Nonparametric Bayesian Methods: Models, Algorithms, and Applications (Part II)

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Electrical Engineering & Computer Science
MIT

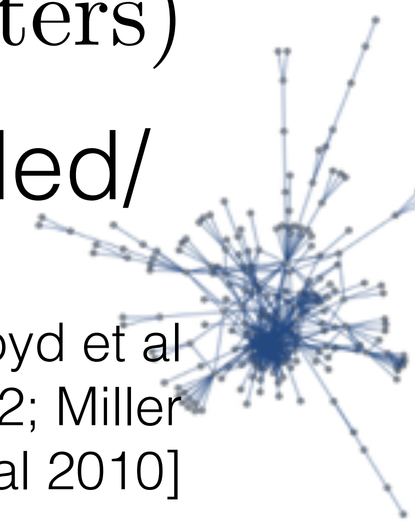
http://www.tamarabroderick.com/tutorials.html

Nonparametric Bayes

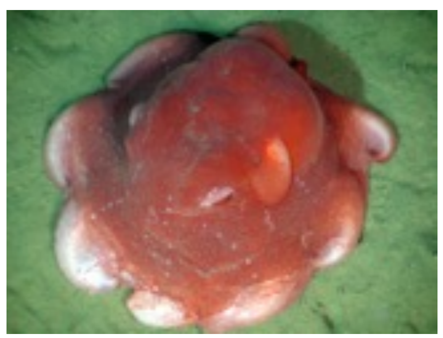
- Bayesian methods that are not parametric
- Bayesian

$$\mathbb{P}(\text{parameters}|\text{data}) \propto \mathbb{P}(\text{data}|\text{parameters})\mathbb{P}(\text{parameters})$$

- Not parametric (i.e. not finite parameter, unbounded/ growing/infinite number of parameters)



[Lloyd et al 2012; Miller et al 2010]



[Ed Bowlby, NOAA]

[Fox et al 2014]

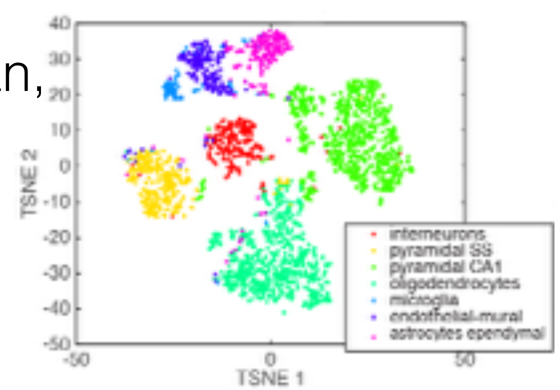
[MIT xPRO]

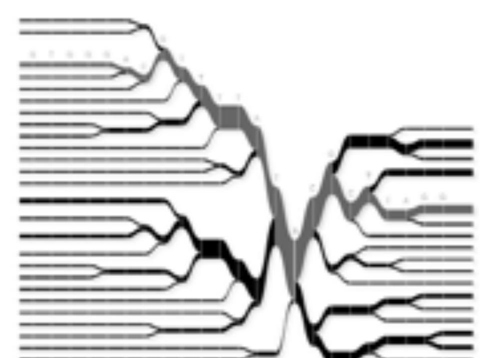
[Lan et al 2015]

[ESO/ L. Calçada/ M. Kornmesser 2017]

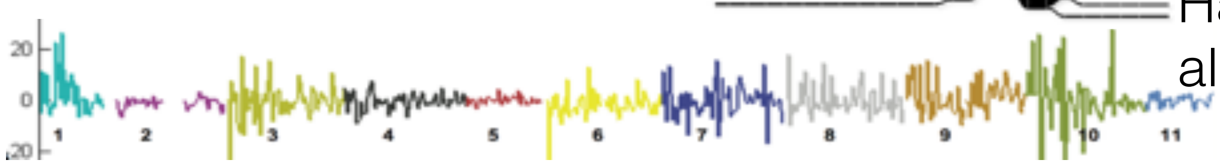
[Del Pozzo et al 2017, 2018]

[Prabhakaran, Azizi, Carr, Pe'er 2016]





[Ewens 1972; Hartl, Clark 2003; Harris et al 2015]



[Saria et al 2010]

[Xu et al 2015; Cassidy et al 2015]

Roadmap

[slides, code: www.tamarabroderick.com/tutorials.html]

- Example problem: clustering
- Example NPBayes model: Dirichlet process
- Chinese restaurant process
- Inference
- Venture further into the wild world of Nonparametric Bayes

- Big questions
  - Why NPBayes?
  - What does an infinite/growing number of parameters really mean (in NPBayes)?
  - Why is NPBayes challenging but practical?

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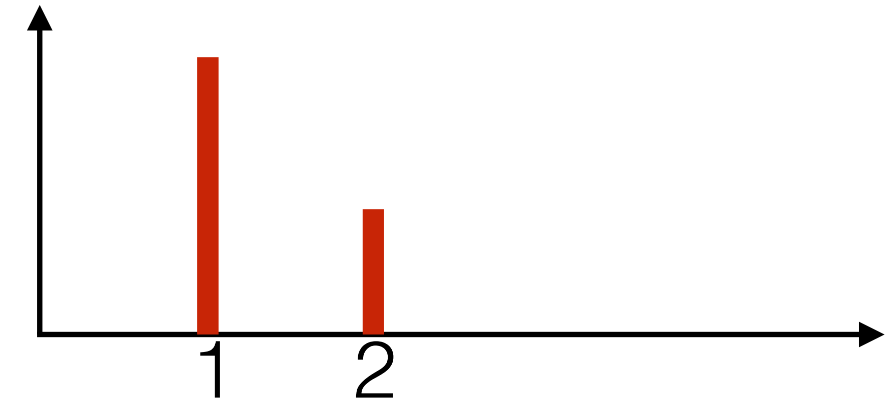
  - Why is NPBayes challenging but practical? Infinite dimensional parameter; more on this today!

Distributions

Distributions

- Beta → random distribution over $1, 2$



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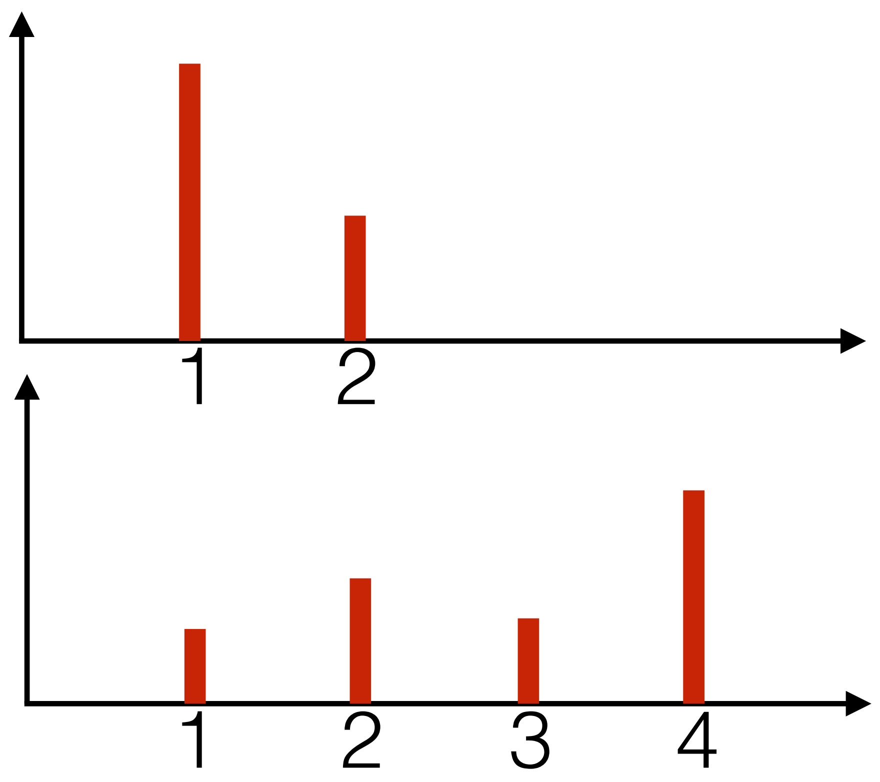

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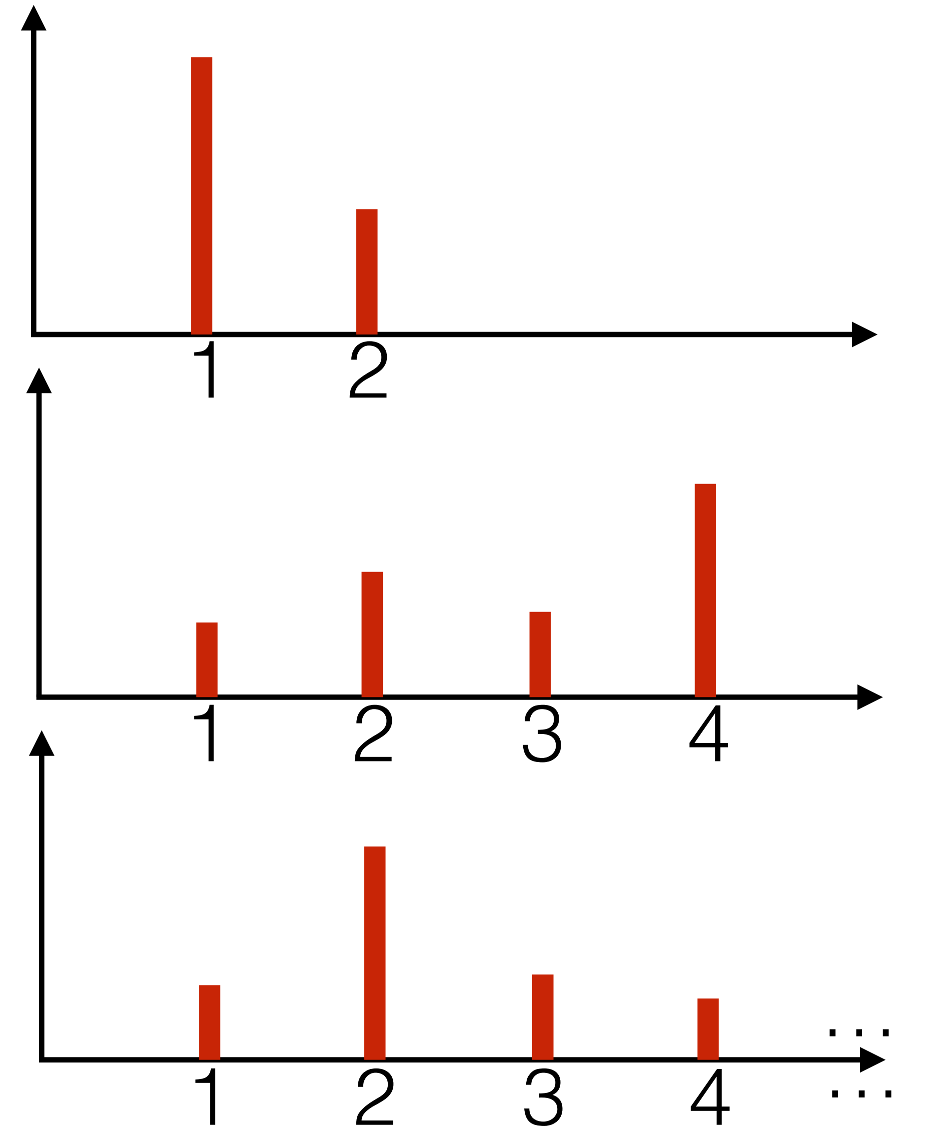

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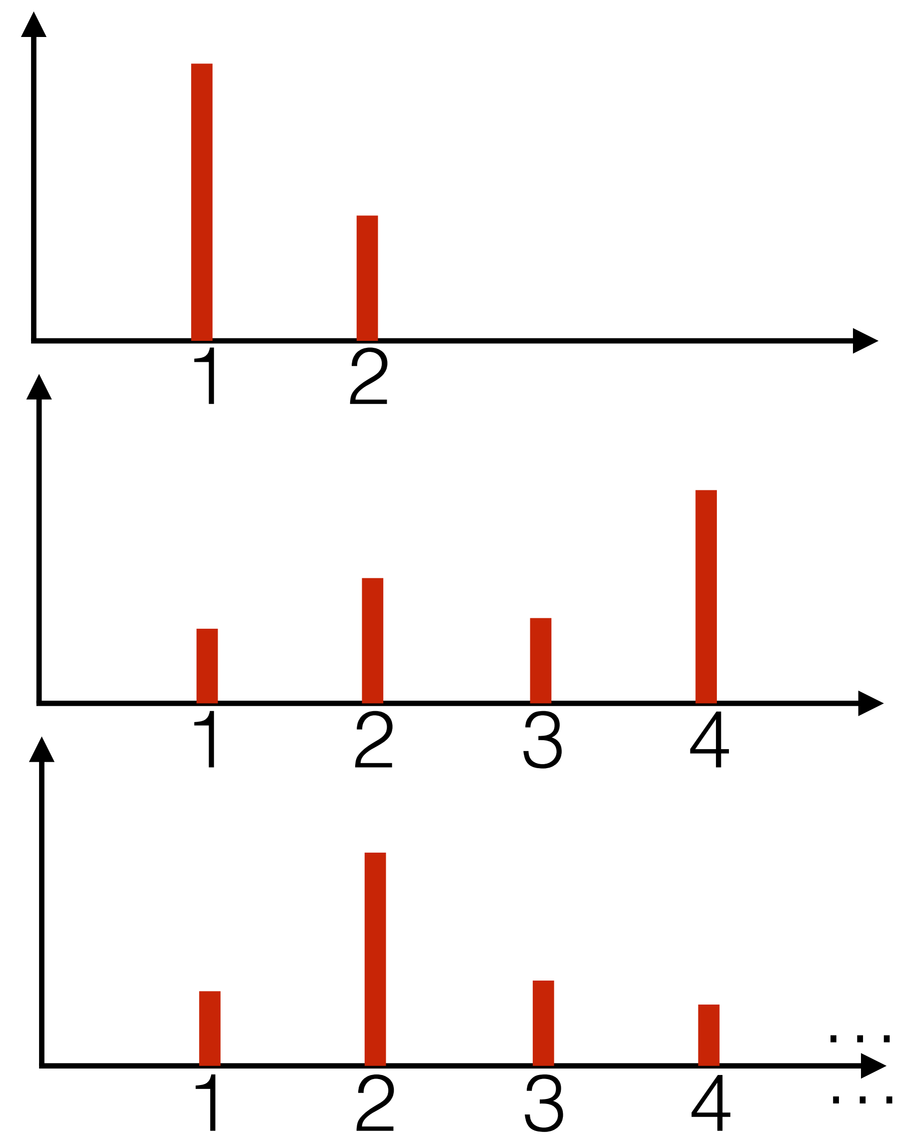

- Infinity of parameters: components
- Growing number of parameters: clusters

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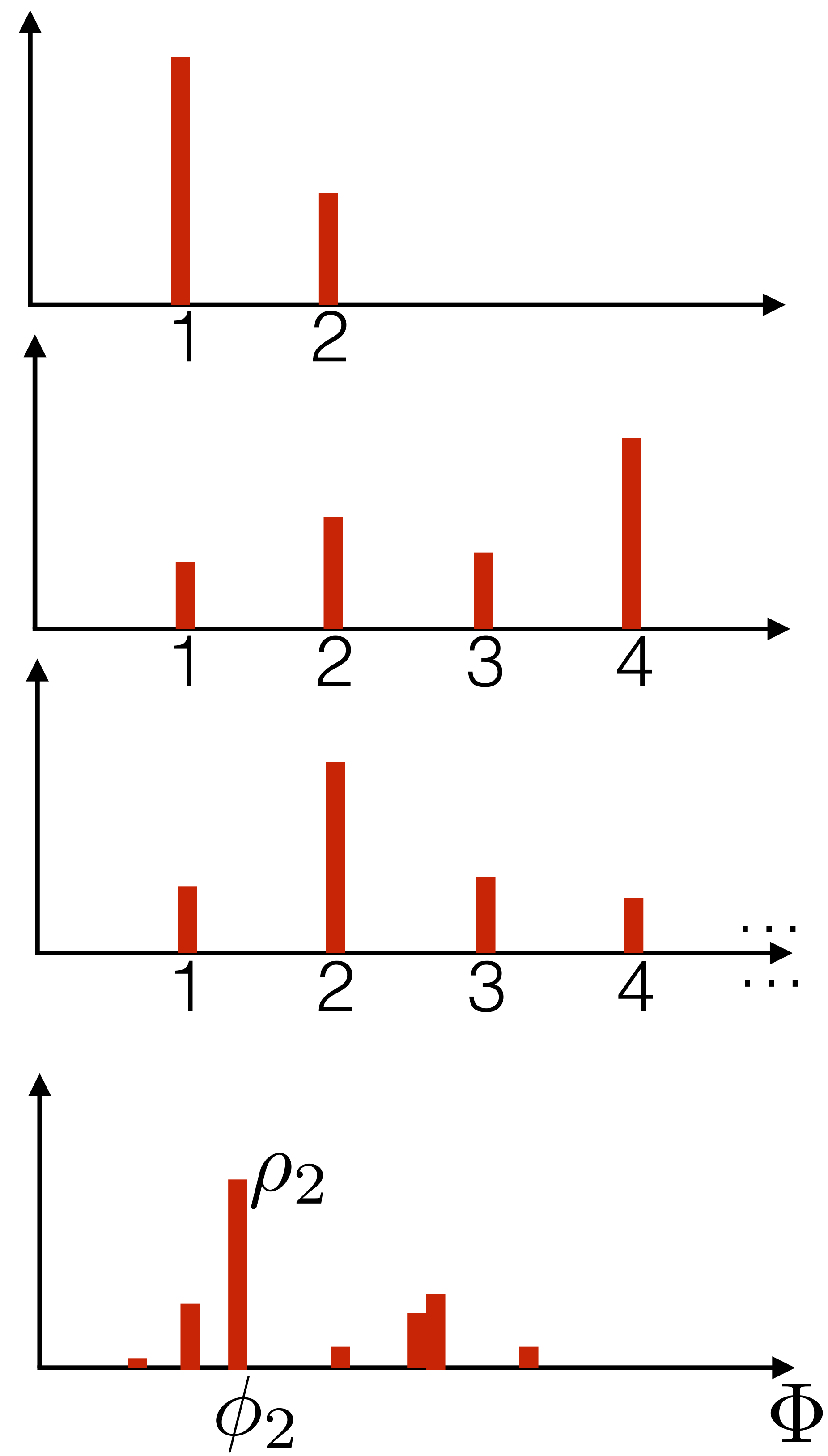

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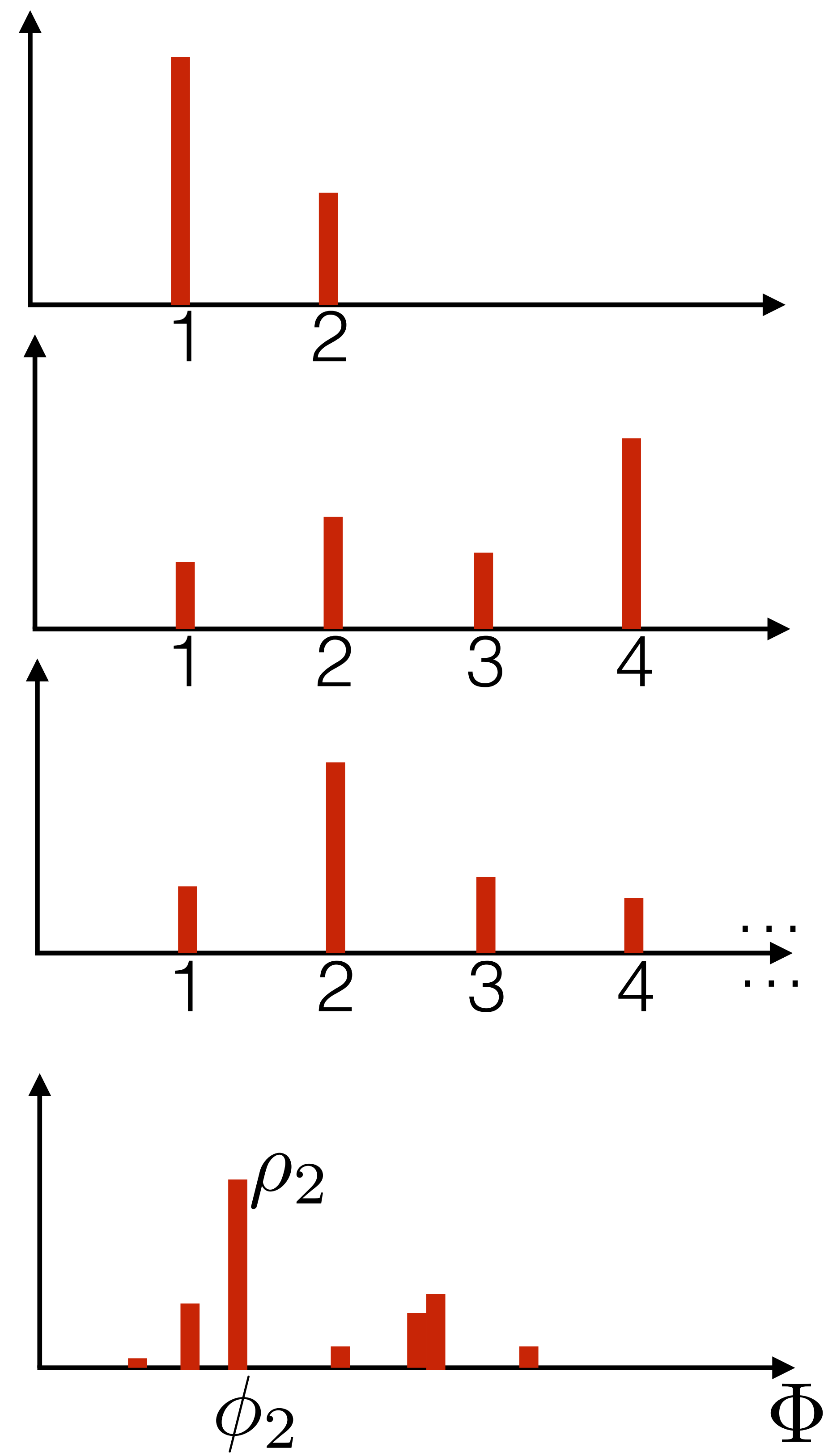

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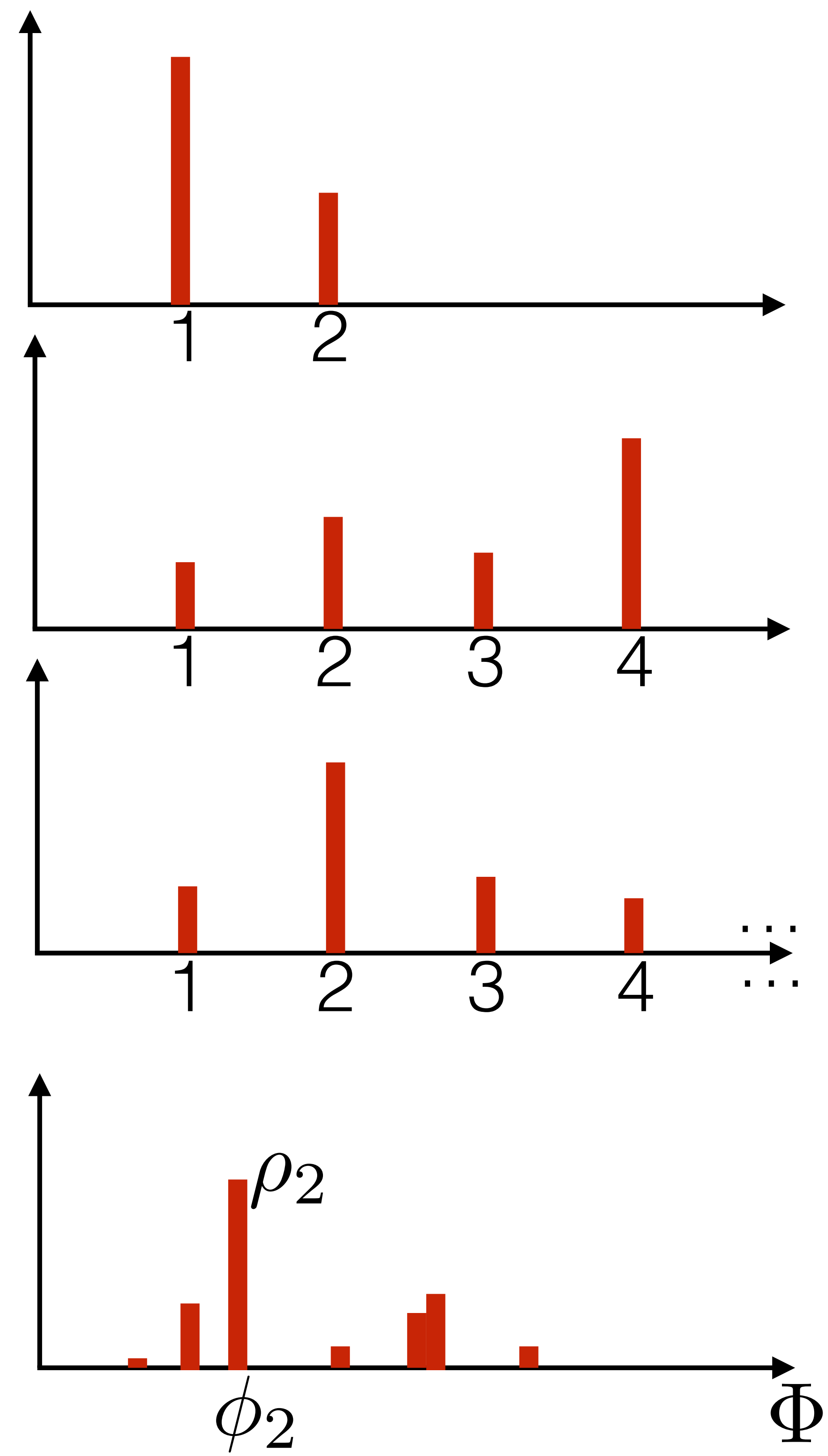

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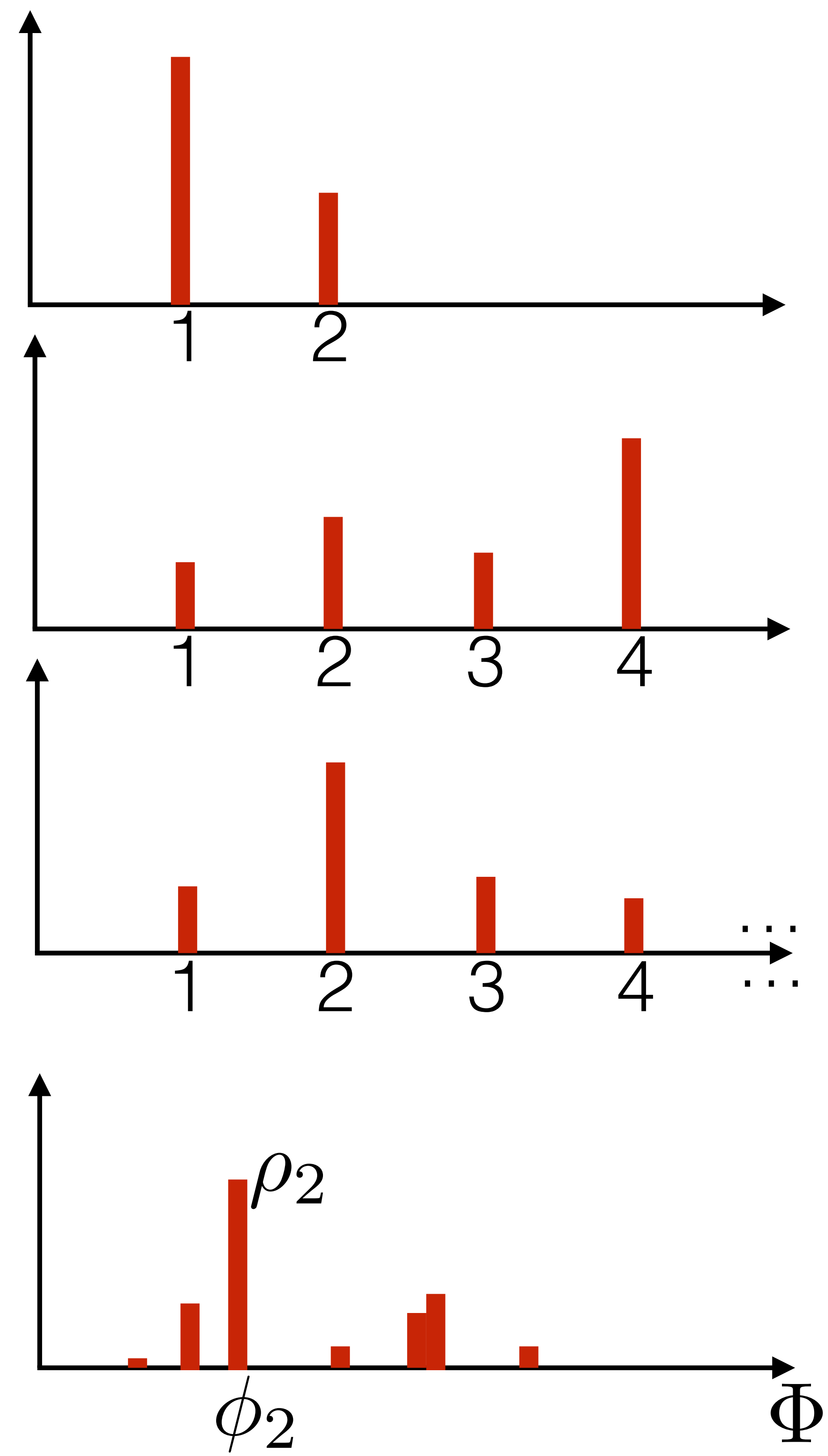

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- **Dirichlet process →** random distribution over Φ:

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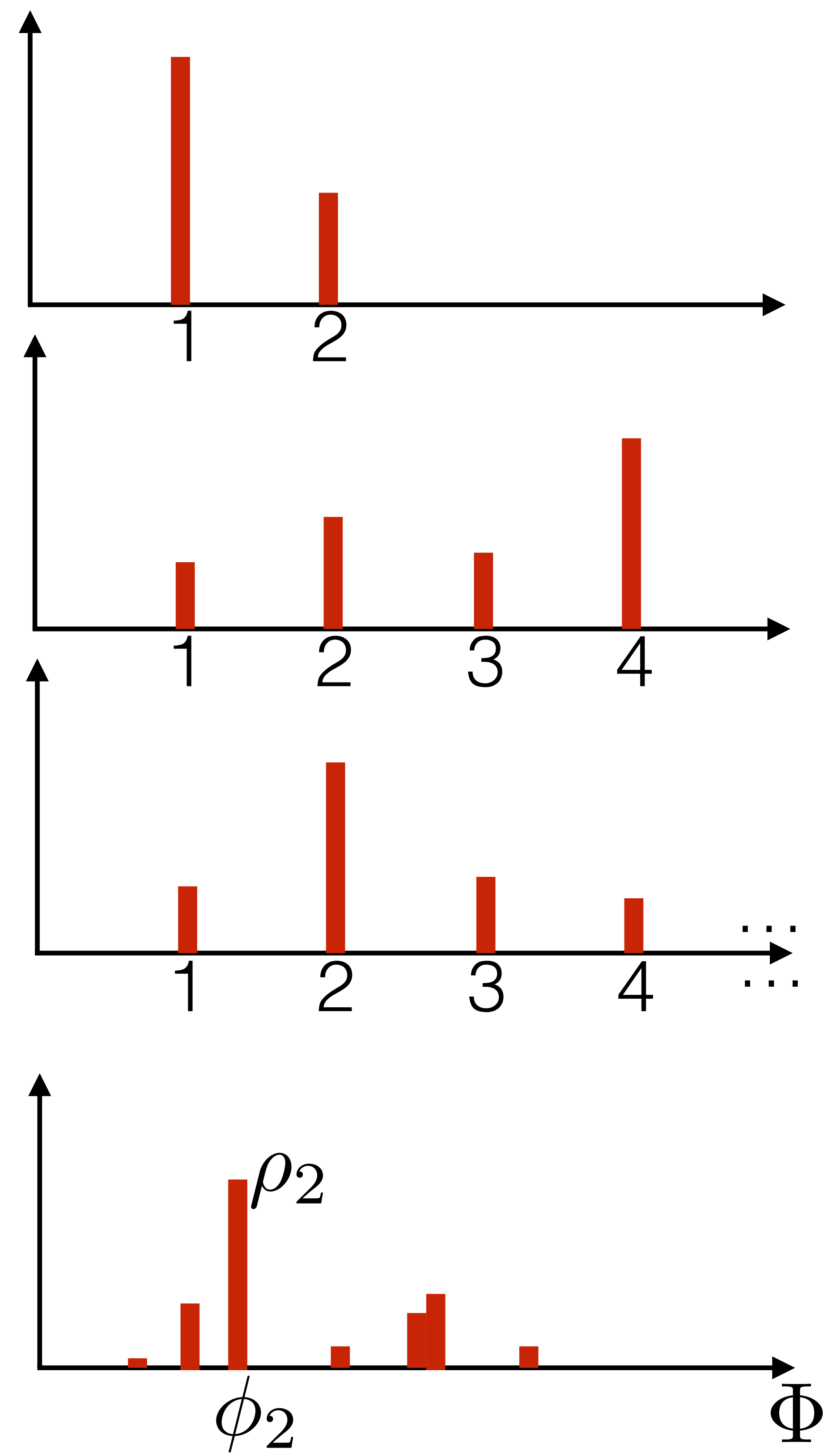

[Ferguson 1973]

Dirichlet process mixture model

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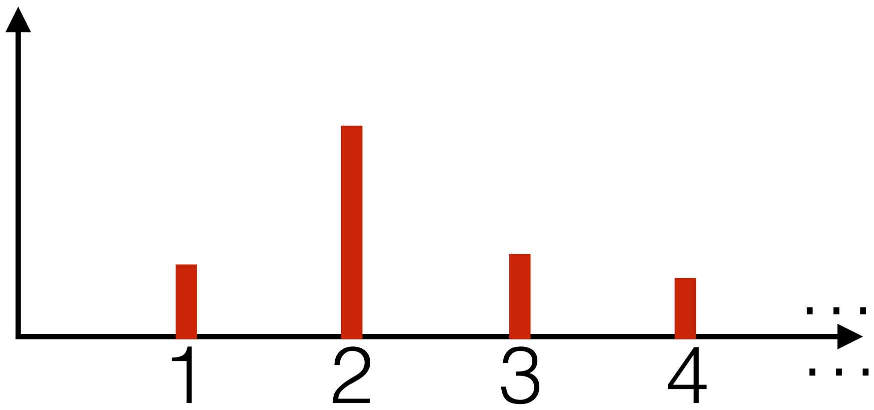

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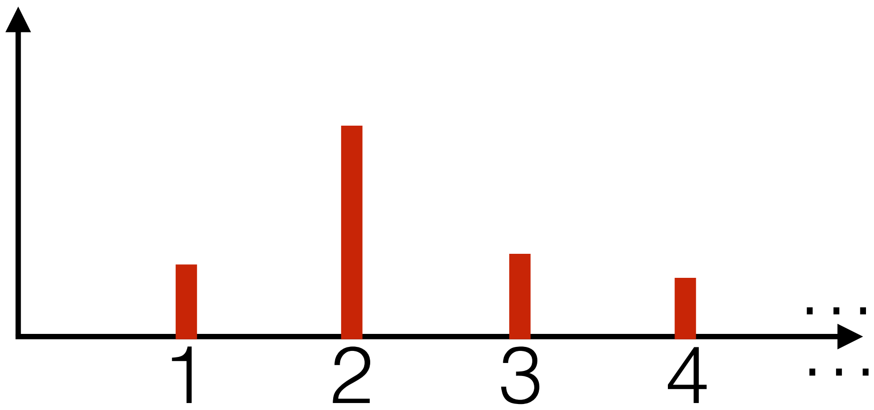

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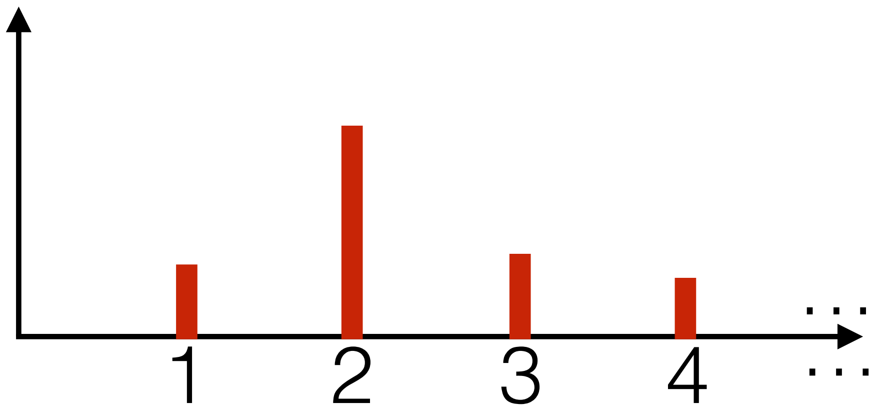

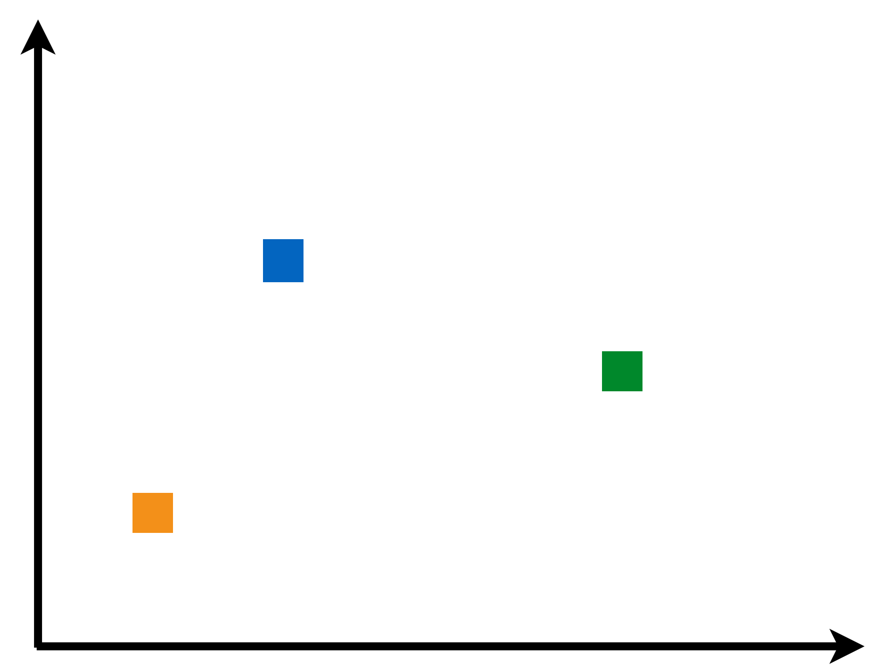

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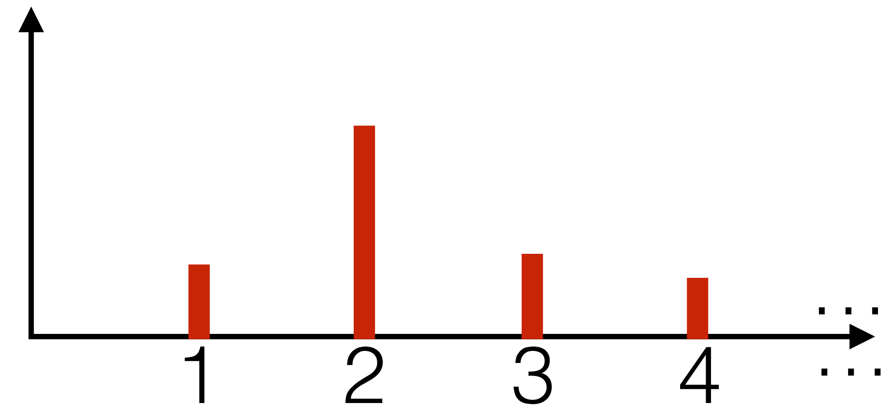

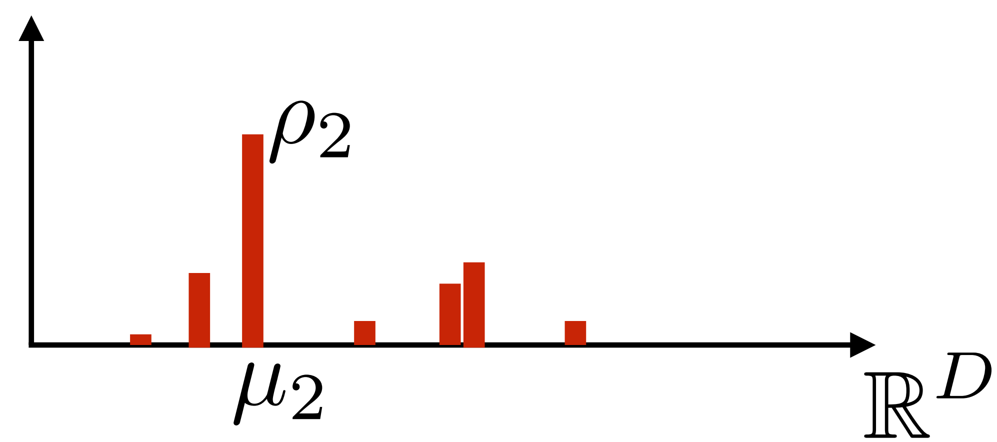

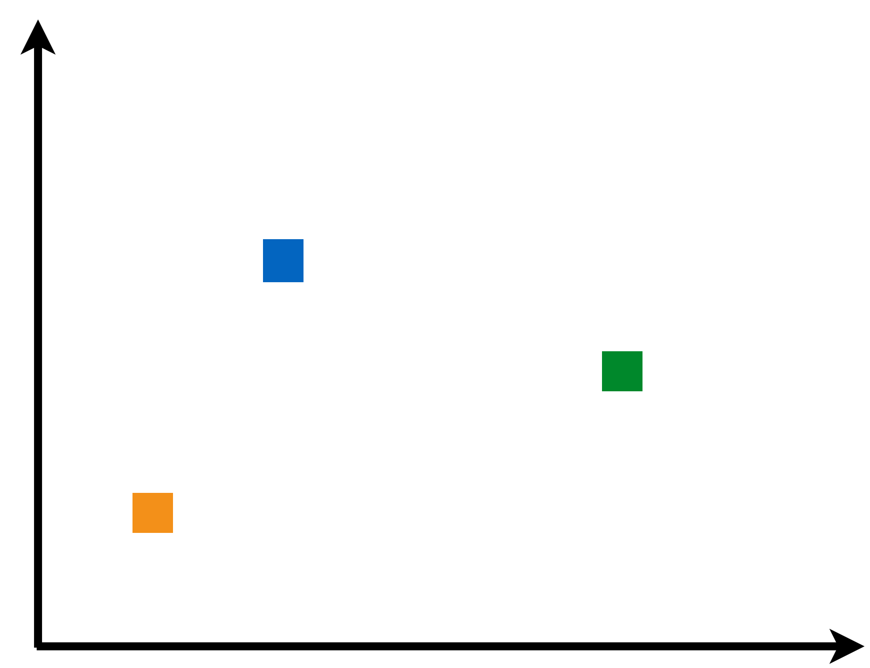

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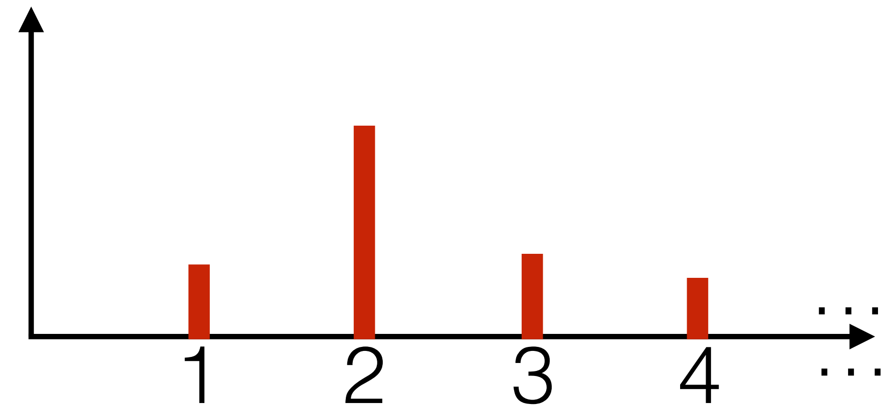

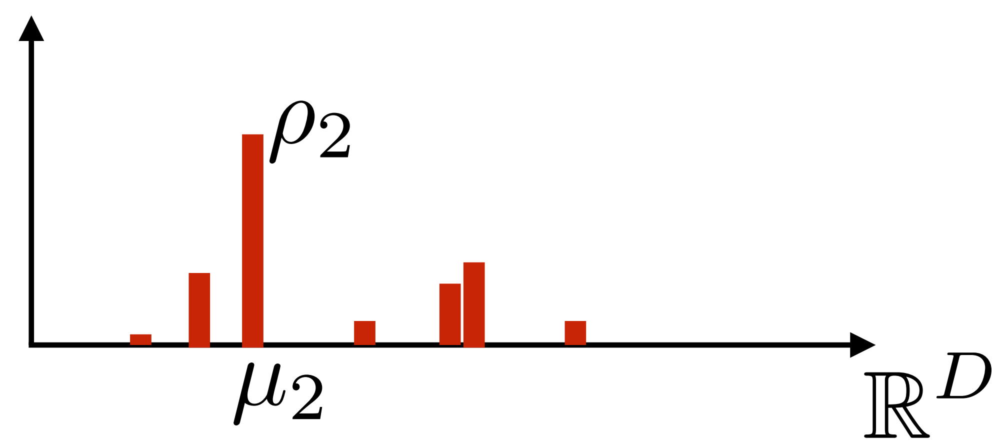

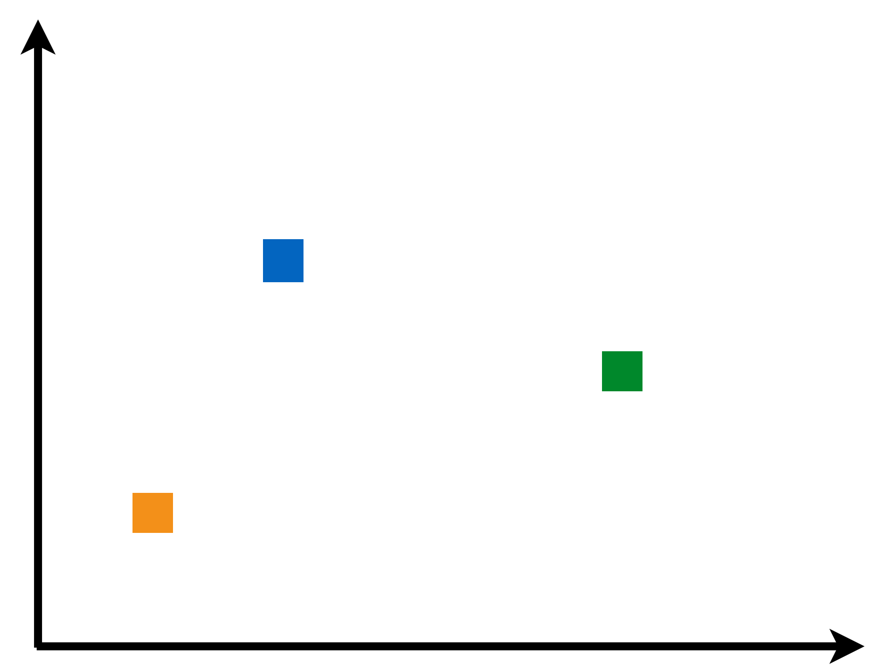

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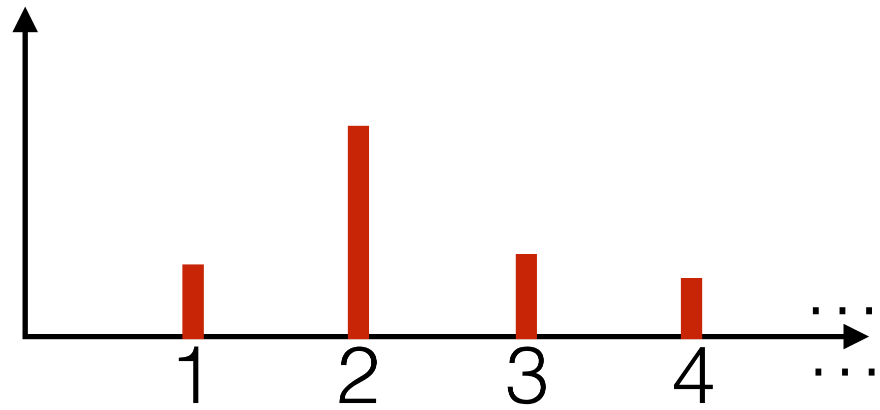

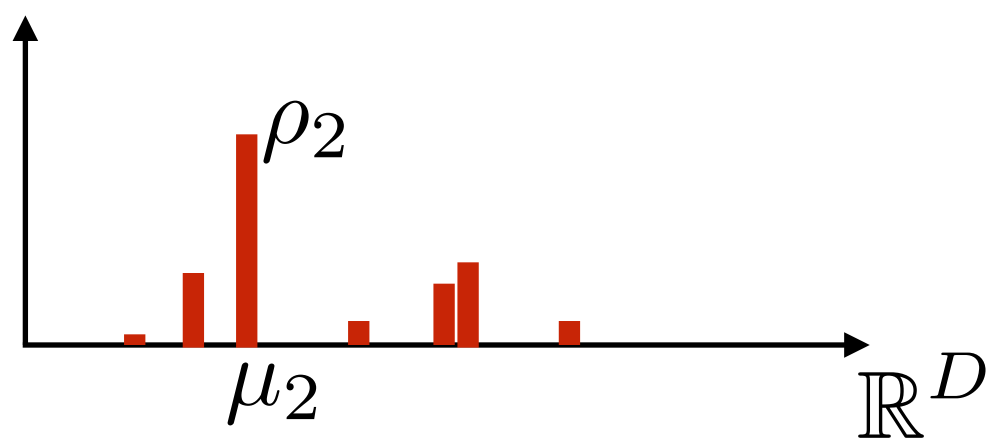

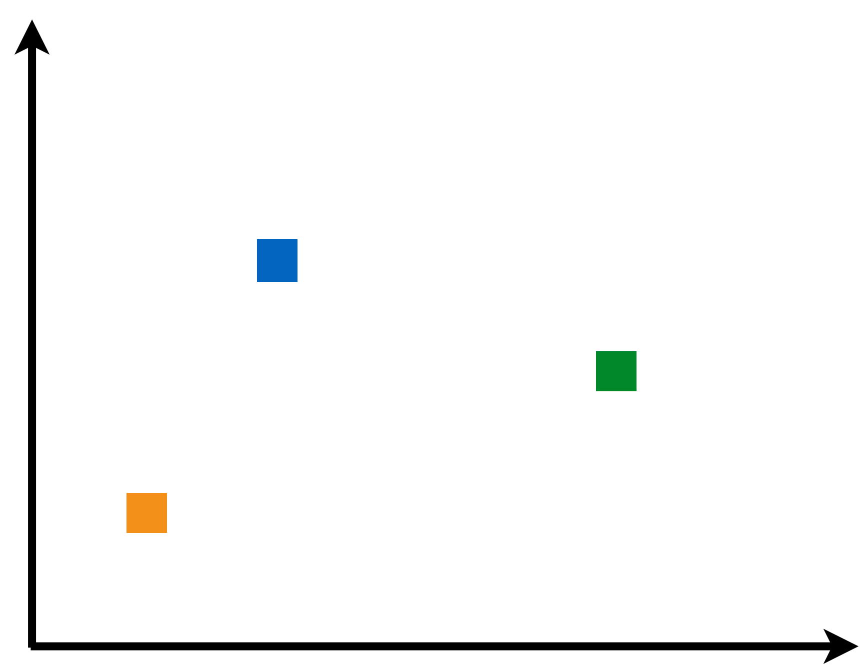

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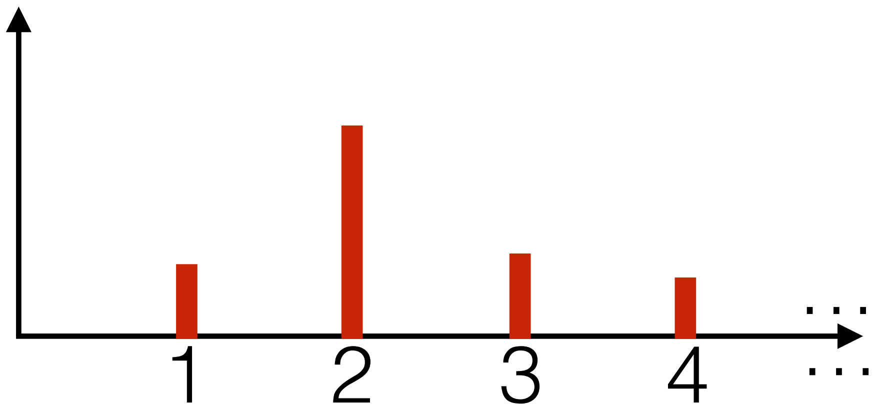

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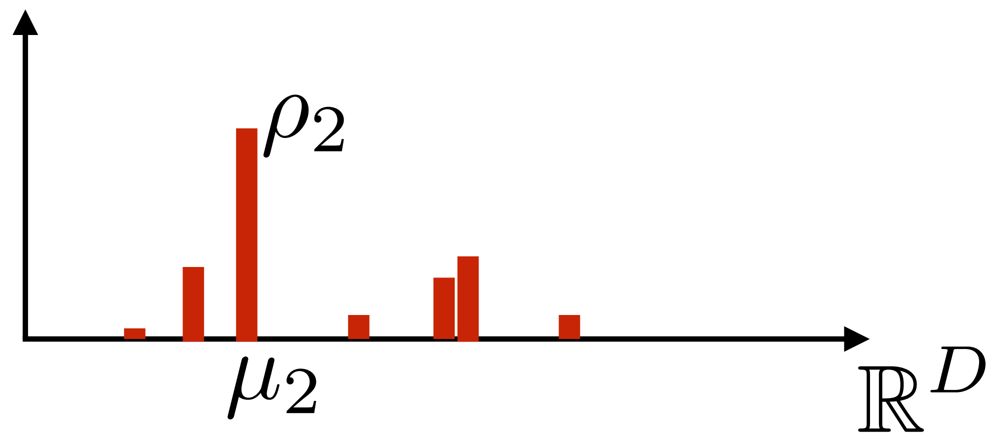

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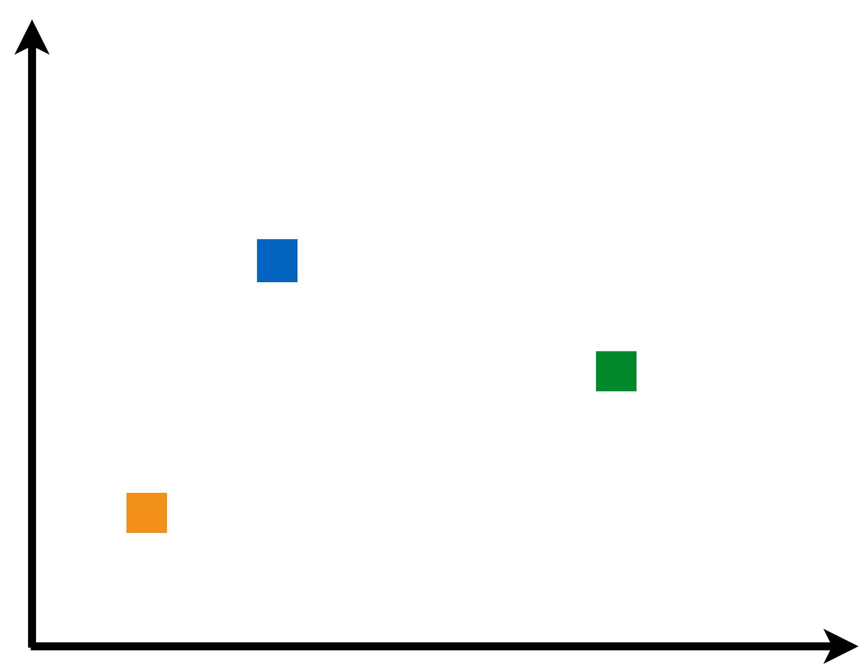

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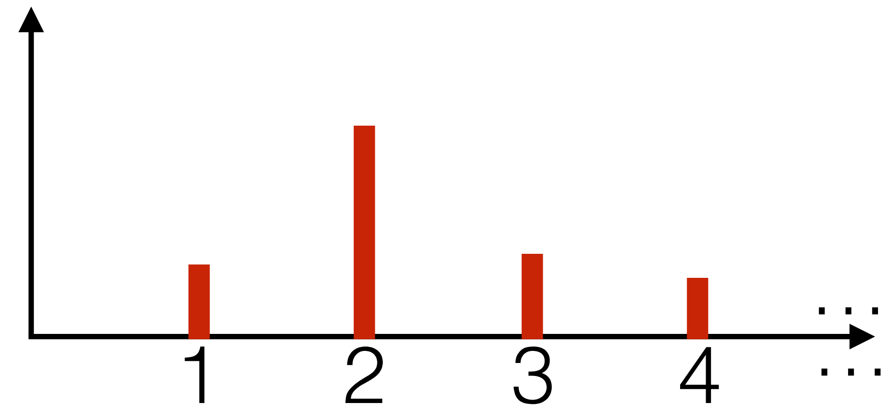

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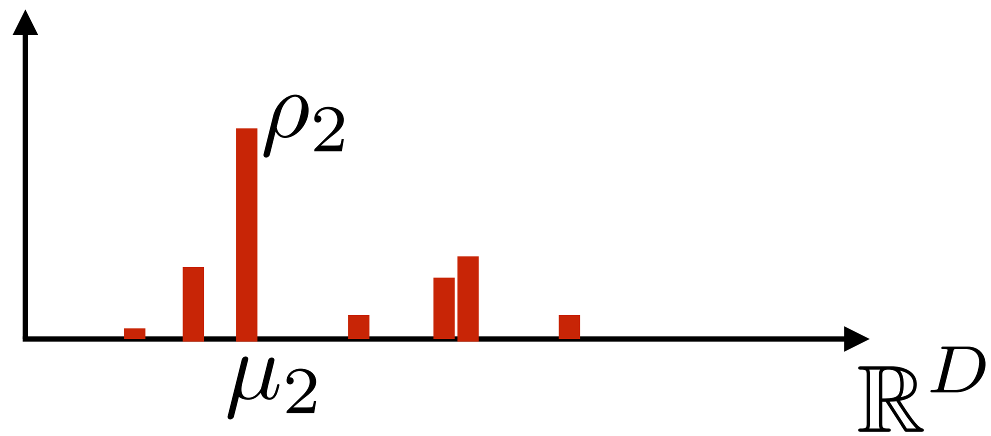

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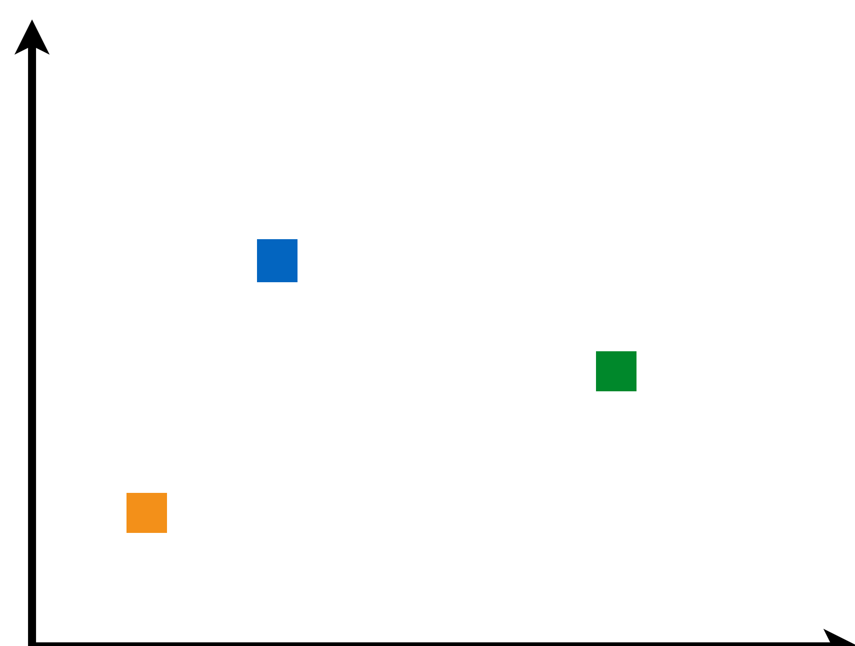

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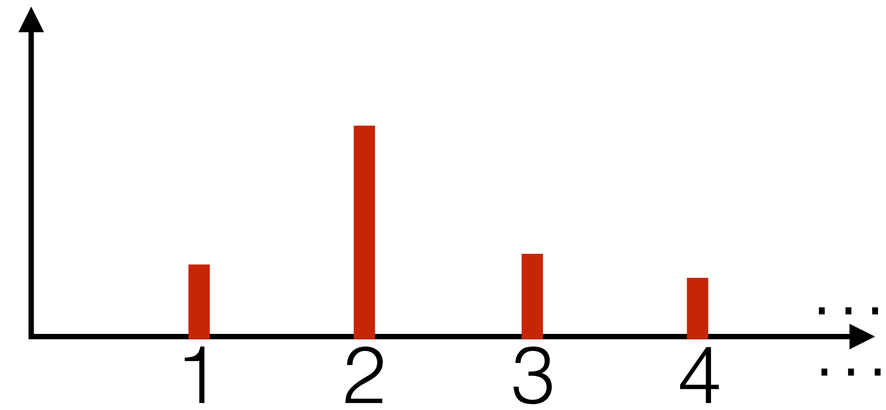

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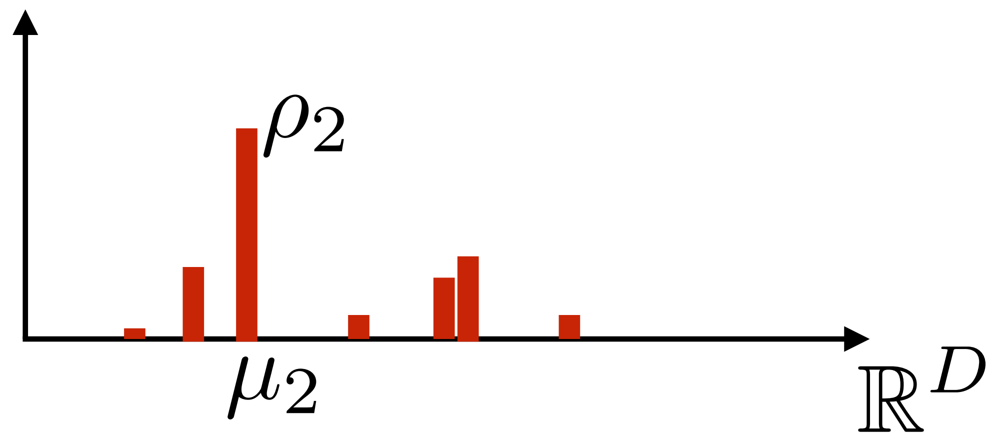

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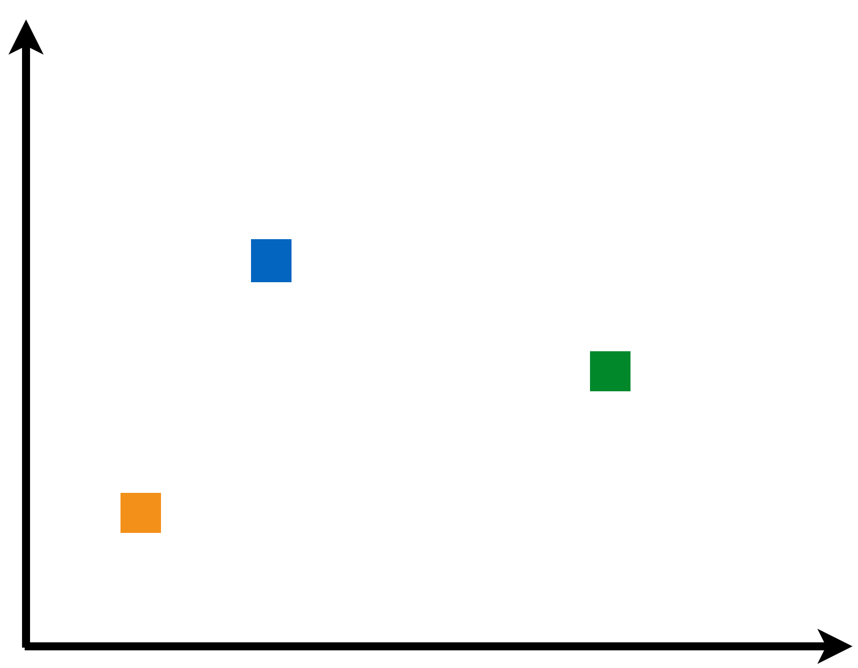

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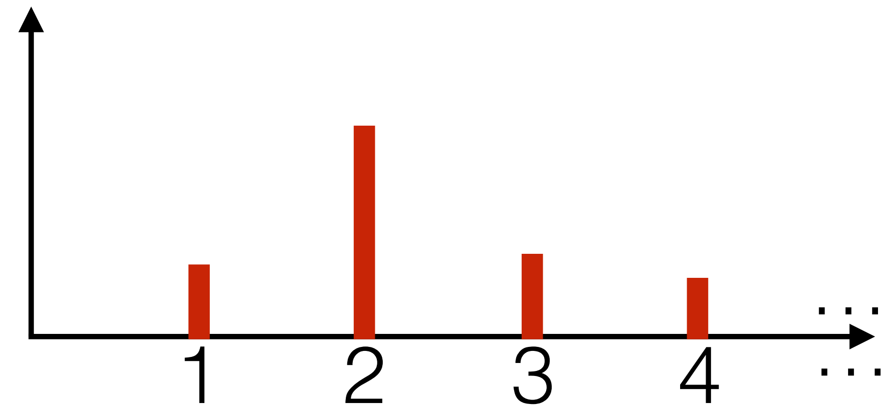

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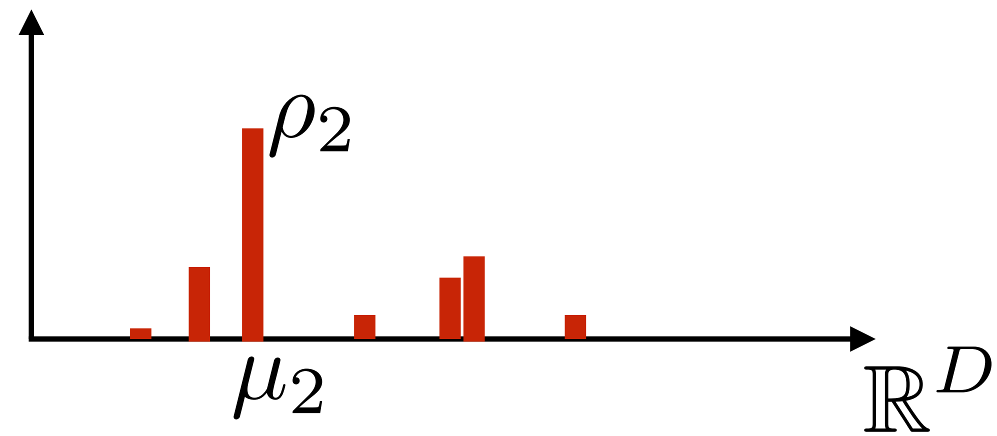

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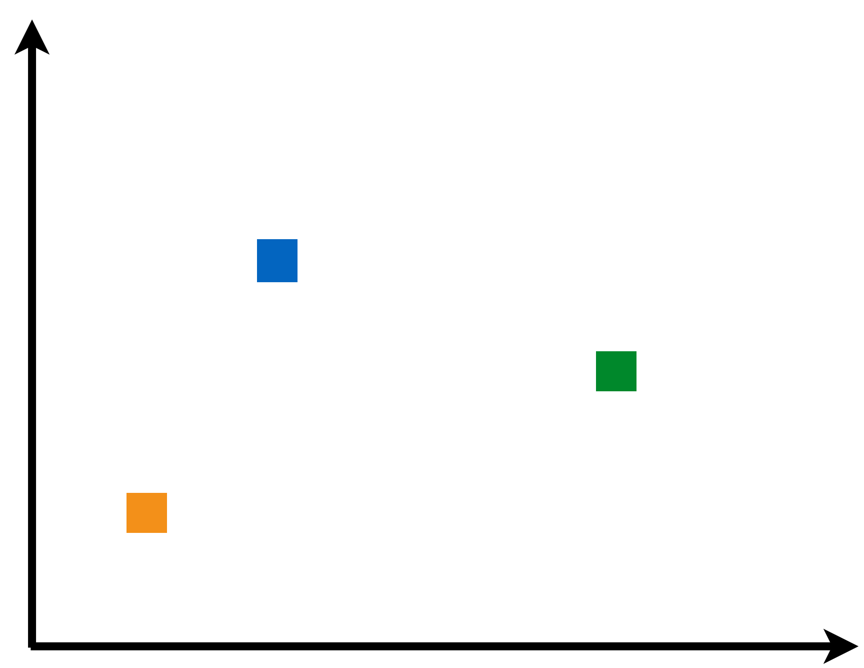

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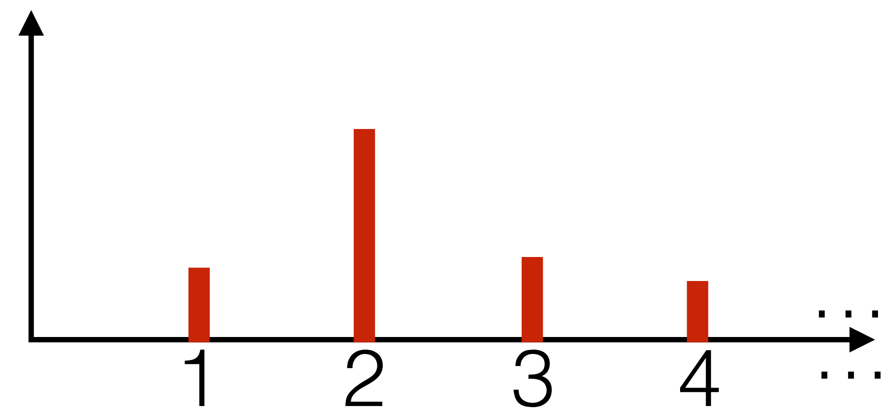

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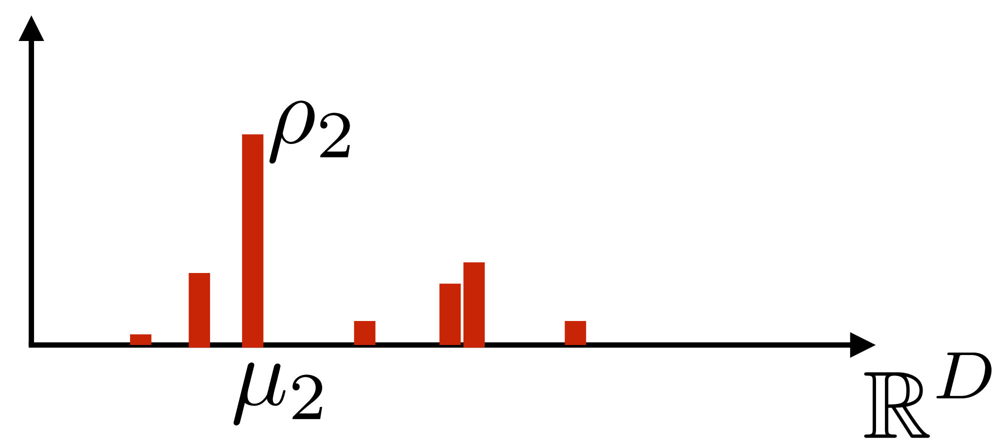

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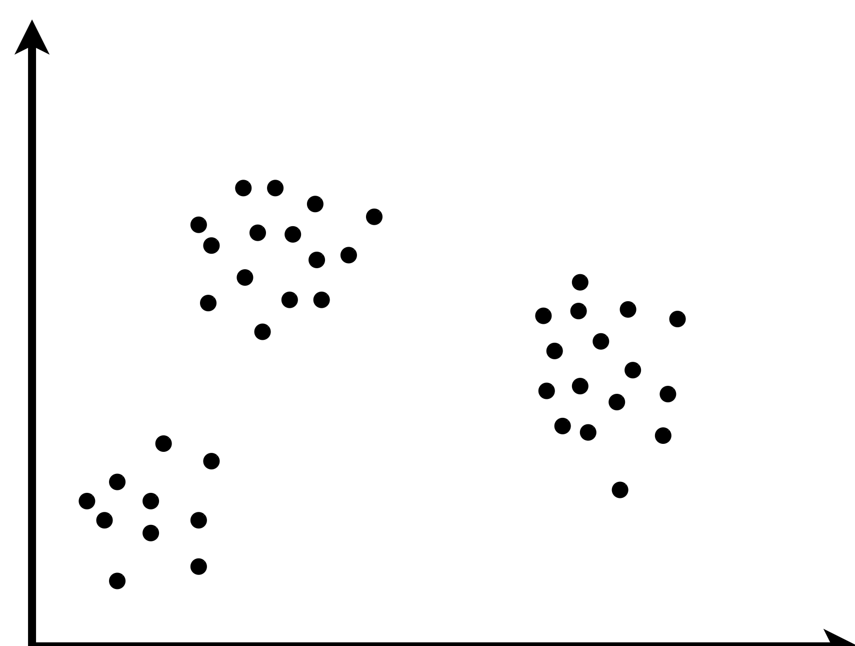

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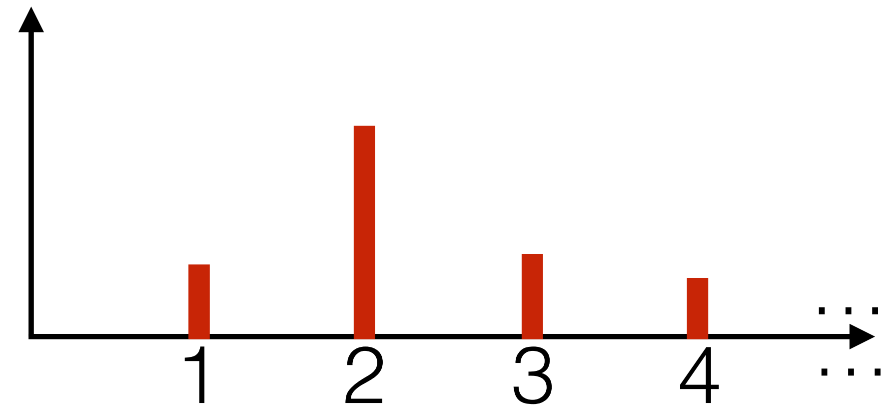

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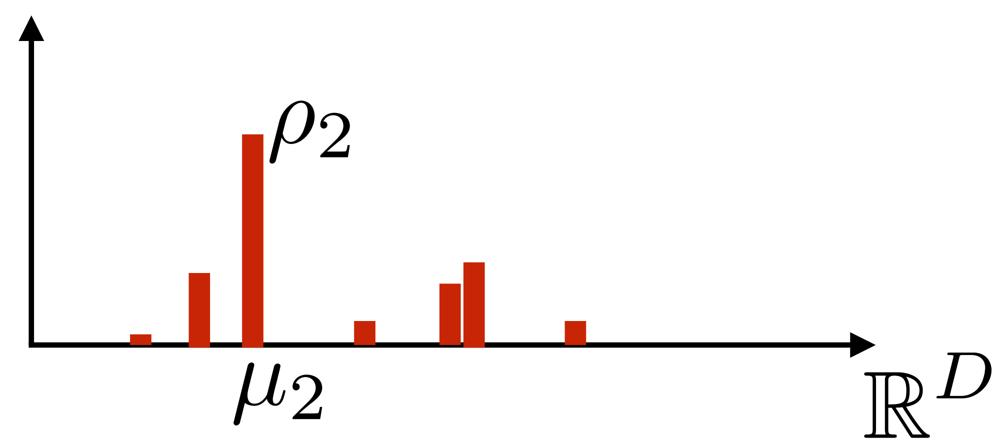

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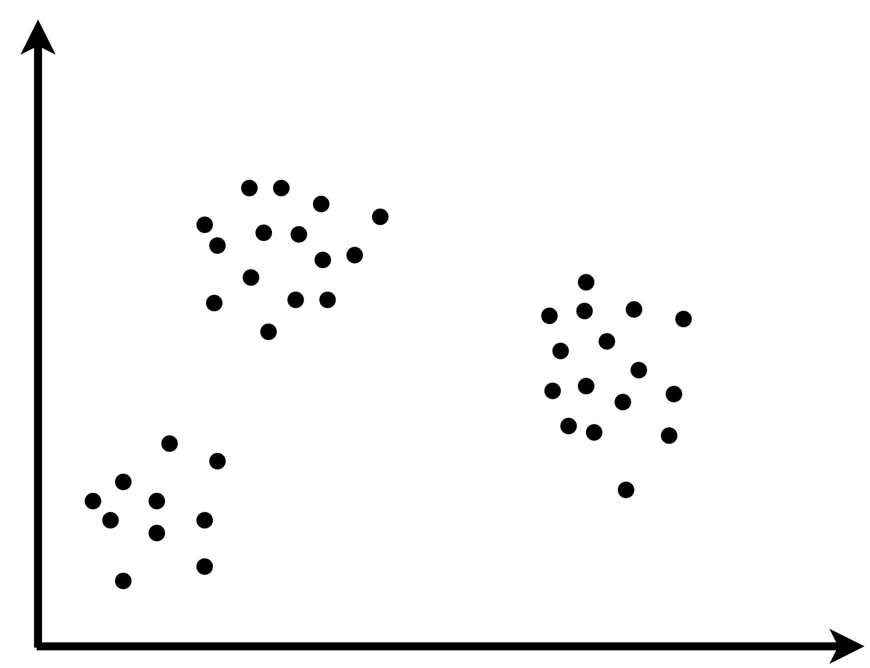
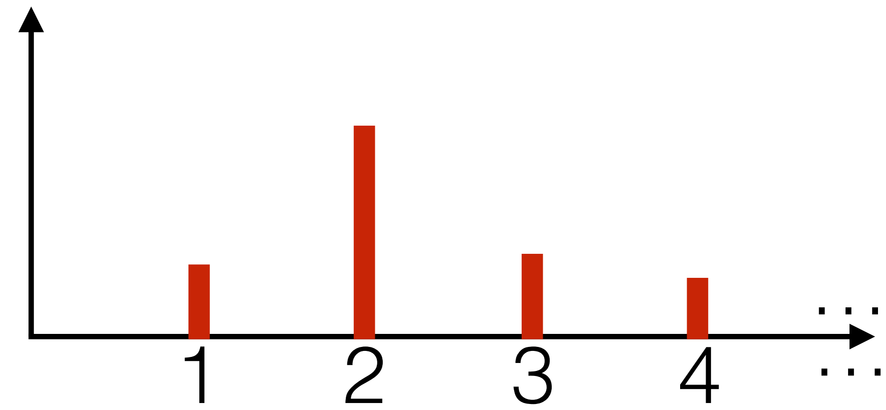


$$x_n \overset{indep}{\sim} \mathcal{N}(\mu_n^*, \Sigma)$$

[demo]

Dirichlet process mixture model

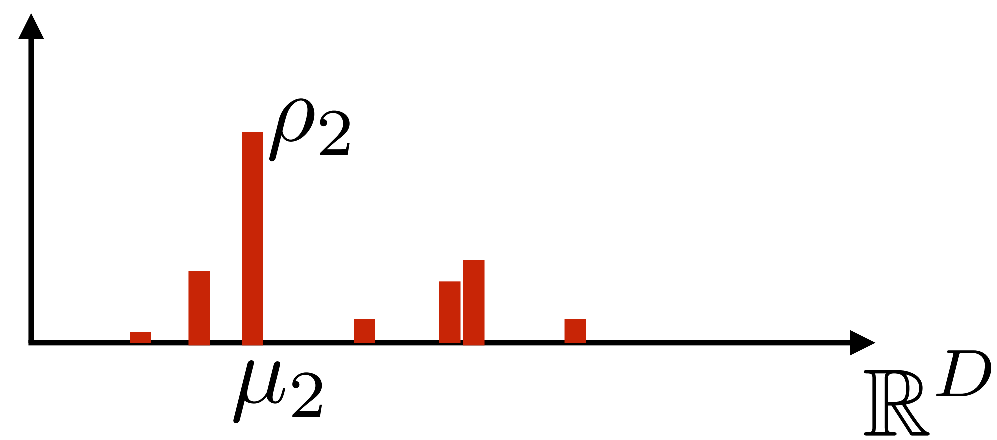
- More generally



$$\rho = (\rho_1, \rho_2, \ldots) \sim \mathrm{GEM}(\alpha)$$

$$\mu_k \stackrel{iid}{\sim} \mathcal{N}(\mu_0, \Sigma_0), k = 1, 2, \ldots$$

- i.e. $G = \sum_{k=1}^{\infty} \rho_k \delta_{\mu_k} \stackrel{d}{=} \mathrm{DP}(\alpha, \mathcal{N}(\mu_0, \Sigma_0))$



$$z_n \stackrel{iid}{\sim} \mathrm{Categorical}(\rho)$$

$$\mu_n^* = \mu_{z_n}$$

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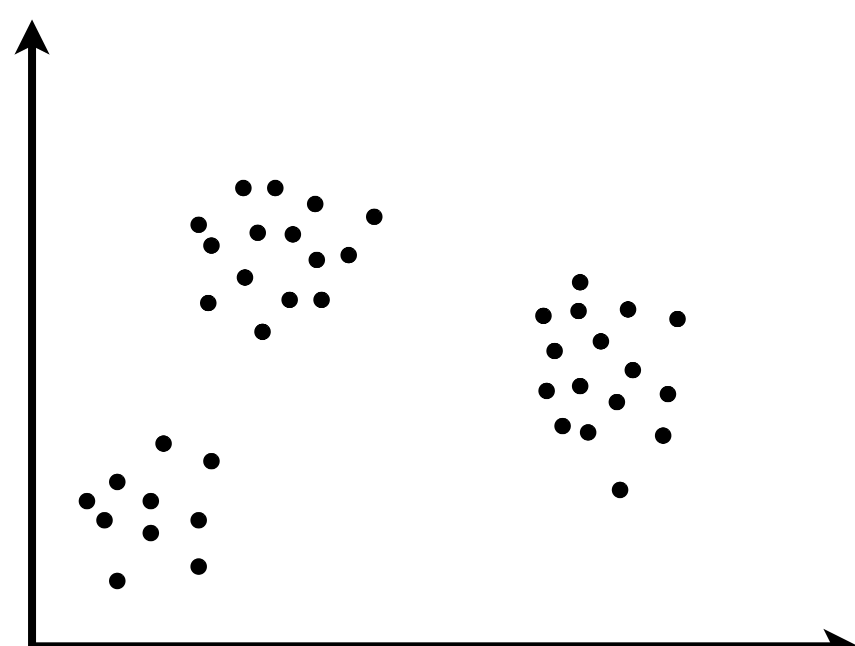
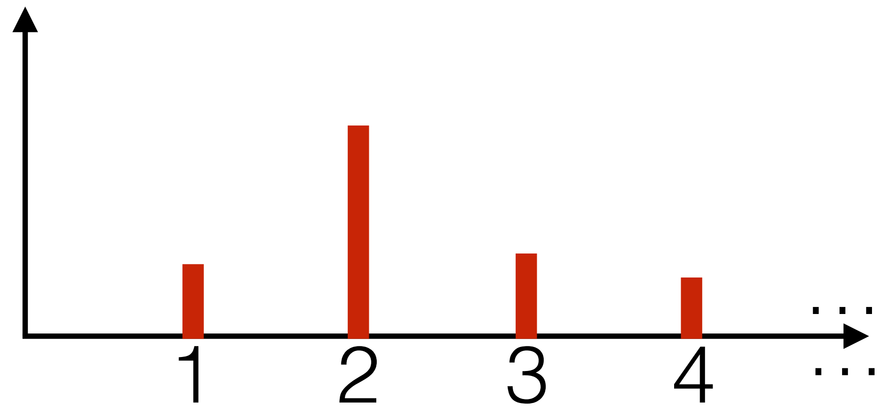
$$x_n \stackrel{indep}{\sim} \mathcal{N}(\mu_n^*, \Sigma)$$



Dirichlet process mixture model



- More generally

$$\rho = (\rho_1, \rho_2, \ldots) \sim \mathrm{GEM}(\alpha)$$

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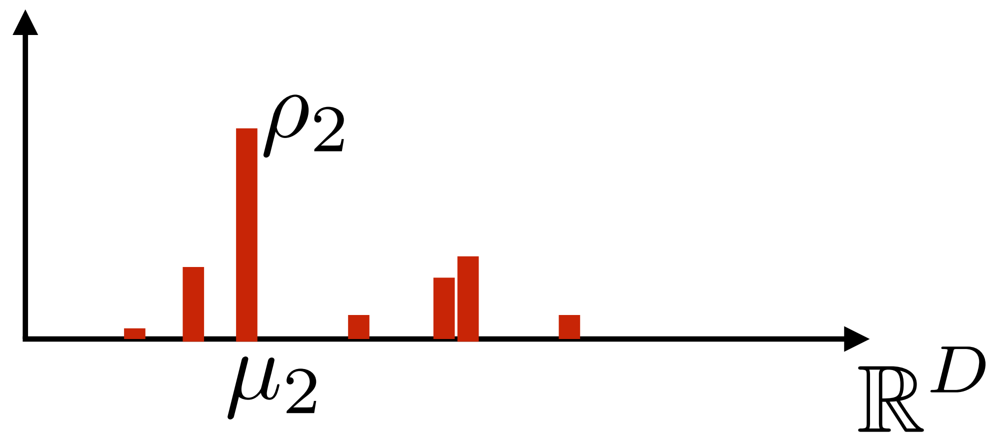

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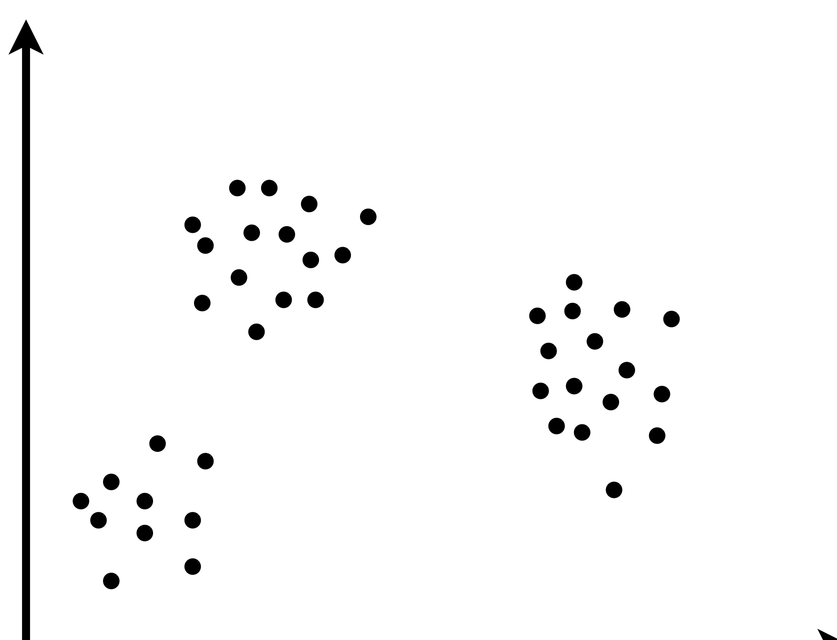

$$x_n \overset{indep}{\sim} \mathcal{N}(\mu_n^*, \Sigma)$$

Dirichlet process mixture model

- More generally

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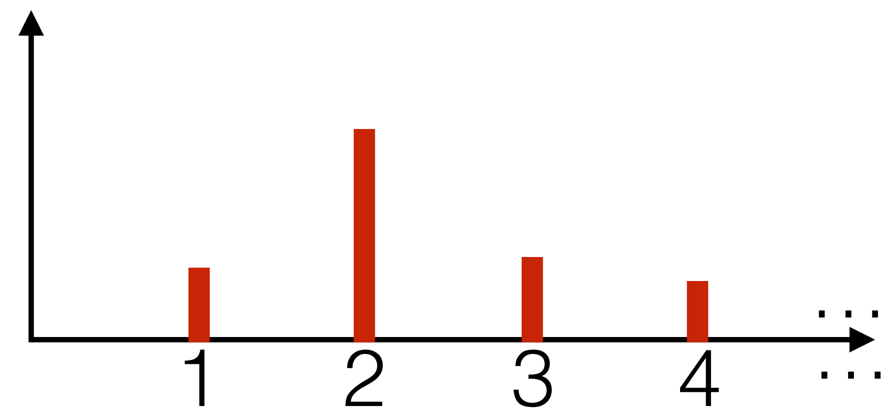

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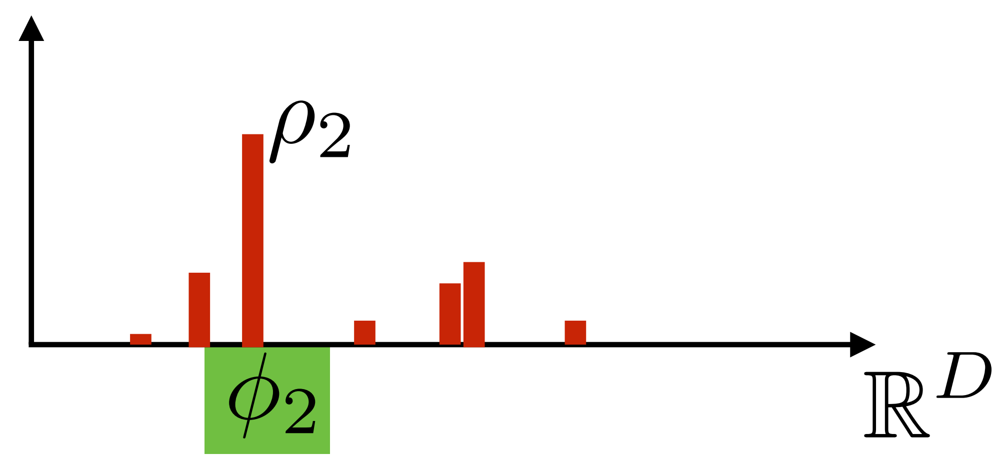

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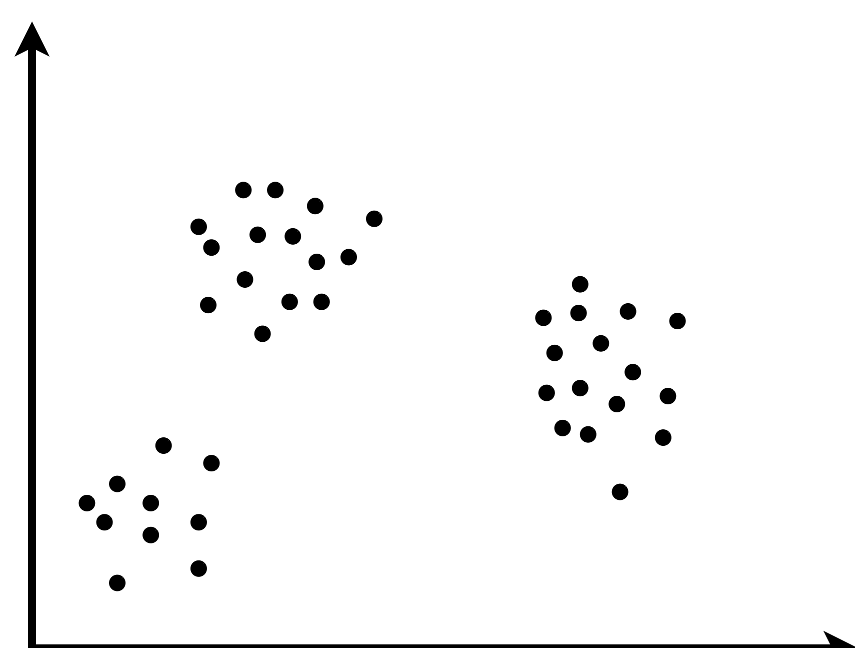

Dirichlet process mixture model

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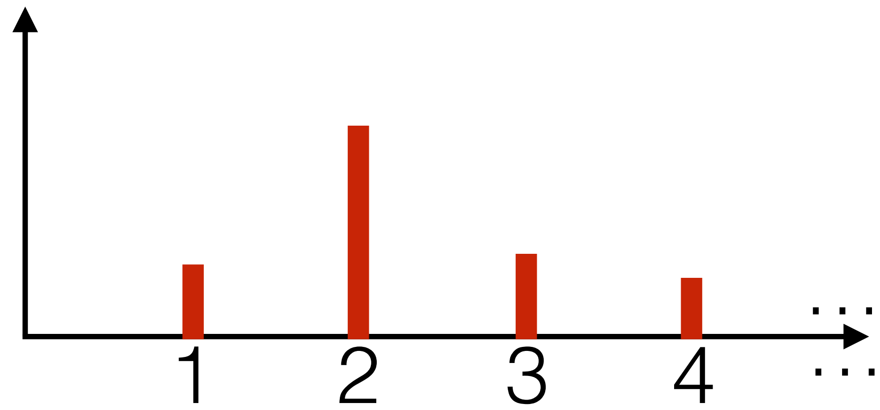

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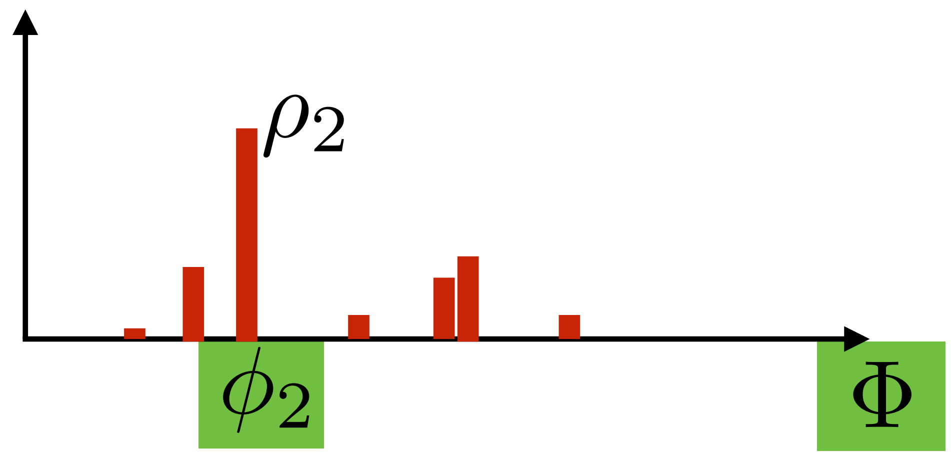

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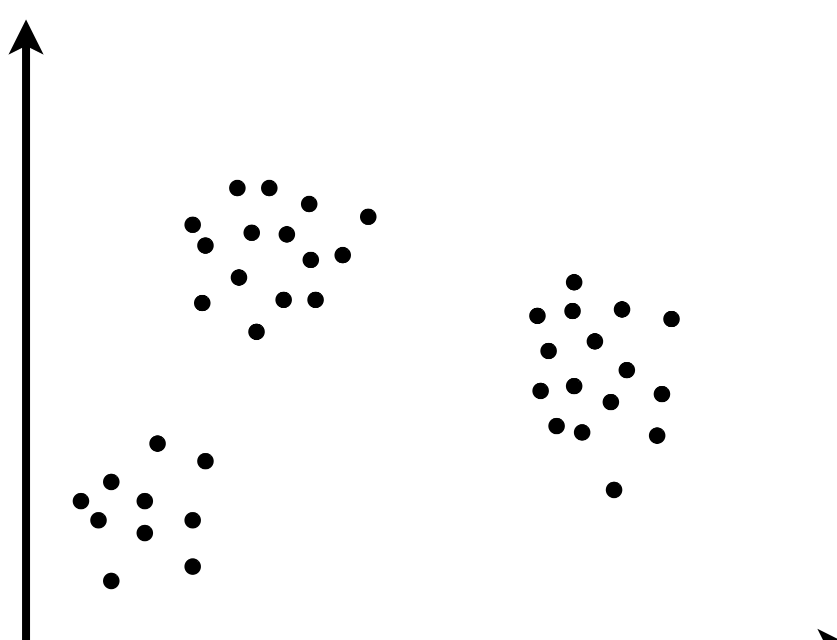

Dirichlet process mixture model

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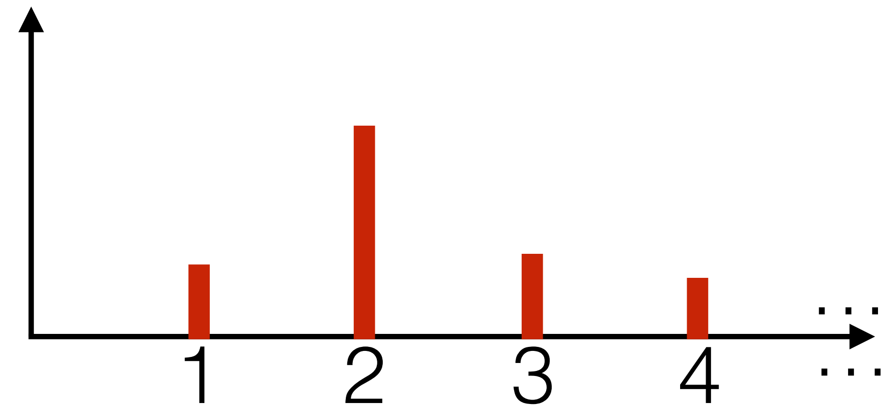

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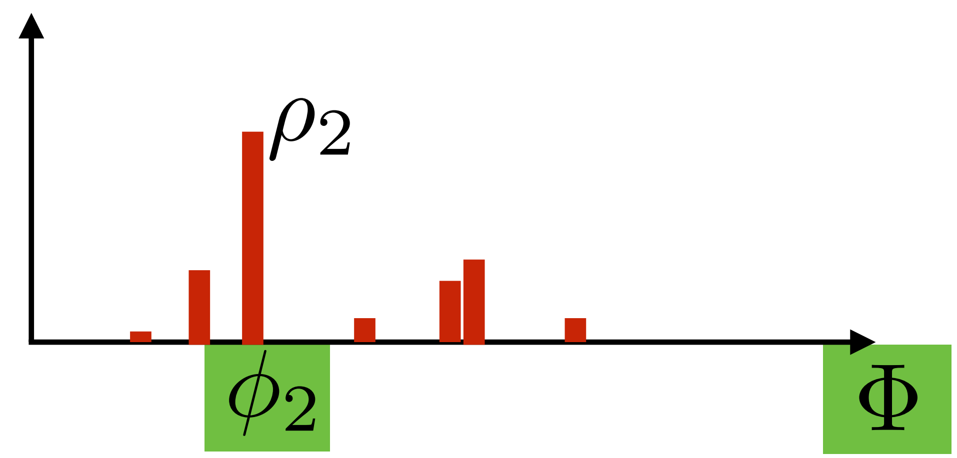

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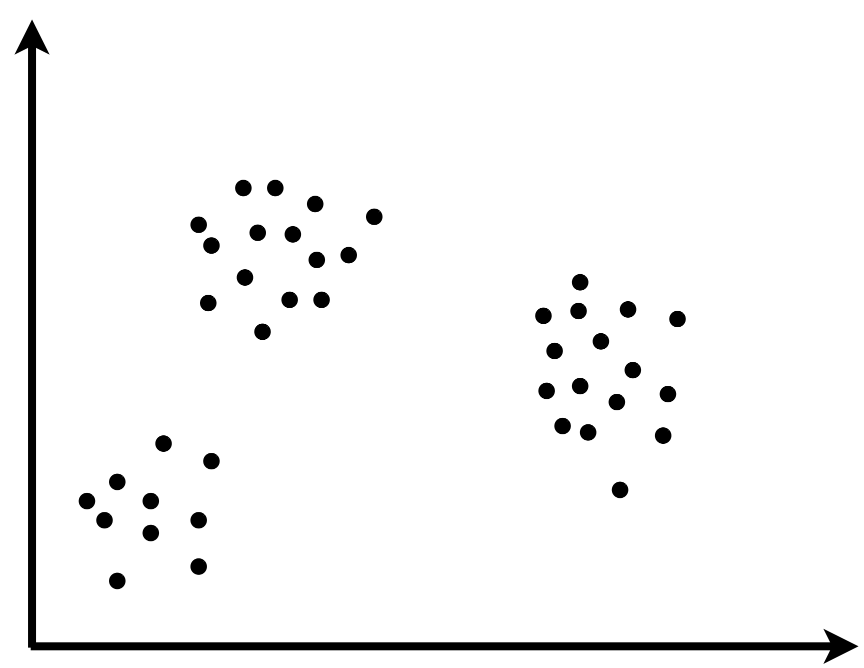
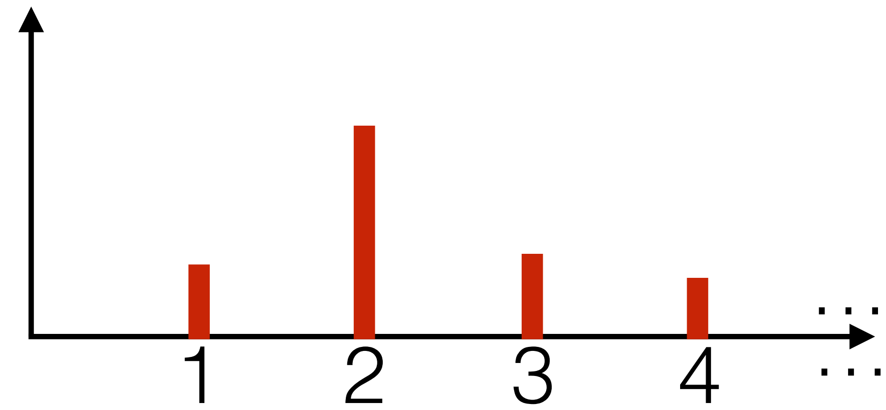


Dirichlet process mixture model



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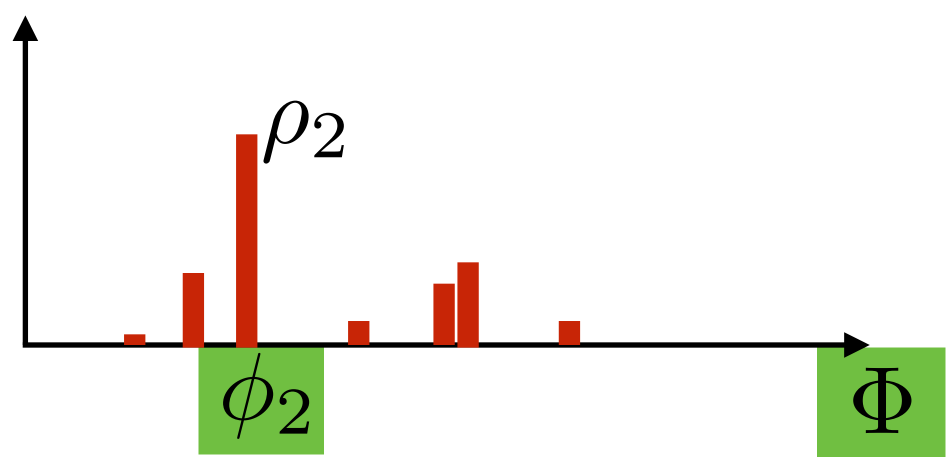

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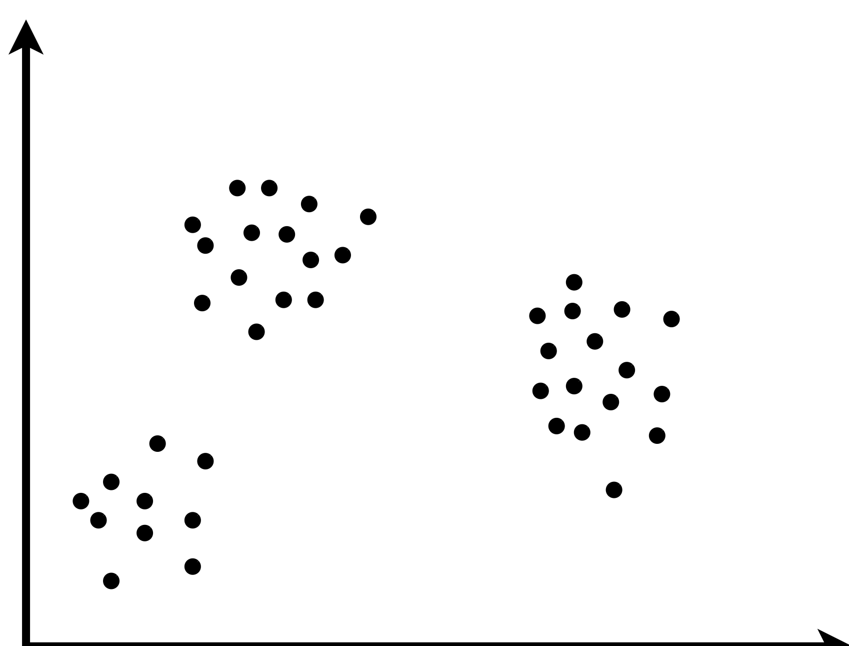

$$x_n \overset{indep}{\sim} \mathcal{N}(\mu_n^*, \Sigma)$$

Dirichlet process mixture model

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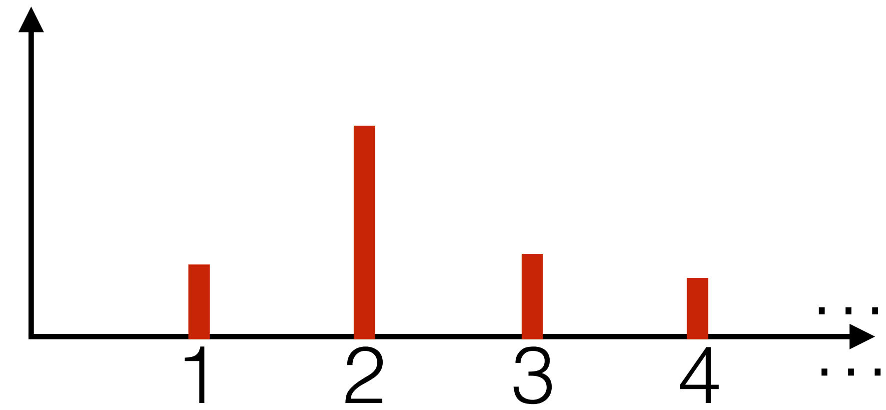

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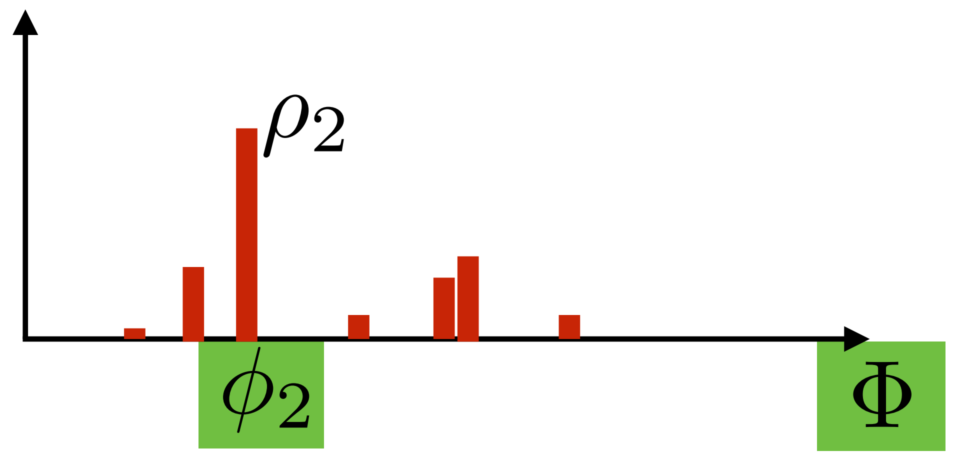

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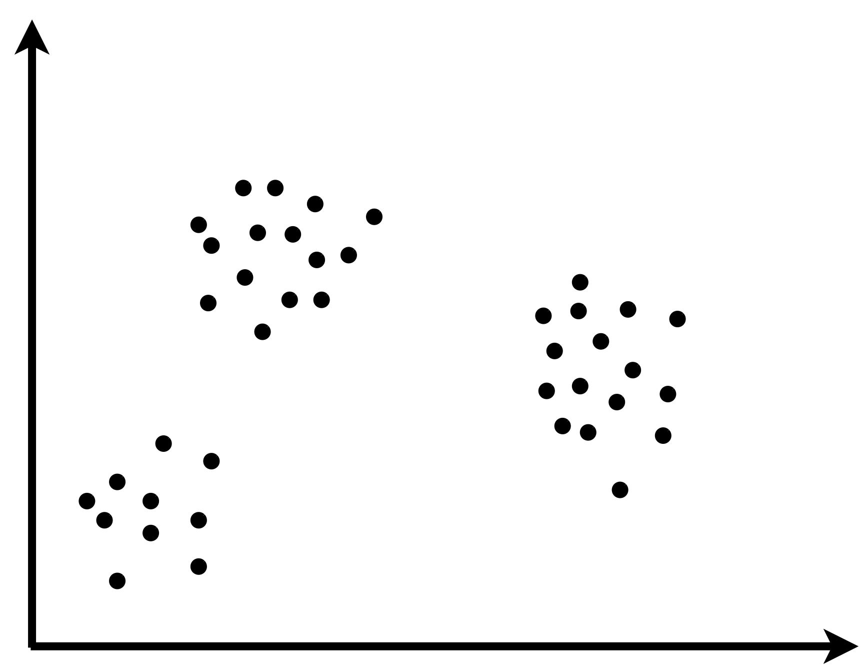

Dirichlet process mixture model

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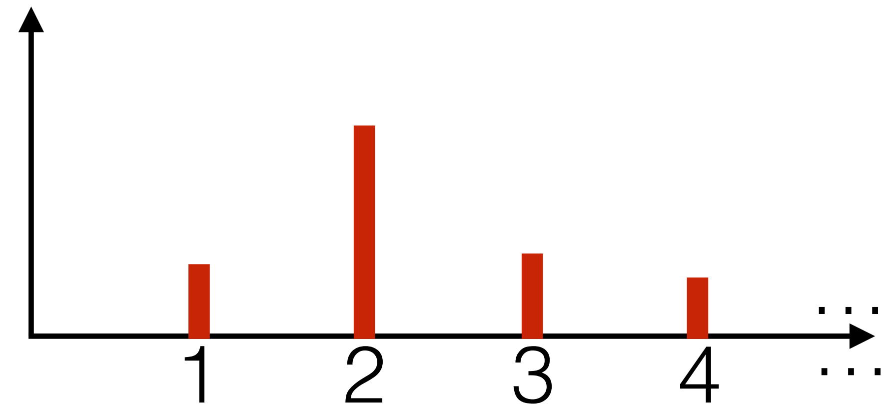

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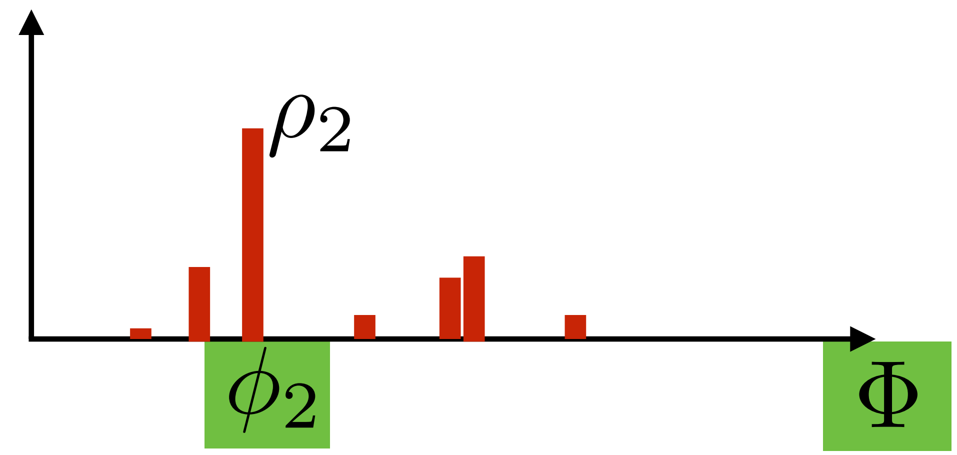

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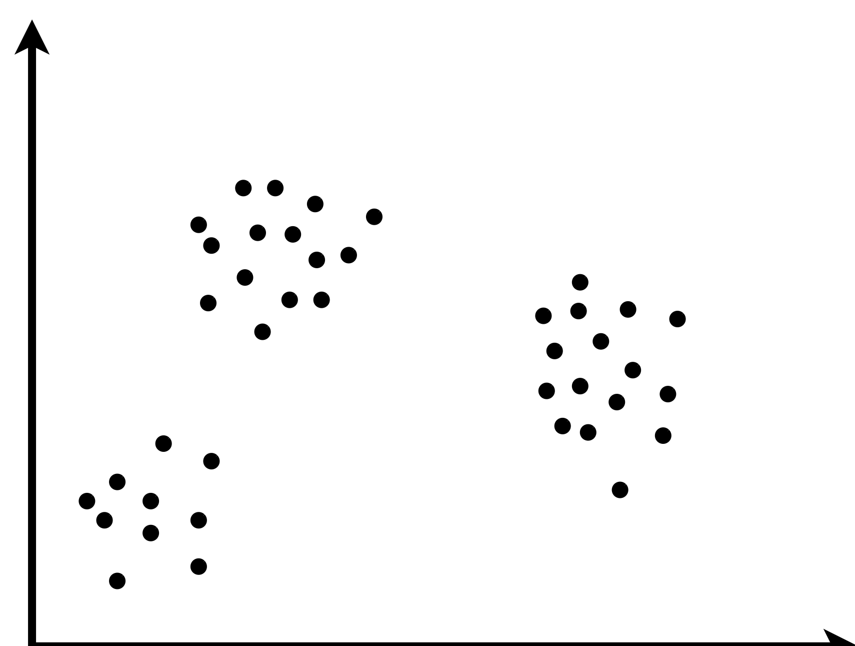

Dirichlet process mixture model

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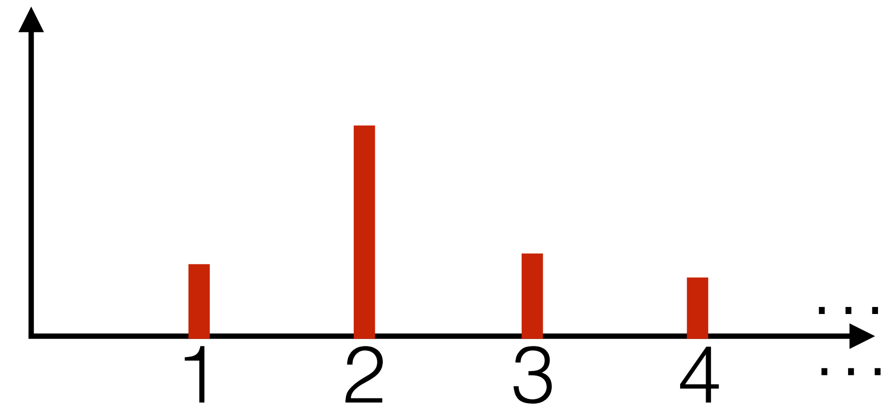

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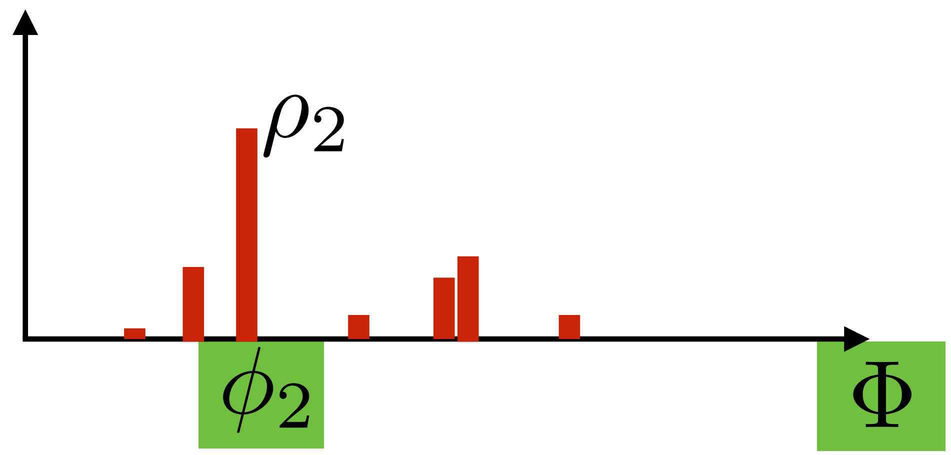

$$z_n \overset{iid}{\sim} \text{Categorical}(\rho)$$

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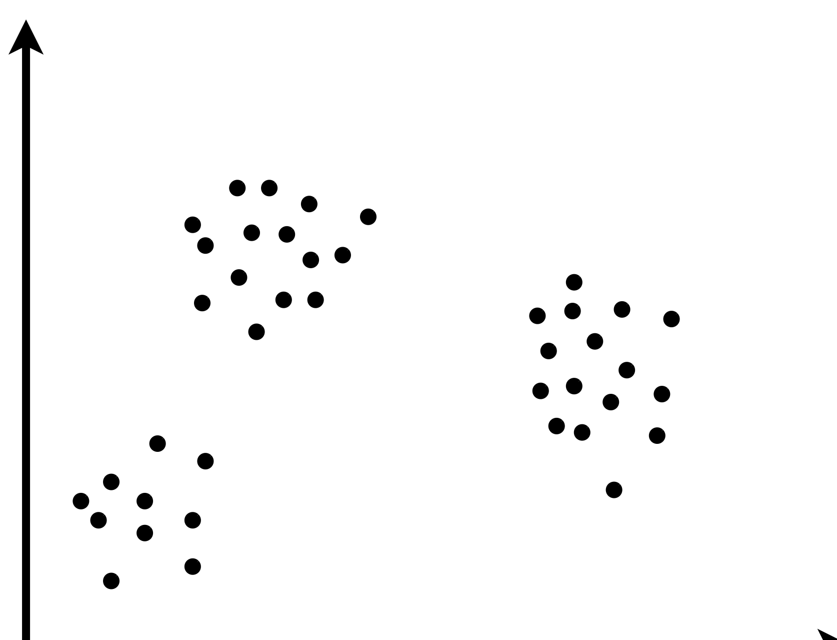

$$x_n \overset{indep}{\sim} F(\theta_n)$$

Dirichlet process mixture model

- More generally

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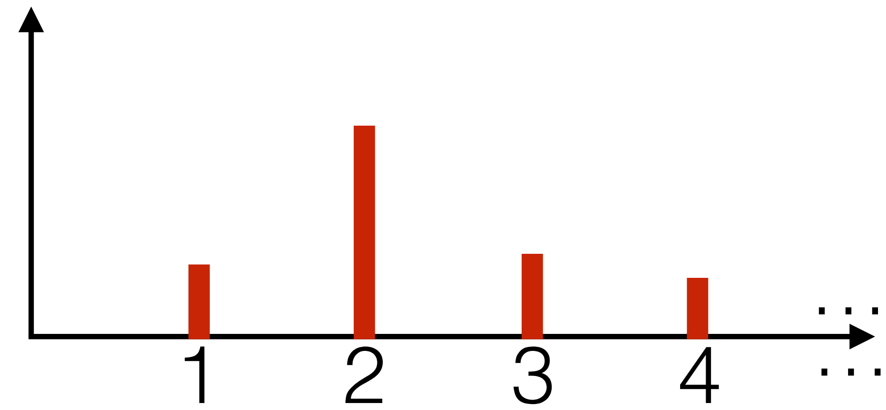

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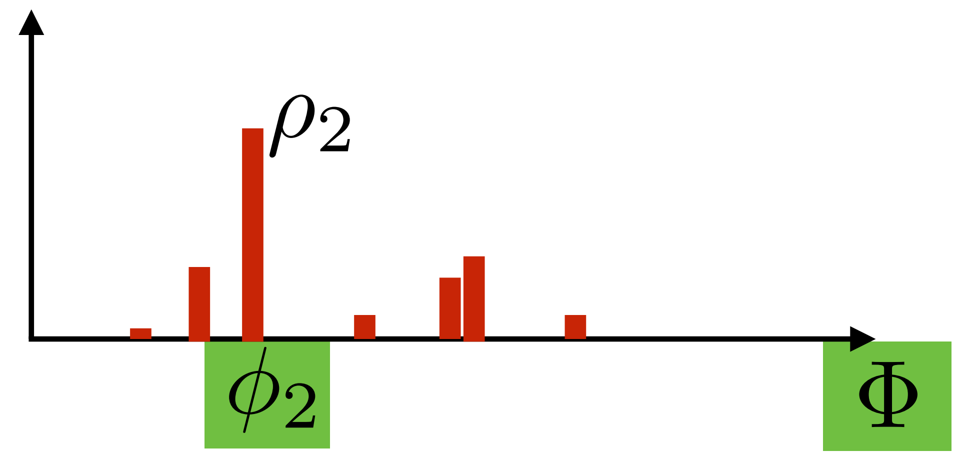

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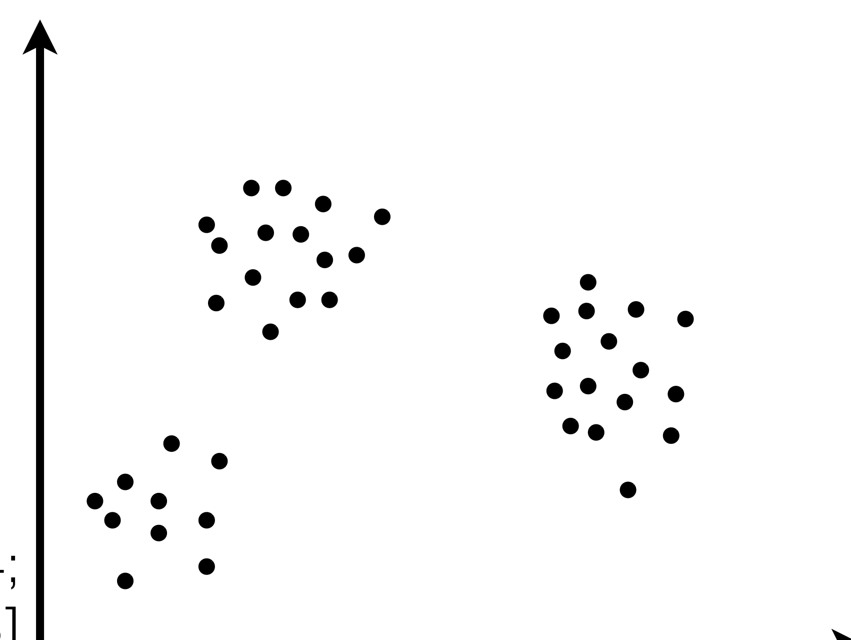
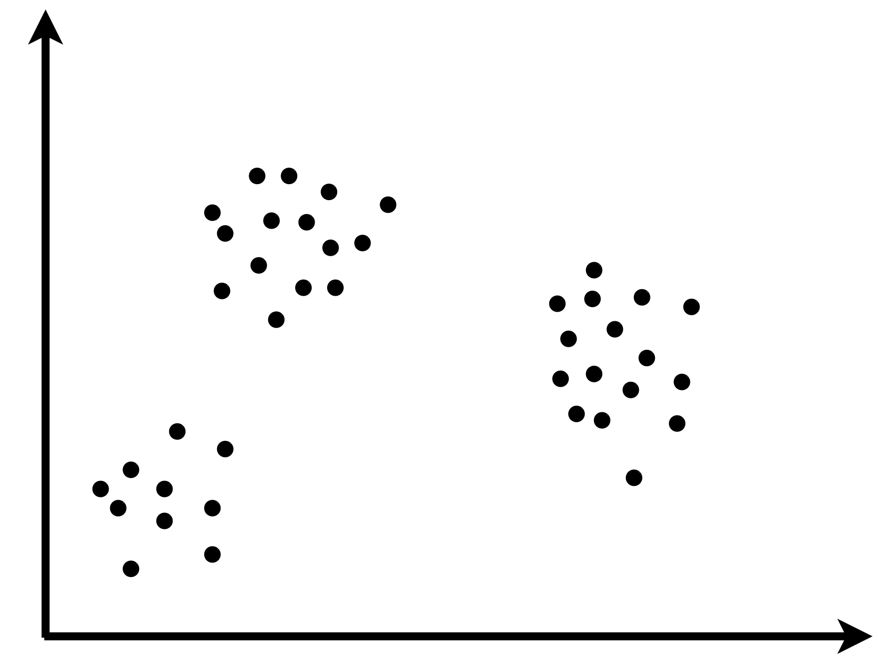


$$x_n \stackrel{indep}{\sim} F(\theta_n)$$

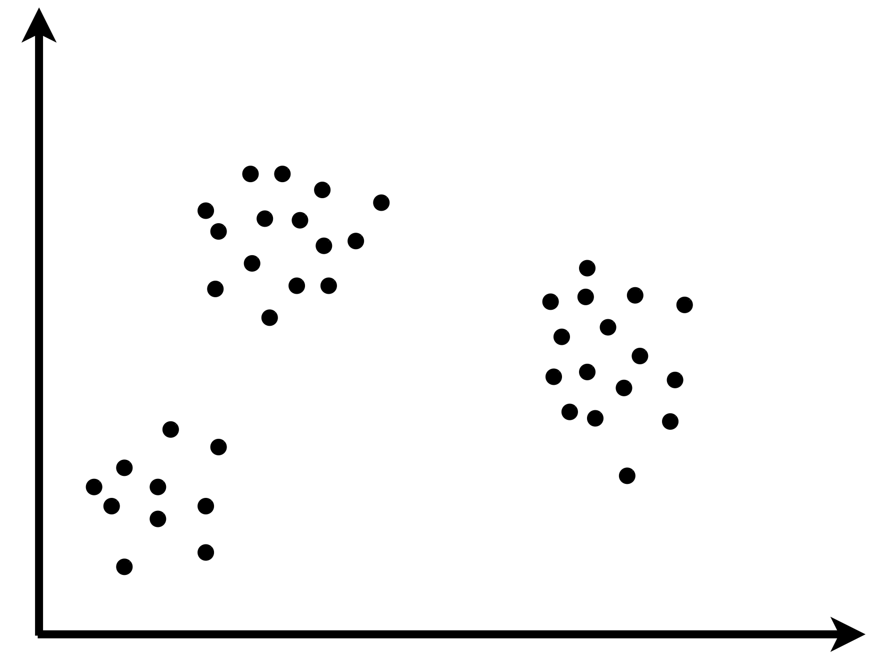
[Antoniak 1974; Ferguson 1983; West, Müller, Escobar 1994; Escobar, West 1995; MacEachern, Müller 1998]

DP or not DP, that is the question



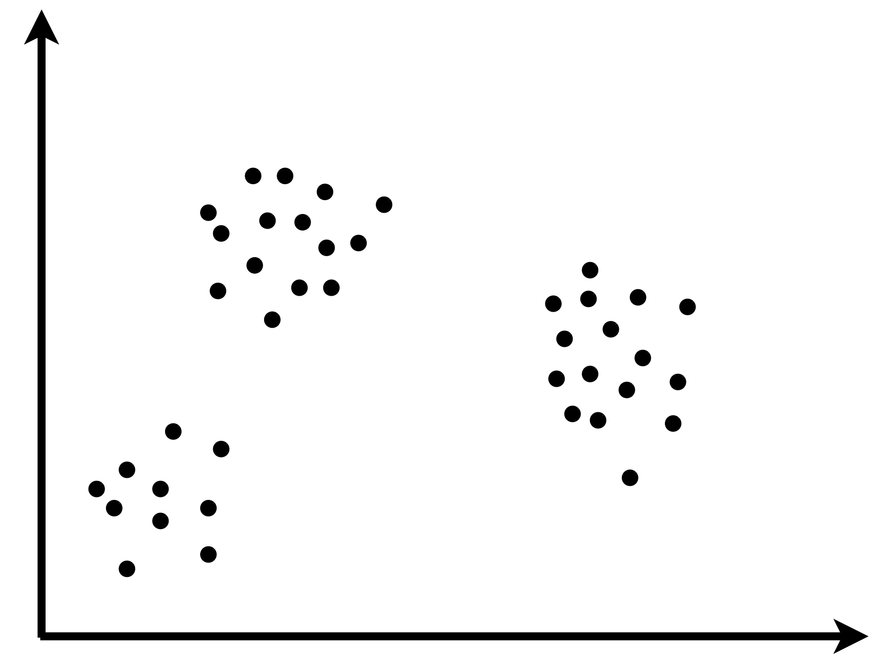
DP or not DP, that is the question

- GEM: 



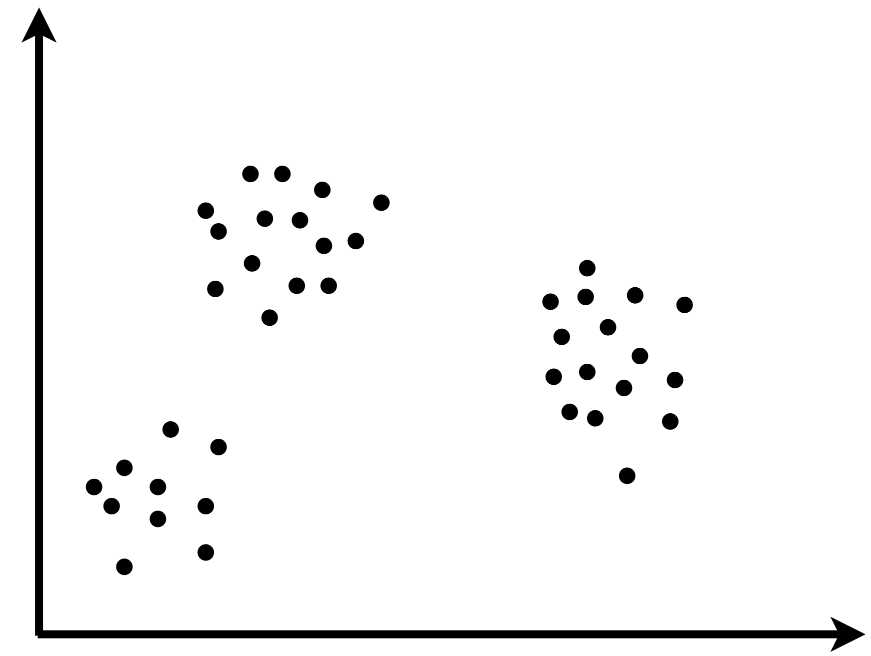
DP or not DP, that is the question

- GEM: 
- Compare to:



DP or not DP, that is the question

- GEM: 
- Compare to:
  - Finite (small *K*) mixture model



DP or not DP, that is the question

- GEM: 
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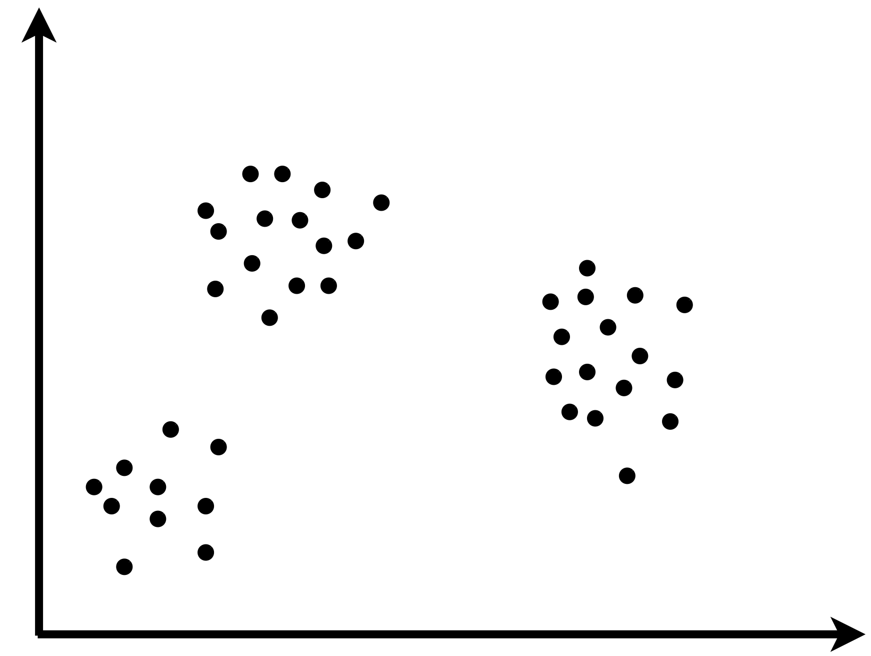
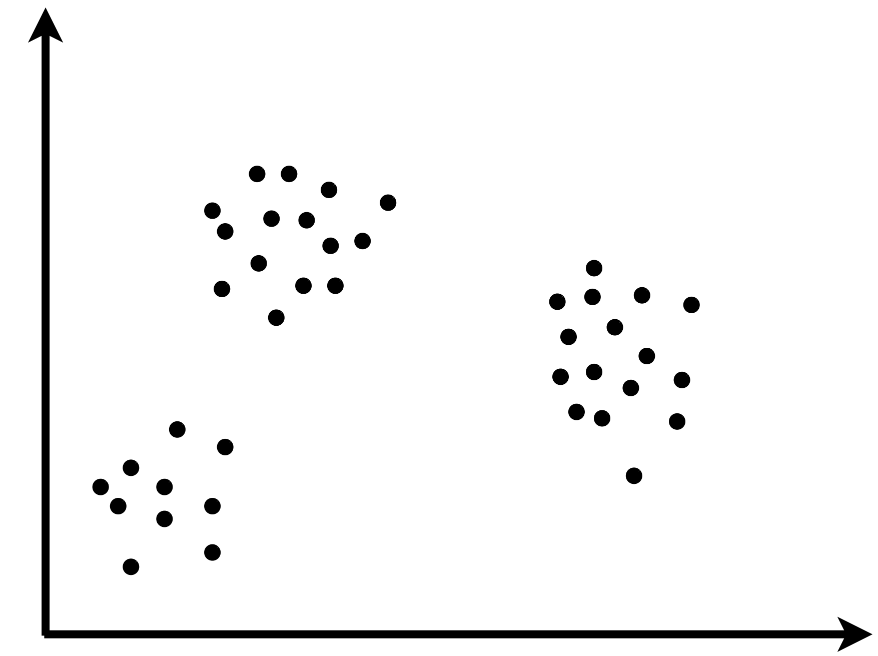


  - Finite (large *K*) mixture model

DP or not DP, that is the question

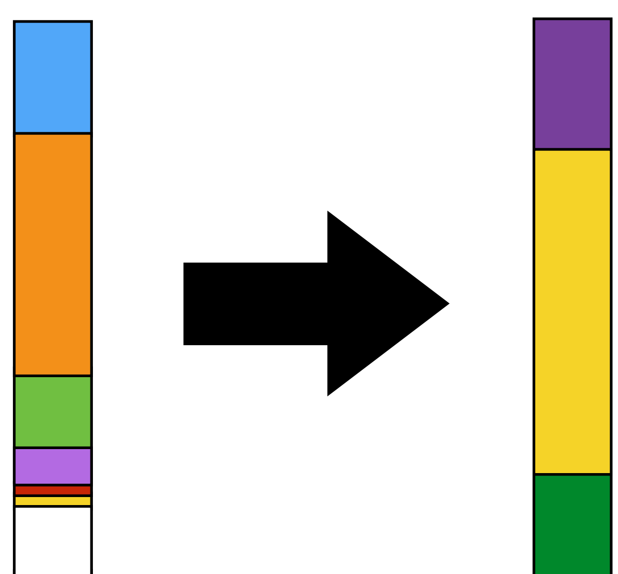
- GEM: 



- Compare to:
  - Finite (small *K*) mixture model

  - Finite (large *K*) mixture model

  - Time series



Roadmap

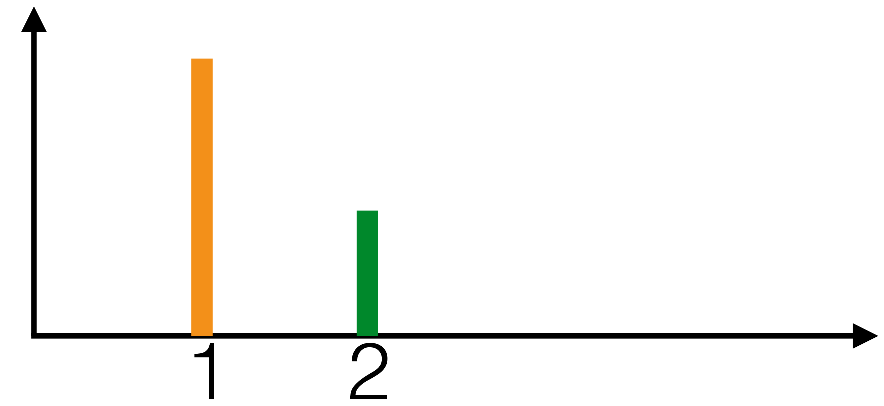
- Example problem: clustering
- Example NPBayes model: Dirichlet process
- Chinese restaurant process
- Inference
- Venture further into the wild world of Nonparametric Bayes

- Big questions
  - Why NPBayes? Learn more as acquire more data
  - What does an infinite/growing number of parameters really mean (in NPBayes)? Components vs. clusters; latent vs. realized
  - Why is NPBayes challenging but practical? Infinite dimensional parameter; more on this today!

Marginal cluster assignments

Marginal cluster assignments

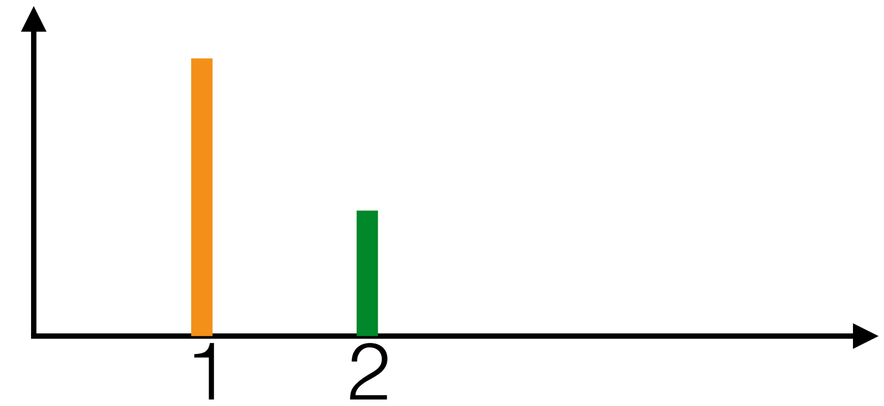
$$\rho_1 \sim \mathrm{Beta}(a_1, a_2), z_n \overset{iid}{\sim} \mathrm{Cat}(\rho_1, \rho_2)$$



Marginal cluster assignments

- Integrate out the frequencies

$\rho_1 \sim \mathrm{Beta}(a_1, a_2), z_n \overset{iid}{\sim} \mathrm{Cat}(\rho_1, \rho_2)$

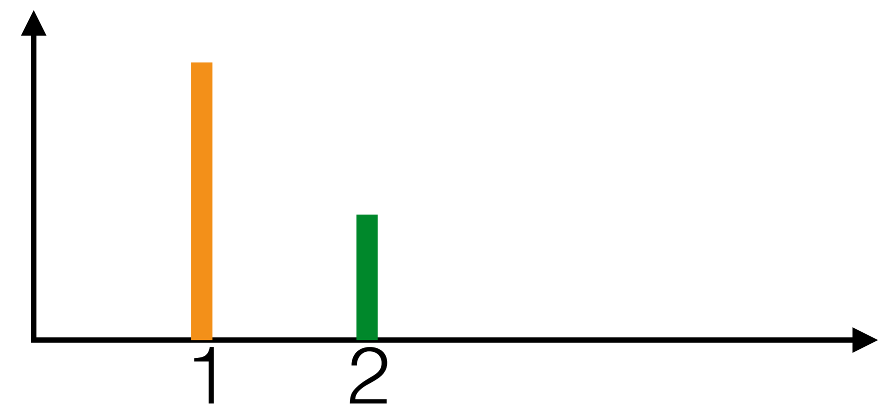


Marginal cluster assignments

- Integrate out the frequencies

$$\rho_1 \sim \mathrm{Beta}(a_1, a_2), z_n \overset{iid}{\sim} \mathrm{Cat}(\rho_1, \rho_2)$$

$$p(z_n = 1 | z_1, \ldots, z_{n-1})$$



Marginal cluster assignments

- Integrate out the frequencies

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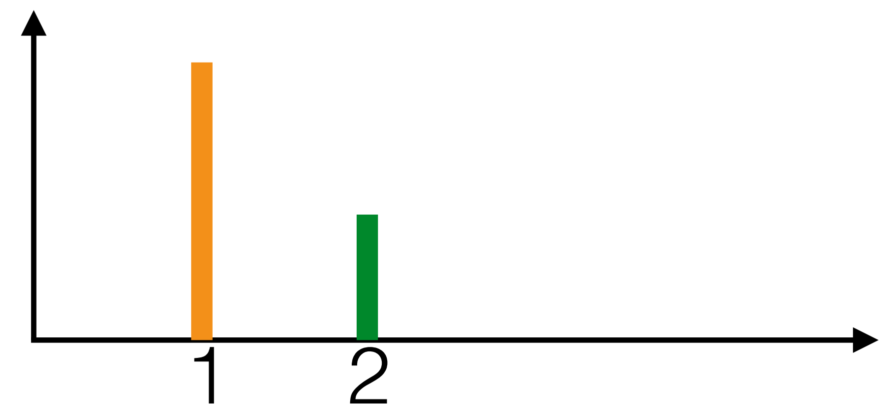
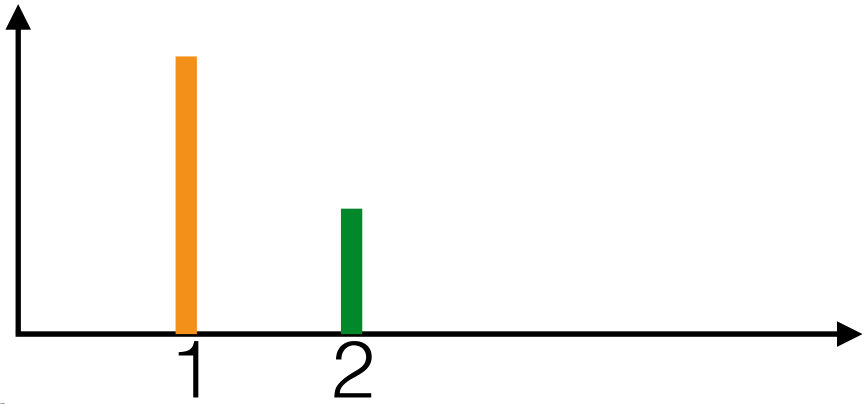
$$p(z_n = 1 | z_1, \ldots, z_{n-1})$$
$$= \int p(z_n = 1, \rho_1 | z_1, \ldots, z_{n-1}) d\rho_1$$



Marginal cluster assignments

- Integrate out the frequencies



$$\rho_1 \sim \text{Beta}(a_1, a_2), z_n \overset{iid}{\sim} \text{Cat}(\rho_1, \rho_2)$$

$$p(z_n = 1 | z_1, \ldots, z_{n-1})$$

$$= \int p(z_n = 1 | \rho_1) p(\rho_1 | z_1, \ldots, z_{n-1}) d\rho_1$$

Marginal cluster assignments

- Integrate out the frequencies

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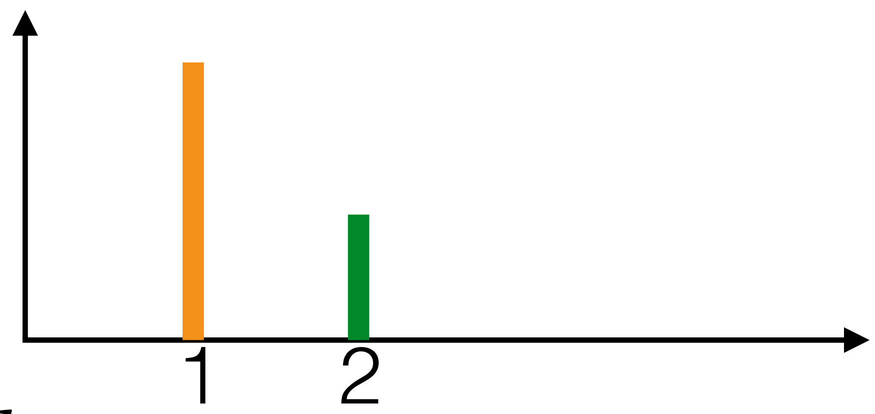

Marginal cluster assignments

- Integrate out the frequencies

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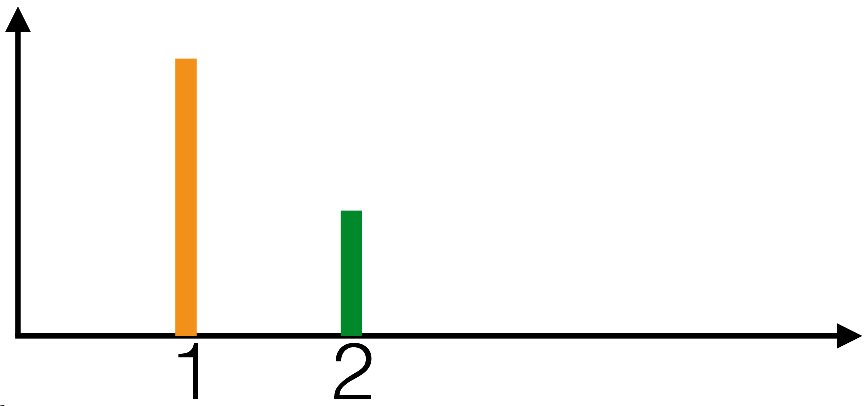

Marginal cluster assignments

- Integrate out the frequencies

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$$p(z_n = 1 | z_1, \ldots, z_{n-1})$$
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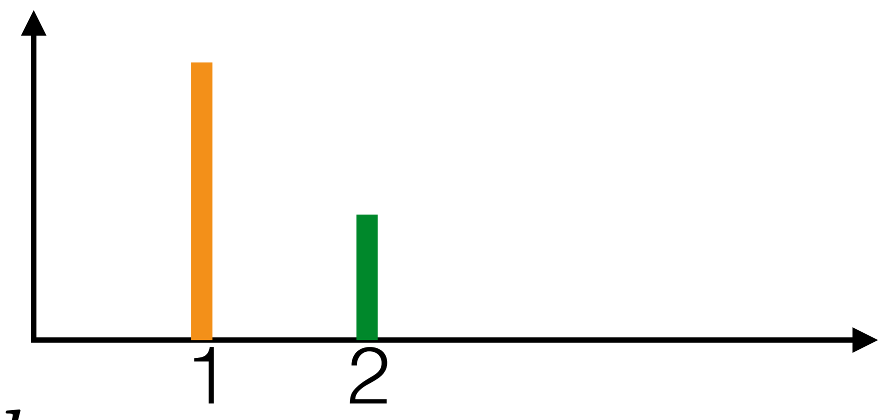
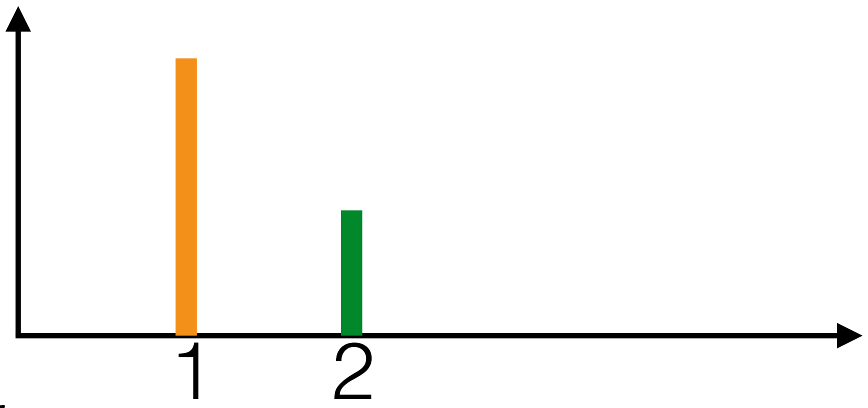


Marginal cluster assignments

- Integrate out the frequencies



$$\rho_1 \sim \mathrm{Beta}(a_1, a_2), z_n \stackrel{iid}{\sim} \mathrm{Cat}(\rho_1, \rho_2)$$

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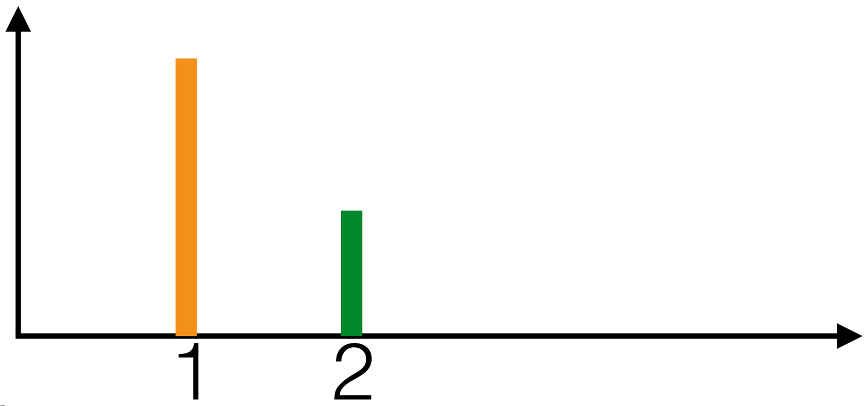

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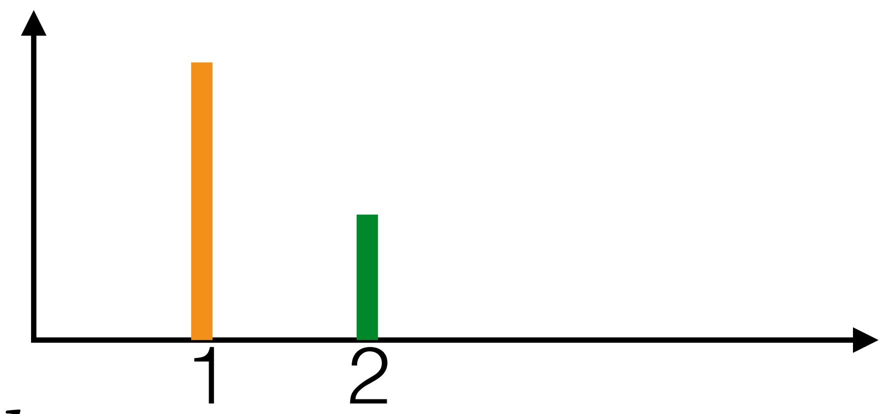

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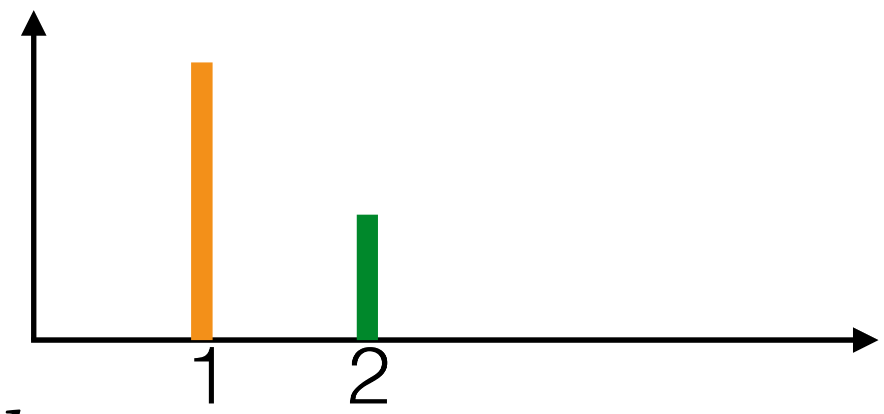

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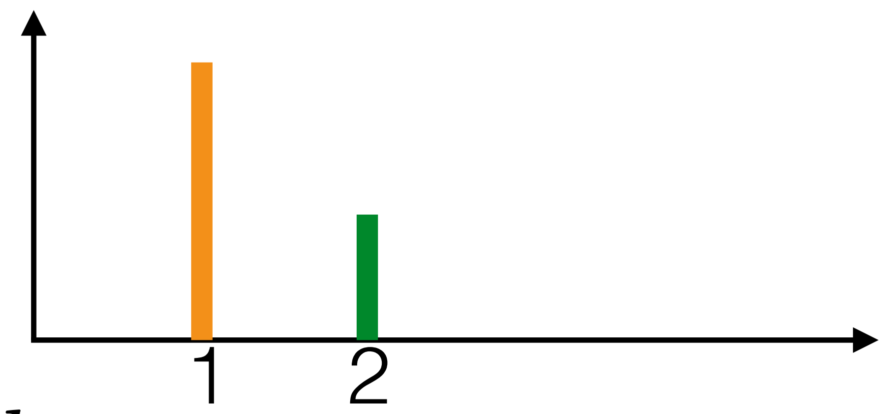

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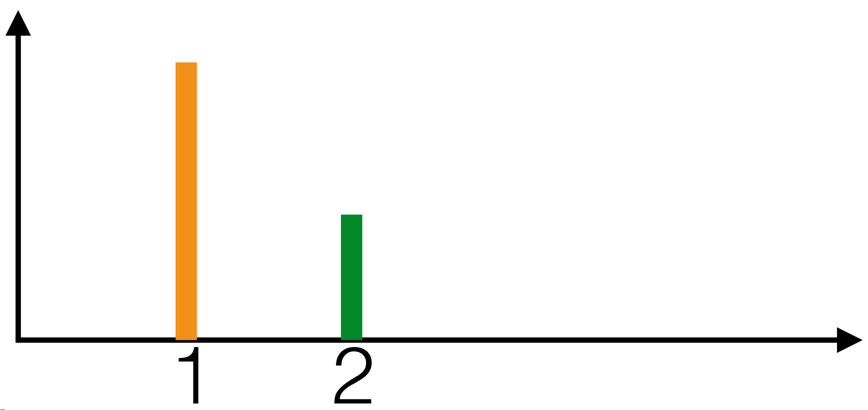

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Recall
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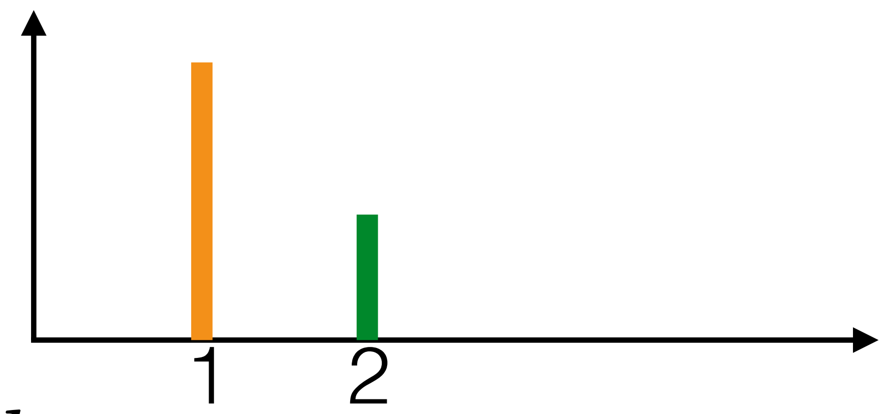

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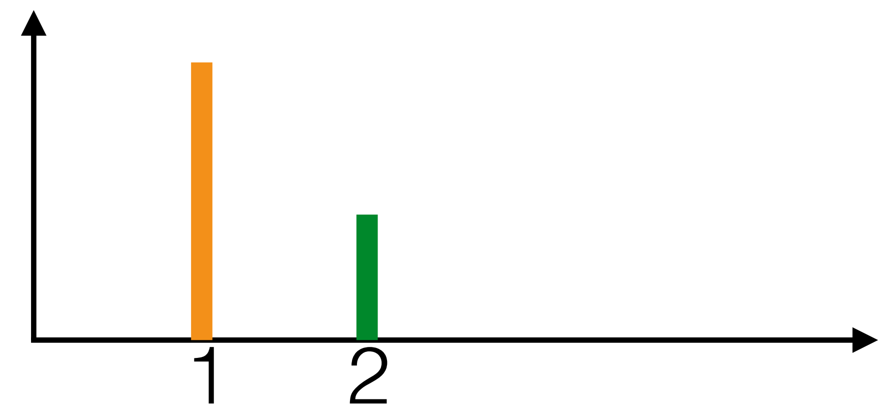

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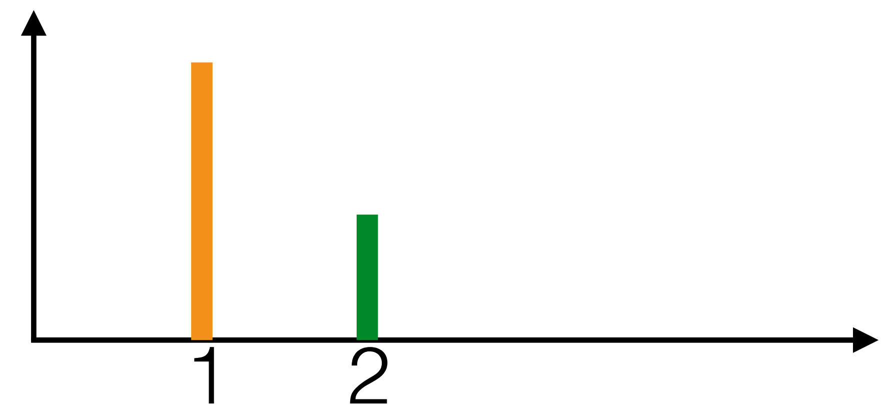

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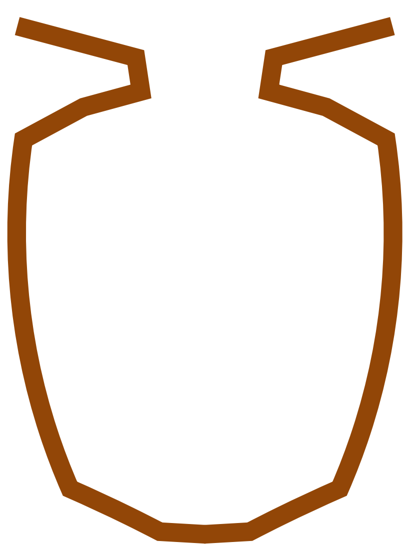

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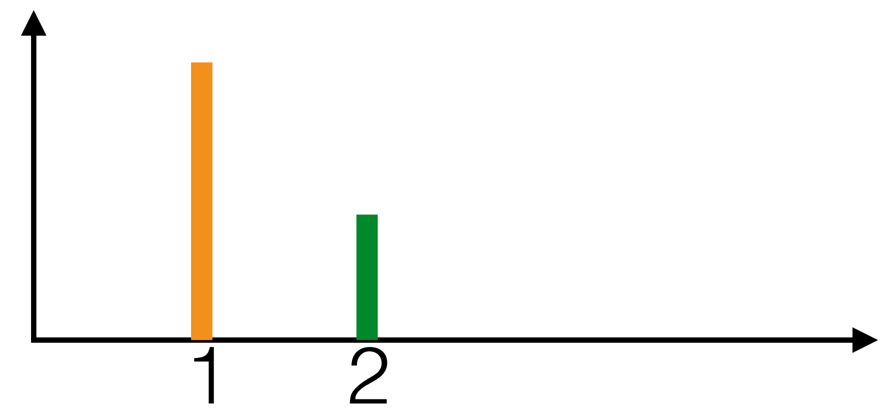

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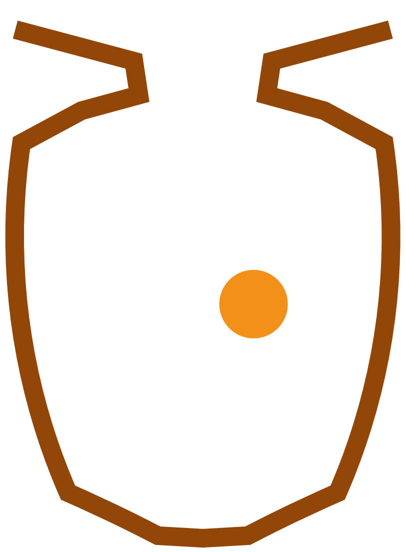

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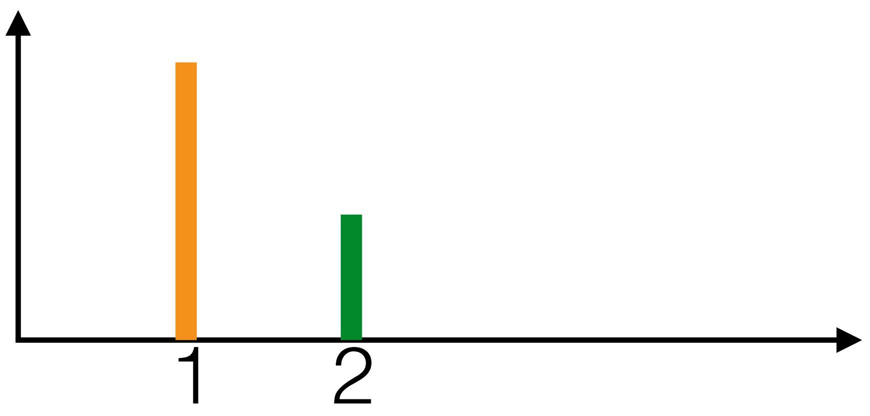

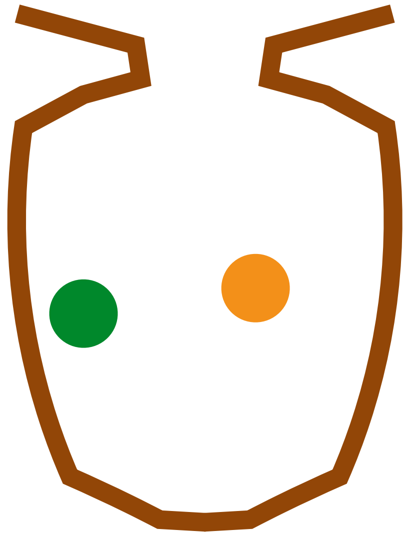

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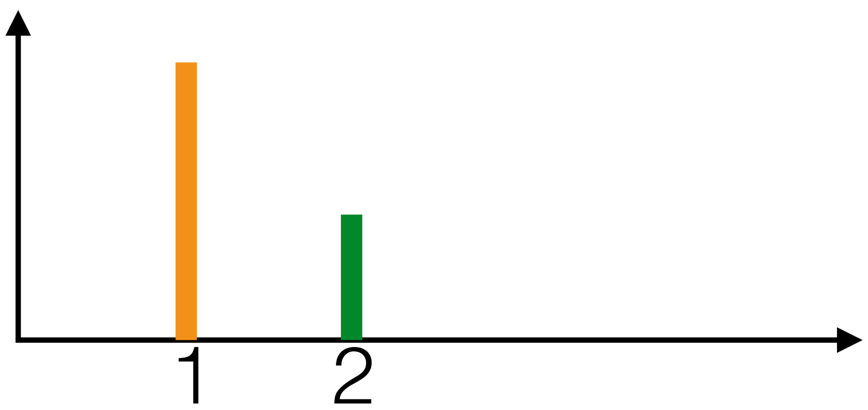

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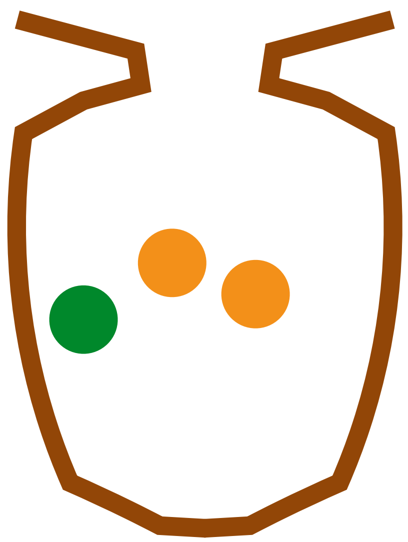

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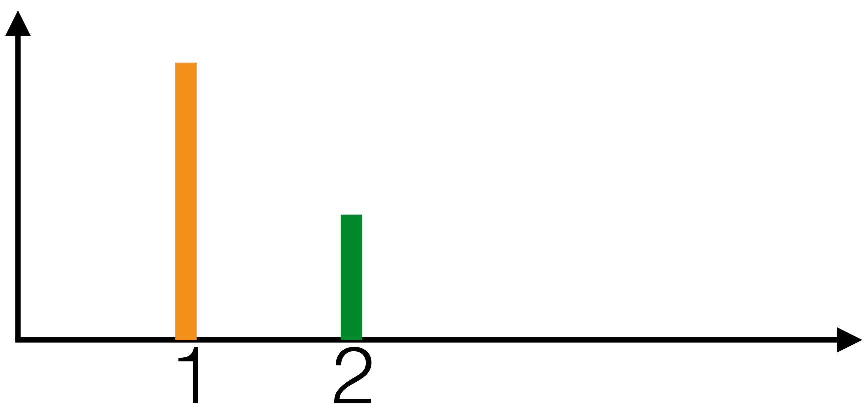

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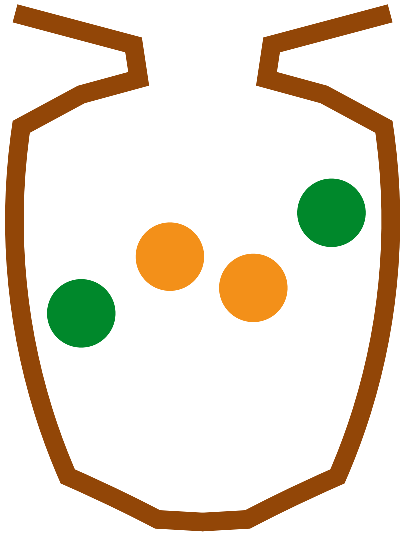

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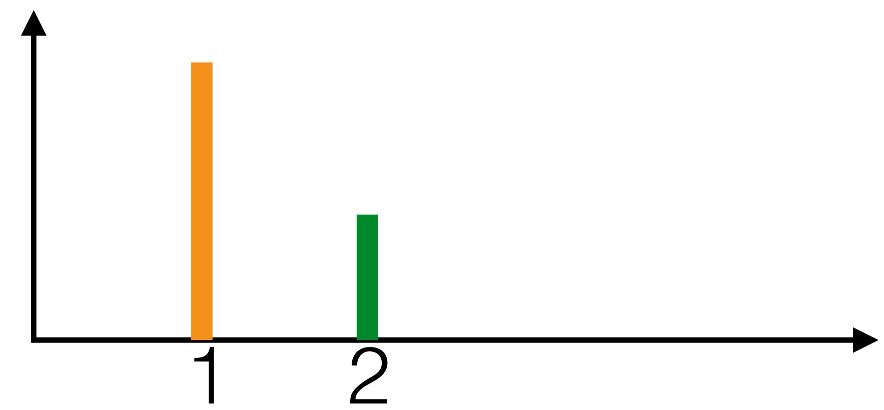

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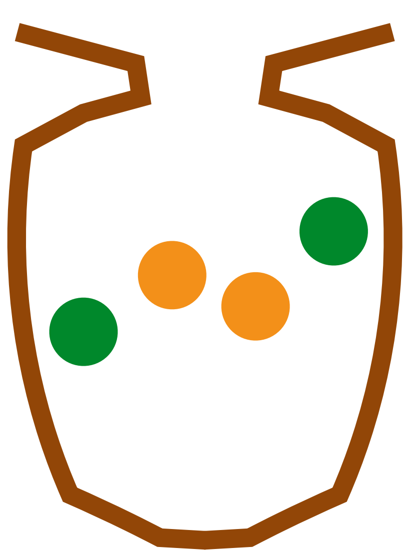

$$\lim_{n \to \infty} \frac{\#\text{ orange}}{\#\text{ total}}$$

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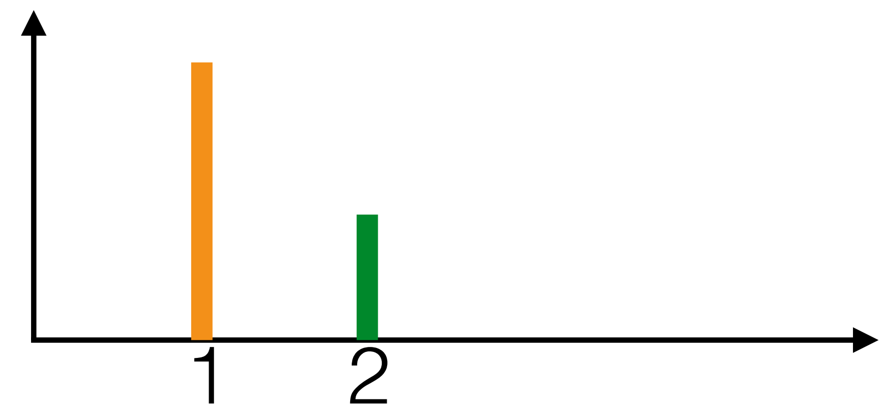

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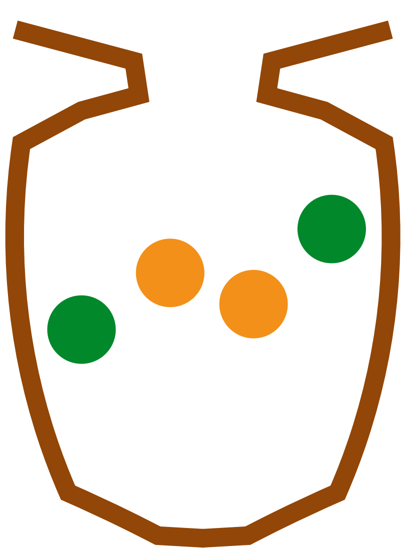

$$\lim_{n \to \infty} \frac{\#\ \text{orange}}{\#\ \text{total}} = \rho_{\text{orange}}$$

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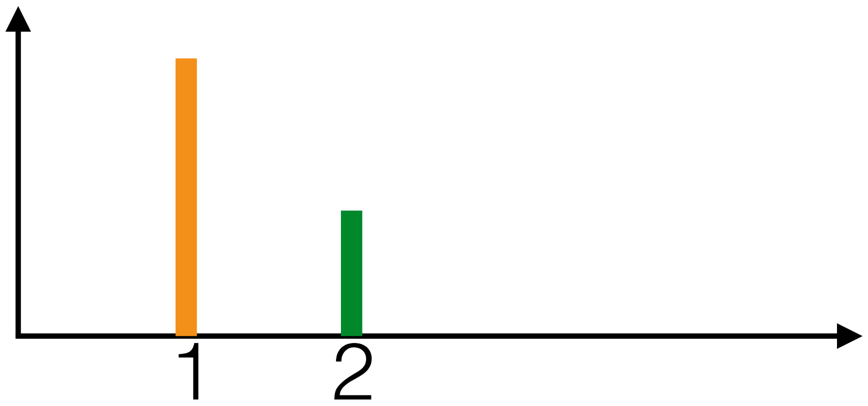

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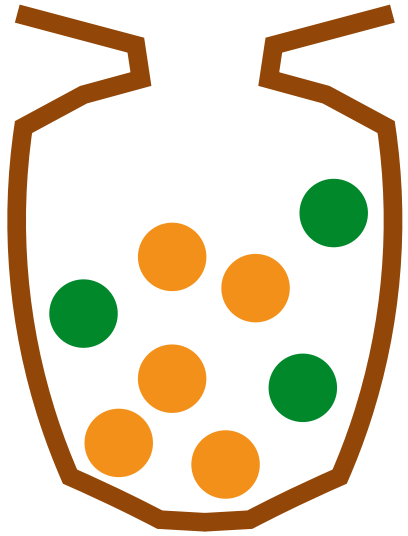

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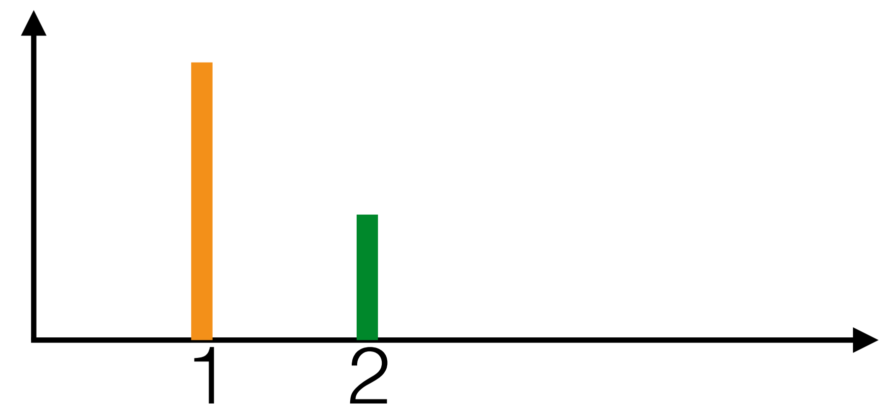

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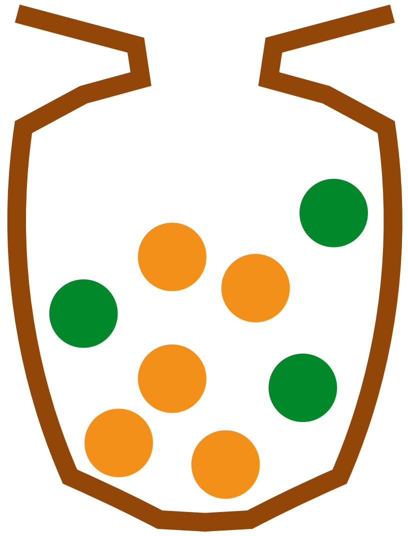

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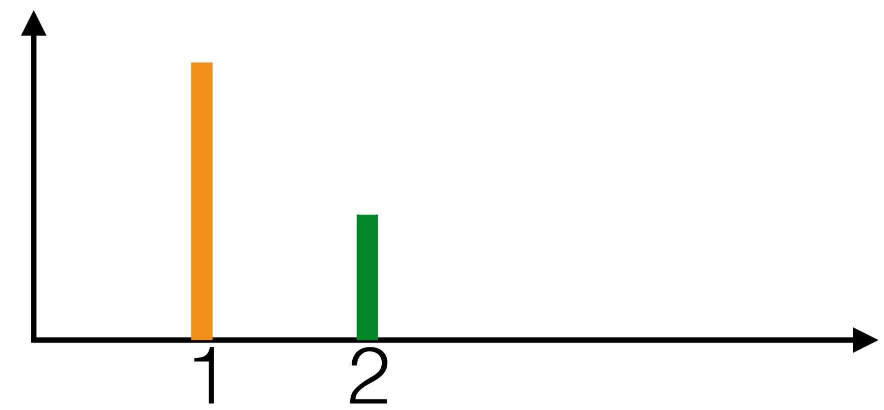

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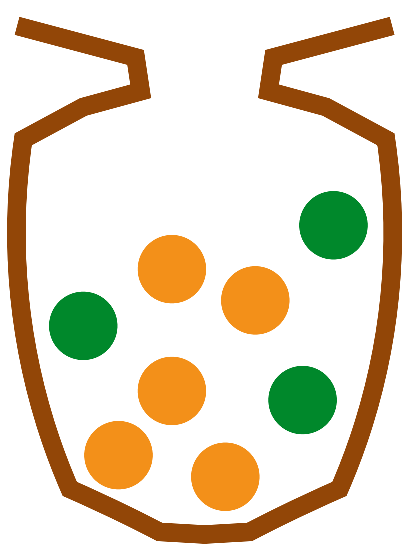

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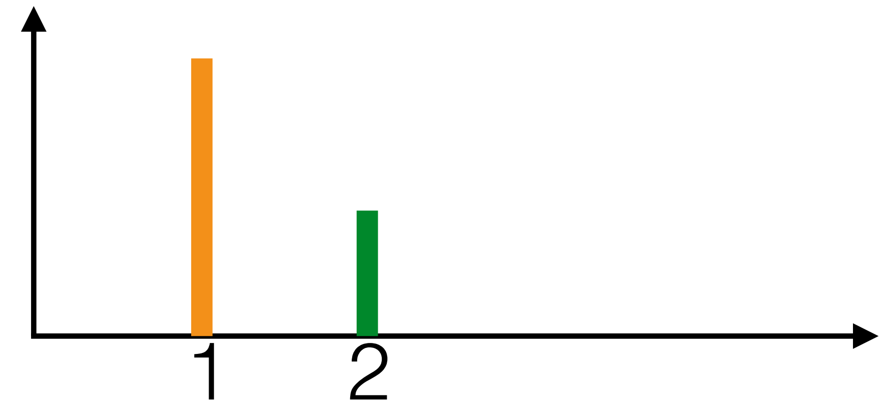

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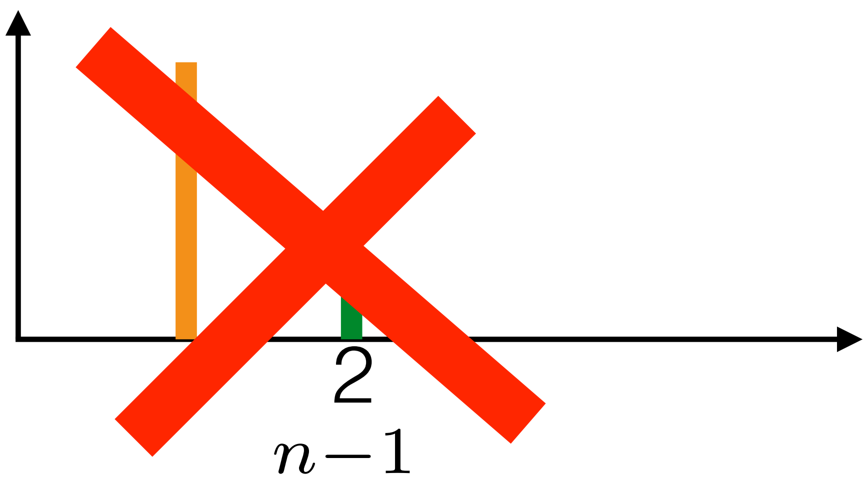

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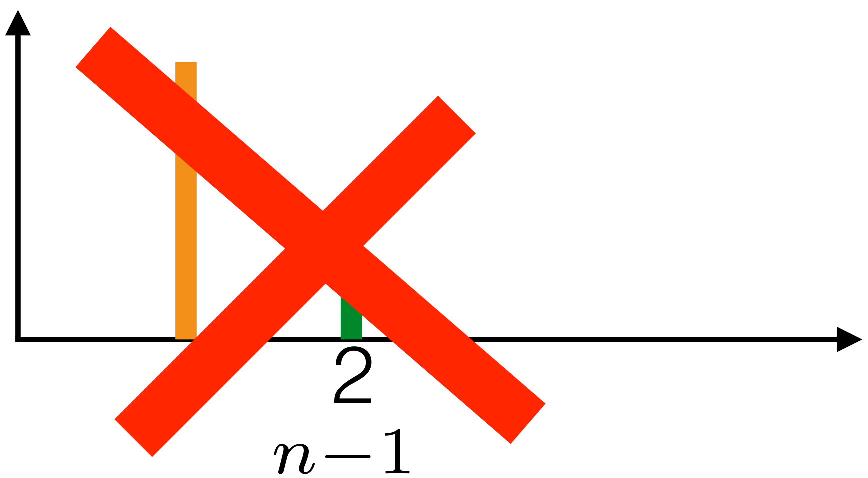

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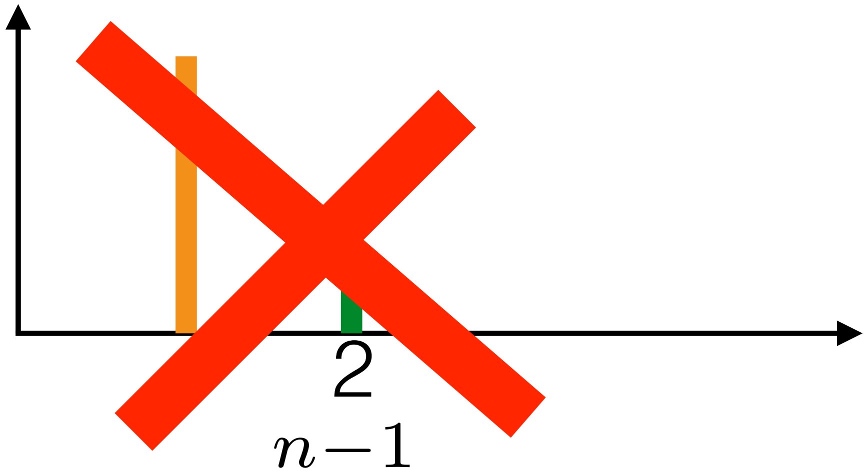
- Pólya urn

Marginal cluster assignments

- Integrate out the frequencies

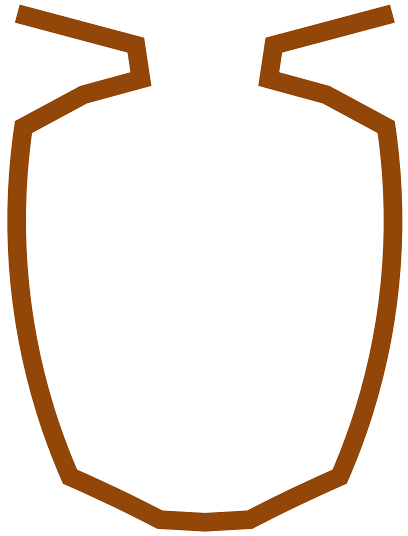
$$\rho_1 \sim \text{Beta}(a_1, a_2),\ z_n \overset{iid}{\sim} \text{Cat}(\rho_1, \rho_2)$$

$$p(z_n = 1 | z_1, \ldots, z_{n-1}) = \frac{a_{1,n}}{a_{1,n} + a_{2,n}}$$



$$a_{1,n} := a_1 + \sum_{m=1}^{n-1} \mathbf{1}\{z_m = 1\},\ a_{2,n} = a_2 + \sum_{m=1}^{n-1} \mathbf{1}\{z_m = 2\}$$

- Pólya urn



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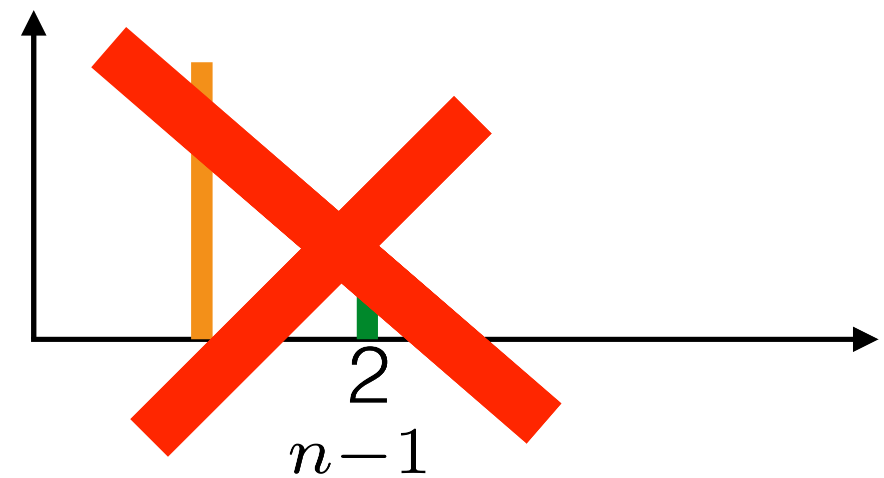

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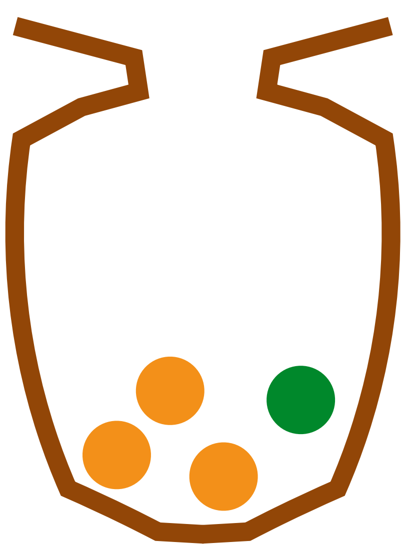

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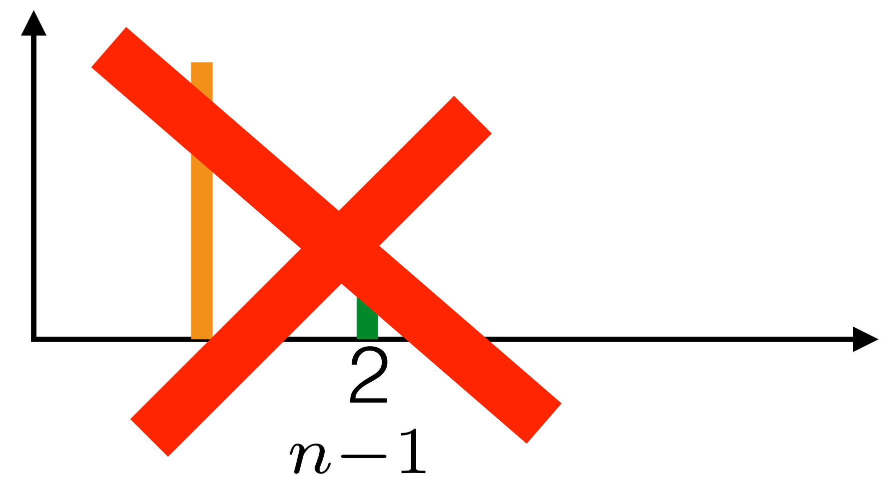

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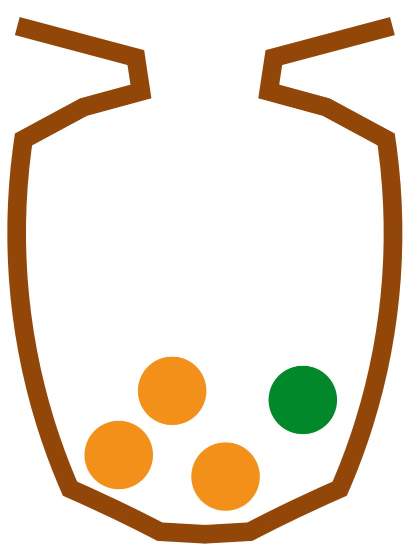

Marginal cluster assignments

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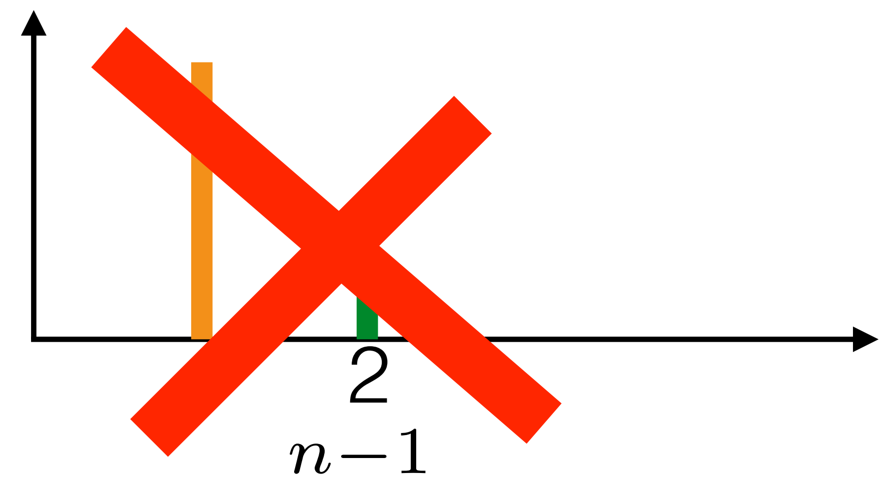

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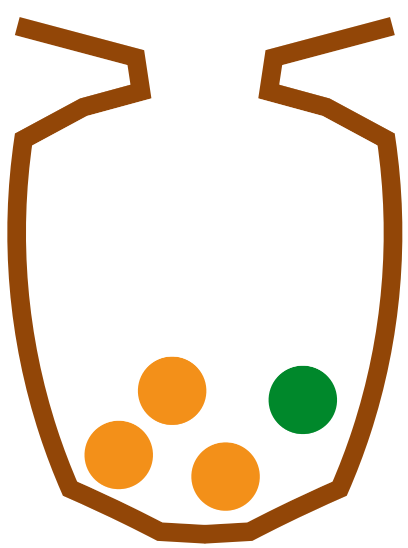

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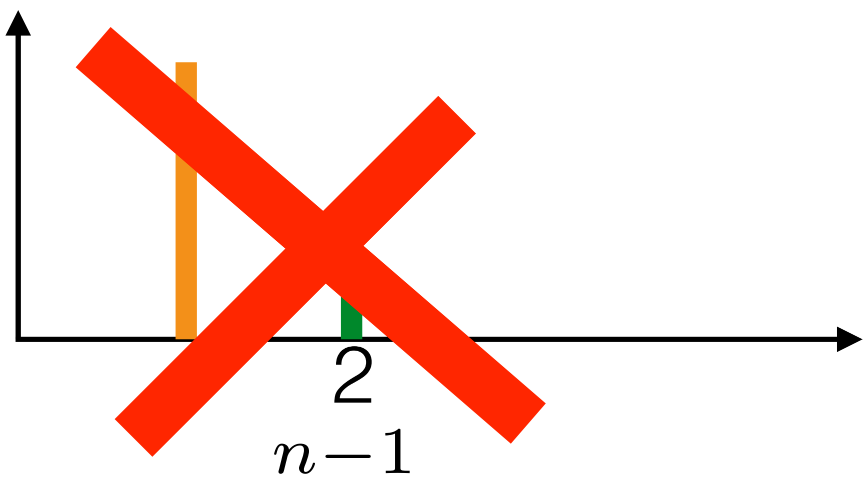

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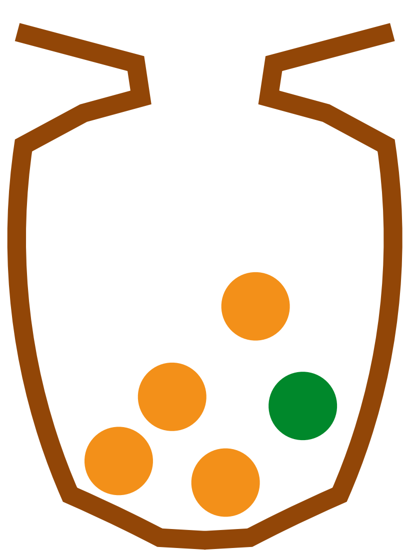

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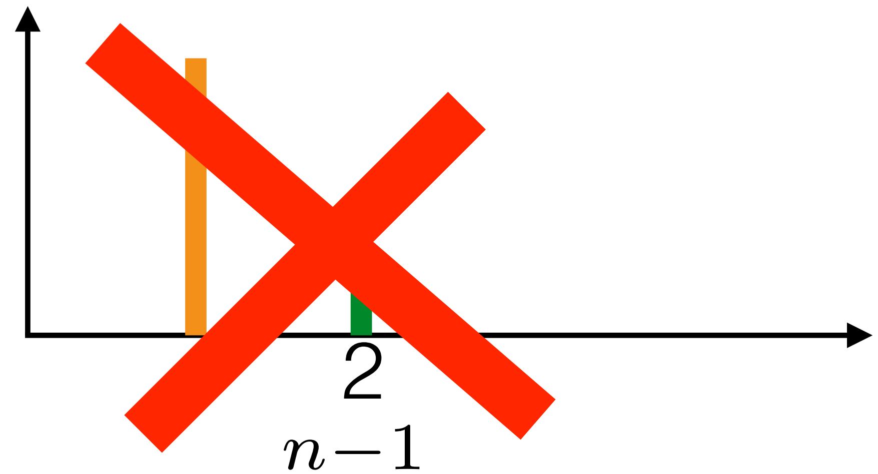

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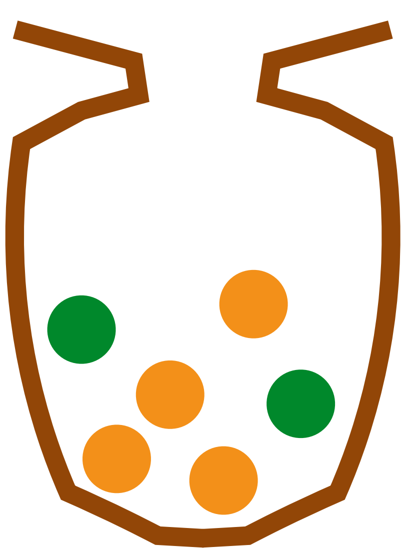

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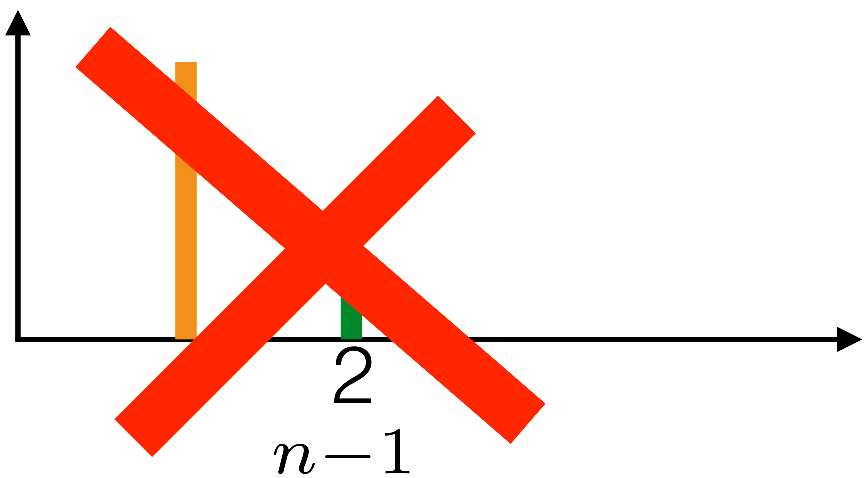

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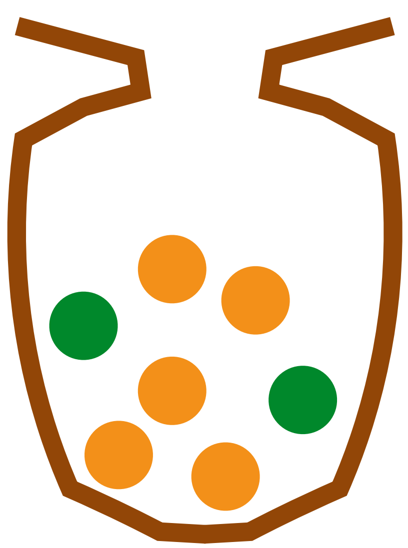

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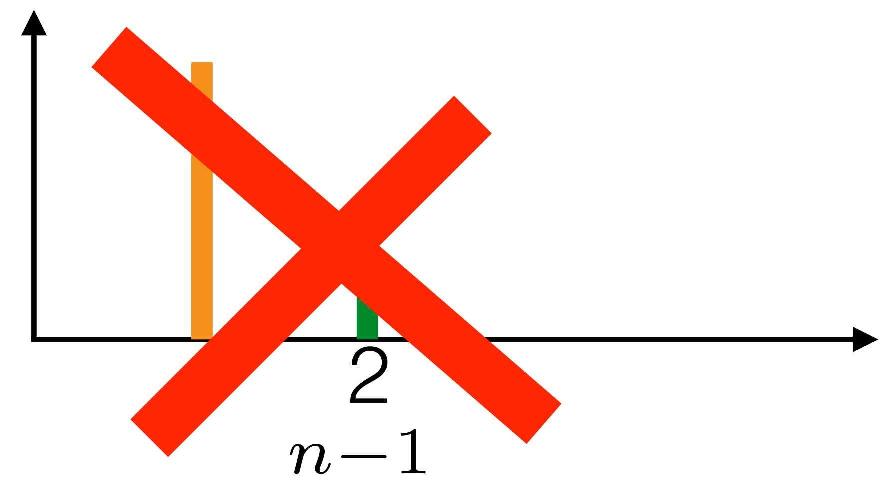

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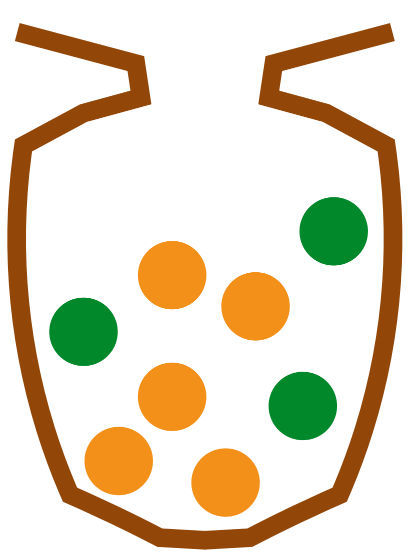
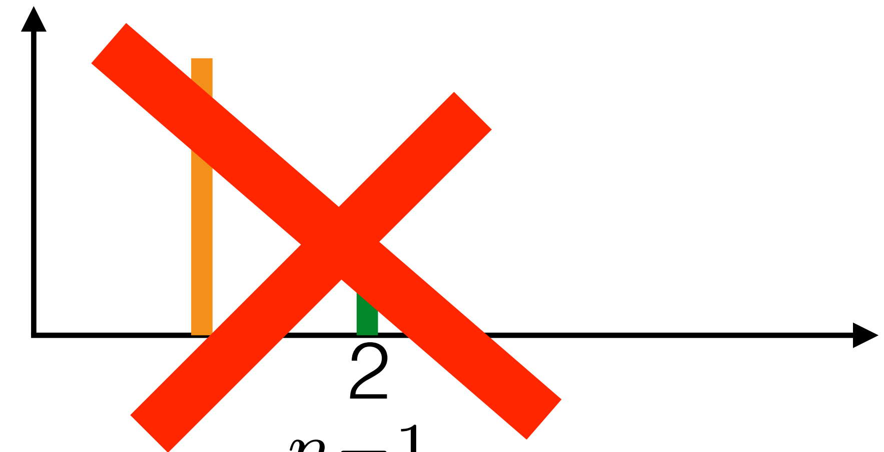


Marginal cluster assignments

- Integrate out the frequencies



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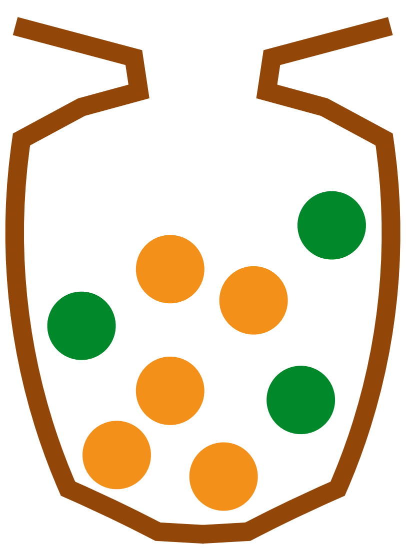

$$\lim_{n \to \infty} \frac{\#\ \text{orange}}{\#\ \text{total}}$$

Marginal cluster assignments

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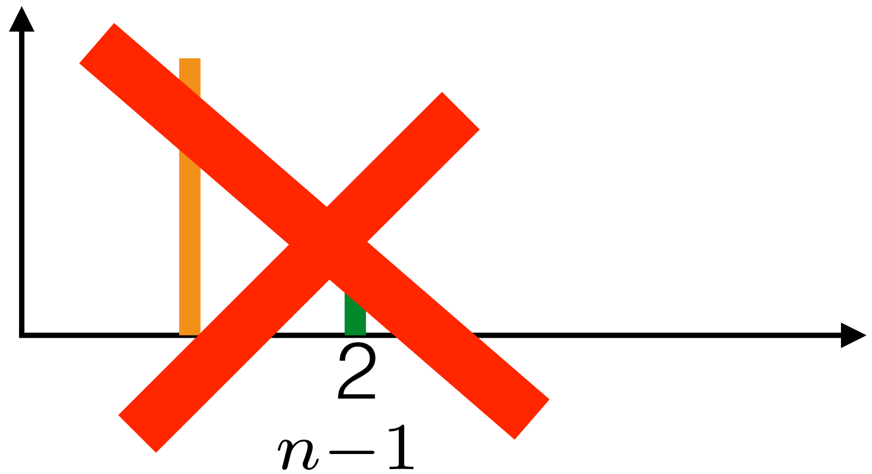

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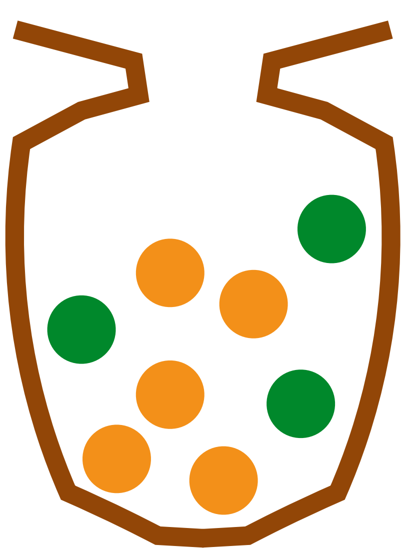

$$\lim_{n \to \infty} \frac{\#\ \text{orange}}{\#\ \text{total}} = \rho_{\text{orange}}$$

Marginal cluster assignments

- Integrate out the frequencies

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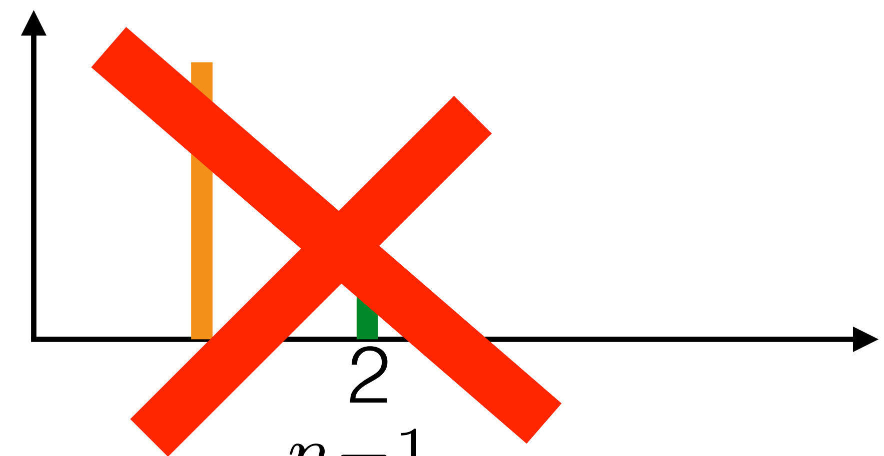

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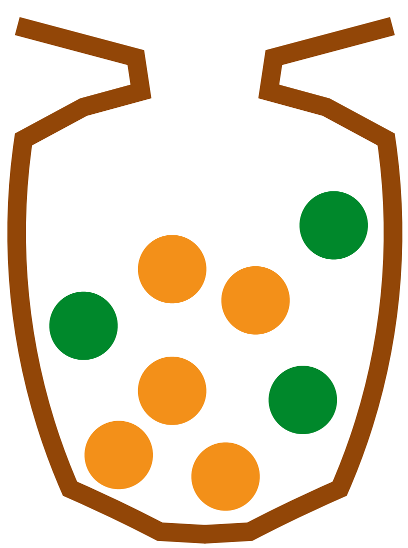
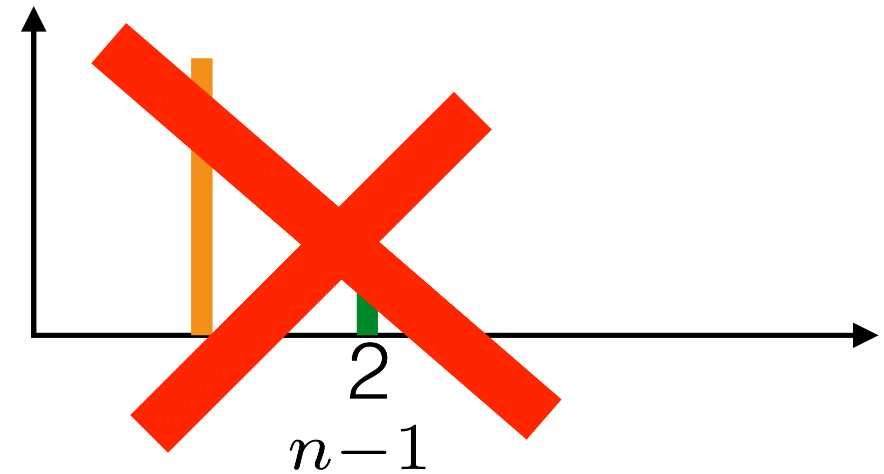


$$\lim_{n \to \infty} \frac{\#\ \text{orange}}{\#\ \text{total}} = \rho_{\text{orange}} \overset{d}{=} \mathrm{Beta}(a_{\text{orange}}, a_{\text{green}})$$

Marginal cluster assignments

- Integrate out the frequencies



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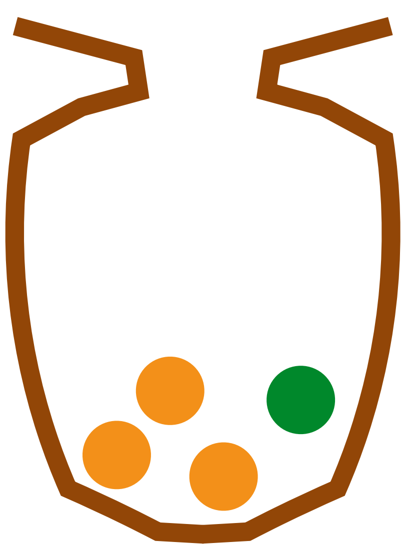

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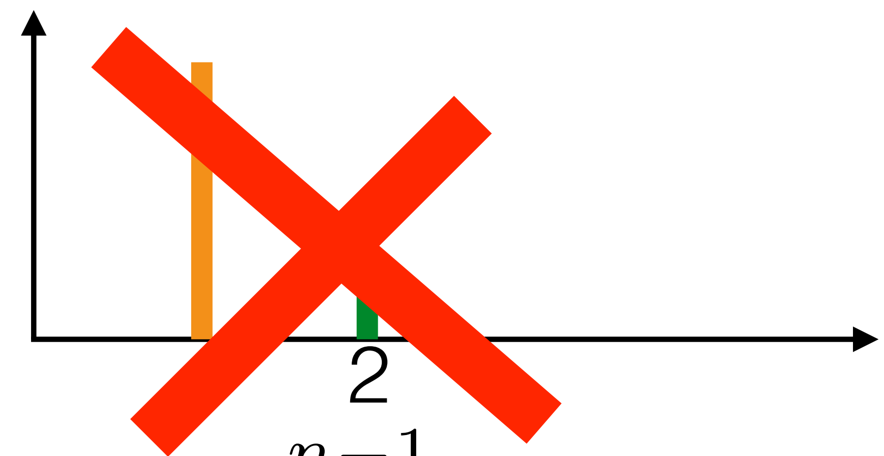

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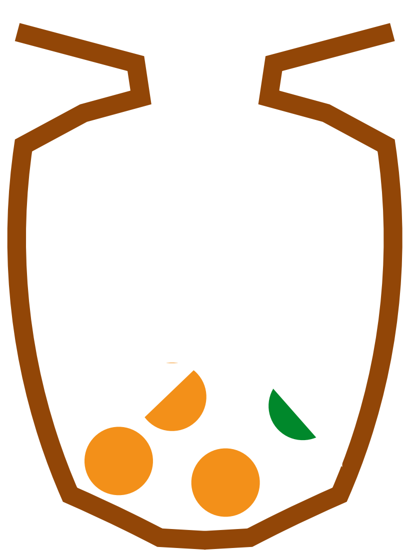

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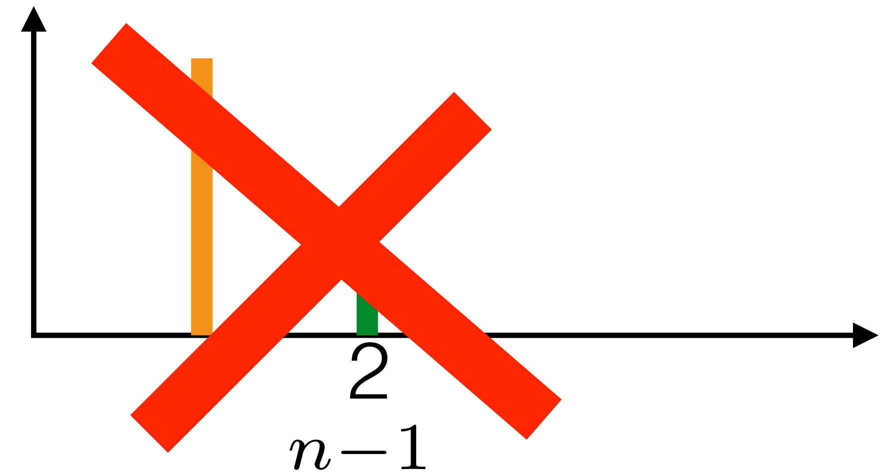

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  - Choose any ball with prob proportional to its mass
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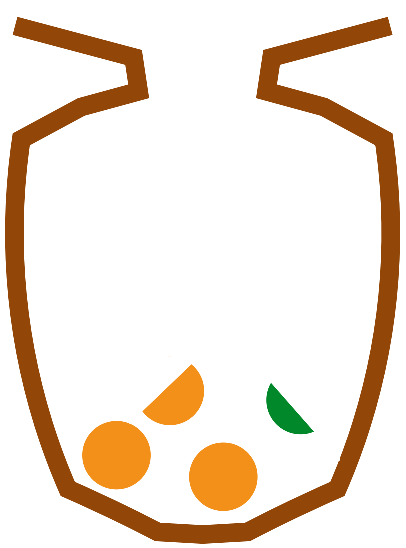

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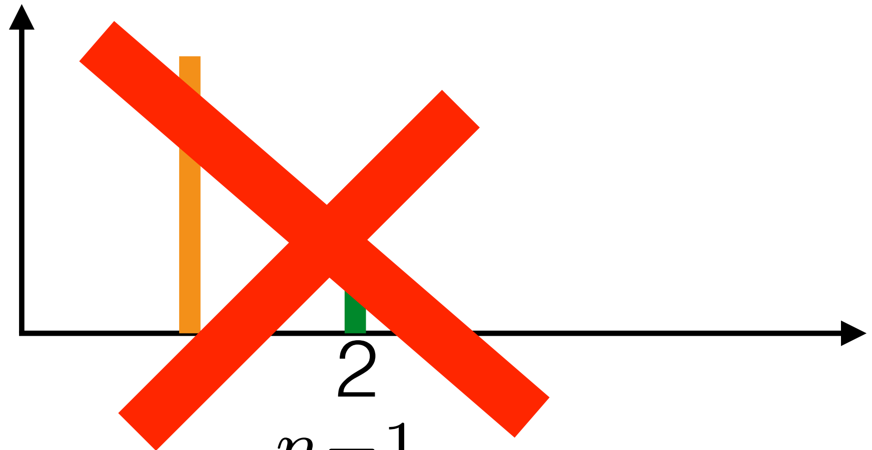

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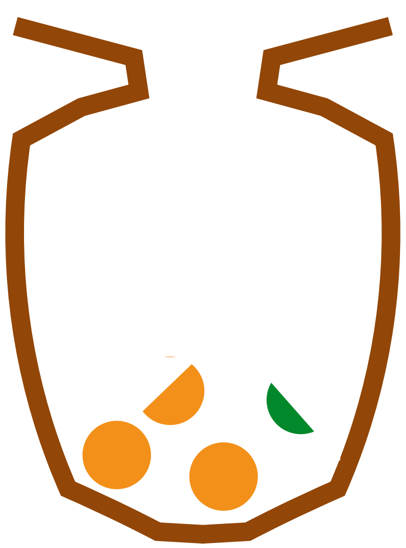

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$$\mathrm{PolyaUrn}(a_{\text{orange}}, a_{\text{green}})$$

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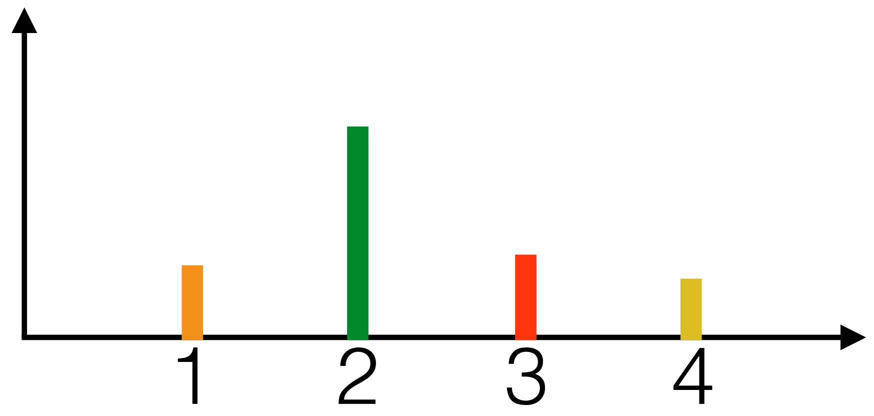

Marginal cluster assignments

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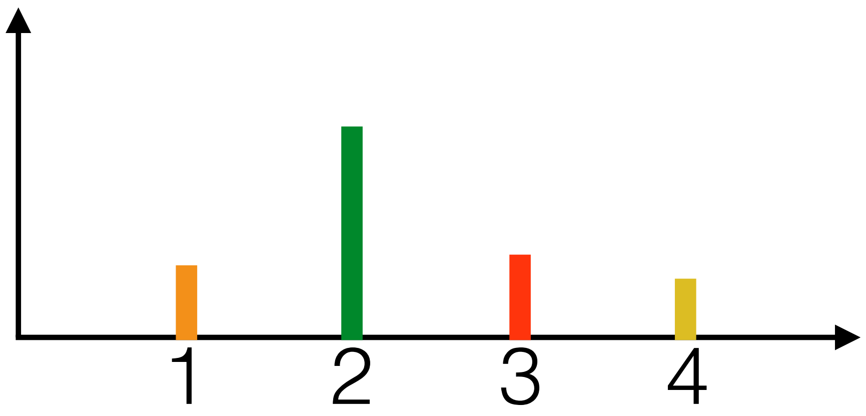
$$\rho_{1:K} \sim \mathrm{Dirichlet}(a_{1:K}), z_n \overset{iid}{\sim} \mathrm{Cat}(\rho_{1:K})$$



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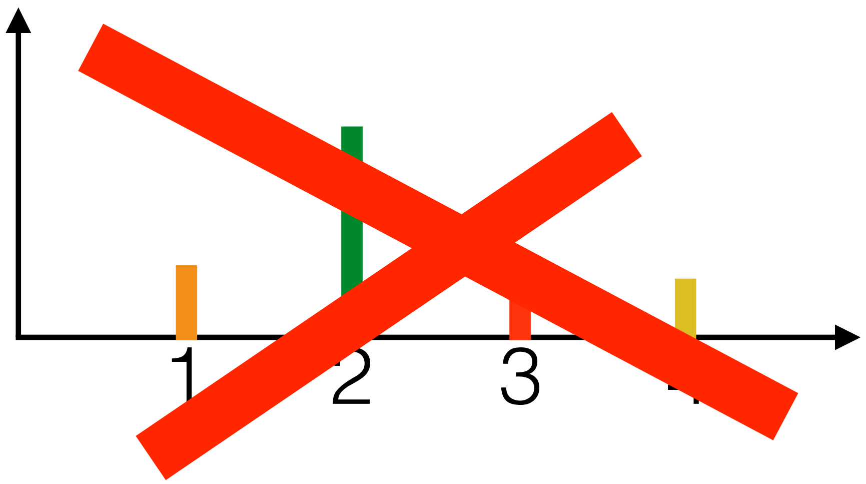

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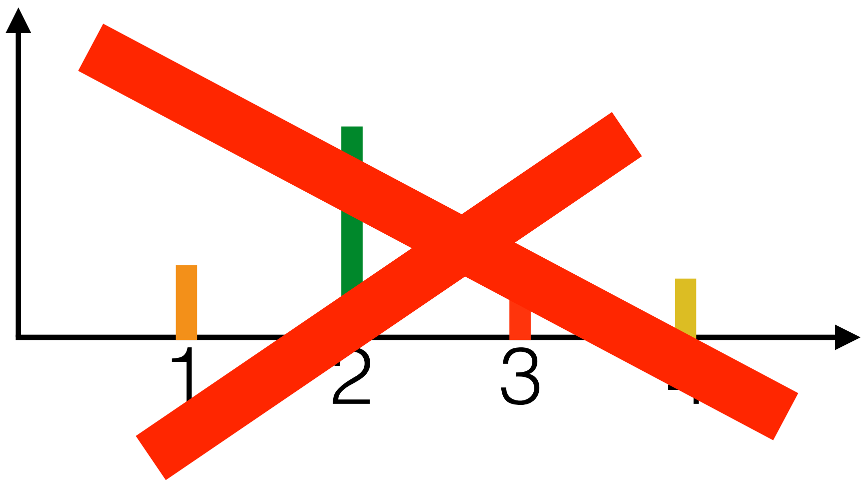

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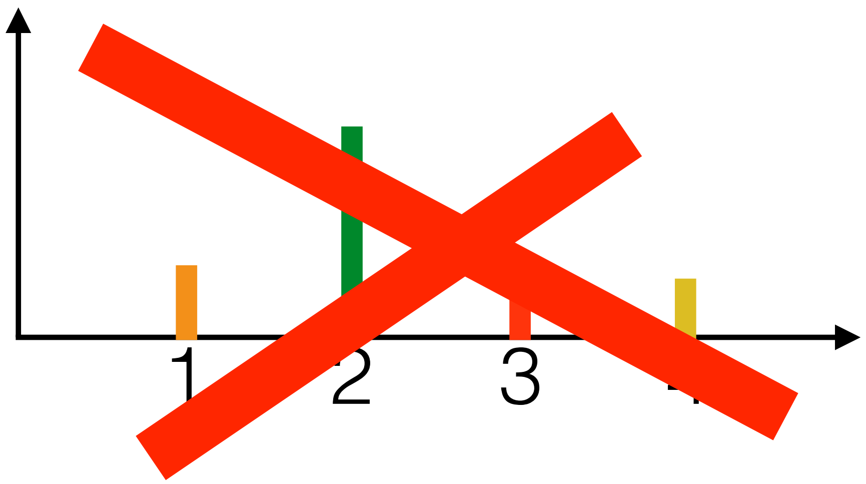

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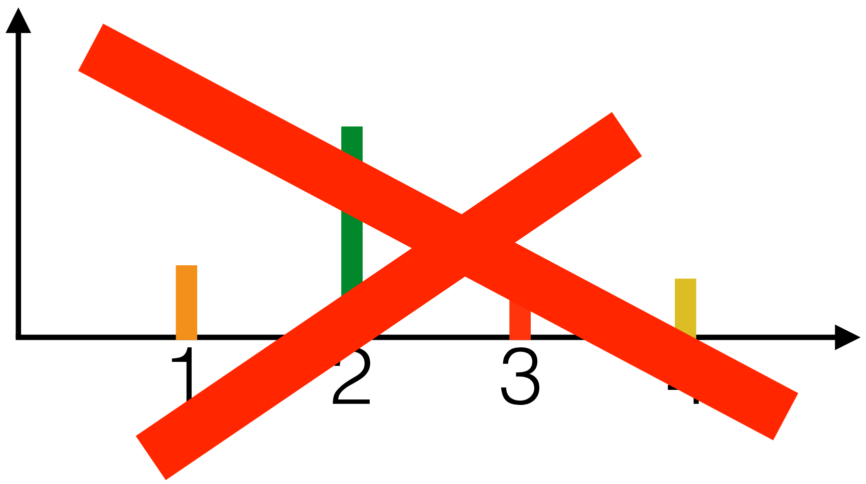

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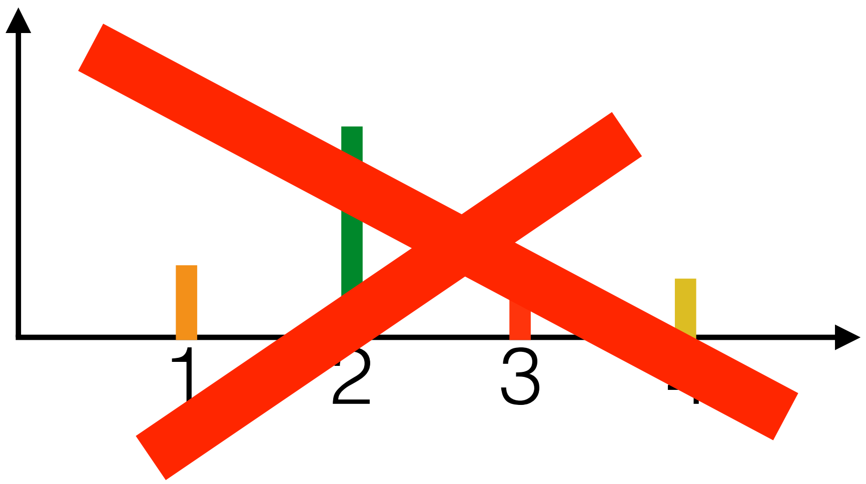

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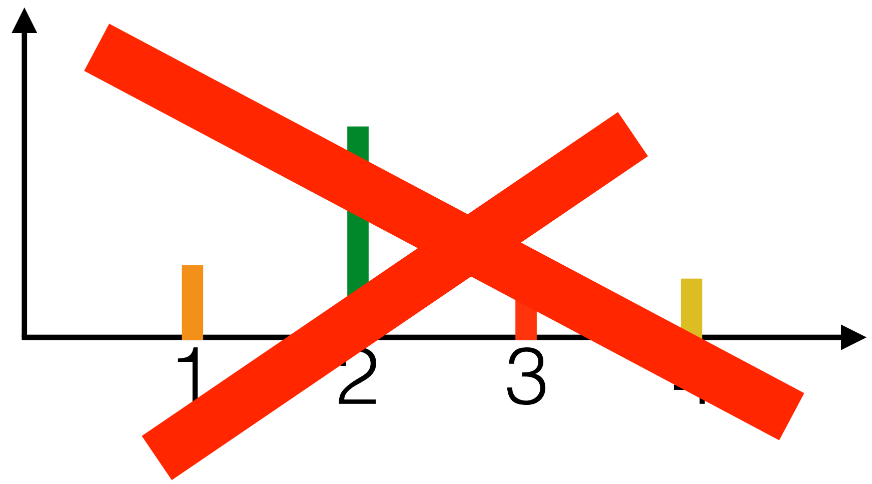


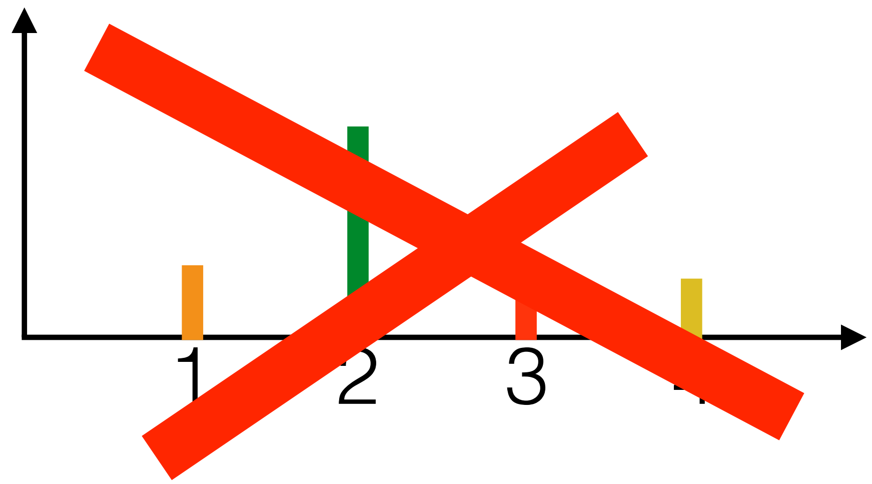
$$p(z_n = k | z_1, \ldots, z_{n-1}) = \frac{a_{k,n}}{\sum_{j=1}^{K} a_{j,n}}$$

$$a_{k,n} := a_k + \sum_{m=1}^{n-1} \mathbf{1}\{z_m = k\}$$

- multivariate Pólya urn
  - Choose any ball with prob proportional to its mass

Marginal cluster assignments

- Integrate out the frequencies



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Marginal cluster assignments

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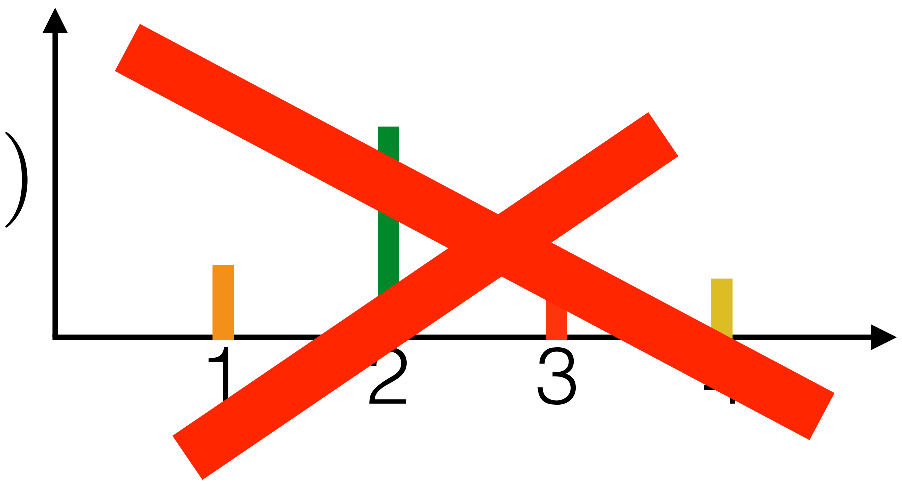

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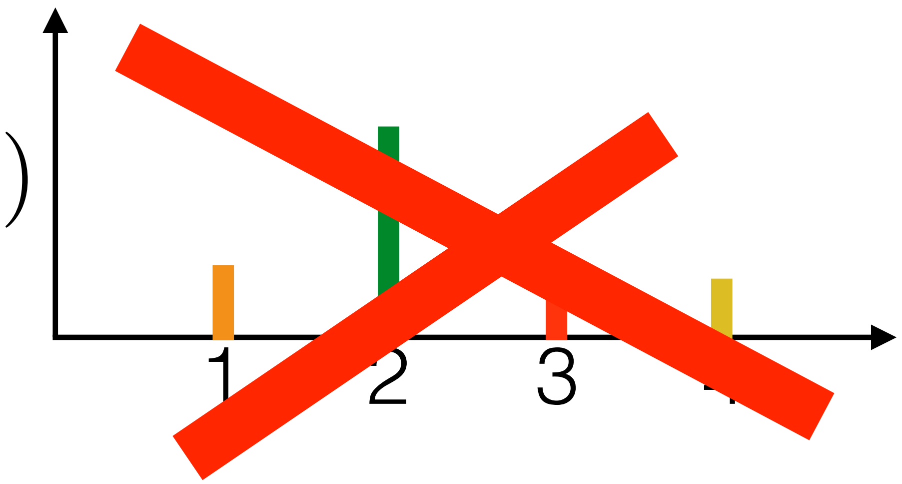
$$\lim_{n \to \infty} \frac{(\#\ \text{orange},\ \#\ \text{green},\ \#\ \text{red},\ \#\ \text{yellow})}{\#\ \text{total}}$$

Marginal cluster assignments

- Integrate out the frequencies



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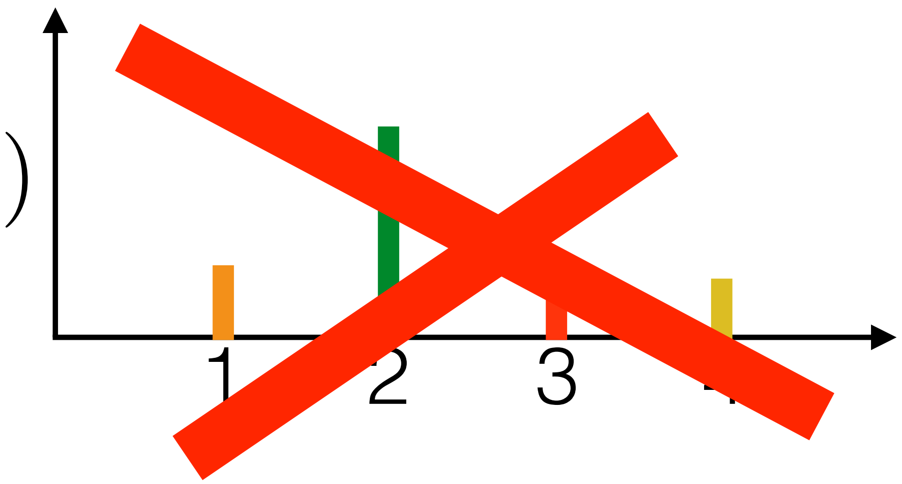
$$\to (\rho_{\text{orange}}, \rho_{\text{green}}, \rho_{\text{red}}, \rho_{\text{yellow}})$$

Marginal cluster assignments

- Integrate out the frequencies



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$$\to (\rho_{\text{orange}}, \rho_{\text{green}}, \rho_{\text{red}}, \rho_{\text{yellow}})$$

$$\overset{d}{=} \text{Dirichlet}(a_{\text{orange}}, a_{\text{green}}, a_{\text{red}}, a_{\text{yellow}})$$

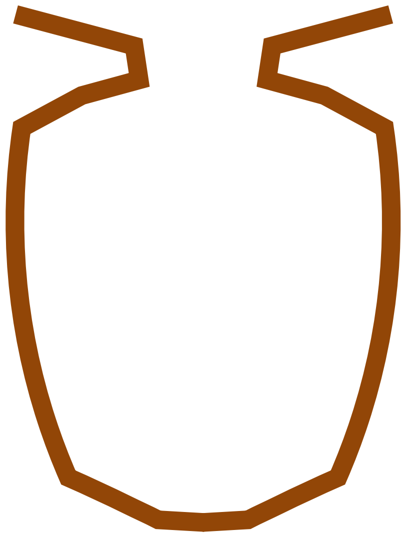
Marginal cluster assignments

- Hoppe urn / Blackwell-MacQueen urn

[Blackwell, MacQueen 1973; Hoppe 1984]

Marginal cluster assignments

- Hoppe urn / Blackwell-MacQueen urn



[Blackwell, MacQueen 1973; Hoppe 1984]

Marginal cluster assignments

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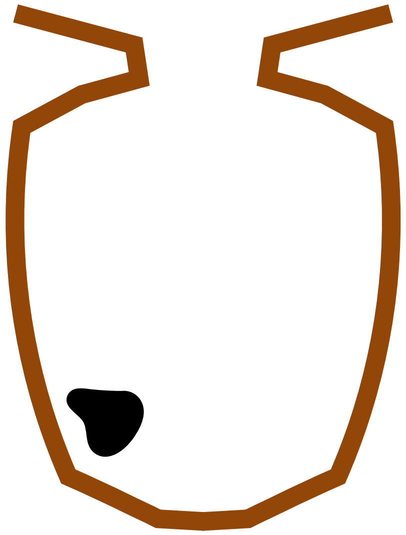

[Blackwell, MacQueen 1973; Hoppe 1984]

Marginal cluster assignments

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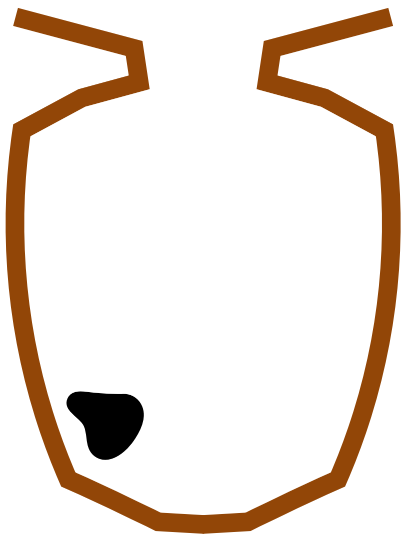
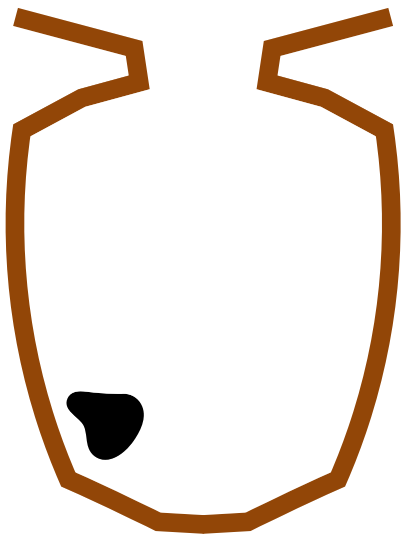


- Choose ball with prob proportional to its mass

[Blackwell, MacQueen 1973; Hoppe 1984]

Marginal cluster assignments

- Hoppe urn / Blackwell-MacQueen urn



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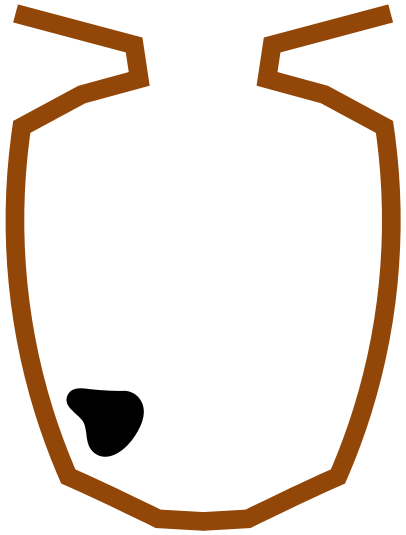
Marginal cluster assignments

- Hoppe urn / Blackwell-MacQueen urn



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Marginal cluster assignments

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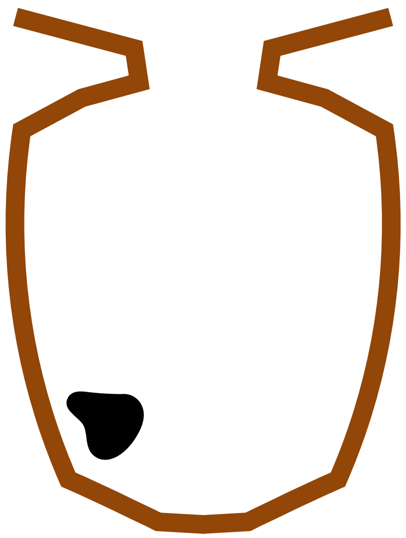

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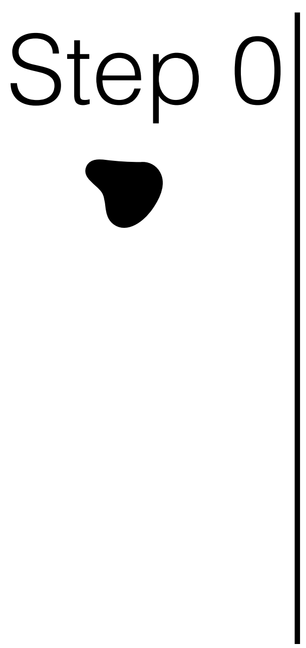
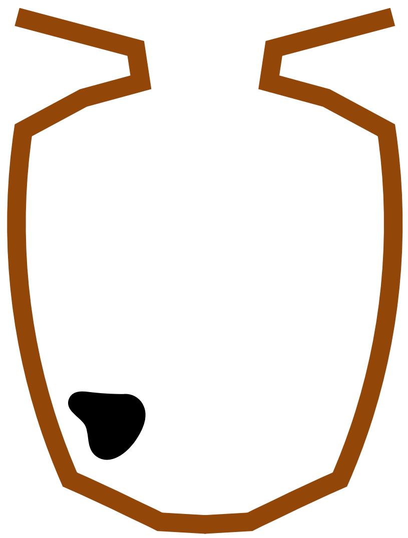


[Blackwell, MacQueen 1973; Hoppe 1984]

Marginal cluster assignments

- Hoppe urn / Blackwell-MacQueen urn



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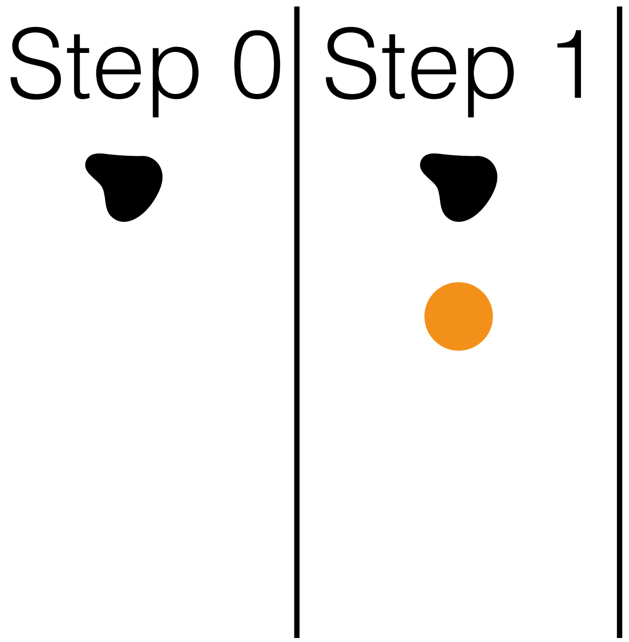
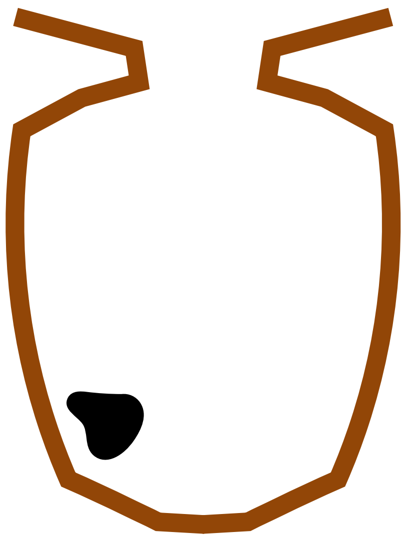


[Blackwell, MacQueen 1973; Hoppe 1984]

Marginal cluster assignments

- Hoppe urn / Blackwell-MacQueen urn



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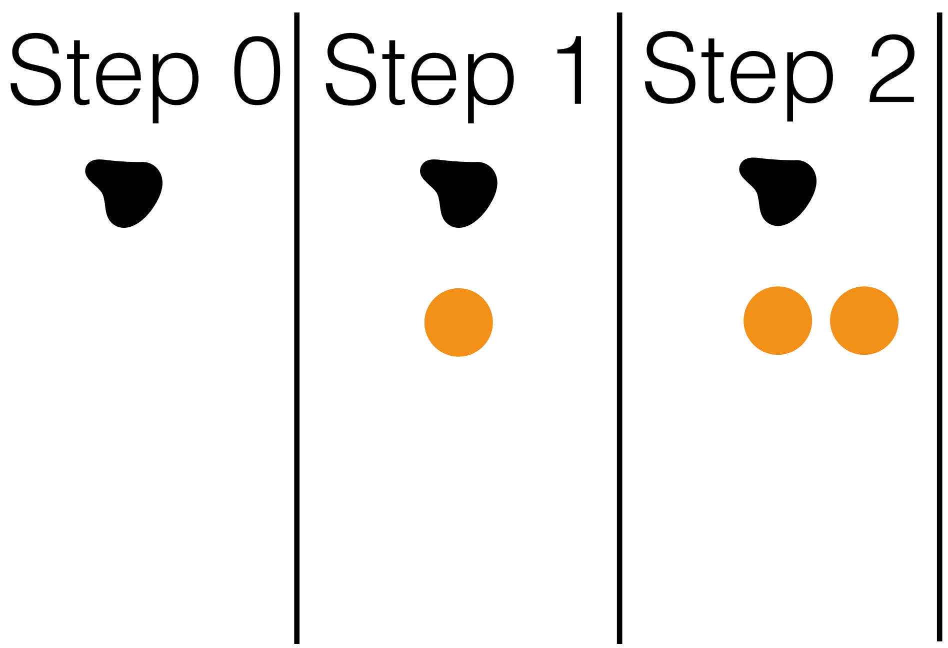
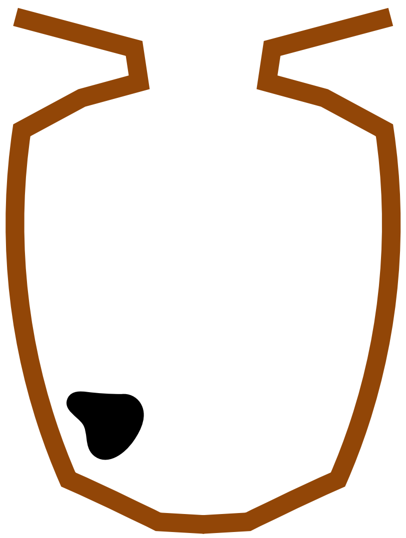


[Blackwell, MacQueen 1973; Hoppe 1984]

Marginal cluster assignments

- Hoppe urn / Blackwell-MacQueen urn



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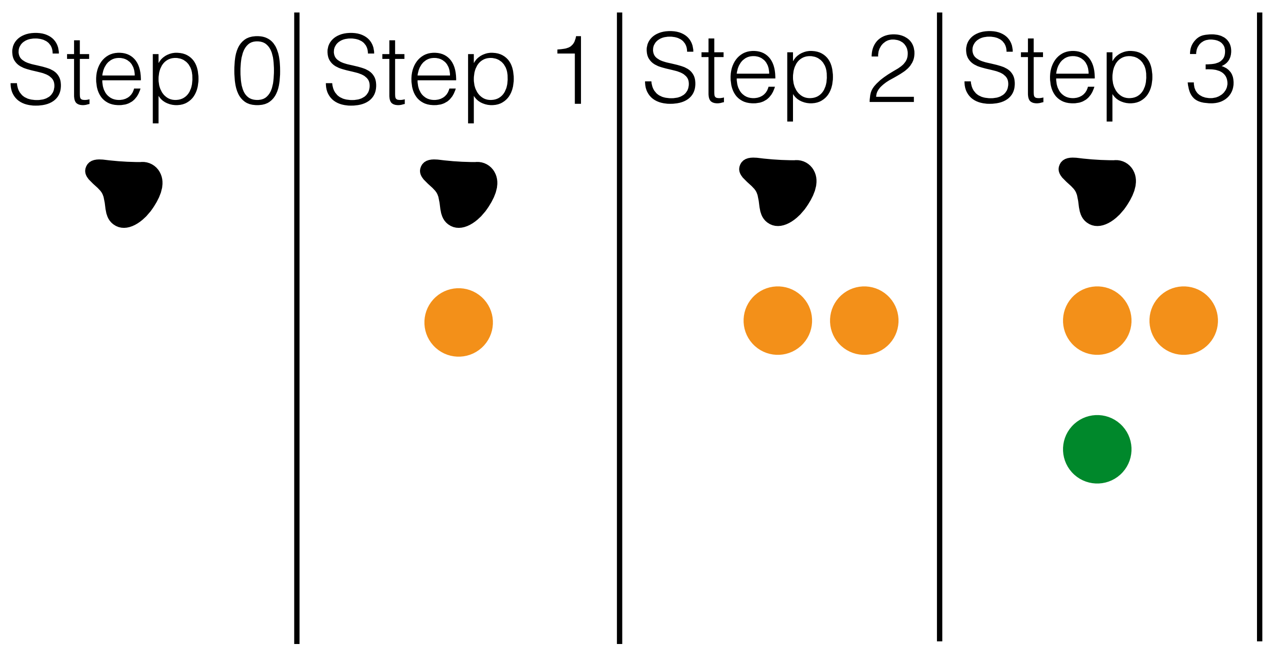

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Marginal cluster assignments

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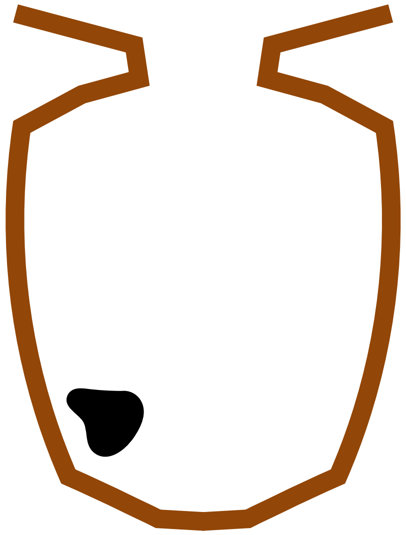

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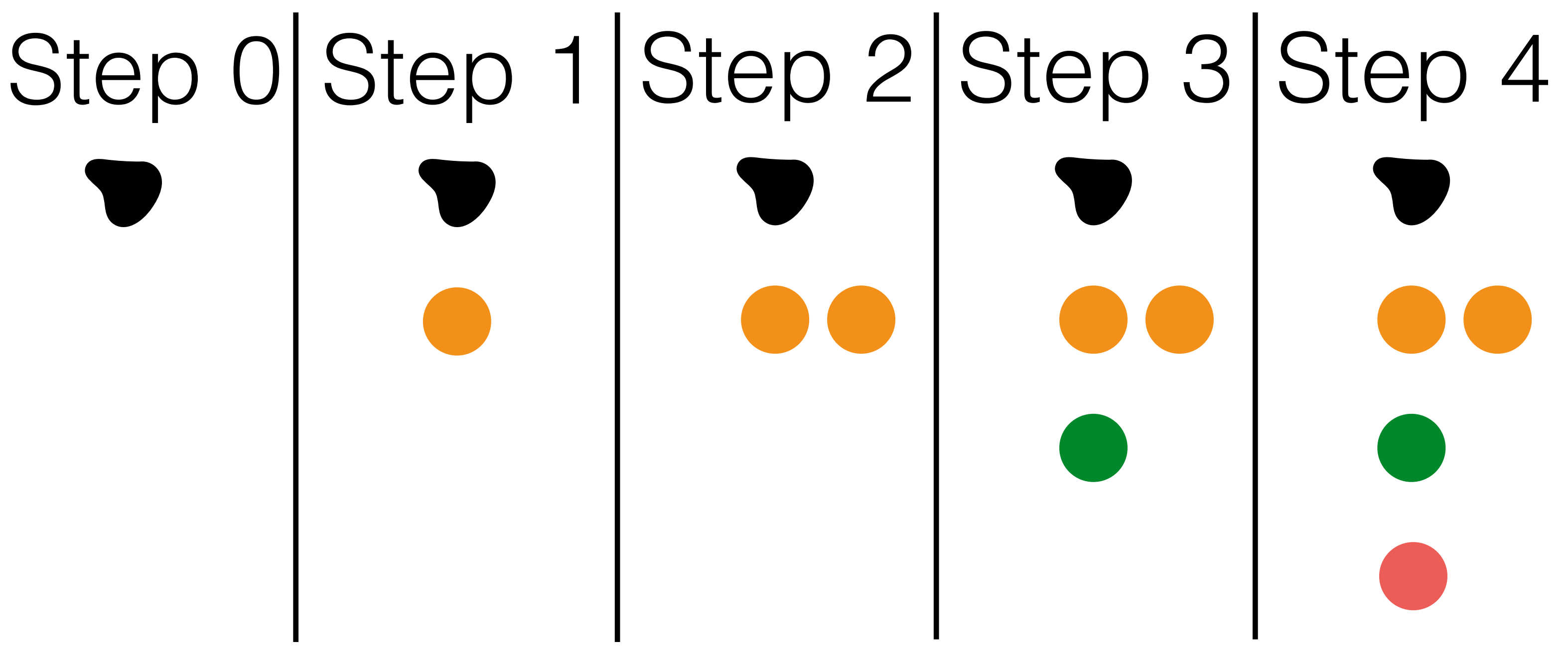

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Marginal cluster assignments

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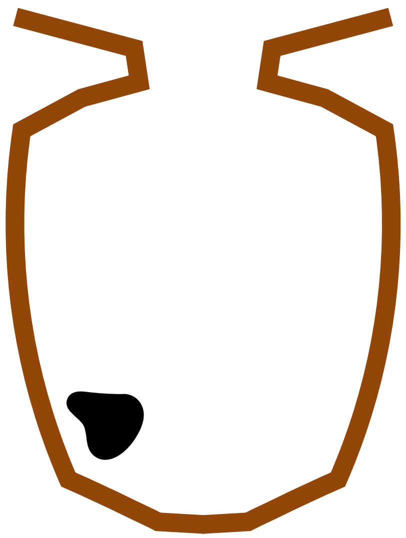

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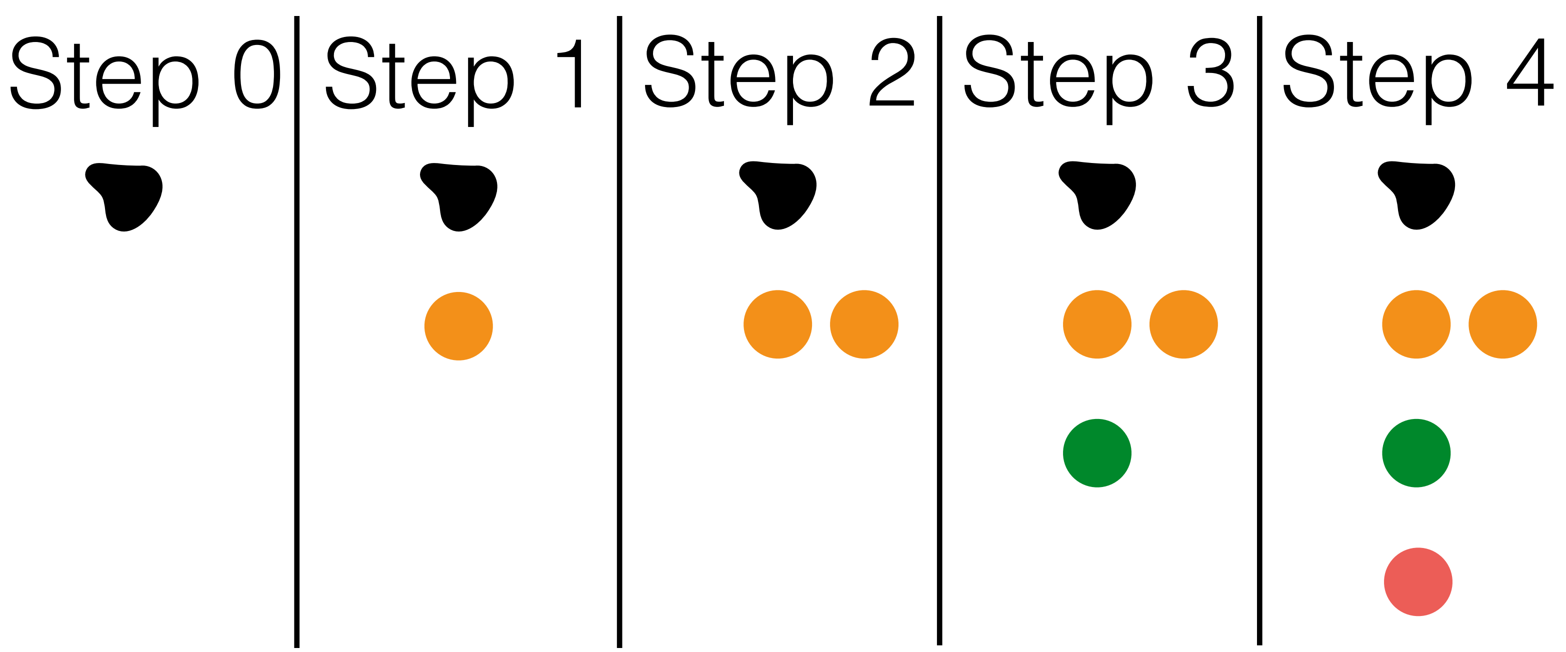
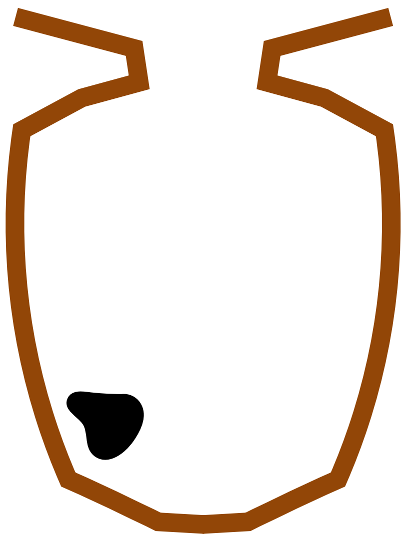


$(\#\text{orange}, \#\text{other})$

Marginal cluster assignments

- Hoppe urn / Blackwell-MacQueen urn



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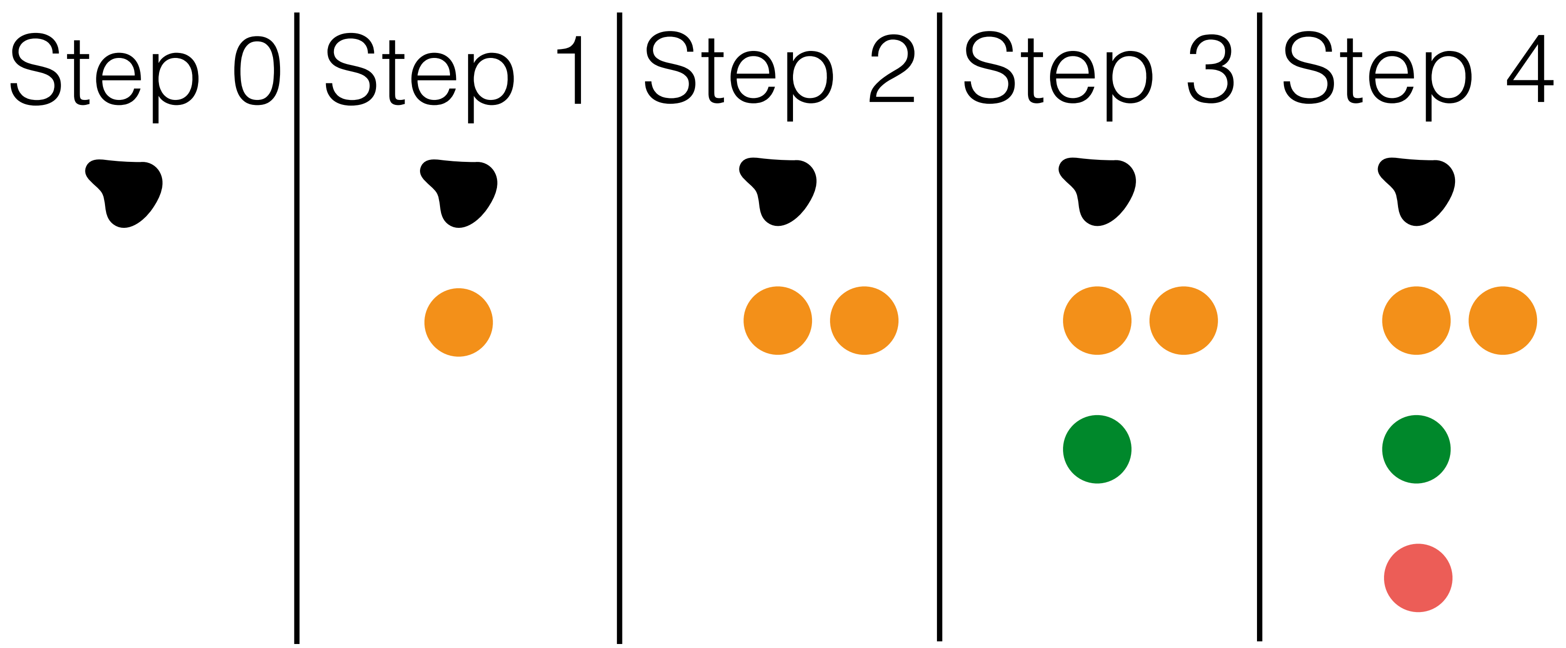
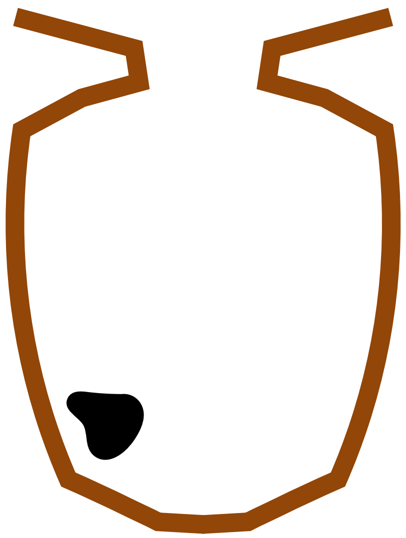


$$(\#\text{orange}, \#\text{other}) = \text{PolyaUrn}(1, \alpha)$$

Marginal cluster assignments

- Hoppe urn / Blackwell-MacQueen urn



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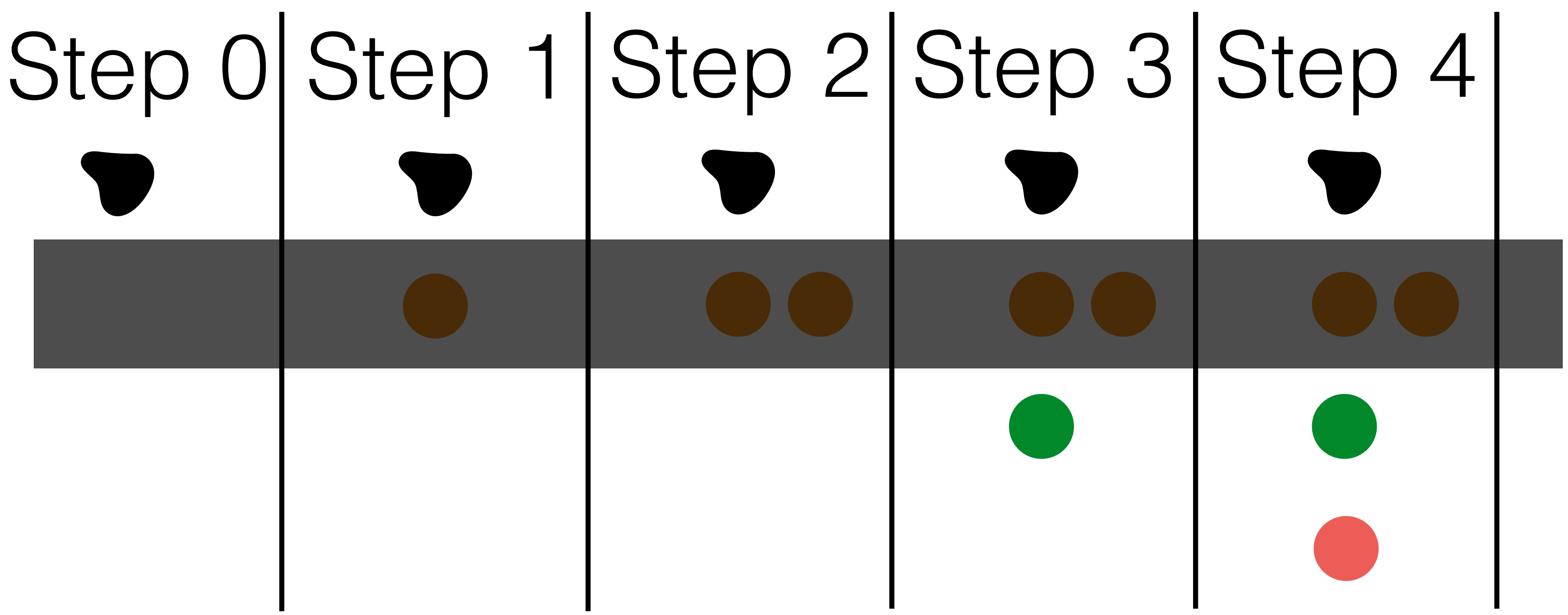
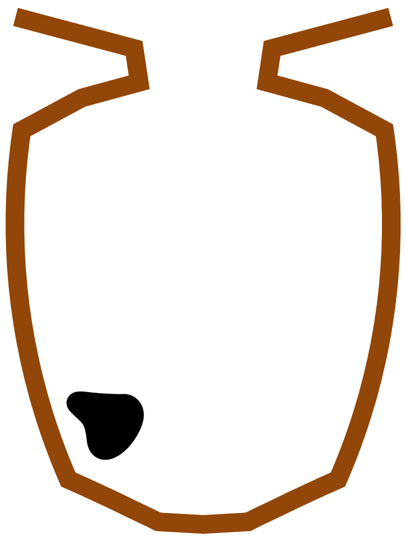


$$(\#\text{orange}, \#\text{other}) = \text{PolyaUrn}(1, \alpha)$$

Marginal cluster assignments

- Hoppe urn / Blackwell-MacQueen urn



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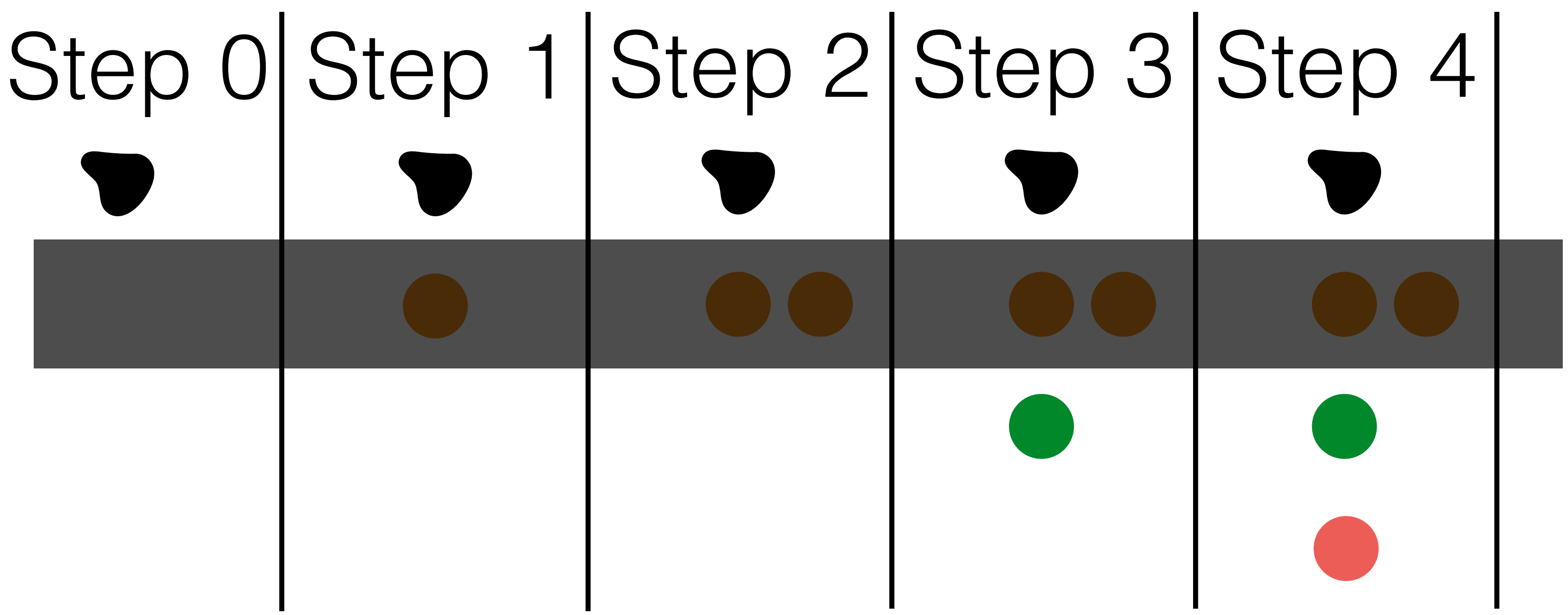
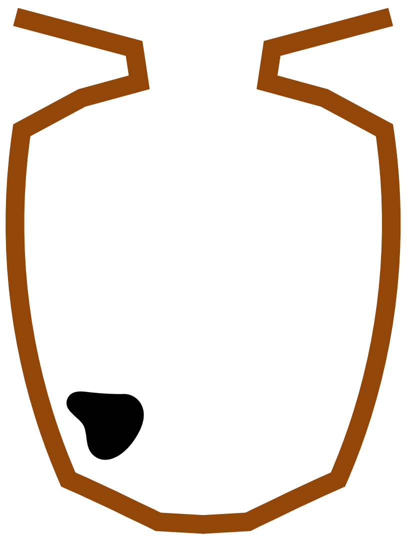


$$(\#\text{orange}, \#\text{other}) = \text{PolyaUrn}(1, \alpha)$$

- not orange: $(\#\text{green}, \#\text{other}) = \text{PolyaUrn}(1, \alpha)$

Marginal cluster assignments

- Hoppe urn / Blackwell-MacQueen urn



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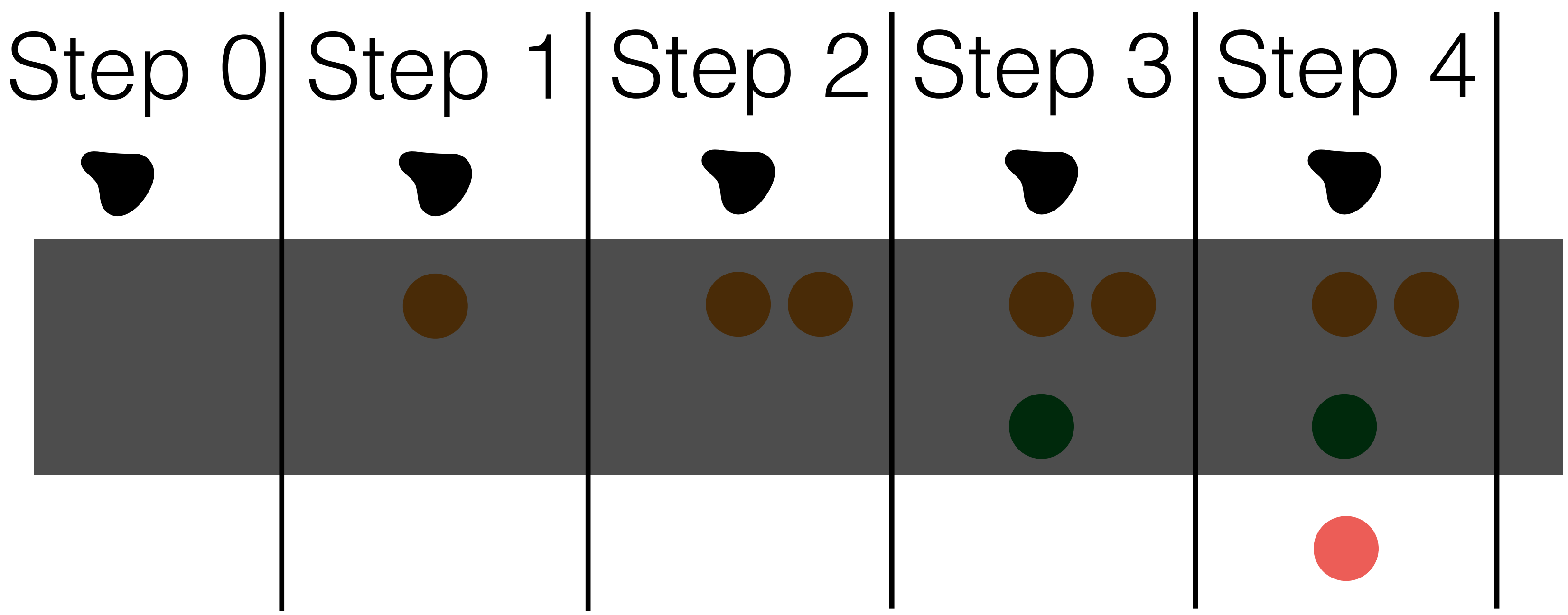

$$(\#\text{orange}, \#\text{other}) = \text{PolyaUrn}(1, \alpha)$$

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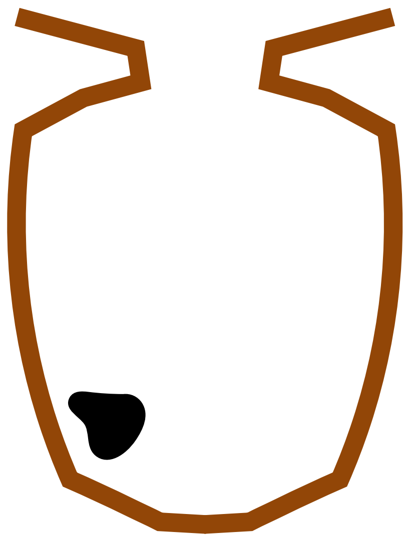
Marginal cluster assignments

- Hoppe urn / Blackwell-MacQueen urn



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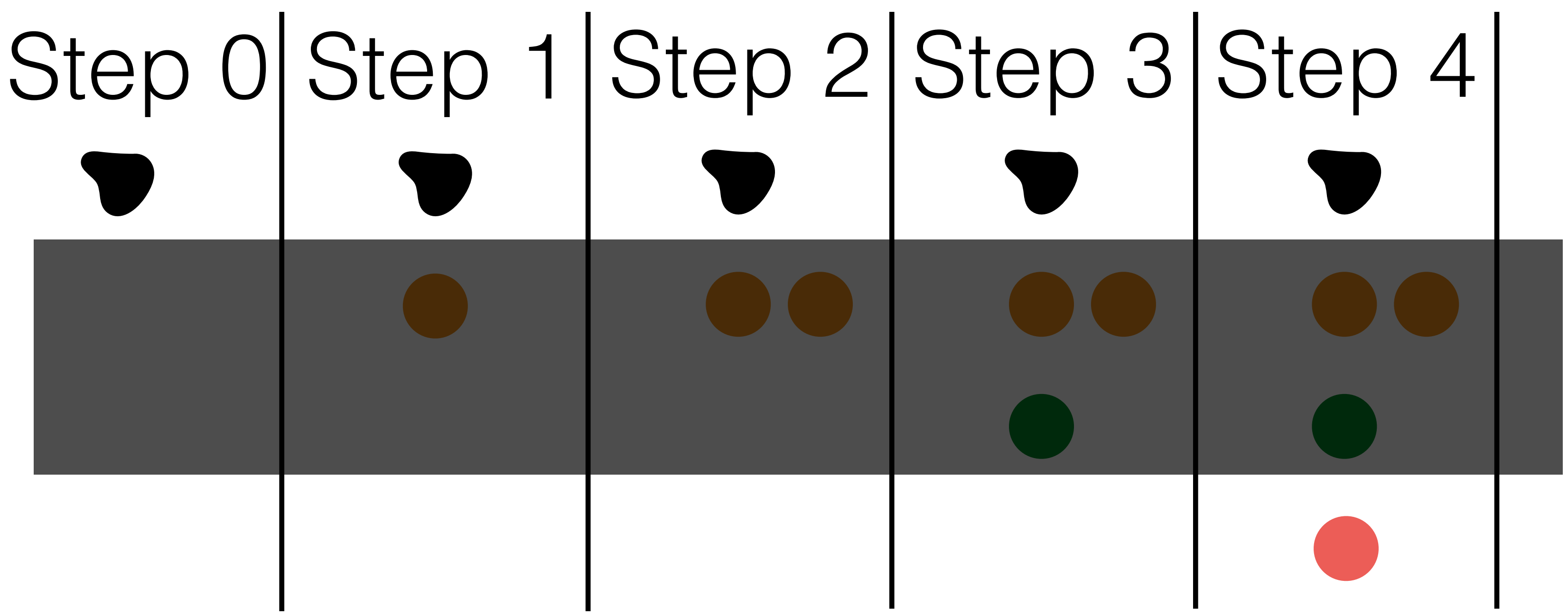

$$(\#\text{orange}, \#\text{other}) = \text{PolyaUrn}(1, \alpha)$$

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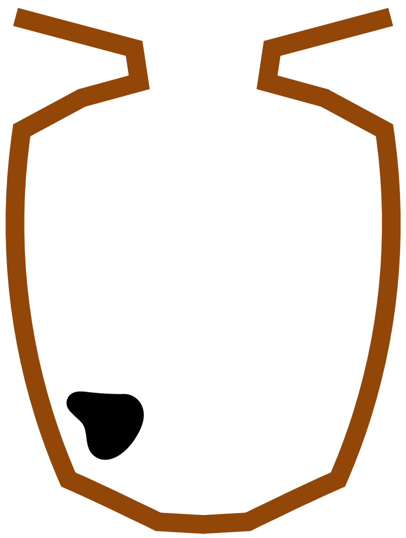
Marginal cluster assignments

- Hoppe urn / Blackwell-MacQueen urn



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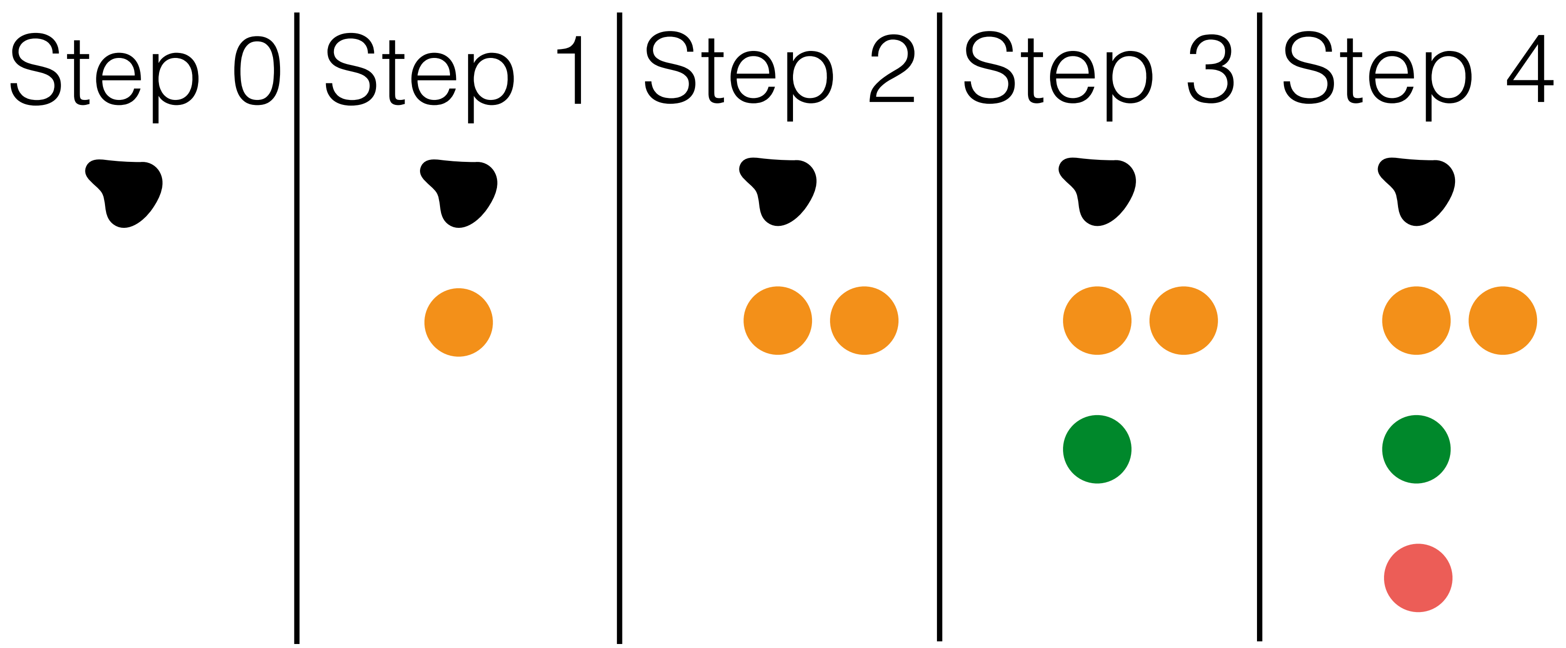

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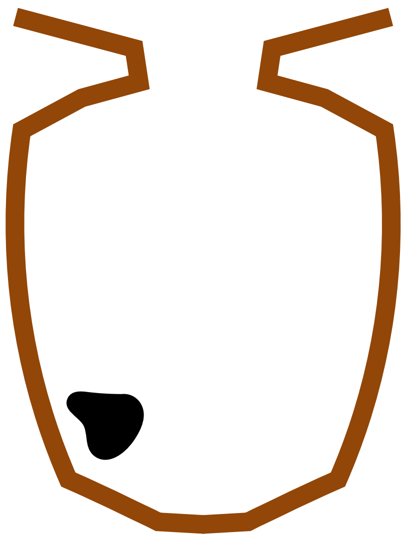
Marginal cluster assignments

- Hoppe urn / Blackwell-MacQueen urn



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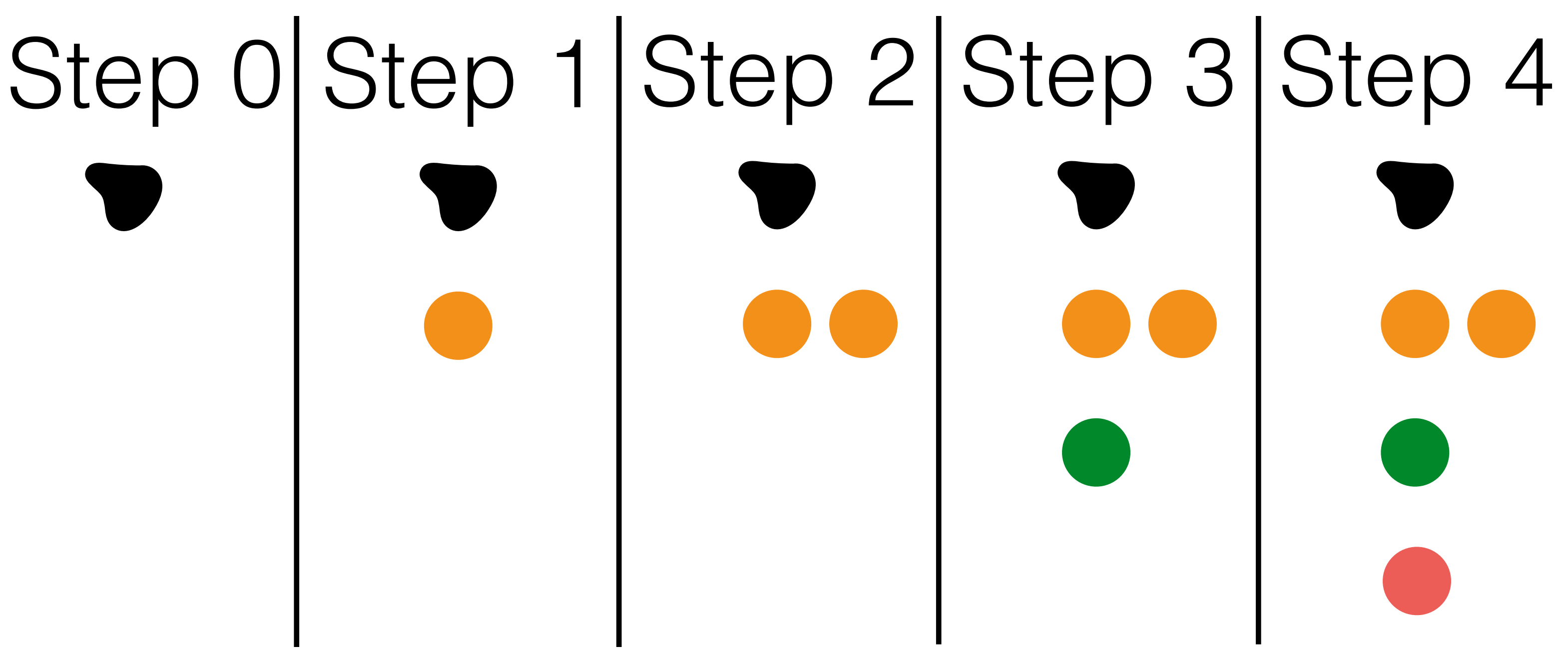

$$V_k \stackrel{iid}{\sim} \text{Beta}(1, \alpha)$$

$$(\#\text{orange}, \#\text{other}) = \text{PolyaUrn}(1, \alpha)$$

- not orange: $(\#\text{green}, \#\text{other}) = \text{PolyaUrn}(1, \alpha)$
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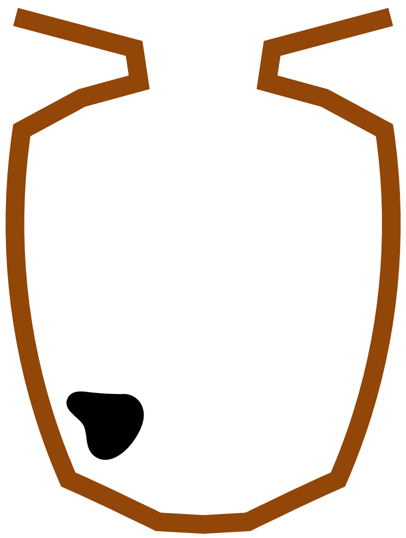
Marginal cluster assignments

- Hoppe urn / Blackwell-MacQueen urn



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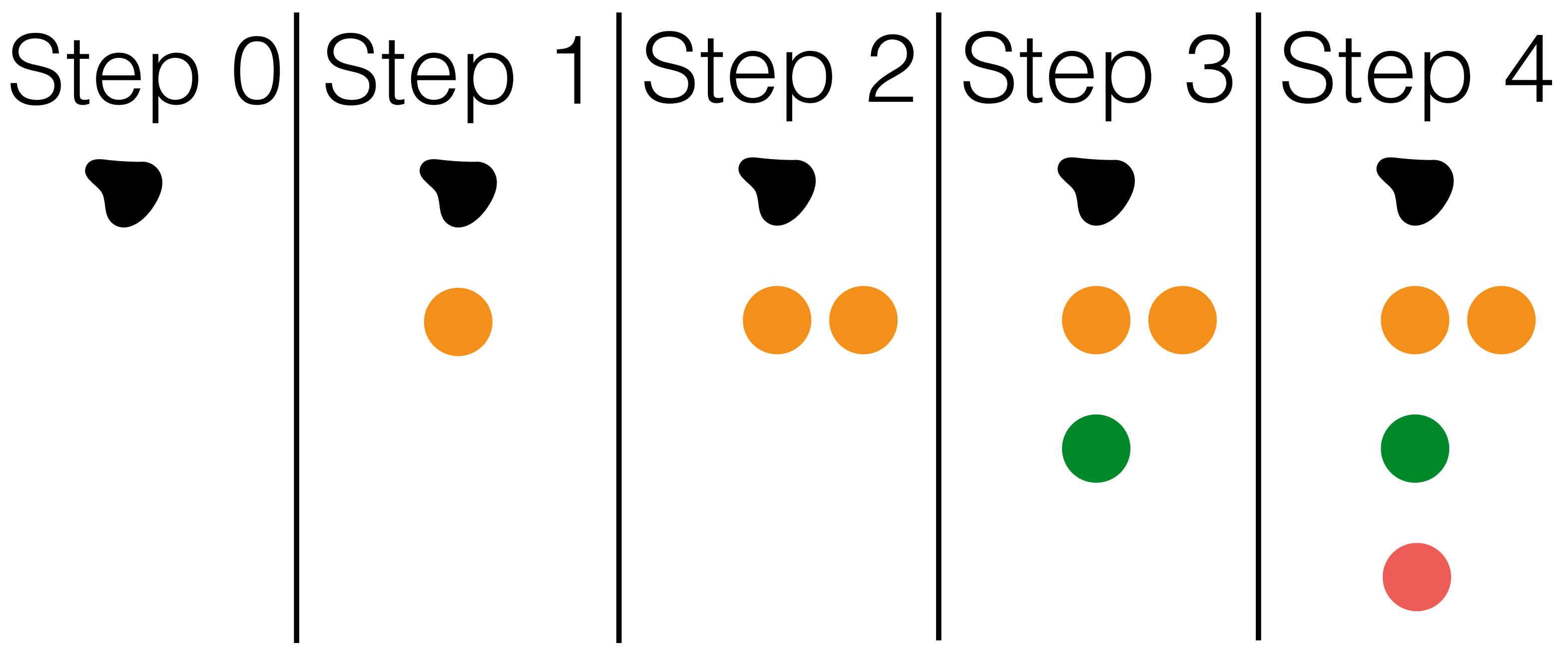

$$V_k \stackrel{iid}{\sim} \text{Beta}(1, \alpha)$$

$$\rho_1 = V_1$$

$$(\#\text{orange}, \#\text{other}) = \text{PolyaUrn}(1, \alpha)$$

- not orange: $(\#\text{green}, \#\text{other}) = \text{PolyaUrn}(1, \alpha)$
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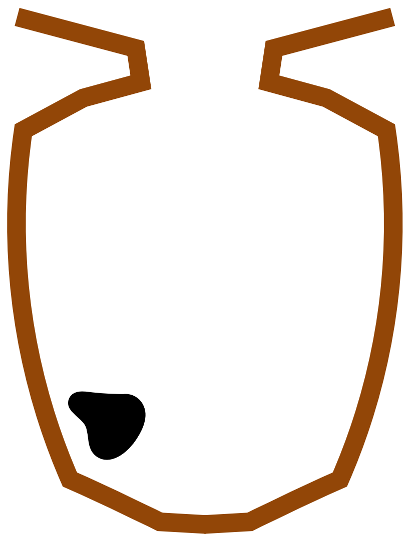
Marginal cluster assignments

- Hoppe urn / Blackwell-MacQueen urn



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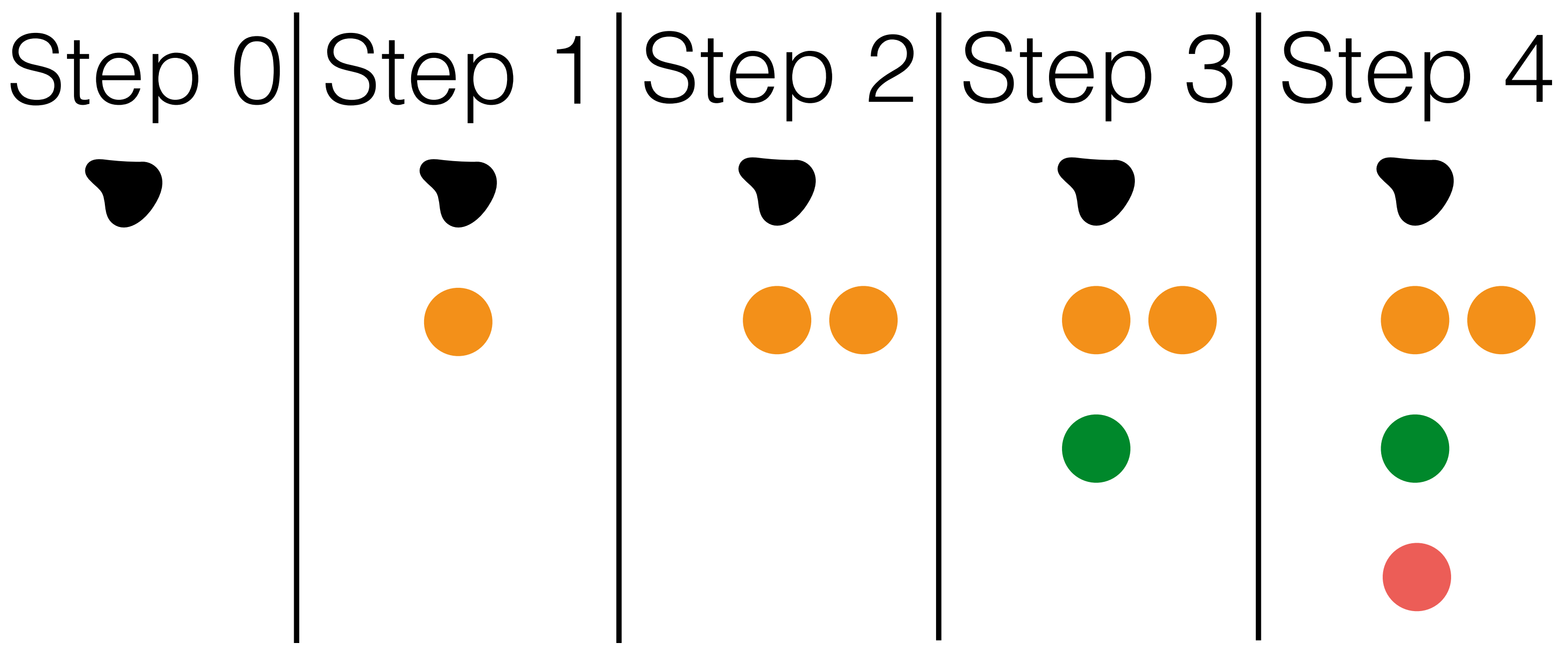

$$V_k \stackrel{iid}{\sim} \mathrm{Beta}(1, \alpha)$$

$$\rho_1 = V_1$$

$$\rho_2 = (1 - V_1)V_2$$

$$(\#\mathrm{orange}, \#\mathrm{other}) = \mathrm{PolyaUrn}(1, \alpha)$$

- not orange: $(\#\mathrm{green}, \#\mathrm{other}) = \mathrm{PolyaUrn}(1, \alpha)$
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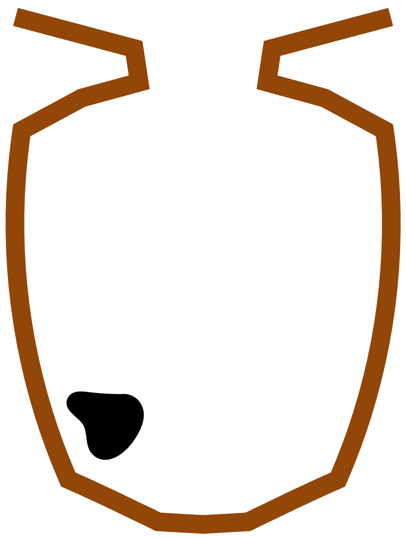
Marginal cluster assignments

- Hoppe urn / Blackwell-MacQueen urn



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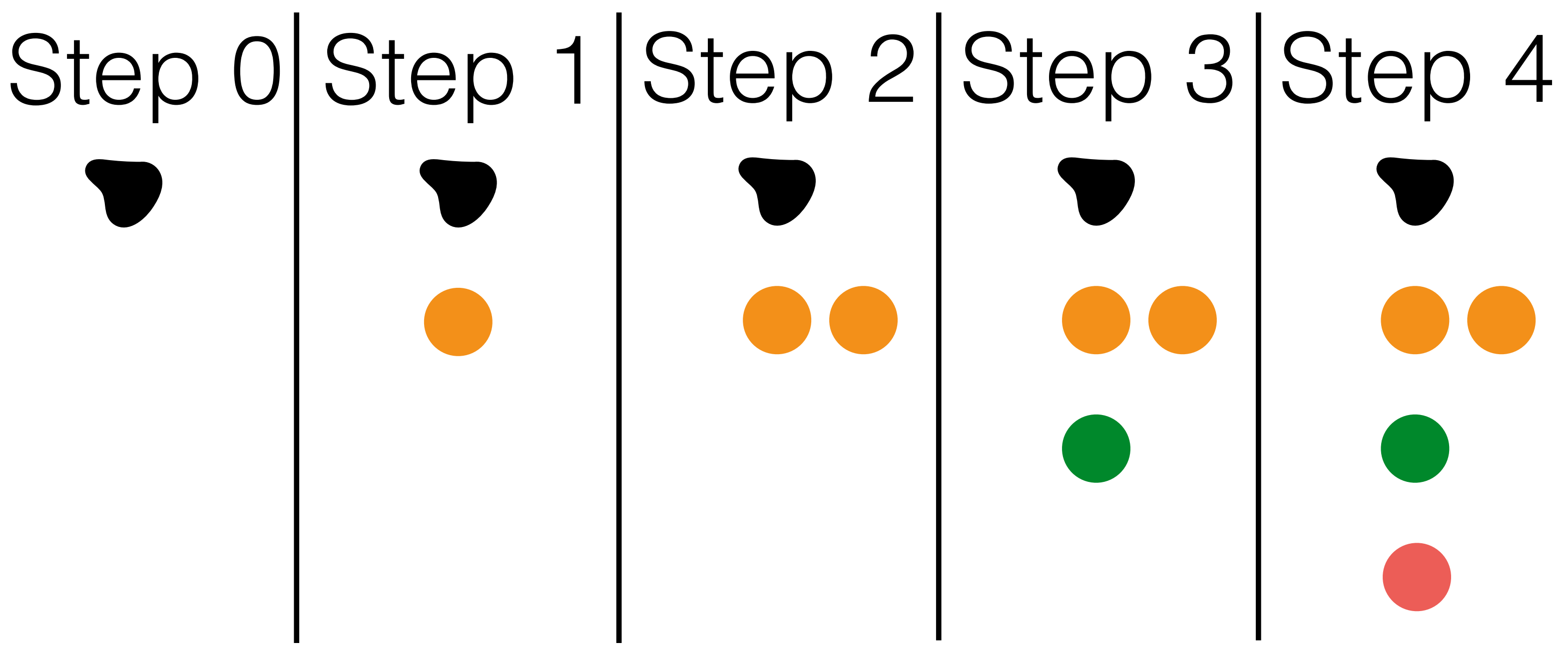


$$V_k \overset{iid}{\sim} \mathrm{Beta}(1, \alpha)$$

$$\rho_1 = V_1$$

$$\rho_2 = (1 - V_1)V_2$$

$$\rho_3 = [\prod_{k=1}^{2} (1 - V_k)]V_3$$

$$(\#\text{orange}, \#\text{other}) = \mathrm{PolyaUrn}(1, \alpha)$$

- not orange: $(\#\text{green}, \#\text{other}) = \mathrm{PolyaUrn}(1, \alpha)$
- not orange, green: $(\#\text{red}, \#\text{other}) = \mathrm{PolyaUrn}(1, \alpha)$

Chinese restaurant process

Chinese restaurant process

- Same thing we just did

Chinese restaurant process

- Same thing we just did
- Each customer walks into the restaurant

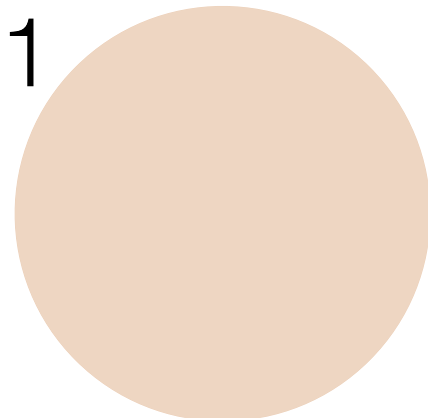
Chinese restaurant process

- Same thing we just did
- Each customer walks into the restaurant
  - Sits at existing table with prob proportional to # people there

Chinese restaurant process

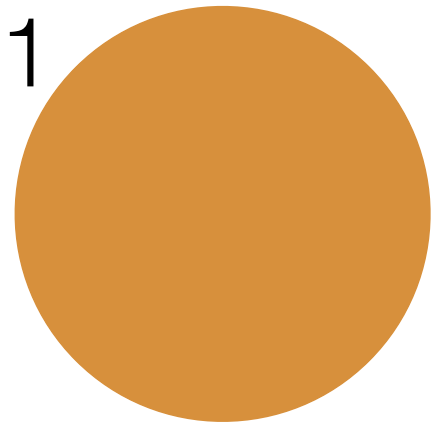
- Same thing we just did
- Each customer walks into the restaurant
    - Sits at existing table with prob proportional to # people there
    - Forms new table with prob proportional to α

Chinese restaurant process



- Same thing we just did
- Each customer walks into the restaurant
  - Sits at existing table with prob proportional to # people there
  - Forms new table with prob proportional to α

Chinese restaurant process



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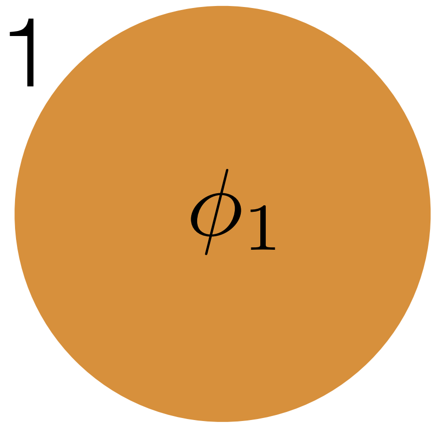

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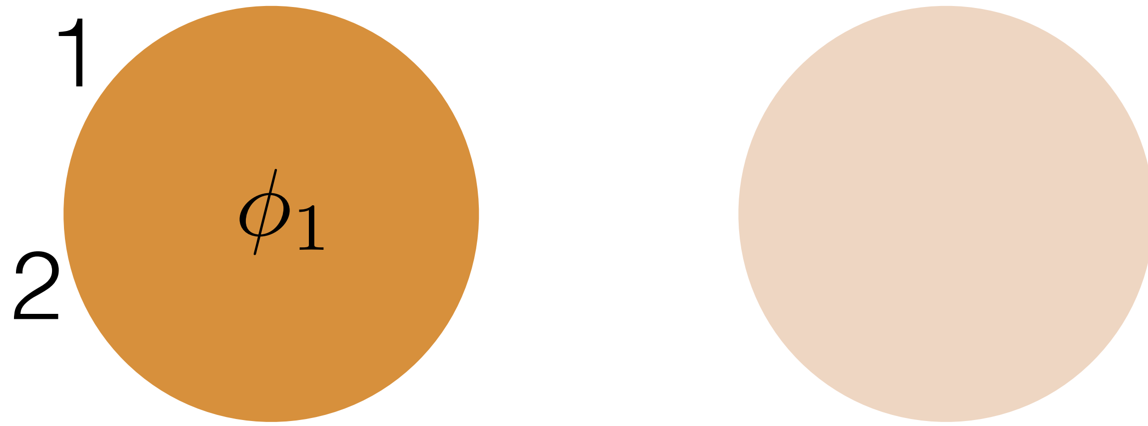

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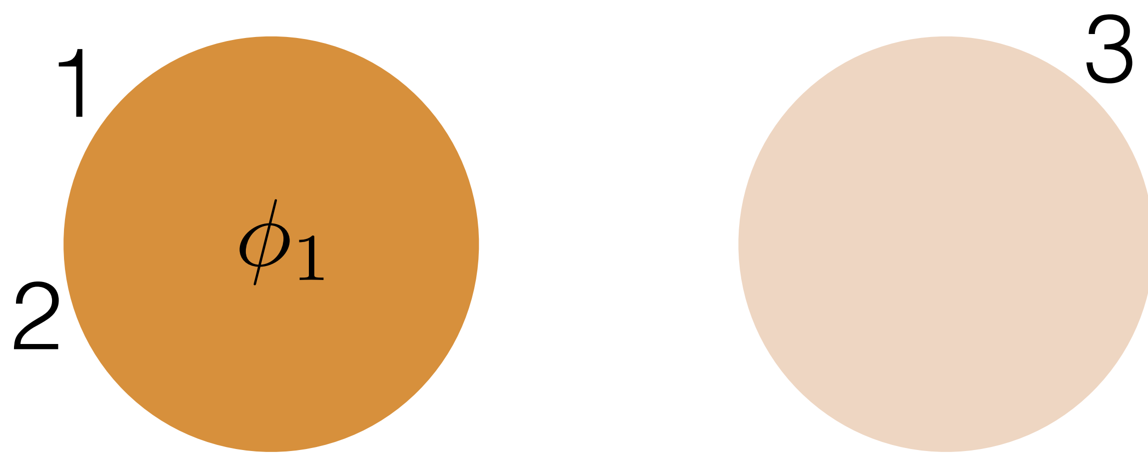

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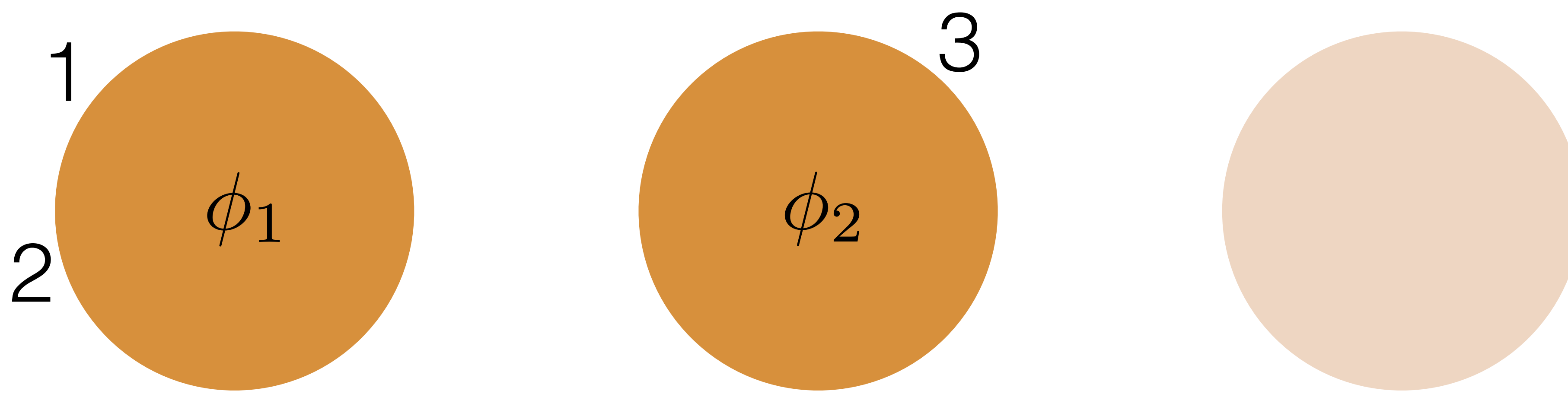

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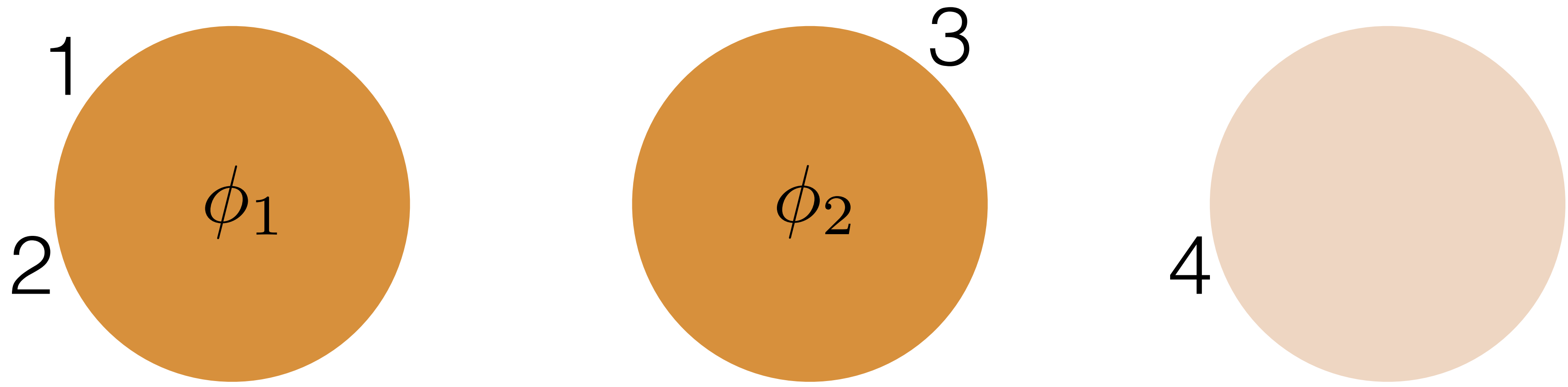

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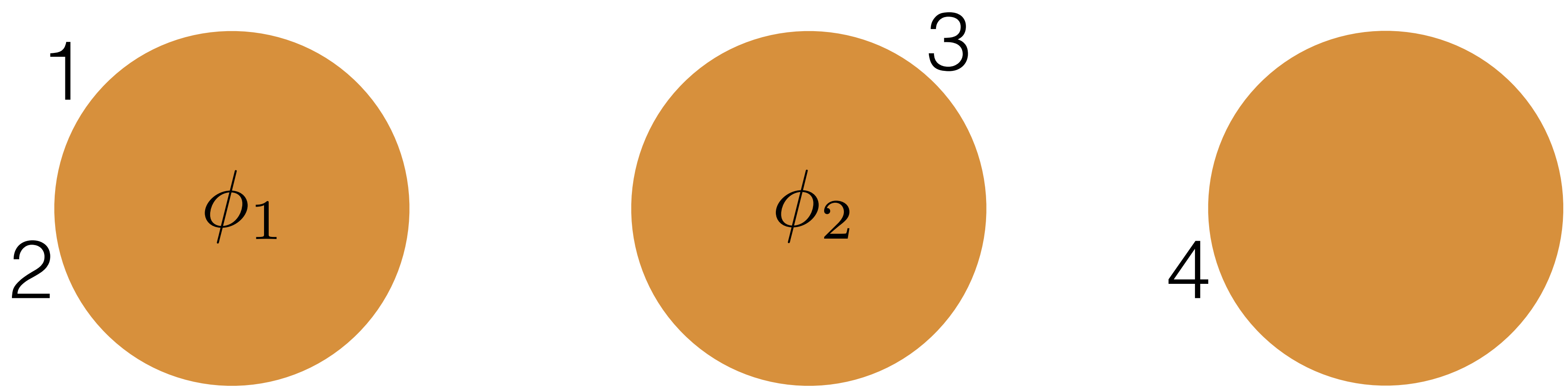

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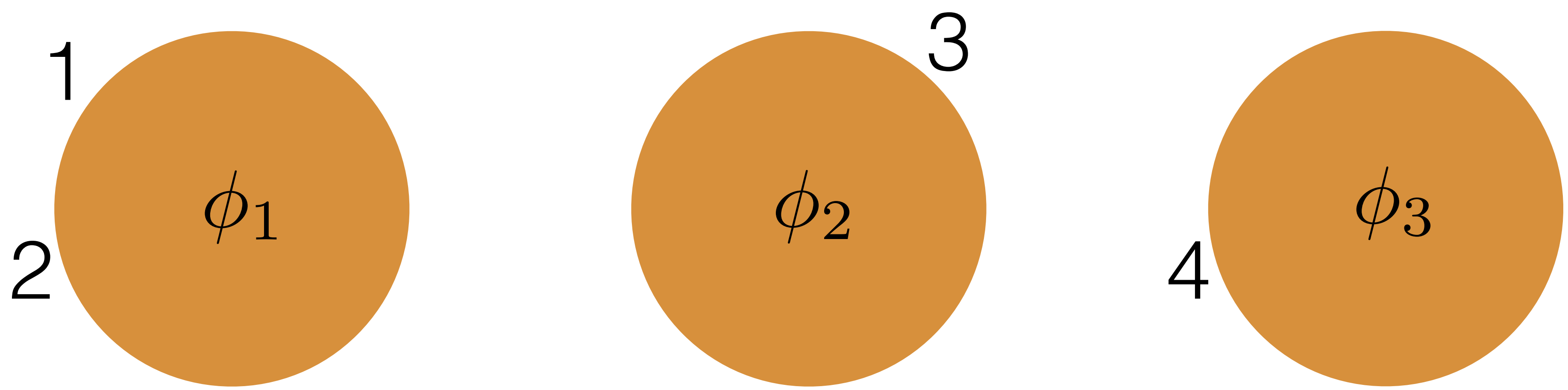

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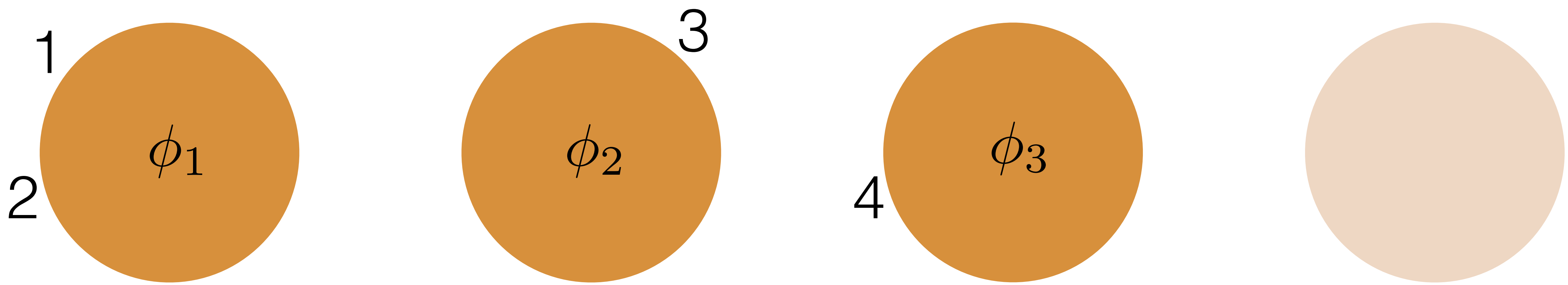

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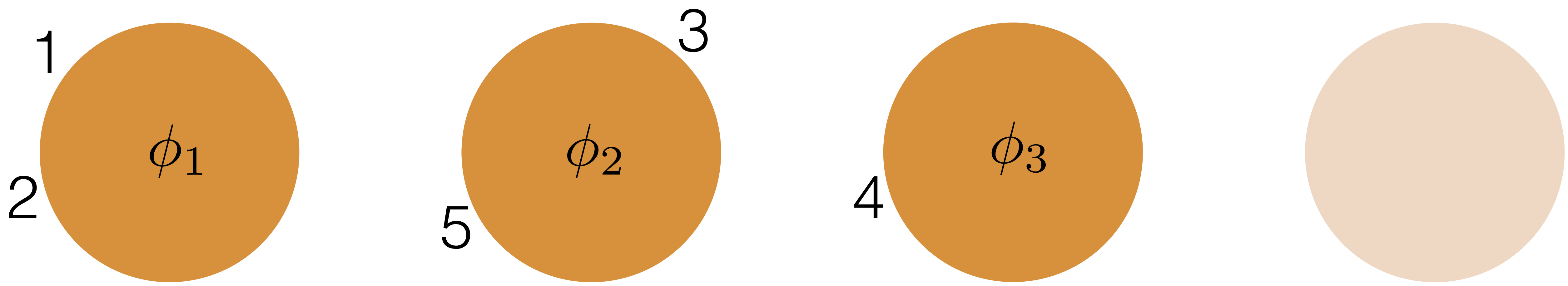

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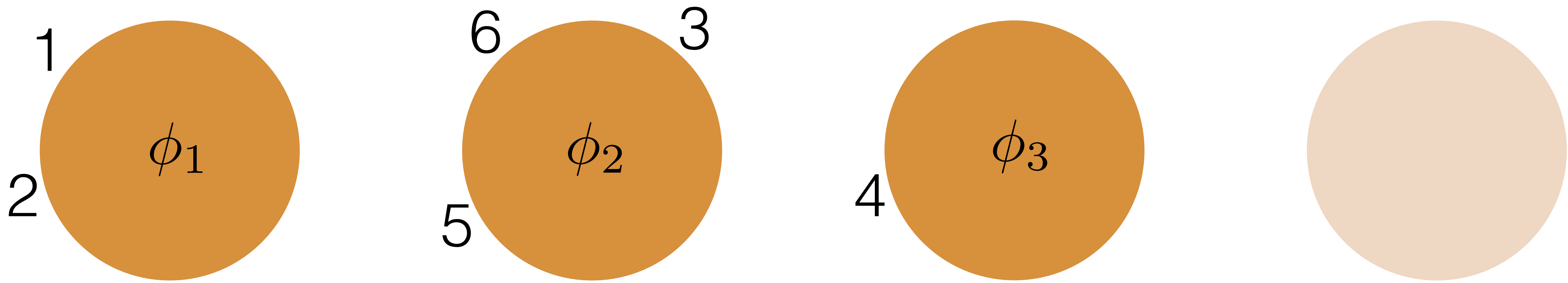

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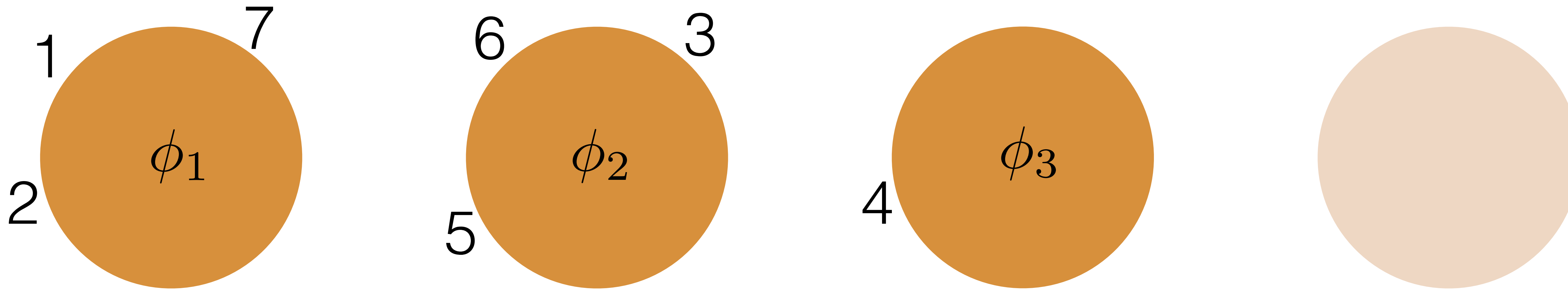

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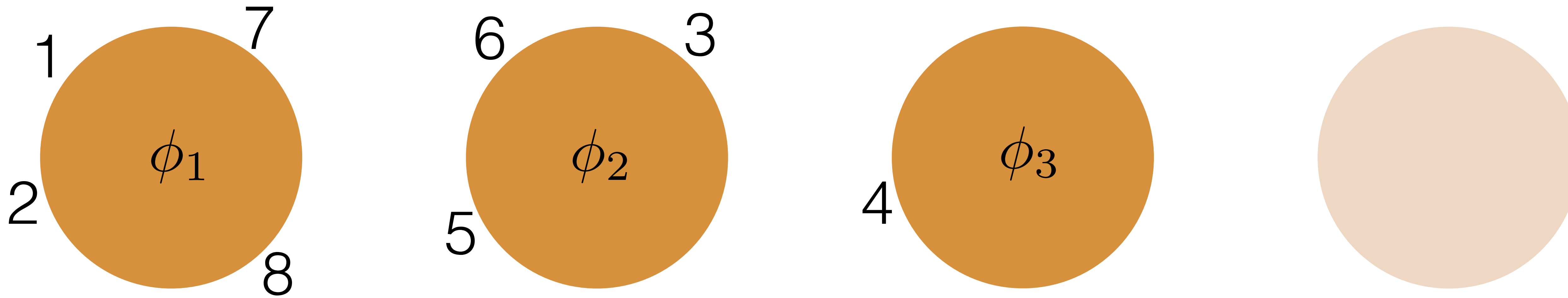

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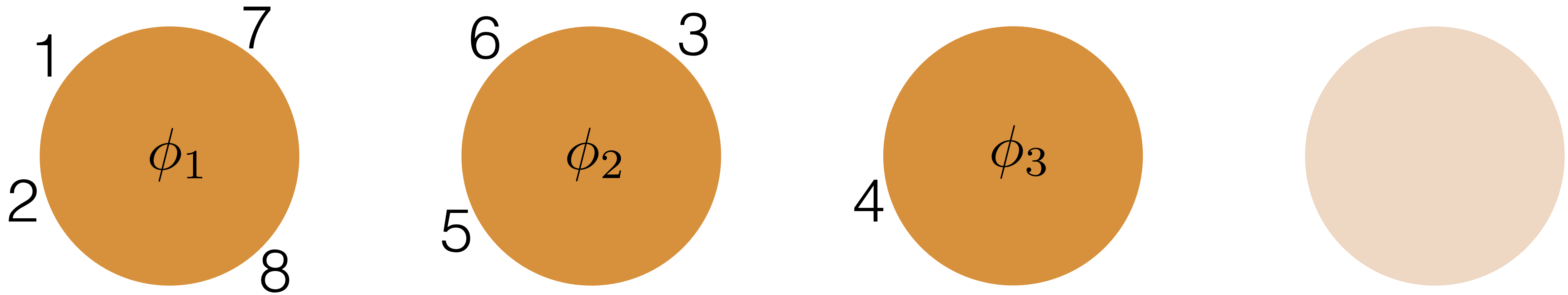

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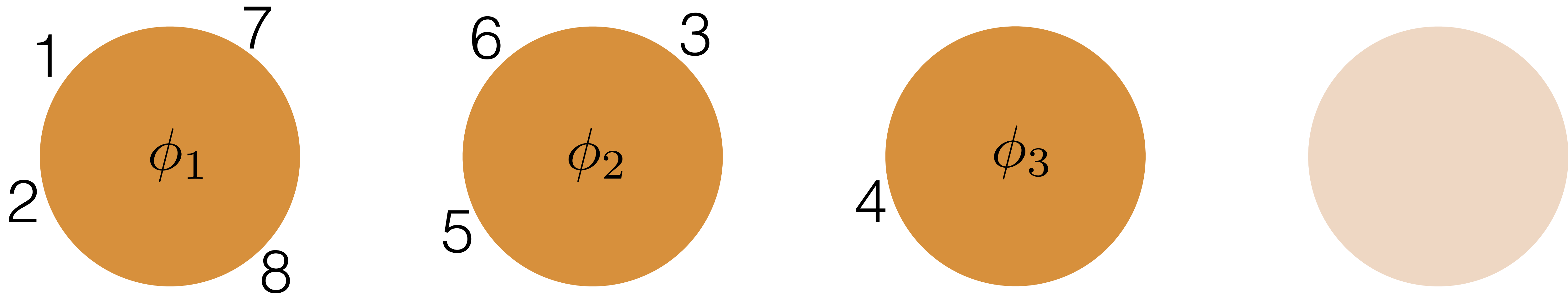
[Aldous 1983]

Chinese restaurant process



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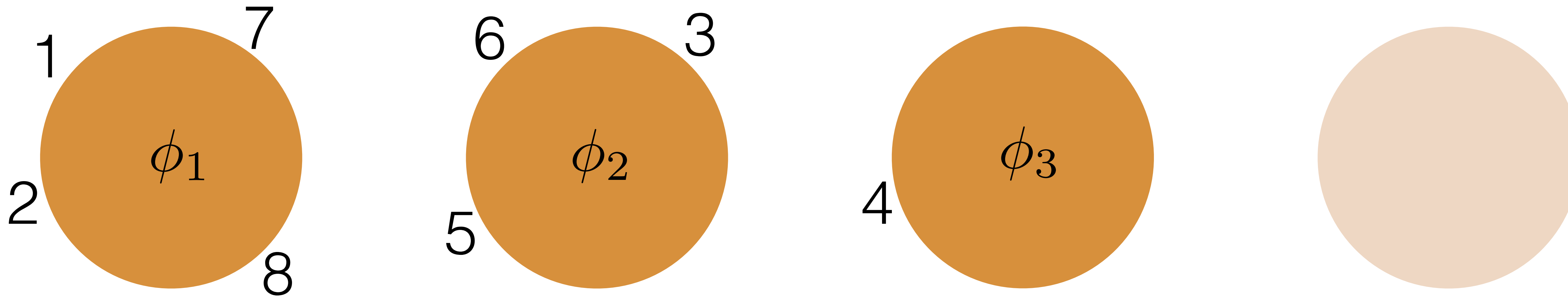

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So far: Dirichlet process, Chinese restaurant process

- Infinity of parameters, growing number of parameters

Roadmap

- Example problem: clustering
- Example NPBayes model: Dirichlet process
- Chinese restaurant process
- Inference
- Venture further into the wild world of Nonparametric Bayes

- Big questions
  - Why NPBayes?
  - What does an infinite/growing number of parameters really mean (in NPBayes)?
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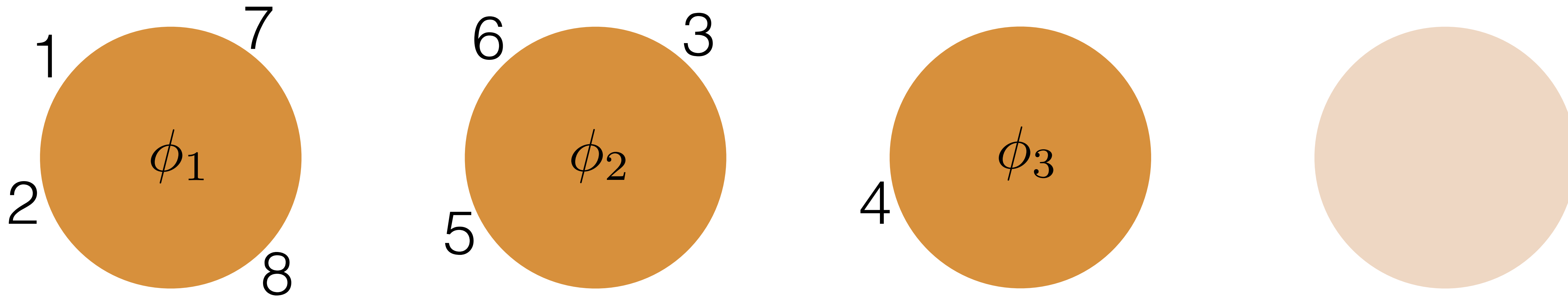
Chinese restaurant process



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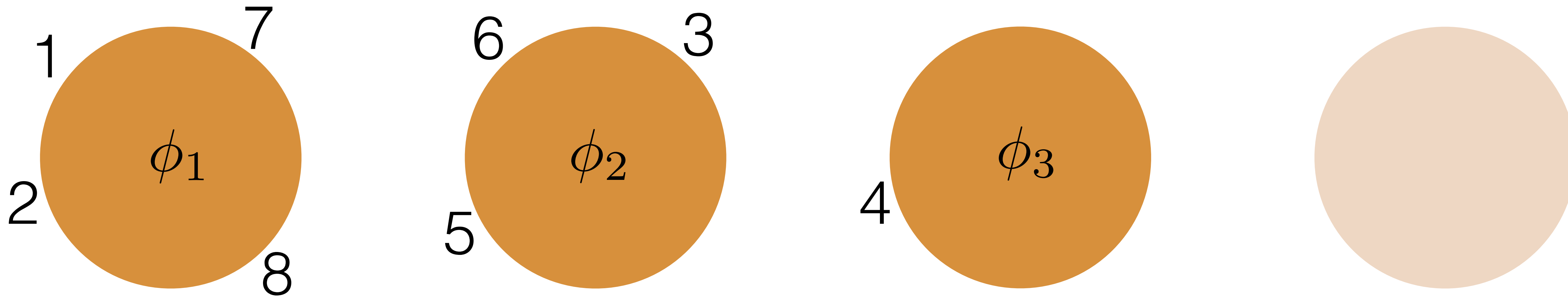

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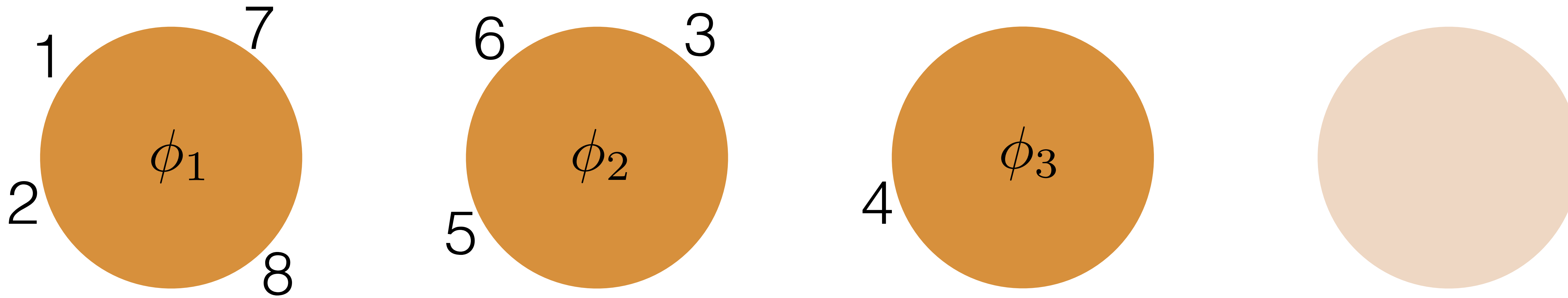
$$z_1 = z_2 = z_7 = z_8 = 1, z_3 = z_5 = z_6 = 2, z_4 = 3$$

Chinese restaurant process



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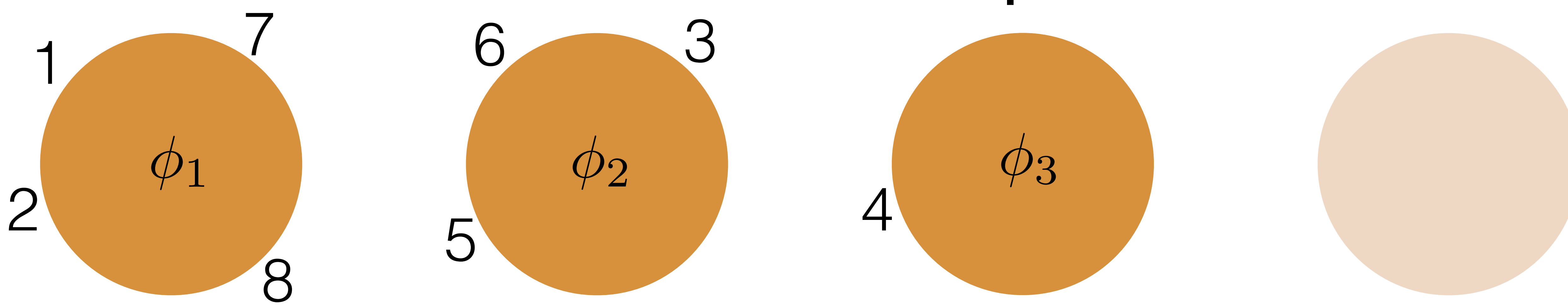
$$z_1 = z_2 = z_7 = z_8 = 1, z_3 = z_5 = z_6 = 2, z_4 = 3$$
$$\Rightarrow \Pi_8 = \{\{1, 2, 7, 8\}, \{3, 5, 6\}, \{4\}\}$$

Chinese restaurant process



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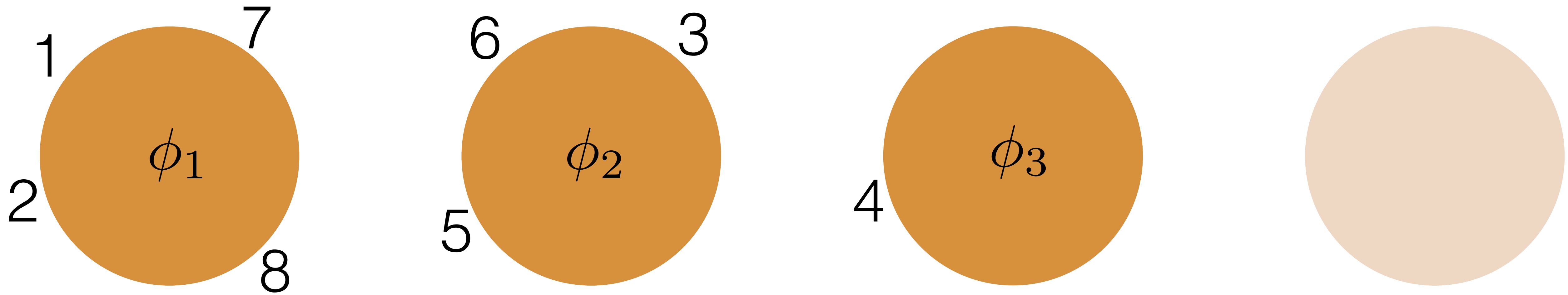
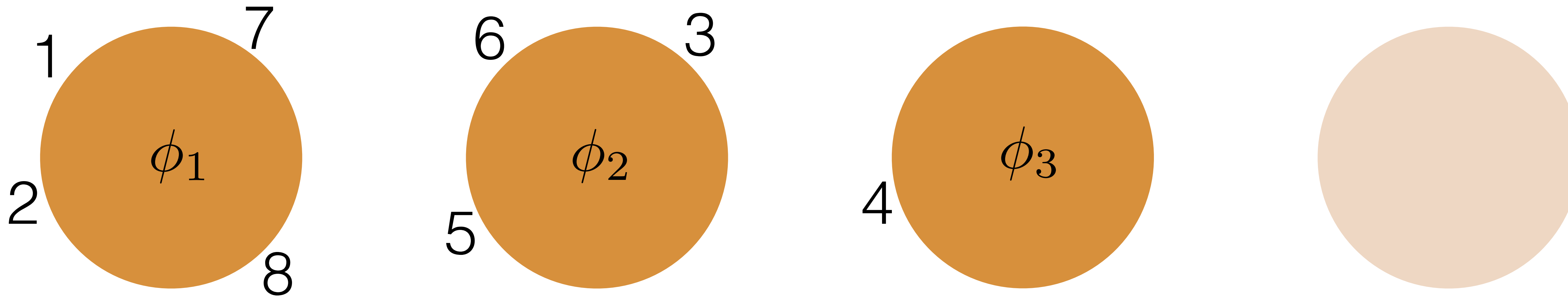
- *Partition of [8]*: set of mutually exclusive & exhaustive sets of $[8] := \{1, \ldots, 8\}$

Chinese restaurant process



- Probability of this seating:

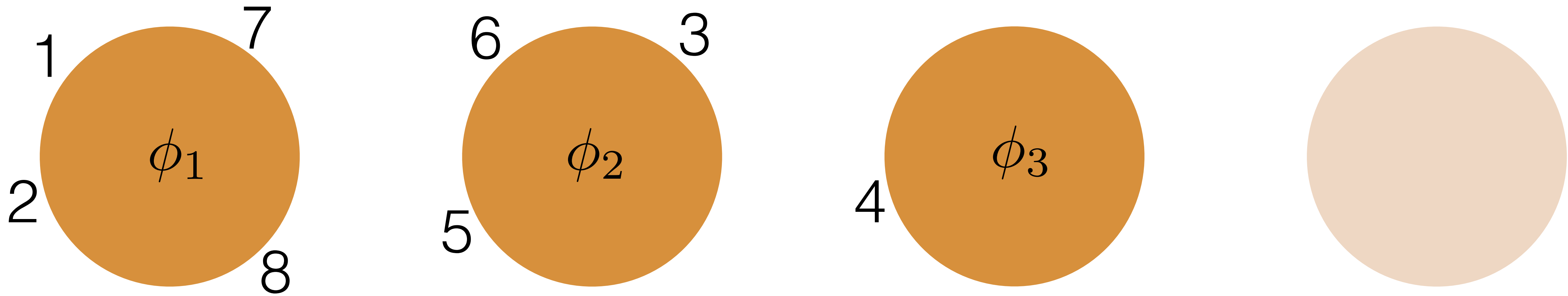
Chinese restaurant process



- Probability of this seating:

$$\frac{\alpha}{\alpha}$$

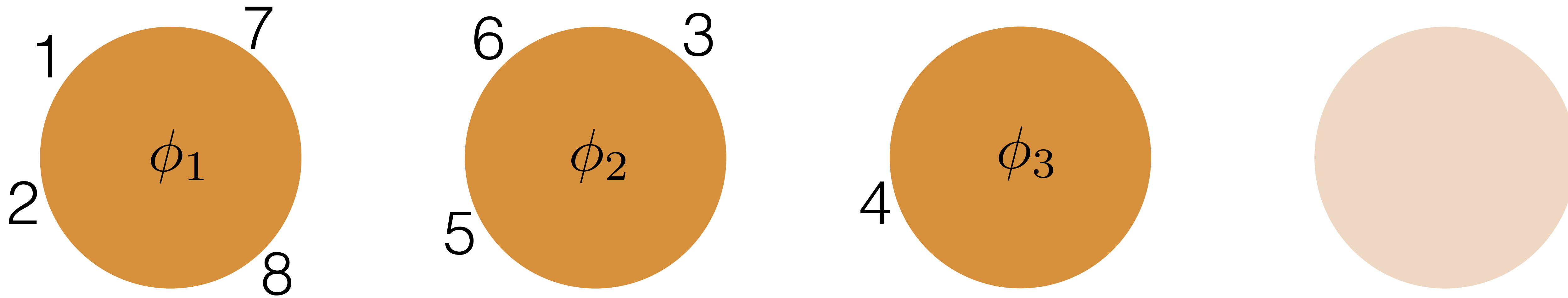
Chinese restaurant process



- Probability of this seating:

$$\frac{\alpha}{\alpha} \cdot \frac{1}{\alpha + 1}$$

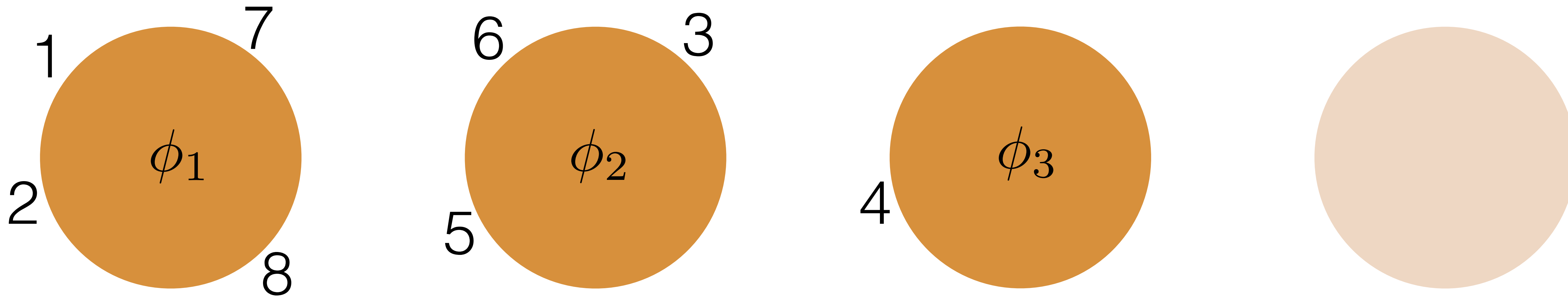
Chinese restaurant process



- Probability of this seating:

$$\frac{\alpha}{\alpha} \cdot \frac{1}{\alpha + 1} \cdot \frac{\alpha}{\alpha + 2}$$

Chinese restaurant process



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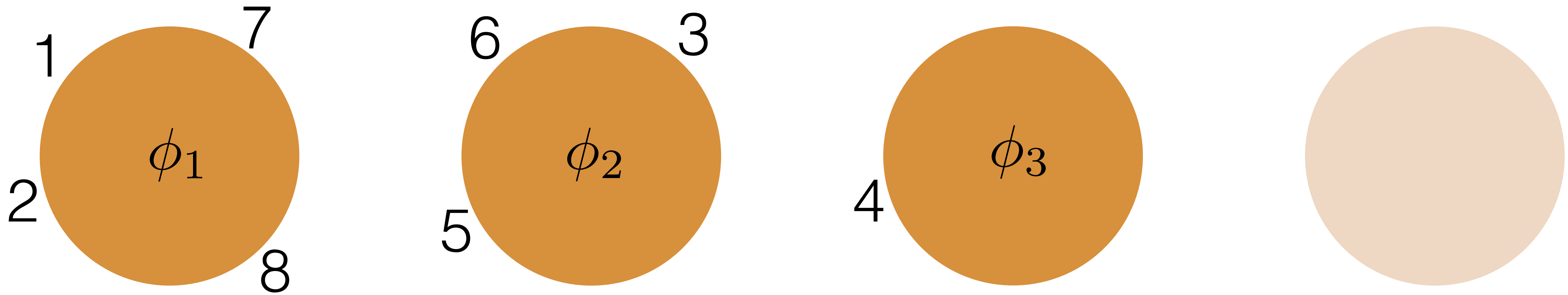
Chinese restaurant process



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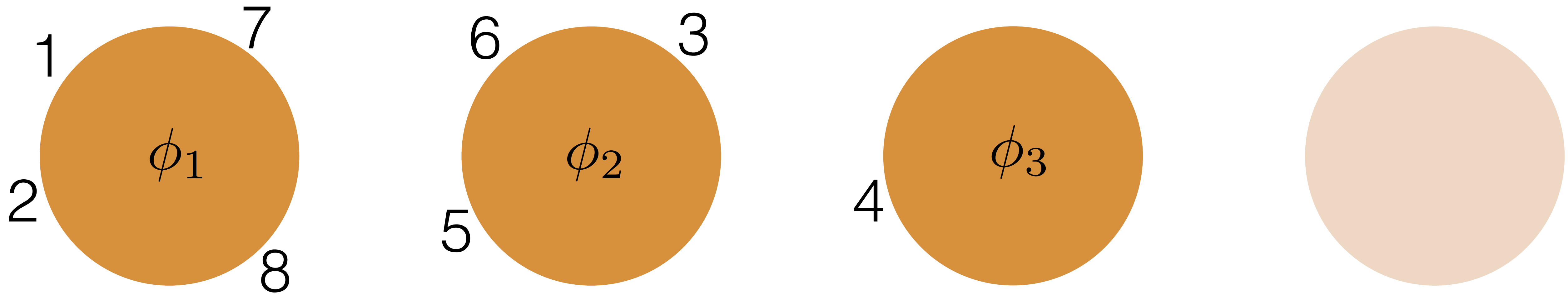
$$\frac{\alpha}{\alpha} \cdot \frac{1}{\alpha+1} \cdot \frac{\alpha}{\alpha+2} \cdot \frac{\alpha}{\alpha+3} \cdot \frac{1}{\alpha+4}$$

Chinese restaurant process



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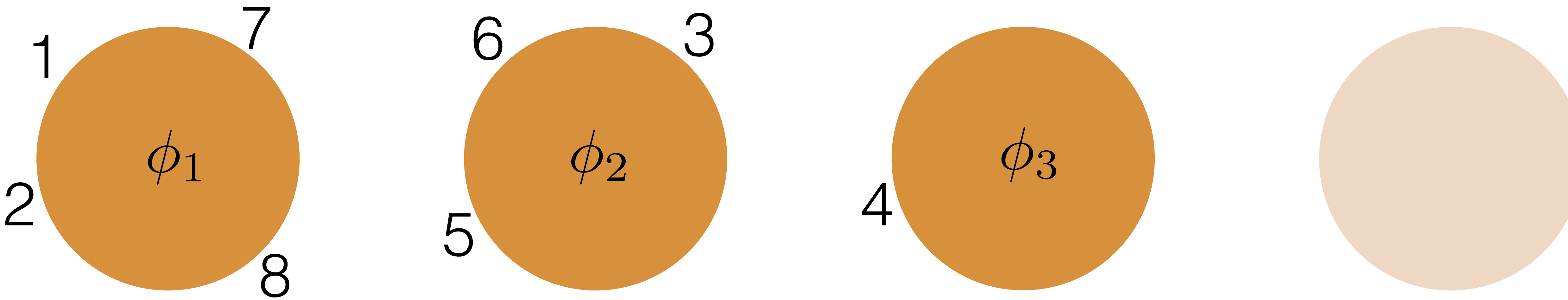
$$\frac{\alpha}{\alpha} \cdot \frac{1}{\alpha+1} \cdot \frac{\alpha}{\alpha+2} \cdot \frac{\alpha}{\alpha+3} \cdot \frac{1}{\alpha+4} \cdot \frac{2}{\alpha+5}$$

Chinese restaurant process



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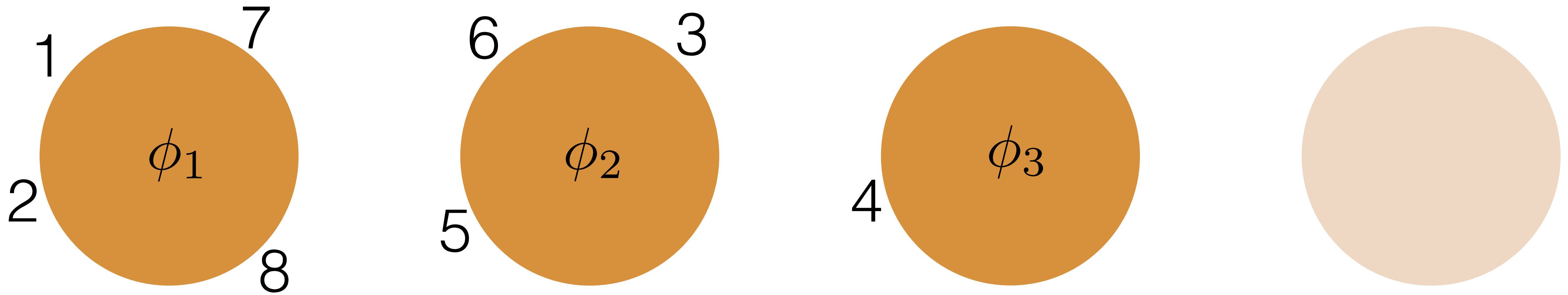
Chinese restaurant process



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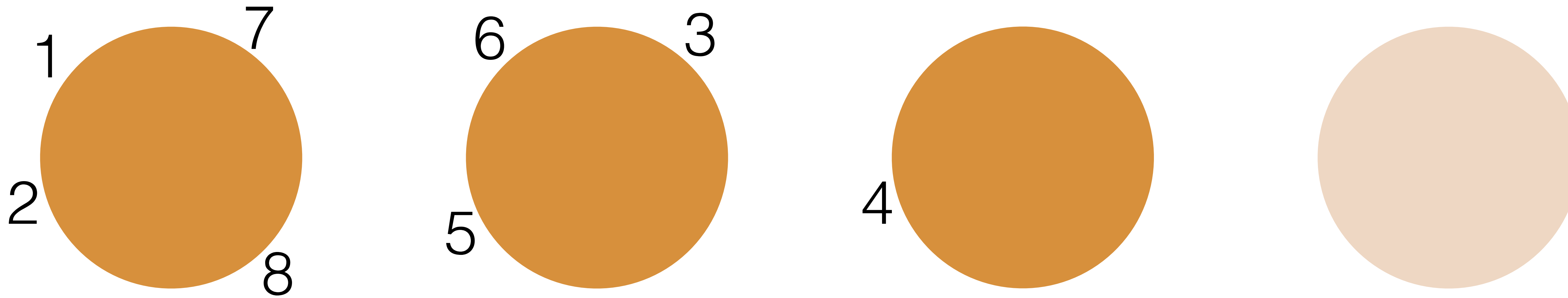
Chinese restaurant process



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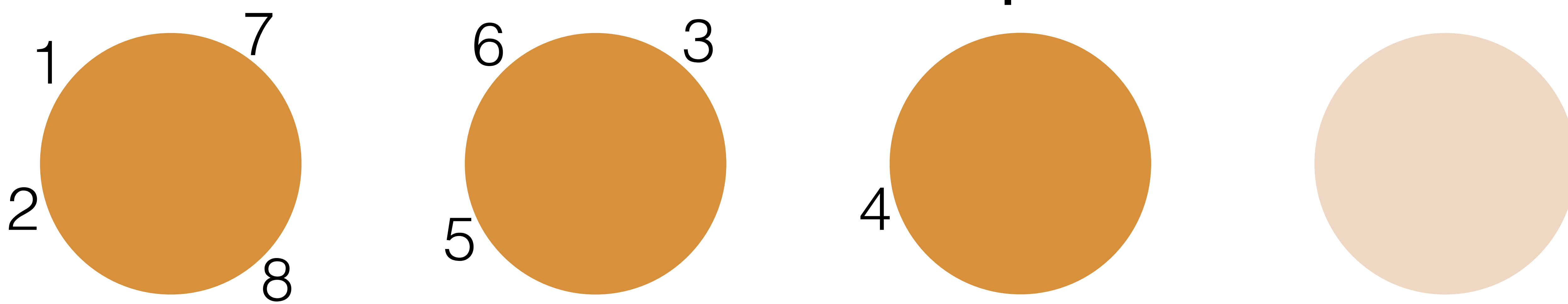
- Probability of *N* customers (K_N tables, n_k at table *k*):

Chinese restaurant process



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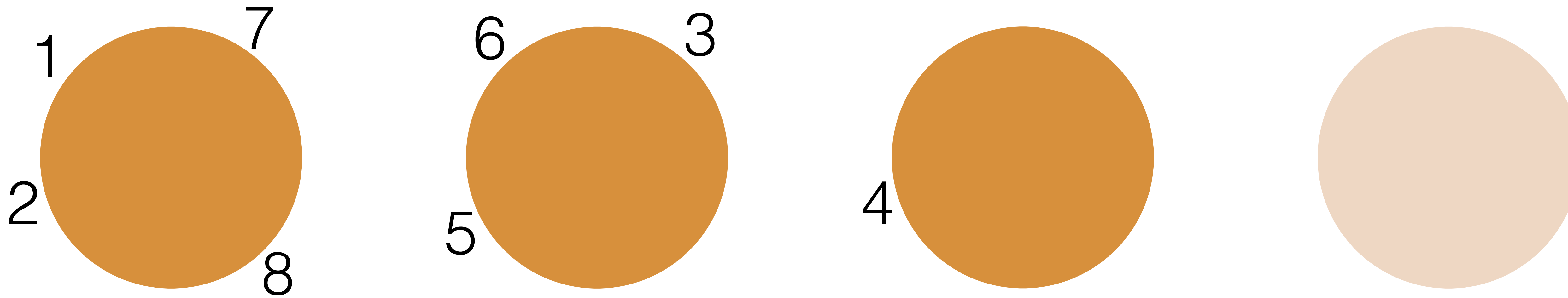

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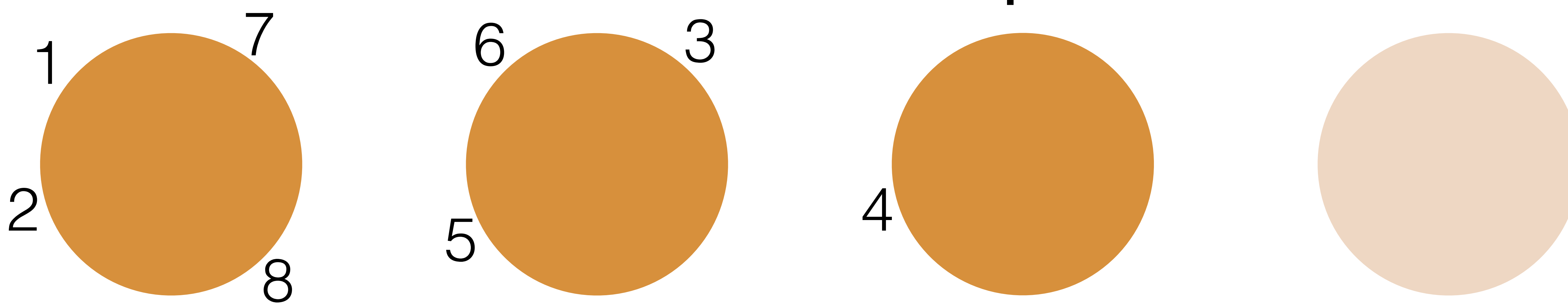
$$\frac{}{\alpha \cdots (\alpha + N - 1)}$$

Chinese restaurant process



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$$\frac{\alpha^{K_N}}{\alpha \cdots (\alpha + N - 1)}$$

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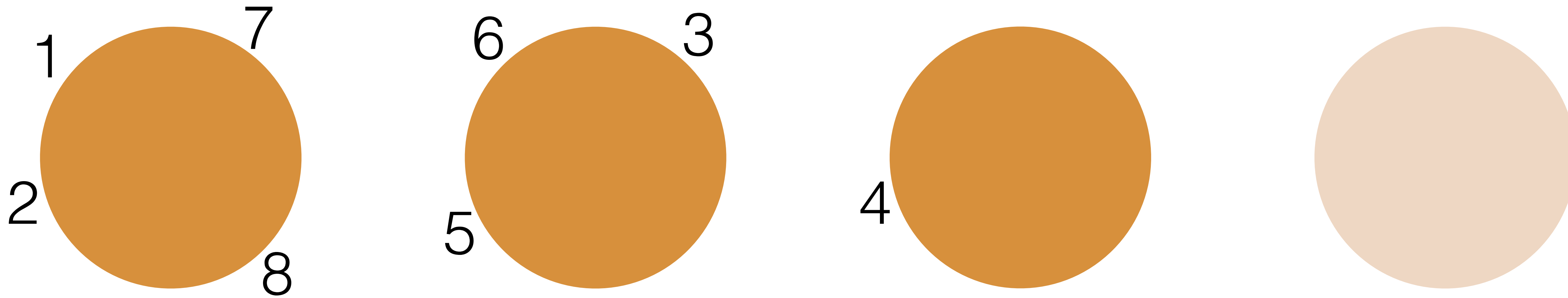

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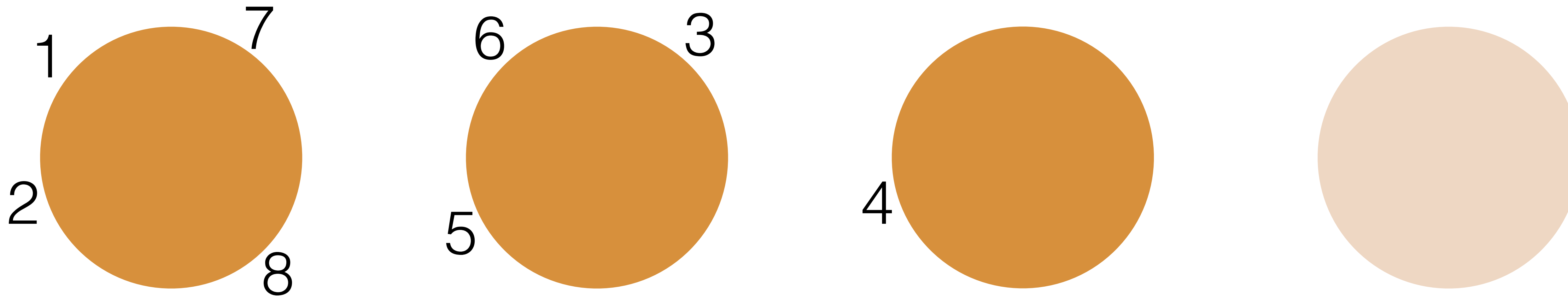

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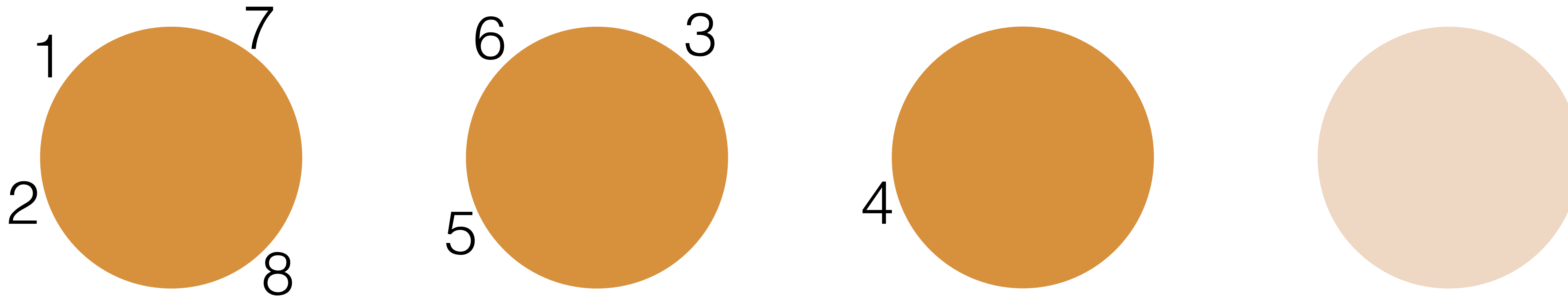

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$$\frac{\alpha^{K_N}}{\alpha \cdots (\alpha + N - 1)}$$

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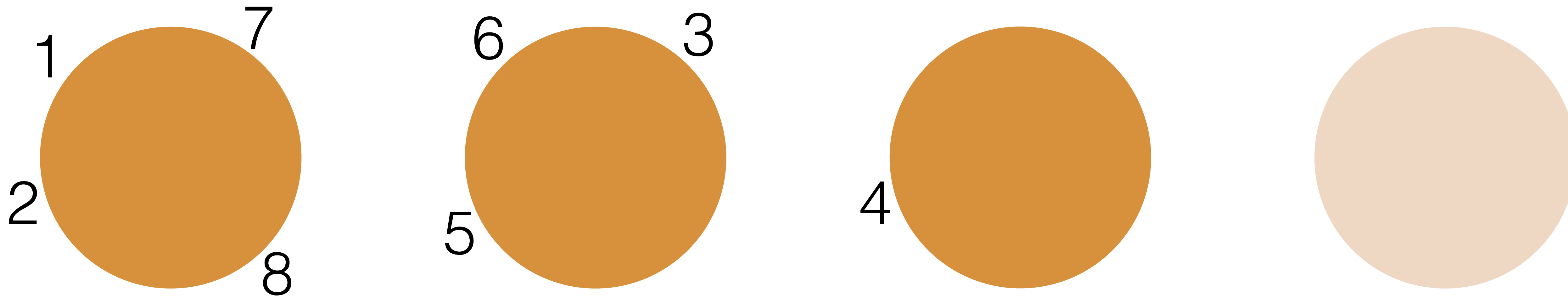

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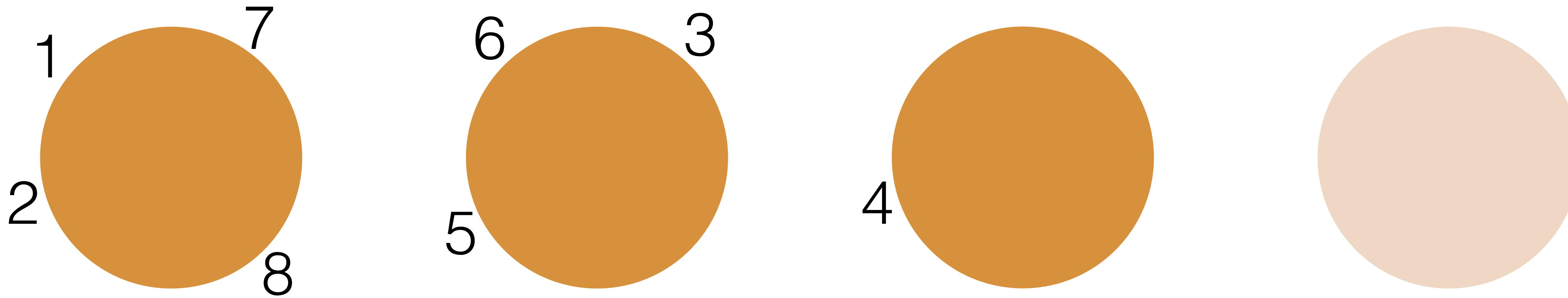

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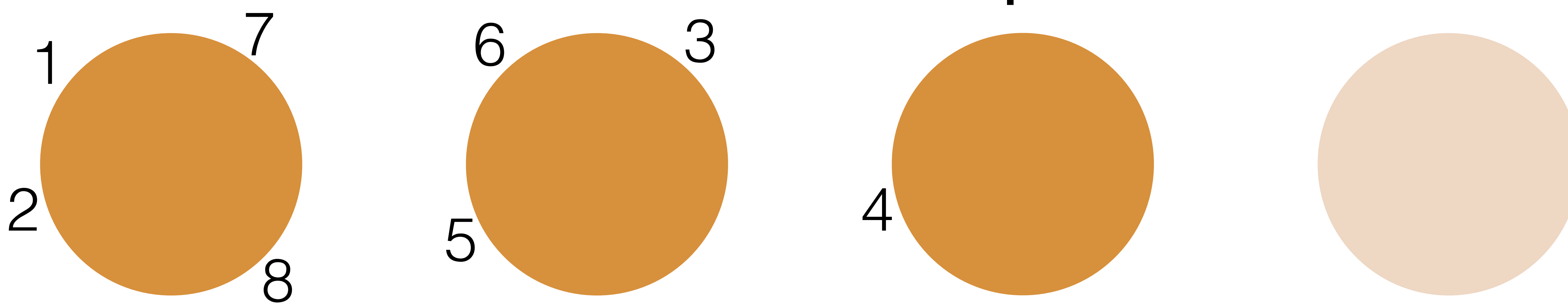
- Probability of *N* customers (K_N tables, n_k at table *k*):

$$\frac{\alpha^{K_N} \prod_{k=1}^{K_N} (n_k - 1)!}{\alpha \cdots (\alpha + N - 1)}$$

Chinese restaurant process



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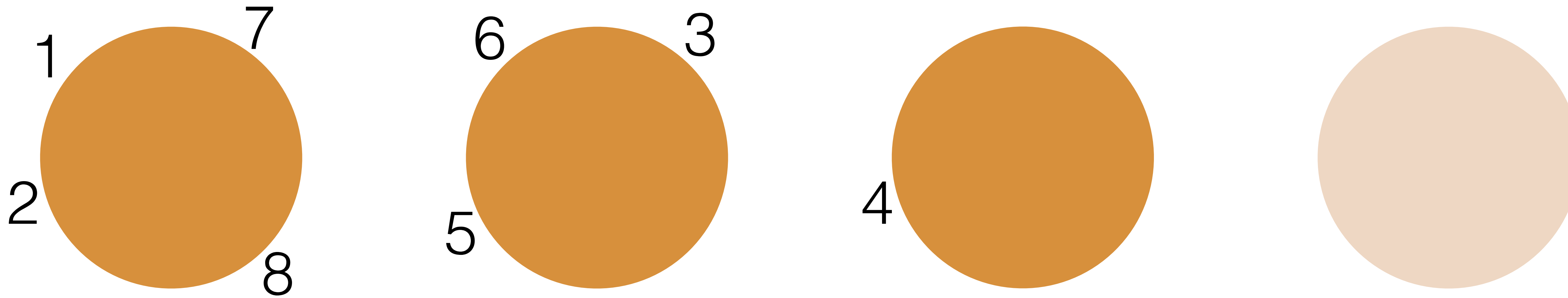
$$\frac{\alpha}{\alpha} \cdot \frac{1}{\alpha+1} \cdot \frac{\alpha}{\alpha+2} \cdot \frac{\alpha}{\alpha+3} \cdot \frac{1}{\alpha+4} \cdot \frac{2}{\alpha+5} \cdot \frac{2}{\alpha+6} \cdot \frac{3}{\alpha+7}$$

- Probability of N customers (K_N tables, $\#C$ at table C):

$$\frac{\alpha^{K_N} \prod_{C \in \Pi_N} (\#C - 1)!}{\alpha \cdots (\alpha + N - 1)}$$

Chinese restaurant process



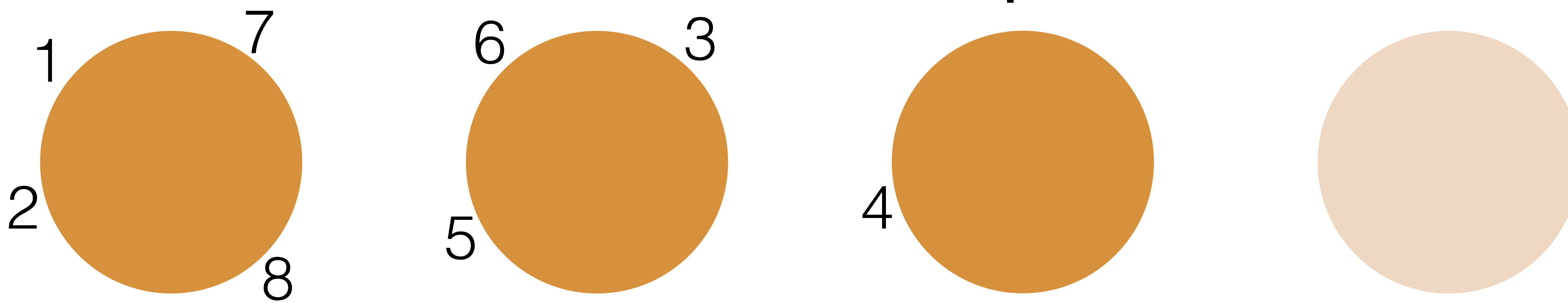
- Probability of this seating:

$$\frac{\alpha}{\alpha} \cdot \frac{1}{\alpha+1} \cdot \frac{\alpha}{\alpha+2} \cdot \frac{\alpha}{\alpha+3} \cdot \frac{1}{\alpha+4} \cdot \frac{2}{\alpha+5} \cdot \frac{2}{\alpha+6} \cdot \frac{3}{\alpha+7}$$

- Probability of *N* customers (K_N tables, #*C* at table *C*):

$$\frac{\alpha^{K_N} \prod_{C \in \Pi_N} (\#C - 1)!}{\alpha \cdots (\alpha + N - 1)} = \mathbb{P}(\Pi_N = \pi_N)$$

Chinese restaurant process



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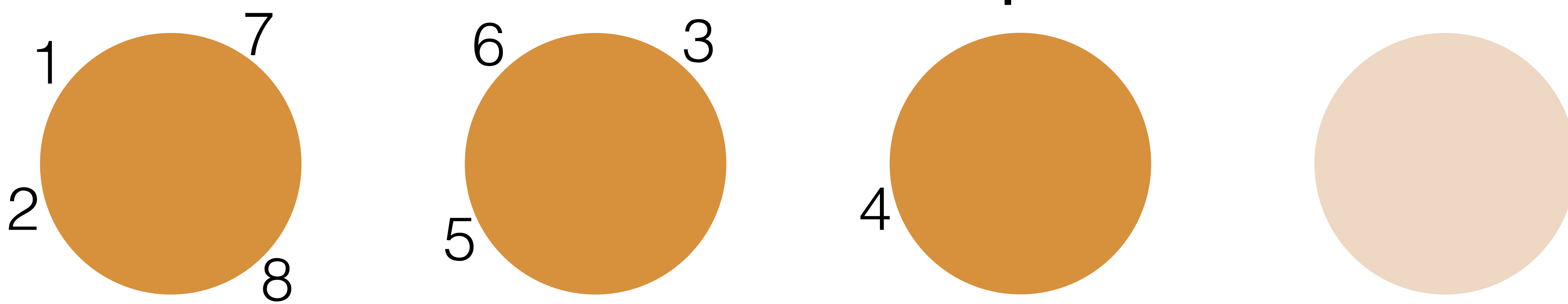
- Prob doesn't depend on customer order: *exchangeable*

Chinese restaurant process



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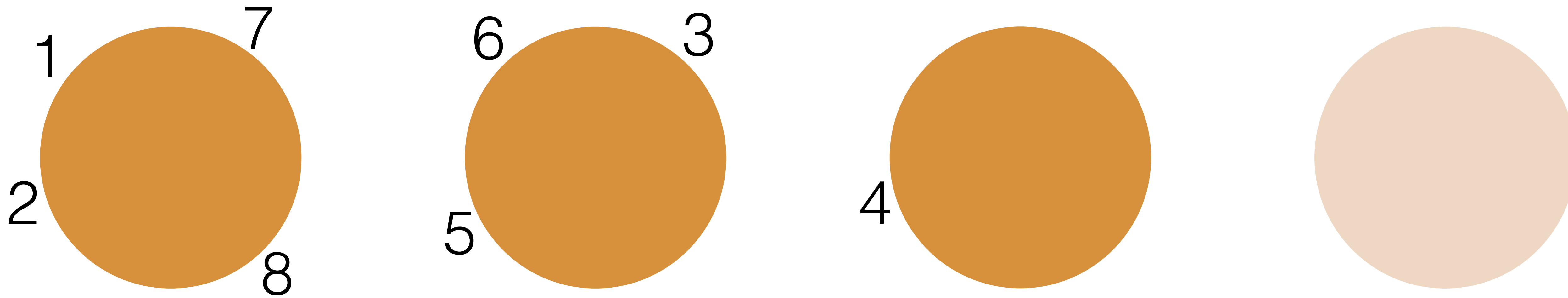
- Prob doesn't depend on customer order: *exchangeable*

$$\mathbb{P}(\Pi_8 = \{\{1,2,7,8\},\{3,5,6\},\{4\}\}) = \mathbb{P}(\Pi_8 = \{\{2,3,8,1\},\{4,6,7\},\{5\}\})$$

Chinese restaurant process



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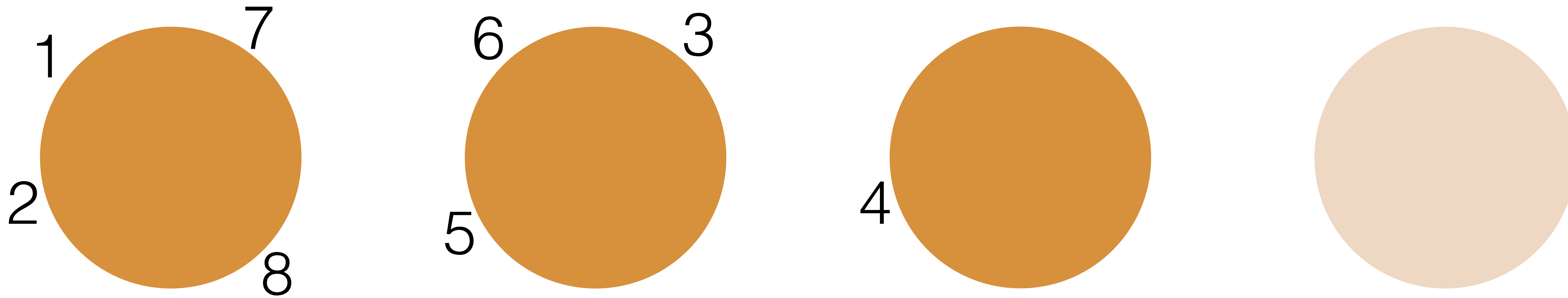
- Can always pretend *n* is the last customer and calculate $p(\Pi_N | \Pi_{N,\text{-}n})$

Chinese restaurant process



- Probability of this seating:

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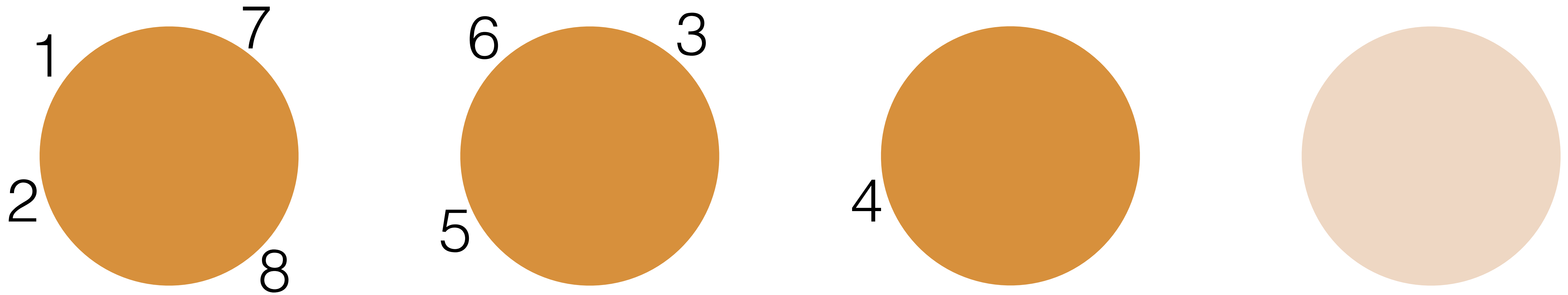
- Can always pretend *n* is the last customer and calculate $p(\Pi_N | \Pi_{N,\text{-}n})$
    - e.g. $\Pi_{8,\text{-}5} = \{\{1,2,7,8\},\{3,6\},\{4\}\}$

Chinese restaurant process



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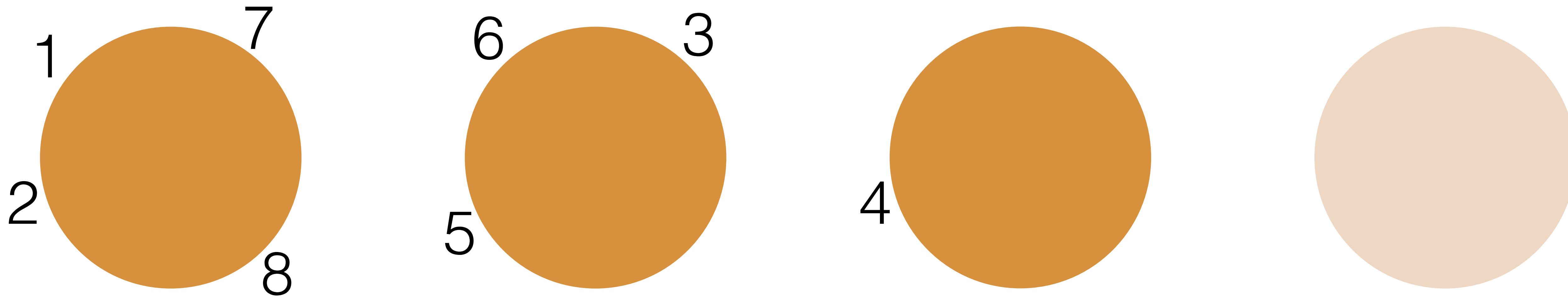
- So:

$$p(\Pi_N | \Pi_{N,-n}) =$$

Chinese restaurant process



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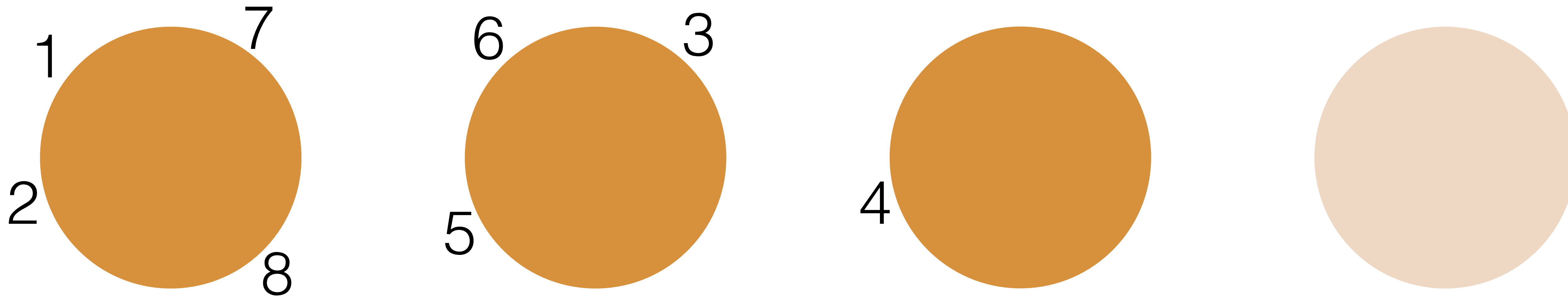
Chinese restaurant process



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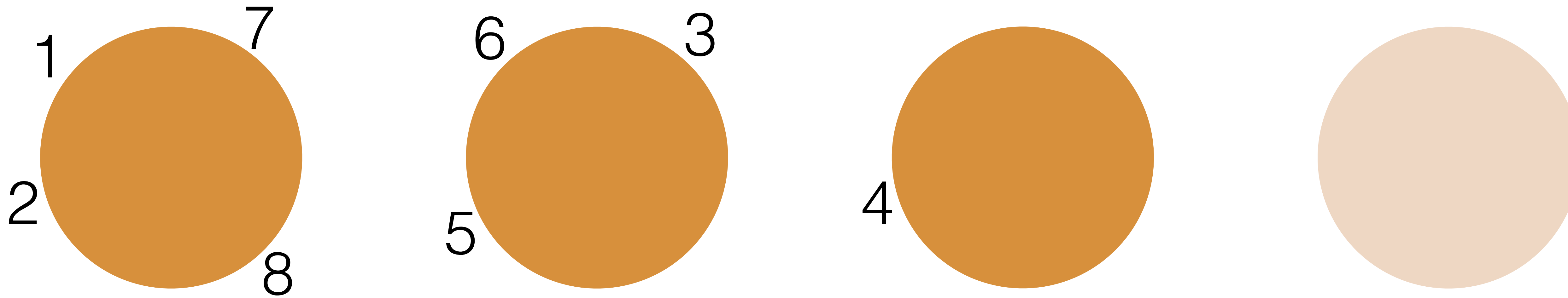
$$p(\Pi_N | \Pi_{N,-n}) = \begin{cases} & \text{if } n \text{ joins cluster } C \\ & \text{if } n \text{ starts a new cluster} \end{cases}$$

Chinese restaurant process



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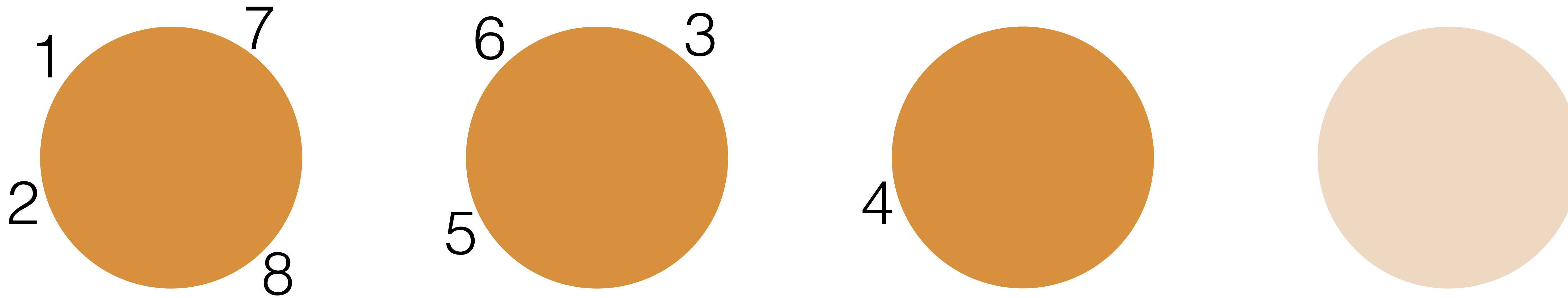
Chinese restaurant process



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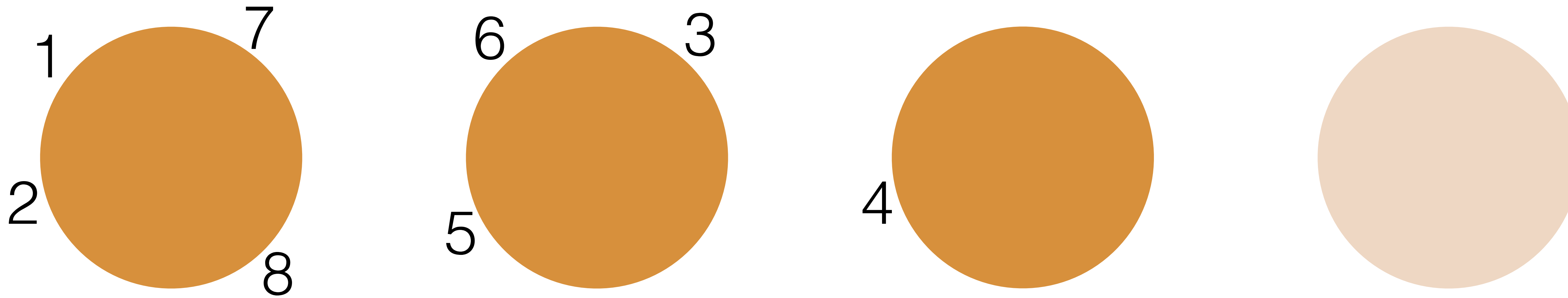
Chinese restaurant process



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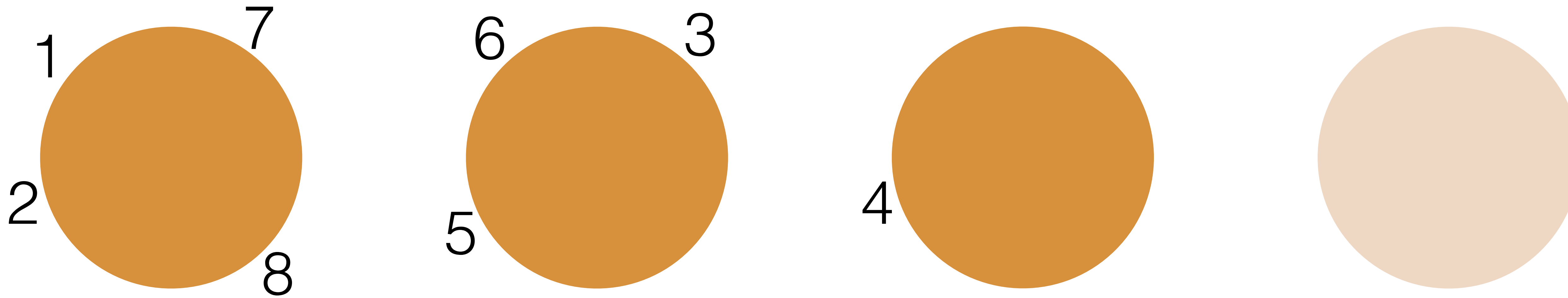
- Gibbs sampling review:

Chinese restaurant process



- Probability of *N* customers (K_N tables, #*C* at table *C*):

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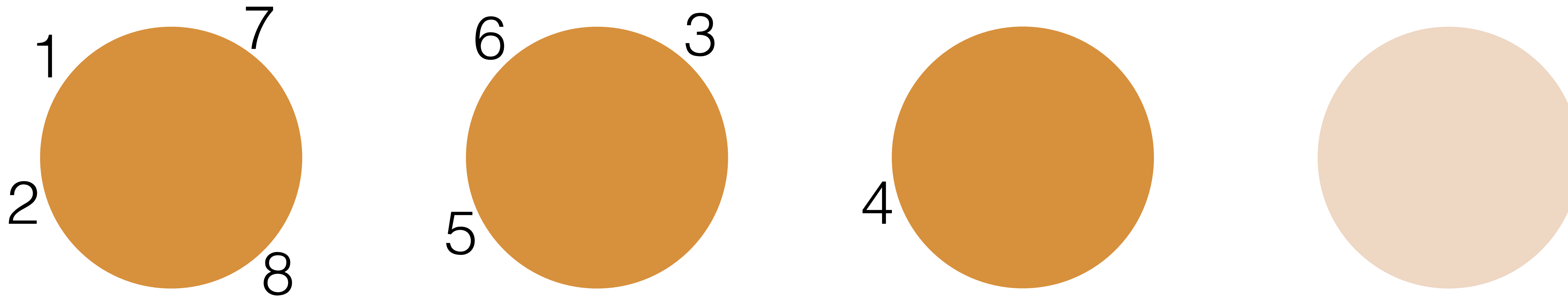
- Gibbs sampling review: target distribution $p(v_1, v_2, v_3)$

Chinese restaurant process



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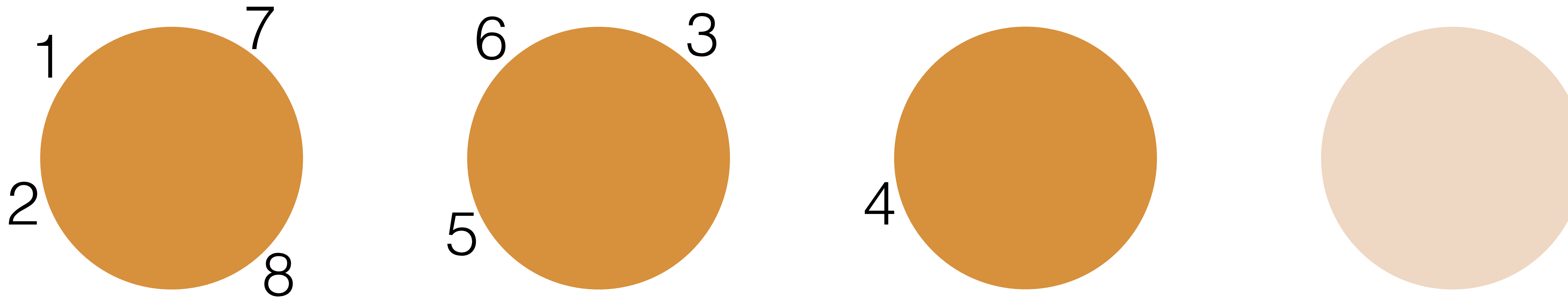
- Gibbs sampling review: target distribution $p(v_1, v_2, v_3)$
  - Start: $v_1^{(0)}, v_2^{(0)}, v_3^{(0)}$

Chinese restaurant process



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- Gibbs sampling review: target distribution $p(v_1, v_2, v_3)$
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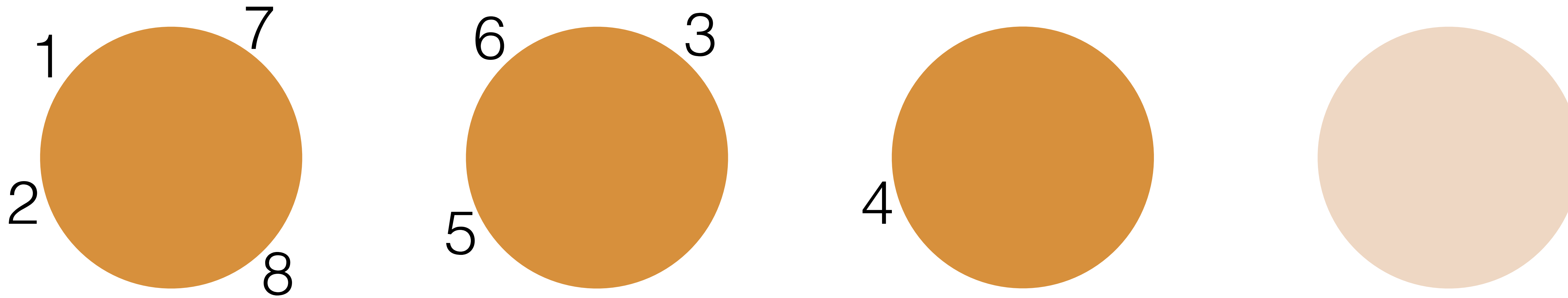
  - t th step: $v_1^{(t)} \sim p(v_1 | v_2^{(t-1)}, v_3^{(t-1)})$

Chinese restaurant process



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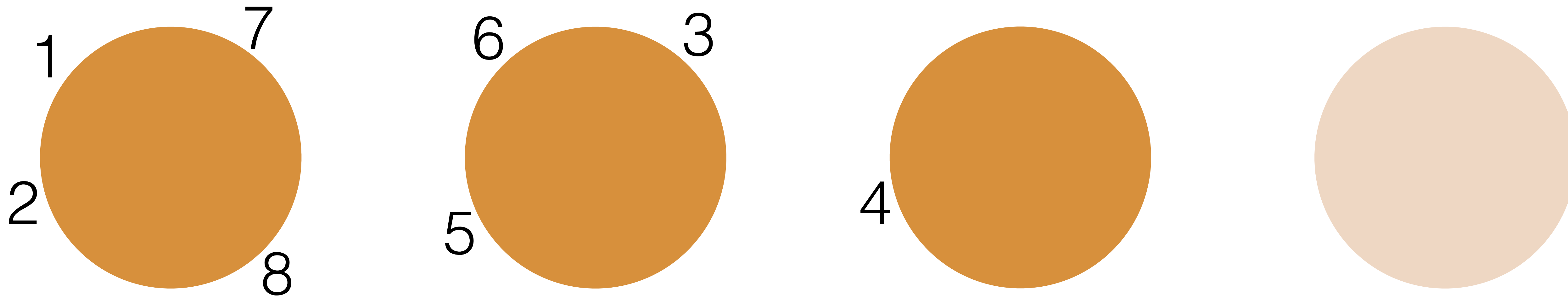
$$v_2^{(t)} \sim p(v_2 | v_1^{(t)}, v_3^{(t-1)})$$

Chinese restaurant process



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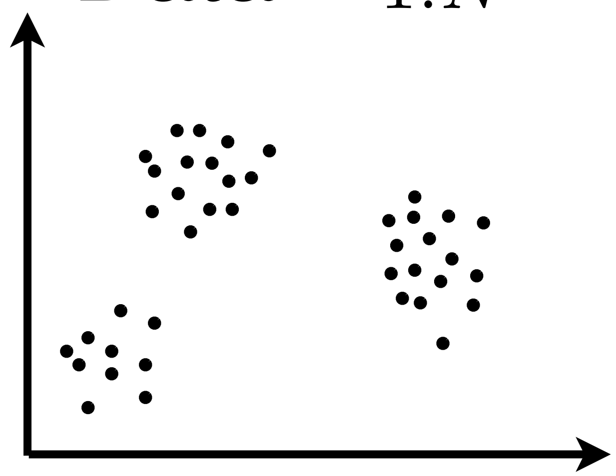
CRP mixture model: inference

CRP mixture model: inference

- Data $x_{1:N}$



CRP mixture model: inference

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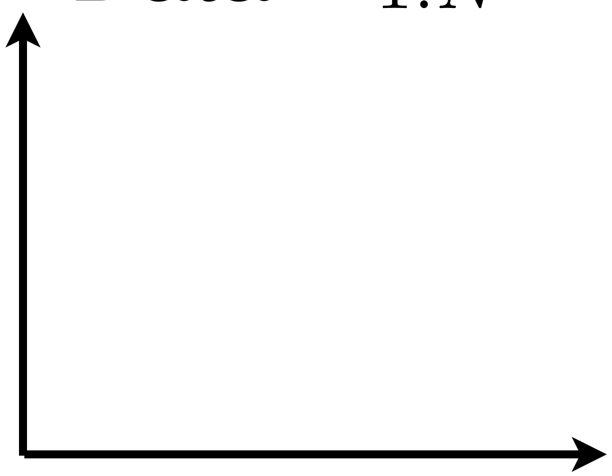

CRP mixture model: inference

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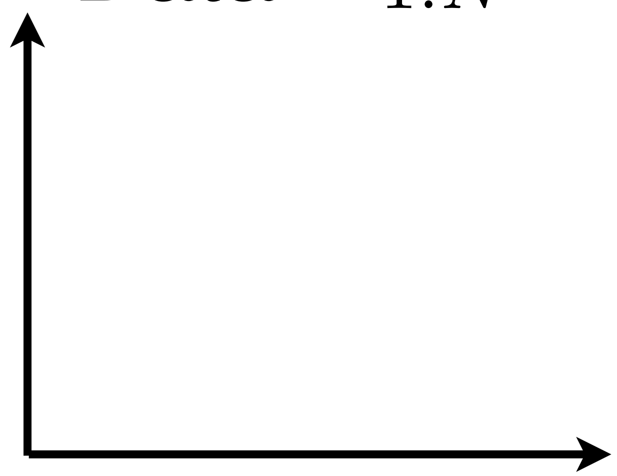
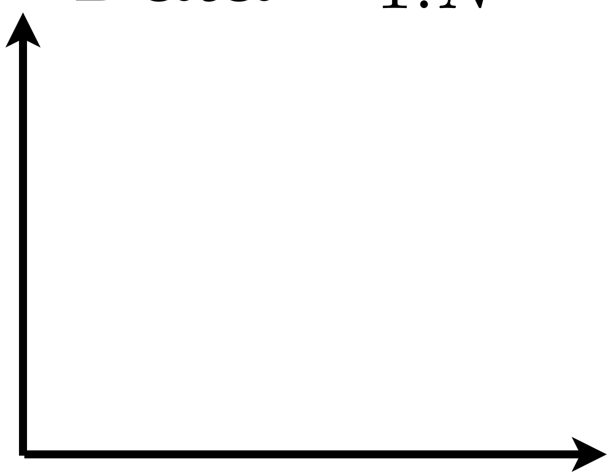


- Generative model

CRP mixture model: inference

- Data $x_{1:N}$

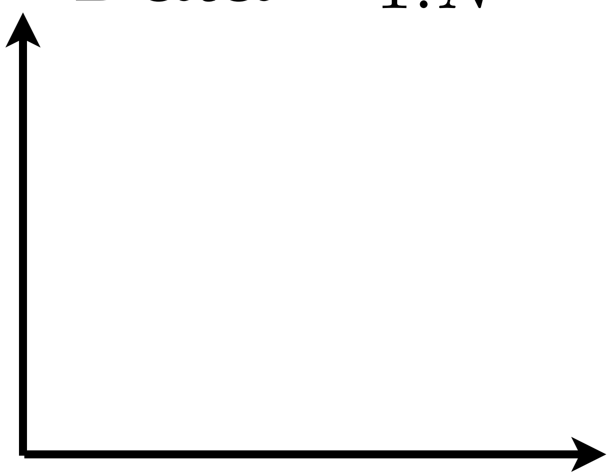


- Generative model

$$\Pi_N \sim \mathrm{CRP}(N, \alpha)$$

CRP mixture model: inference

- Data $x_{1:N}$



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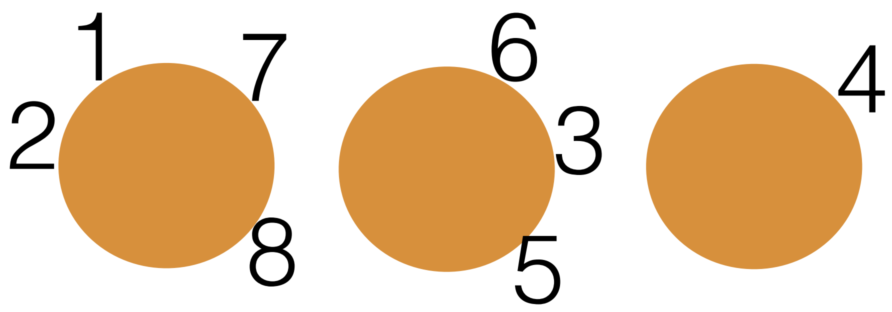

CRP mixture model: inference

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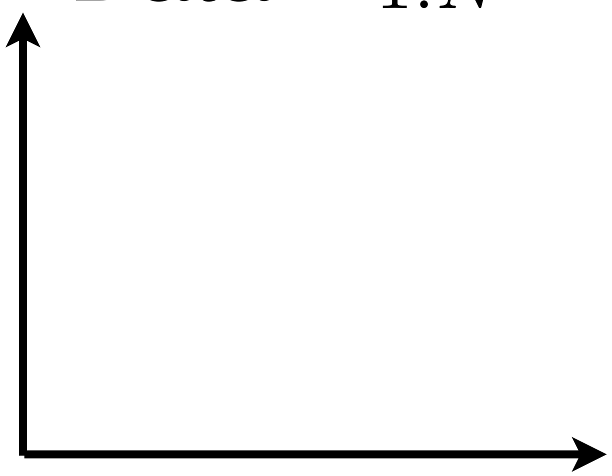

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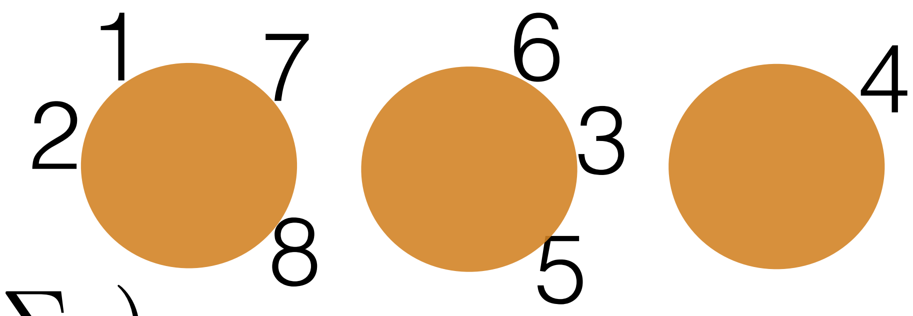

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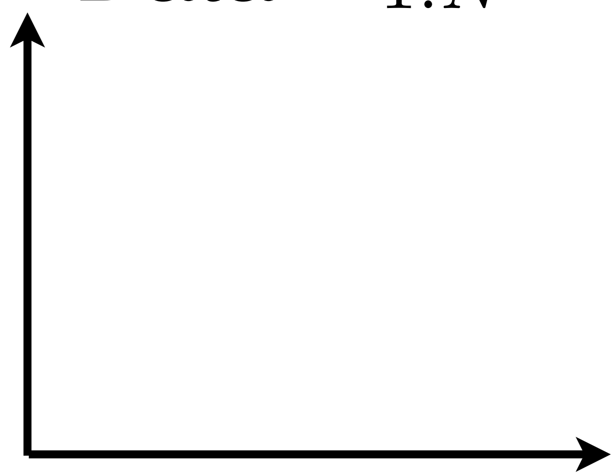

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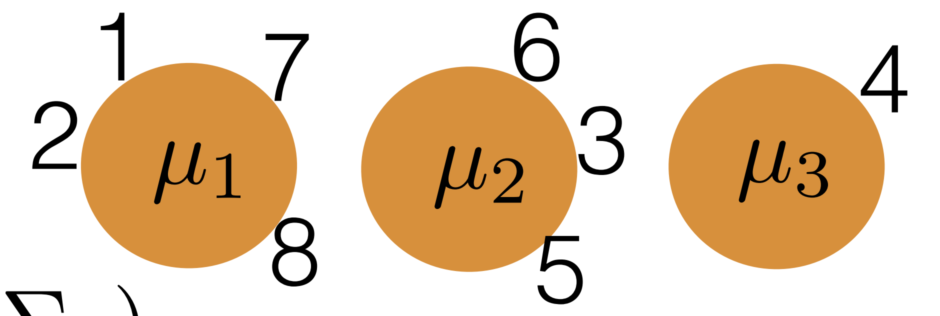

CRP mixture model: inference

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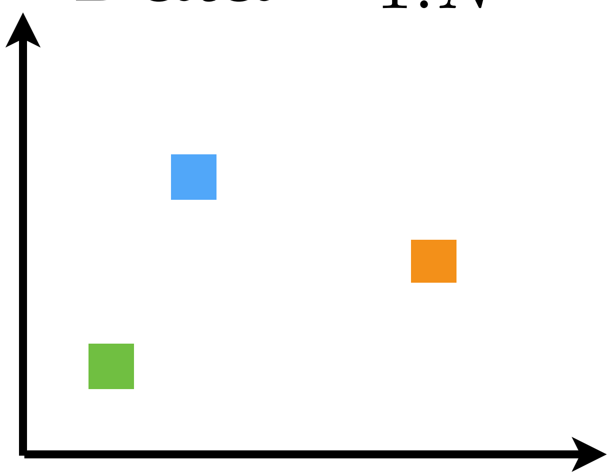

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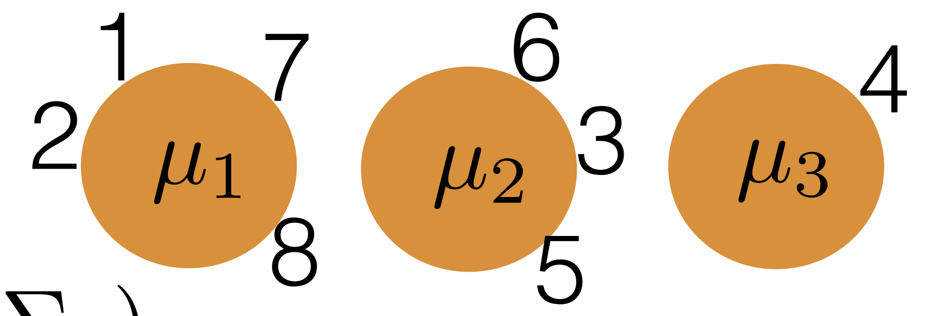

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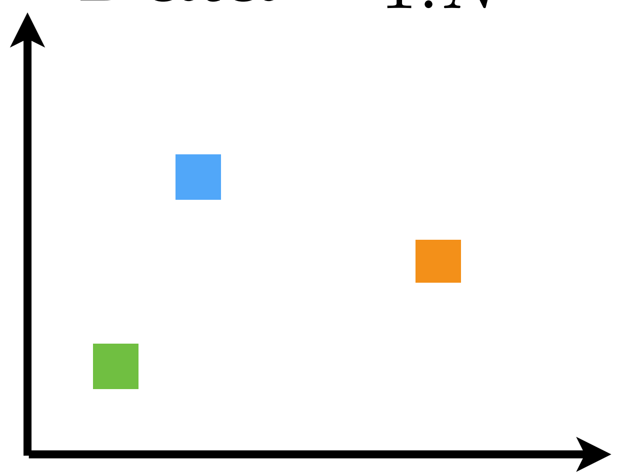

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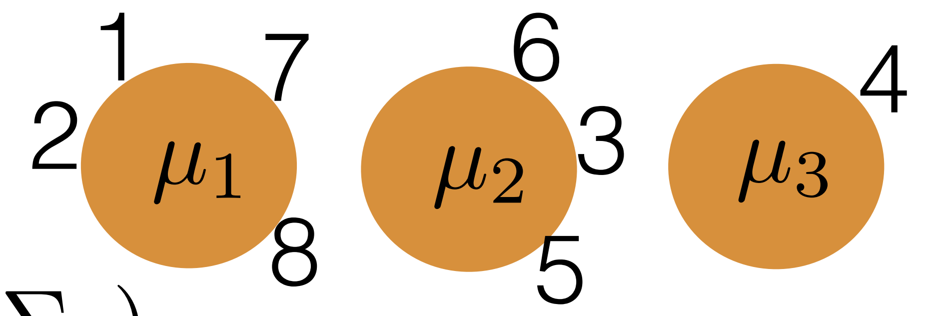
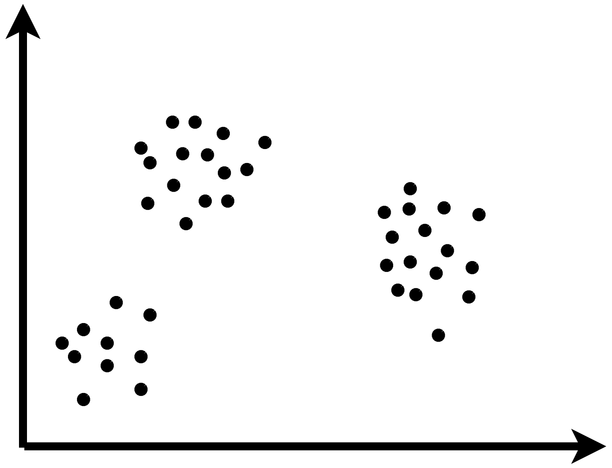


CRP mixture model: inference

- Data $x_{1:N}$



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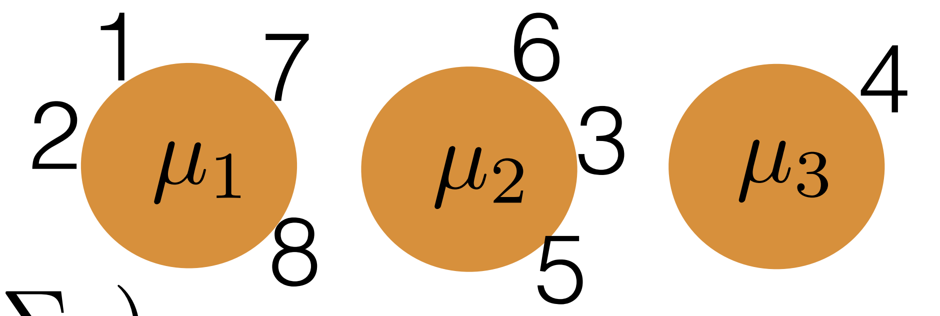
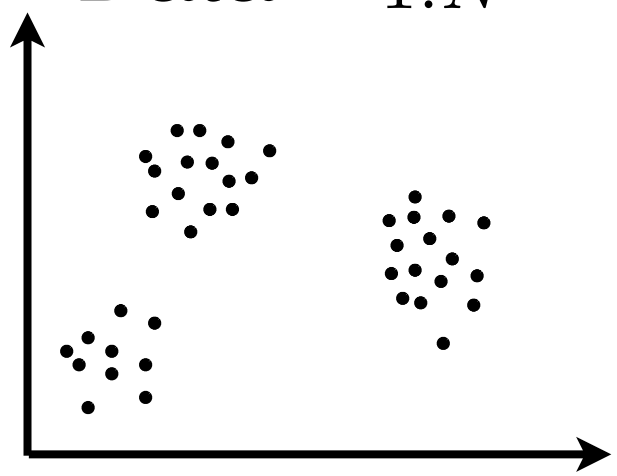


CRP mixture model: inference

- Data $x_{1:N}$



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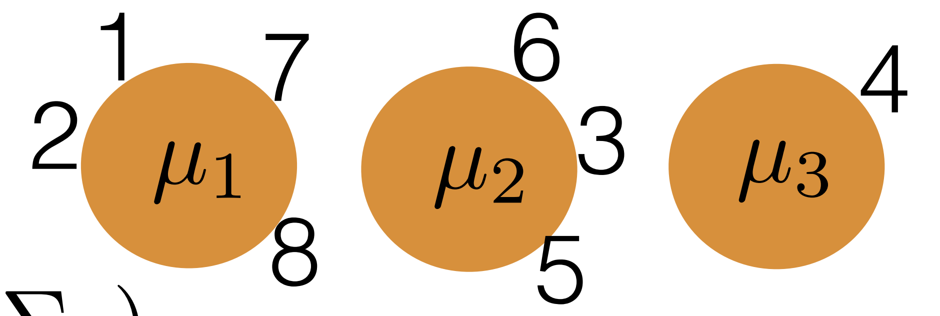
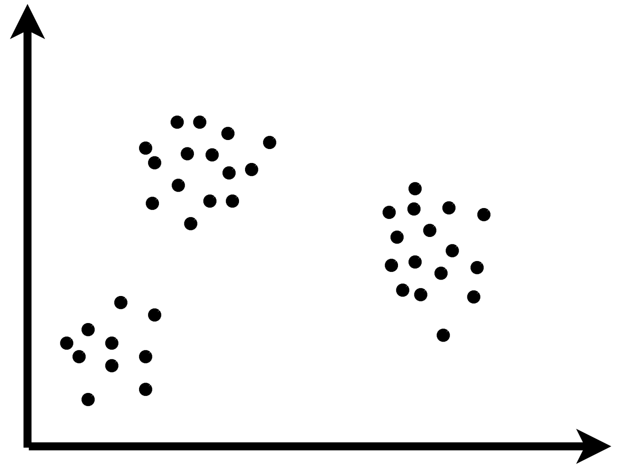


- Want: posterior

CRP mixture model: inference

- Data $x_{1:N}$



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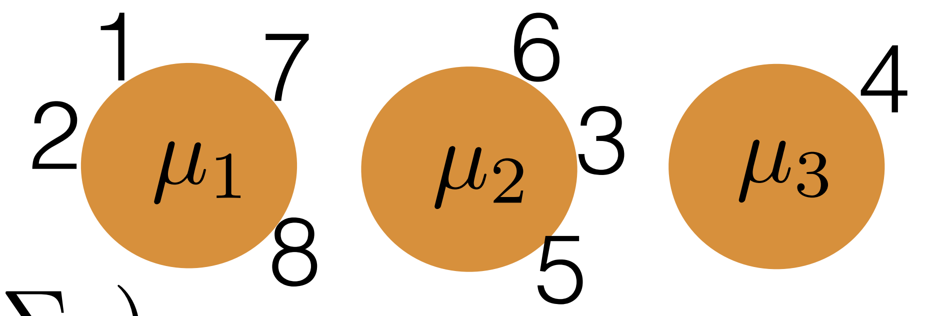

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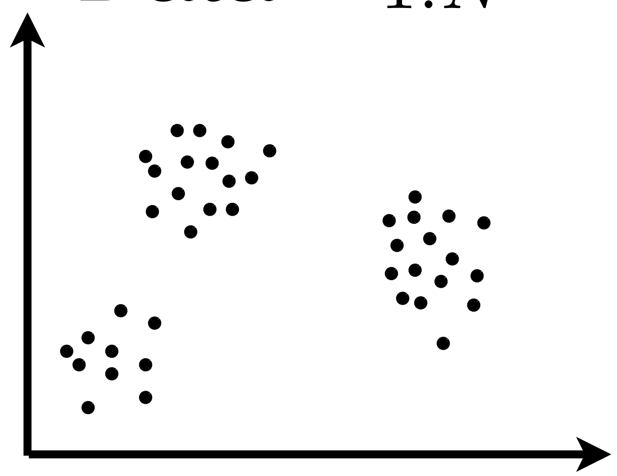
CRP mixture model: inference

- Data $x_{1:N}$



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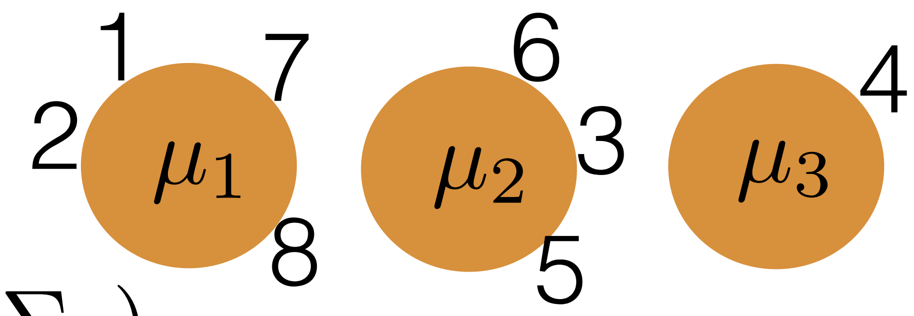

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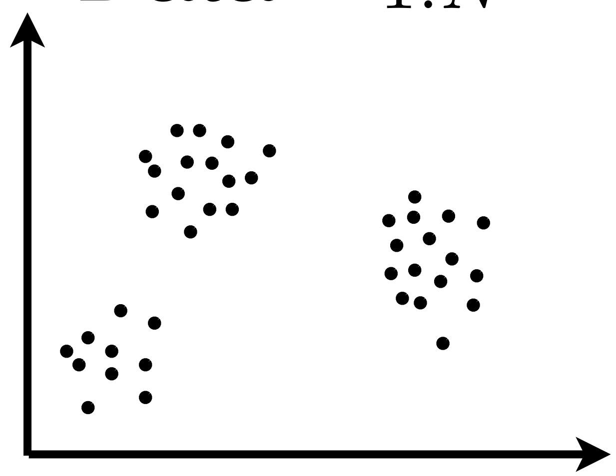

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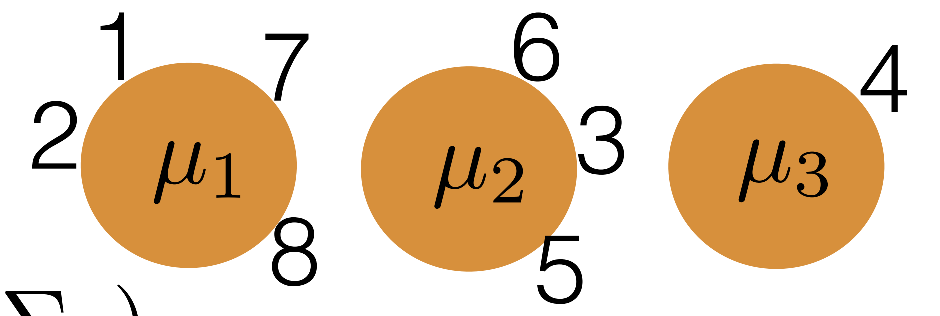

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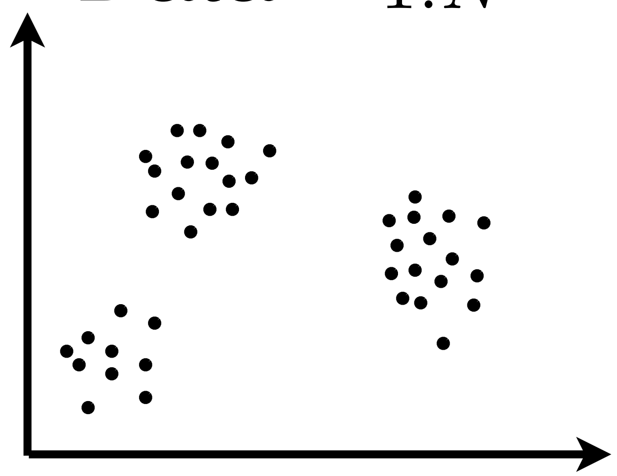

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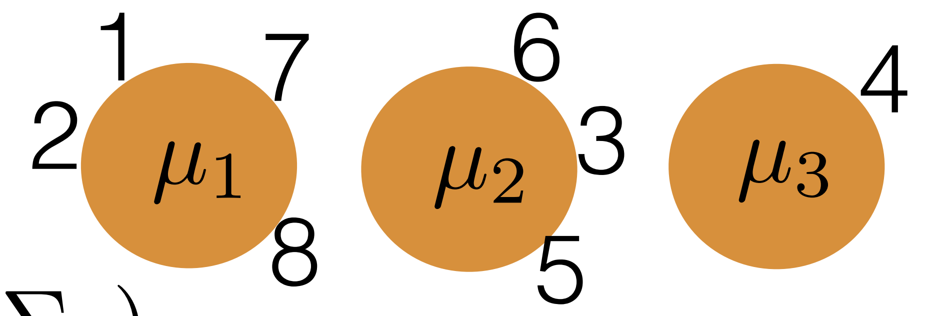

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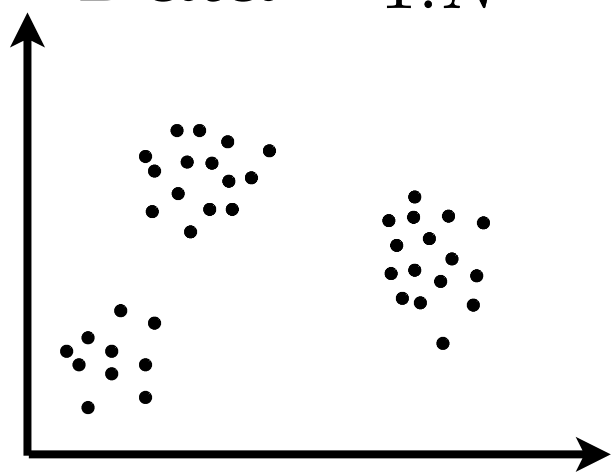

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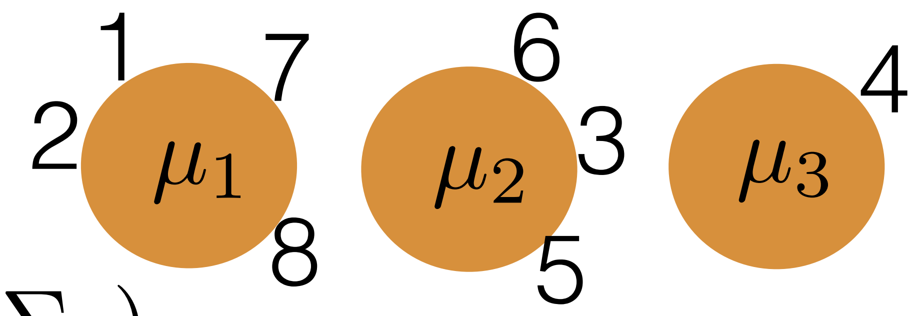

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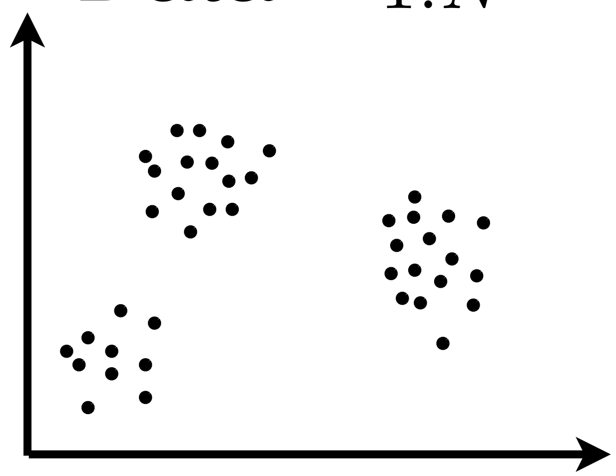

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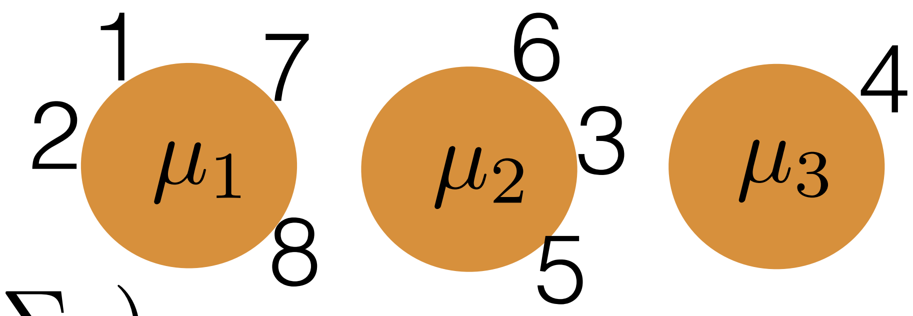

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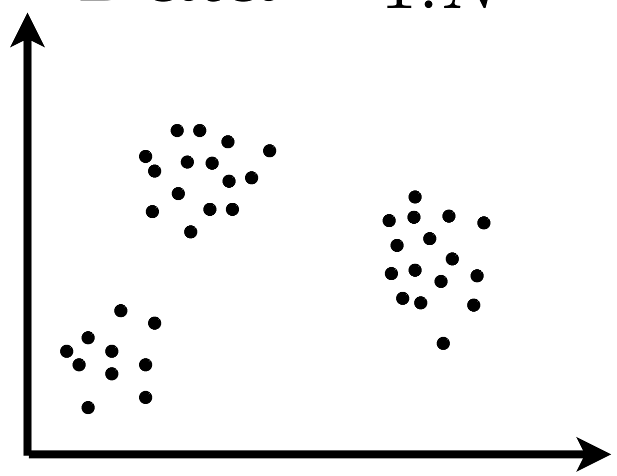

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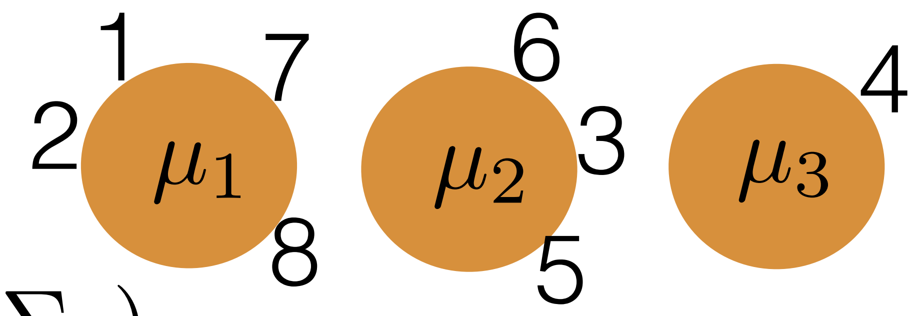

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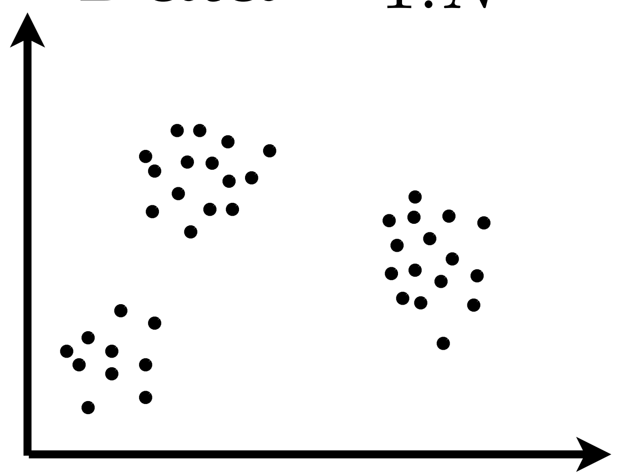

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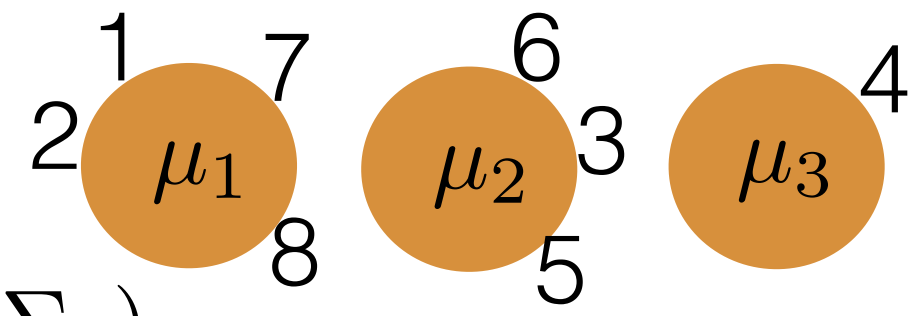

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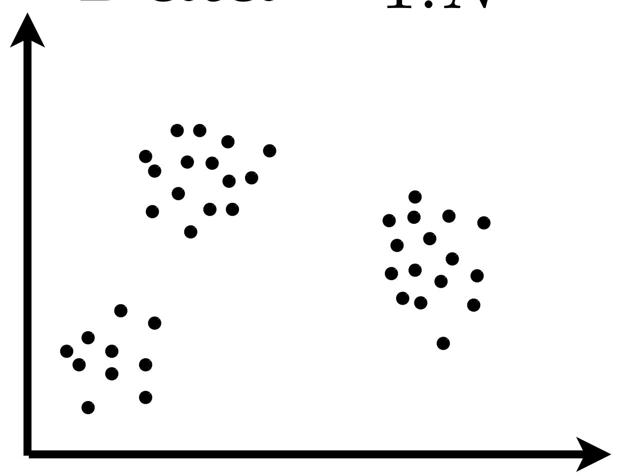

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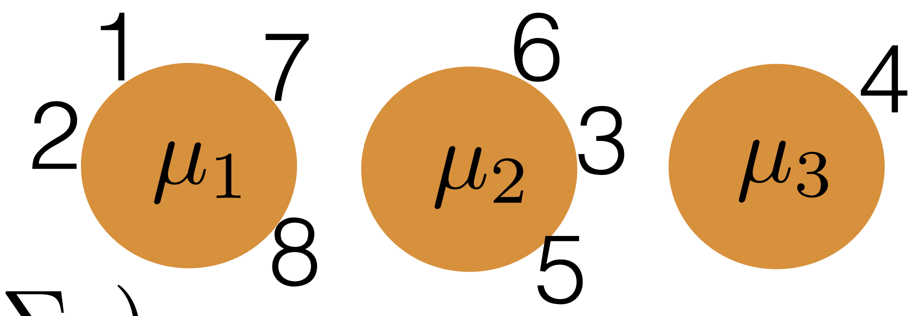

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CRP mixture model: inference

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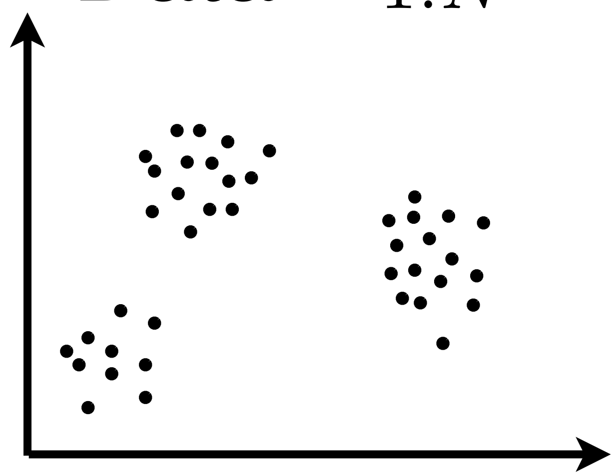

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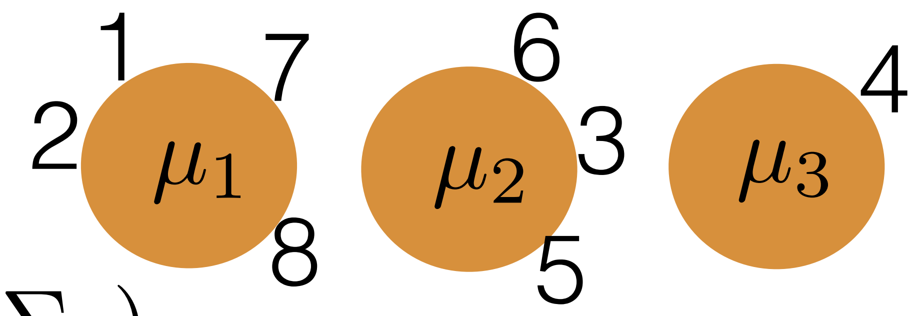

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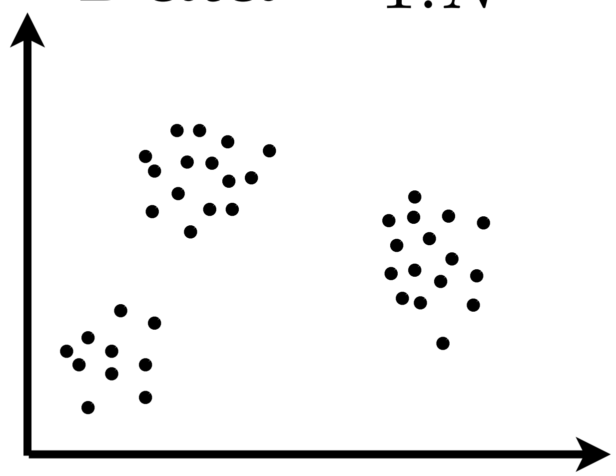

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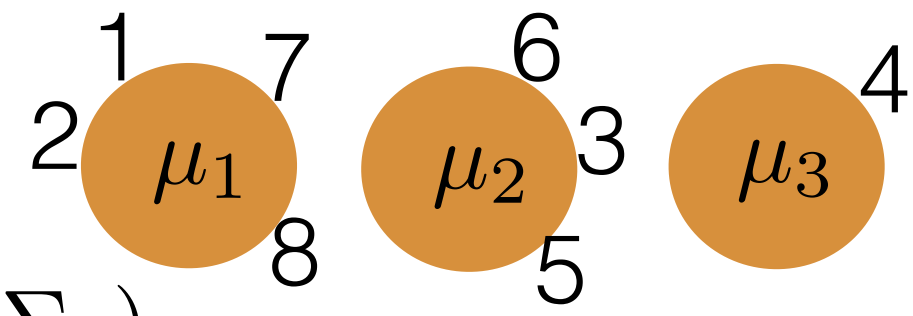

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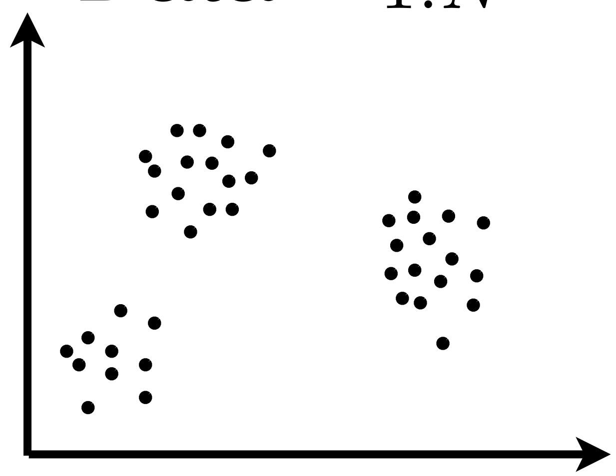

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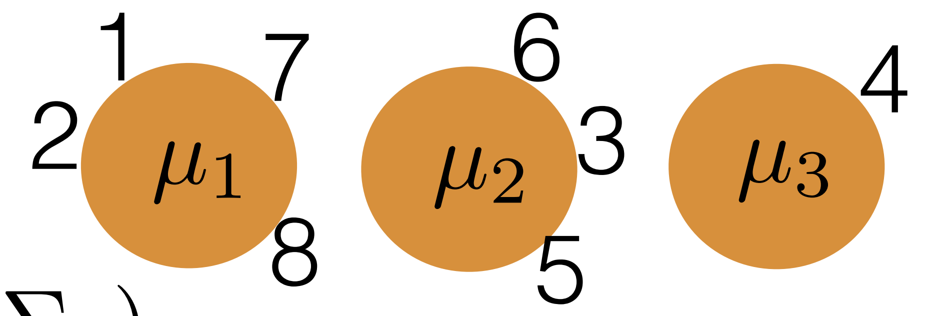

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[MacEachern 1994; Neal 1992; Neal 2000]

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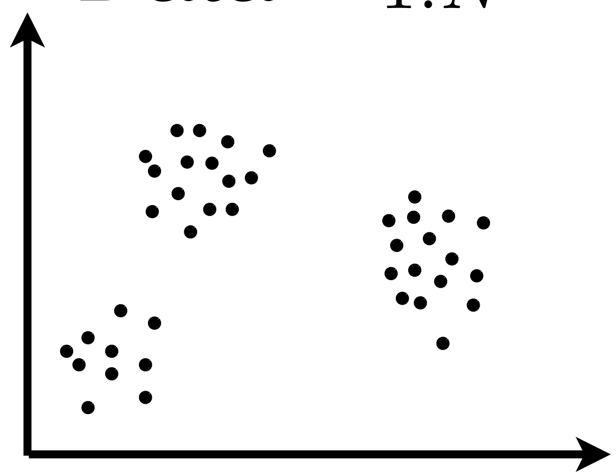

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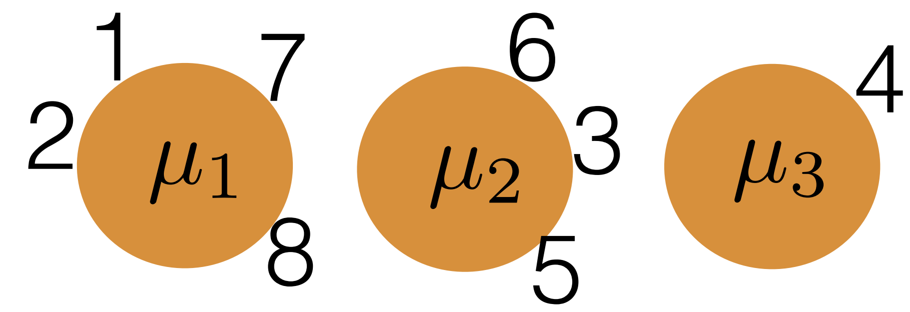

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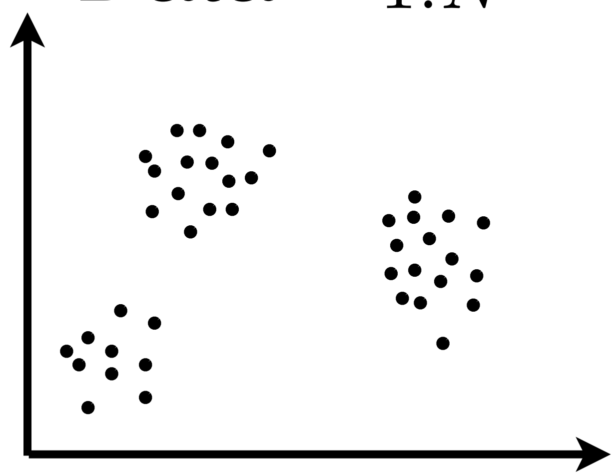

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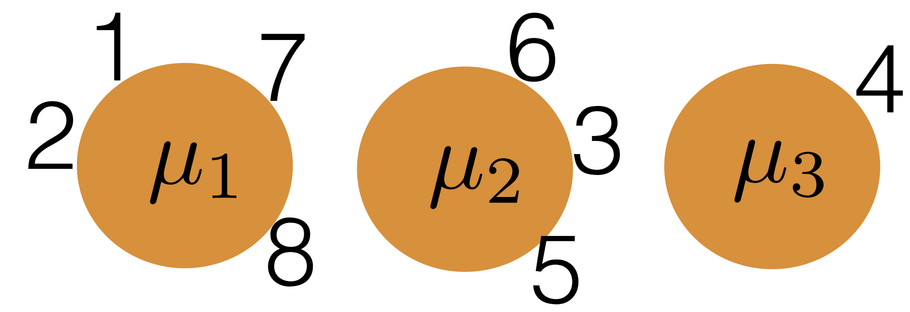

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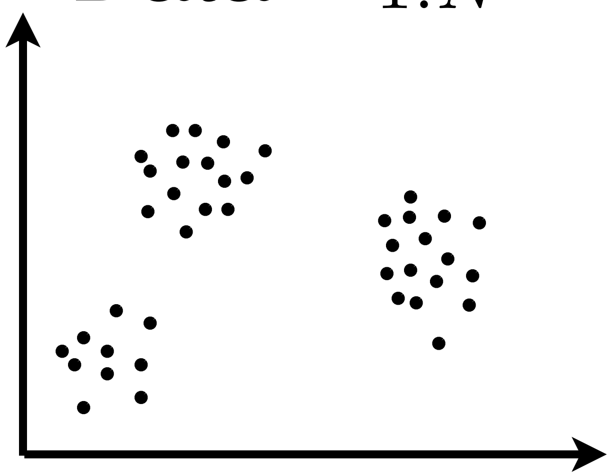

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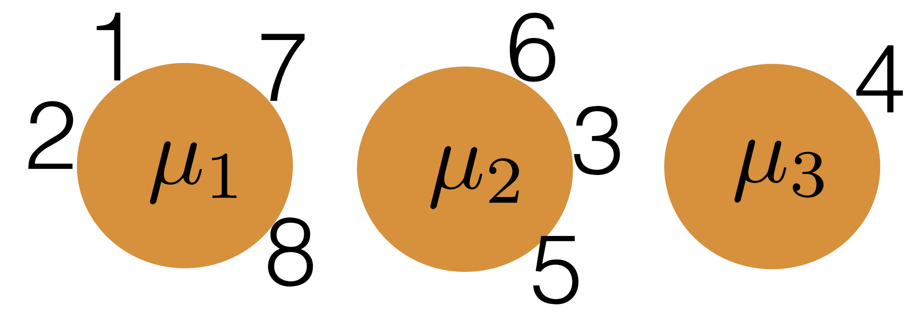

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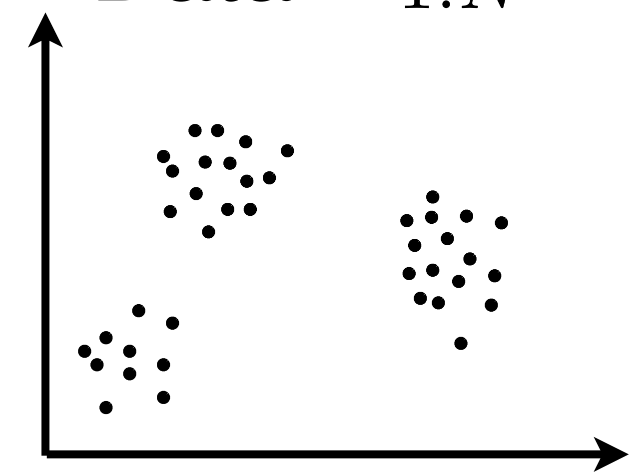

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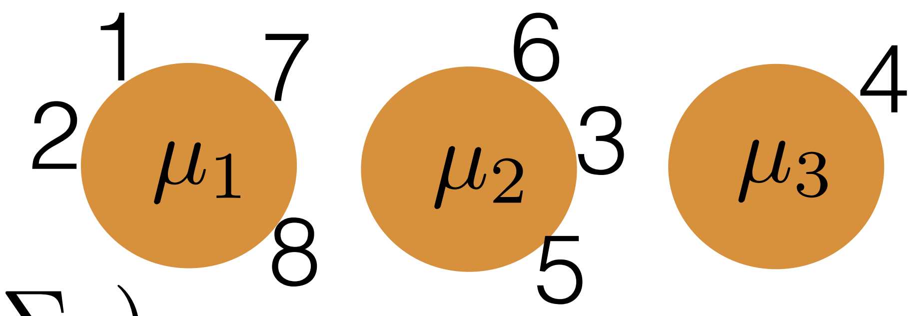

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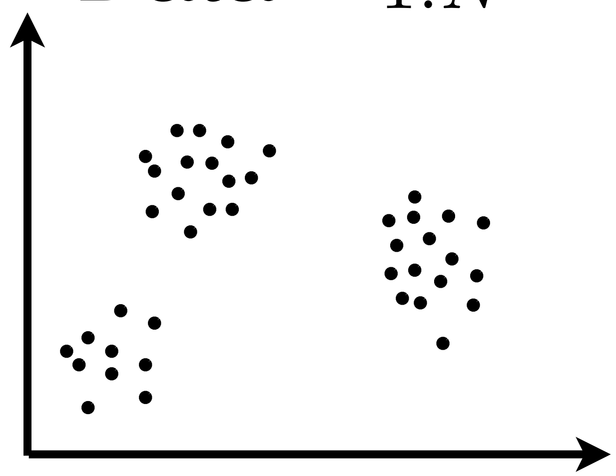

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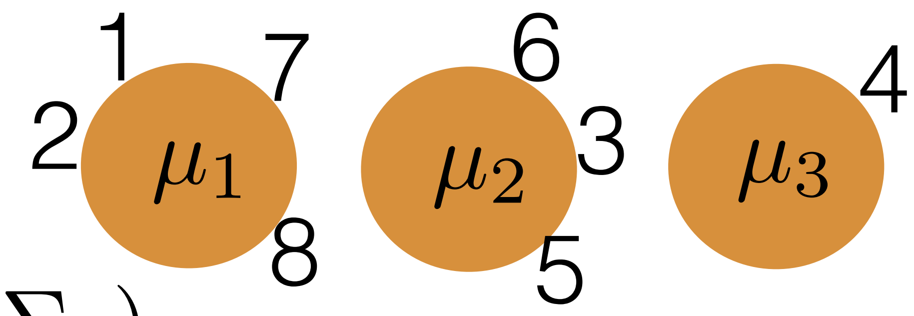

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[demo]

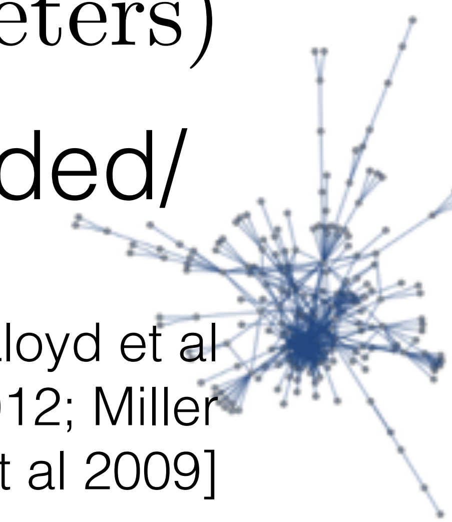
[MacEachern 1994; Neal 1992; Neal 2000]

Nonparametric Bayes

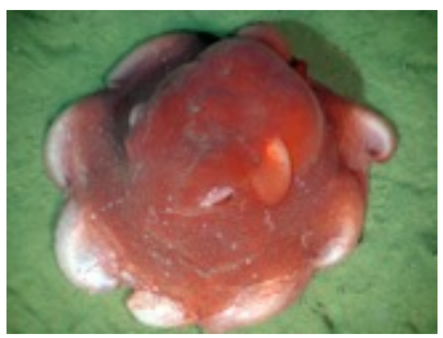
- Bayesian methods that are not parametric
- Bayesian

$$\mathbb{P}(\text{parameters}|\text{data}) \propto \mathbb{P}(\text{data}|\text{parameters})\mathbb{P}(\text{parameters})$$

- Not parametric (i.e. not finite parameter, unbounded/ growing/infinite number of parameters)



[Lloyd et al 2012; Miller et al 2009]



[Ed Bowlby, NOAA]

[Fox et al 2014]

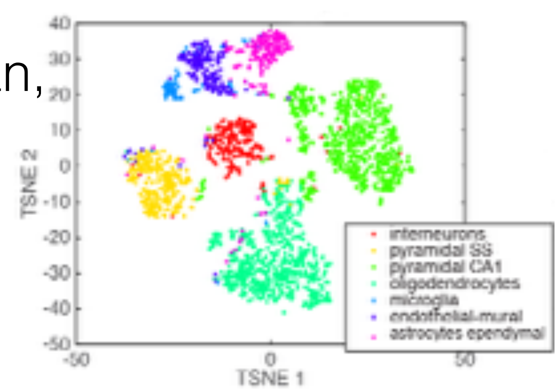
[MIT xPRO]

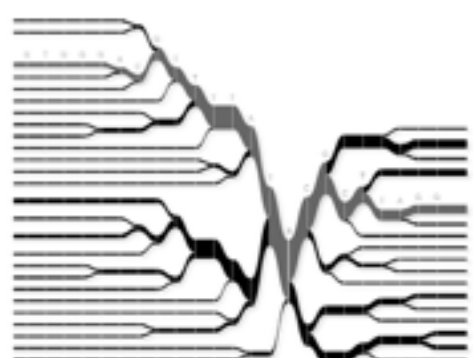
[Lan et al 2015]

[ESO/ L. Calçada/ M. Kornmesser 2017]

[Del Pozzo et al 2017, 2018]

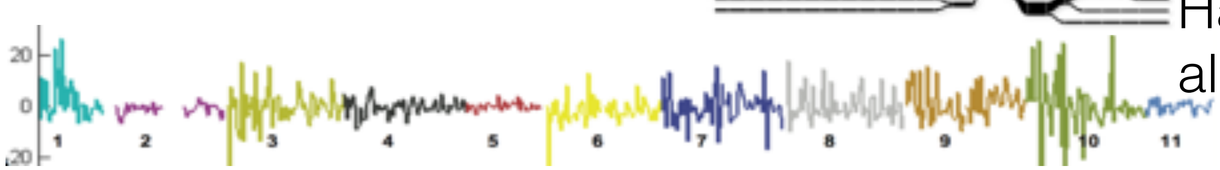
[Prabhakaran, Azizi, Carr, Pe'er 2016]





[Ewens 1972; Hartl, Clark 2003; Harris et al 2017]

[Saria et al 2010]



[Xu et al 2015]

[Cassidy et al 2015]

Roadmap

[slides, code:
www.tamarabroderick.com/tutorials.html]

- Example problem: clustering
- Example NPBayes model: Dirichlet process
- Chinese restaurant process
- Inference
- Venture further into the wild world of Nonparametric Bayes

- Big questions
  - Why NPBayes? Learn more as acquire more data
  - What does an infinite/growing number of parameters really mean (in NPBayes)? Components vs. clusters; latent vs. realized
  - Why is NPBayes challenging but practical? Infinite dimensional parameter, but finitely many parameters realized

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