Gaussian Processes for Regression: Models, Algorithms, and Applications

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Associate Professor
MIT

http://www.tamarabroderick.com/tutorials.html

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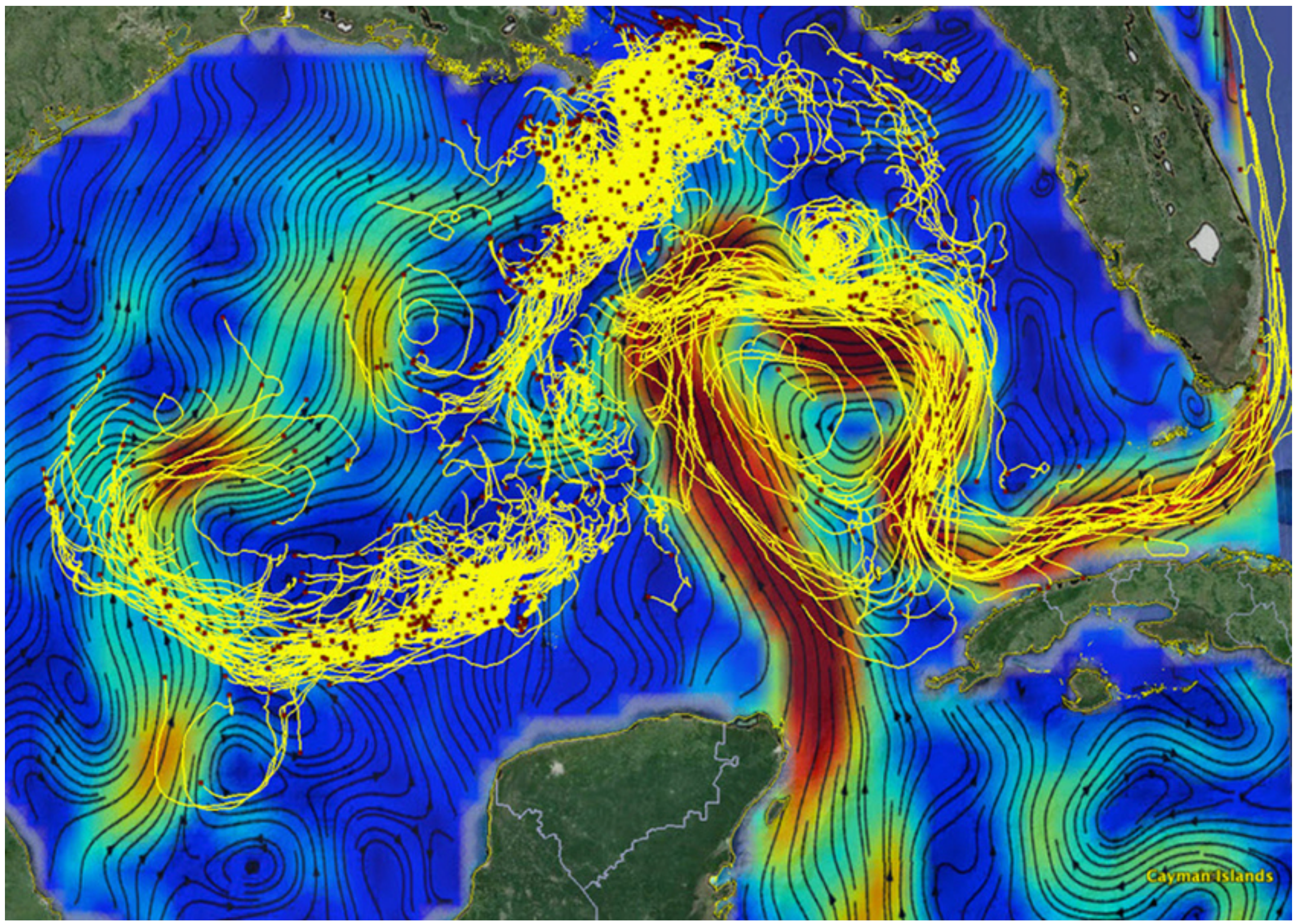

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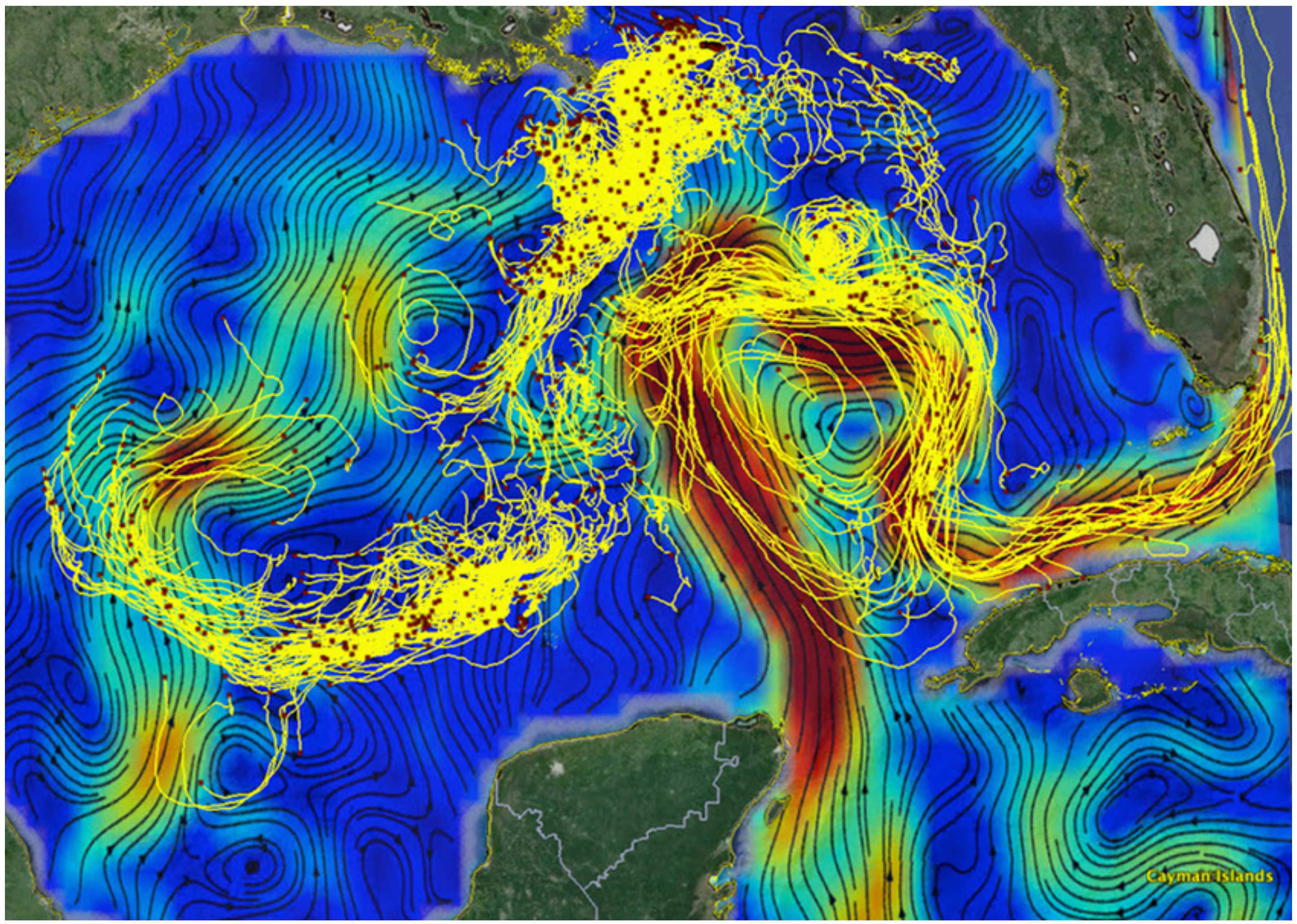

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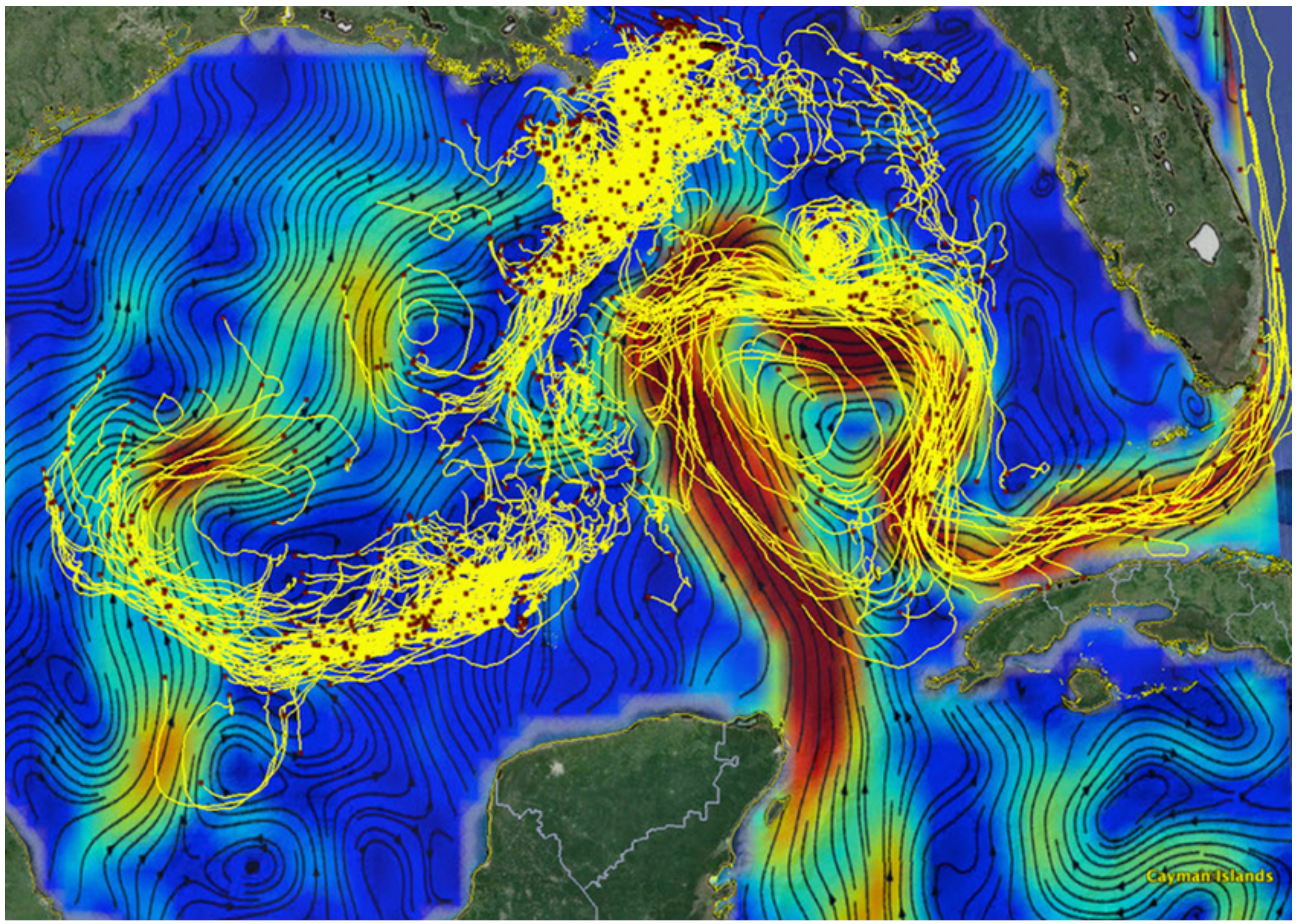

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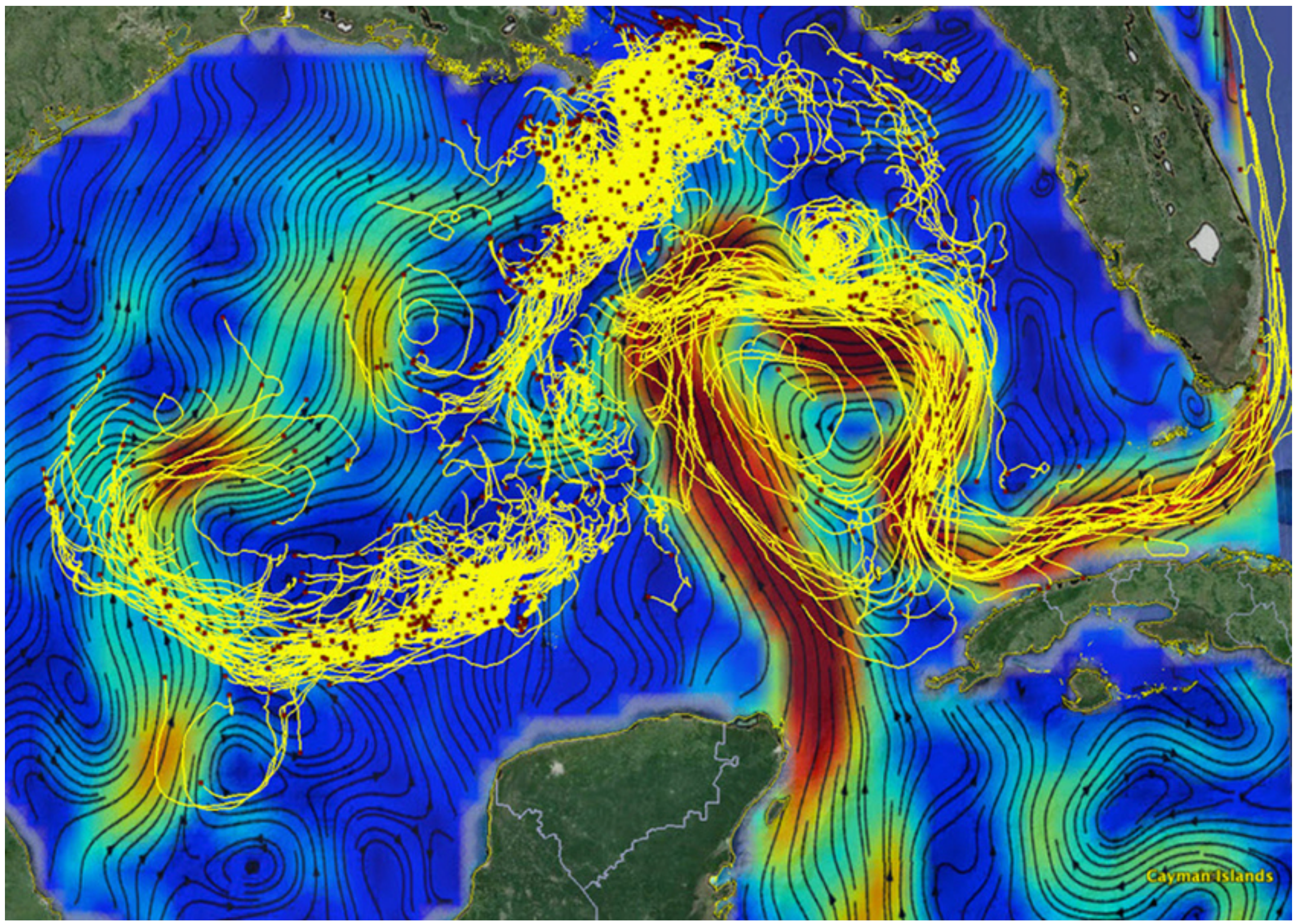

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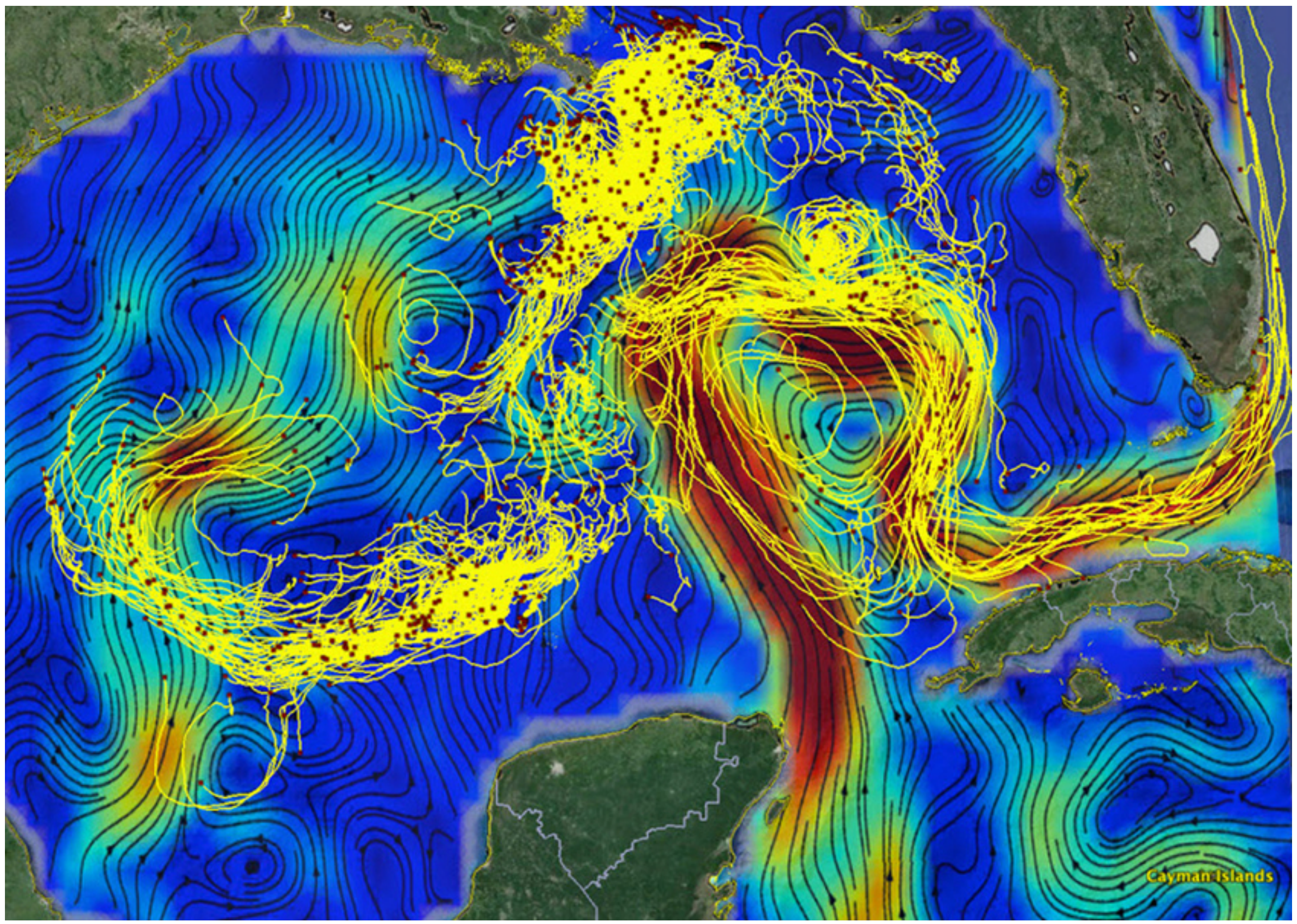

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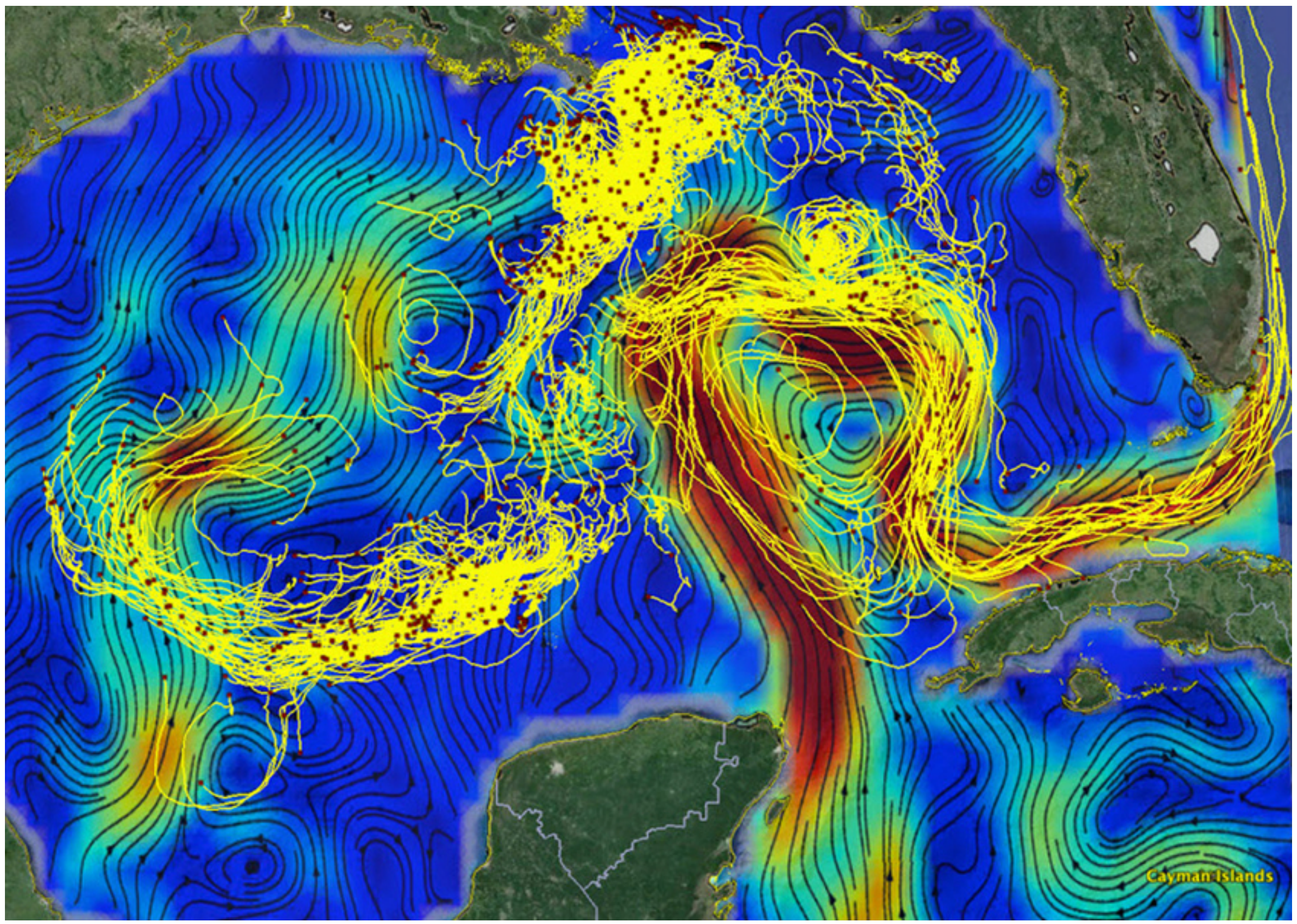

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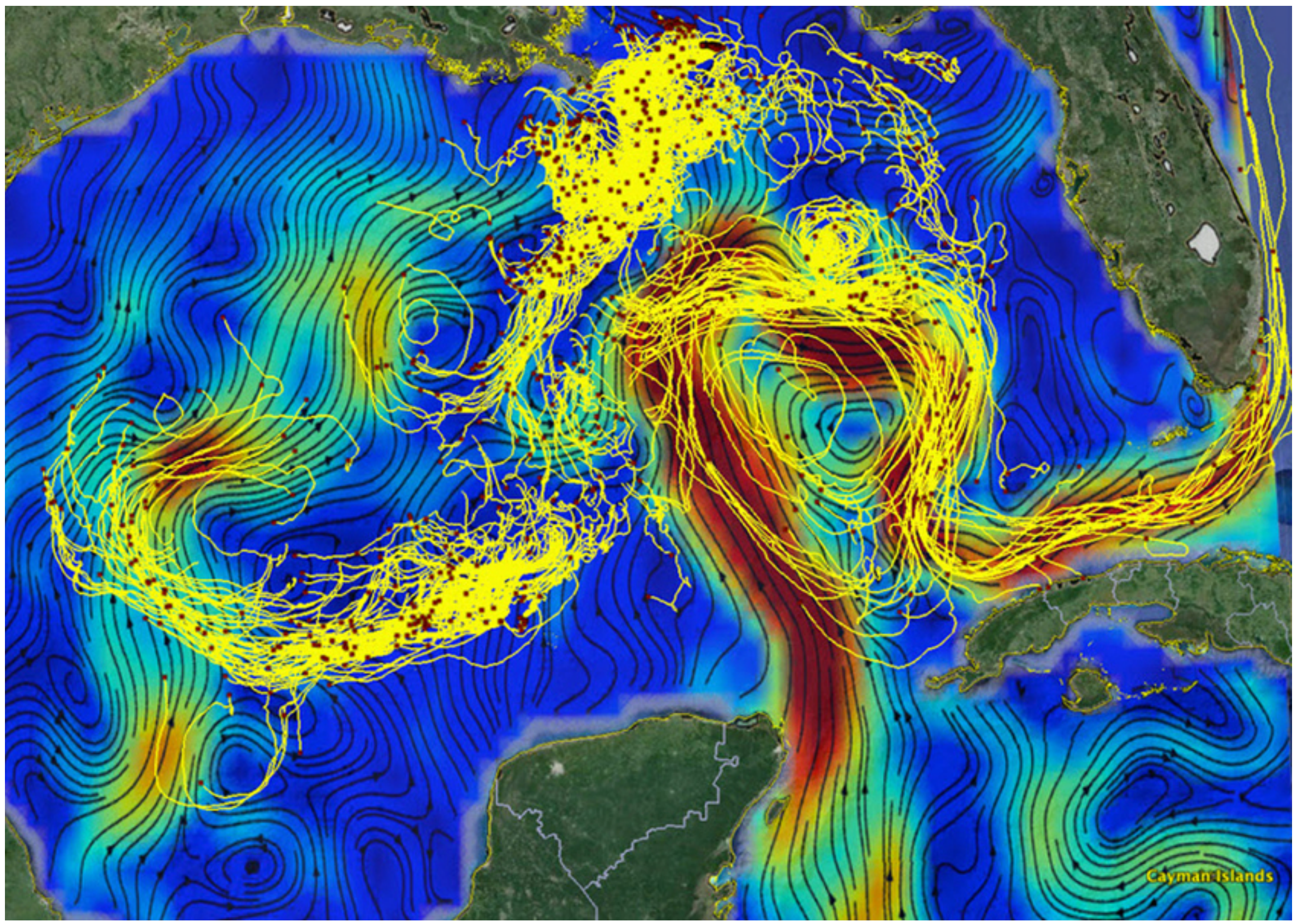

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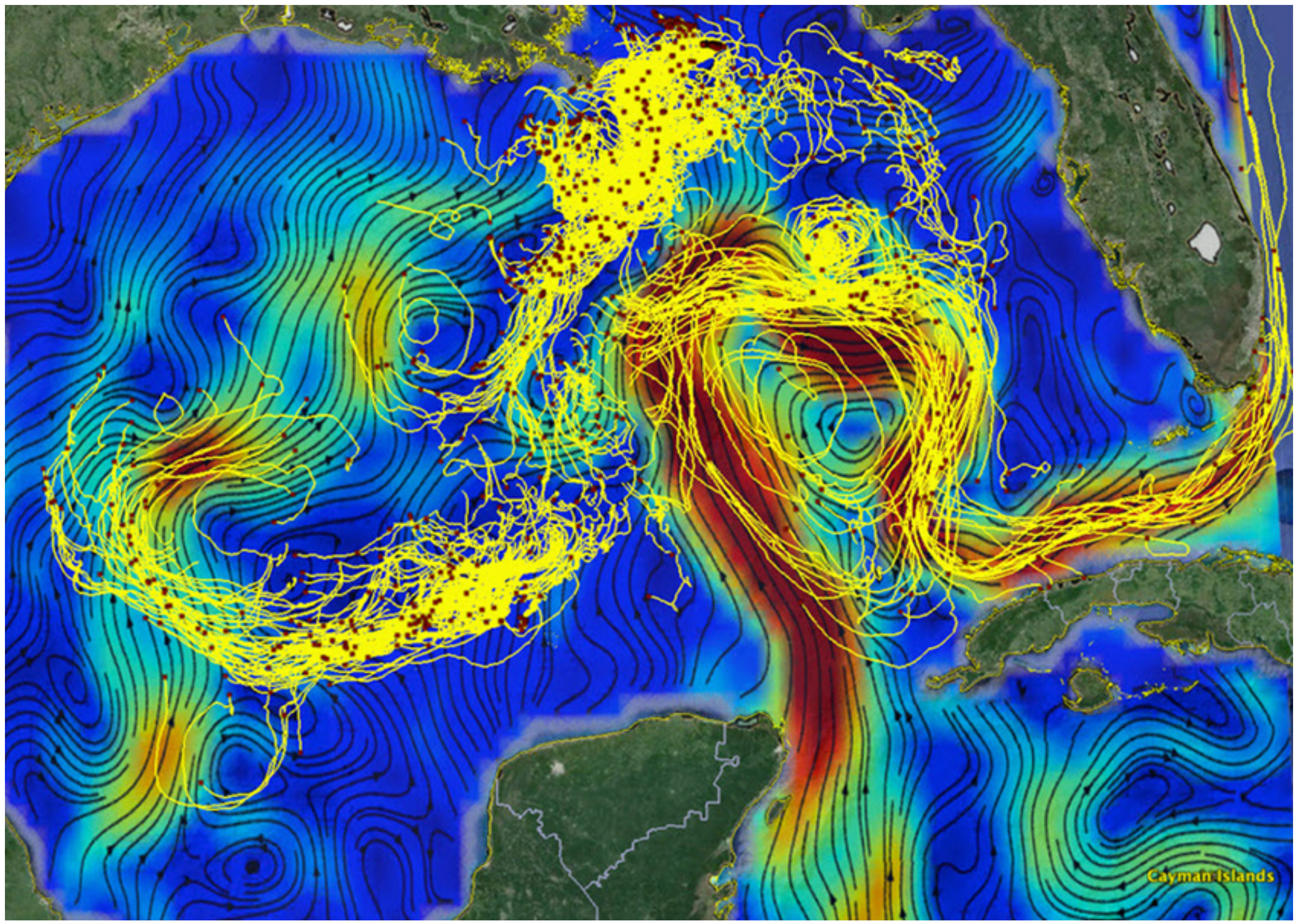

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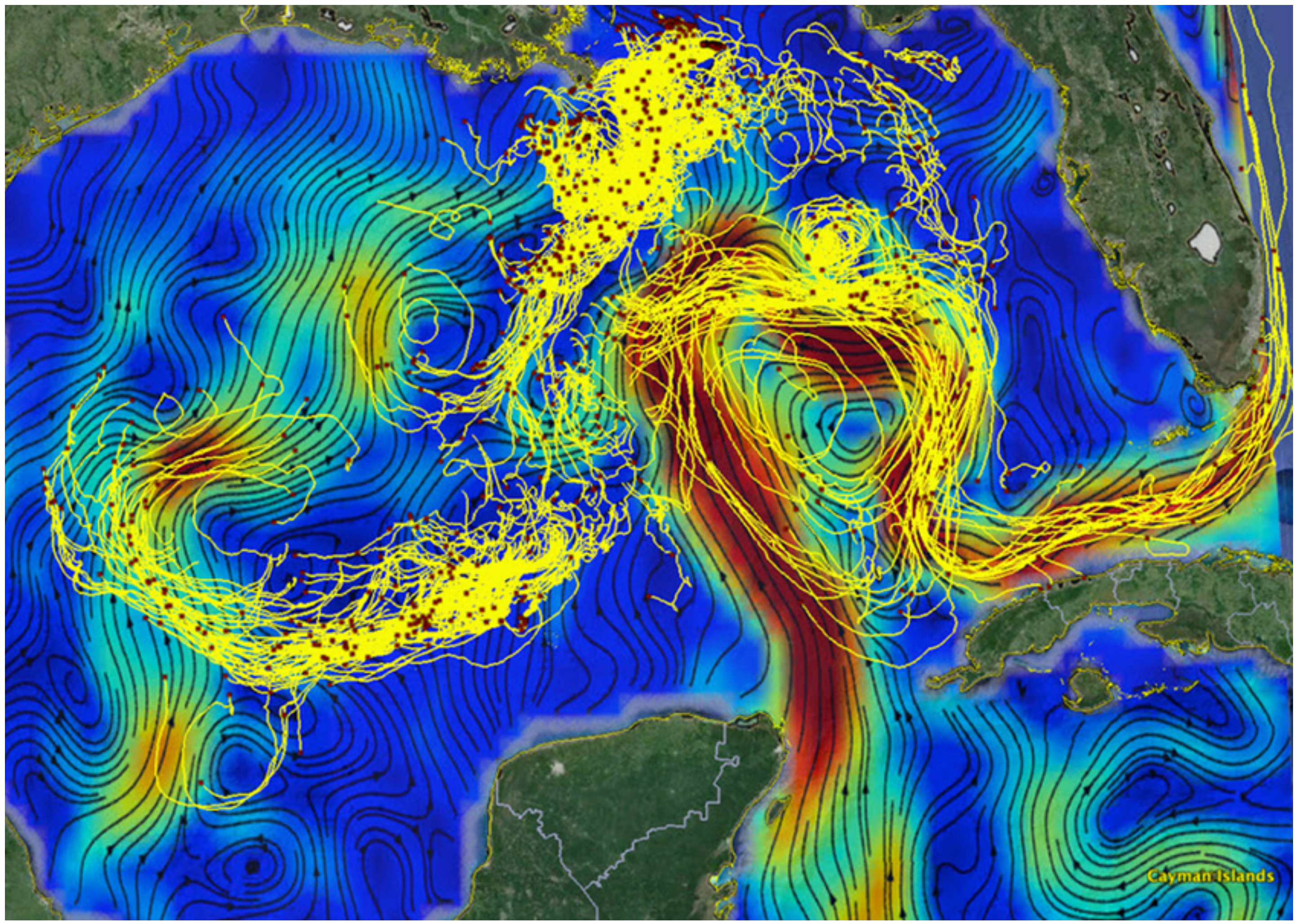

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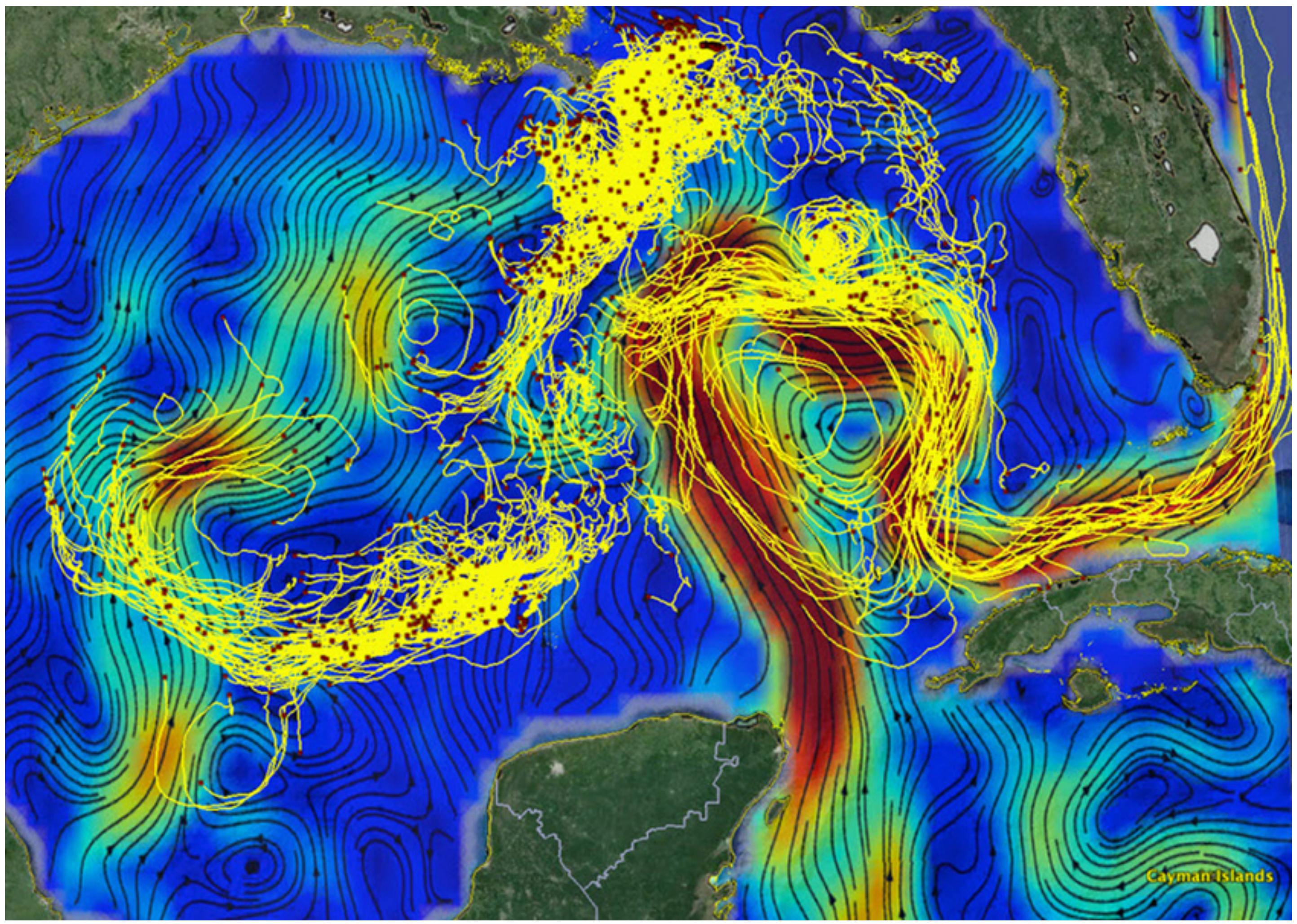

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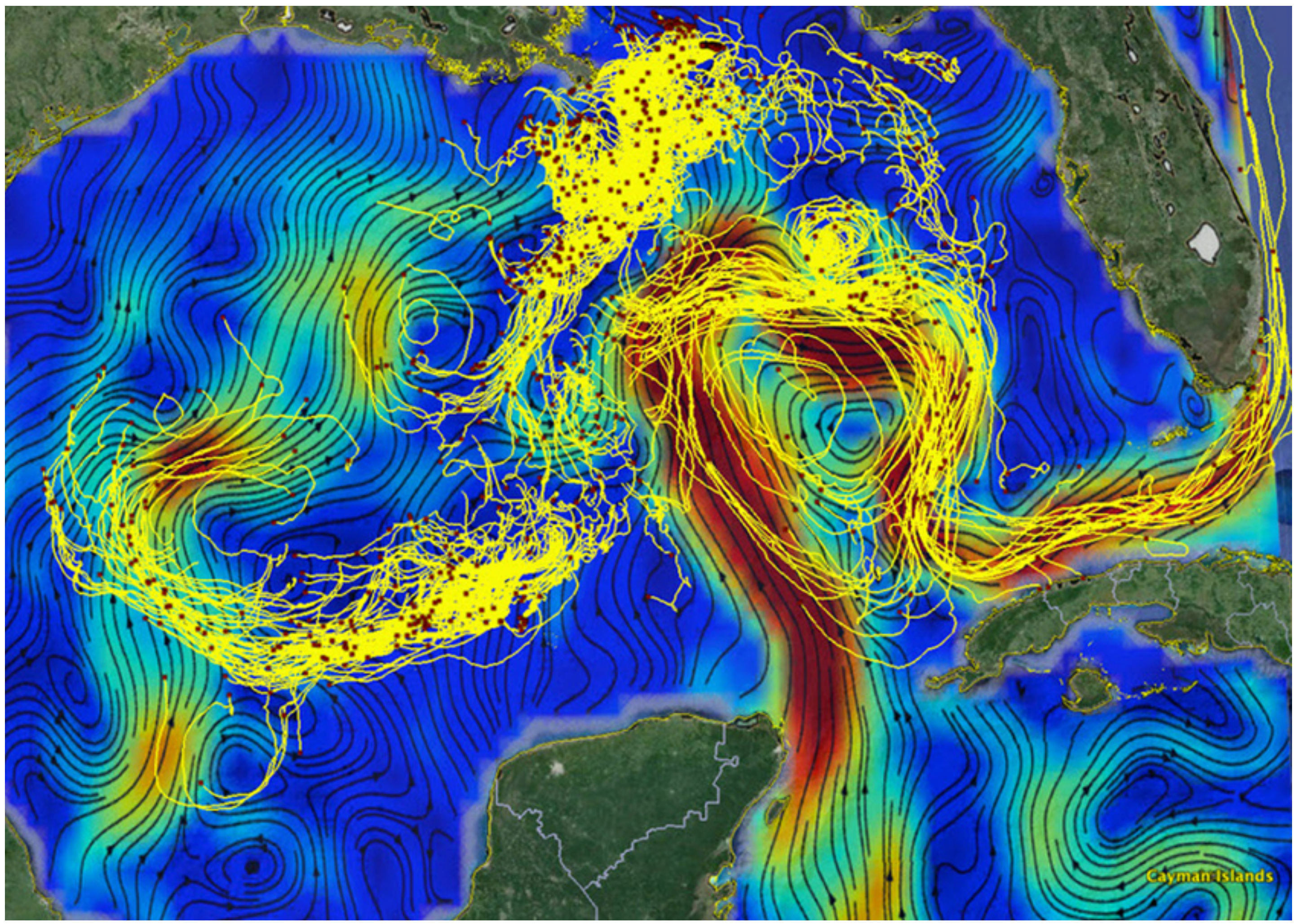

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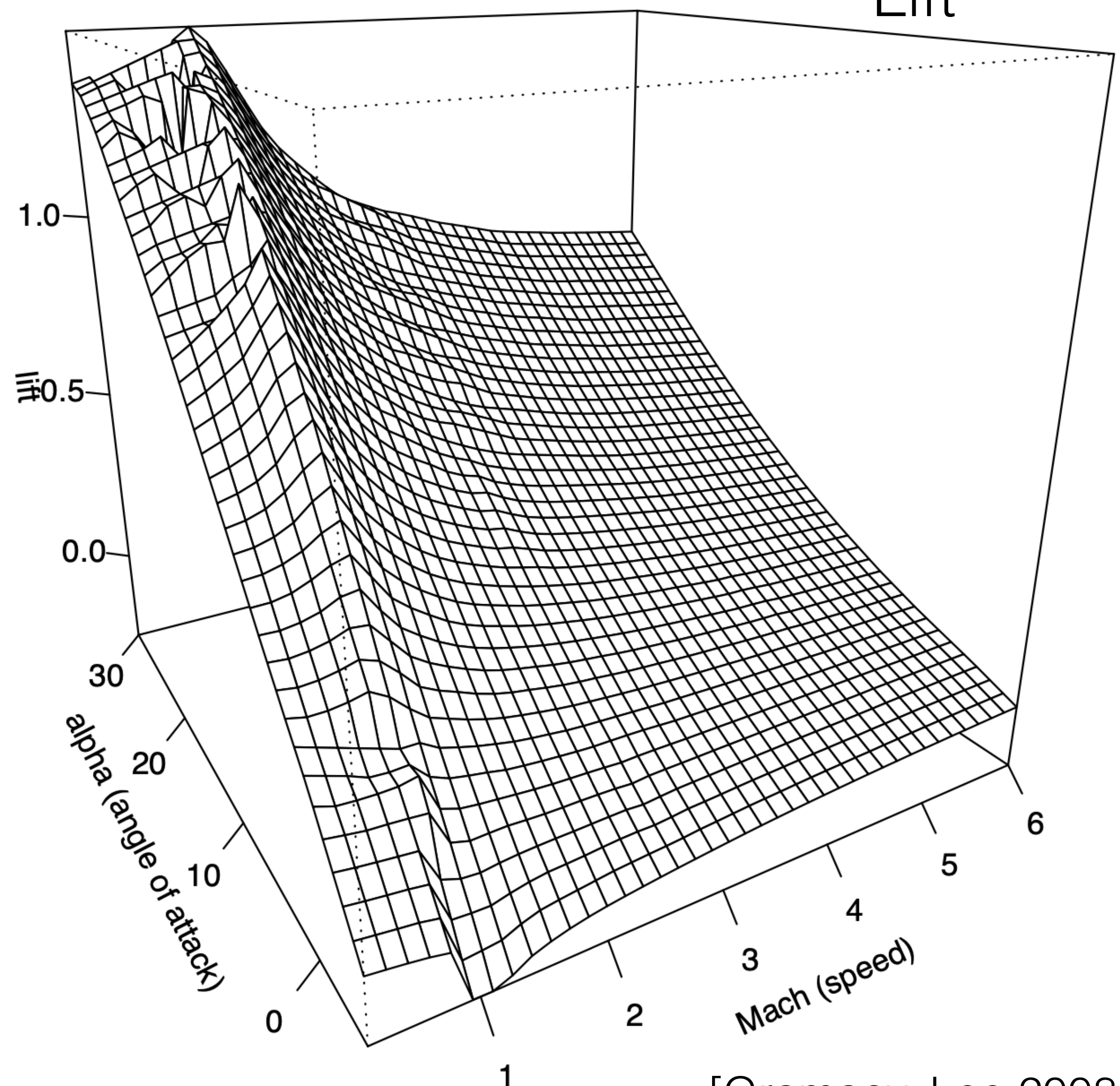

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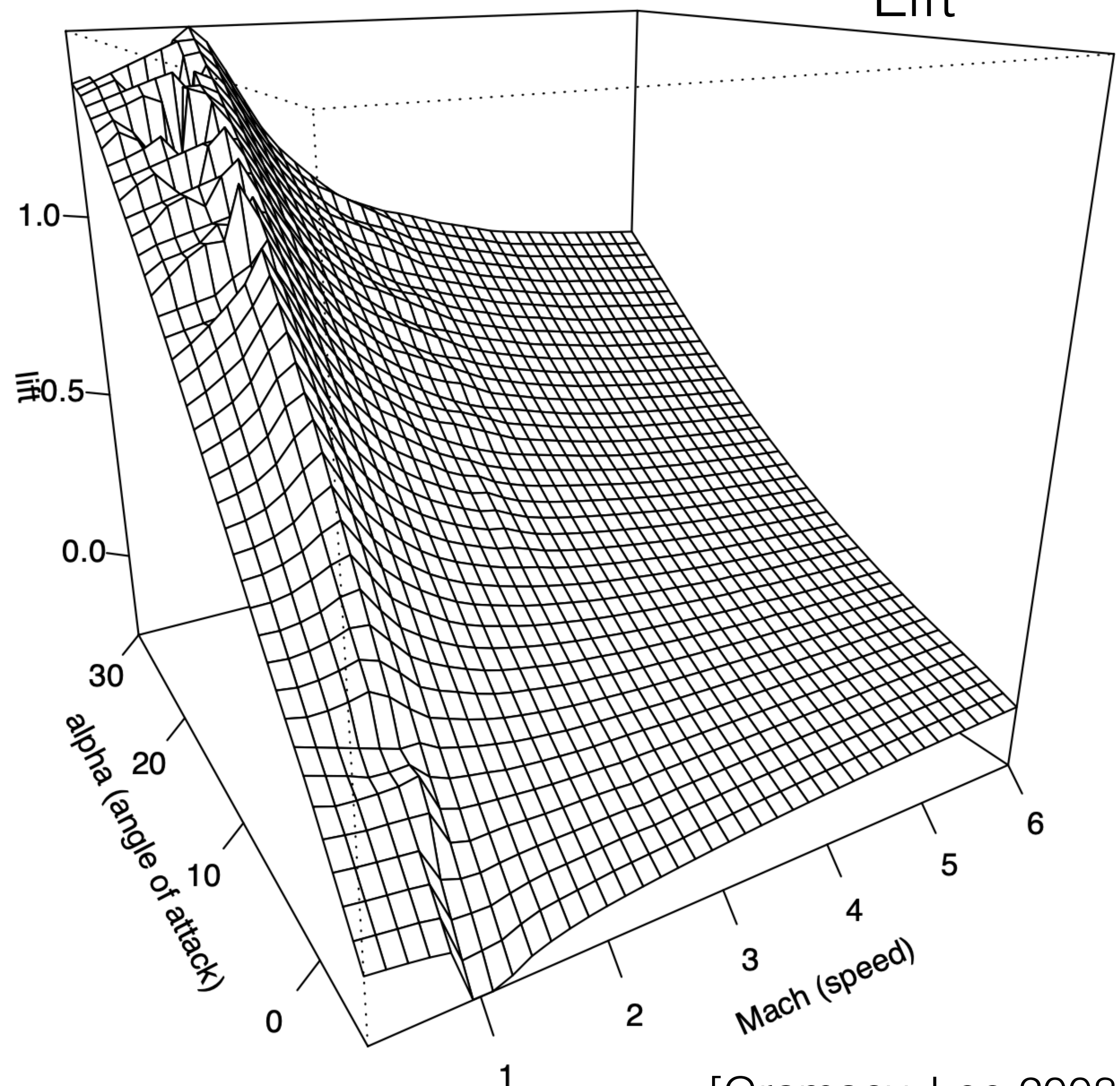

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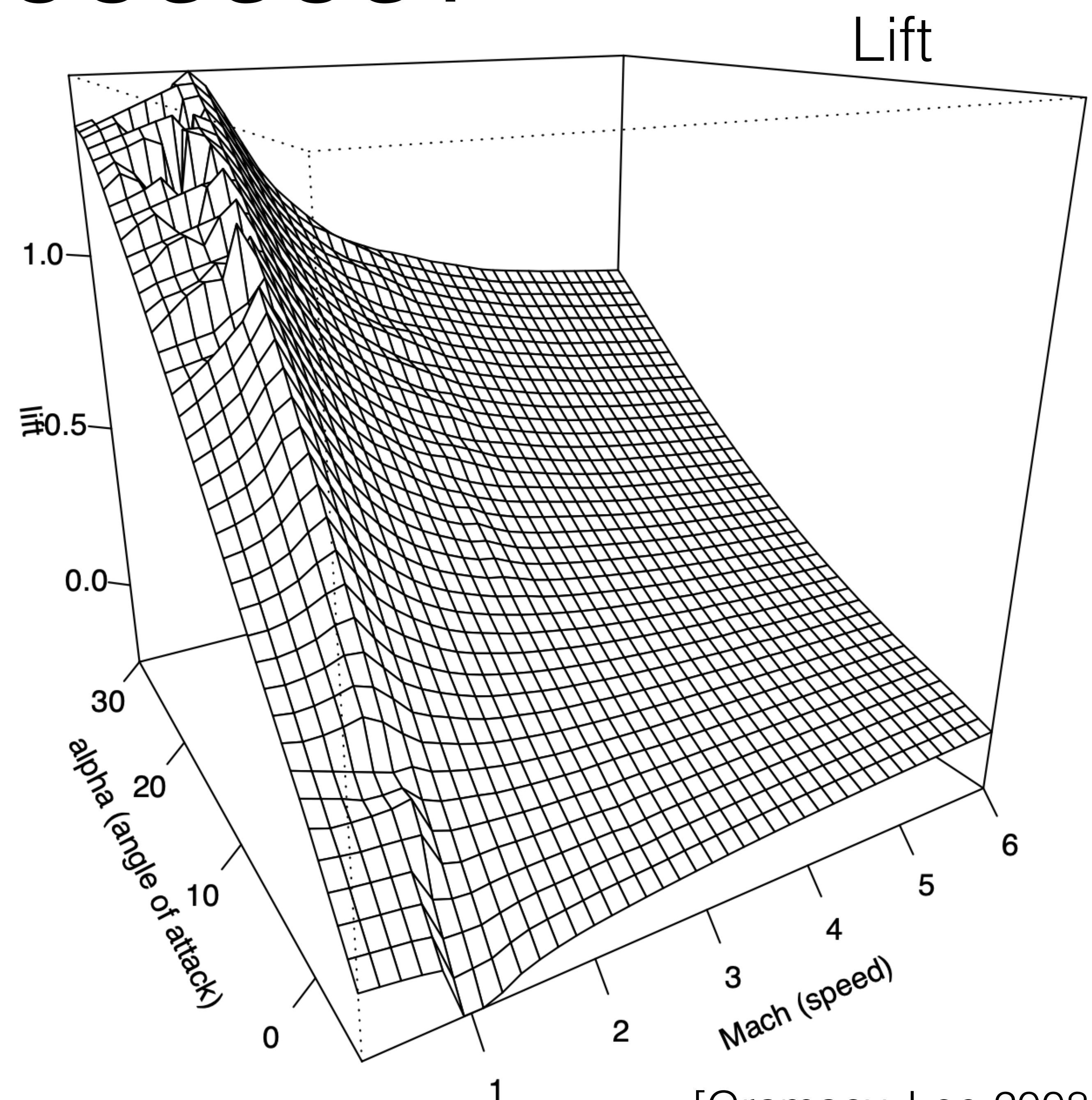

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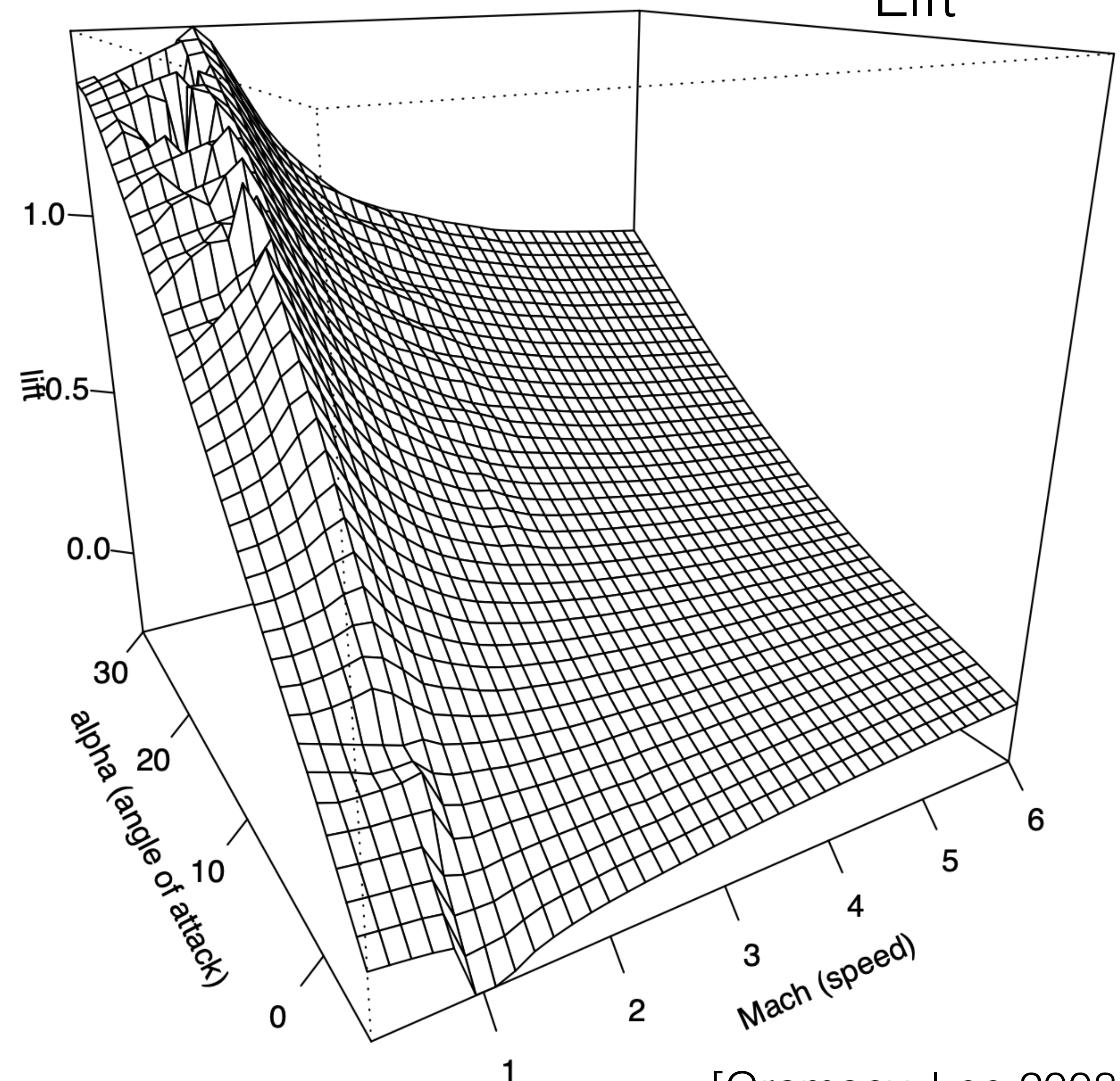

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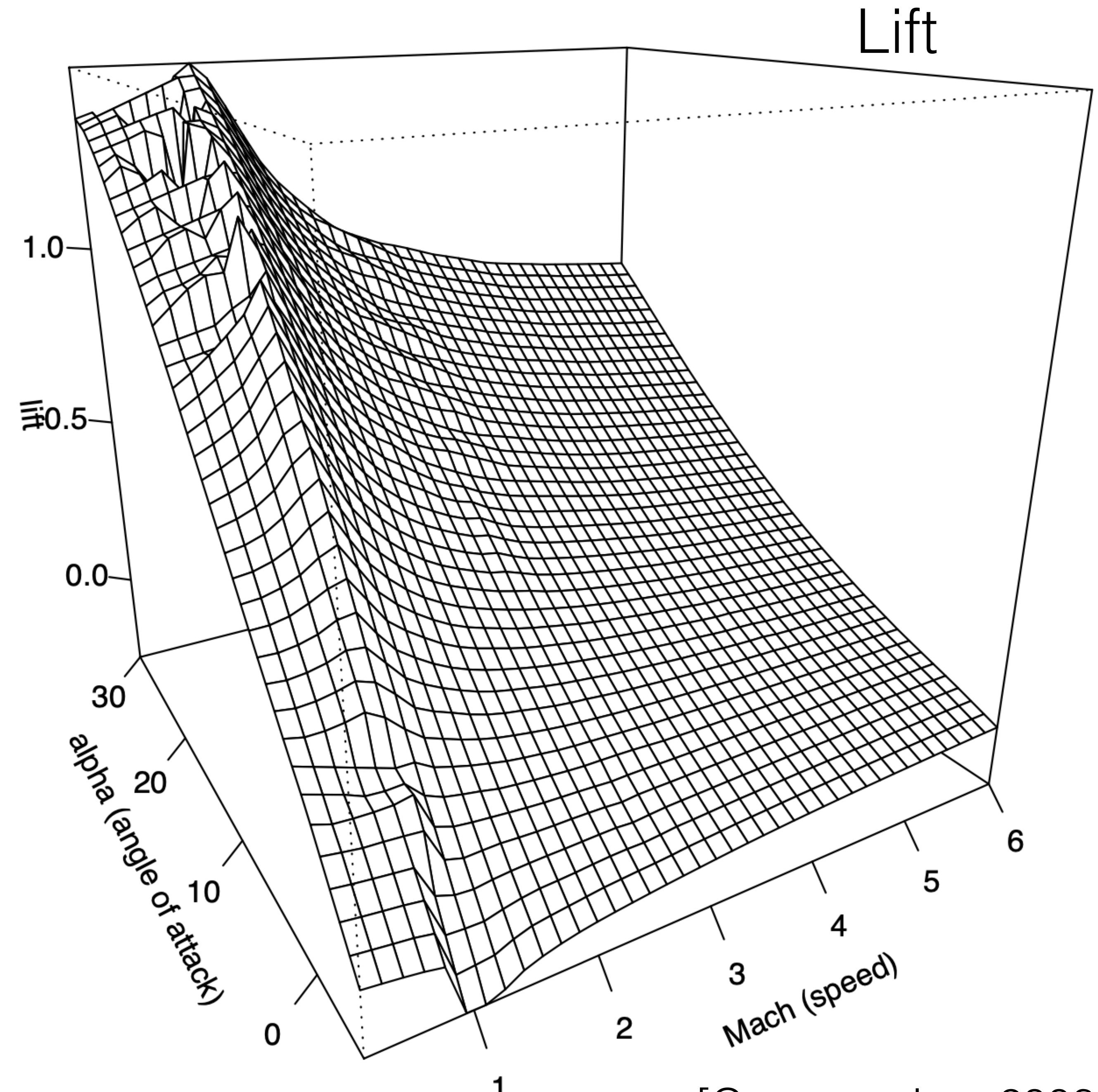

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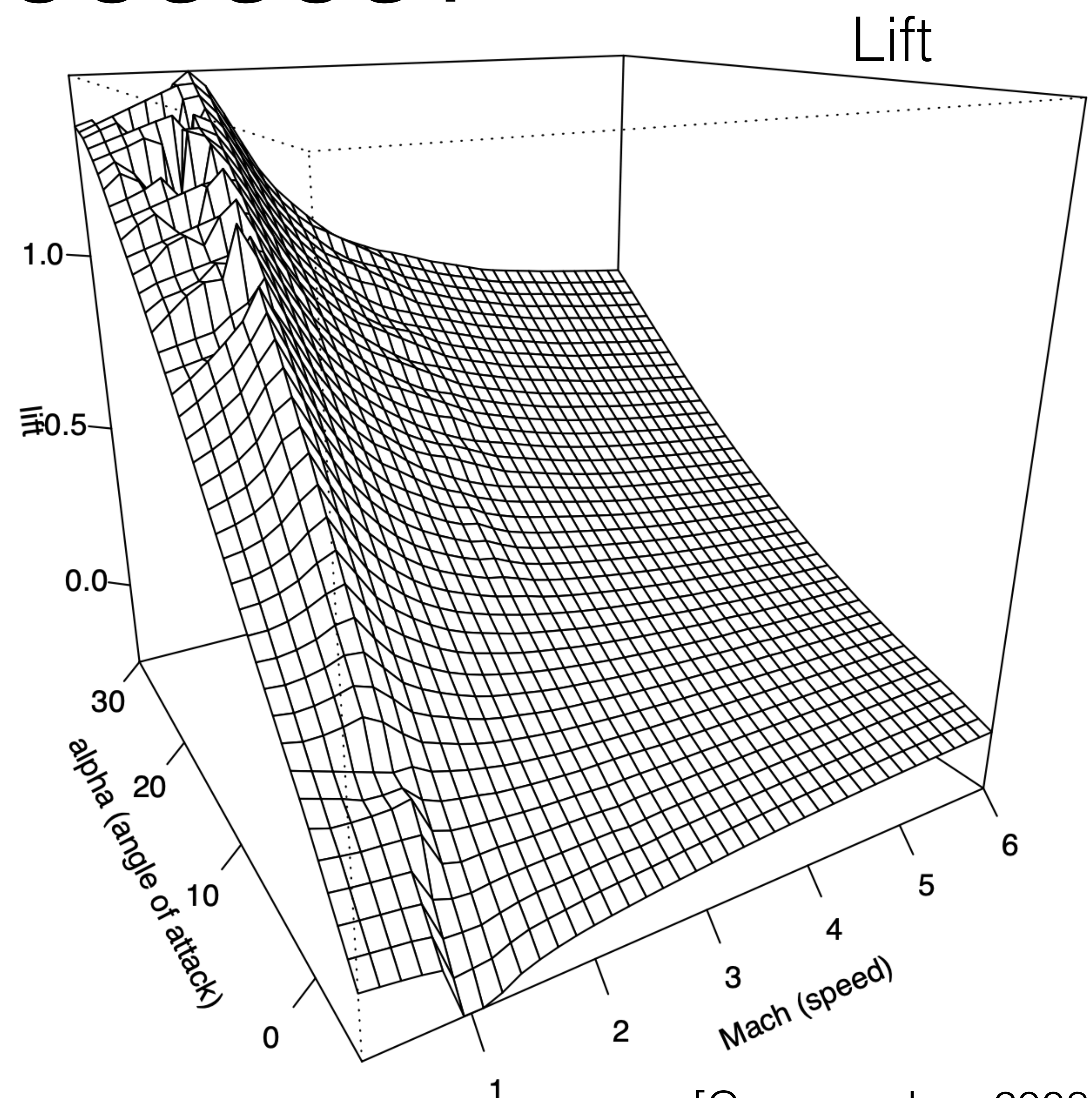

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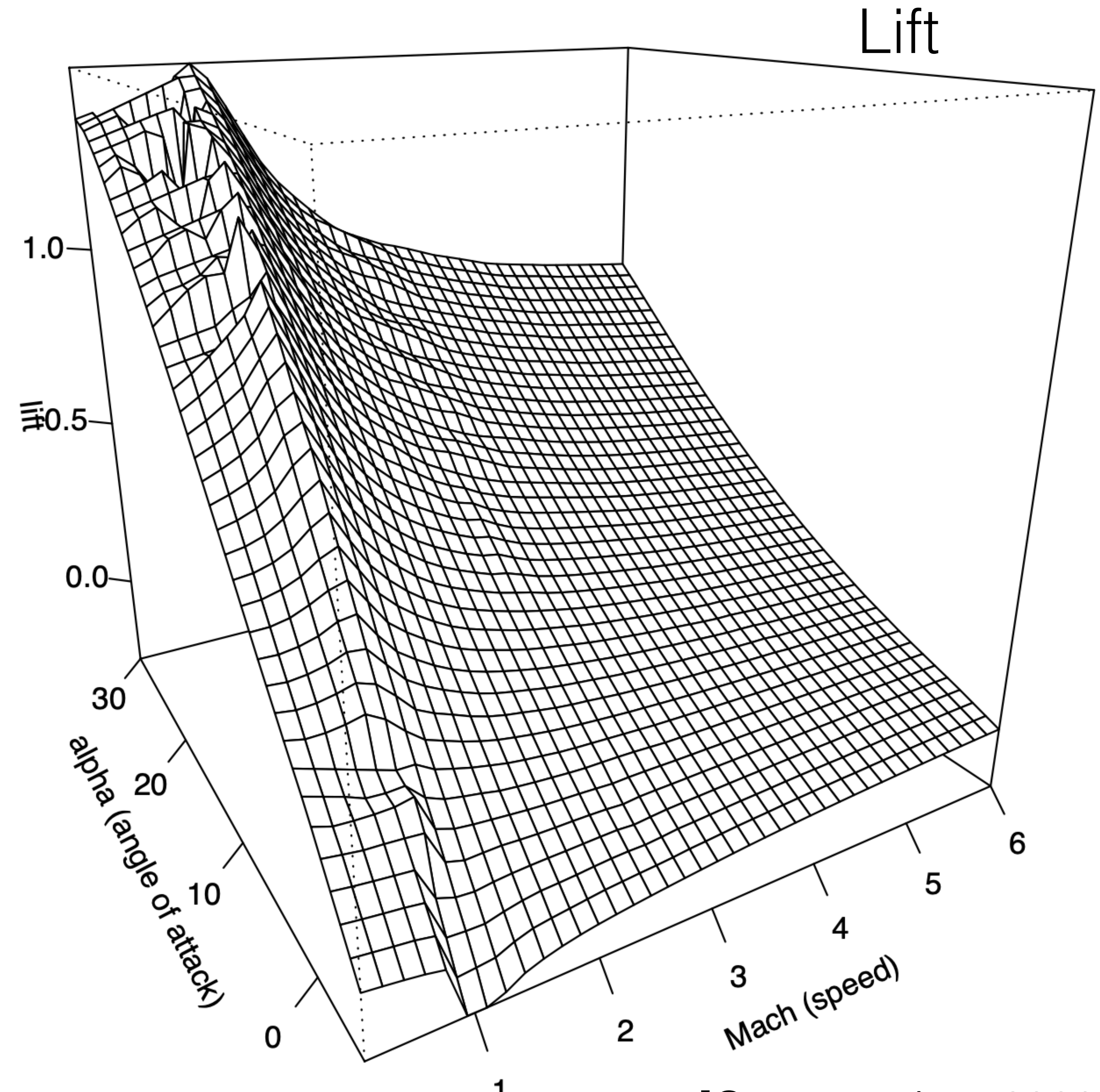

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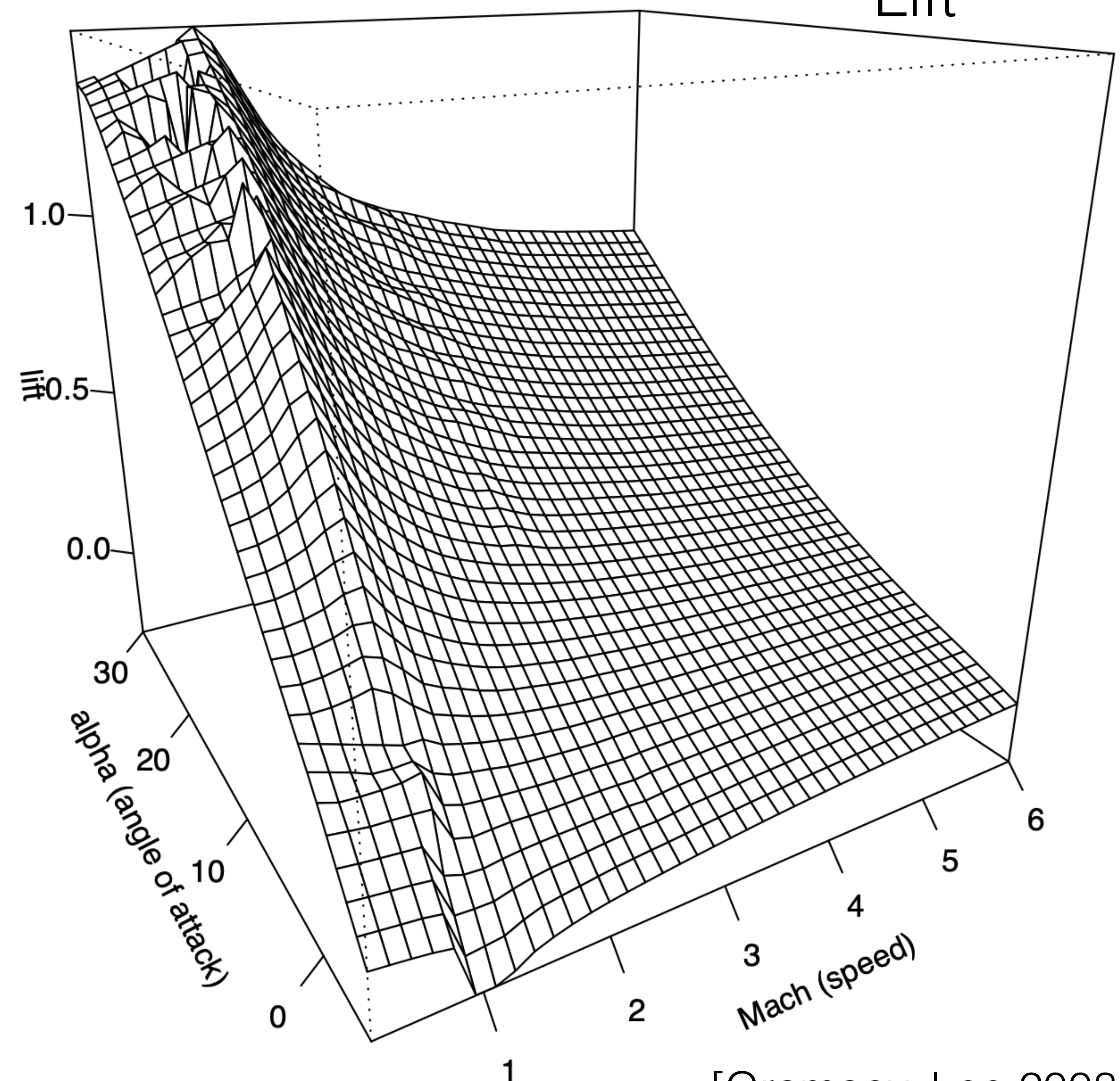

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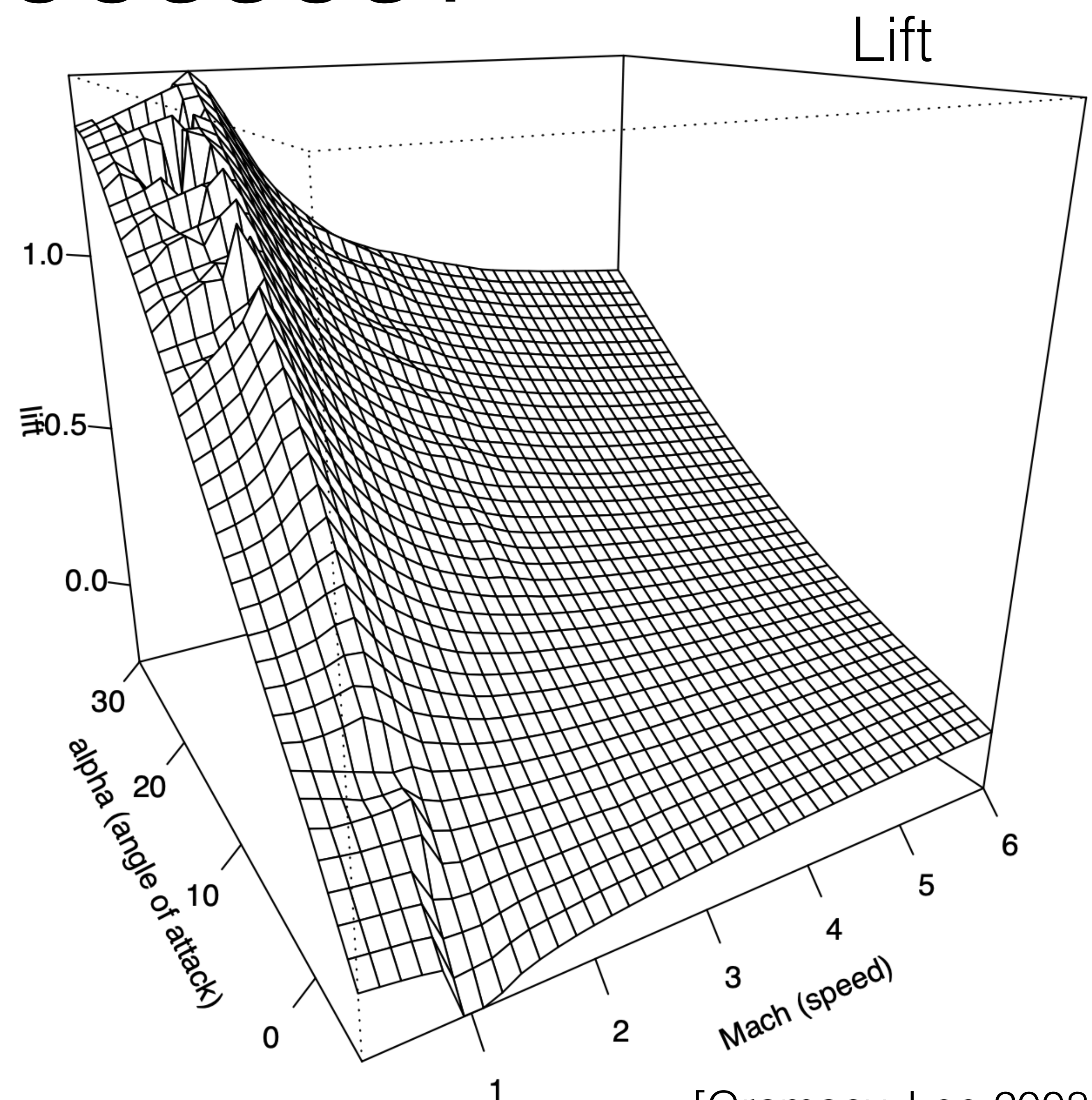

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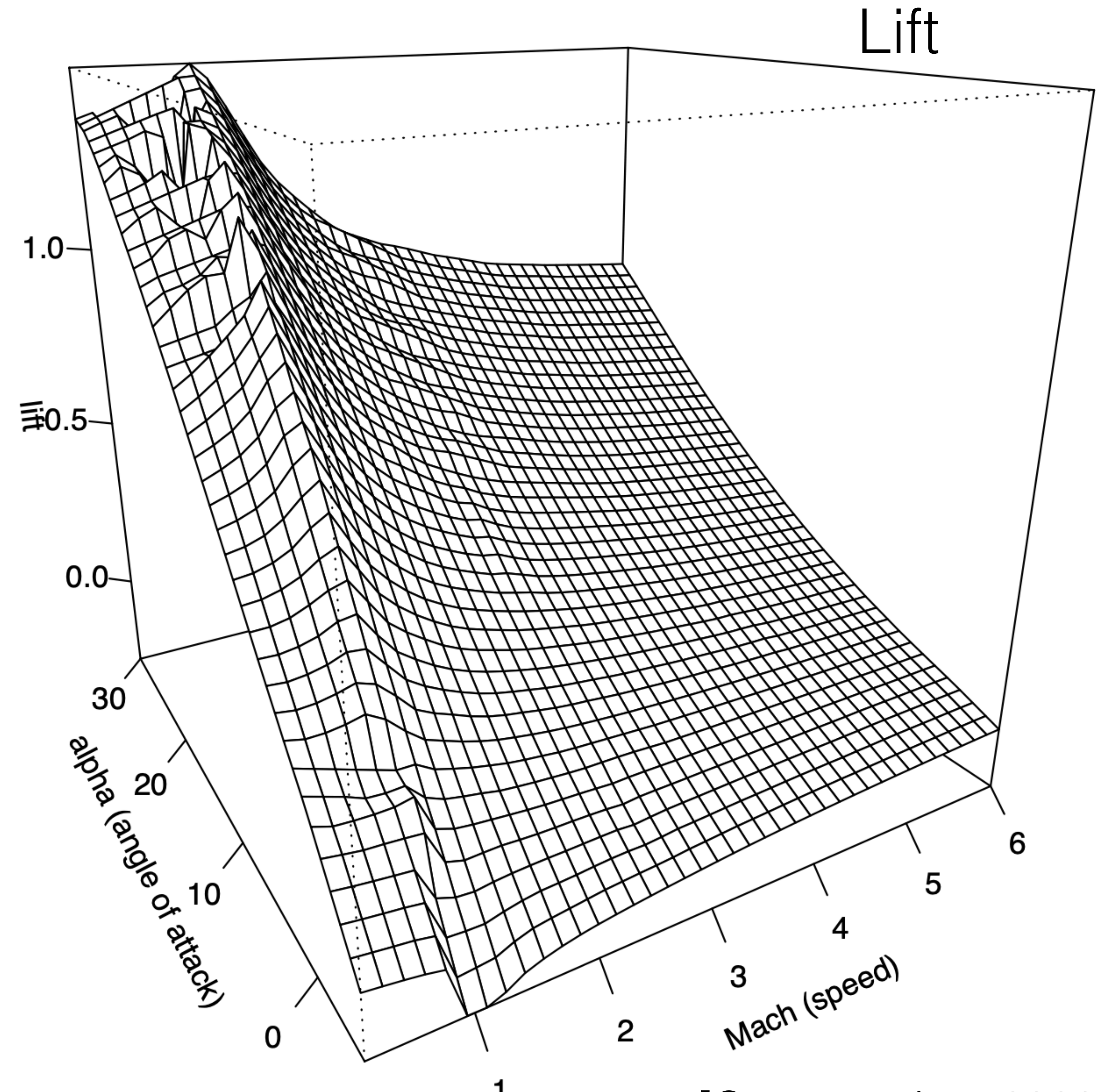

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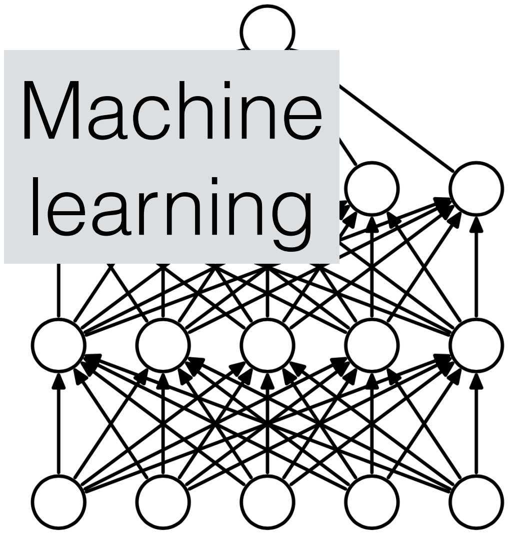

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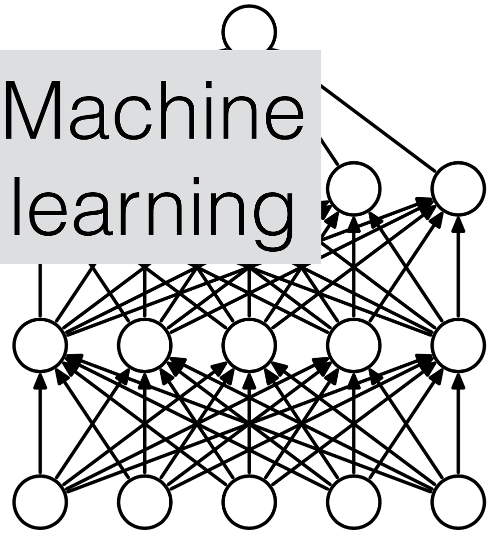

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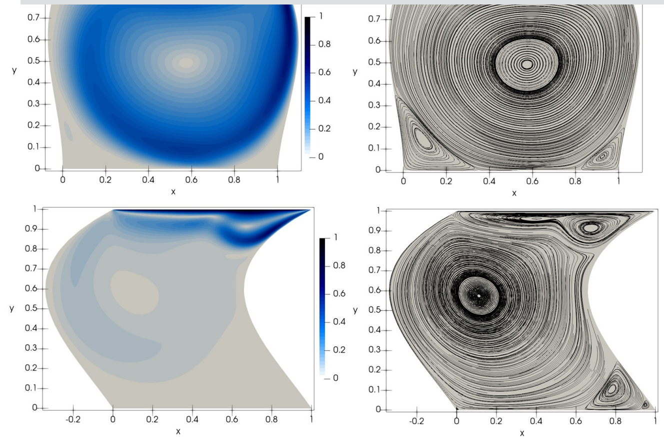

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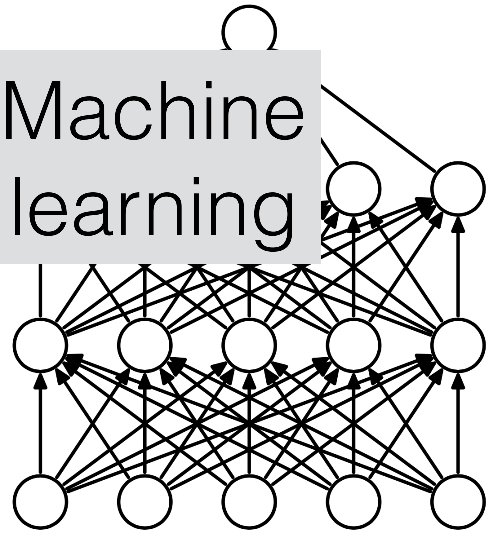

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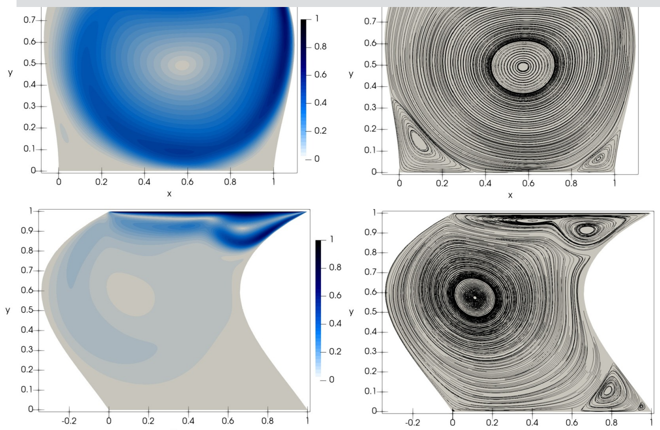

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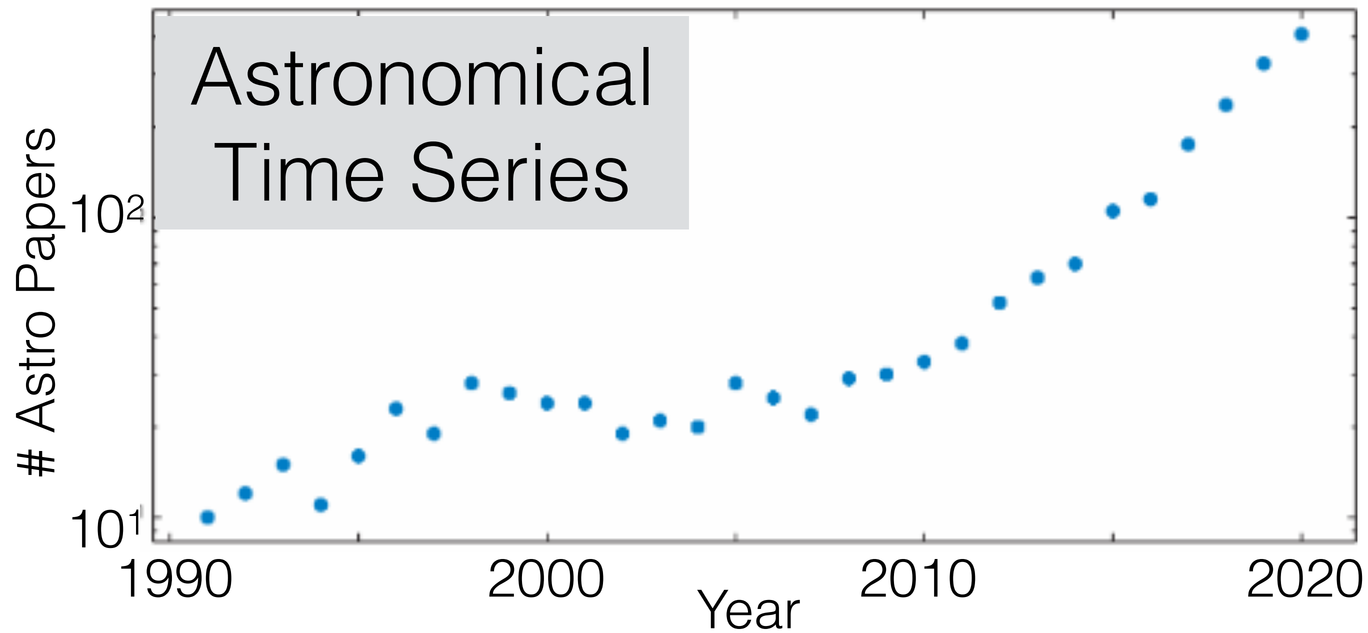
Astrophysics Data System search for “Gaussian process”

Astronomical Time Series



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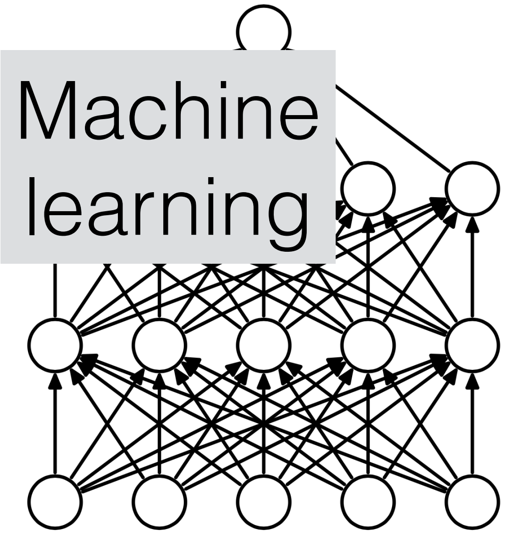

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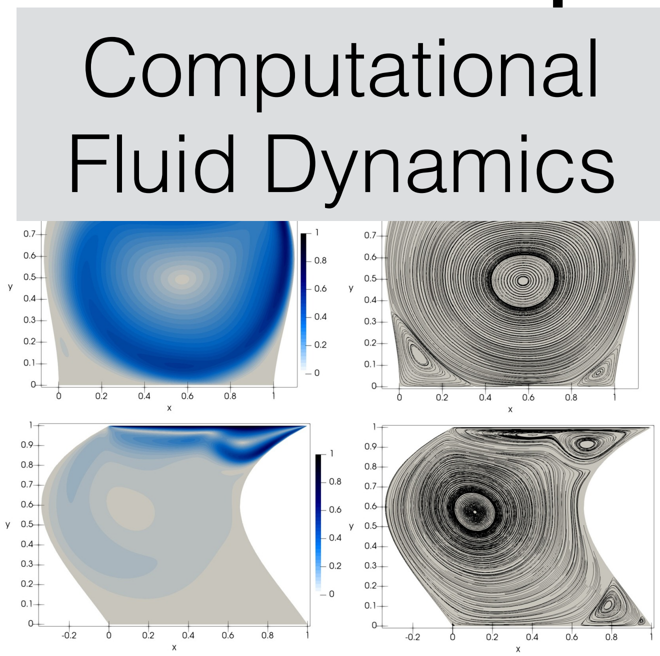

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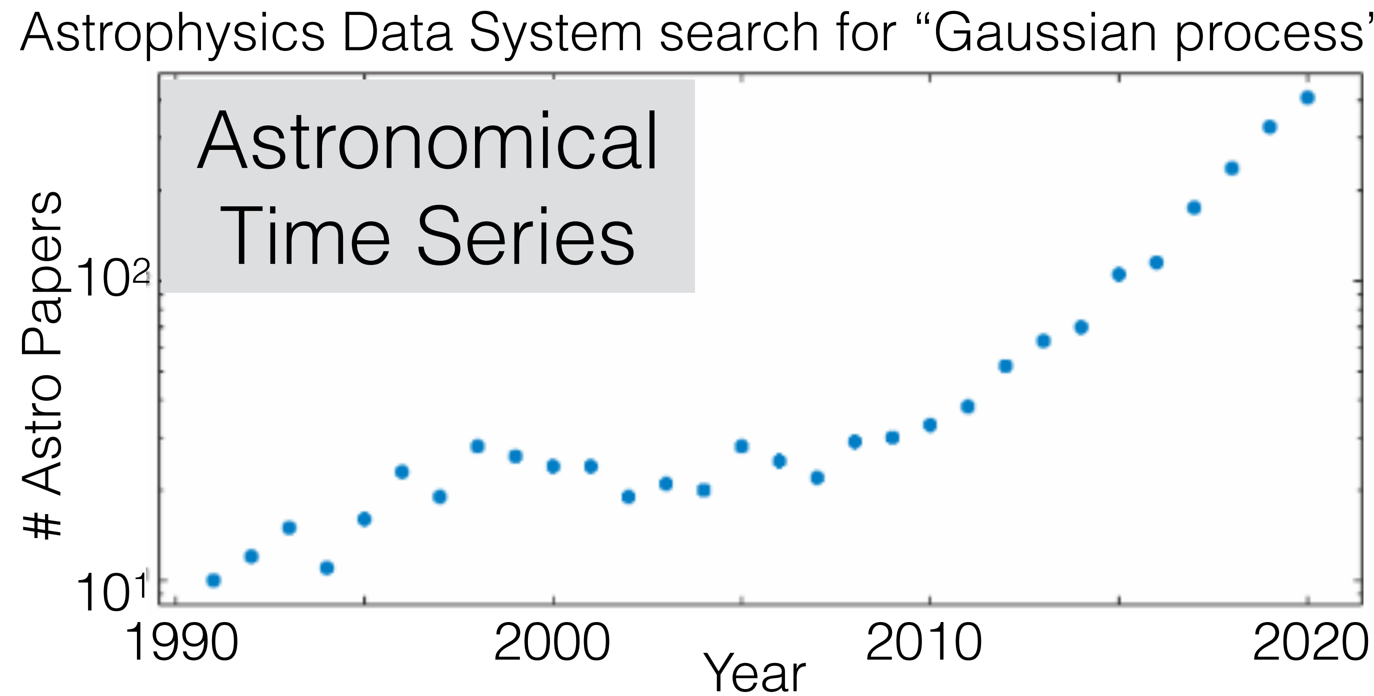

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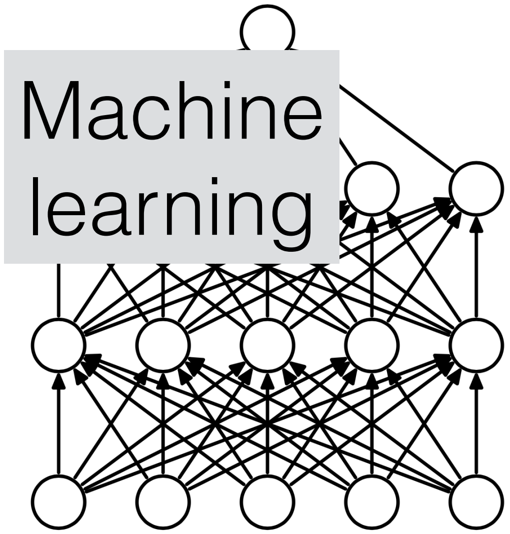

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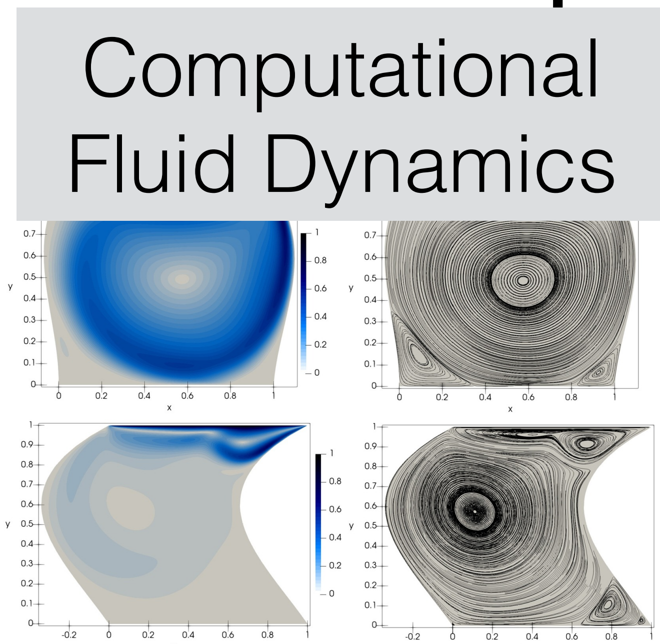

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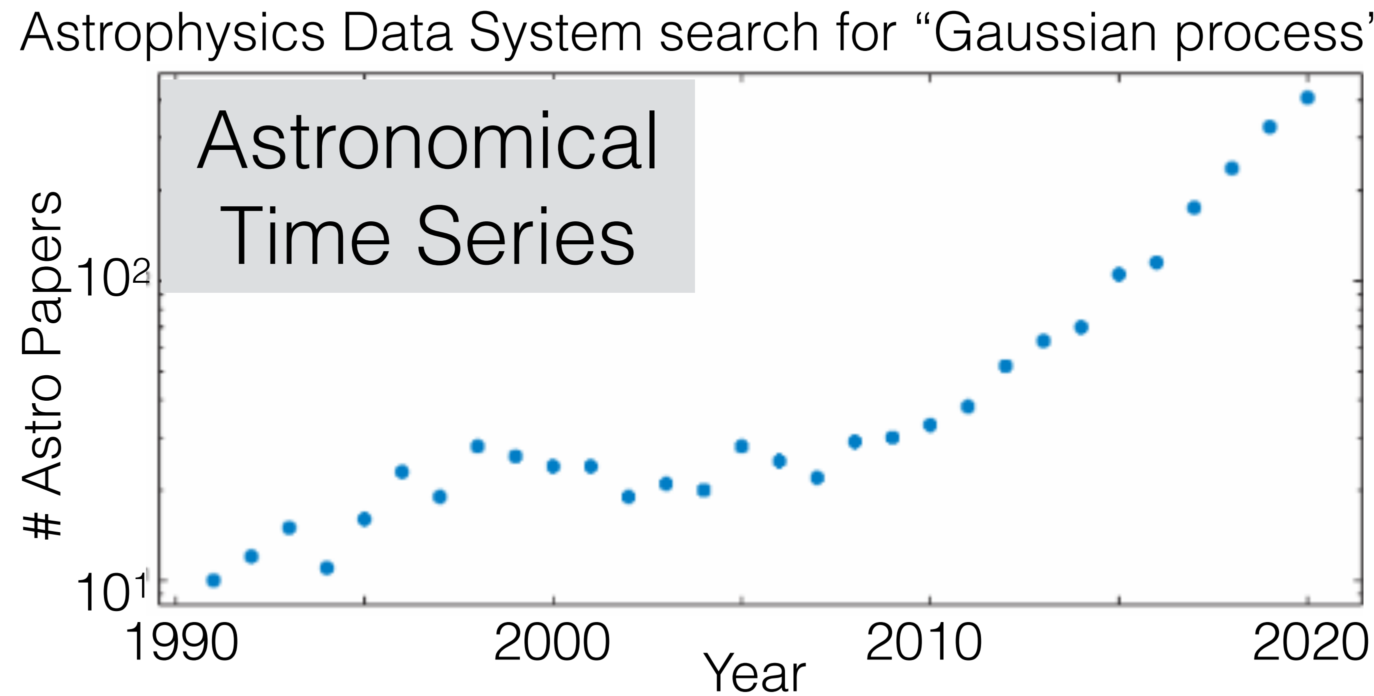

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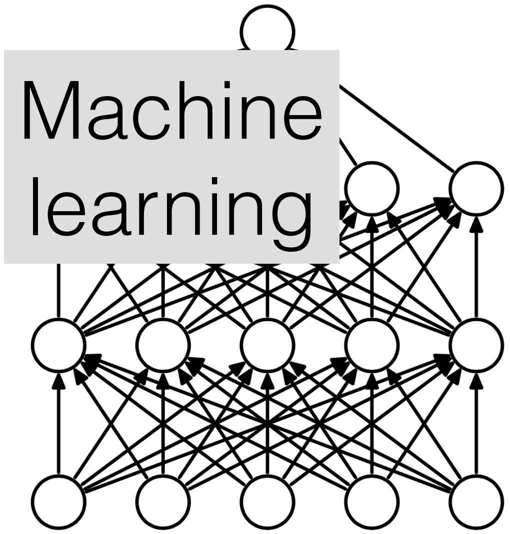

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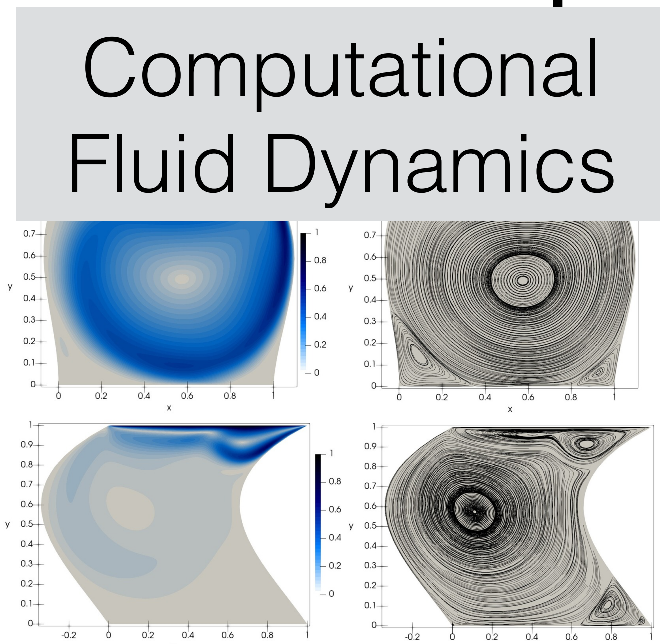

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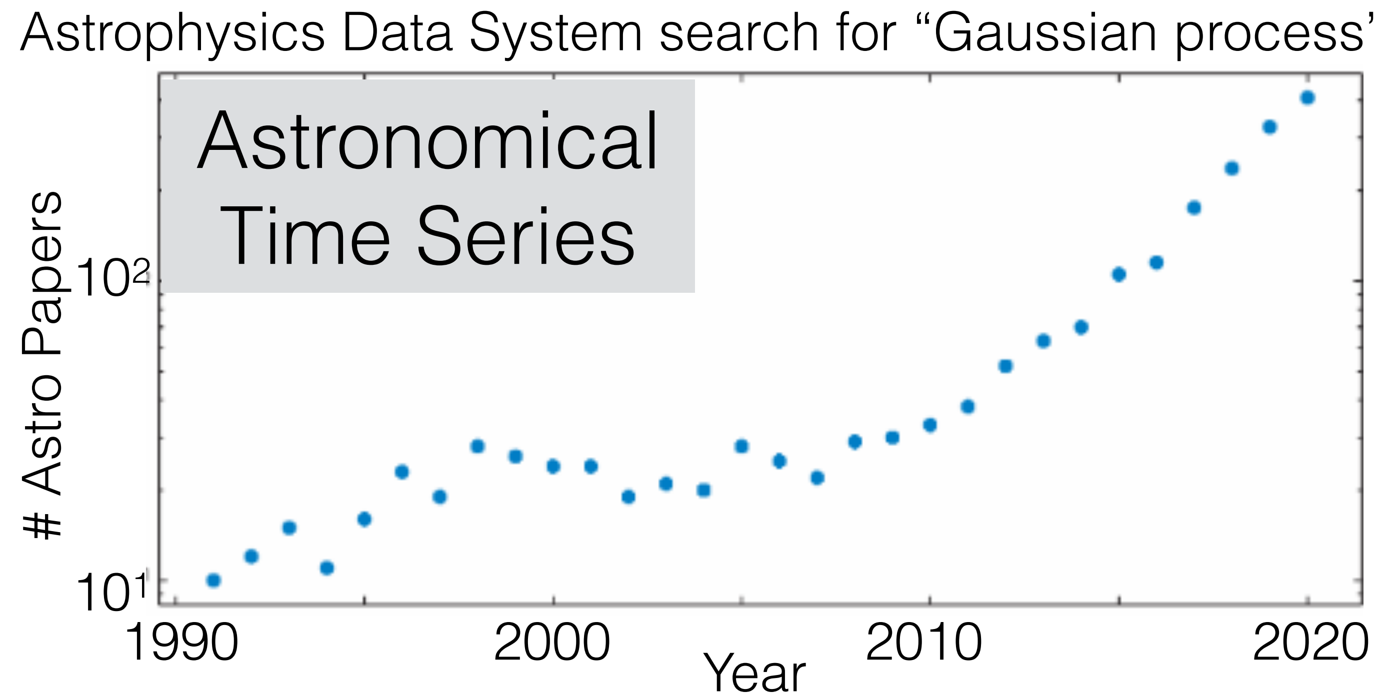

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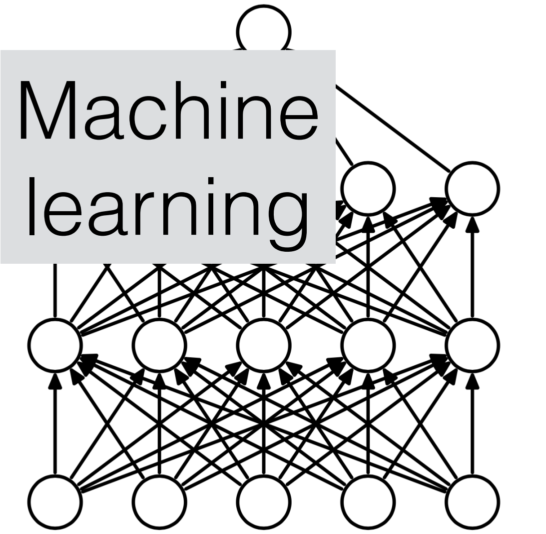

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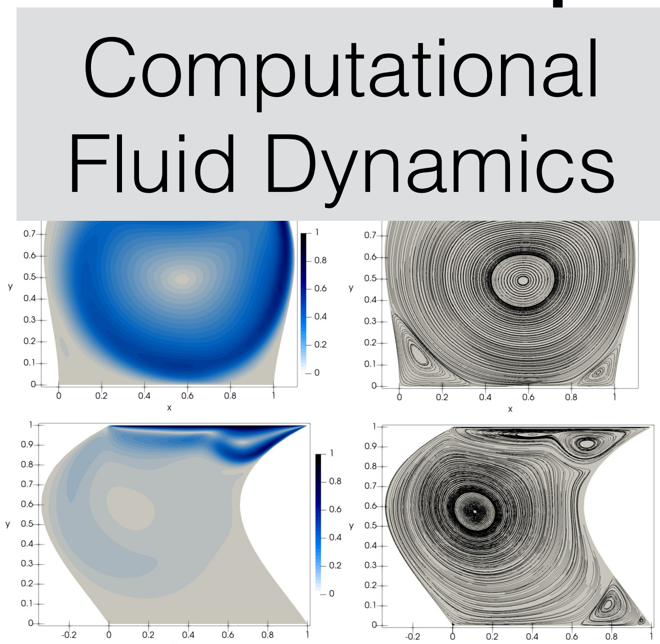

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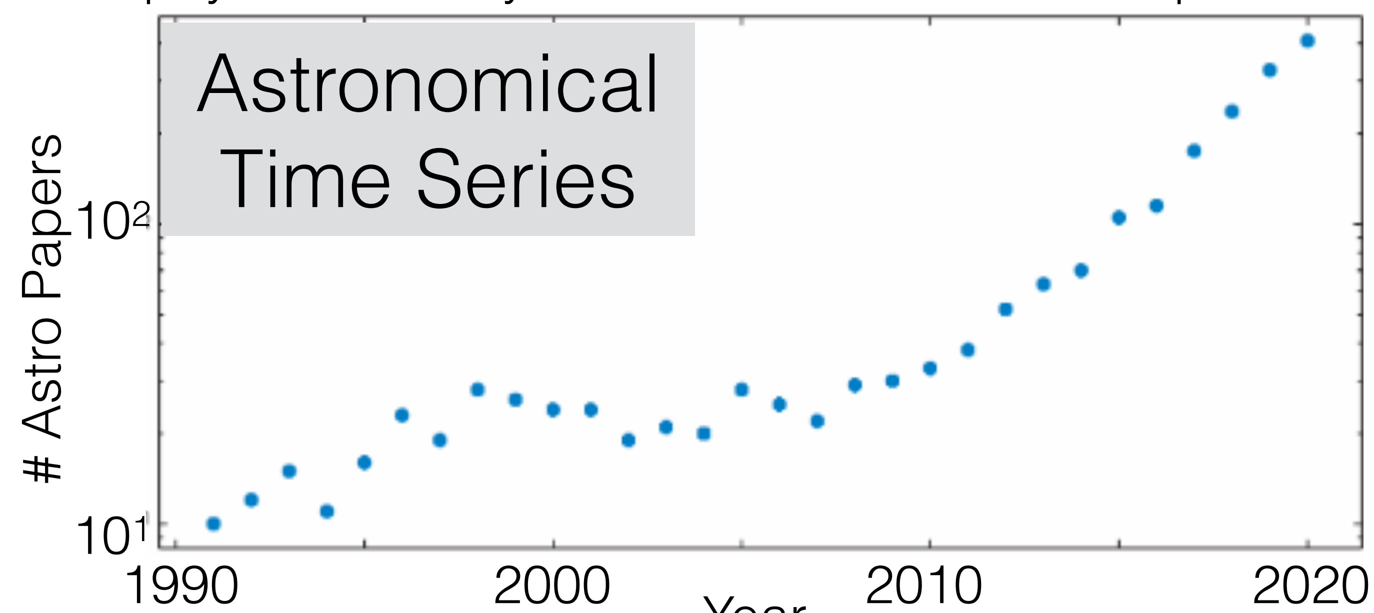

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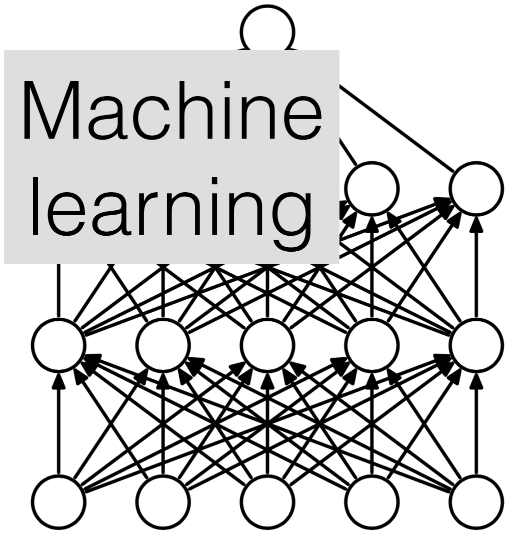

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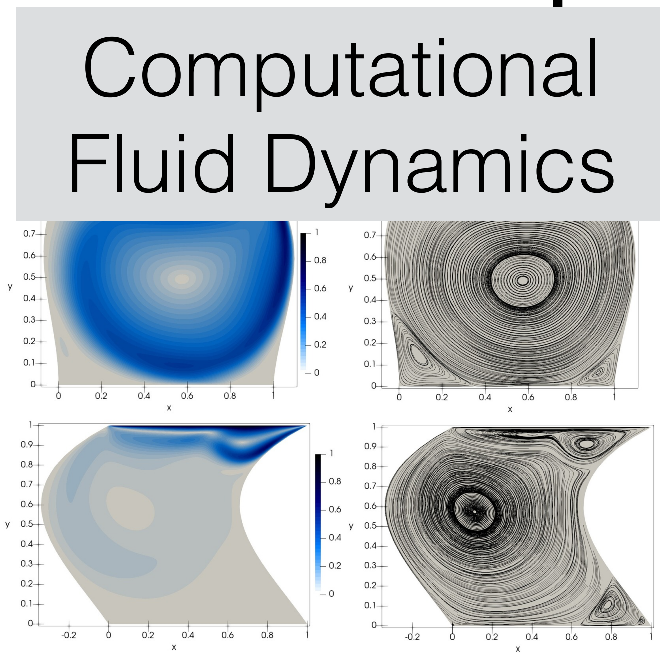

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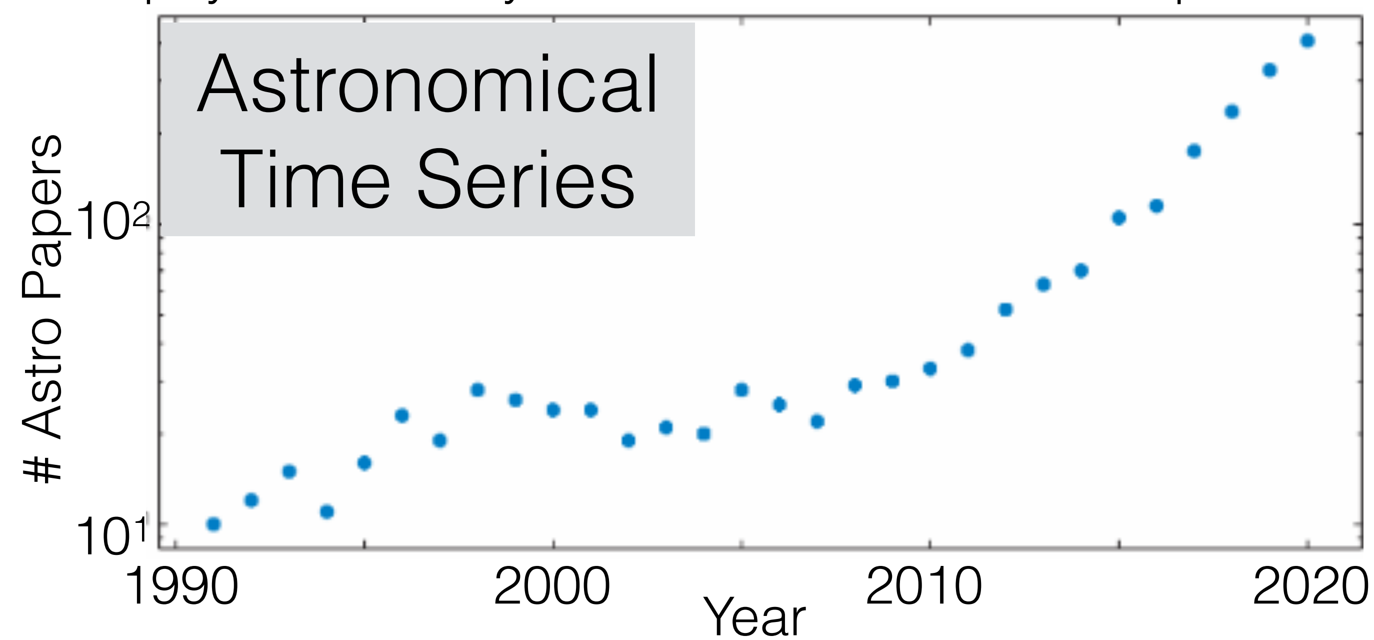

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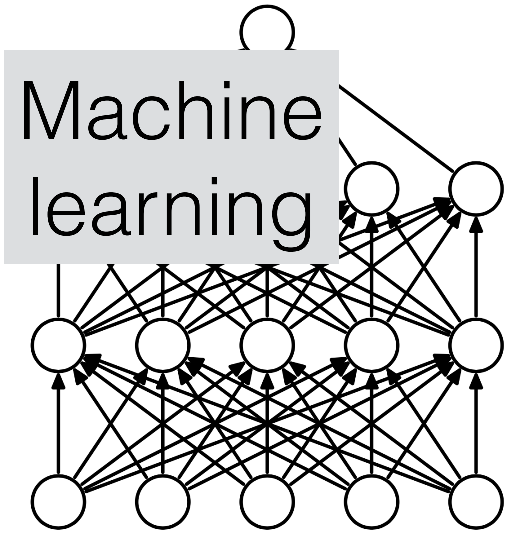

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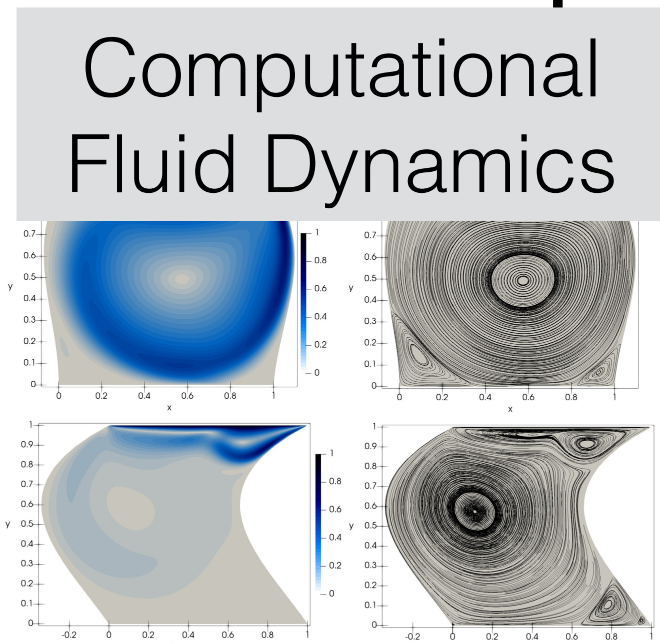

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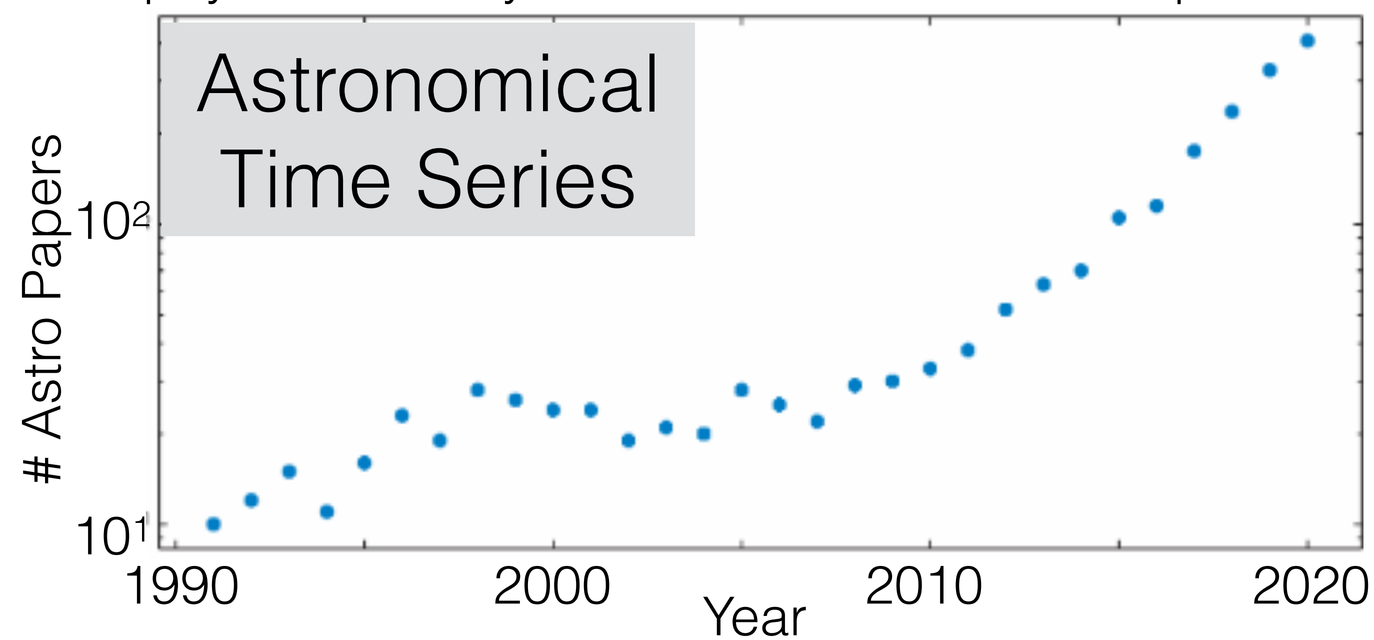

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Bonus benefits:

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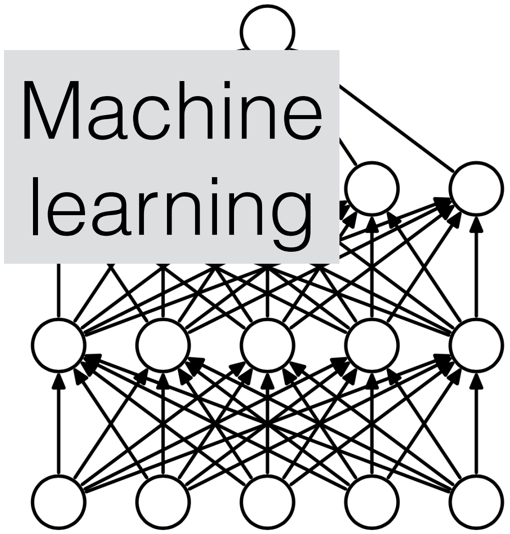

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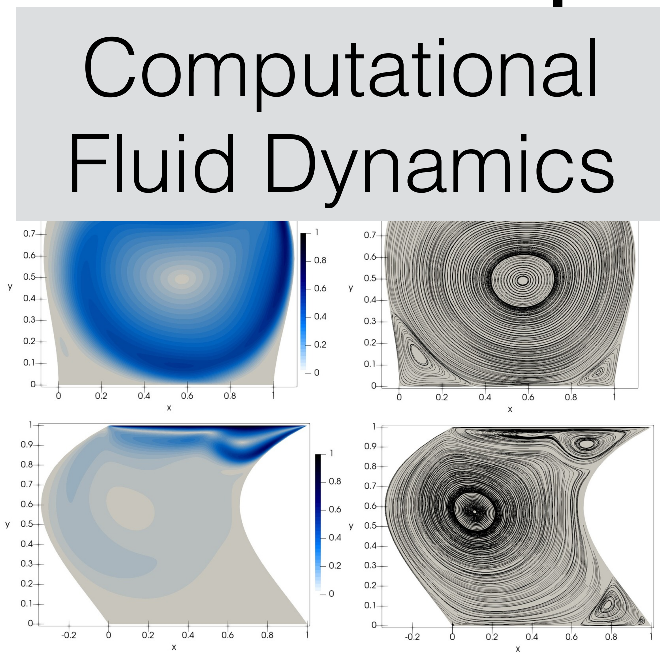
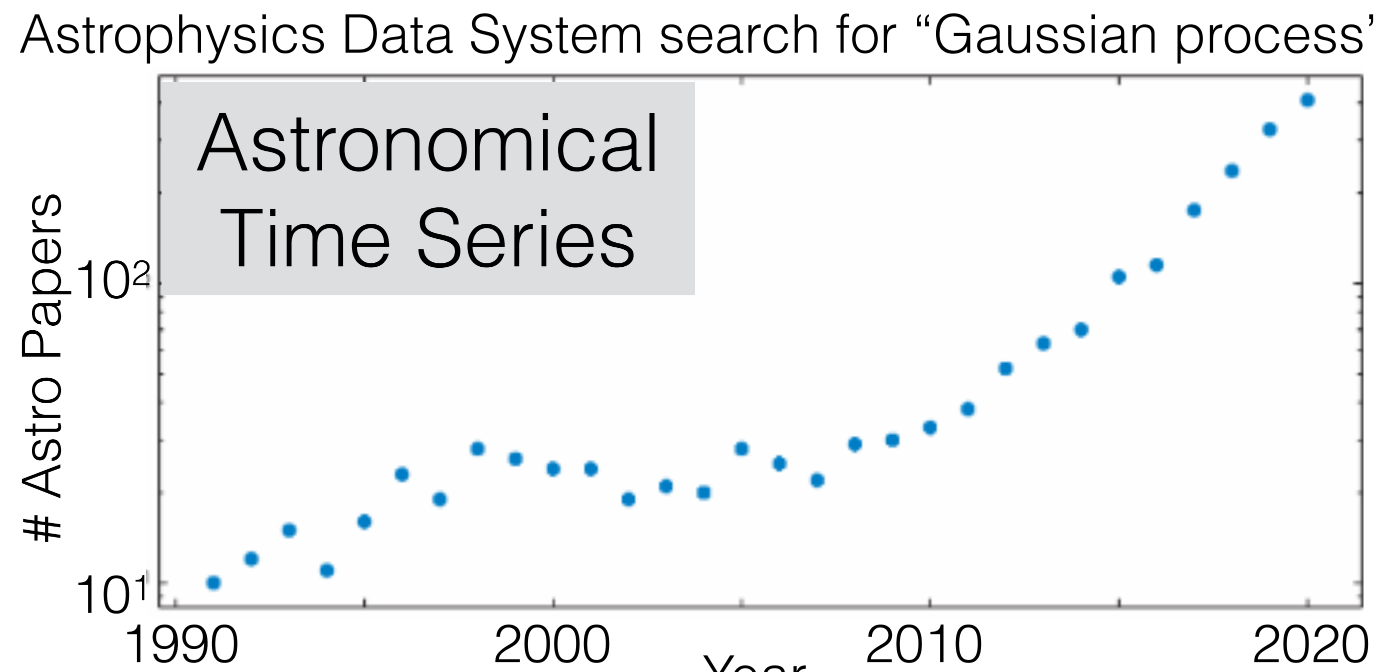


[Morita+ 2022]



[Aigrain, Foreman-Mackey 2023]

A recurring motif:

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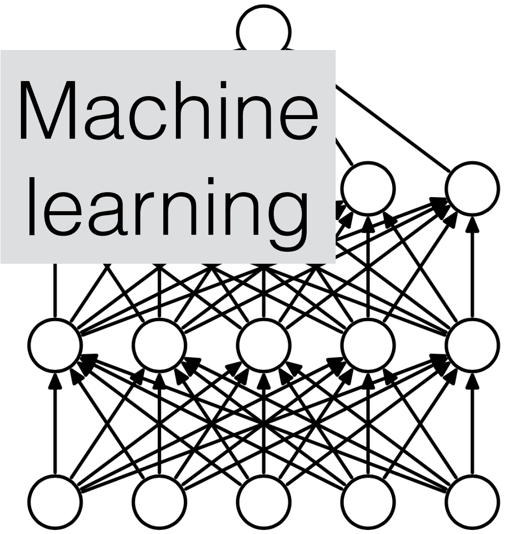

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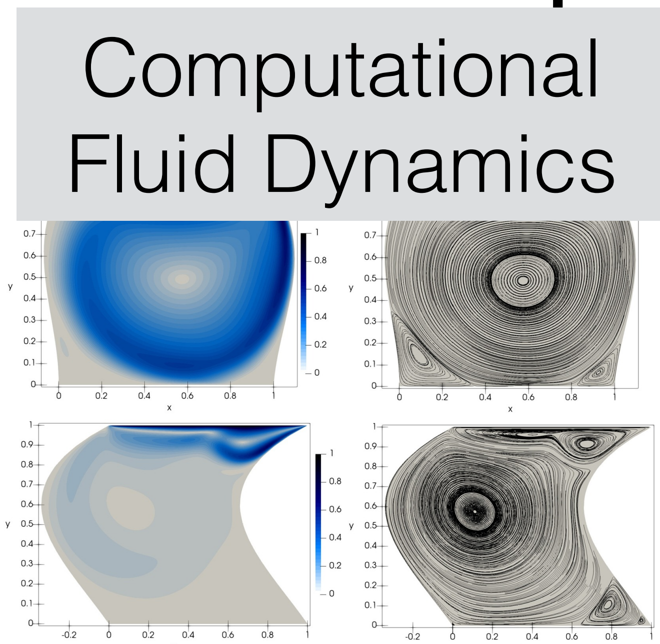

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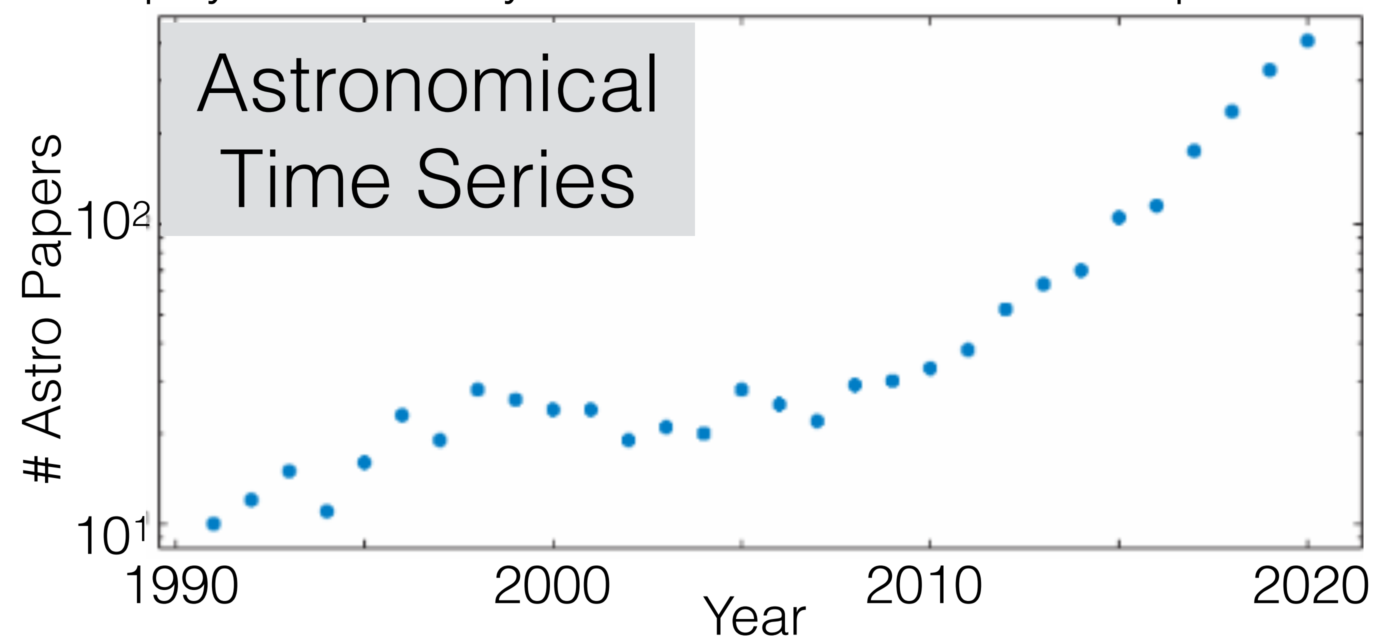
Astrophysics Data System search for “Gaussian process”



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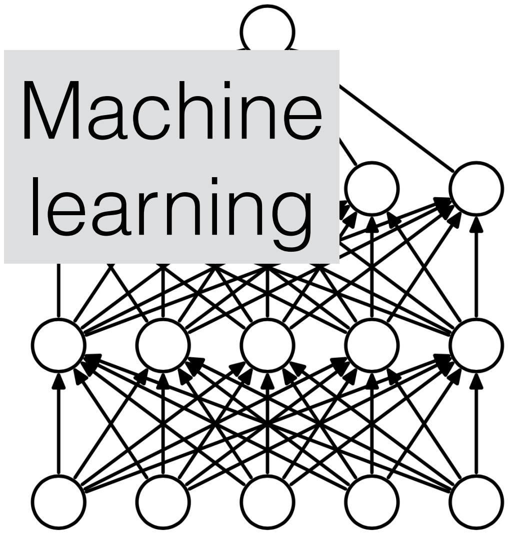

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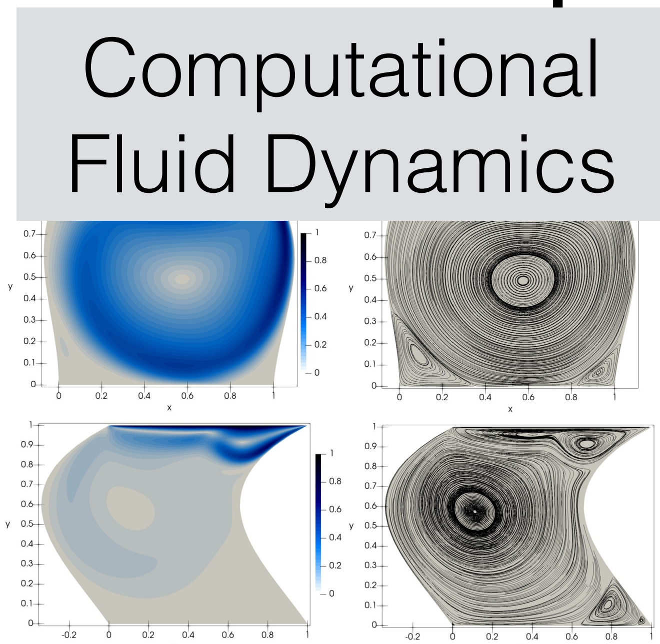

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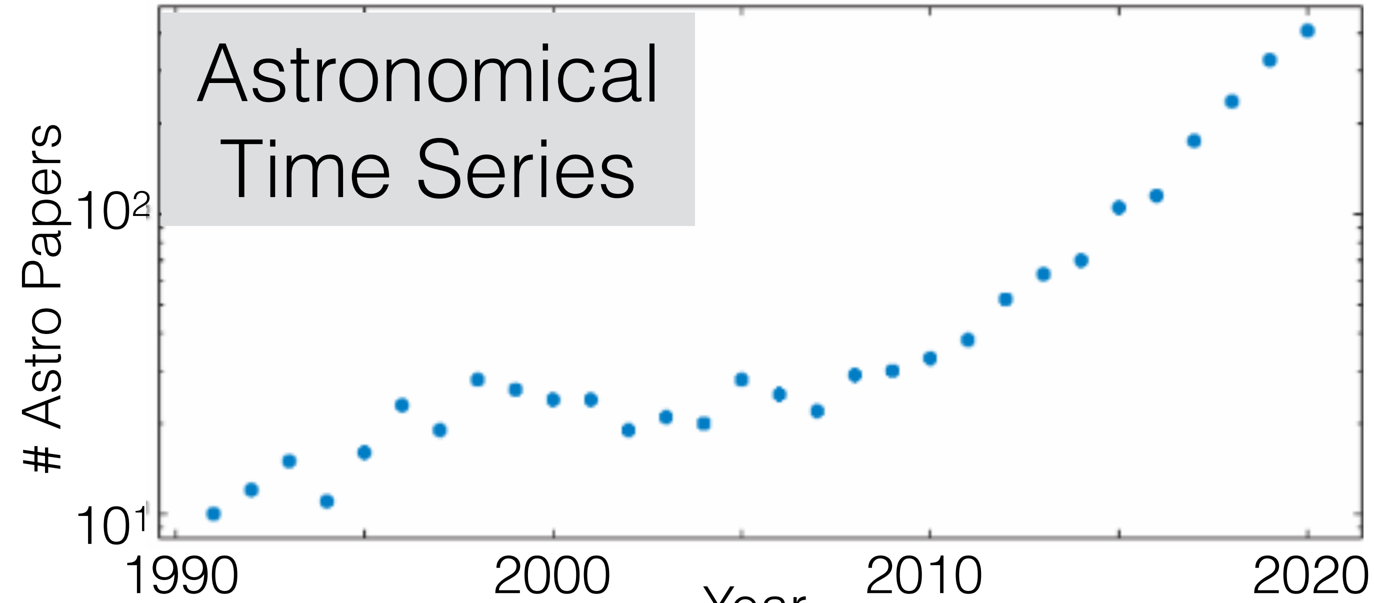

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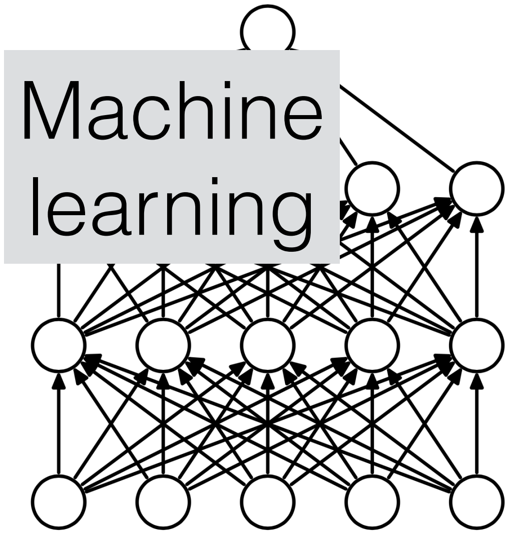

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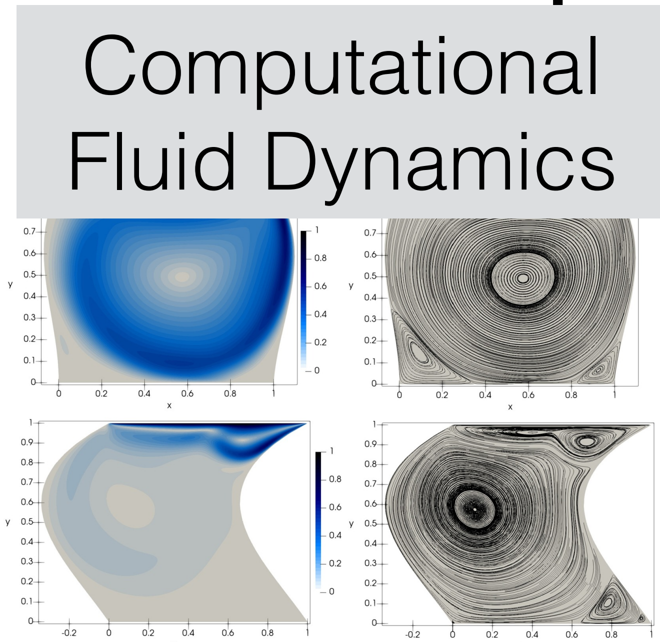

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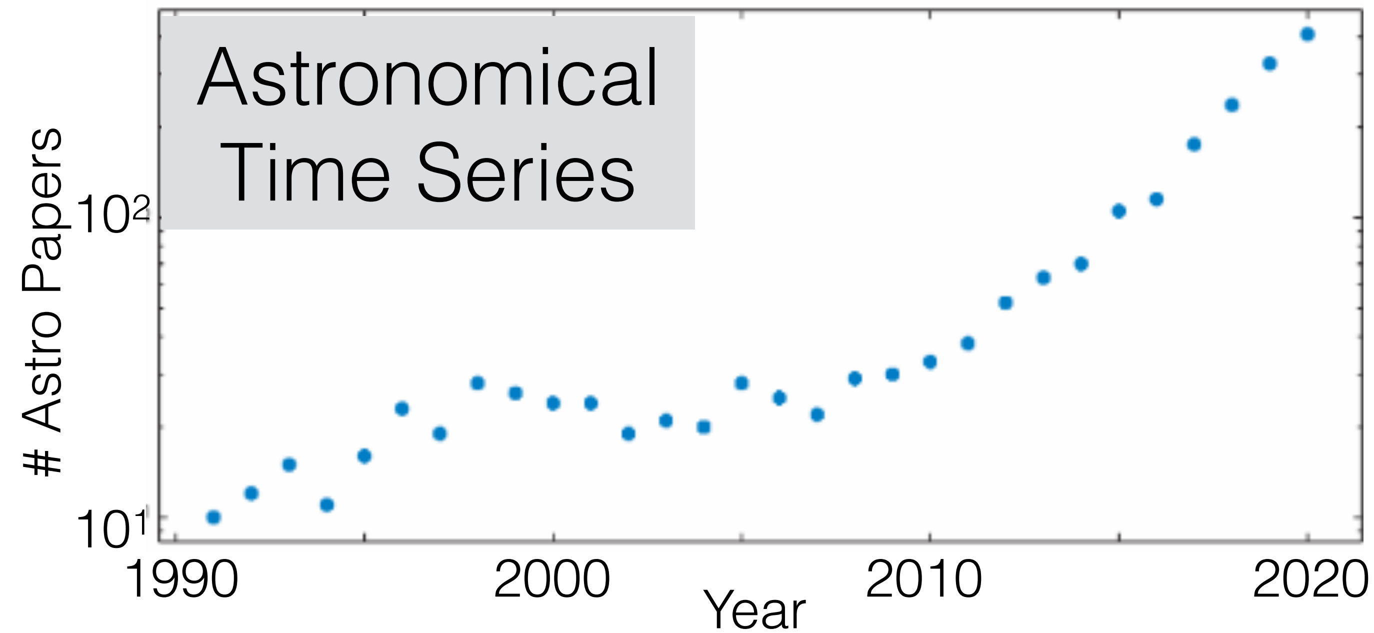

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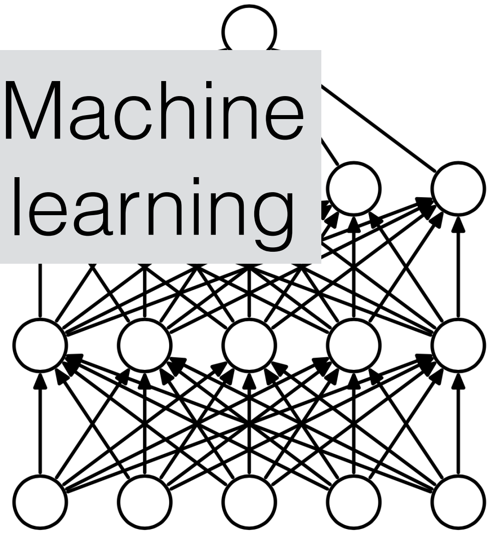
see also "kriging," "optimal interpolation (OI)"

Machine learning



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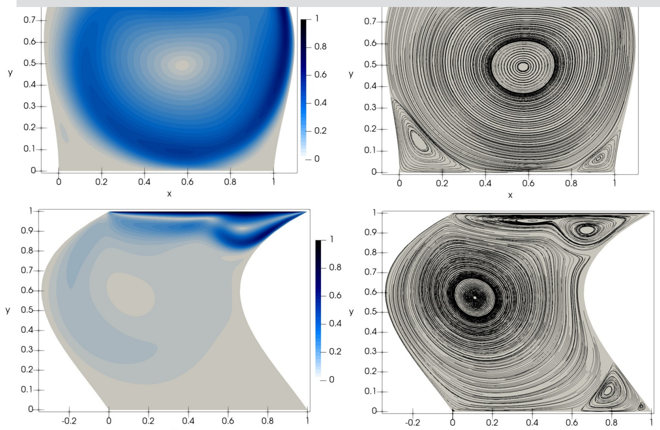
Computational Fluid Dynamics



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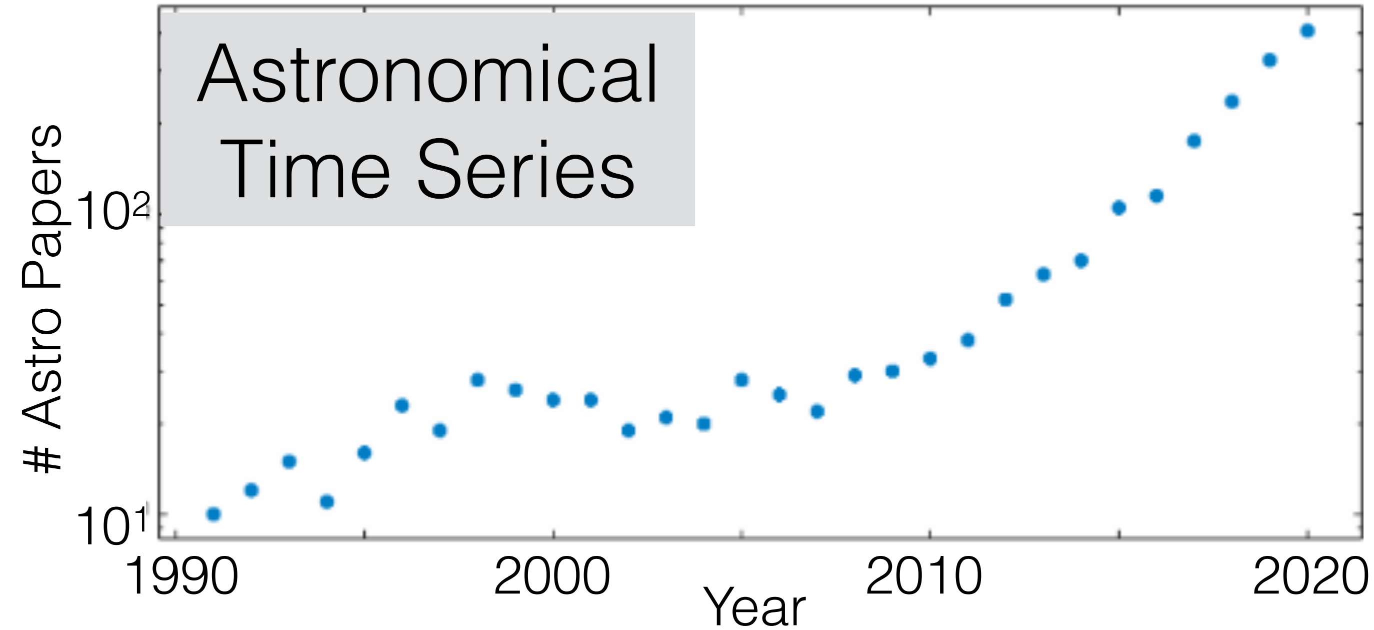
Astronomical Time Series



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Roadmap

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  - Learn the skills to be responsible users of standard GPs (transferable to other ML/AI methods)

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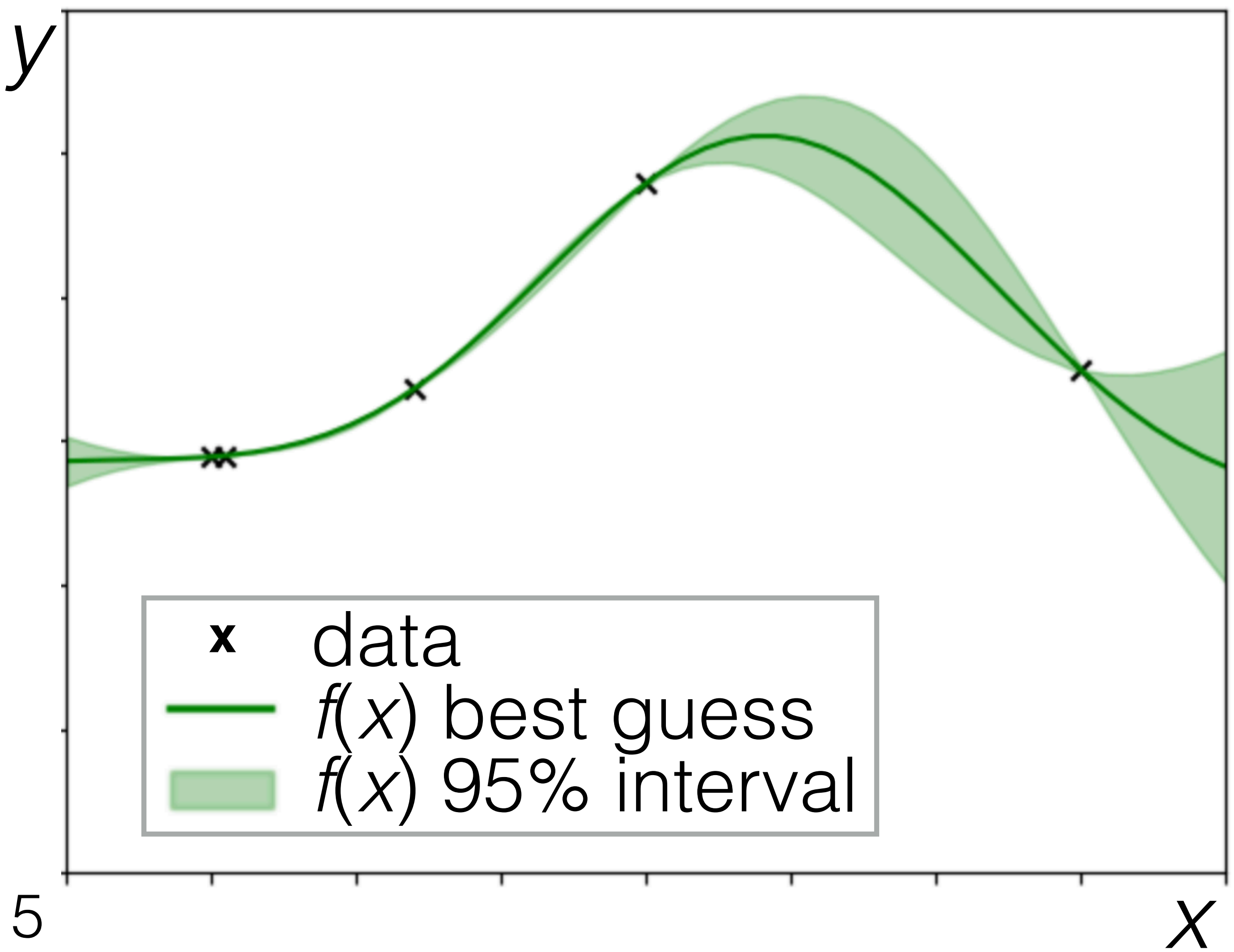

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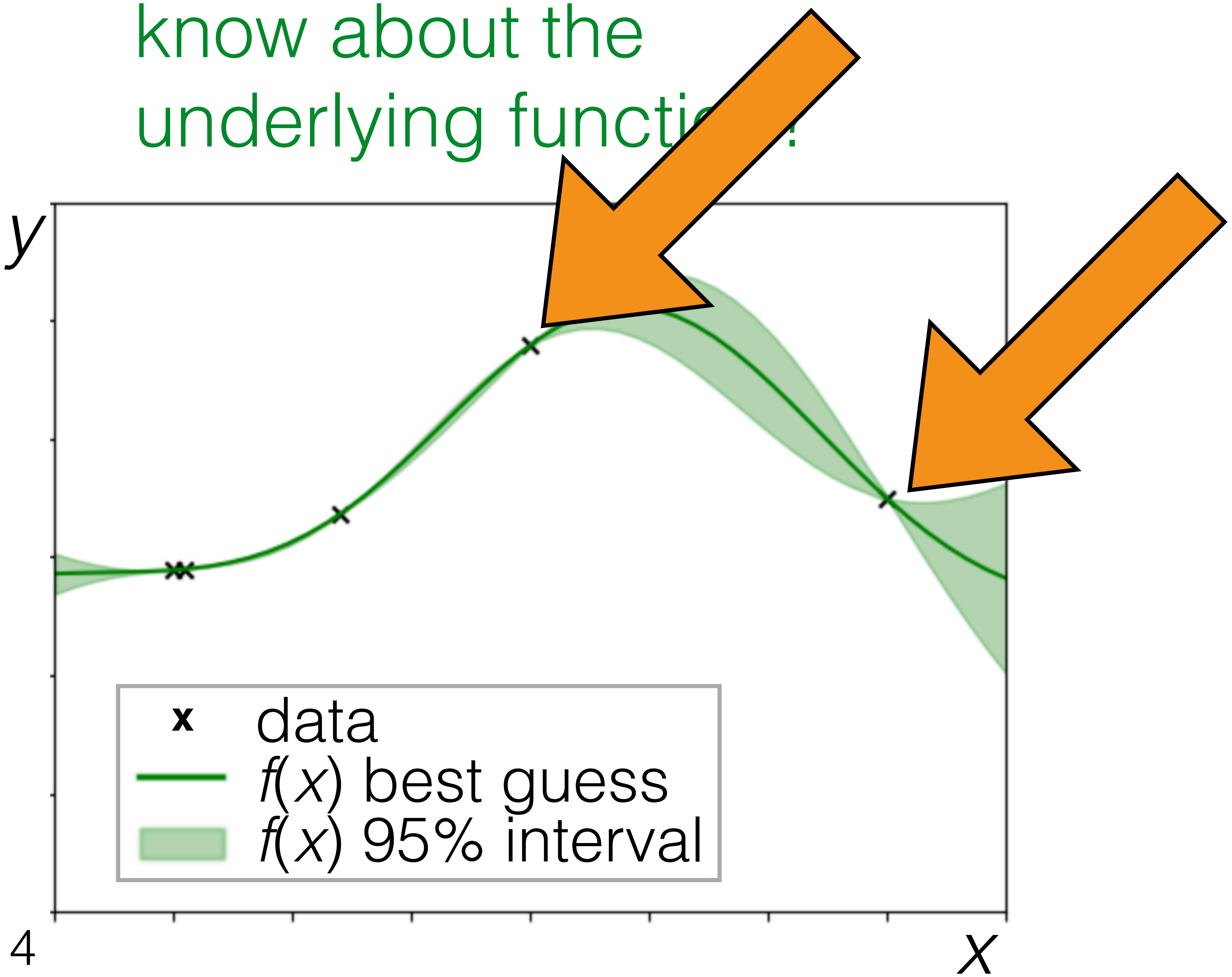

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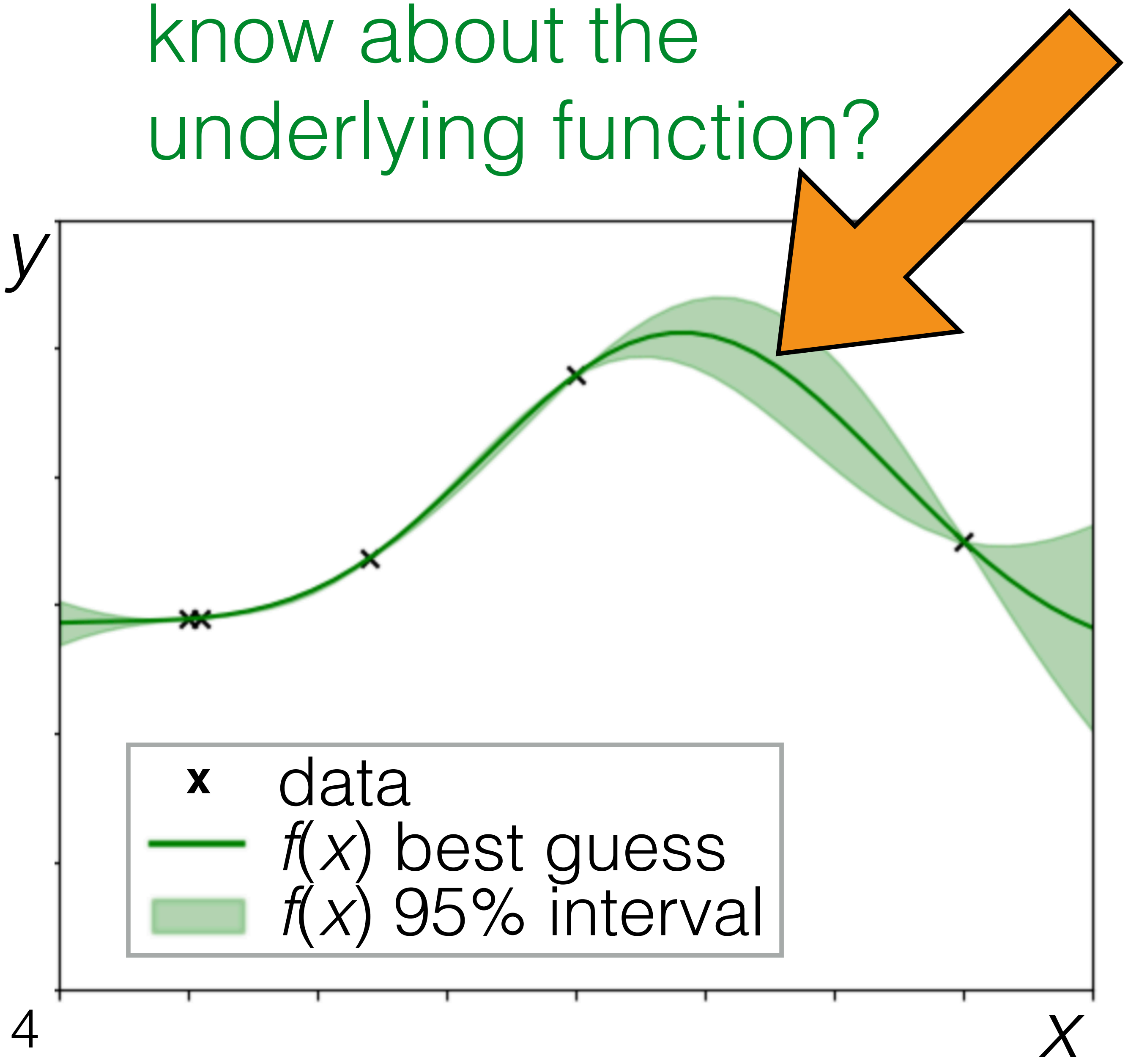

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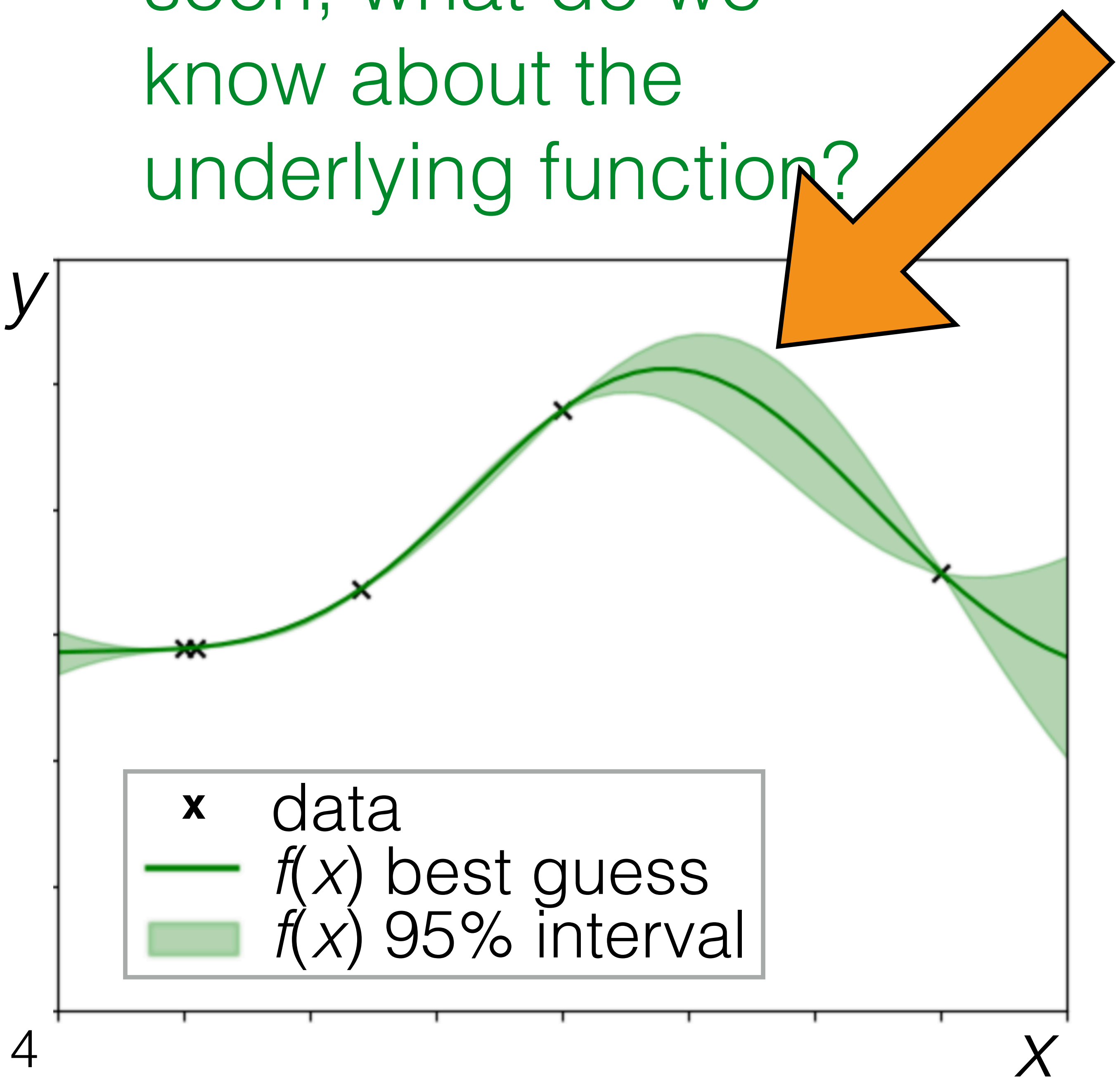

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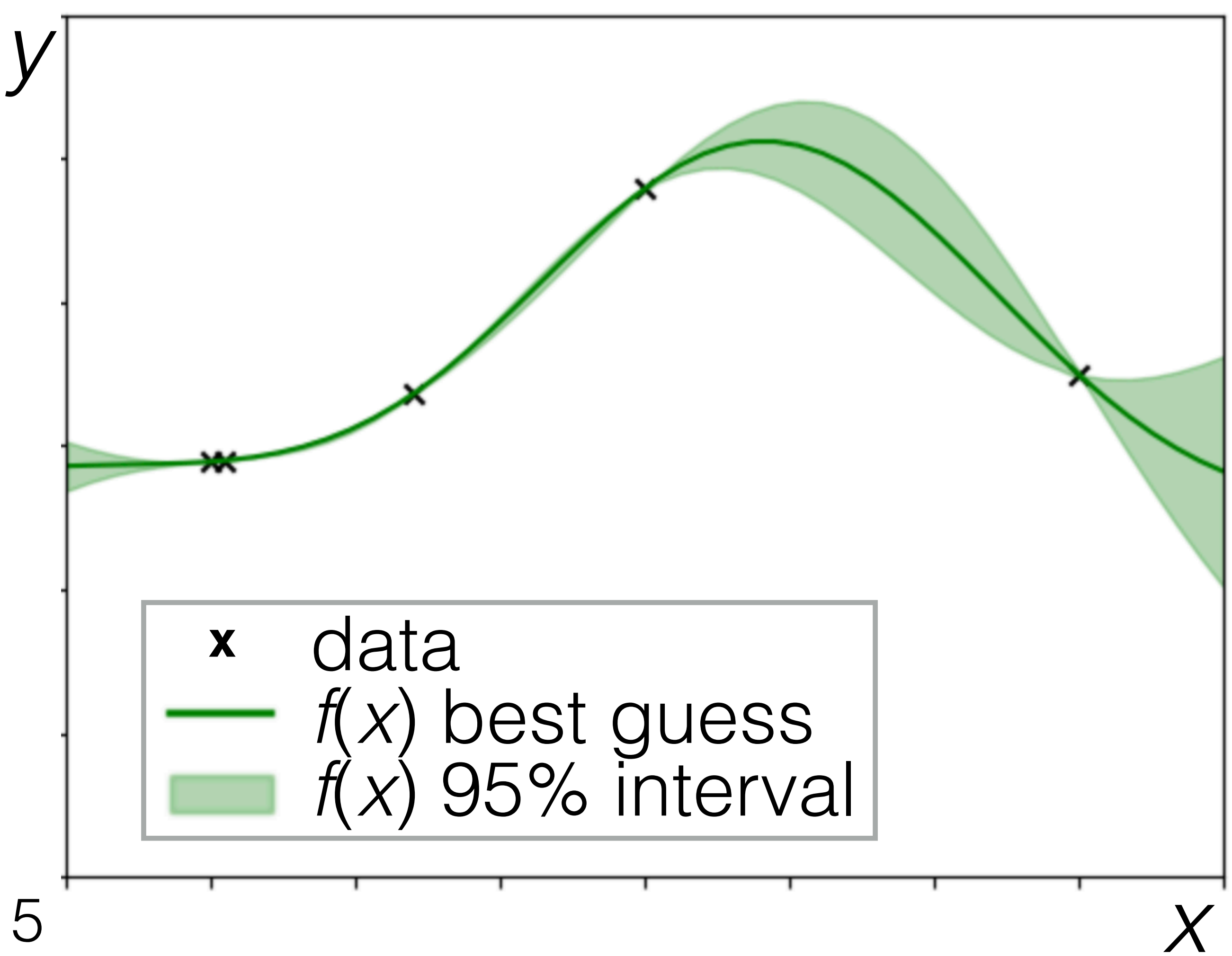

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- $p(\text{unknowns} \mid \text{data}) \propto p(\text{data} \mid \text{unknowns})\, p(\text{unknowns})$

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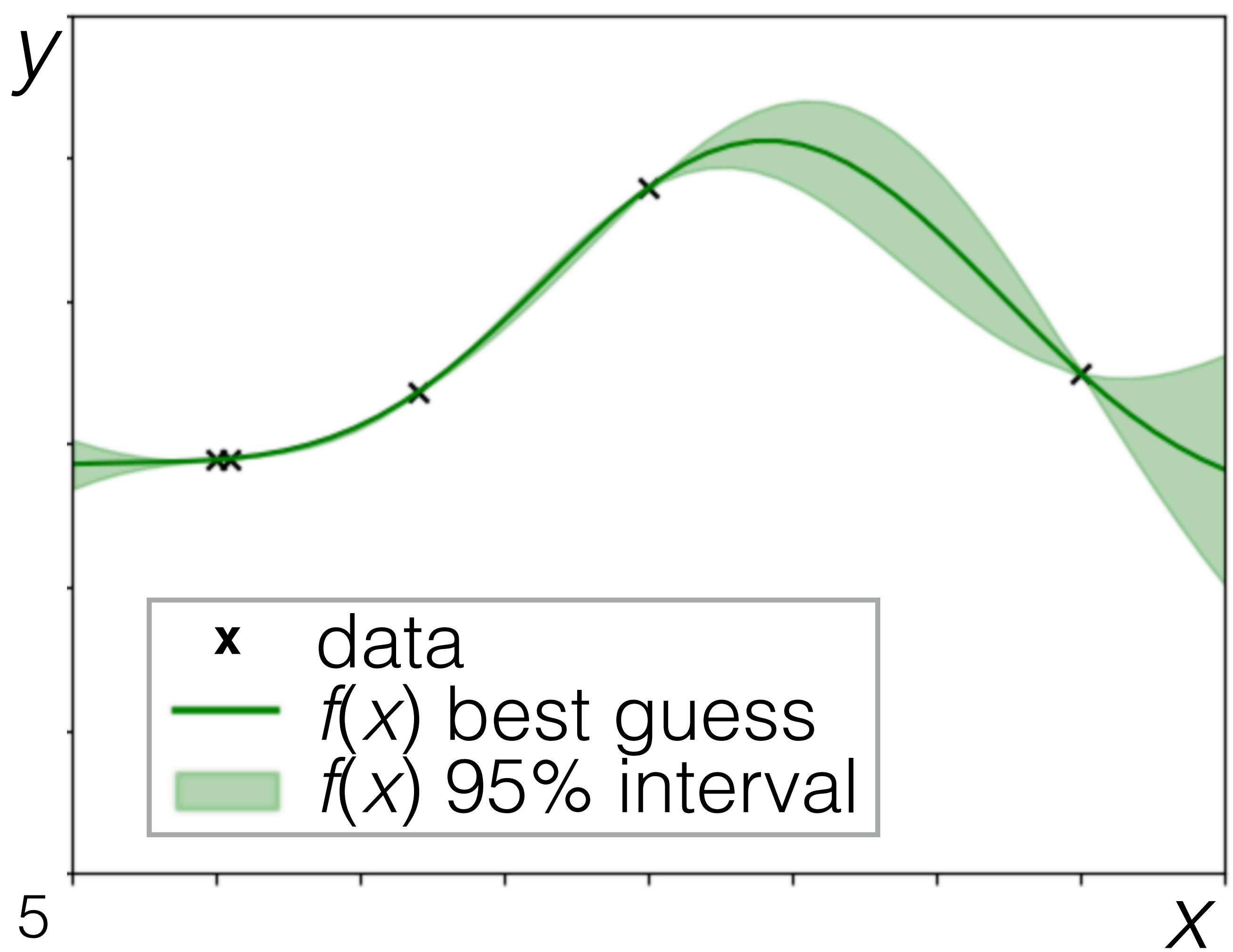
A (statistical) model that can generate functions and data of interest



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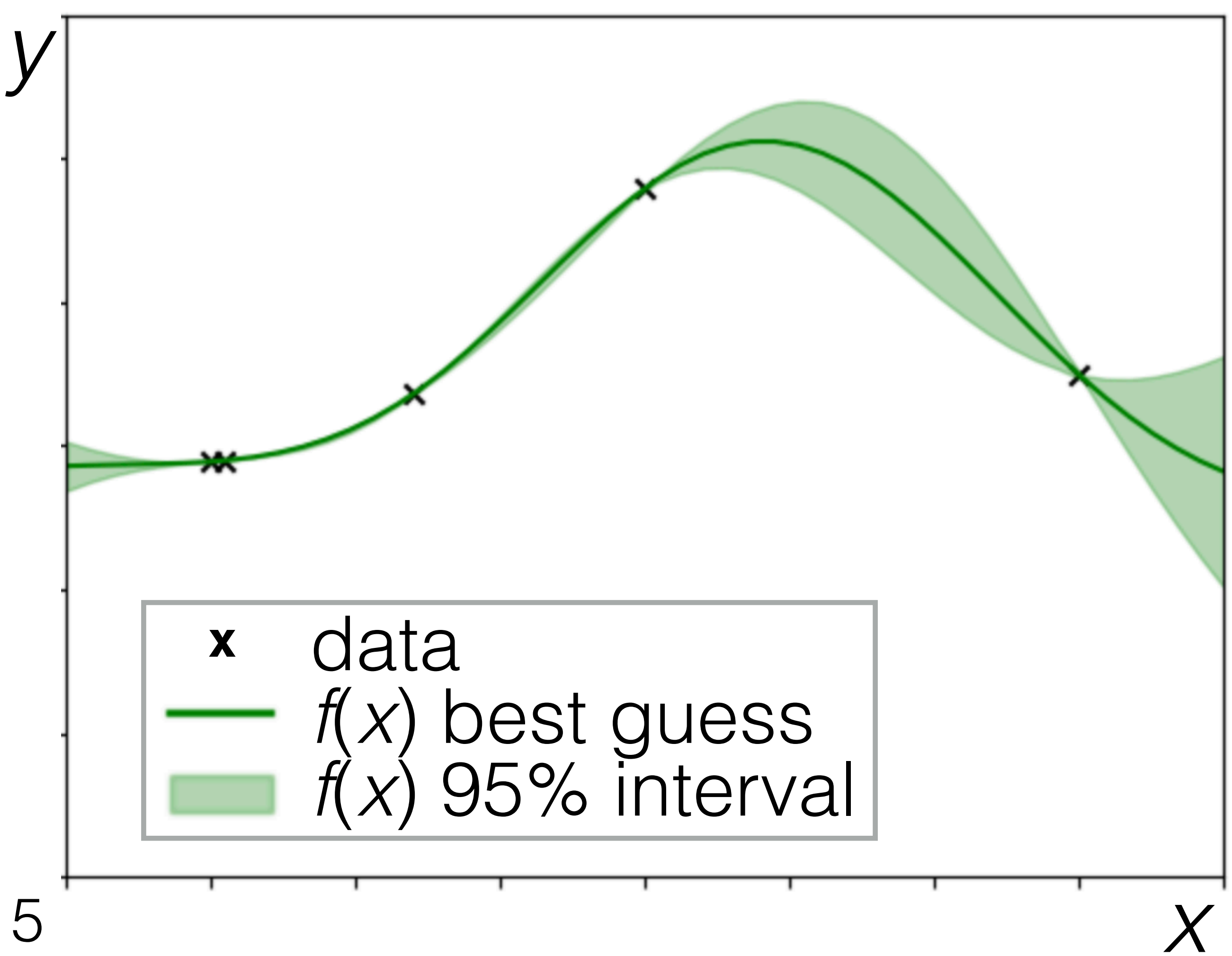

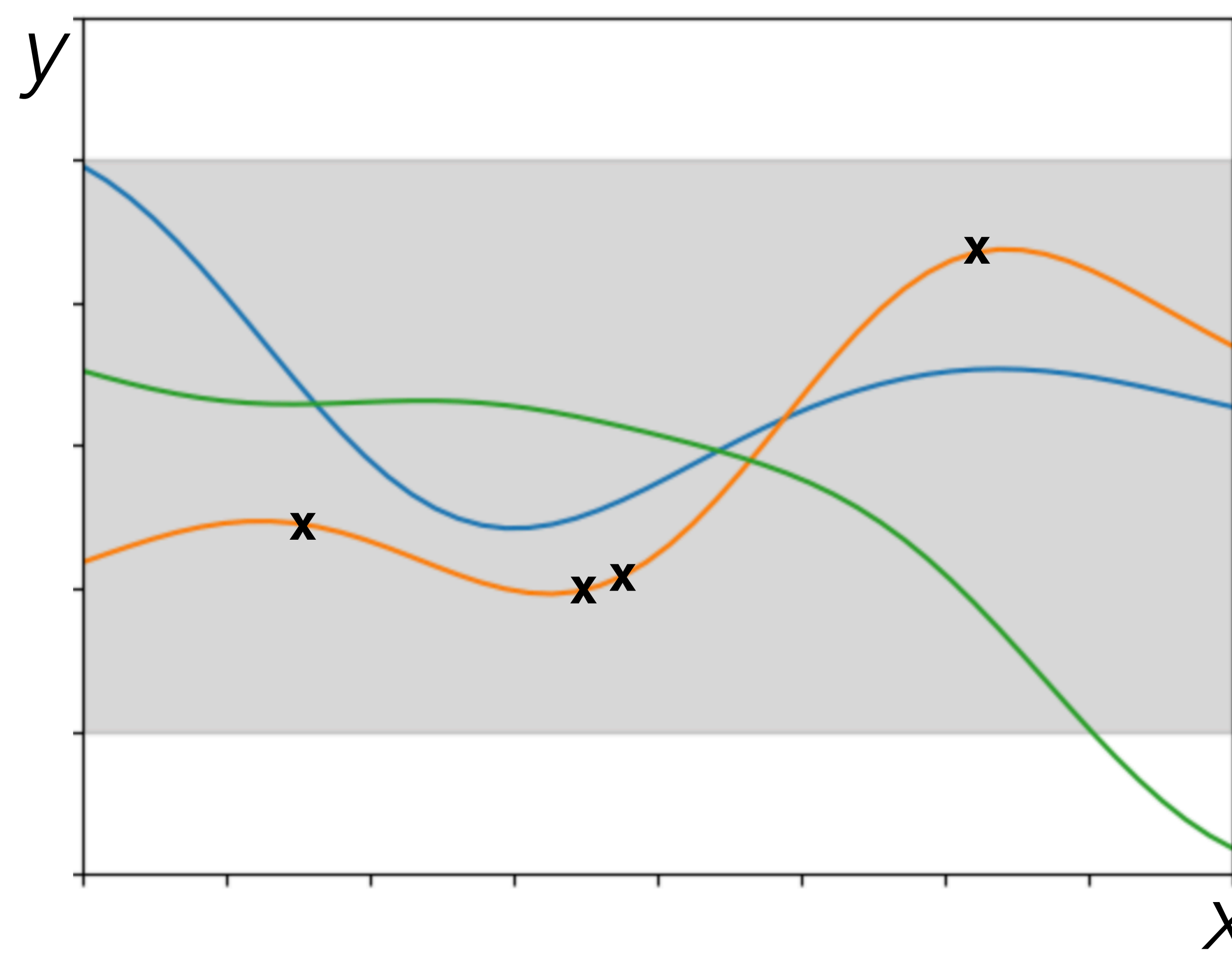

Univariate Gaussian distribution review

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$$\mathcal{N}(y|\mu,\sigma^2) = (2\pi\sigma^2)^{-1/2} \exp\left[\frac{-(y-\mu)^2}{\sigma^2}\right]$$

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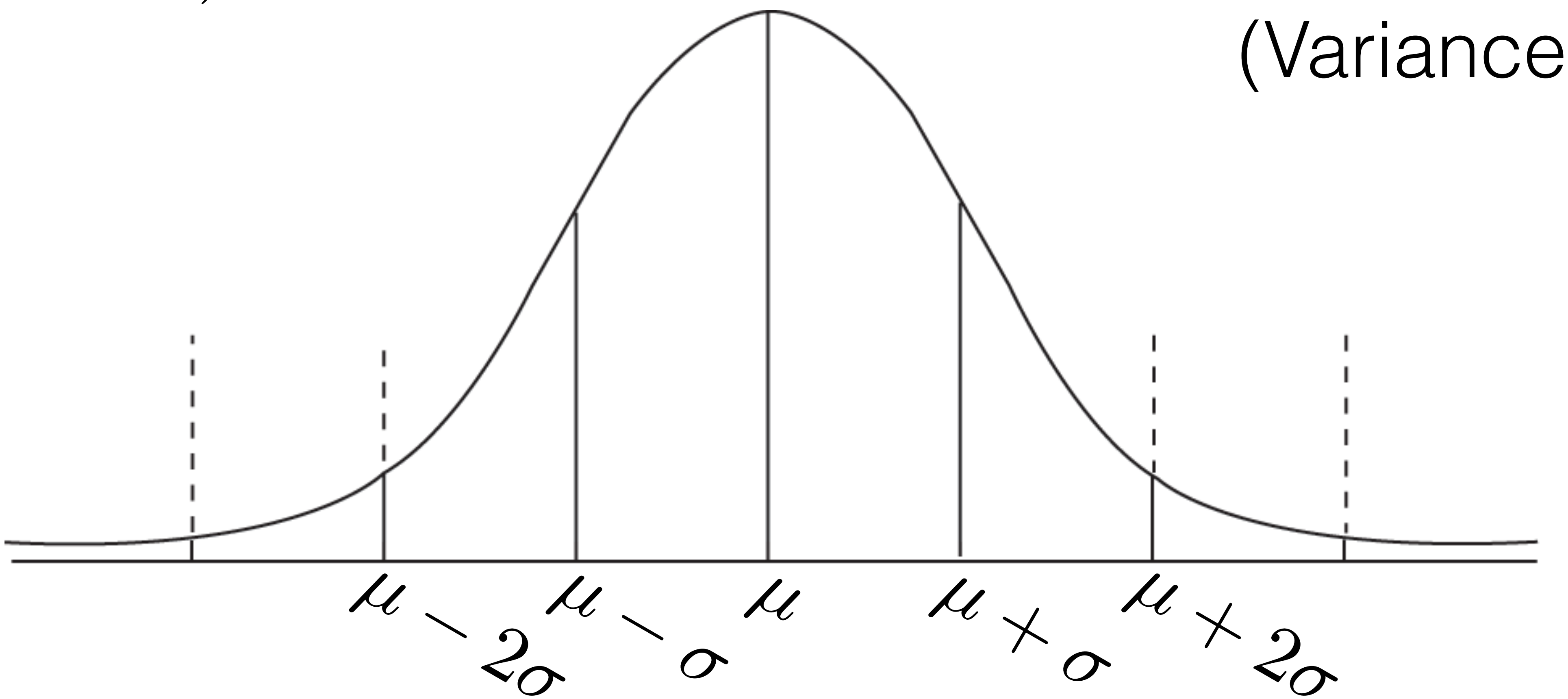

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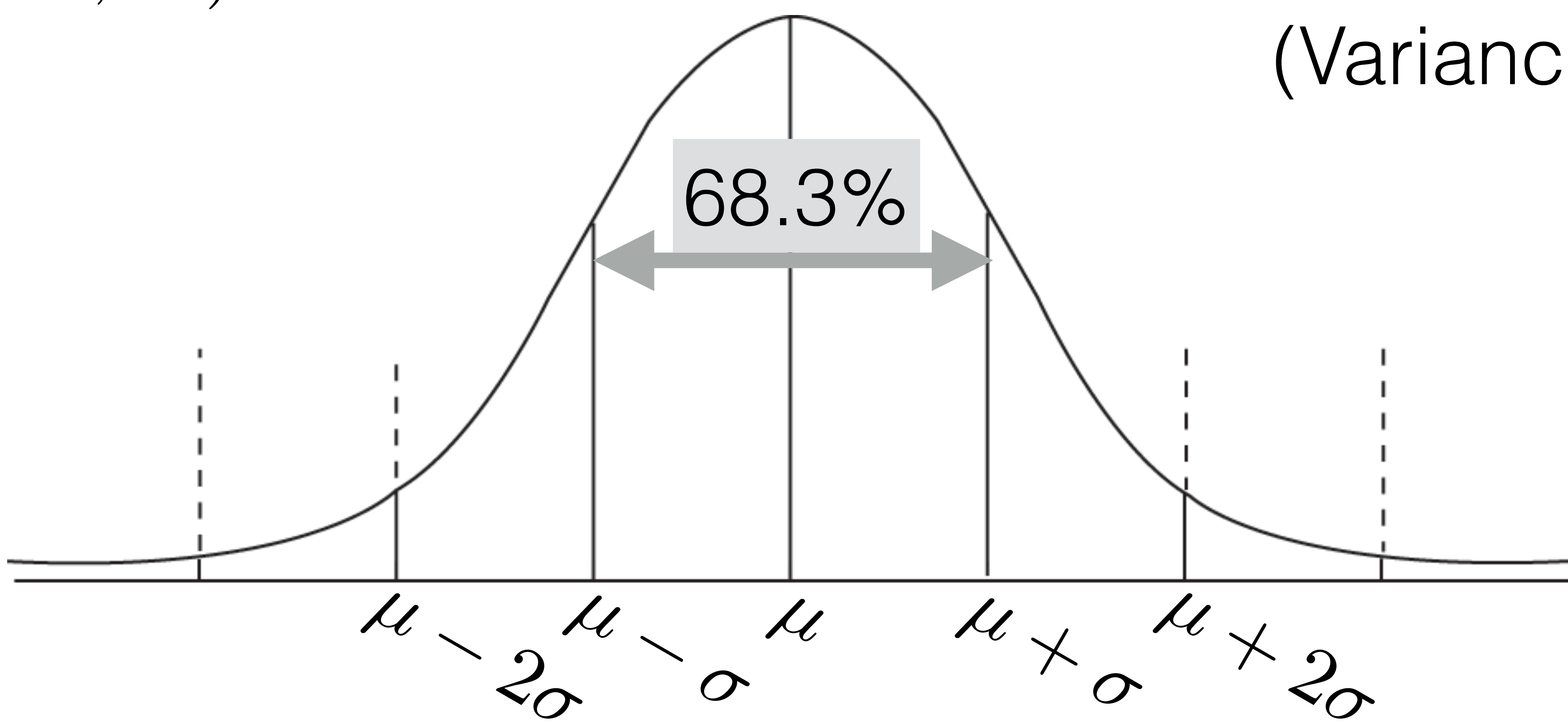

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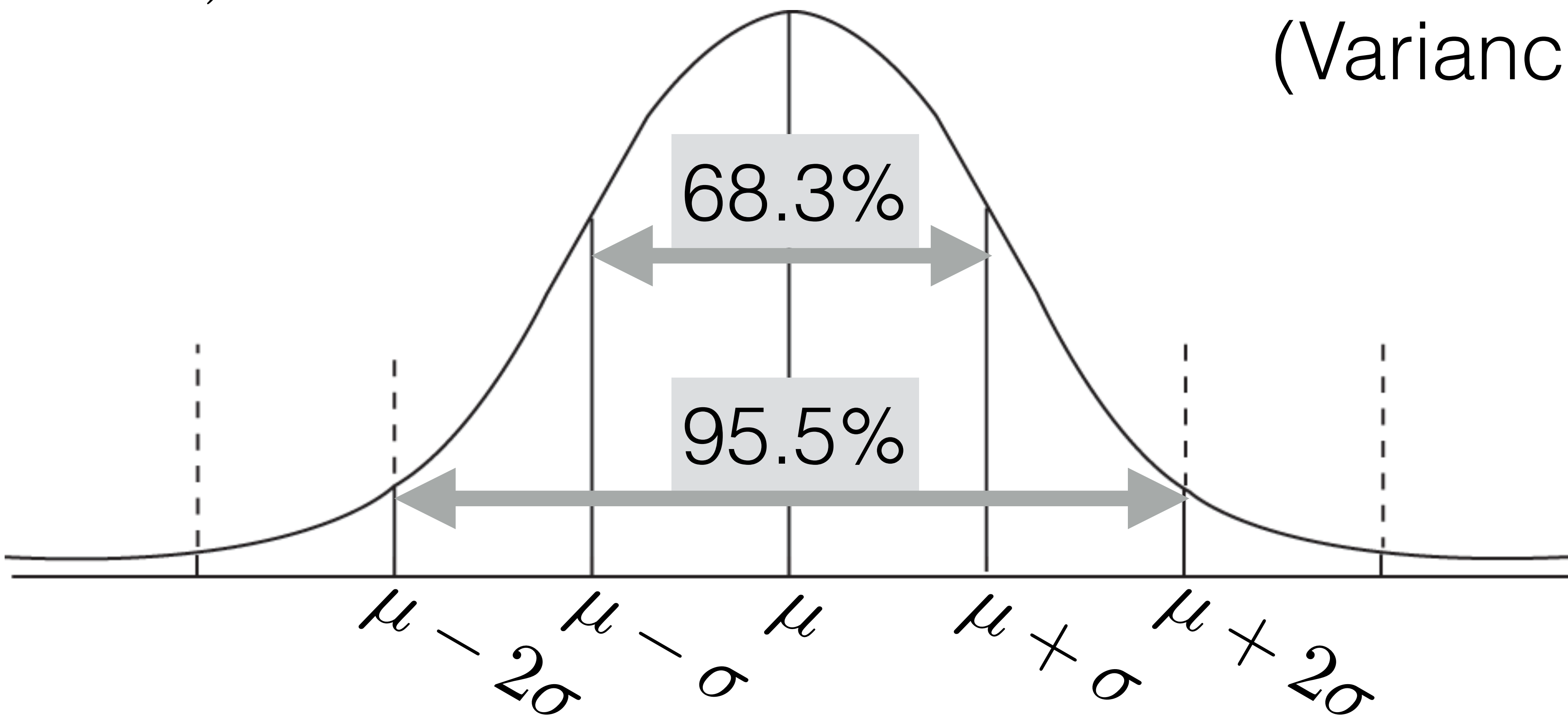

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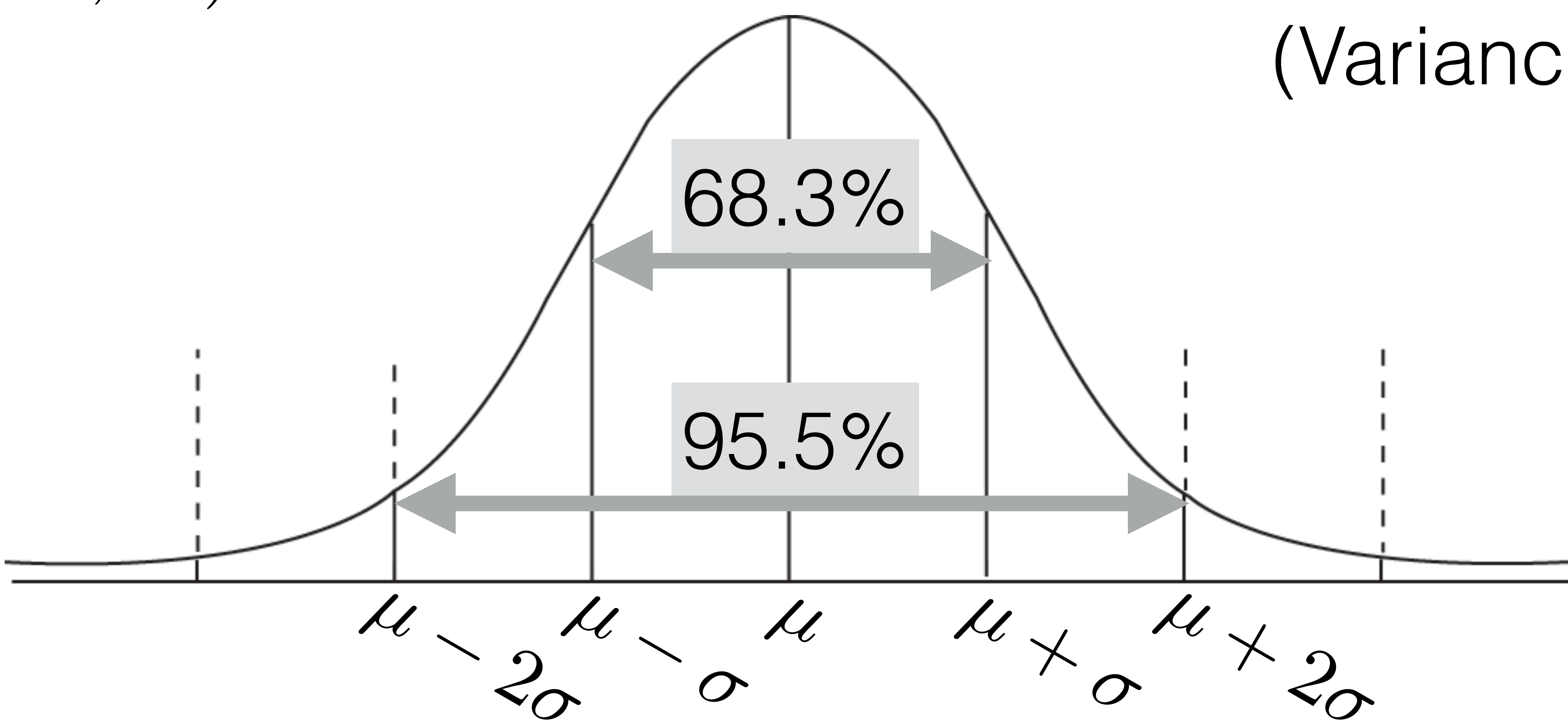

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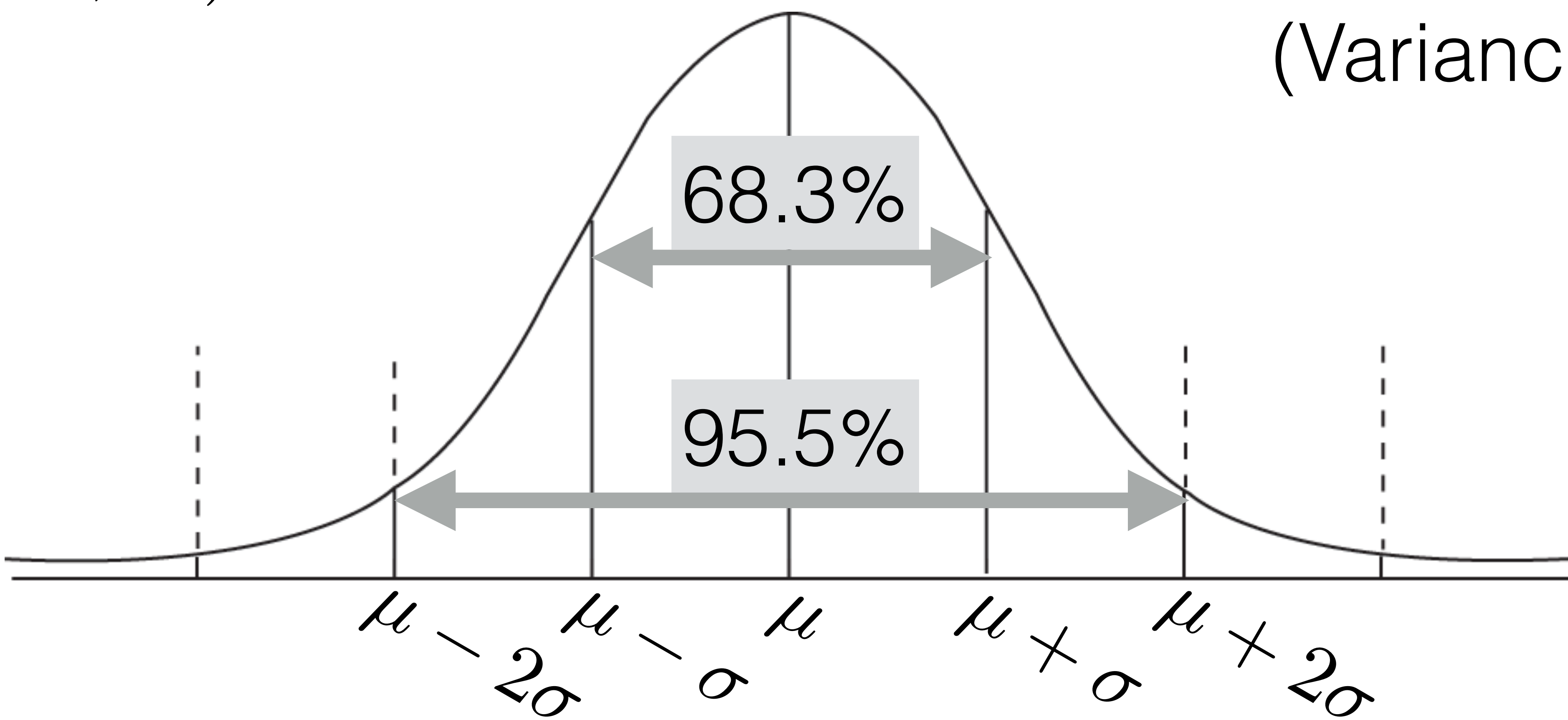

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[demo]

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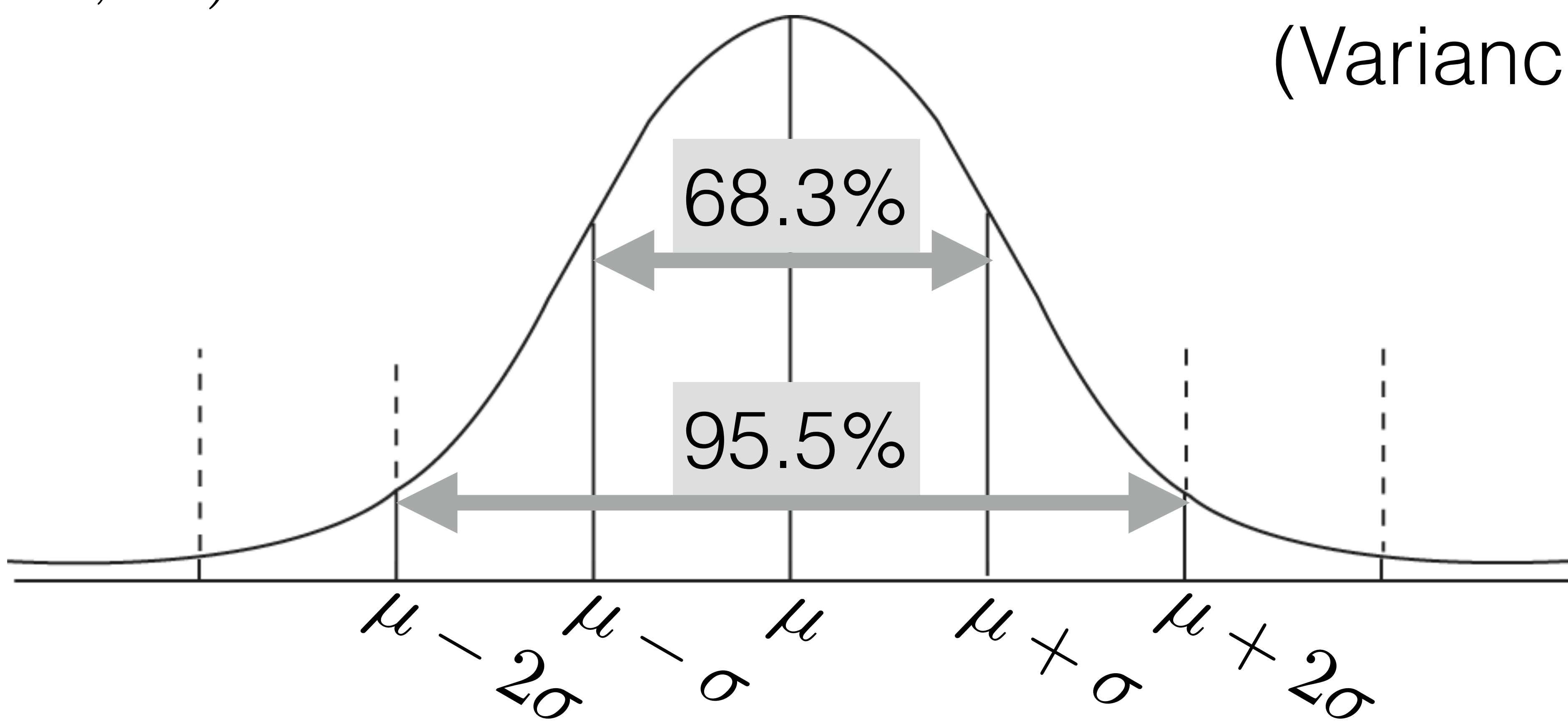

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[demo]

- If $Y \sim \mathcal{N}(0,1)$, then $Y + \mu \sim \mathcal{N}(\mu, 1)$

$$\sigma Y \sim \mathcal{N}(0, \sigma^2)$$

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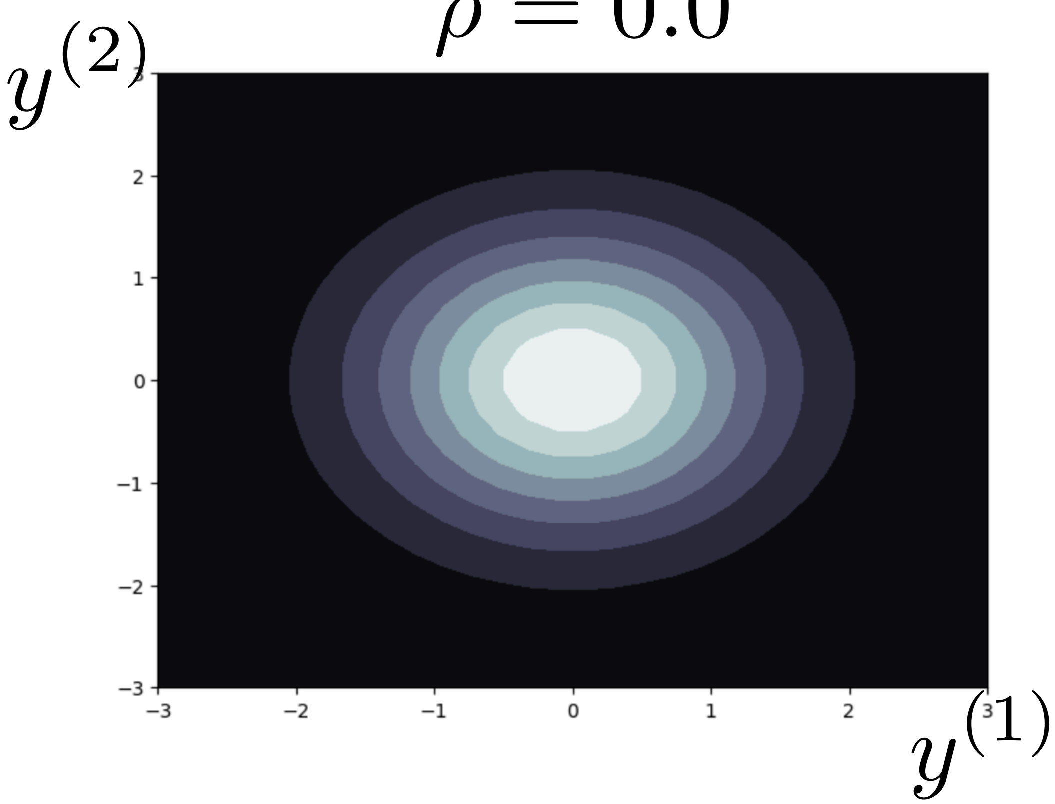

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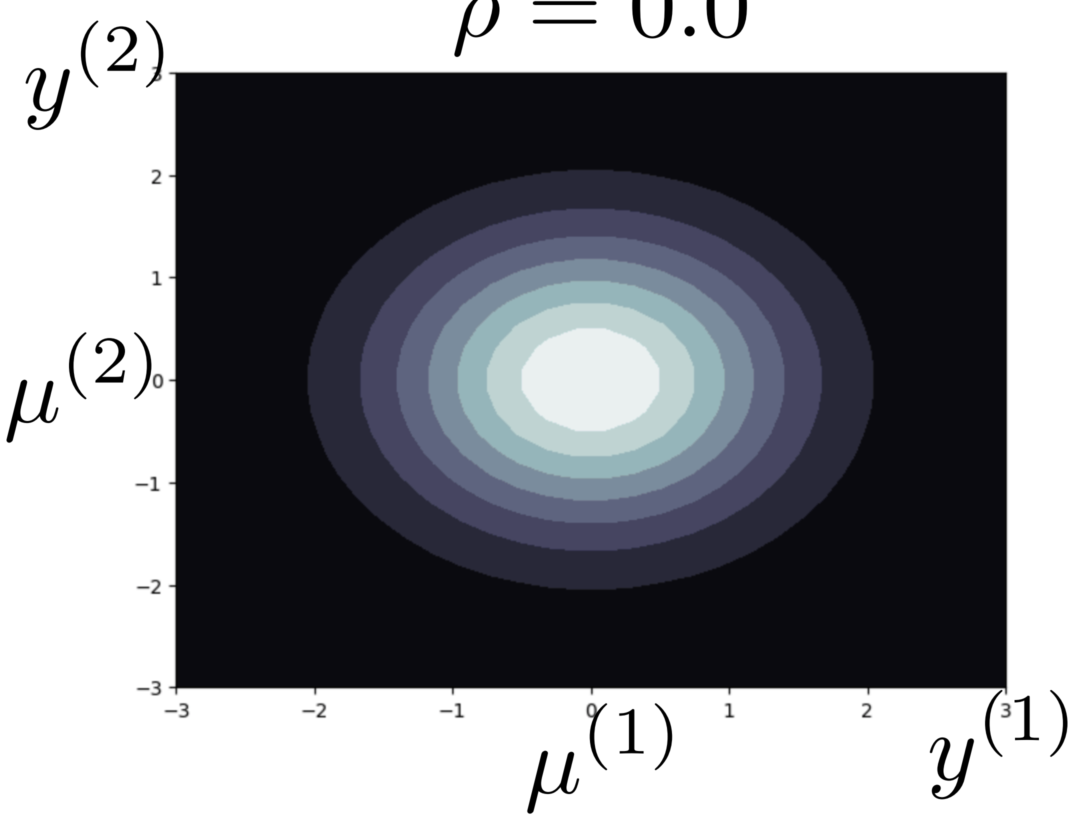

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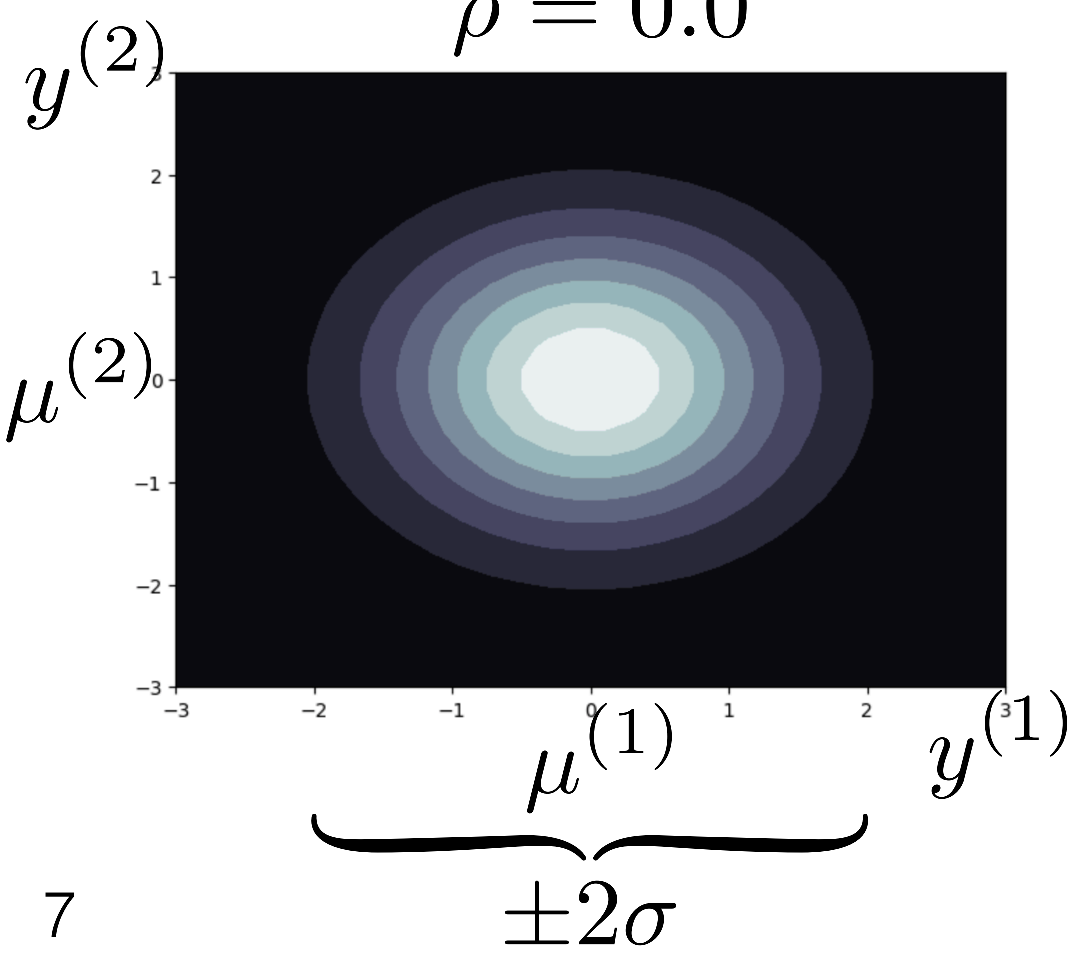

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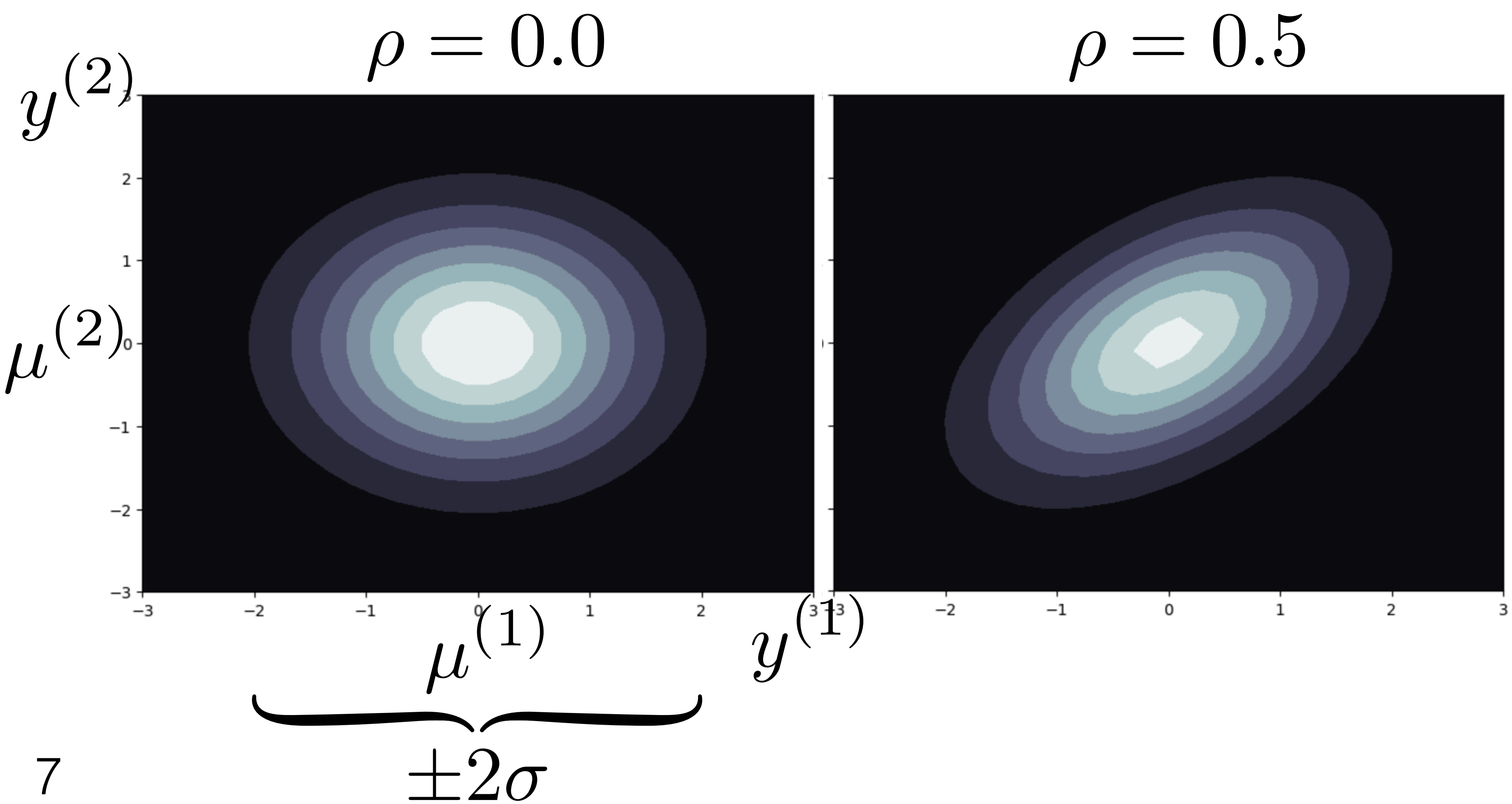

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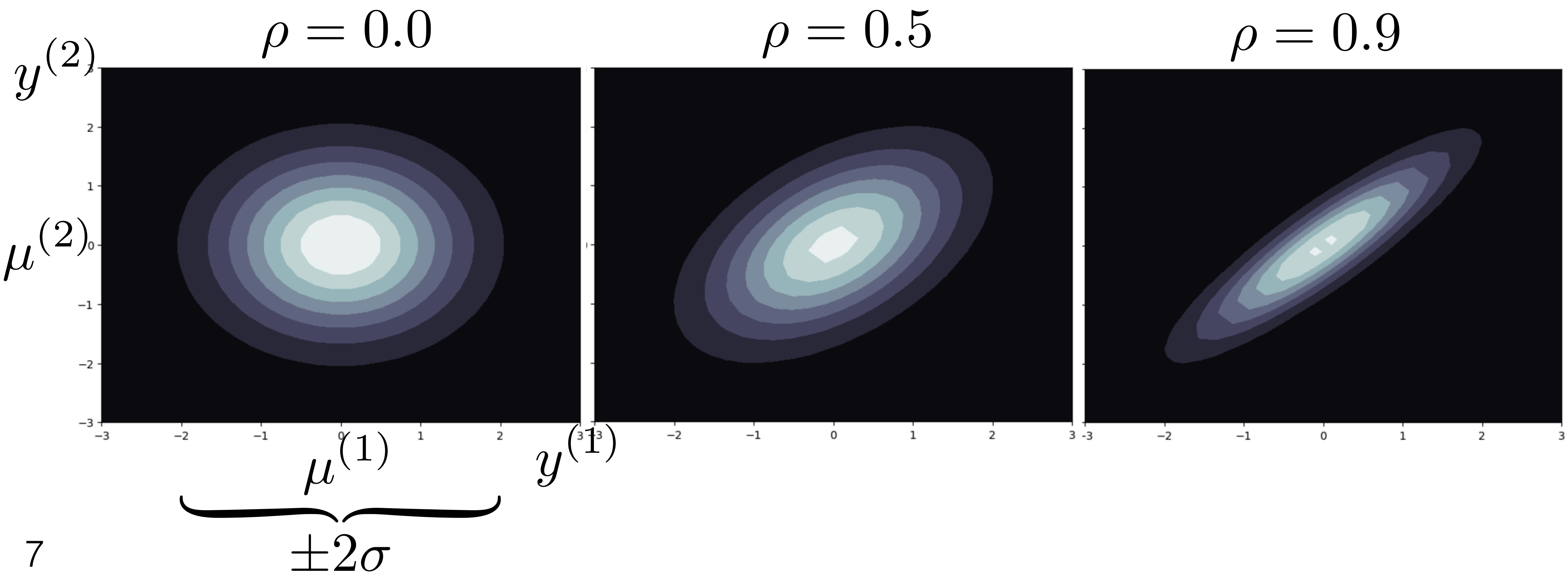

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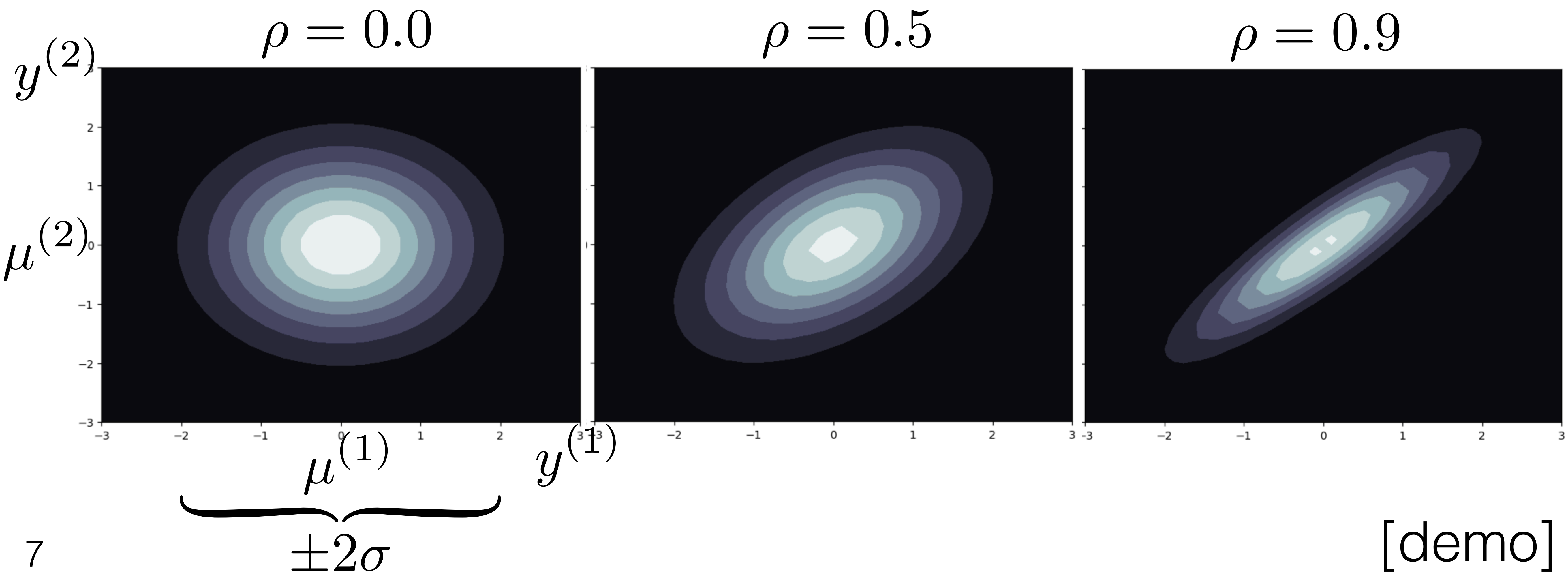

[demo]

Multivariate Gaussian

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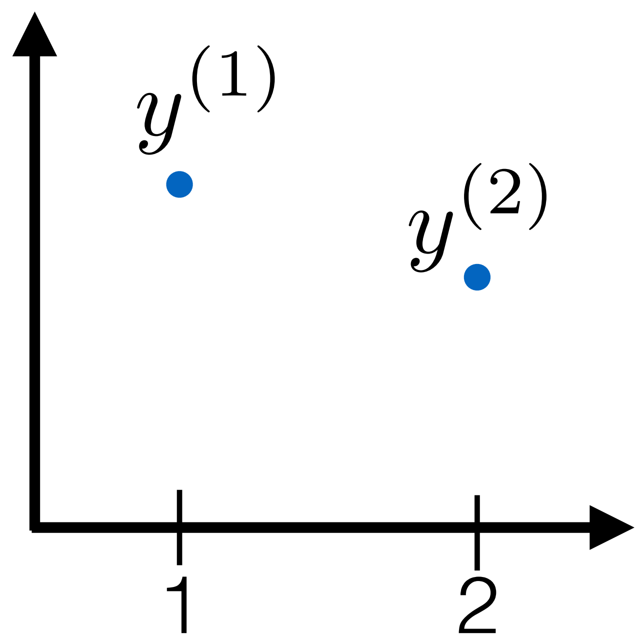

Multivariate Gaussian using locations

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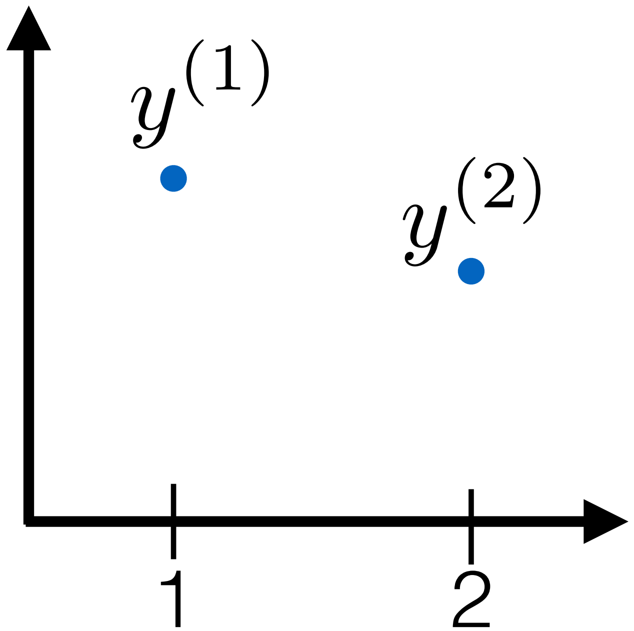

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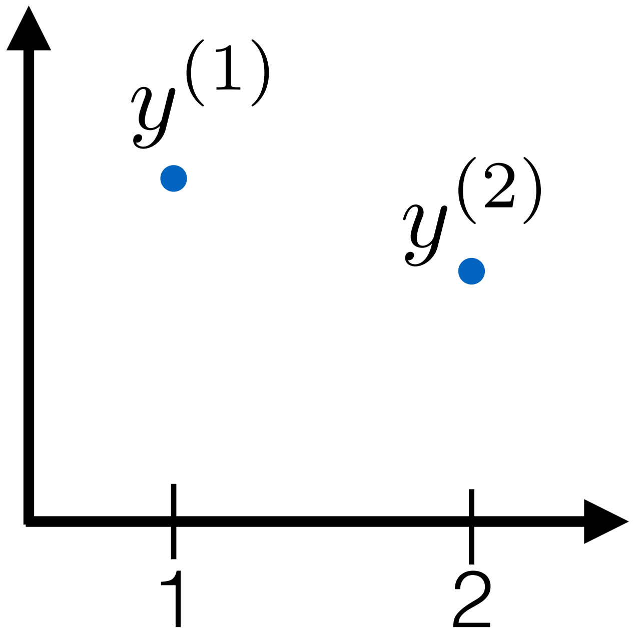

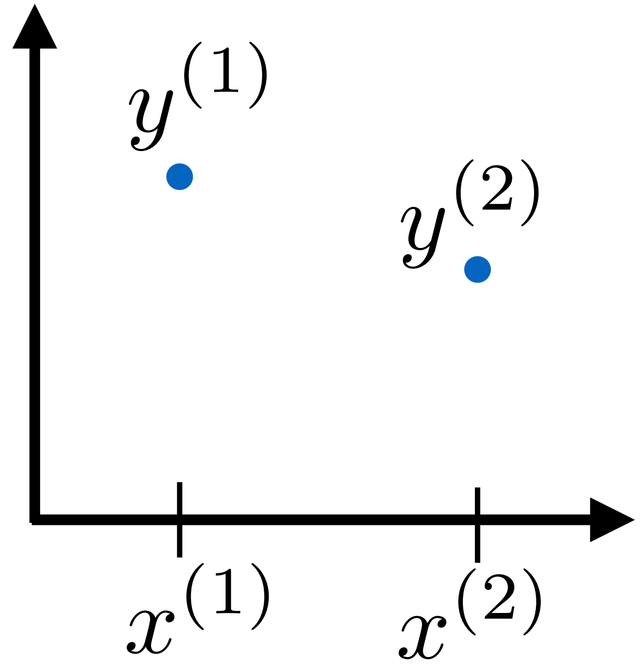

Multivariate Gaussian using locations

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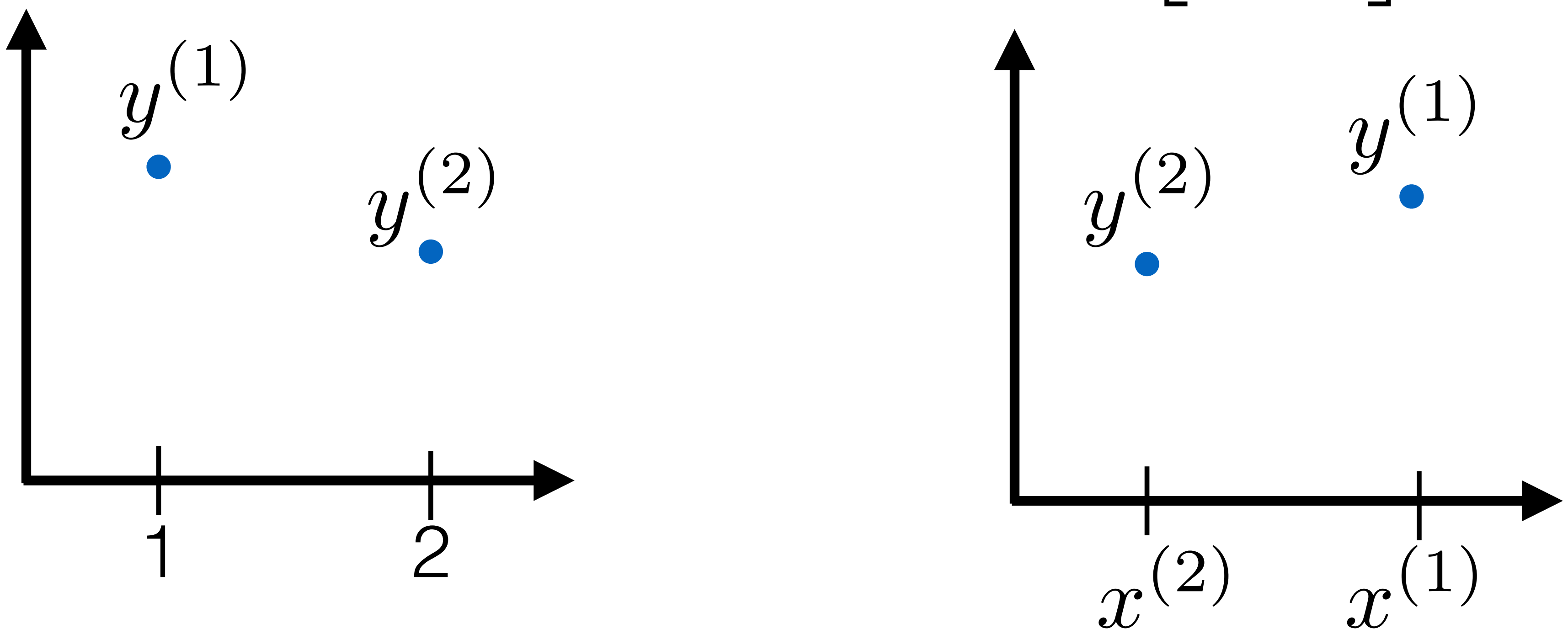

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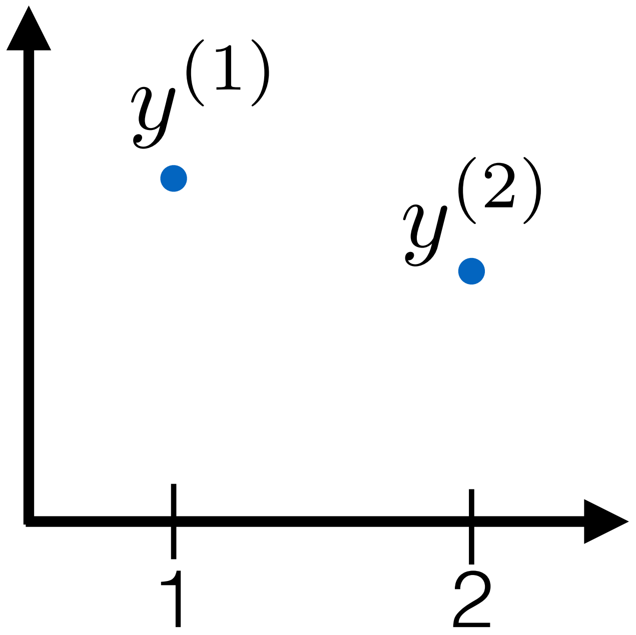

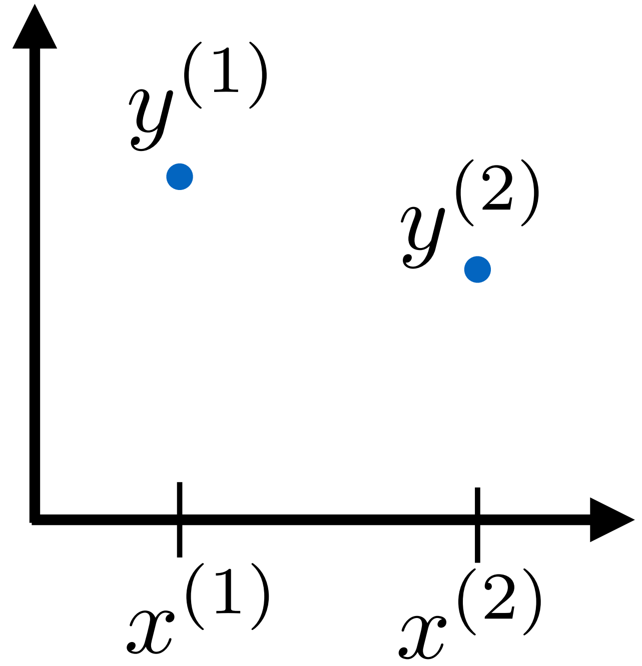

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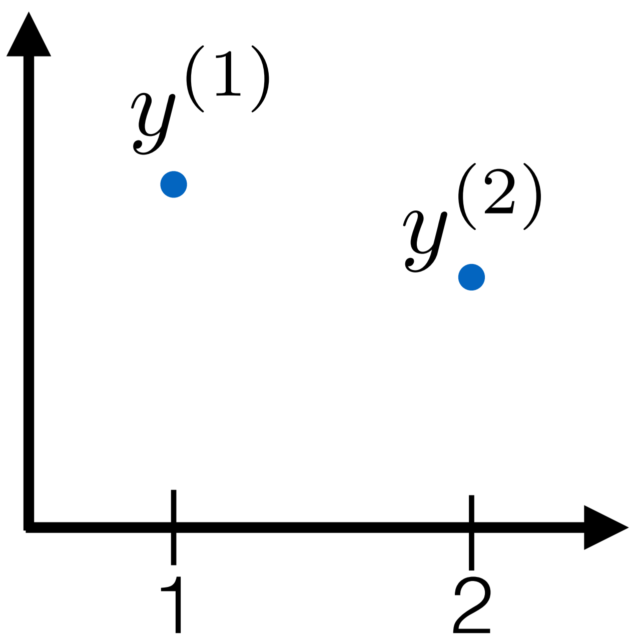

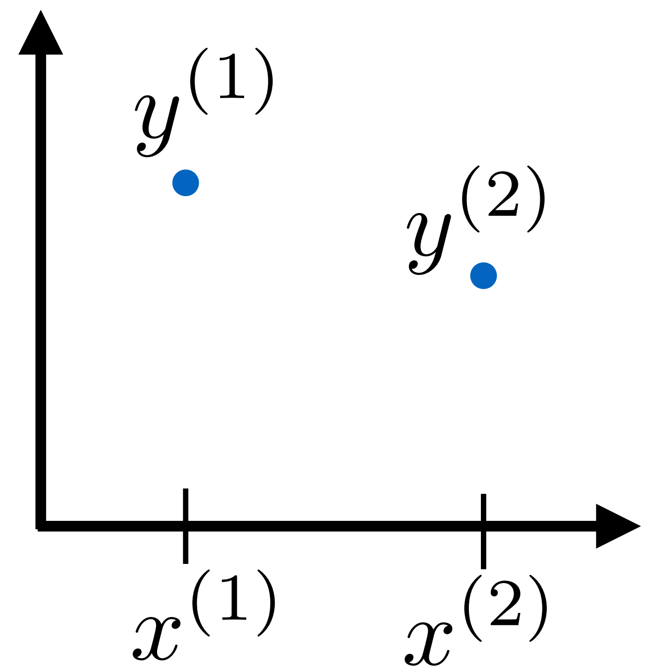

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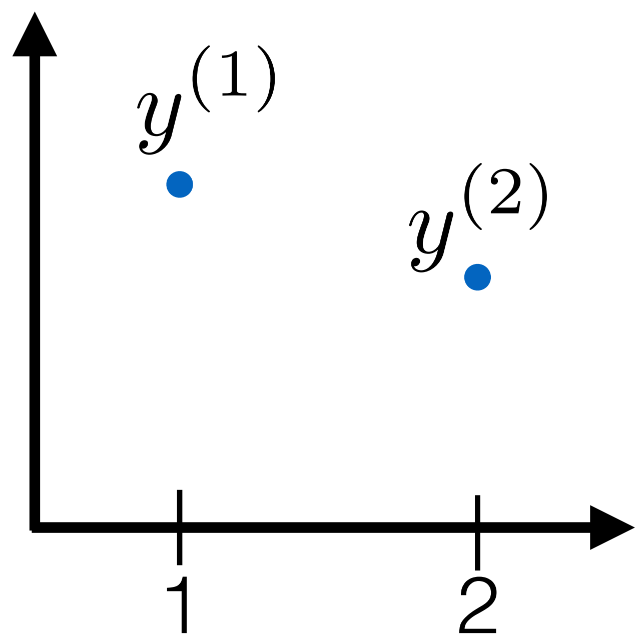

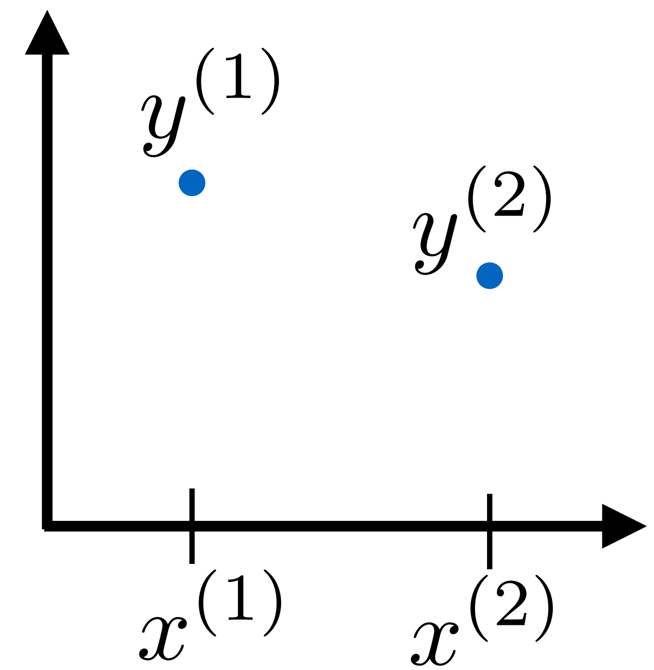

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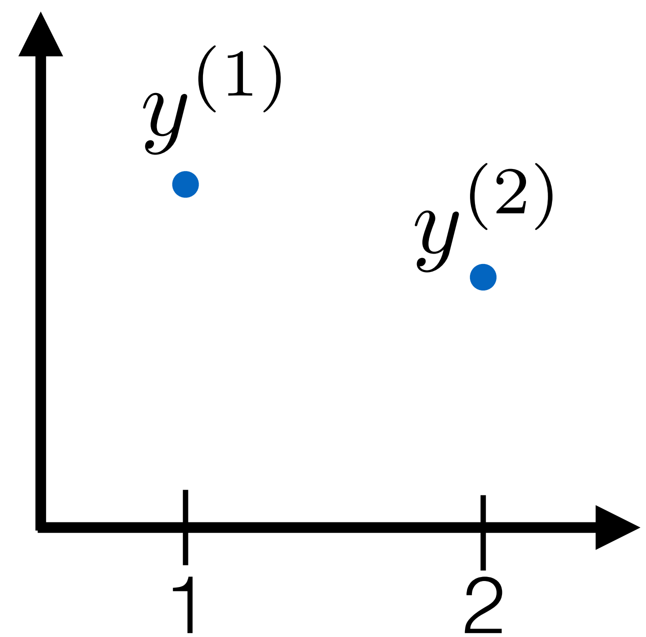

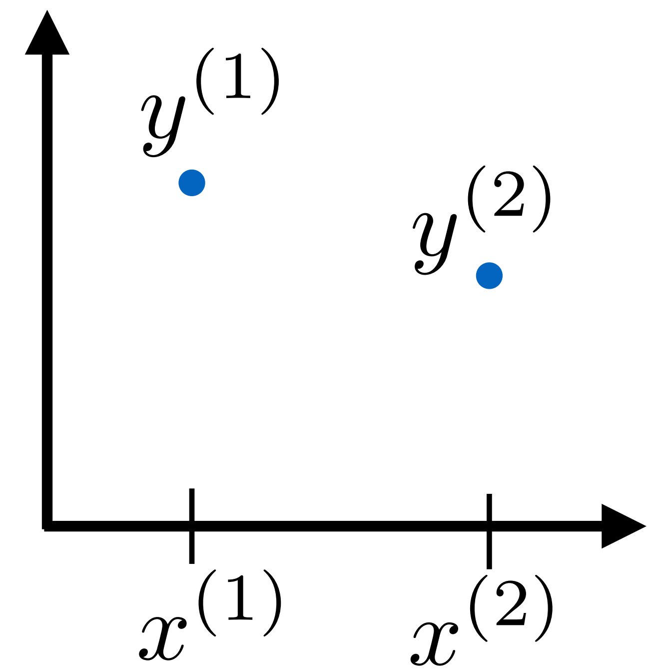

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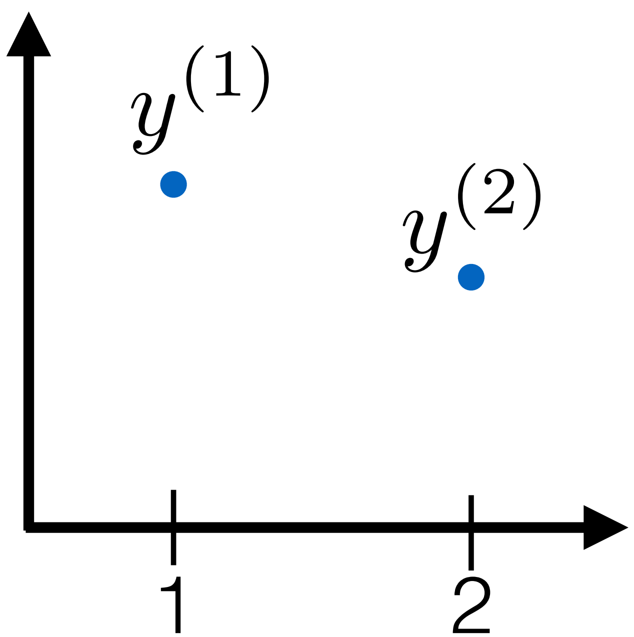

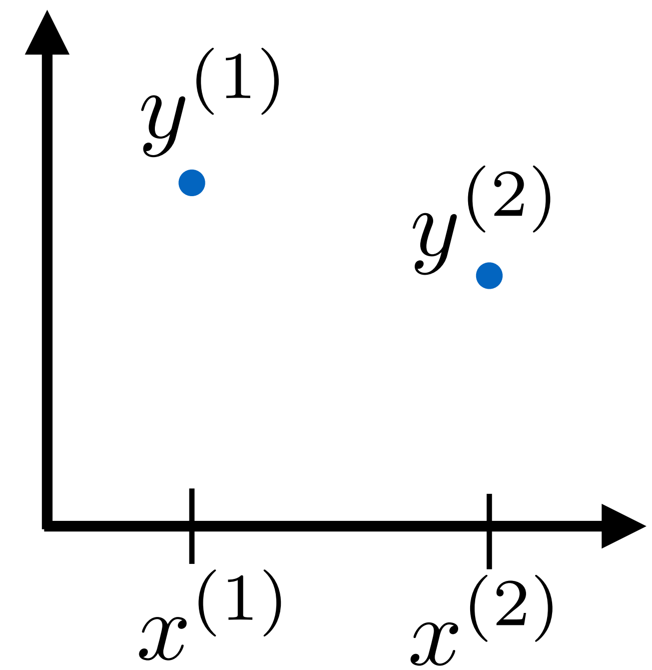

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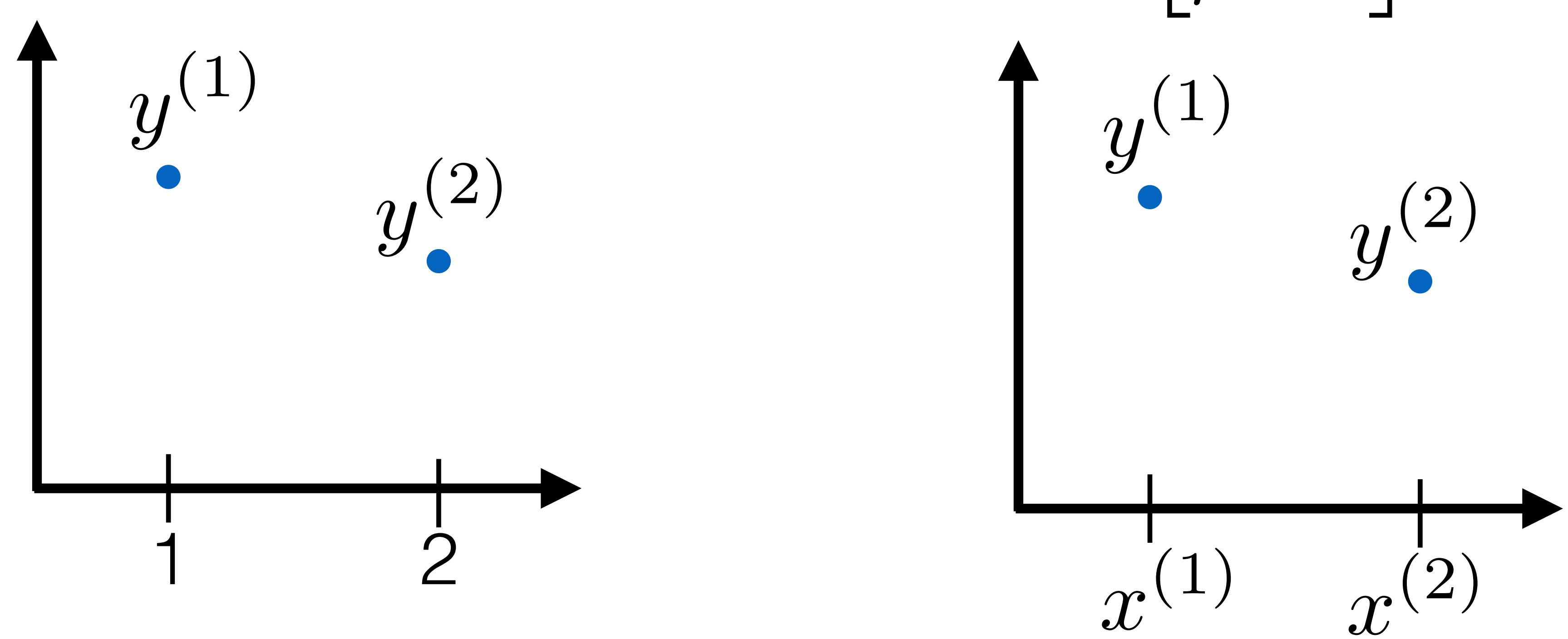

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[demo]

Multivariate Gaussian using locations

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[demo1, demo2]

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We just drew random functions from a type of Gaussian process that is very commonly used in practice!