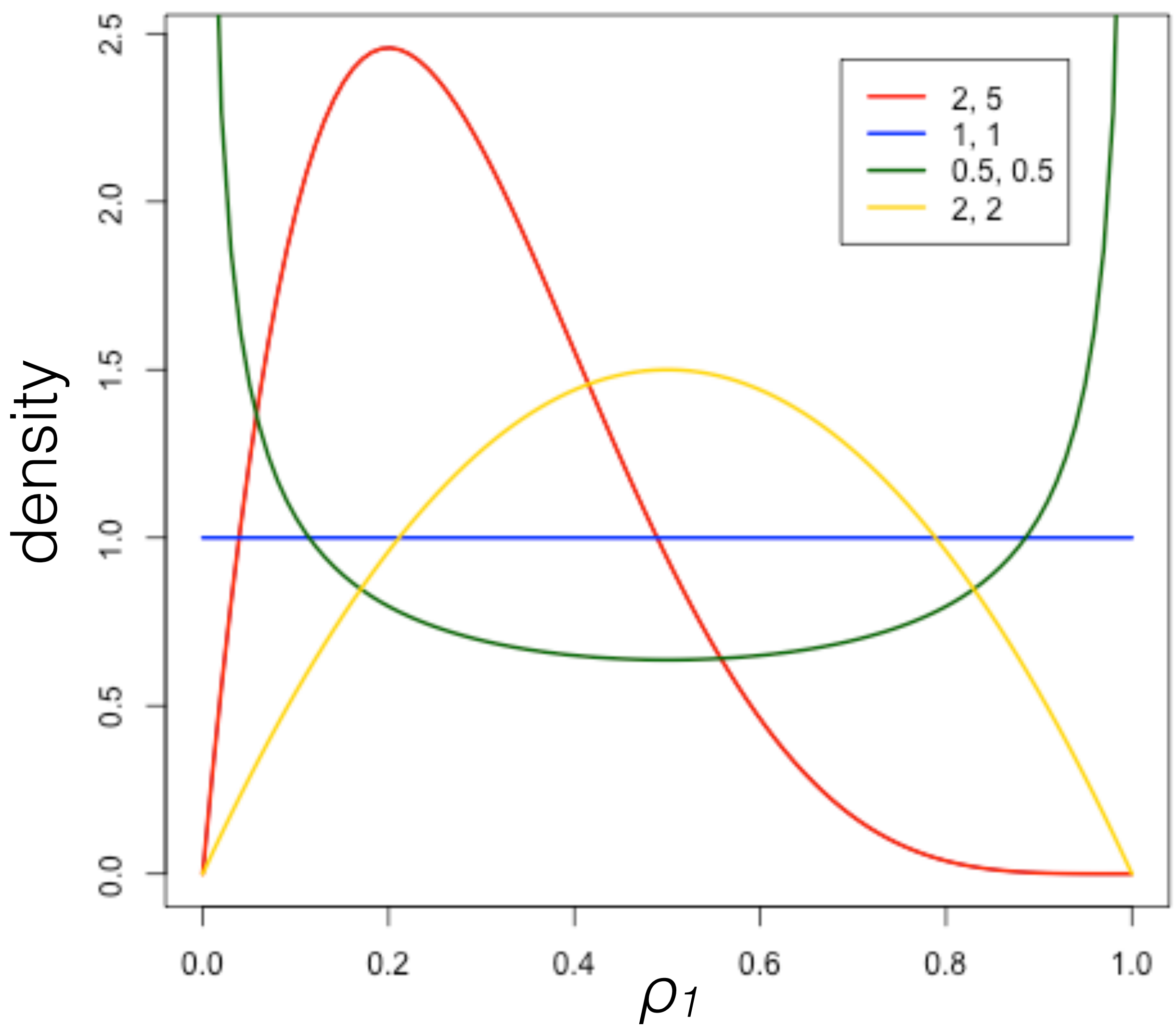


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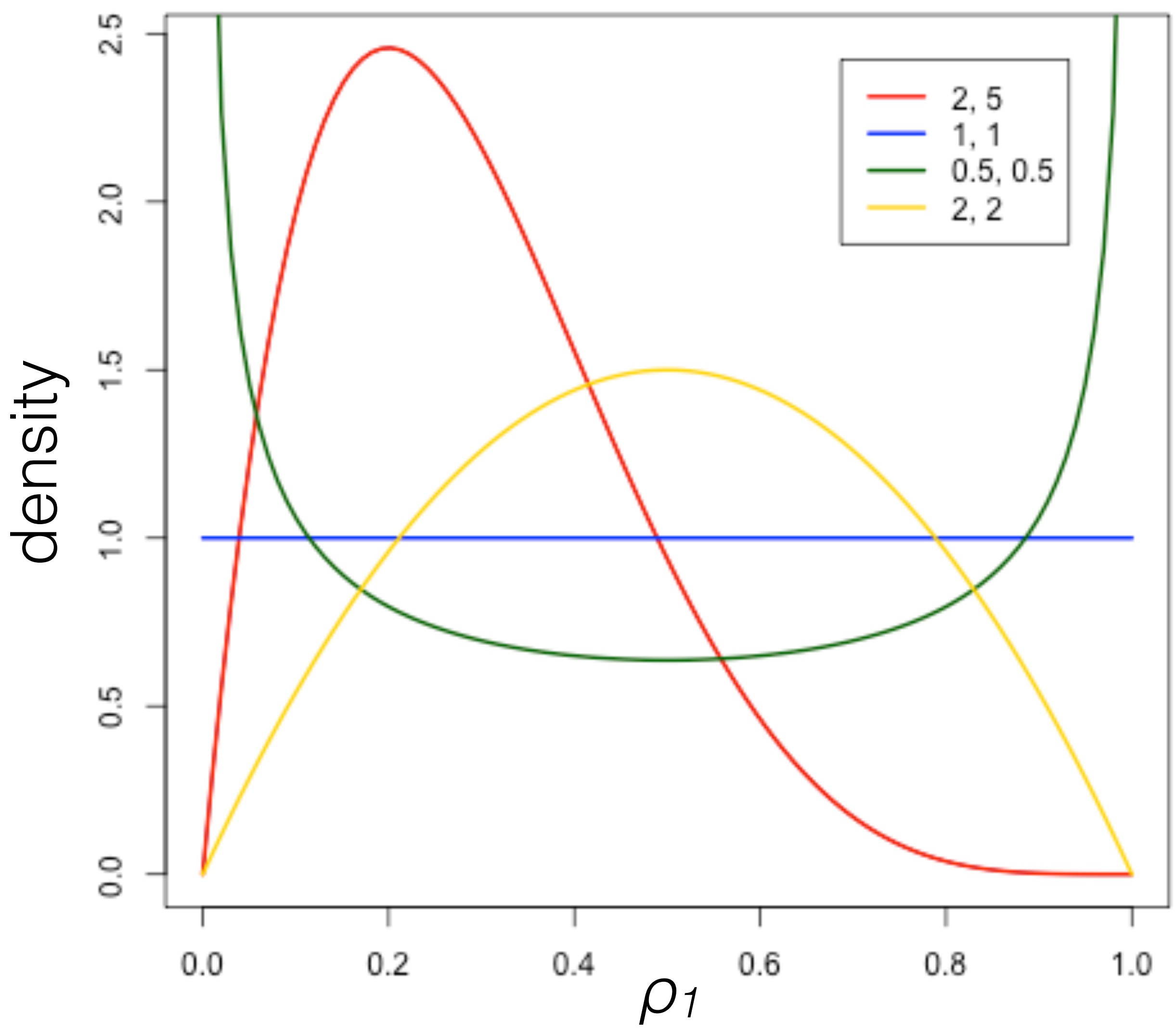


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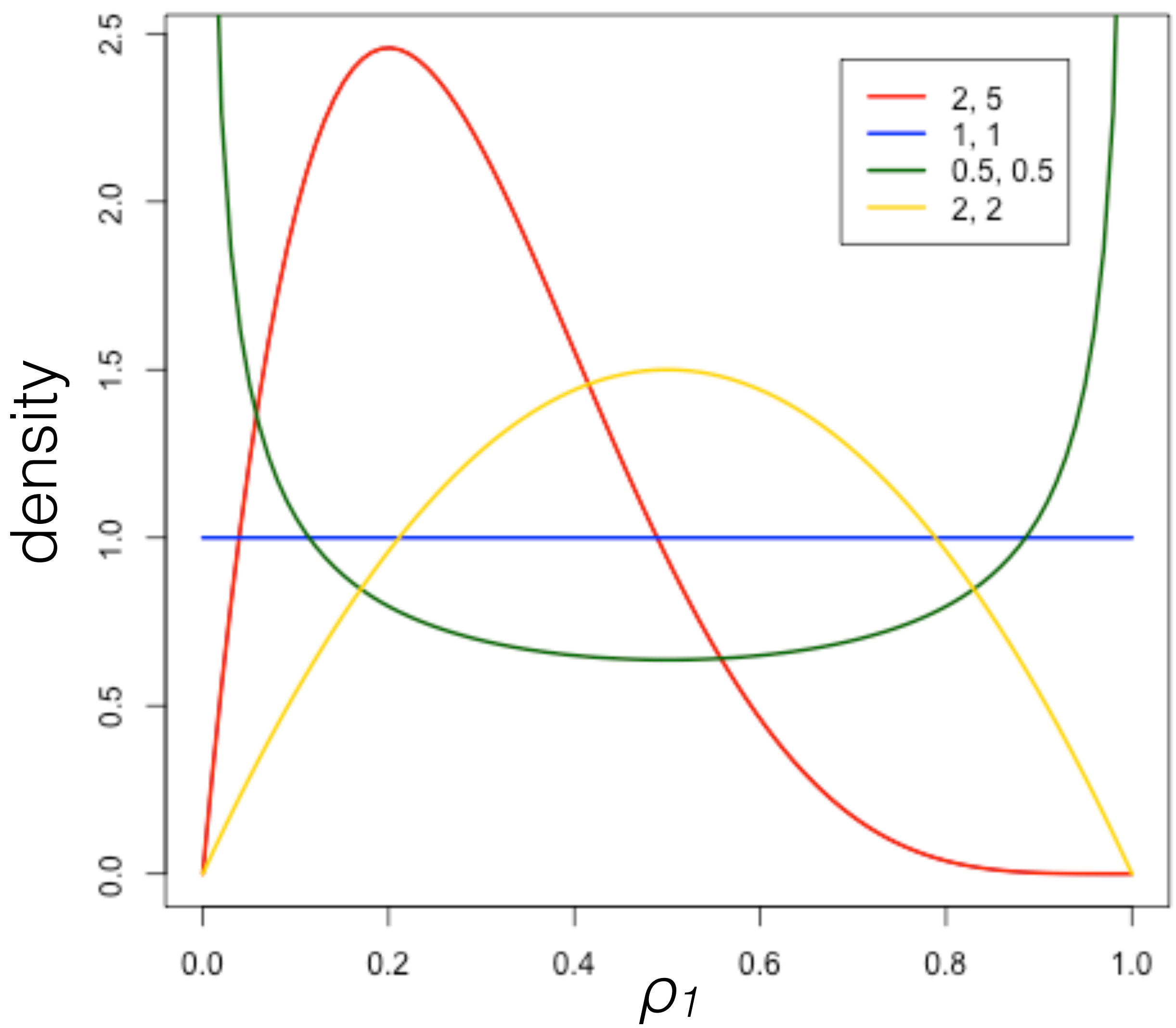


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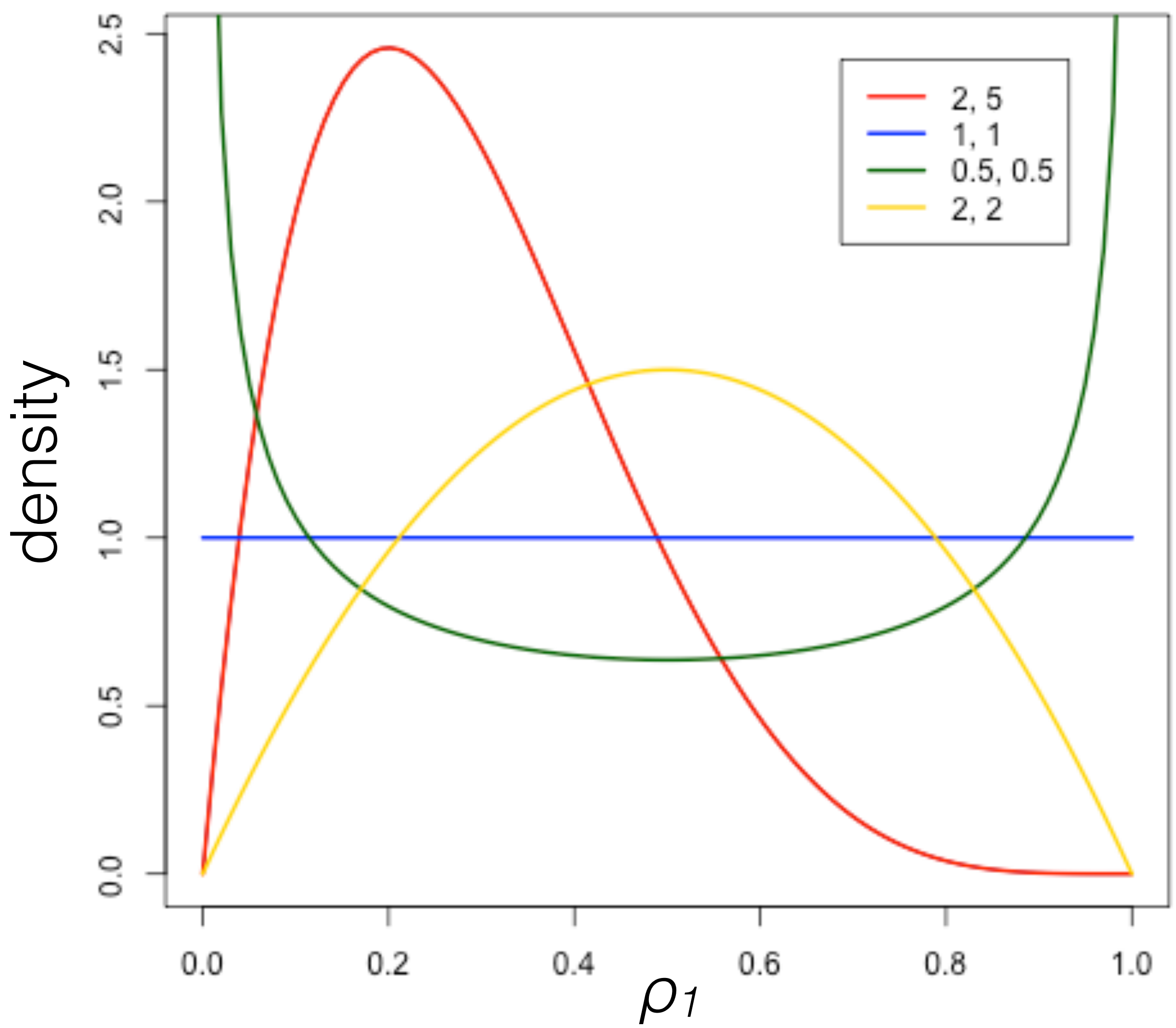


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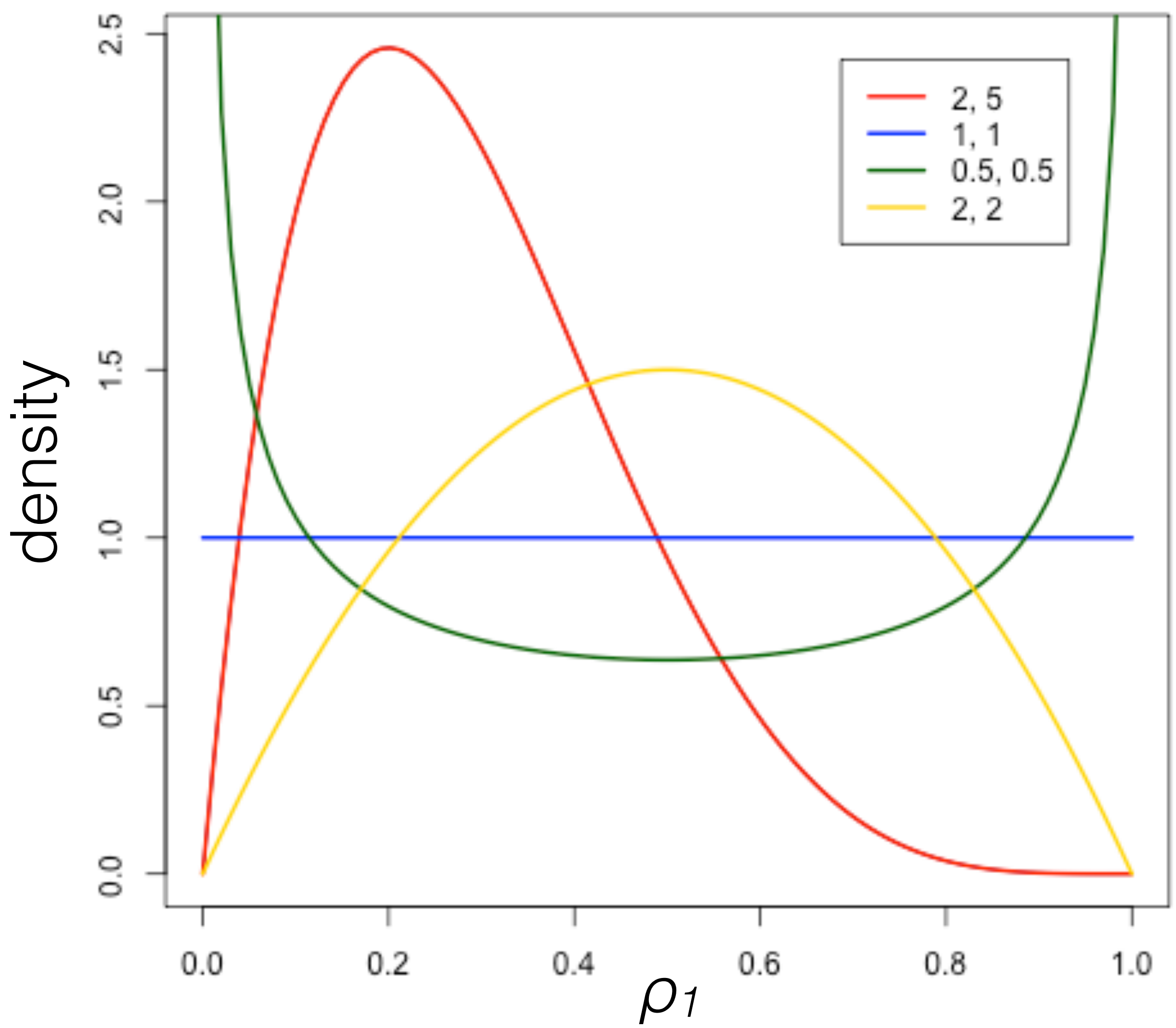


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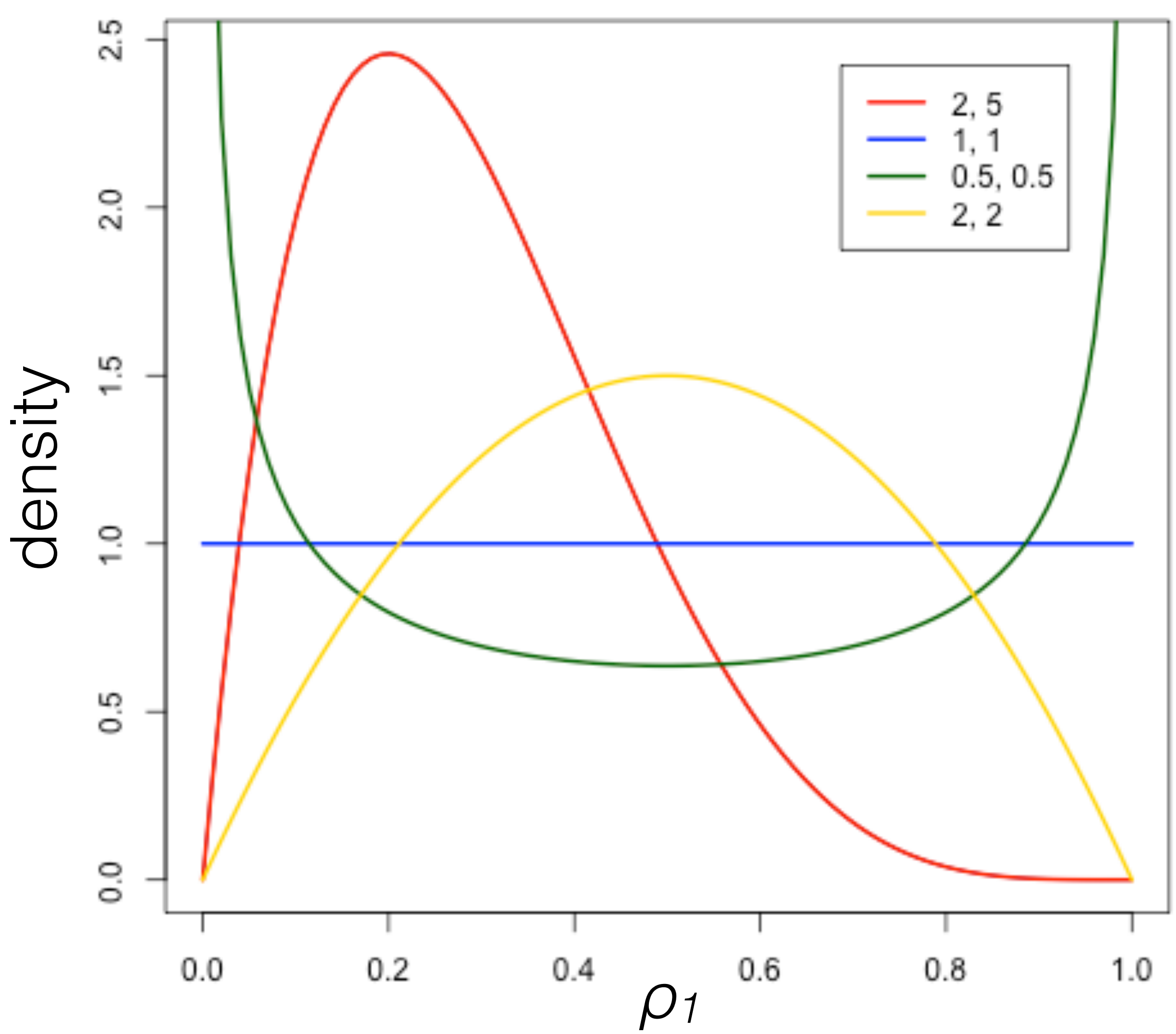


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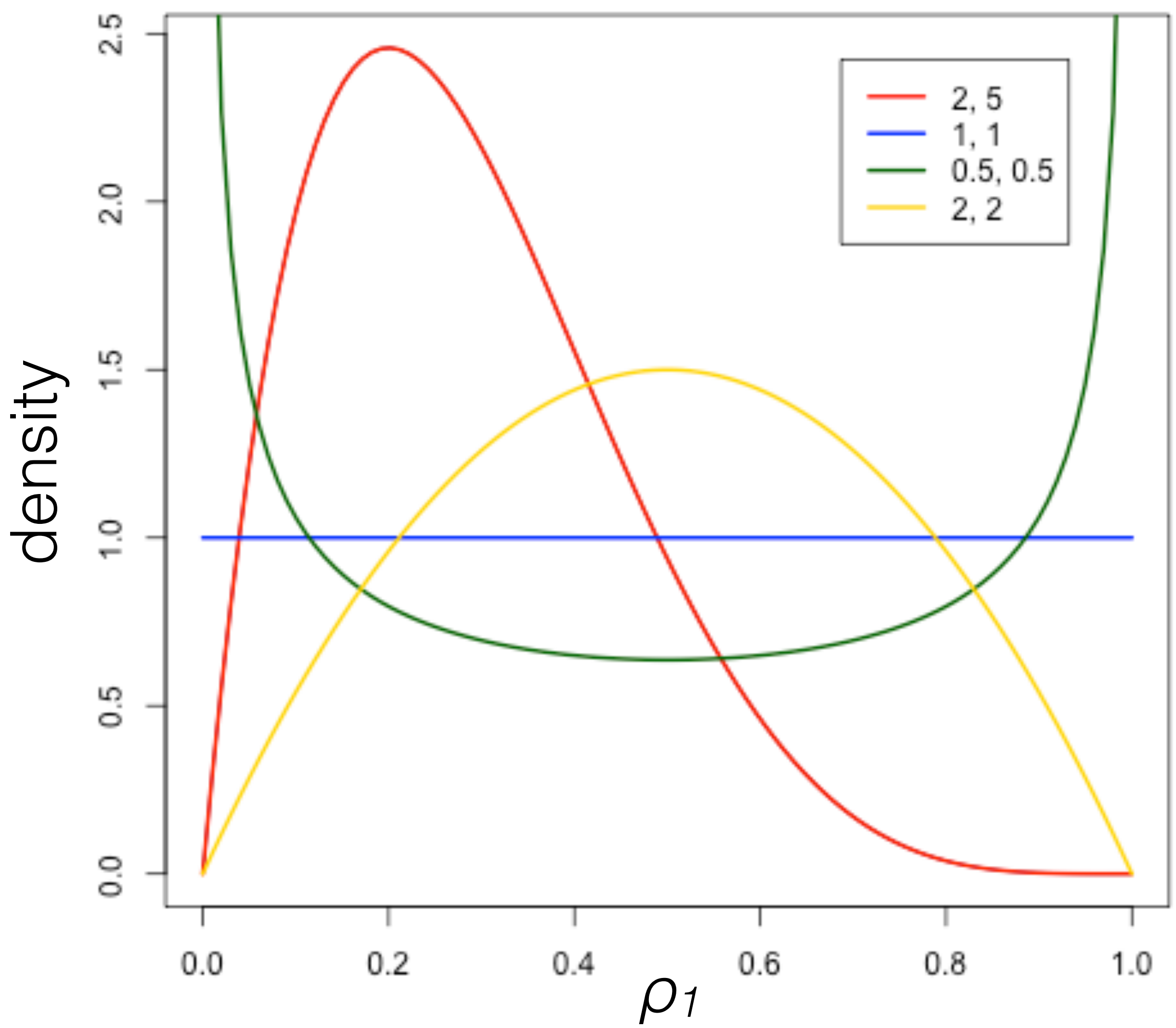


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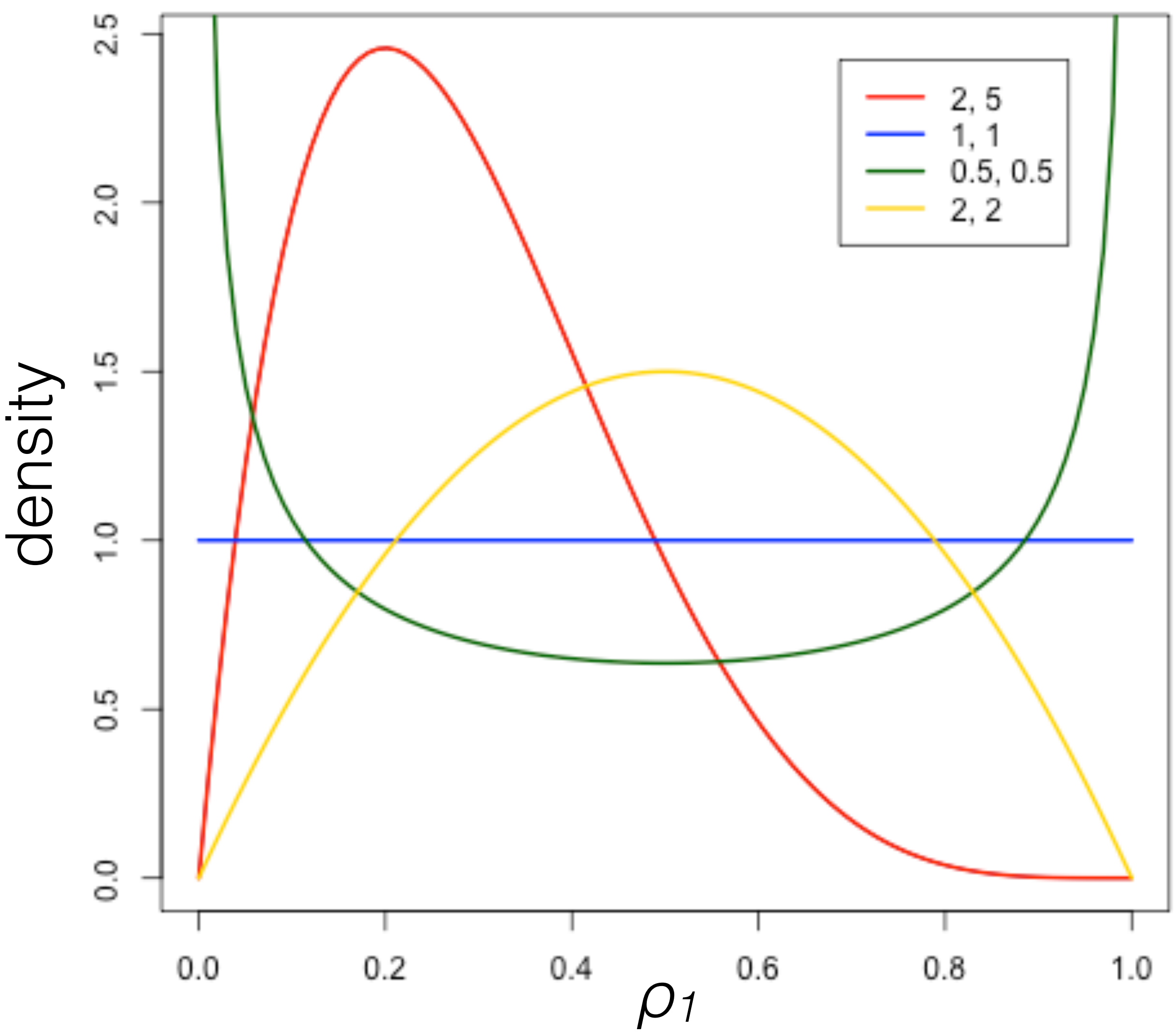


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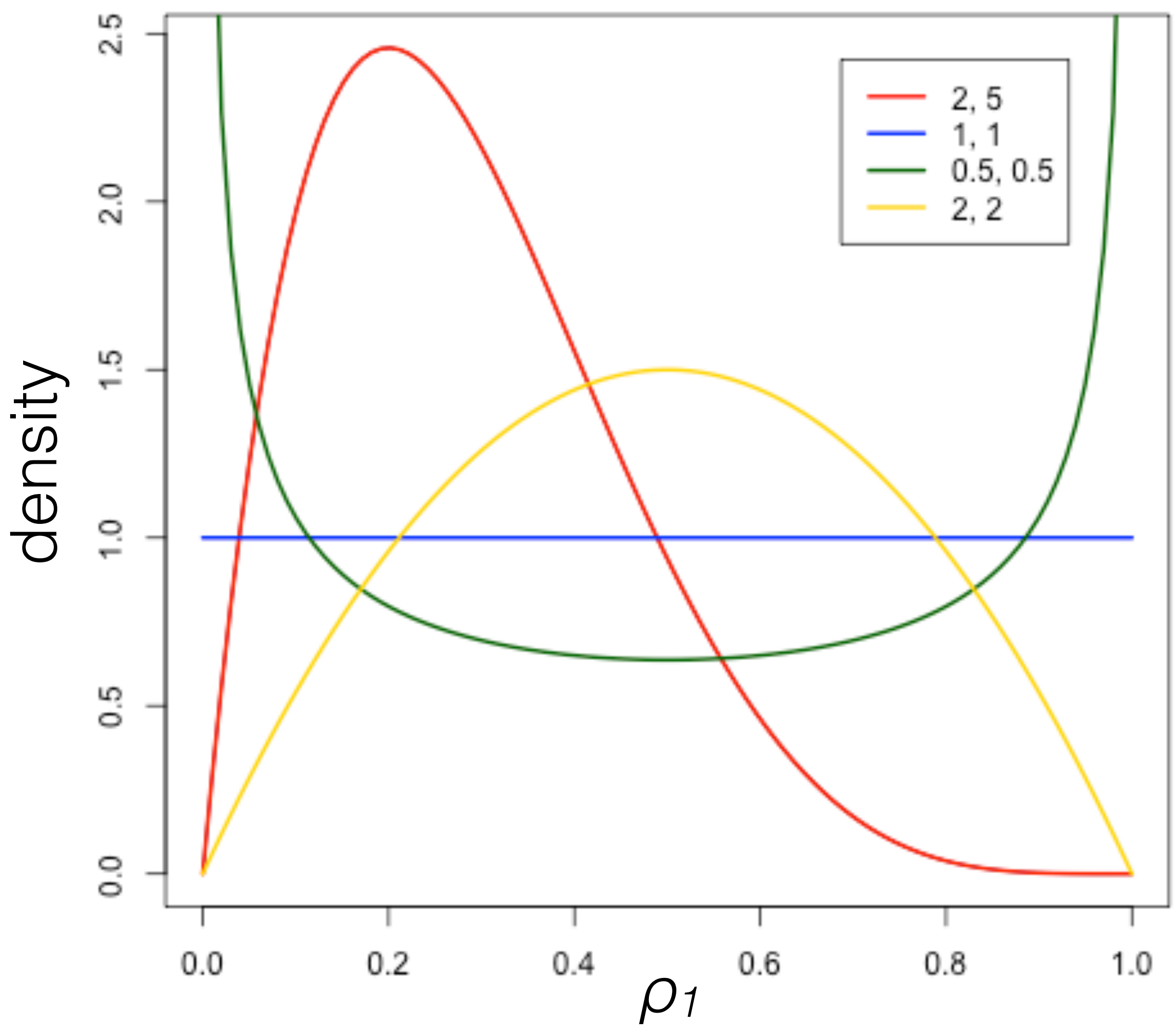


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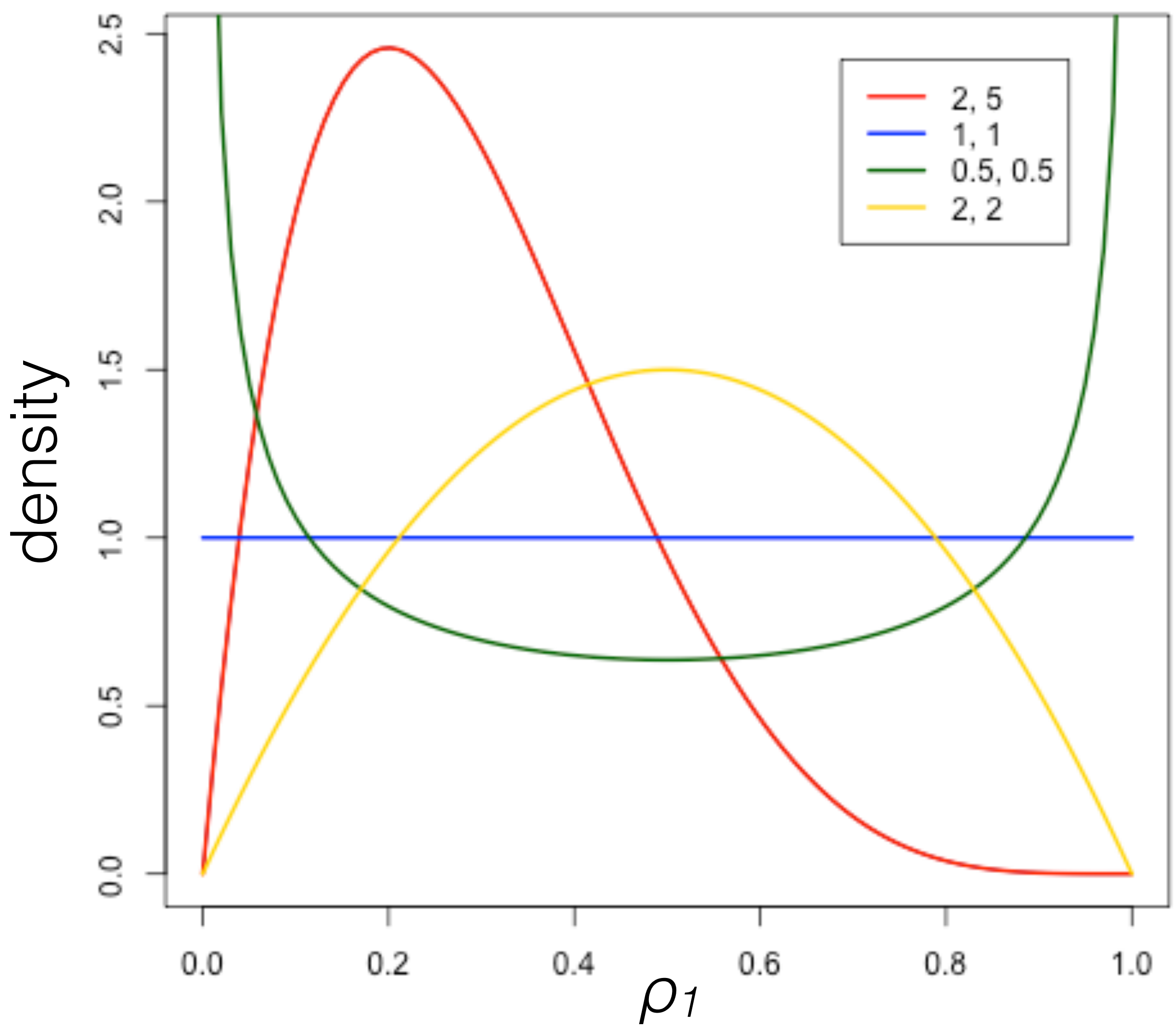


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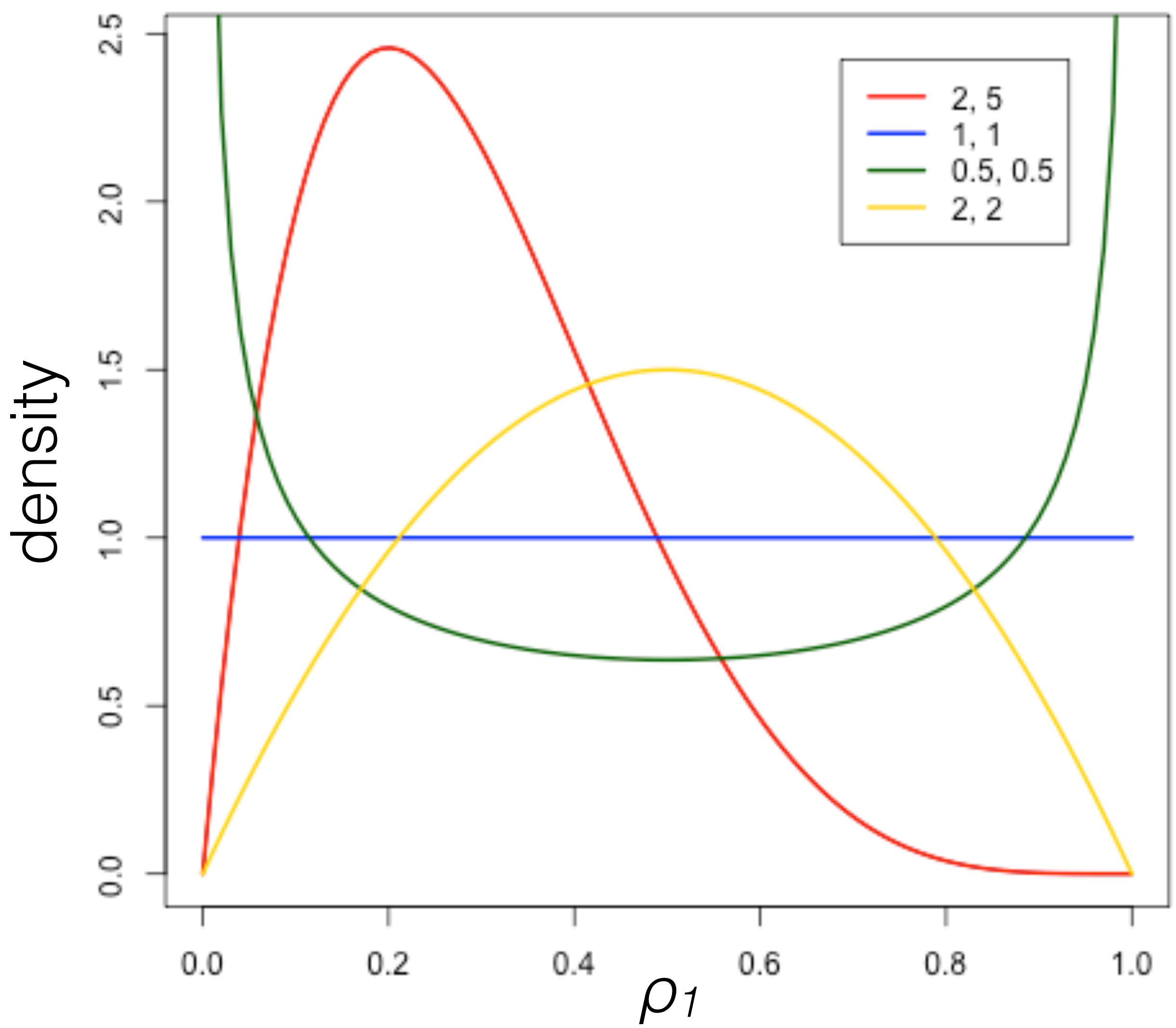


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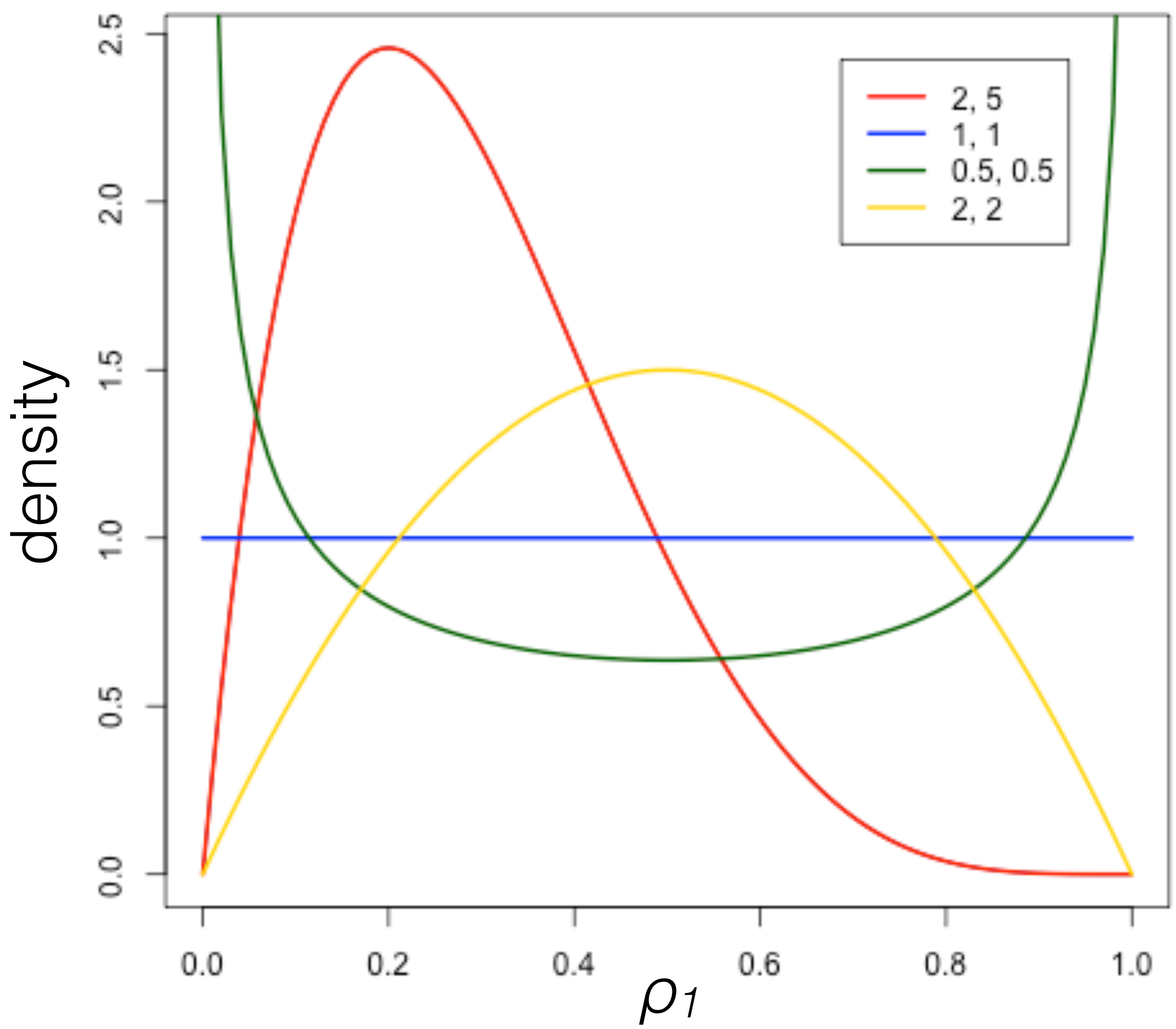


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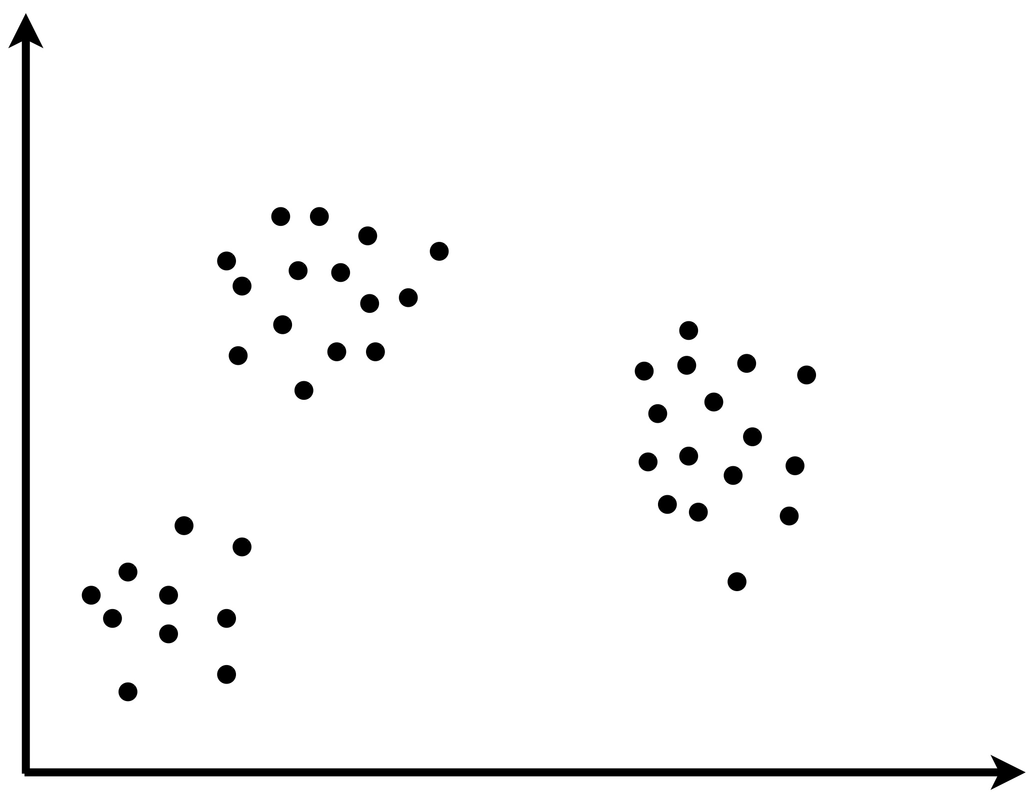

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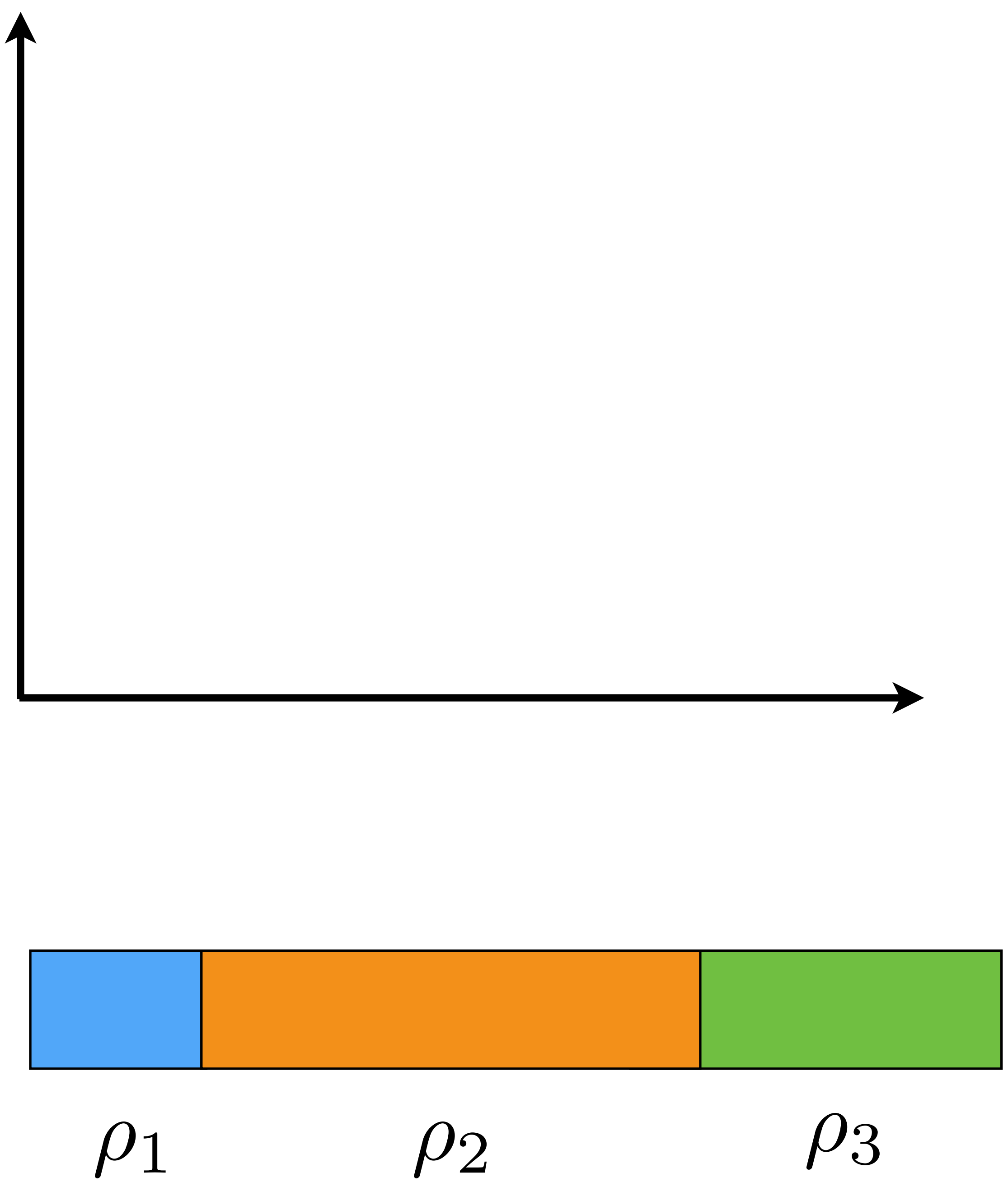


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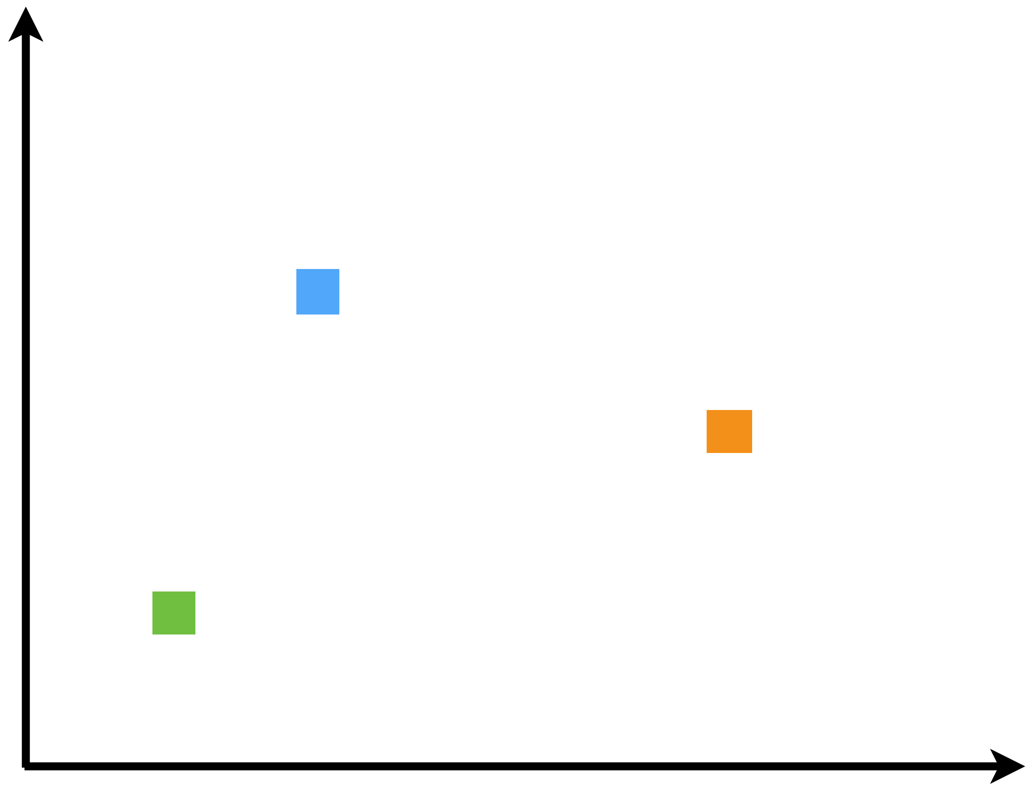

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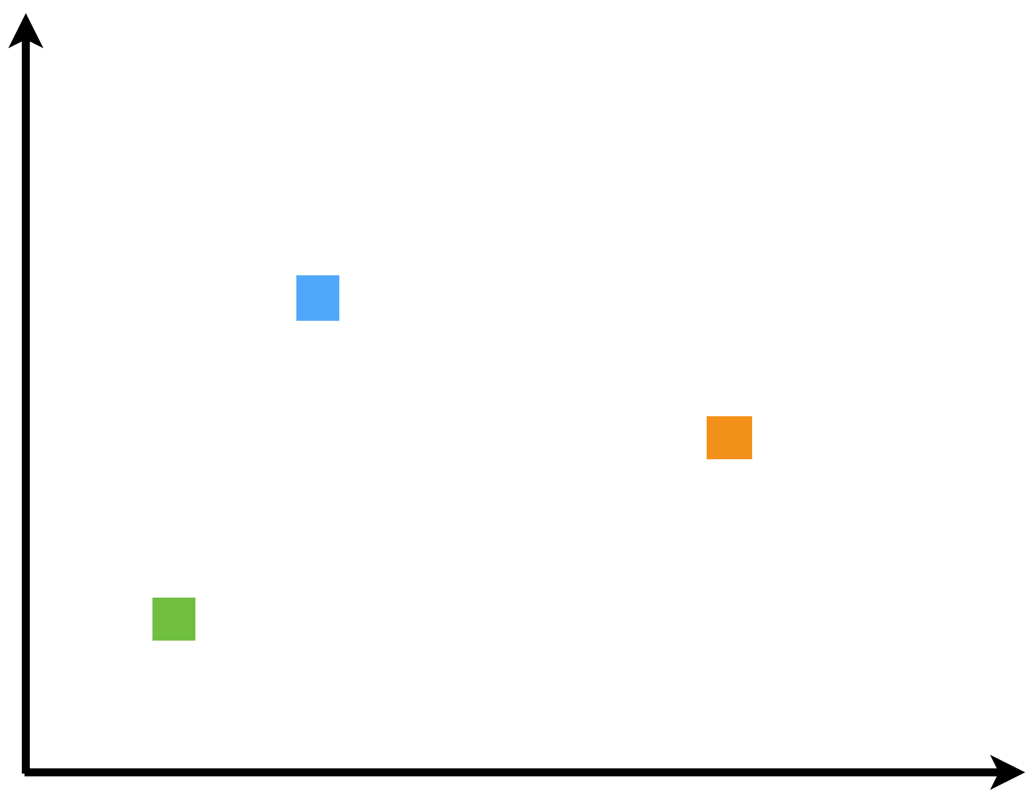

- Finite Gaussian mixture model (*K* clusters)

$$\rho_{1:K} \sim \text{Dirichlet}(a_{1:K})$$

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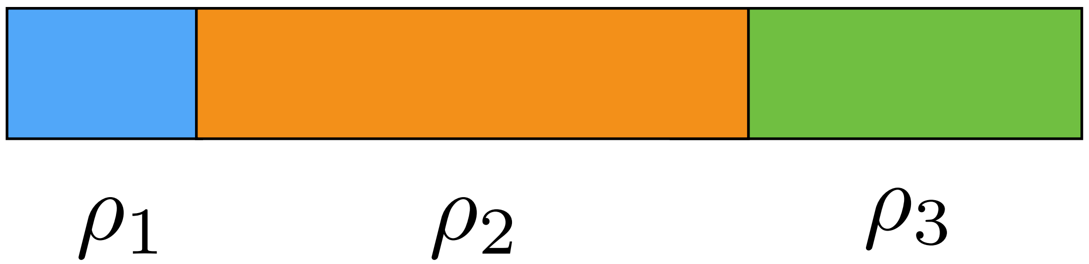

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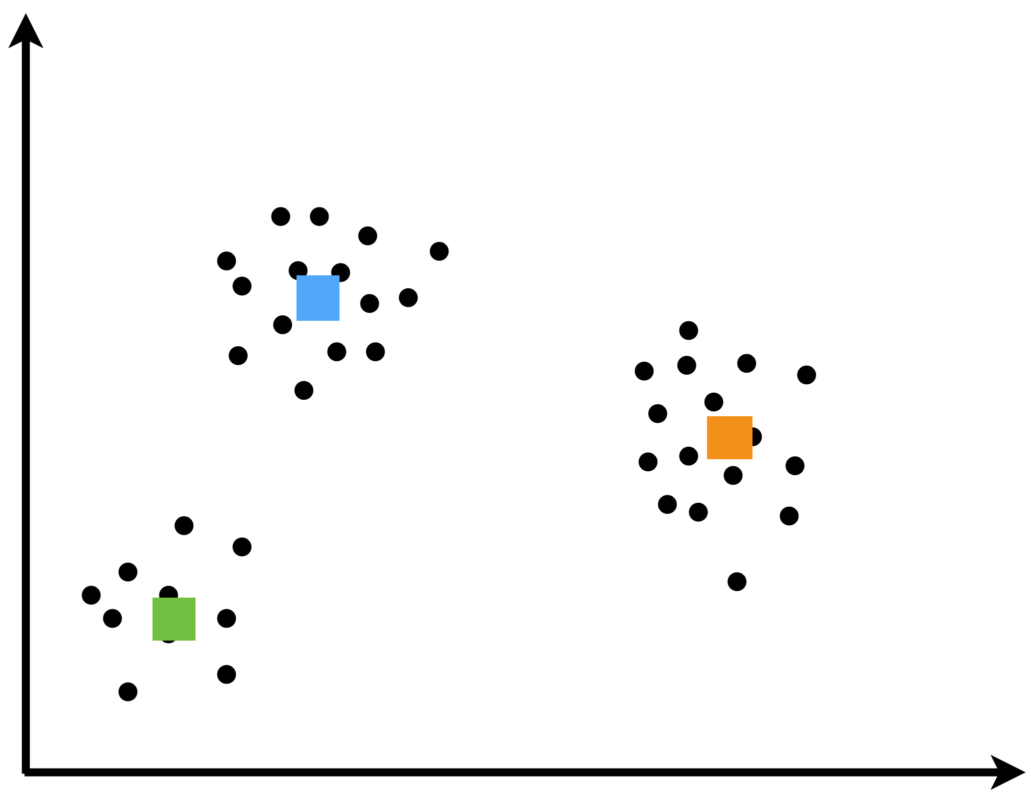

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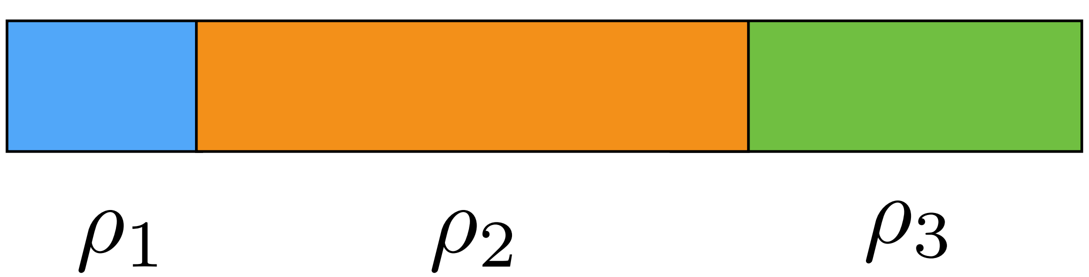

# Dirichlet distribution review

$$\mathrm{Dirichlet}(\rho_{1:K}|a_{1:K}) = \frac{\Gamma(\sum_{k=1}^{K} a_k)}{\prod_{k=1}^{K} \Gamma(a_k)} \prod_{k=1}^{K} \rho_k^{a_k - 1} \qquad a_k > 0$$

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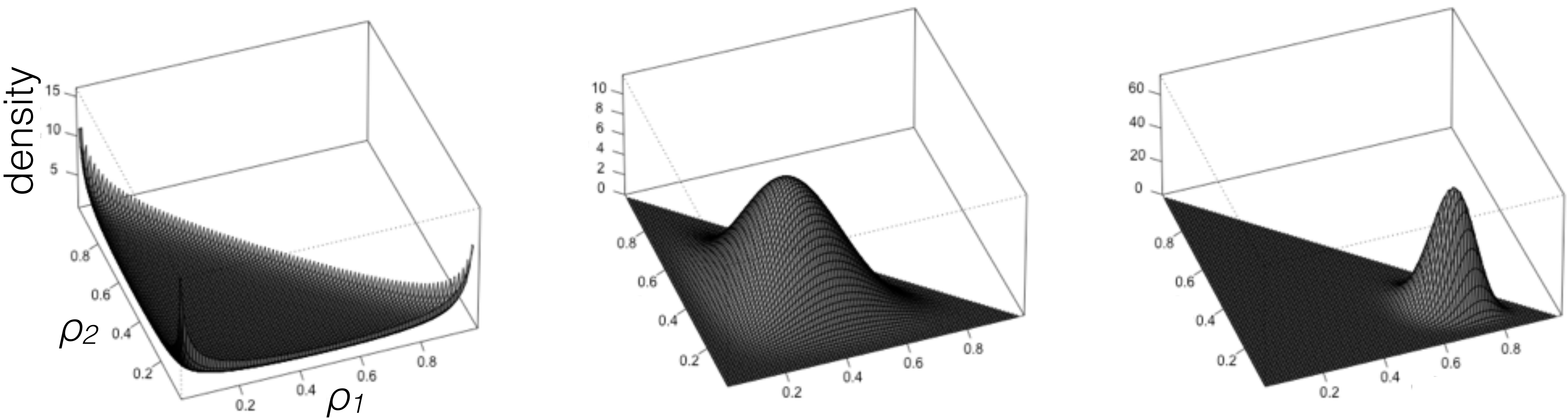


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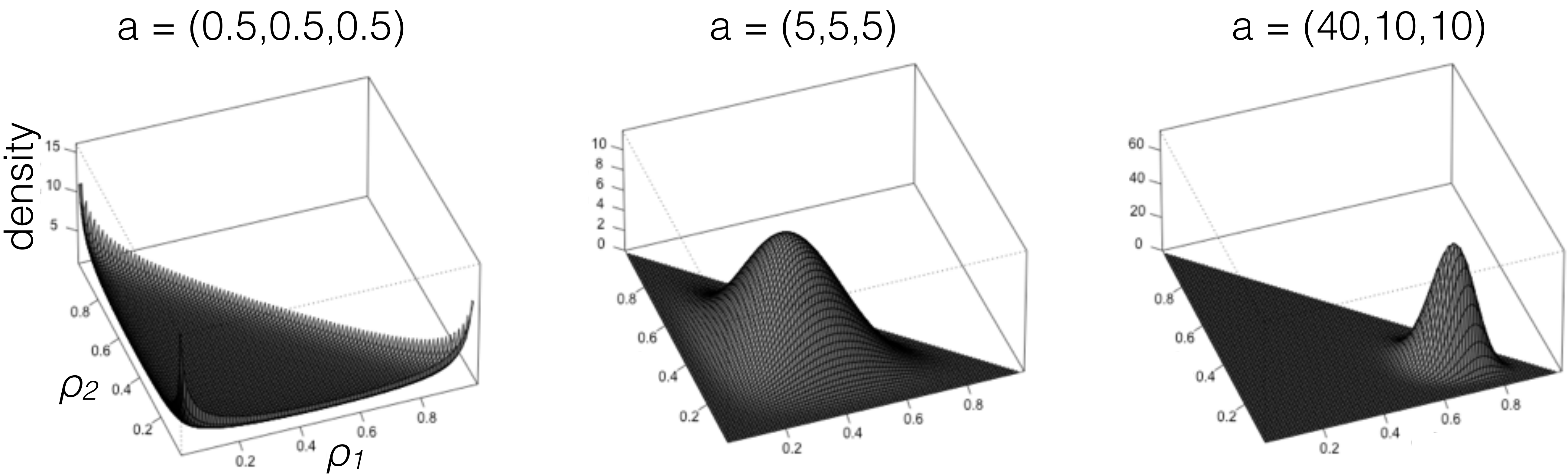


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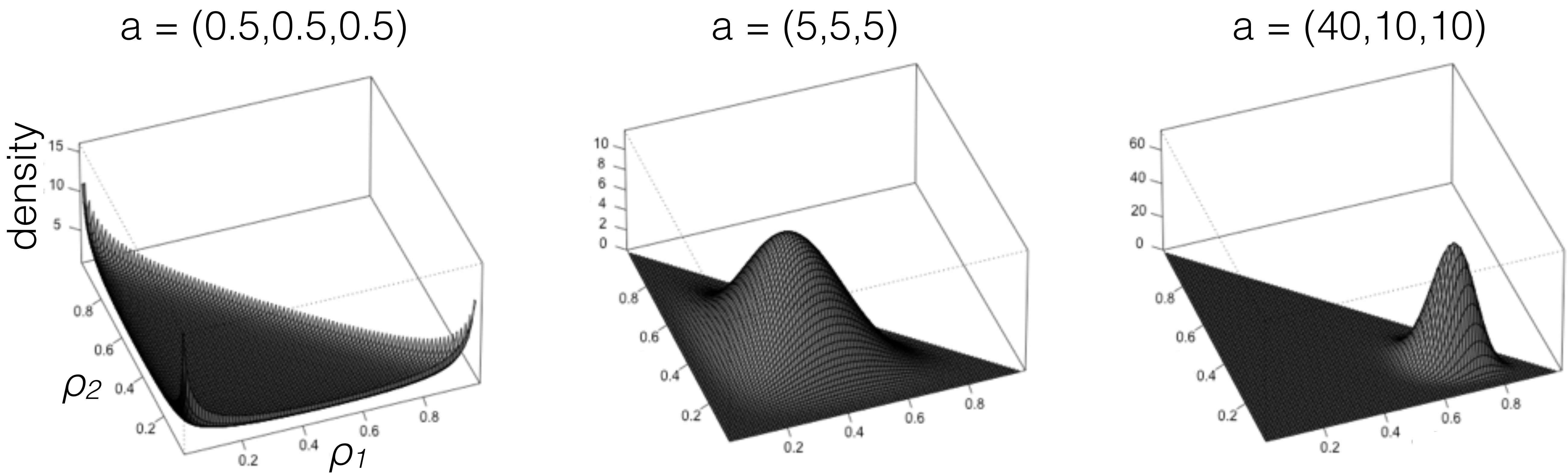


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[demo]

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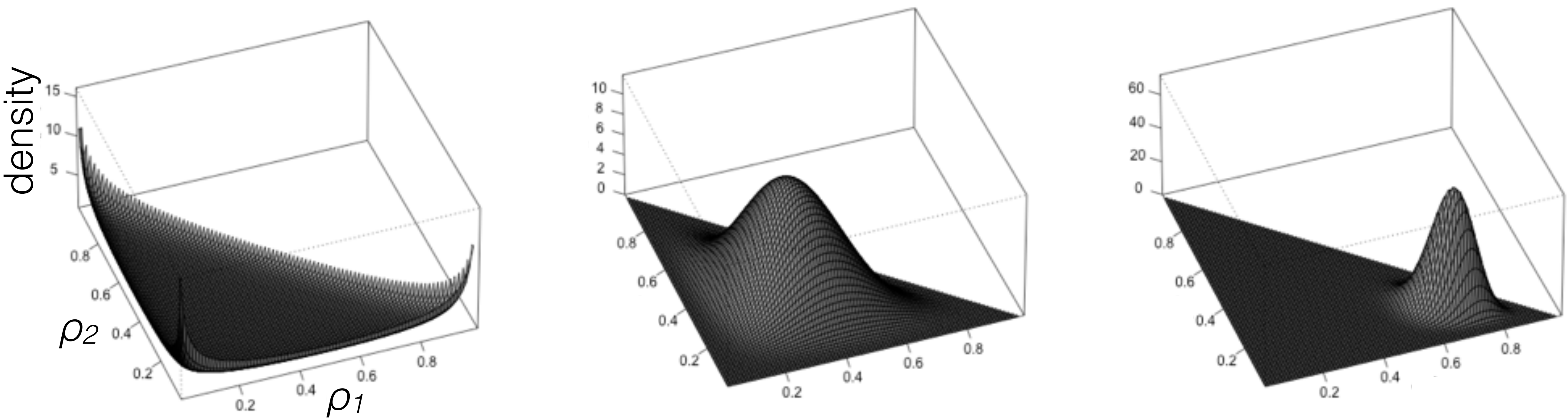


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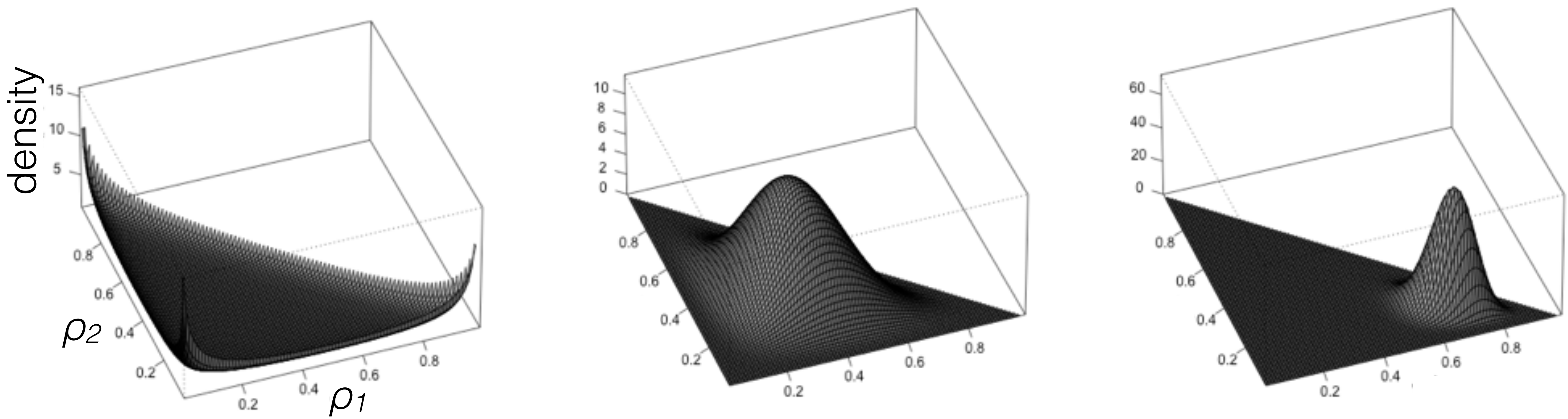


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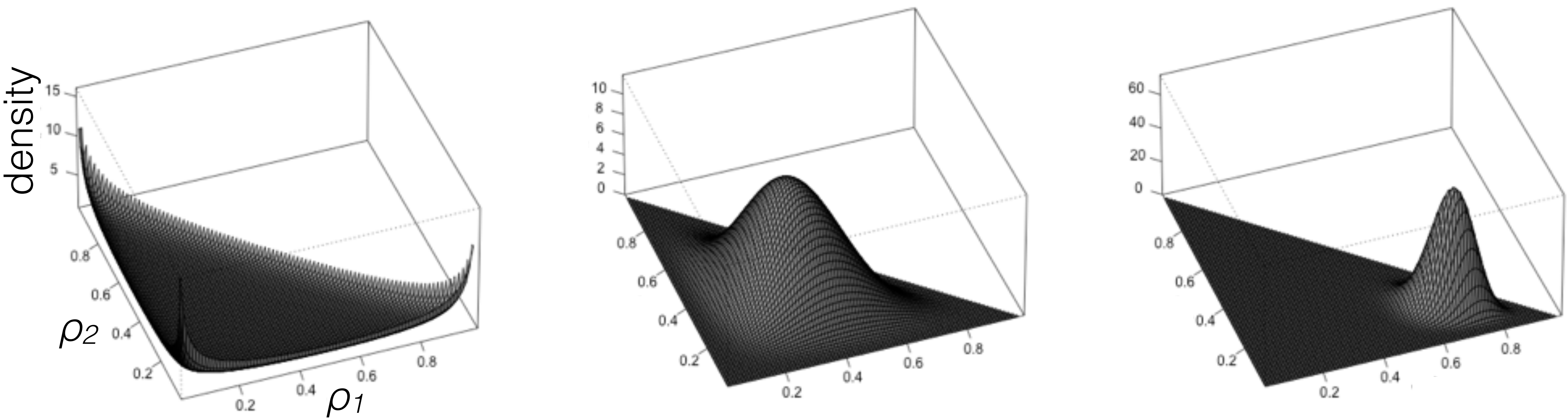


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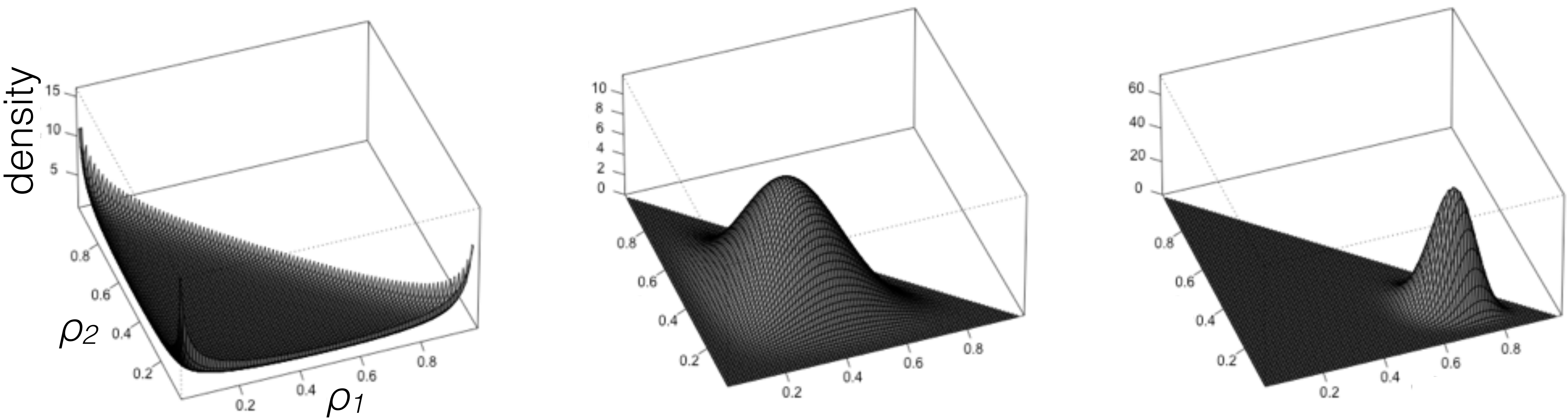


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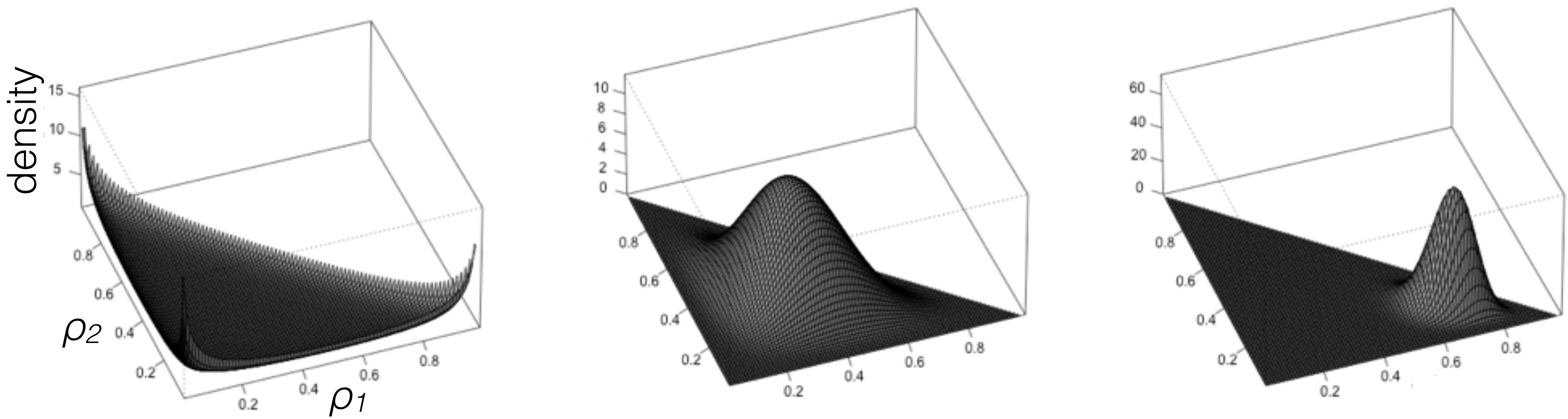


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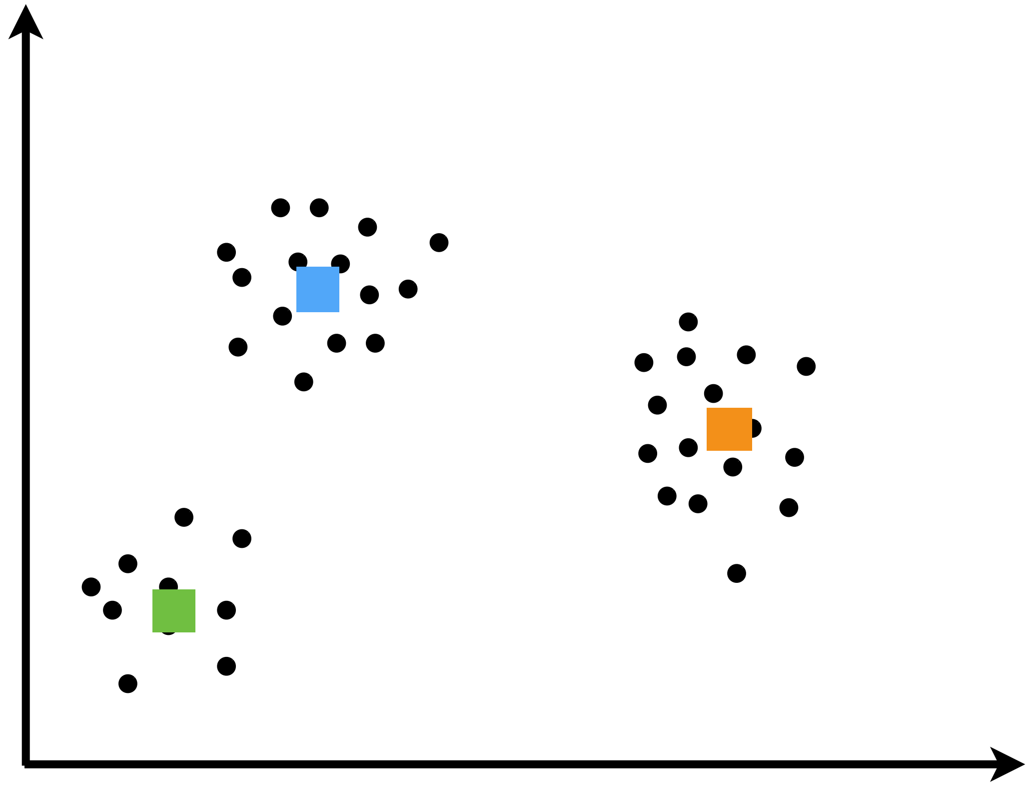

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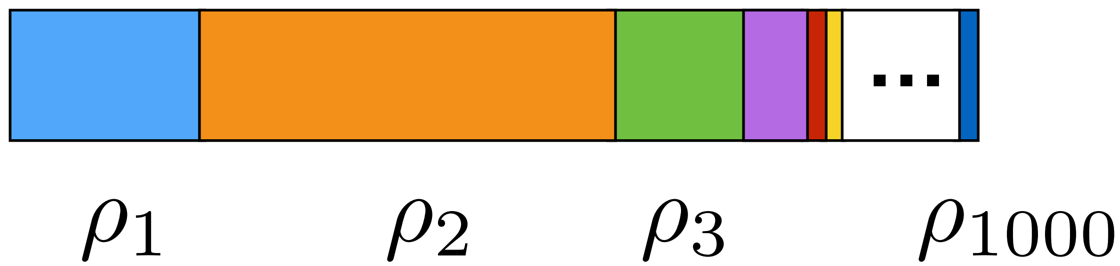

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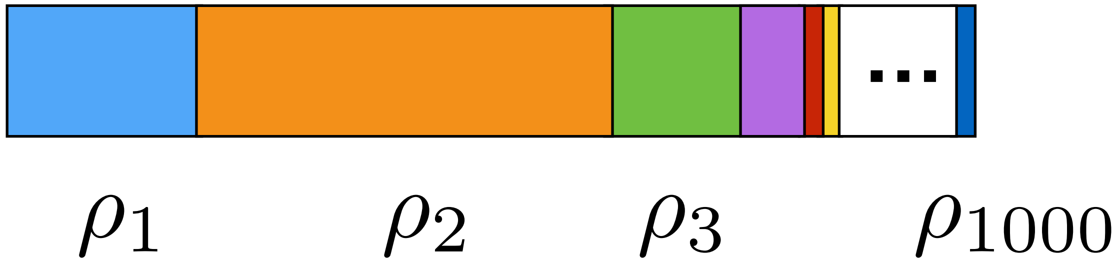

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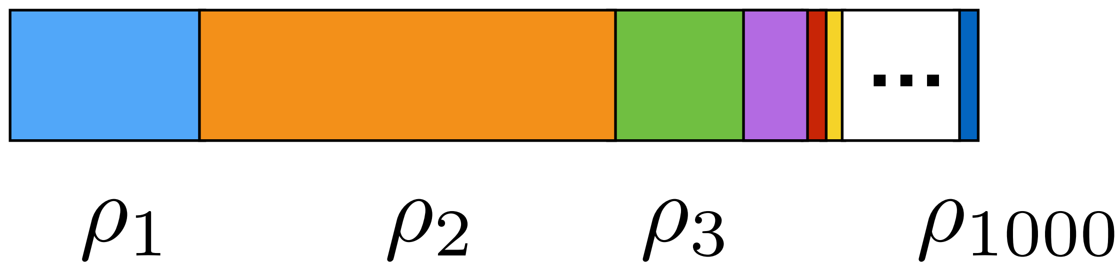


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$\rho_1$ $\rho_2$ $\rho_3$ $\rho_{1000}$

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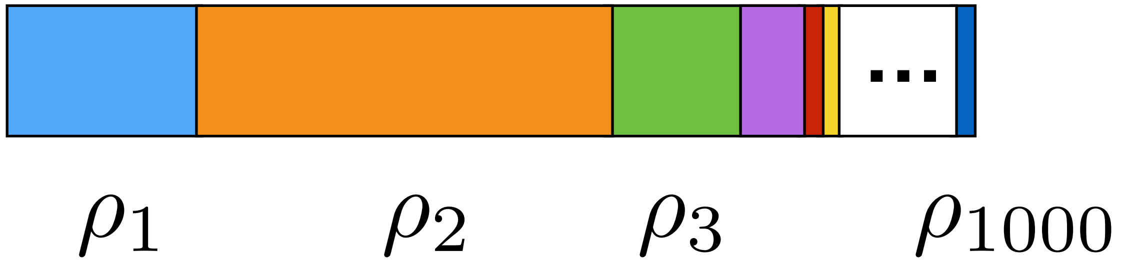


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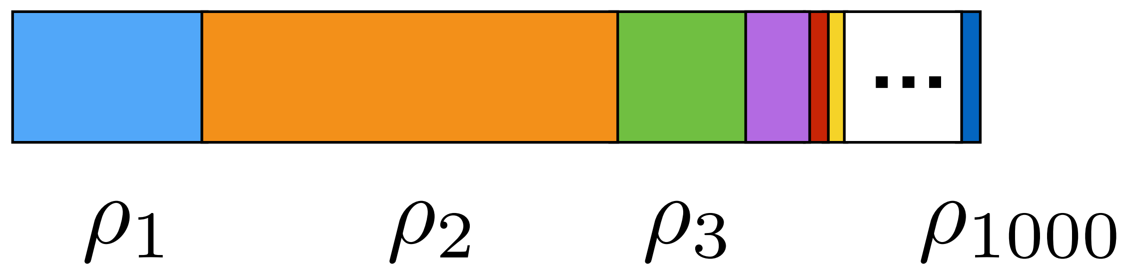


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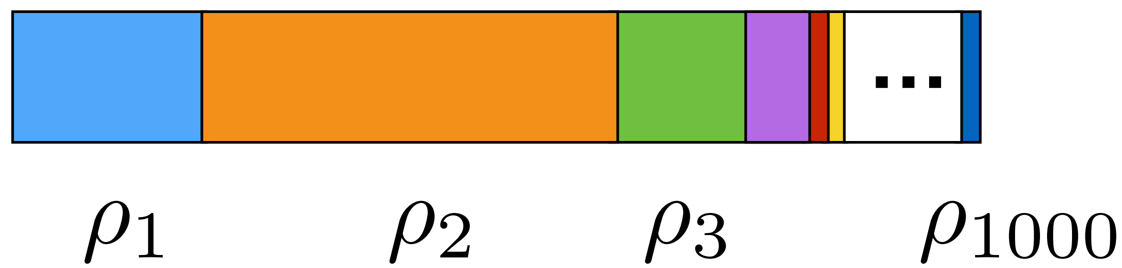


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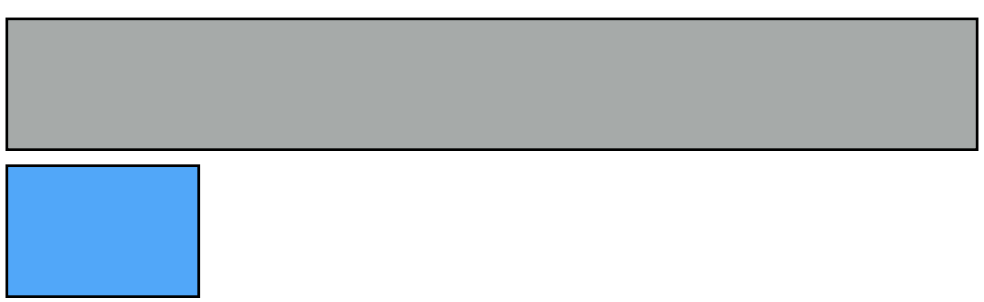

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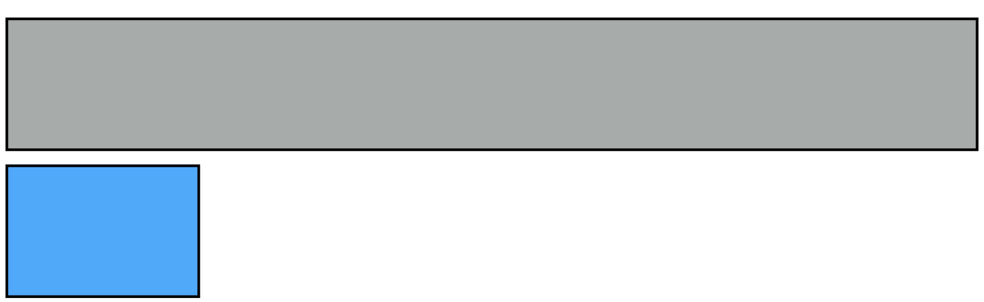

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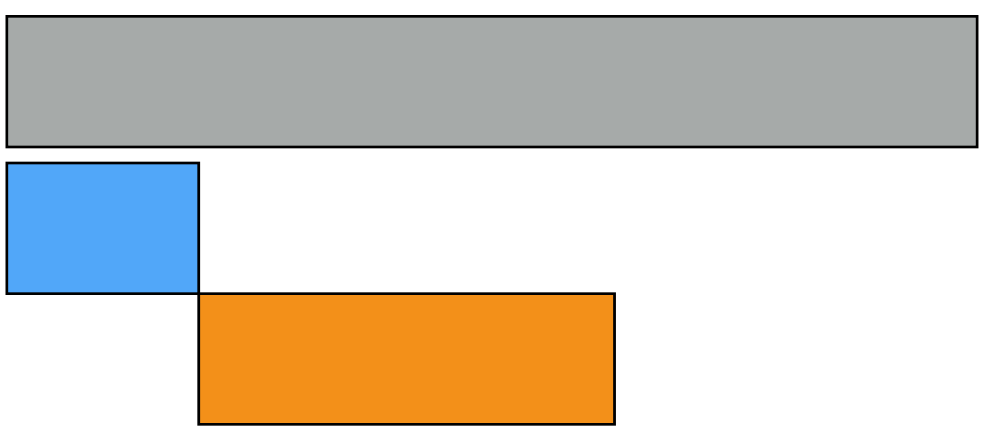

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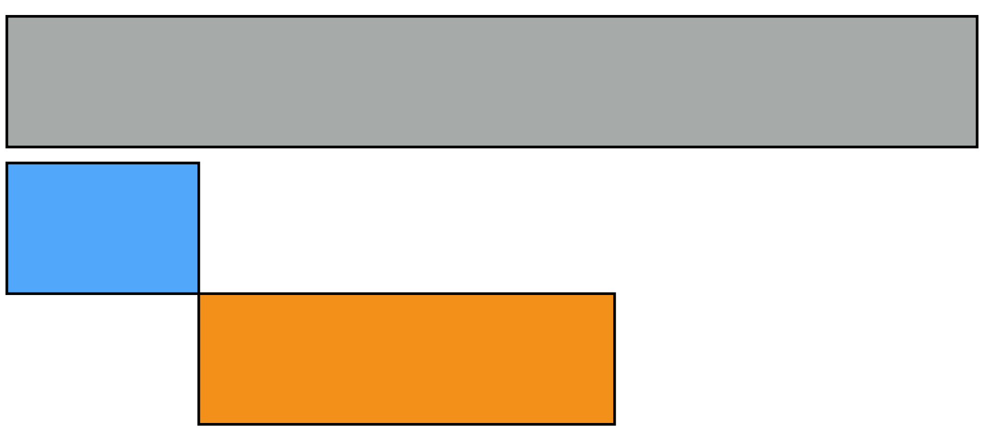

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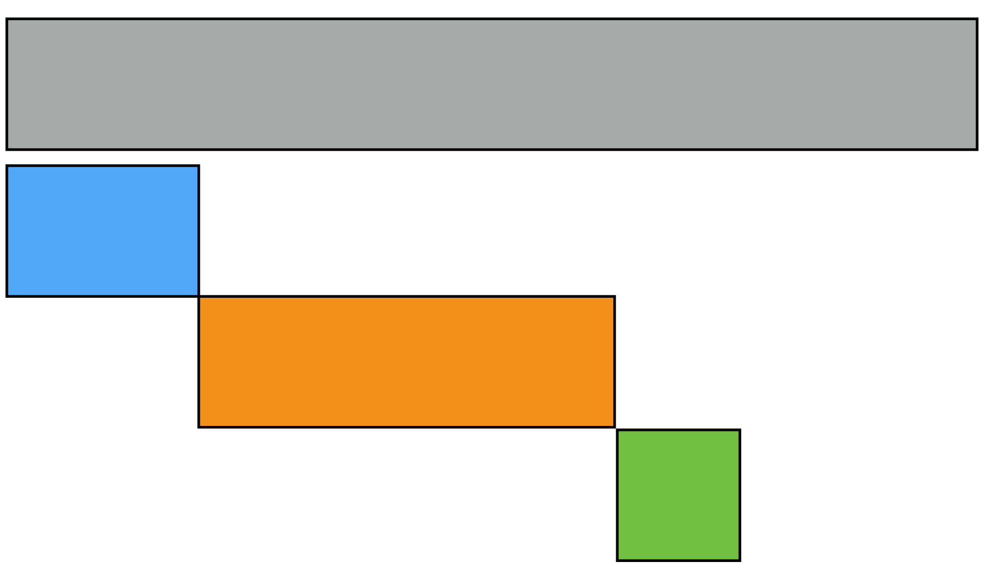

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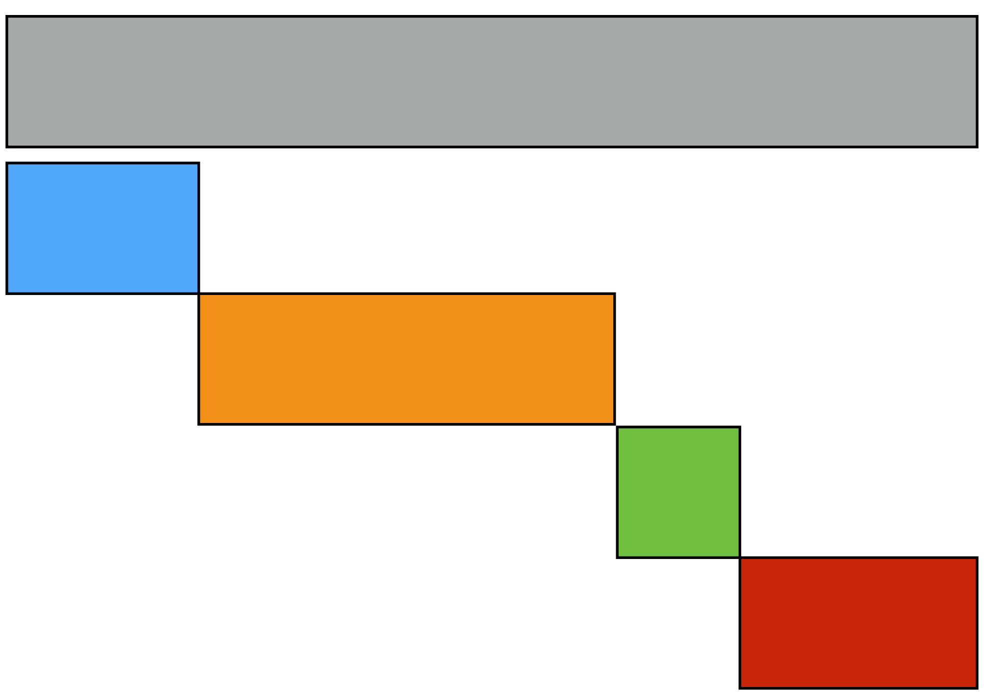
$$\Leftrightarrow \rho_1 \stackrel{d}{=} \mathrm{Beta}\left(a_1, \sum_{k=1}^{K} a_k - a_1\right) \perp\!\!\!\perp \tfrac{(\rho_2, \ldots, \rho_K)}{1-\rho_1} \stackrel{d}{=} \mathrm{Dirichlet}(a_2, \ldots, a_K)$$



- "Stick breaking"

$$V_1 \sim \mathrm{Beta}(a_1, a_2 + a_3 + a_4) \qquad \rho_1 = V_1$$

$$V_2 \sim \mathrm{Beta}(a_2, a_3 + a_4) \qquad \rho_2 = (1 - V_1)V_2$$

$$V_3 \sim \mathrm{Beta}(a_3, a_4) \qquad \rho_3 = (1 - V_1)(1 - V_2)V_3$$

$$\rho_4 = 1 - \sum_{k=1}^{3} \rho_k$$

# Choosing $K = \infty$

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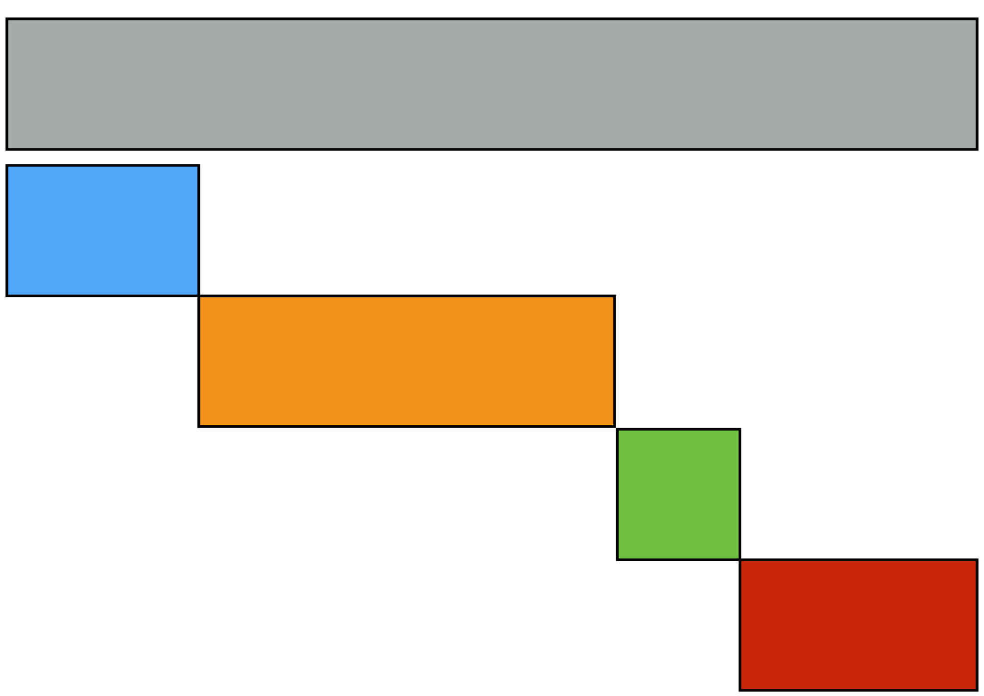
- How to generate $K = \infty$ strictly positive frequencies that sum to one?



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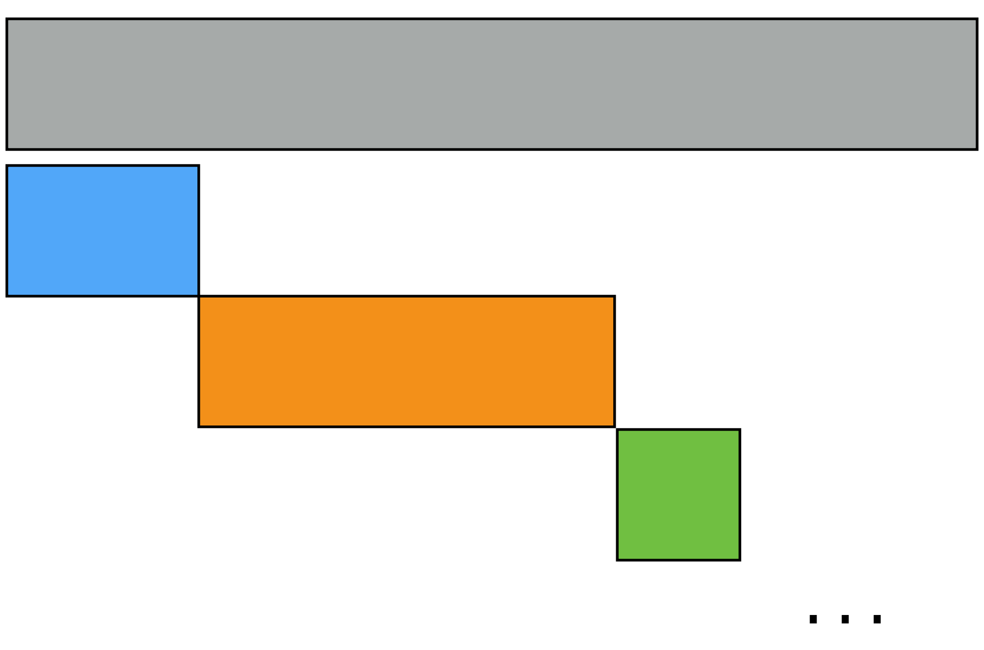

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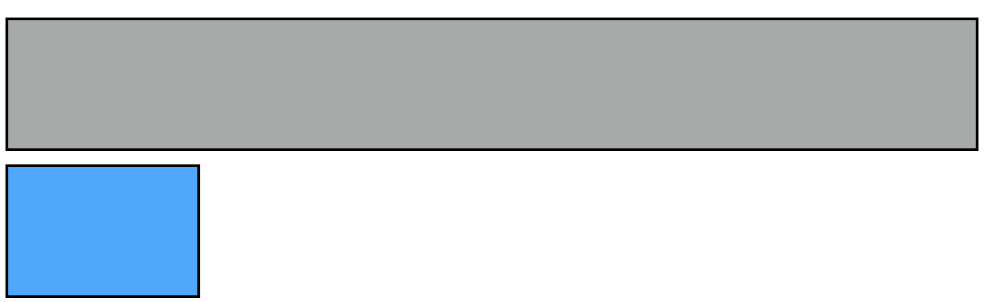

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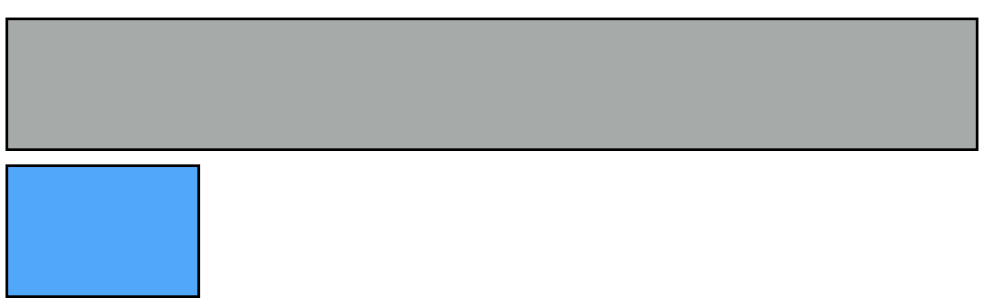

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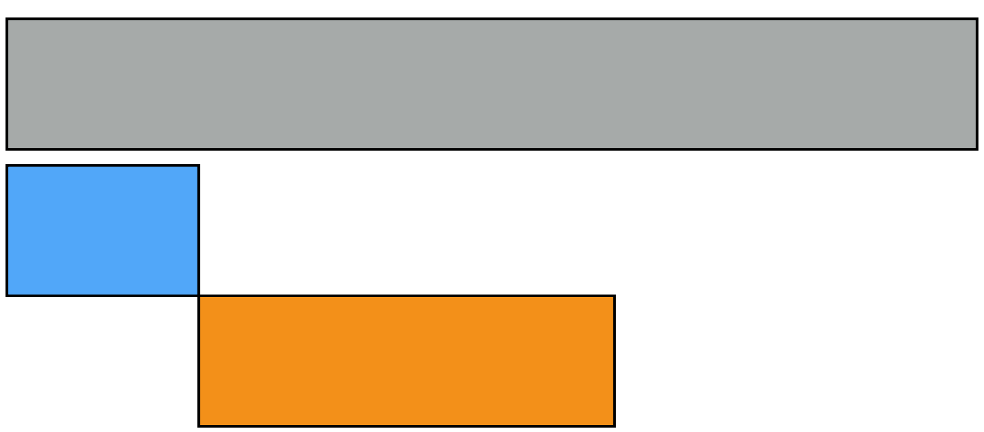

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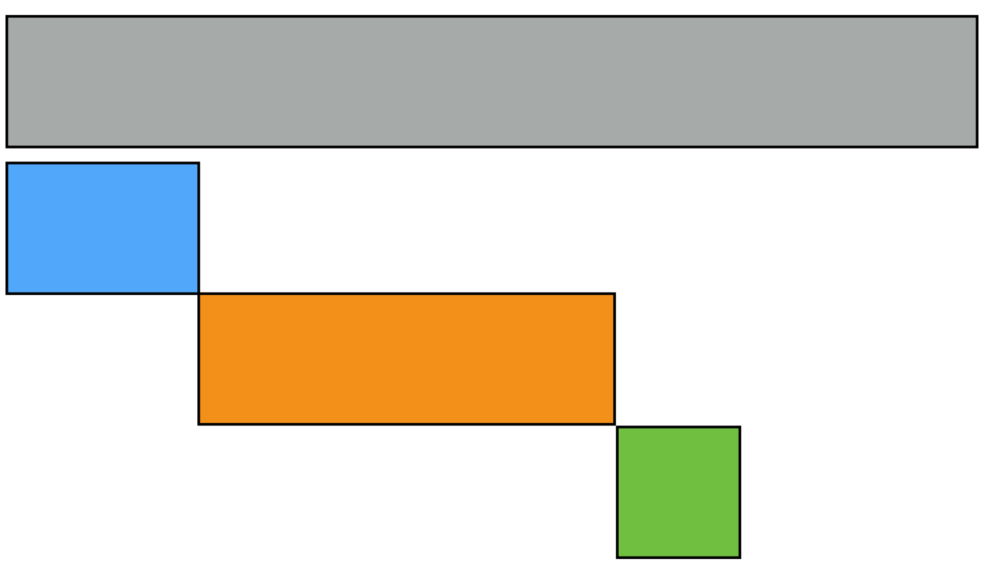

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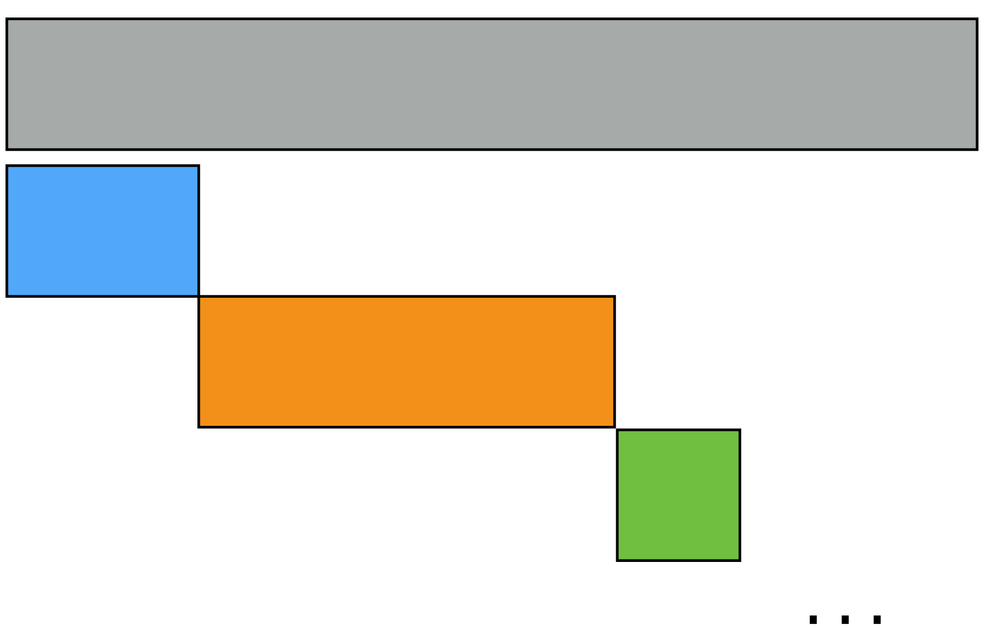

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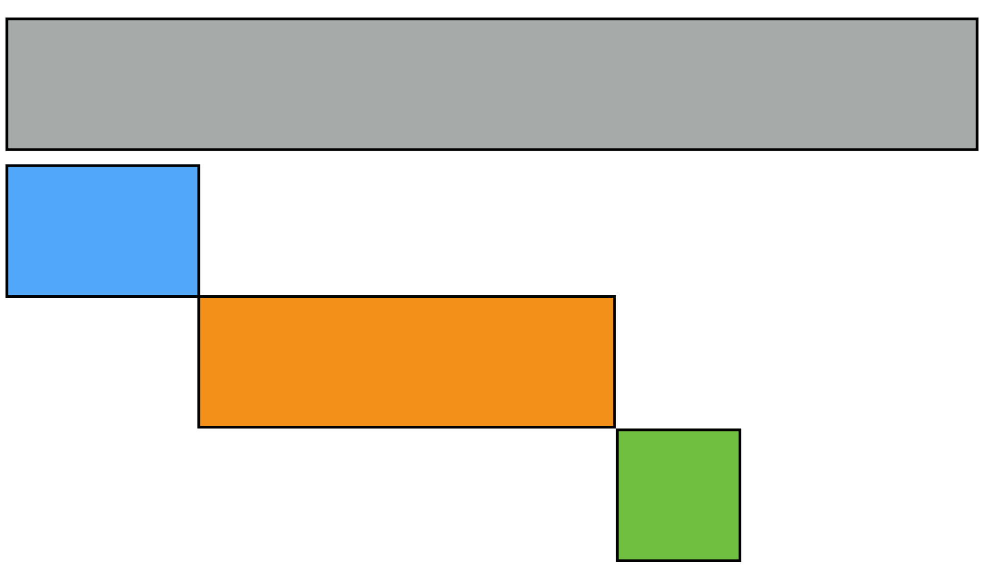

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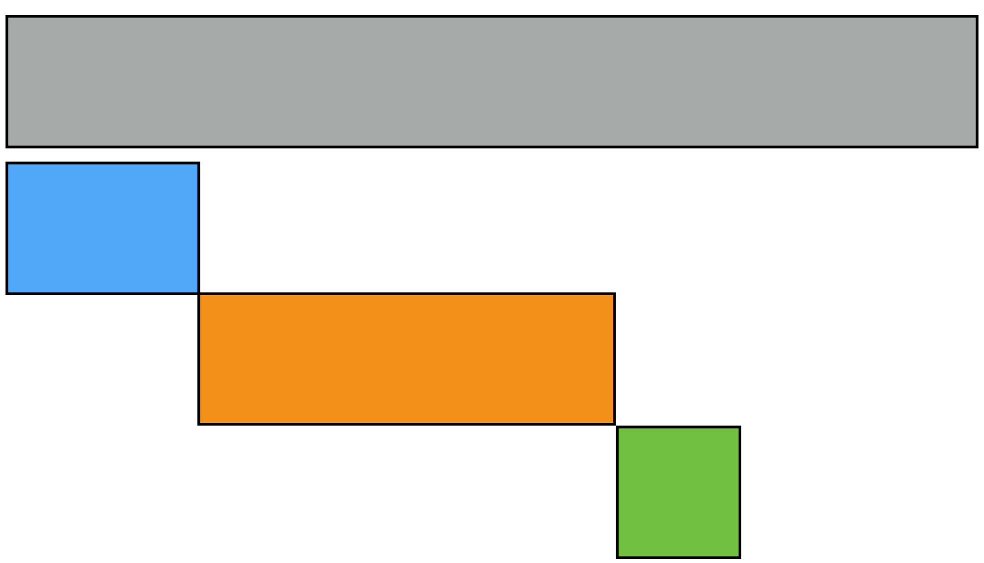

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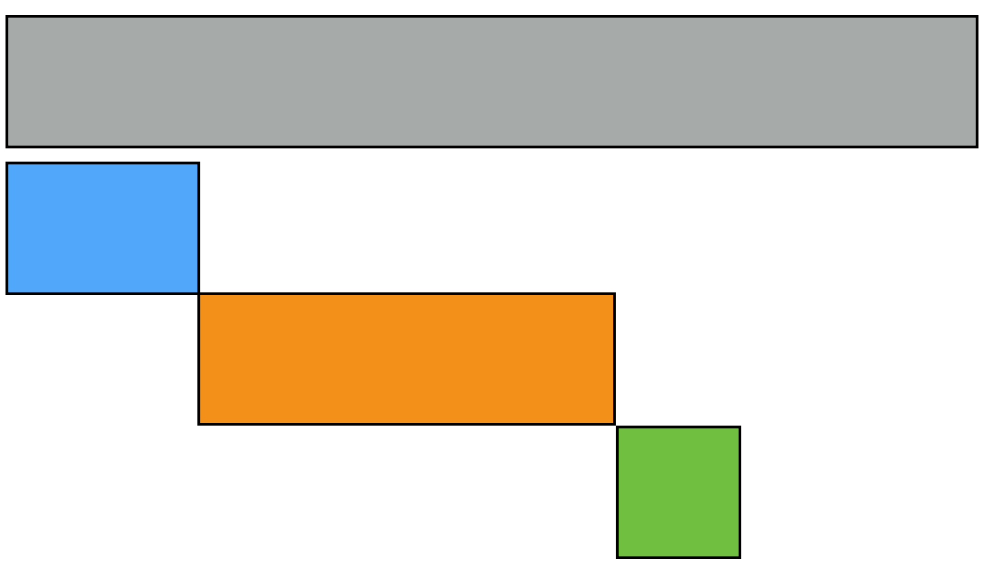

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[van der Vaart, Ghosal 2017]

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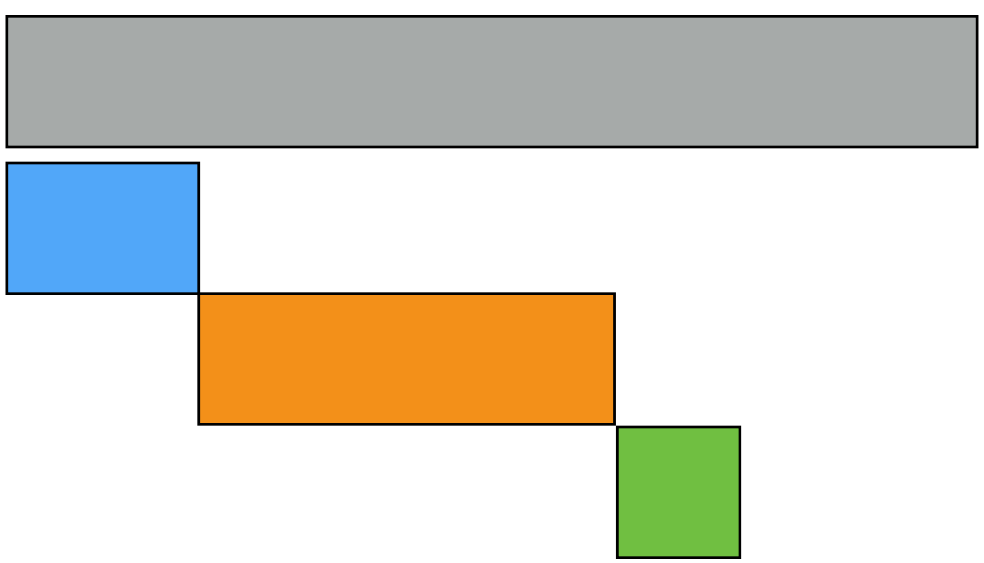

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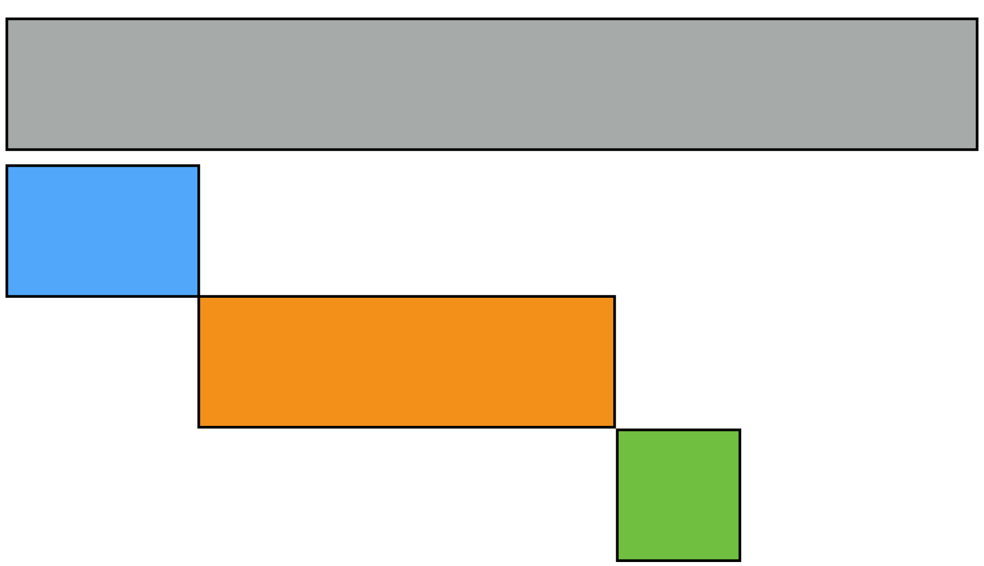

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[McCloskey 1965; Engen 1975; Patil and Taillie 1977; Ewens 1987; Sethuraman 1994; van der Vaart, Ghosal 2017]

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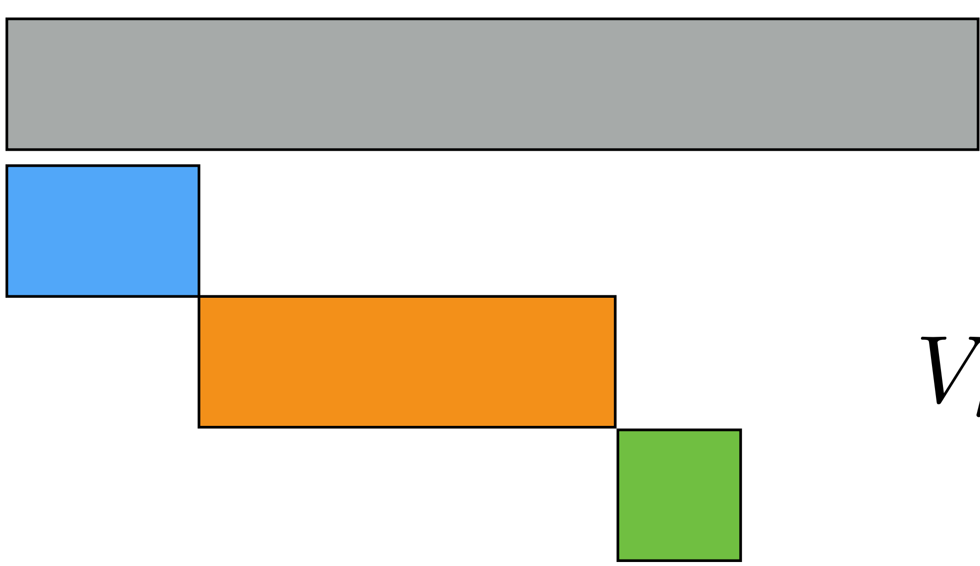

$\ldots$

$$V_k \overset{iid}{\sim} \mathrm{Beta}(1, \alpha) \qquad \rho_k = \left[\prod_{j=1}^{k-1} (1 - V_j)\right] V_k$$

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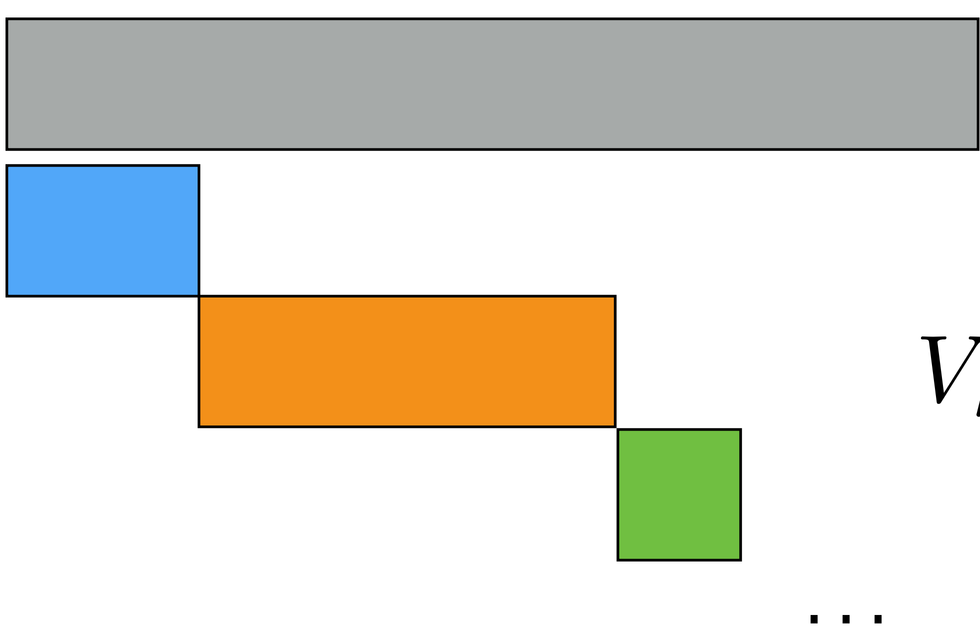

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[demo]

[McCloskey 1965; Engen 1975; Patil and Taillie 1977; Ewens 1987; Sethuraman 1994; van der Vaart, Ghosal 2017]

# Distributions

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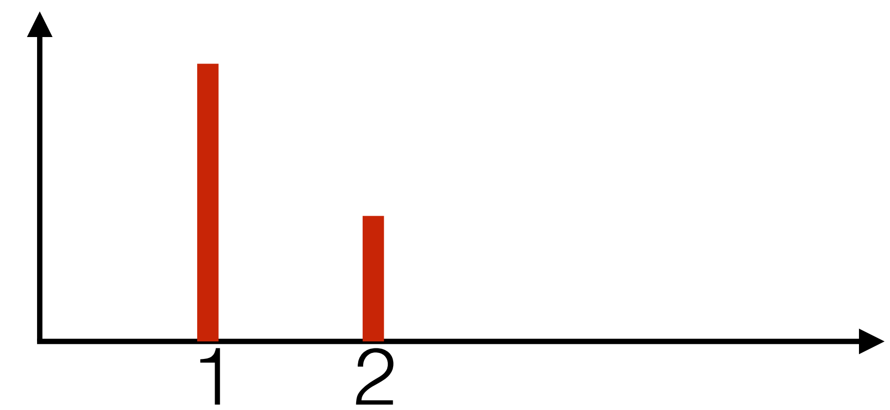
- Beta → random distribution over $1, 2$



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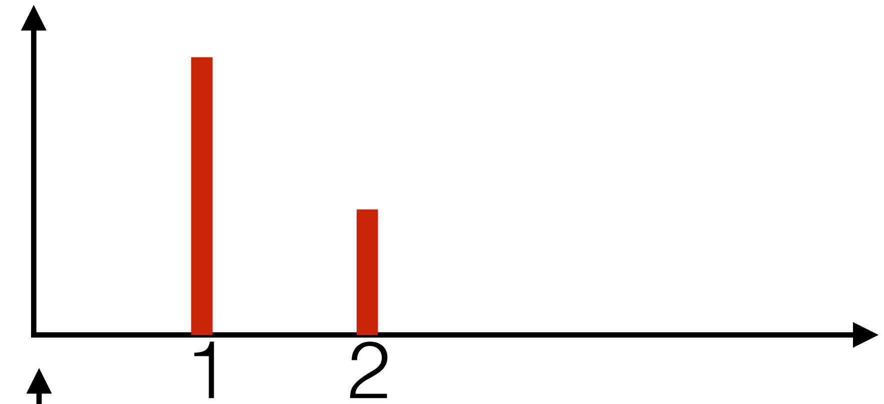


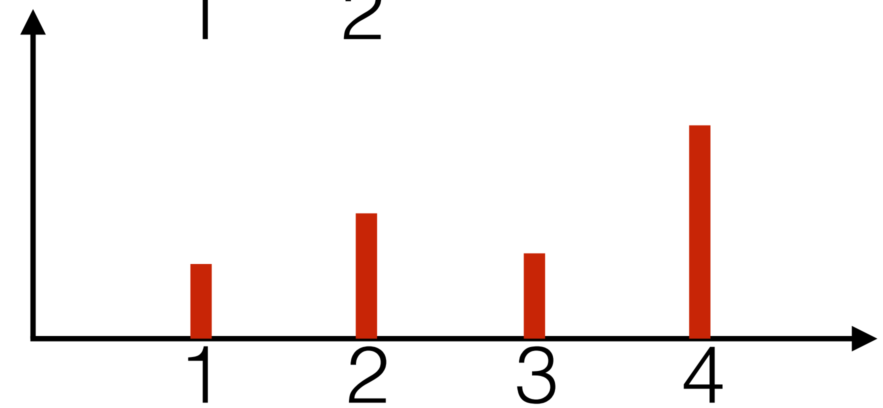

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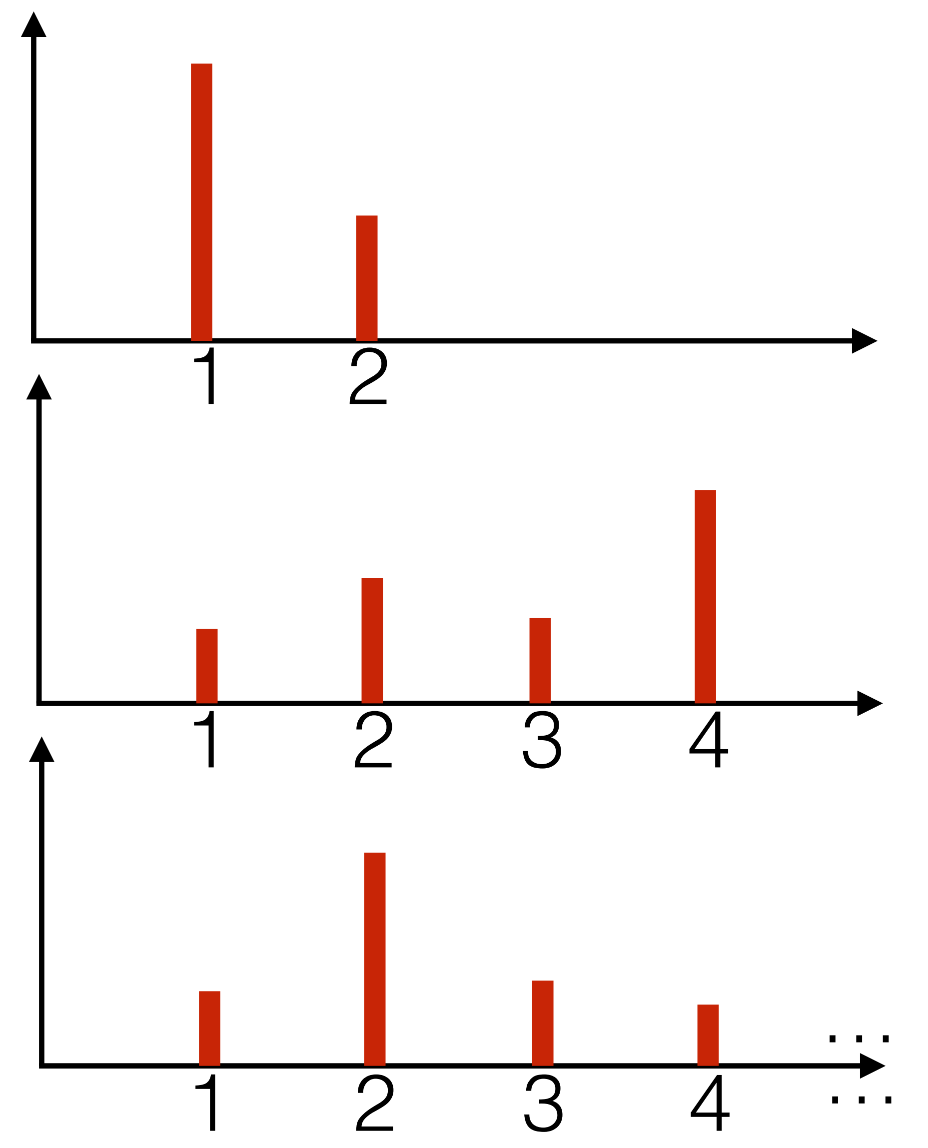

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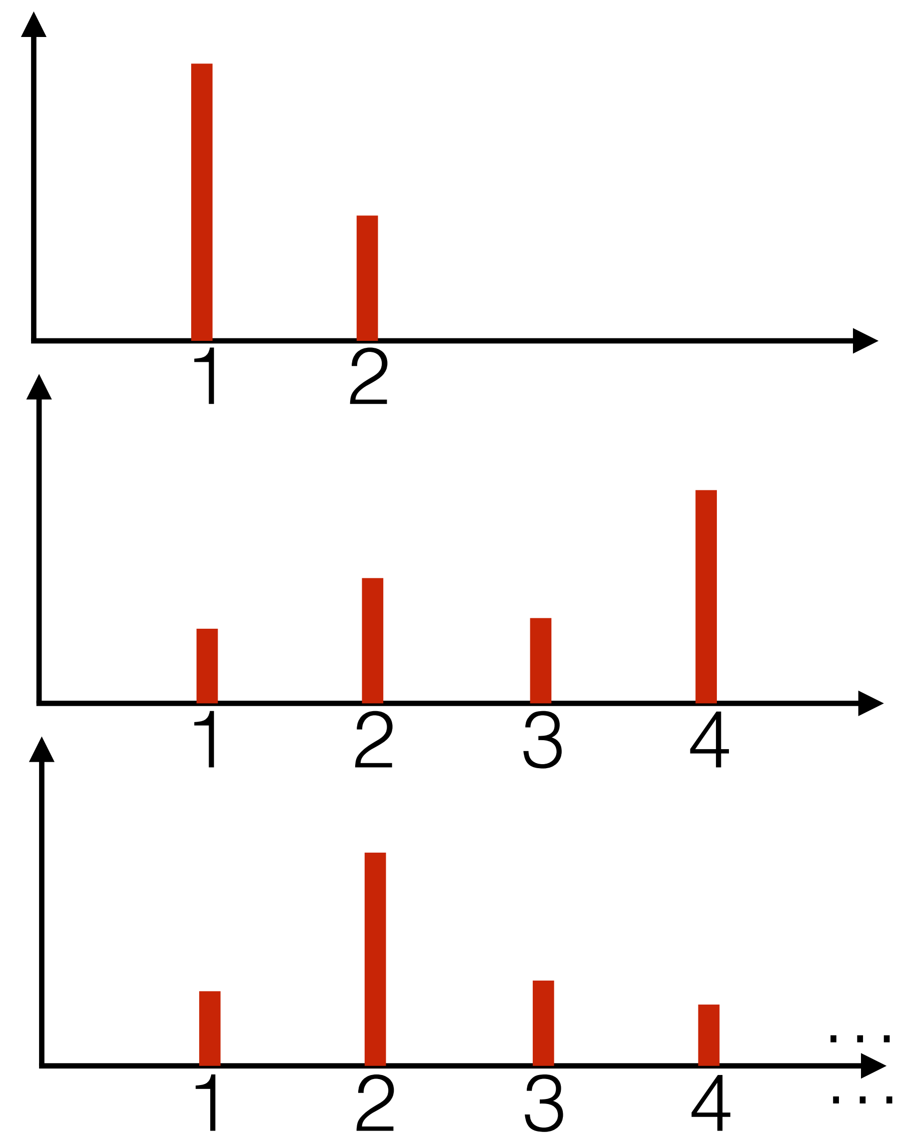


- Infinity of parameters: components
- Growing number of parameters: clusters

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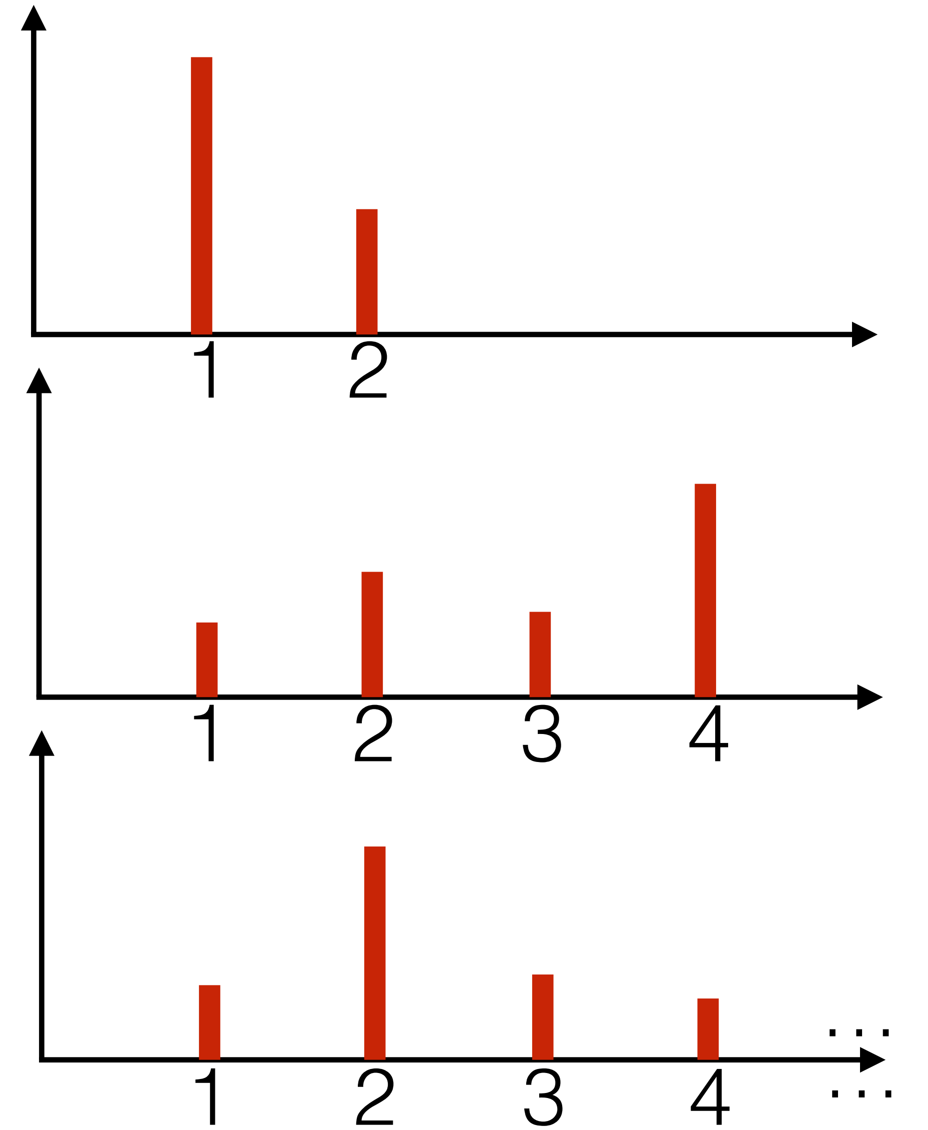

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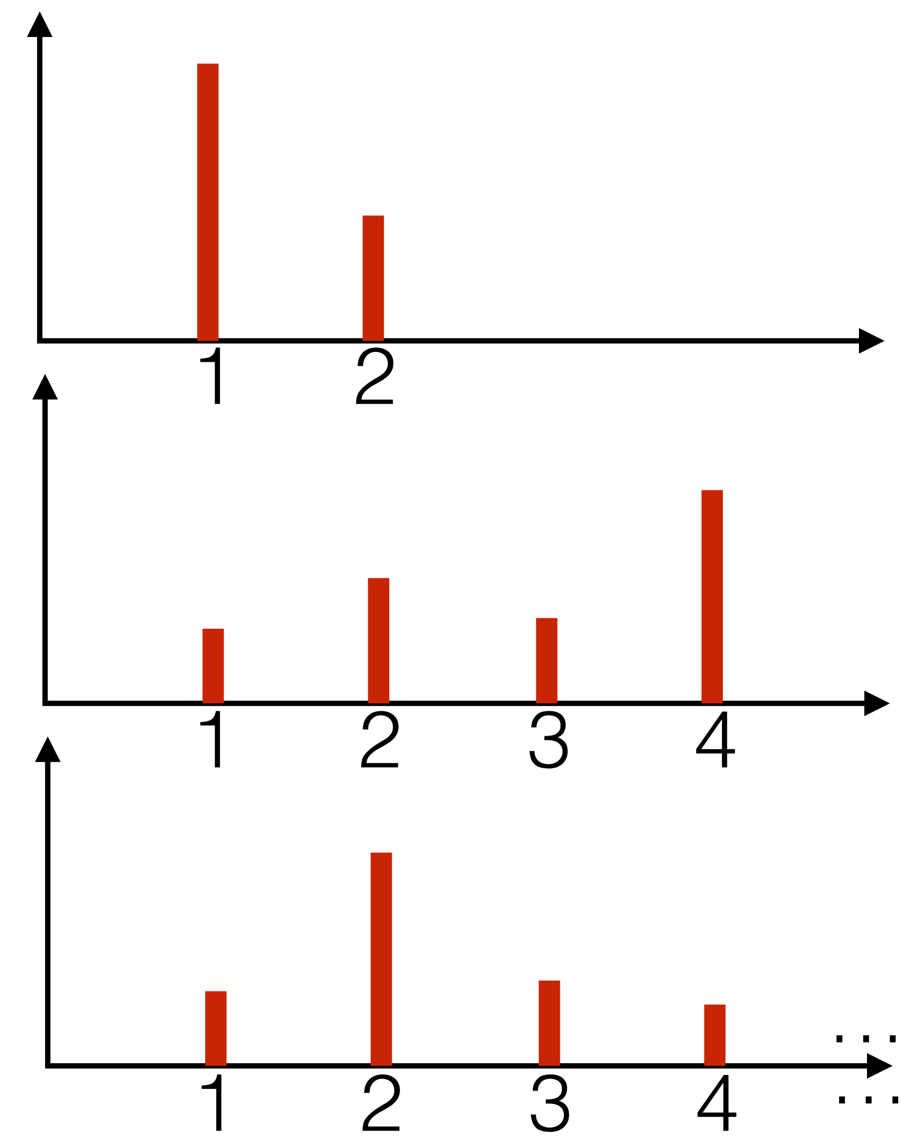


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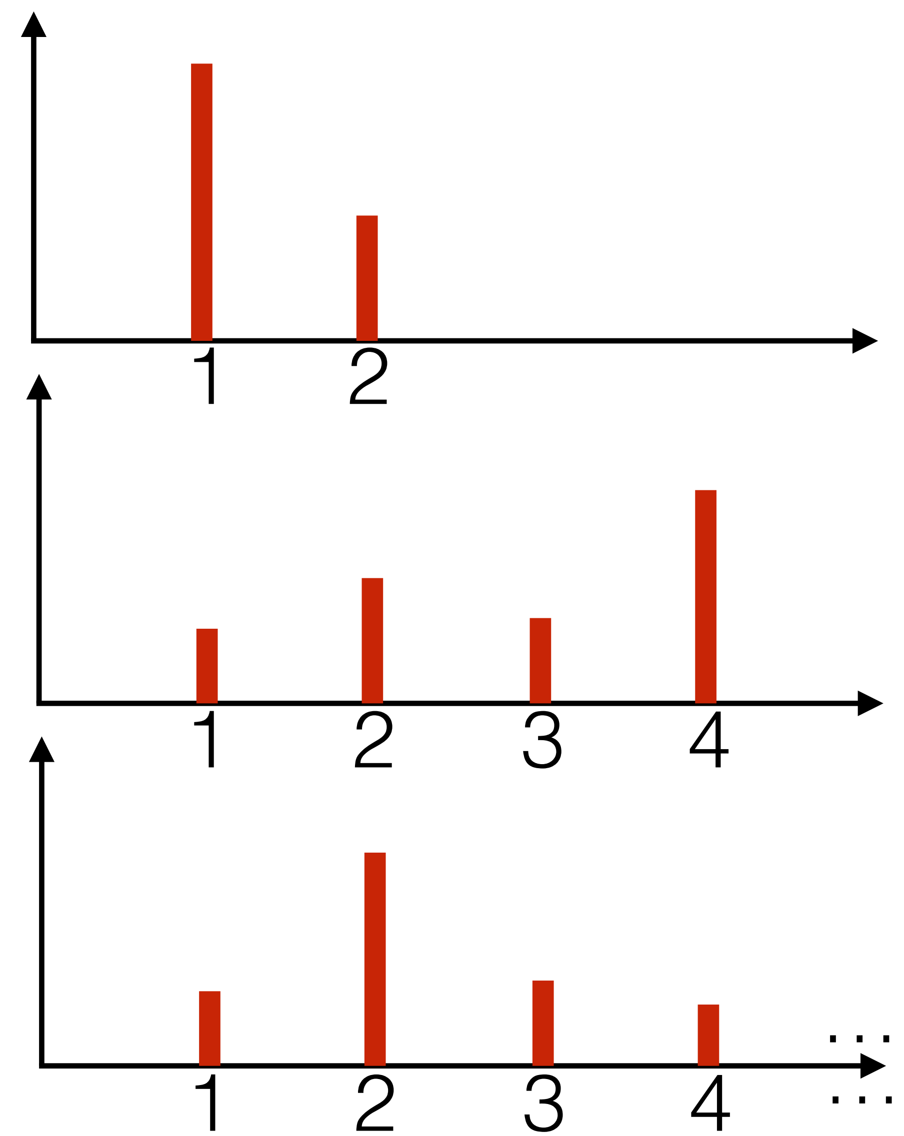


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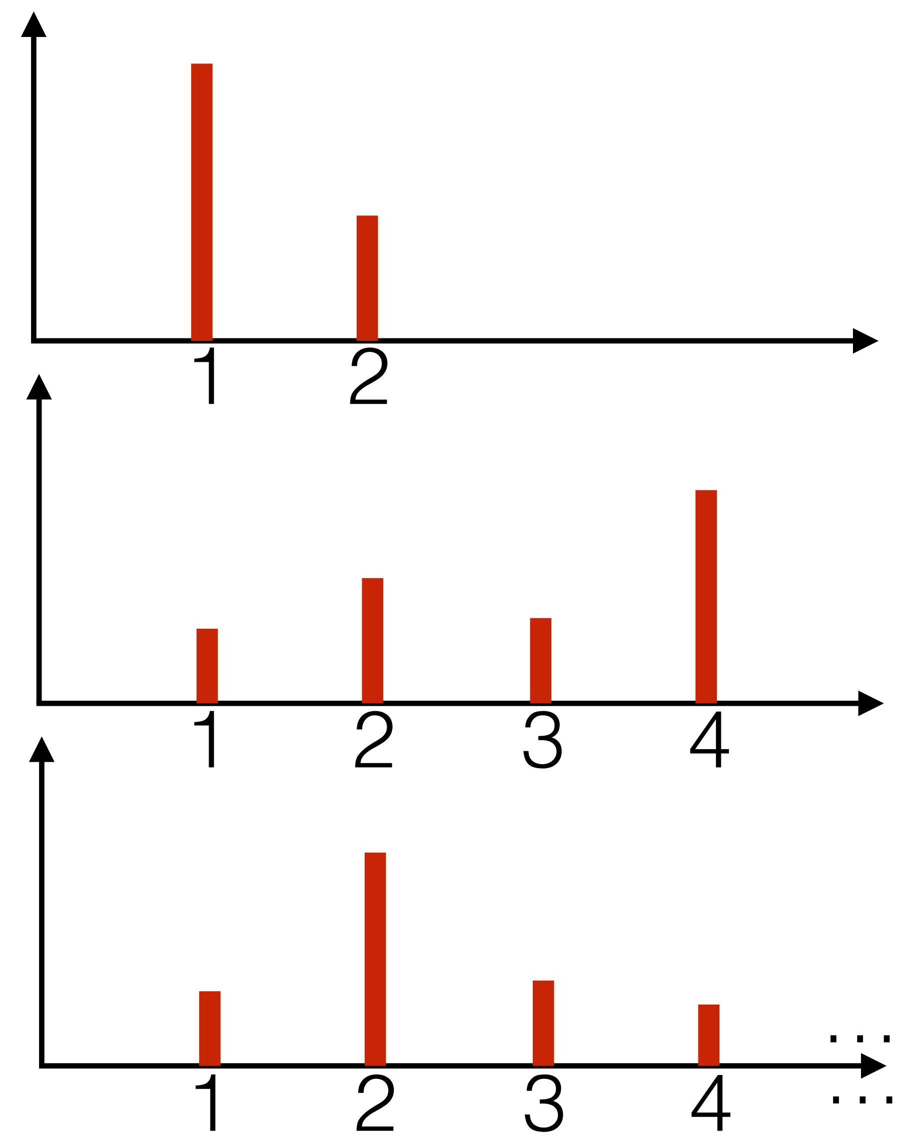


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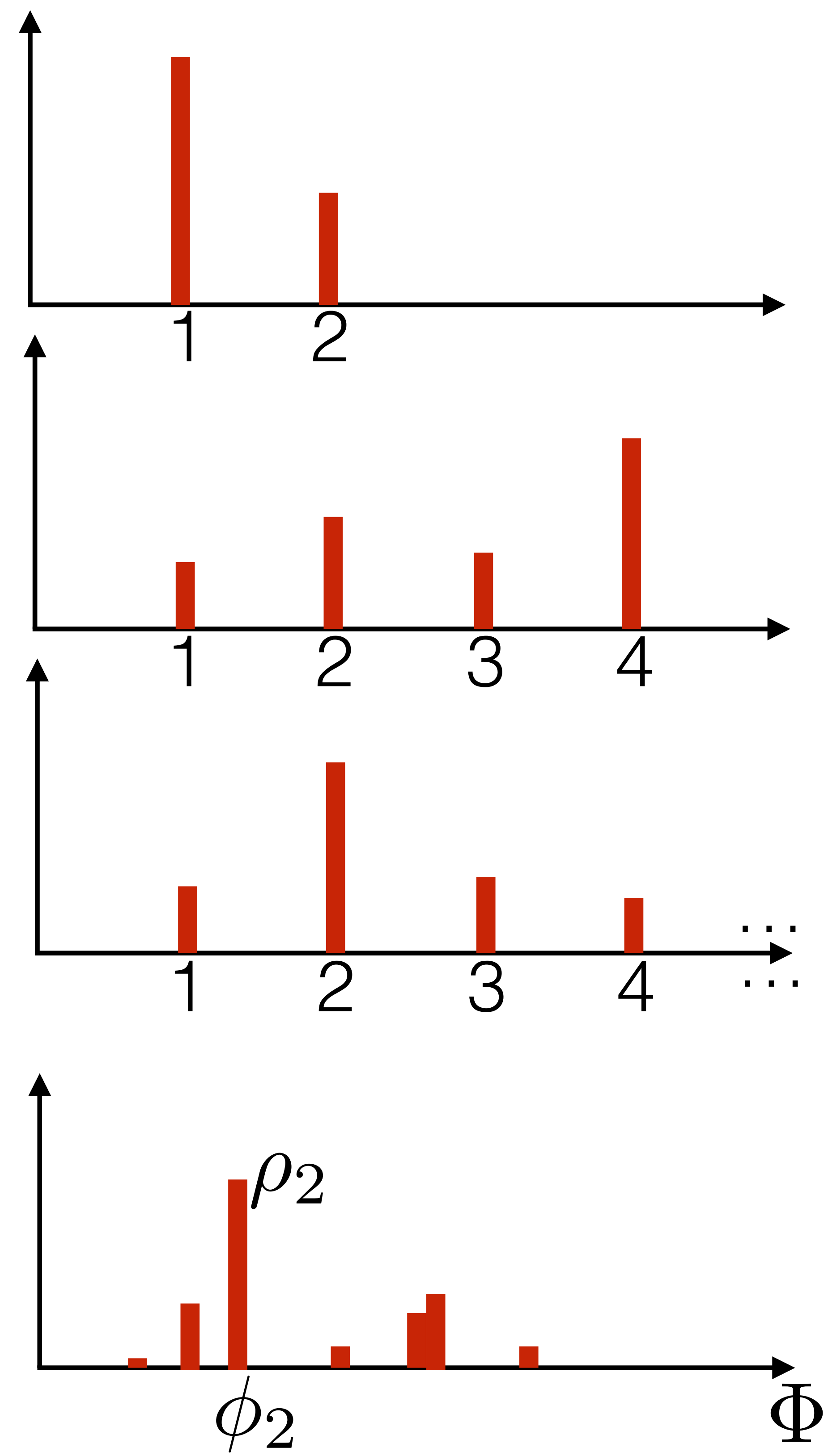


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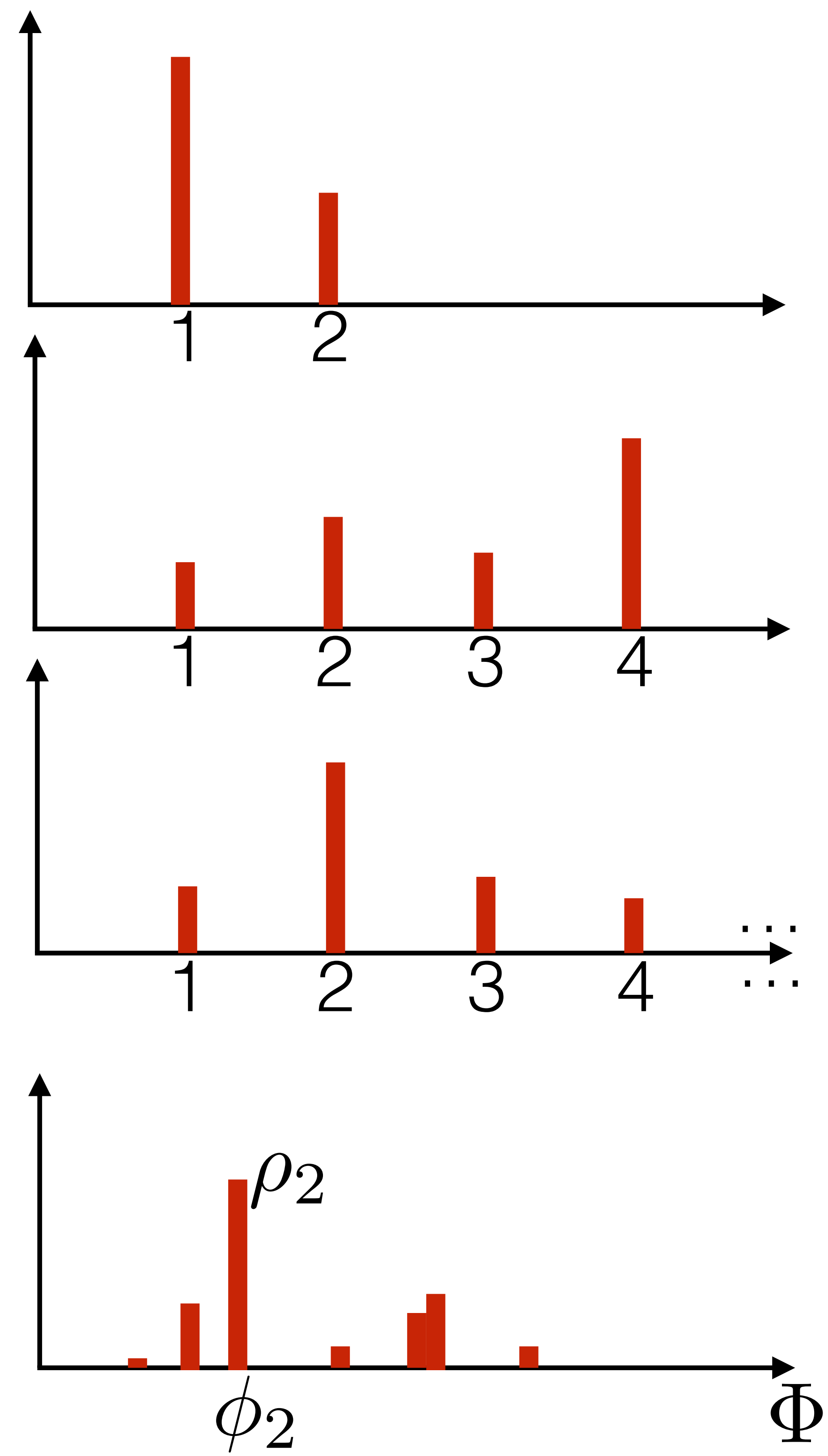


[Ferguson 1973]

# Dirichlet process mixture model

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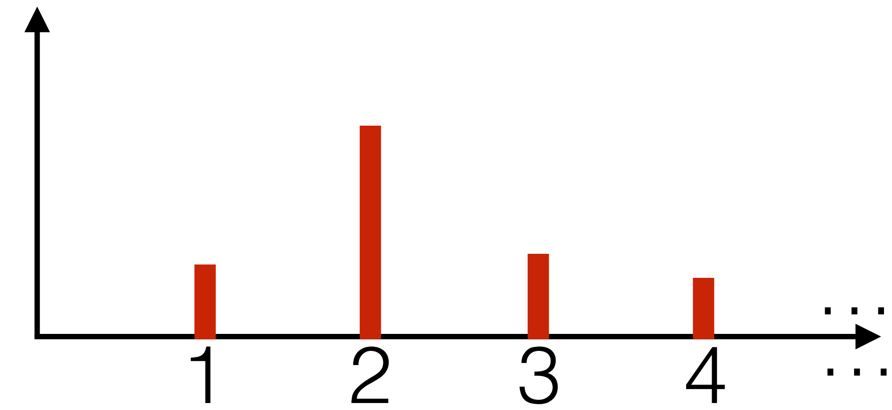

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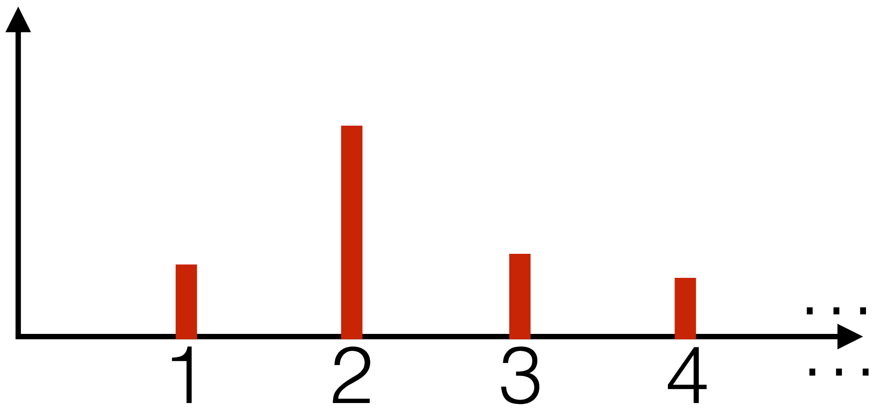

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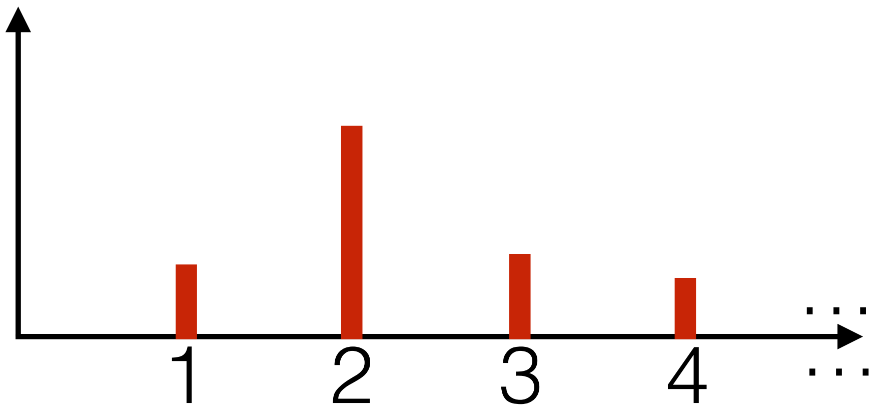


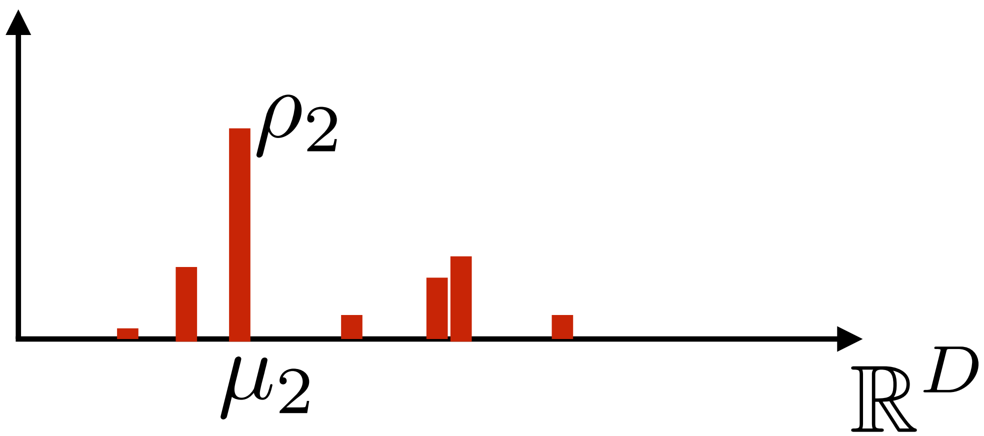

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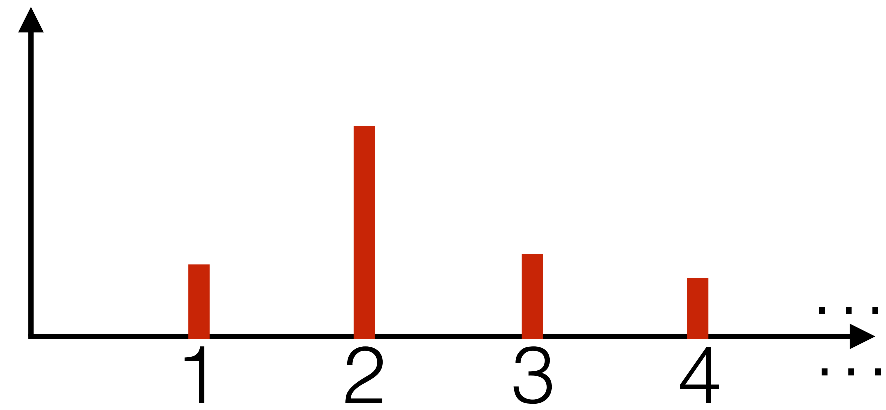


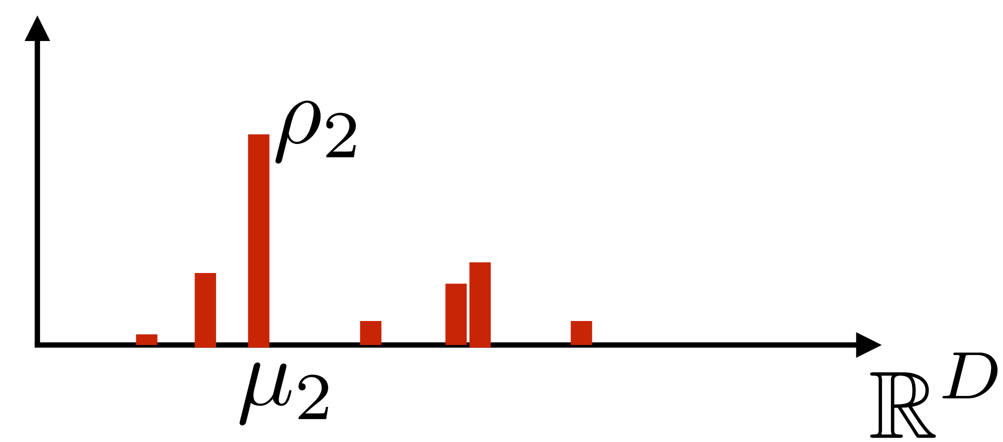

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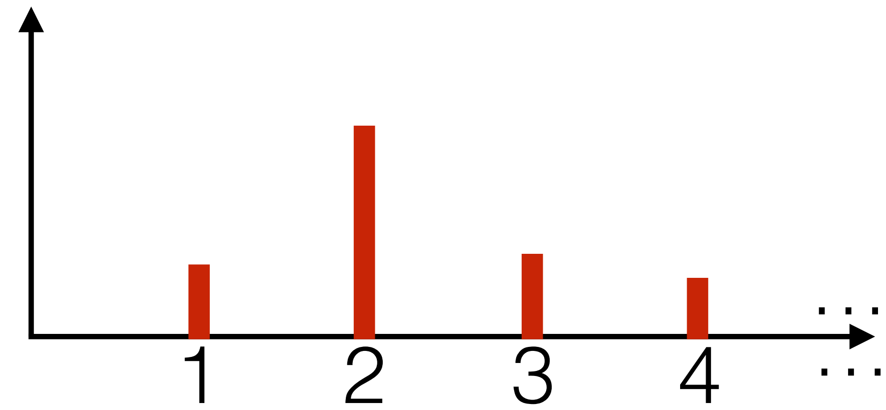
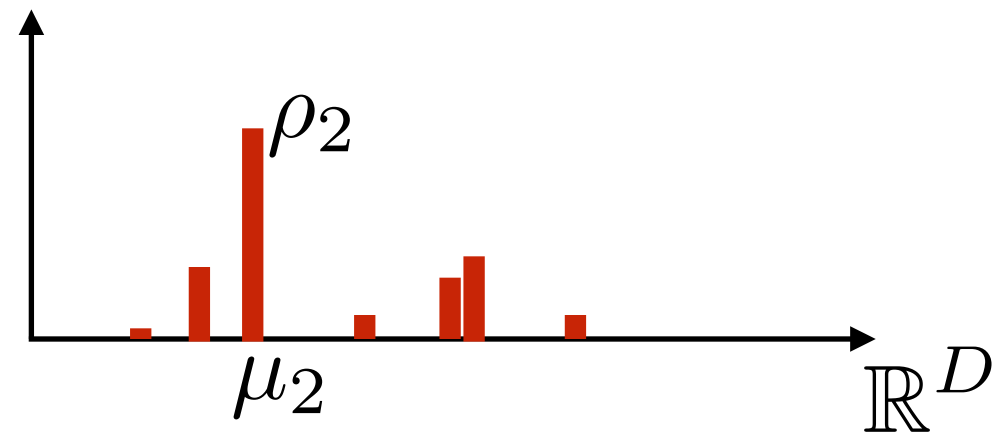



- i.e. $G = \sum_{k=1}^{\infty} \rho_k \delta_{\mu_k} \overset{d}{=} \mathrm{DP}(\alpha, \mathcal{N}(\mu_0, \Sigma_0))$



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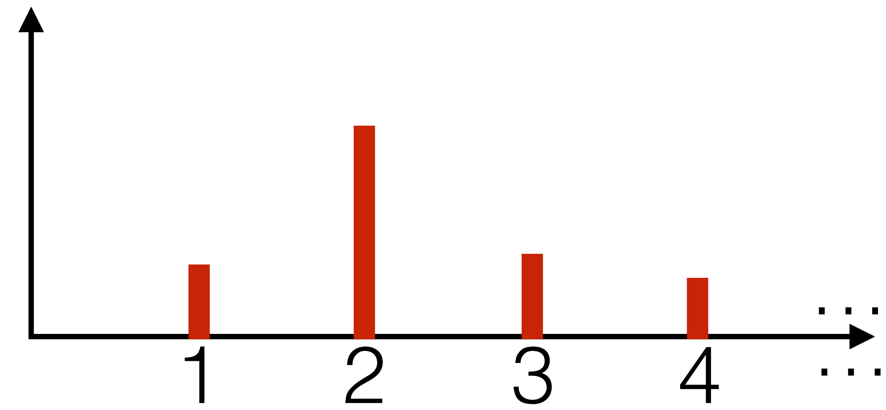


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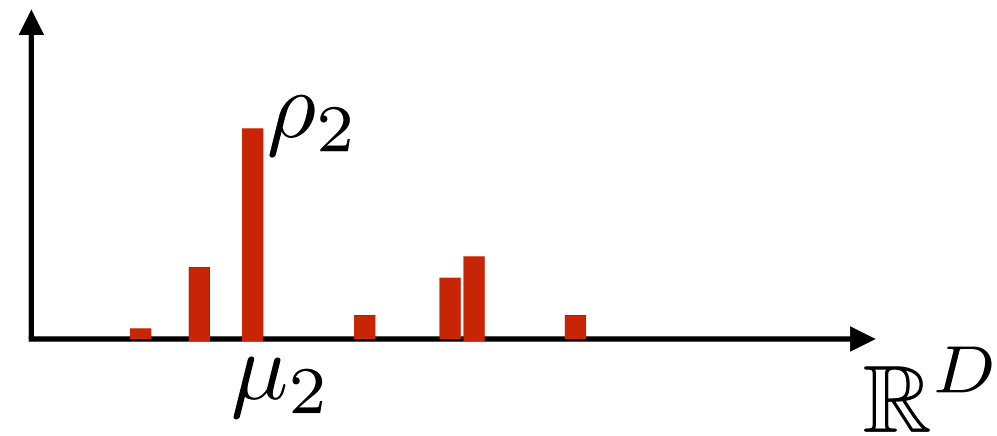


$$z_n \overset{iid}{\sim} \mathrm{Categorical}(\rho)$$

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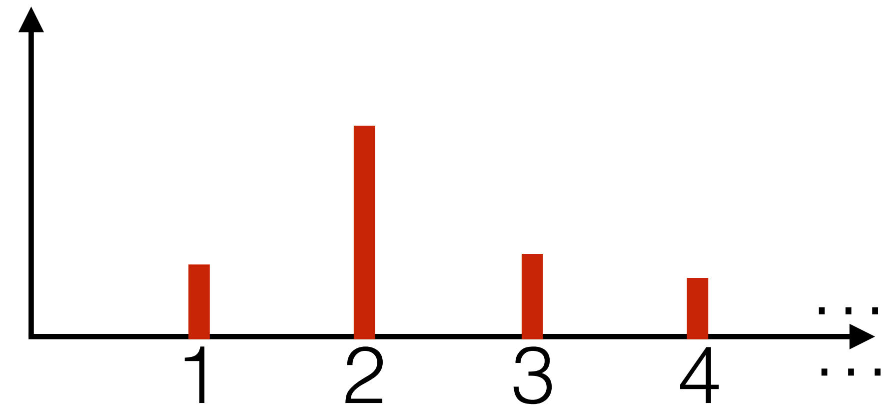
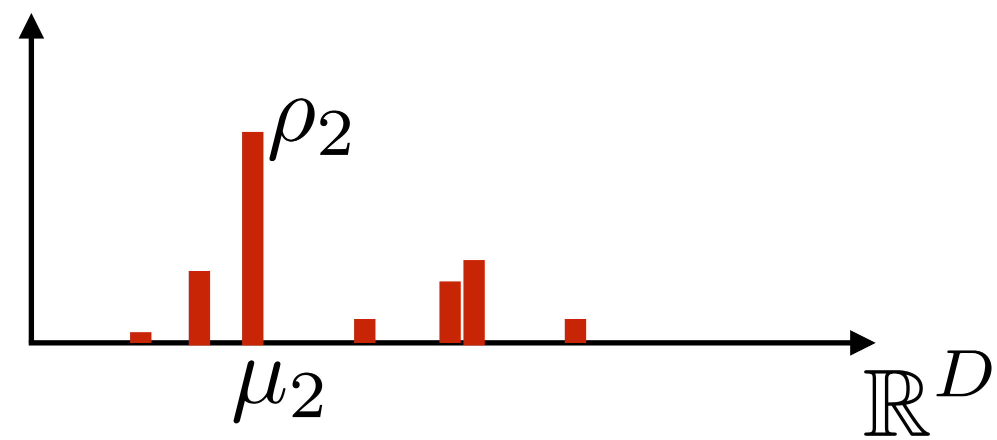



- i.e. $G = \sum_{k=1}^{\infty} \rho_k \delta_{\mu_k} \overset{d}{=} \mathrm{DP}(\alpha, \mathcal{N}(\mu_0, \Sigma_0))$

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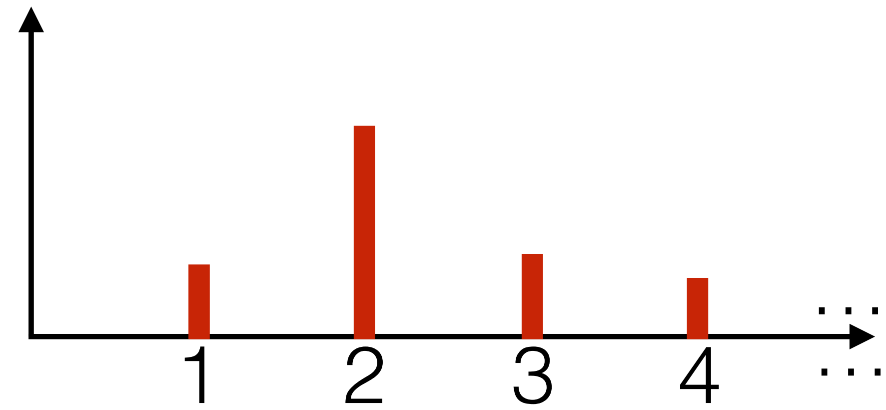
# Dirichlet process mixture model

- Gaussian mixture model

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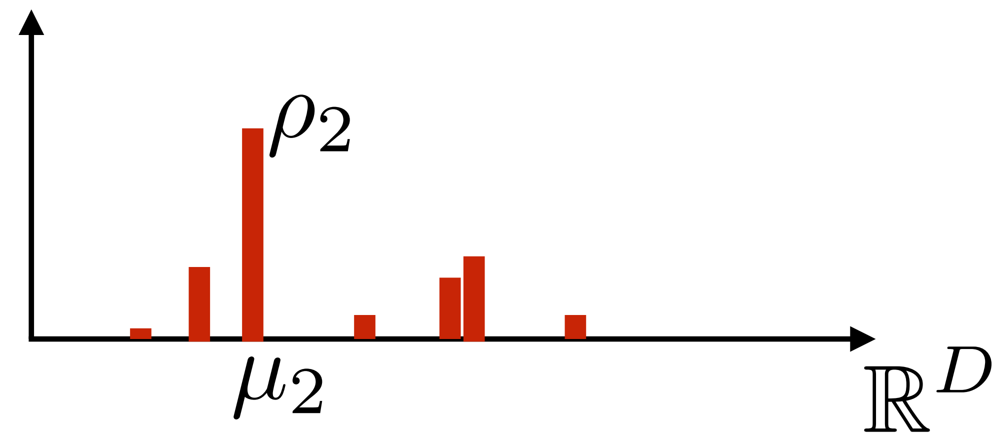


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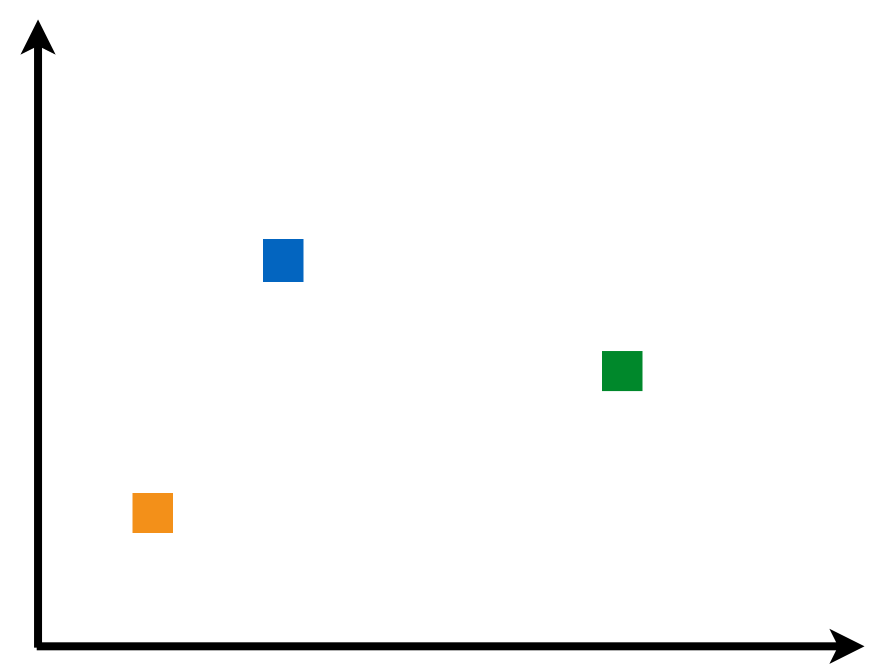

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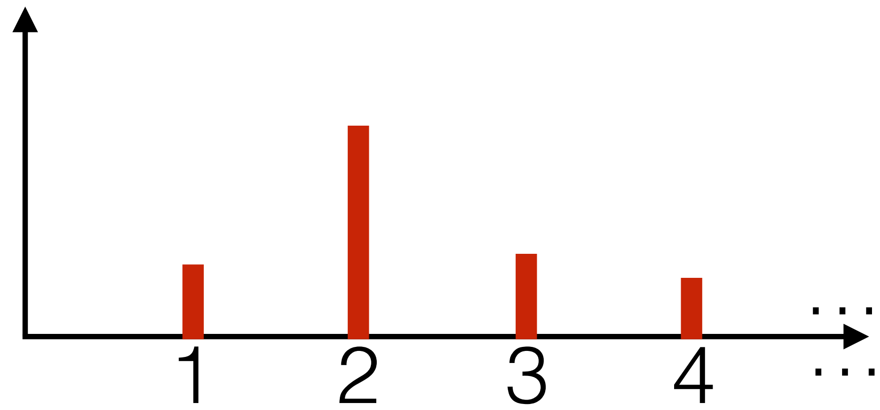


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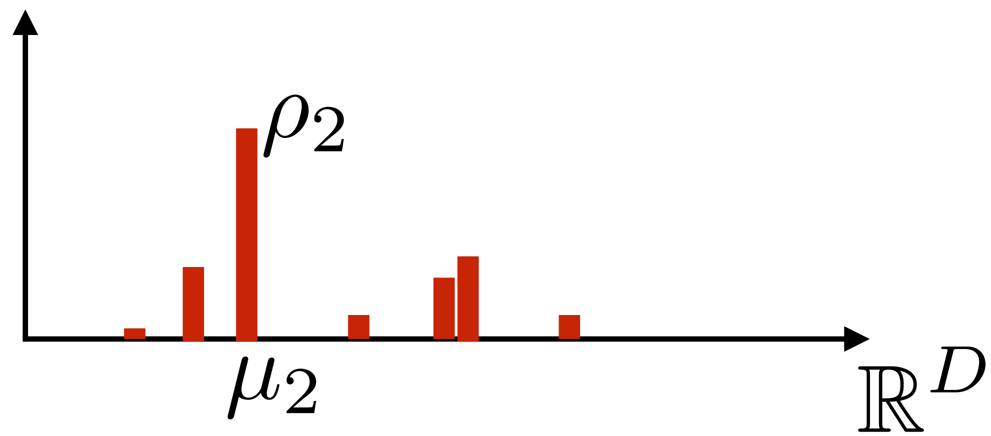


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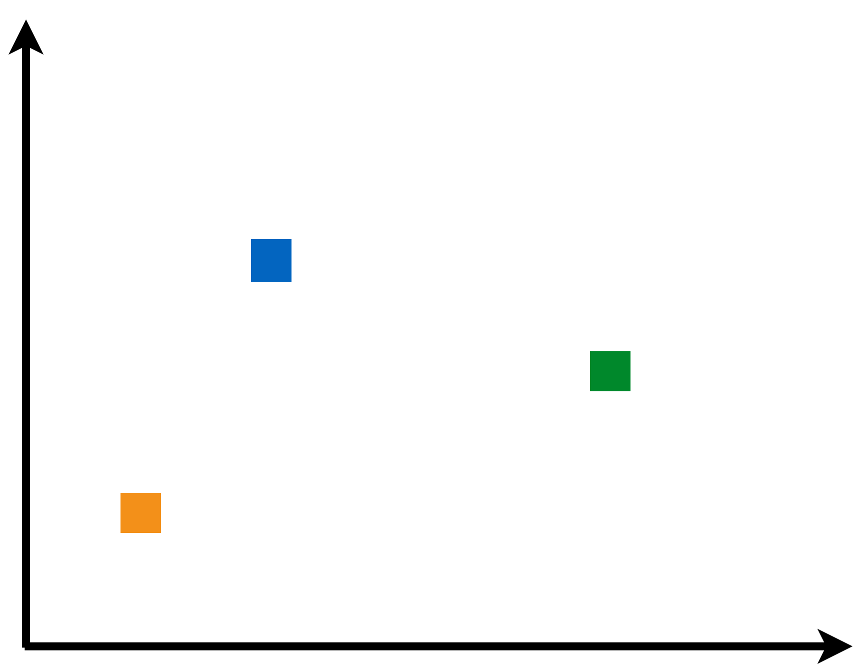

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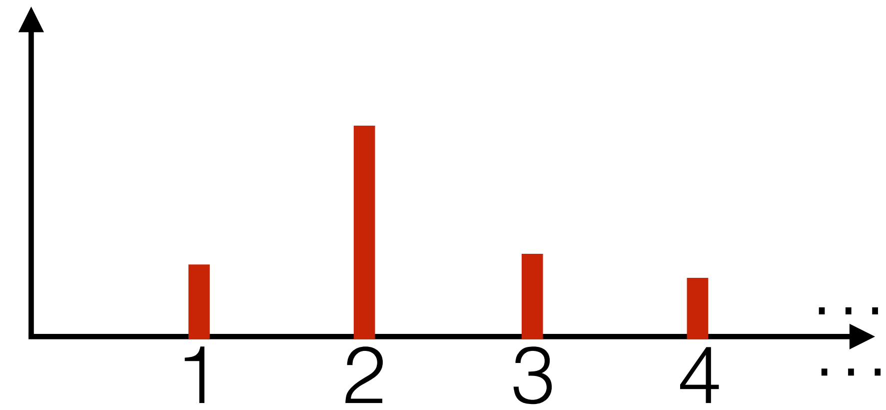


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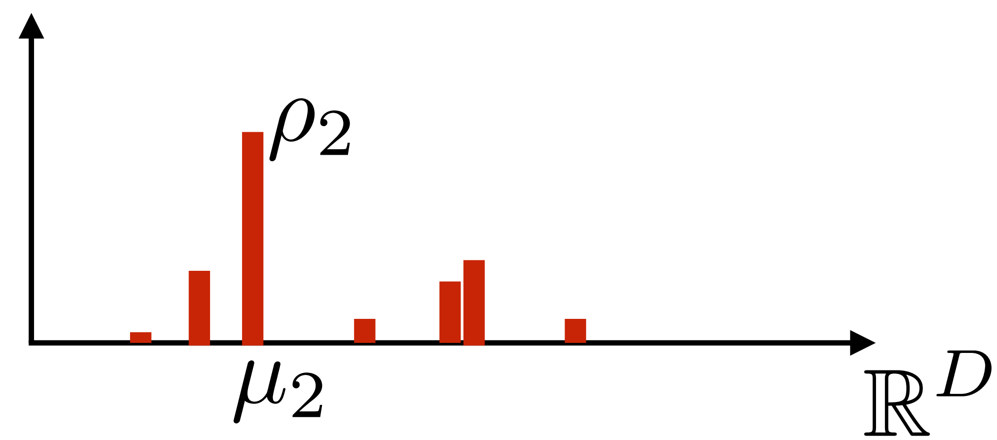


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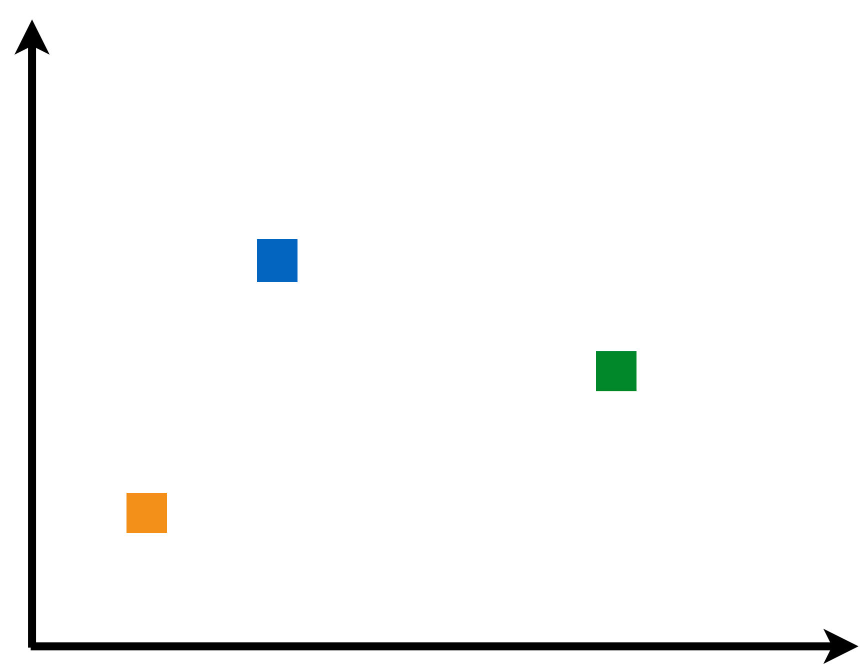

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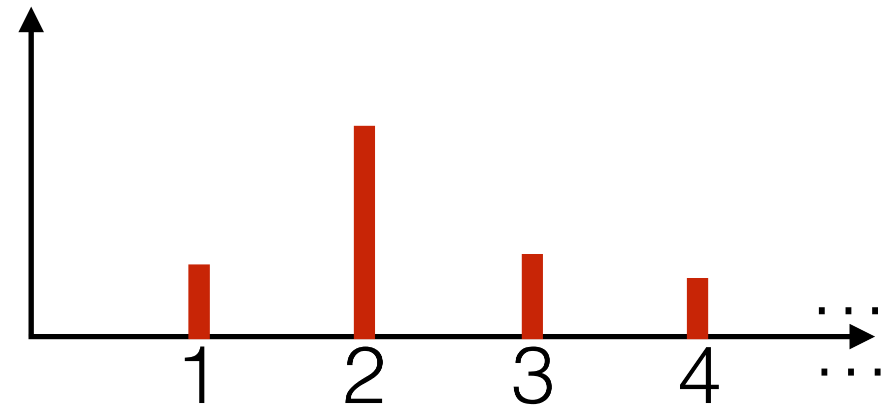


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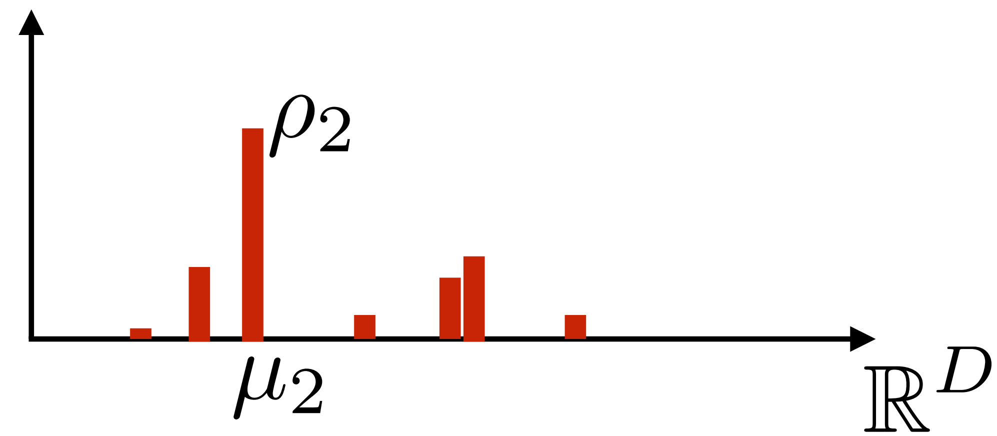


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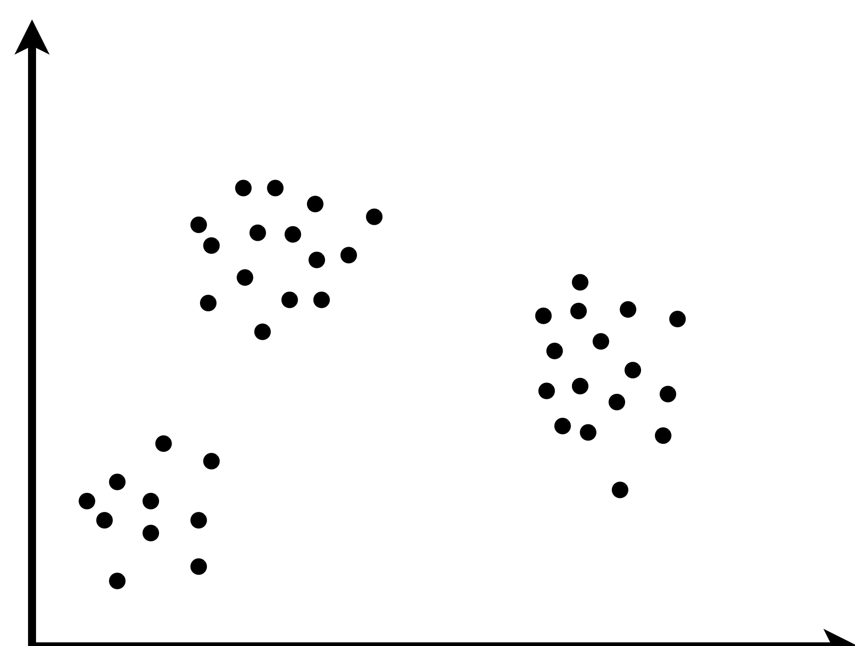

# Roadmap

- Example problem: clustering
- Example NPBayes model: Dirichlet process
- Chinese restaurant process
- Inference
- Venture further into the wild world of Nonparametric Bayes

- Big questions
  - Why NPBayes?
  - What does an infinite/growing number of parameters really mean (in NPBayes)?
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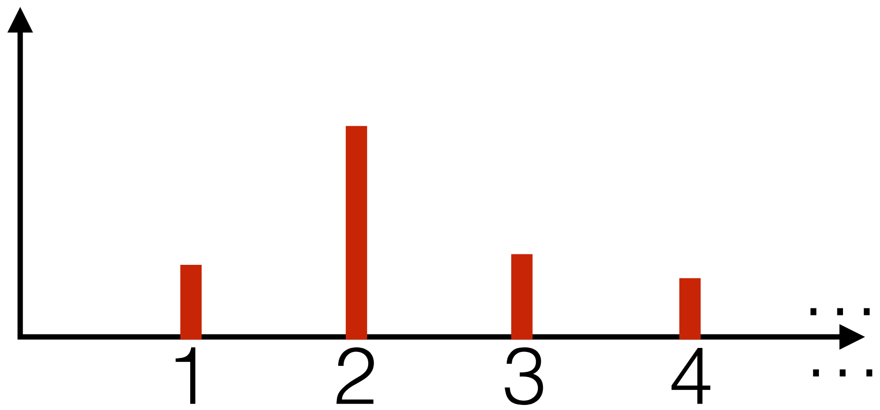
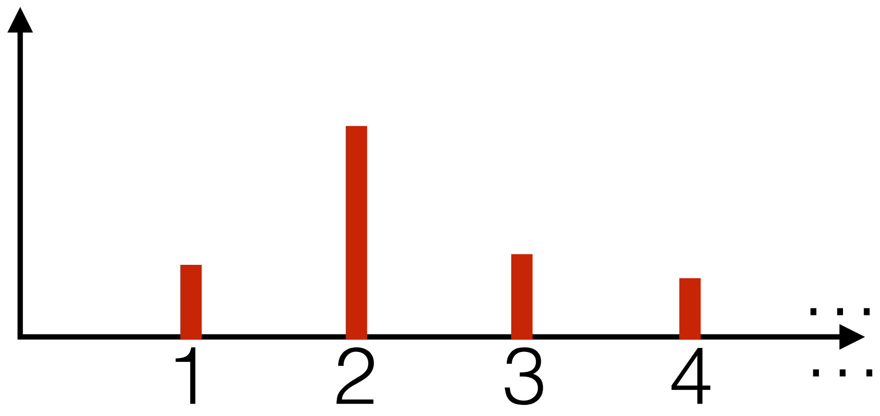
# Exercises



# Exercises

[slides, code: www.tamarabroderick.com/tutorials.html]



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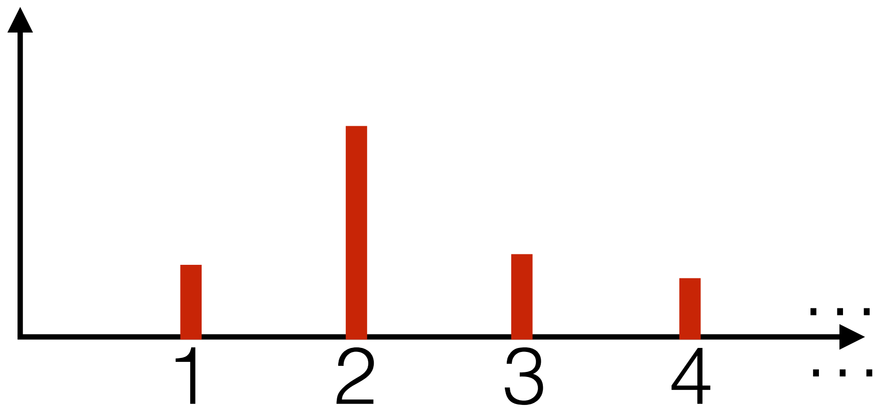

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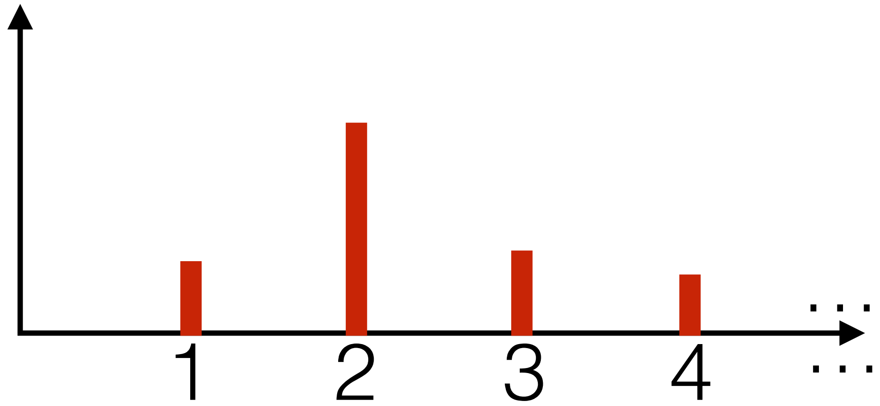

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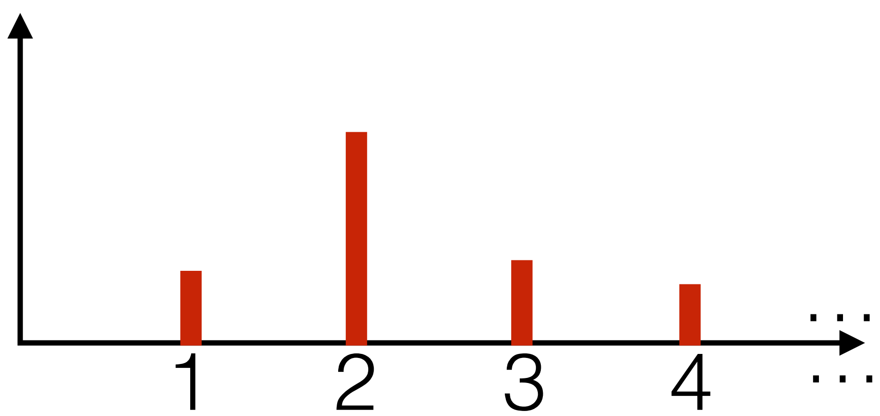

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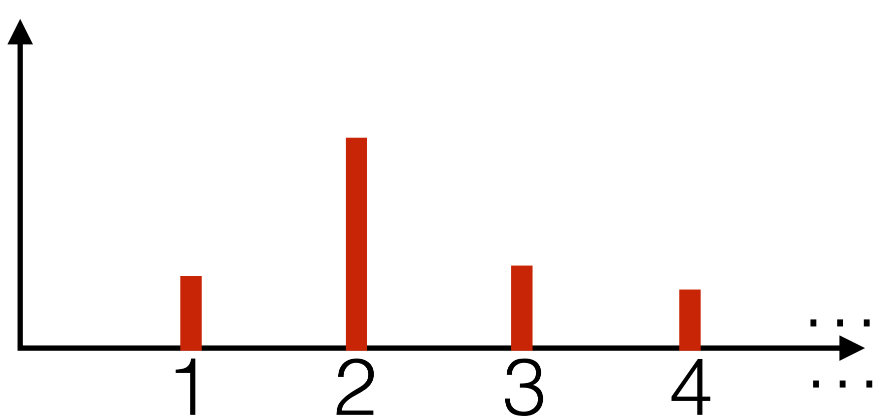

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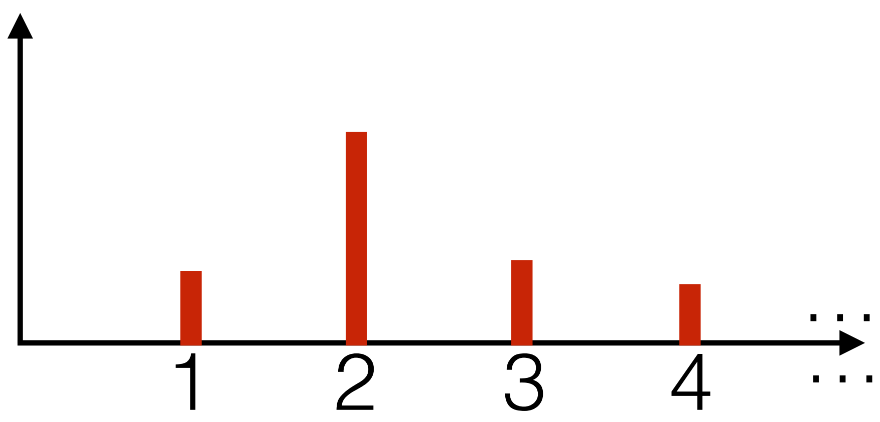

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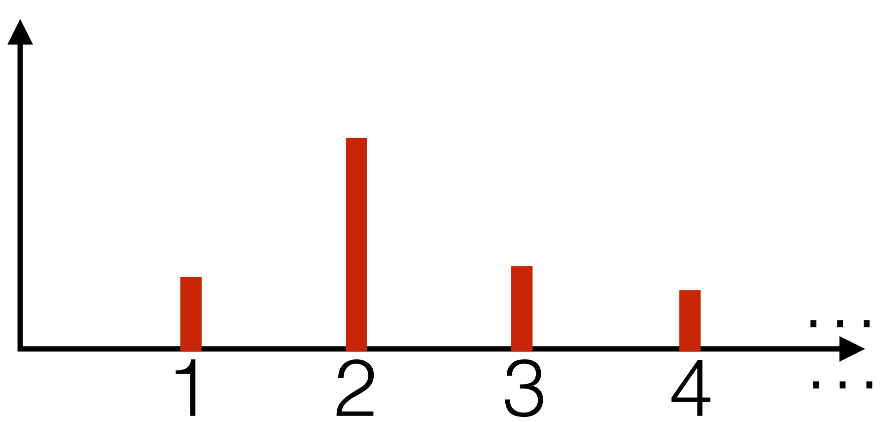


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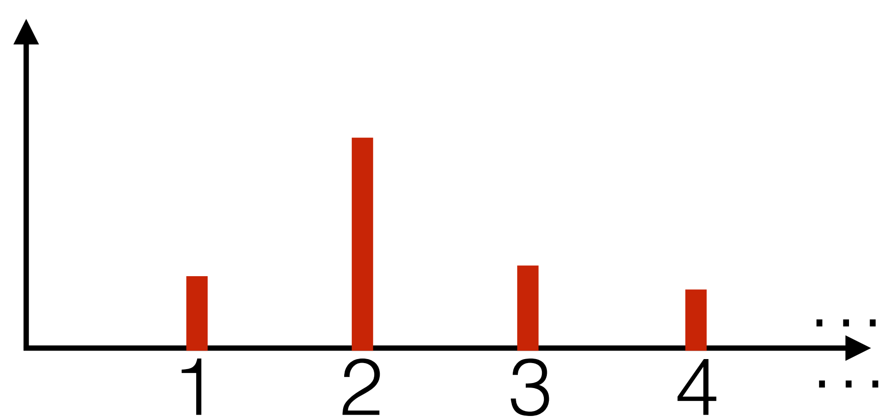


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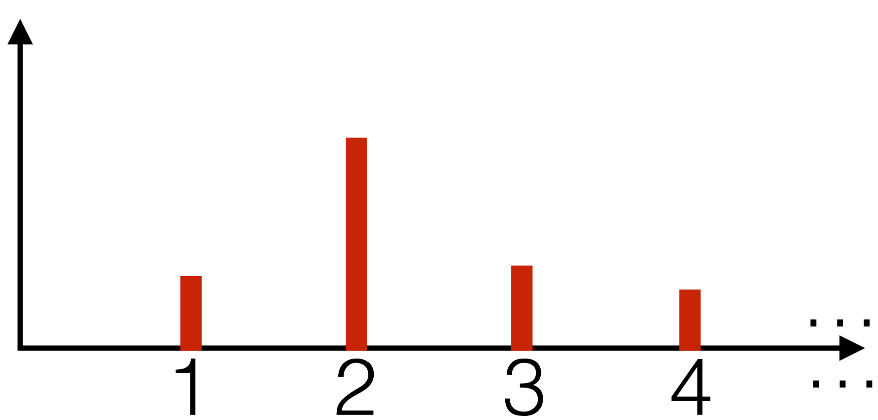


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- How does the distribution of # clusters at *N* change with $\alpha$?

# References

A full reference list is provided at the end of the “Part II” slides.